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Nonspherical Perturbations of Relativistic Gravitational Collapse. I. Scalar and Gravitational Perturbations*

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When a nearly spherical star gravitationally collapses through its event horizon, it cannot leave behind a static gravitational field with nonspherical perturbations. The dynamics of these perturbations during collapse is studied with a scalar-field analog. Computations in comoving coordinates show that the field neither vanishes nor becomes singular as the star falls inside its gravitational radius. The scalar field on the surface of the star must vary as $a_1+a_2\exp(-t/2M)$ due to time dilation. An analysis is presented of the evolution of the exterior scalar field, based on a simple wave equation containing a space-time—curvature-induced potential barrier. This barrier is shown to be impenetrable to zero-frequency waves and thus a_1, the final value of the field on the surface of the star, is not manifested in the exterior; the exterior field vanishes. The detailed nature of the falloff of the field depends on backscattering off the potential. It is shown that an initially static l pole dies out as $t^{-(2l+2)}$. If there is no initial l pole but one develops during the collapse it must fall off as $t^{-(2l+3)}$. Wave equations with curvature-induced potential barriers have been derived by Regge and Wheeler and by Zerilli for gravitational perturbations. With these equations the analysis of gravitational perturbations is precisely the same as for the scalar ones. In particular, gravitational multipole perturbations (with $l\geq 2$) fall off at large t as $t^{-(2l+2)}$ or $t^{-(2l+3)}$, depending on initial conditions. (In an accompanying paper it is shown that this result applies as well to the radiatable multipoles of a zero-rest-mass field of any integer spin.)

I. INTRODUCTION

A. The Problem and Its History

A central role in relativistic astrophysics is played by the Schwarzschild geometry and by the line element

$$ds^2=(1-2M/r)dt^2 - (1-2M/r)^{-1}dr^2-r^2(d\theta^2+\sin^2\theta\, d\phi^2). \quad (1)$$

(We use units in which $c=1, G=1$.) The most interesting and characteristic feature of this line element is its singular behavior at the gravitational radius $r=2M$. On the one hand, we know that the $r=2M$ surface does have very important properties; it is an event horizon and the limit of a family of trapped surfaces. But on the other hand, transforming this line element to a freely falling coordinate system reveals that there are no local pathologies at $r=2M$; the geometry of space-time is quite smooth there.

The most important astrophysical consequence of the properties of the $r=2M$ surface is the inevitability of the catastrophic collapse of a star, once it is inside its gravitational radius. The absence of geometric pathologies at $r=2M$ in the Schwarzschild geometry implies that no anomalously large forces should develop in the star to prevent it from falling inside its gravitational radius. This expectation has been fully confirmed by several calculations.[1,2]

Whether or not catastrophic collapse can be considered as a possible phenomenon for real astrophysical objects depends on the resolution of a recent controversy: Is our picture of gravitational collapse an idiosyncracy of perfect spherical symmetry? The correctness of our qualitative picture is supported by the argument[3,4] that initial aspherical perturbations of a body should remain small during collapse through the gravitational radius, since there are no strong tidal forces there. If the perturbations of the body remain small, then the perturbations of the geometry, and of the whole collapse process, should also remain small.

Because of the nature of the event horizon, we should then expect the following: (i) The gravitational field outside the event horizon should be asymptotically stationary at large t. (ii) At large t, a distant observer "sees" the star as it is at the moment it crosses the event horizon. We expect, therefore, that the geometry left behind is a stationary geometry with aspherical perturbations. It has been shown, however, that such stationary perturbations cannot be well behaved at the event horizon and at spatial infinity.[4] This indicates that, for our picture to be correct, the star must rid itself of all bumps before falling through $r=2M$. But if that is true in all cases, there would have to be pathologically large forces at the event horizon, contrary to our expectations.

These difficulties have encouraged the viewpoint that the $r = 2M$ surface *does* have important *local* properties. Arguments have been given[5–8] to show that initially small perturbations become large without bound, stopping the collapse or destroying the event horizon. All these arguments have relied heavily on speculations regarding stationary solutions.

The opposite viewpoint was first championed by Doroshkevitch, Zel'dovich, and Novikov[4] and by Novikov.[9] The most conclusive evidence that has been given for this viewpoint, that collapse with perturbations is qualitatively like collapse without them, is the work of de la Cruz, Chase, and Israel.[10] They have numerically followed the electromagnetic and gravitational perturbations outside a perturbed collapsing thin shell. Their computations show that no singularity develops to halt the collapse and that the perturbations in the exterior fields die out at large times. It is the goal of the present work to analyze the evolution of perturbation fields in somewhat greater generality and to explain, in physical terms, how singularities are avoided.[11]

B. Outline and Conclusions

In this paper we use a first-order perturbation analysis to see whether initially small asymmetries can greatly affect the collapse process. This approach is quite sufficient to resolve the problem. If, on the one hand, the perturbations grow without bound our results will be meaningless but we will be able to conclude that our present picture of gravitational collapse is wrong. If, on the other hand, the perturbations remain small, the approach is justified. Since the paradox of singular stationary perturbations occurs for first-order perturbations, as well as in the full theory, then if the asymmetries do remain small, we shall be able to see how the paradox is avoided.

In principle the problem is the straightforward one of putting perturbations on a collapsing star just as, for example, Thorne and Campolattaro[12] put them on a static star. In practice the complications of coordinate system, gauge freedom, and many metric components to keep track of make such an approach discouragingly difficult.

There is good reason to suspect that the paradox is a result of properties of the event horizon, and that it should occur for many kinds of perturbations – not only gravitational perturbations. In fact, it is known that these same difficulties arise for electromagnetic perturbations,[13,14] for other integer-spin massless fields,[15] and for scalar fields.[16] In most of this paper we exploit the simplicity of a scalar-field analog.

Section II contains the formulation of such a massless-scalar-field analog and shows that the static perturbations are singular. A modification of this scalar field to a Klein-Gordon field gives some interesting insights into the nature of the singularities.

Our investigation of the scalar field is divided into two main parts: (i) the "local problem," i.e., the study of the behavior of the scalar field in and near a star that collapses from an initially static configuration, containing a source for the scalar field; (ii) the evolution of the scalar field in the Schwarzschild exterior. In Sec. II the local problem is analyzed by a detailed calculation, using comoving coordinates, of a physically reasonable collapse situation. The resulting dynamic equations give no indication that $r = 2M$ has any special local significance, for the evolution of the field. Numerical integrations of those equations confirm this; the scalar field in the star remains finite as the star falls through its gravitational radius.

Section III deals with the second part of the problem, the field in the exterior and the resolution of the paradox. It is shown that a description of the dynamics using the Schwarzschild time t, and the r^* coordinate of Regge and Wheeler,

$$r^* \equiv r + 2M\ln(r/2M - 1) + \text{constant},$$

leads to a simple picture of the propagation of scalar waves in the Schwarzschild geometry. In this picture the curvature of space-time gives rise to a potential barrier which is transparent to high-frequency waves but impenetrable to those of zero frequency. It is precisely this impenetrability which gives rise to the paradox and which resolves it.

The resolution of the paradox is simply this: The field on the surface of the star can be considered a source for the field in the exterior. Due to time dilation between the surface and distant observers, the field on the surface must be asymptotically stationary in terms of Schwarzschild time. The field on the surface then approaches some stationary final value, *but this final value cannot be manifested in the exterior solution.* The curvature potential prevents a distant observer from ever seeing it. For large time the exterior field is then sourceless and the field radiates itself away, vanishing at $t \to \infty$.

The simple nature of the process of the field radiating itself away is somewhat obscured by the complicated details of the curvature potential, so these ideas are presented first for a very idealized model barrier. This idealization permits exact calculations and results in an exterior field that vanishes exponentially in time, at large time. For the nonidealized problem, the outgoing wave front signalling the onset of collapse is partially backscattered. The resulting ingoing radiation prevents a quick exponential falloff of the field. The rate of falloff depends on the details of the wave front, and hence on the initial conditions on the perturbation field. If a static l-pole field is present outside the star, prior to the onset of collapse, the field will fall off as $t^{-(2l+2)}$. If there is no field initially outside the star, but an l-pole perturbation develops during the collapse process, it will fall off as $t^{-(2l+3)}$ at large time.

The final justification of the scalar analog is given in Sec. IV. Curvature-type potential equations have been derived by Regge and Wheeler[17] for odd-parity gravitational perturbations, and by Zerilli[18] for the even-parity ones. The difference between these gravitational equations and our scalar equation is only in the details of the potential. In Sec. IV, the Regge-Wheeler and Zerilli equations are discussed and it is shown that their solutions at large times are precisely the same as those of the scalar field equation. In particular, radiatable gravitational multipoles avoid the singularities of the static solution by vanishing as $t^{-(2l+2)}$ or $t^{-(2l+3)}$ just as their scalar counterparts do. The motivation for studying the scalar problem is, therefore, much greater than if the scalar field were only a plausible analog.

Certain details of Sec. IV are left to an accompanying paper (hereafter referred to as Paper II). In that paper it is also shown that radiatable multipoles of any integer-spin field satisfy a curvature-potential type equation, and fall off as $t^{-(2l+2)}$ or $t^{-(2l+3)}$.

[An earlier form of this paper, which was circulated as a preprint, contained an important error. The wave-front expansion, the details of which are given in the Appendix, was done incorrectly, giving a wrong expression for the ingoing backscatter. As a consequence, it was claimed that initially static l-poles fall off as $(\ln t)/t^{2l+3}$ rather than as $t^{-(2l+2)}$.]

II. THE SCALAR ANALOG

A. The Paradox

The scalar analog will consist of the following. We imagine a scalar field Φ, coupled to a scalar charge density j with some coupling constant κ, and obeying a wave equation

$$\Phi^{;\nu}{}_{;\nu} = -\kappa j\,. \tag{2}$$

There are other possible choices for the wave equation; at the end of this section we will consider others and see that our results are the same for any reasonable choice. The curvature of the geometry appears in the Christoffel symbols used to form the covariant derivatives in (2). We expect a contribution to the curvature due to the stress-energy of the scalar field such as

$$T_{\mu\nu} = \Phi_{,\mu}\Phi_{,\nu} - \tfrac{1}{2} g_{\mu\nu}\Phi_{,\alpha}\Phi^{,\alpha}.$$

The great advantage of studying a nongravitational field is that we can ignore the contribution of the field energy to the geometry; *throughout this section we use the unperturbed Schwarzschild geometry*. This is easily justified since we can imagine a limiting process in which j is scaled by a small number ϵ, then $\Phi \sim \epsilon$ and perturbations of the geometry $\sim\epsilon^2$. Of course, it may be that Φ or its gradient (in nonpathological coordinates) become large without bound at some point, in which case we must abandon the perturbation scheme.

The situation we consider is that of a star whose matter is scalar-charged. On an initial Cauchy hypersurface, on which the star is still outside its gravitational field, the star is a source of an exterior scalar field. The first, most critical, question we must ask is: Can the star collapse leaving an asymptotically static scalar field behind, or as in the case of gravitational and electromagnetic multipoles, must the scalar field either radiate away or greatly modify the collapse?

If the star collapses, it leaves behind the familiar Schwarzschild geometry described by line element (1). If a static[19] Φ field is left behind by the collapse, it must satisfy

$$-\Phi_{,\nu}{}^{;\nu} = \left(1-\frac{2M}{r}\right)\frac{d^2\Phi}{dr^2} + \left(\frac{2}{r}-\frac{2M}{r^2}\right)\frac{d\Phi}{dr} - l(l+1)\frac{\Phi}{r^2} = 0 \tag{3}$$

for an l pole. To analyze[20] this we introduce the convenient r^* radial coordinate of Regge and Wheeler,[17]

$$r^* = r + 2M\ln(r/2M-1) + \text{constant}. \tag{4}$$

Note that $r \approx r^*$, for $r \gg M$, and that the event horizon $r = 2m$ is at $r^* = -\infty$. In terms of r^* derivatives, Eq. (3) becomes

$$\left(1-\frac{2M}{r}\right)^{-1}\frac{d^2\Phi}{dr^{*2}} + 2r^{-1}\frac{d\Phi}{dr^*} - l(l+1)\frac{\Phi}{r^2} = 0. \tag{5}$$

The asymptotic solutions at large r^* are the usual flat-space forms,

$$\Phi \sim r^{*l} \quad\text{or}\quad \Phi \sim r^{*-(l+1)} \quad\text{at } r^* = +\infty. \tag{6a}$$

And near the event horizon they are

$$\Phi \sim r^* \text{ or } \Phi \sim \text{constant at } r^* = -\infty. \tag{6b}$$

The solution $\Phi \sim r^*$ is unacceptable at $r^* = -\infty$. Specifically, the scalar field's stress-energy and its force on the charge carriers would be unbounded in a comoving frame. The solution $\Phi \sim r^{*l}$ at $r^* = +\infty$ is obviously pathological.

The question then is whether we can connect the well-behaved solutions at $r^* = -\infty$ and $r^* = +\infty$. If we are to connect a constant at one end to a decreasing solution at the other there must be a point of inflection ($d^2\Phi/dr^{*2} = 0$) at which the signs of Φ and $d\Phi/dr^*$ are opposite; this is clearly incompatible with (5). (This analysis is patterned after that of Vishveshwara.[21])

The monopole case, which is just as important here as the higher-multipole cases, since the scalar field can radiate in an $l=0$ mode, is somewhat special in that both solutions at $r^* = +\infty$ are well behaved. For this case, in fact, we have the simple exact solutions

$$\Phi = \ln(1 - 2M/r) \quad \text{or } \Phi = \text{constant}. \tag{7}$$

The solution Φ = constant is trivial and the $\ln(1 - 2M/r)$ solution has the expected $\Phi \sim r^*$ behavior at $r^* = -\infty$.[16]

If the paradox is indeed a manifestation of the special nature of the event horizon, the precise form of the wave equation should not be too important. In flat space-time we consider the generalization of the free-field equation

$$\Box\Phi + C(x^\mu)\Phi = 0, \tag{8a}$$

where

$$\Box \equiv \partial_t^{\,2} - \nabla^2. \tag{8b}$$

In the absence of other fields, translational invariance demands that C be constant and we have the usual Klein-Gordon equation. When we make the usual replacement of ordinary partial derivatives by covariant derivatives we find that in (5) the coefficient of Φ is now

$$-[r^{-2}l(l+1) + C(x^\mu)]. \tag{9}$$

The emergence of the paradox depends on the sign of C. Usually C in the Klein-Gordon equation is taken to be m^2, where m is the mass of the particle mediating the field, in this case our derivation of the necessary singularities of the static solution is unaffected.

It is intriguing that imaginary-mass particles have nonsingular static solutions, since imaginary masses are sometimes associated with faster-than-light motion. More specifically if $C(\chi^\mu)$ is negative in some region of space-time, high-frequency wave packets have group velocities greater than c in that region. It is just such faster-than-light effects that might be expected to rob the $r = 2M$ surface of its properties as a one-way membrane for information propagation.

One other point must be mentioned. In generalizing the wave equation from the laboratory to curved space-time, it is possible that other curvature effects come in, in addition to the covariant derivatives. In particular, those who consider conformal invariance to be compelling would write the free-field equation as

$$\Phi_{;\nu}{}^{;\nu} + \tfrac{1}{6}R\Phi = 0.$$

Since $R=0$ in the vacuum exterior, this modification is of no concern.[22]

B. The Local Problem

Our approach to the problem of the scalar field's evolution can conveniently be divided into two parts. In the first, the "local problem," the evolution of the field in the star and on its surface is followed, up to the point at which the surface passes through the event horizon and is causally disconnected from external observers. The results are then used as an input for the second part: the evolution of the exterior field. The local problem is also important because it resolves the question of whether perturbations remain small, and whether a first-order perturbation calculation is sufficient.

Some important work has already been done on the local problem for gravitational perturbations: the computer integrations by de la Cruz, Chase, and Israel[10] and the analysis by Novikov.[9] In view of the uncertainty still surrounding the question of the behavior of fields at the event horizon, it was deemed useful to follow the evolution of the field in the local problem numerically, with a computer.

The problem is set up in a way that allows an unambiguous interpretation of the results. The background problem is the collapse of a momentarily static uniform pressureless star first described by Oppenheimer and Snyder.[1,23] On the initial $t=0$ Cauchy hypersurface the Φ field is chosen to be static ($d\Phi/dt = 0$, $d^2\Phi/dt^2 = 0$) in the exterior; a stationary observer sees this field remain static until information about the collapse reaches him.

The Friedmann line element,

$$ds^2 = d\tau^2 - a^2(\eta)[d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\; d\phi^2)], \tag{10a}$$

with

$$a(\eta) = \tfrac{1}{2}a_0(1 + \cos\eta), \tag{10b}$$

$$\tau = \tfrac{1}{2}a_0(\eta + \sin\eta), \tag{10c}$$

describes the geometry of the interior of an Oppenheimer-Snyder star of density

$$\rho = 3a_0/8\pi a^3(\eta). \tag{11}$$

If the maximum χ (i.e., that for the stellar surface) is χ_0, then the mass and radius of the star's surface are

$$M = \tfrac{1}{2}a_0 \sin^3\chi_0, \tag{12}$$

$$r_{\mathscr{s}} = a(\eta)\sin\chi_0. \tag{13}$$

At $\eta = 0$ the star is momentarily static and is about to begin its free-fall collapse.

The geometry outside the star is that of Schwarzschild, but we must avoid Schwarzschild's coordinates because of their poor description of the region $r = 2M$. Instead, we choose "comoving," i.e., "synchronous," coordinates. For a vacuum this means a system in which points with fixed spatial coordinate values move on timelike geodesics, and for which the time coordinate is the proper time along these geodesics.

The general comoving spherically symmetric, vacuum line element[24] is

$$ds^2 = dT^2 - \frac{(\partial r/\partial R)^2}{1+2E(R)}\, dR^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{14a}$$

where $r(R, T)$ is derived as the solution of

$$\frac{\partial r}{\partial T} = -\left[\frac{2M}{r} + 2E(R)\right]^{1/2}. \tag{14b}$$

There are two arbitrary functions here, $E(R)$ and $r(R, T=0)$, corresponding to our initial choice of velocity for our observers and to the initial scale for R. By choosing $r(R, T=0) = R$, we give the R coordinates the physical interpretation of the initial radius (i.e., Schwarzschild radial coordinate) from which an observer starts falling. We must, of course, cut the geometry off at the surface of the star, which is initially $R = R_0 = r_{\text{initial}} = a_0 \sin\chi_0$. We choose $E(R) = -M/R$ so that our observers are all initially static.

Since $(\partial r/\partial T)_{R=R_{sf}} = 0$ and $(\partial r/\partial \tau)_{r=r_{sf}} = 0$ the world lines of the shells $r = r_{sf}$ and $R = R_{sf}$ are initially tangent and they are both geodesics. Thus, a consequence of our choice of $E(R)$ is that the star's surface always remains at $R = R_0$, and since both τ and T are proper time on this geodesic, the boundary between the interior and the exterior, we hereafter use only the symbol τ as the time coordinate in both regions.

The background coordinates are pictured in Fig. 1 where we use, as in calculations to follow, the specific choices $\chi_0 = \frac{1}{4}\pi$ and $a_0 = 4\sqrt{2}M$, so that the star starts collapsing from $r_{sf} = 4M$. The function $r(R, \tau)$ is transcendental but is smooth, having no pathologies where the stellar surface crosses the event horizon. For our choice of $E(R)$, $r(R, 0)$, χ_0, and a_0 it is approximately

$$r(R, \tau) \approx \frac{R}{1 + 0.7(2M)\tau^2/R^3} \tag{15}$$

throughout the region of interest of the variables R, τ.

In the interior we must evaluate Eq. (2) in terms of the coordinates of line element (10). For an l-pole field the result is

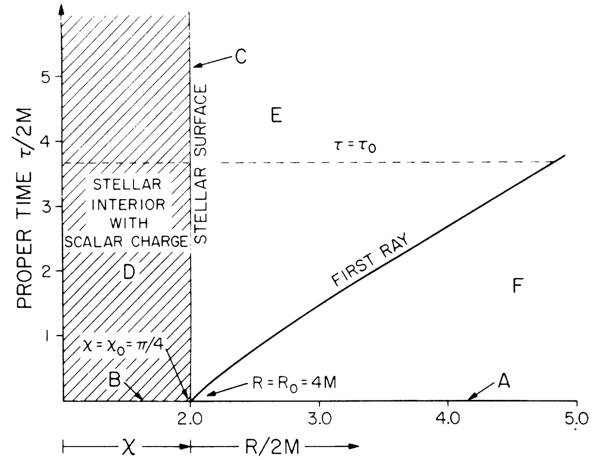

FIG. 1. The "local problem" pictured in the comoving coordinates τ, R, and χ. At $\tau = \tau_0 = 2\sqrt{2}(1+\frac{1}{2}\pi)M$ the stellar surface passes through the event horizon. The details of the local calculation are explained in Table I.

$$\frac{[a^2(\eta)\Phi_{,\eta}]_{,\eta}}{a^2(\eta)} - \frac{(\sin^2\chi\,\Phi_{,\chi})_{,\chi}}{\sin^2\chi} + \frac{l(l+1)\Phi}{\sin^2\chi} = -\kappa a^2(\eta) j. \tag{16}$$

In the exterior ($R \geq R_{sf}$), in terms of the comoving coordinates of (14), Eq. (2) becomes

$$(r^2 r' \Phi_{,\tau})_{,\tau} - (1 - 2M/R)^{1/2}\{[r^2(1-2M/R)^{1/2}/r']\Phi_{,R}\}_{,R} + r' l(l+1)\Psi = 0, \tag{17a}$$

where

$$r' = \partial r/\partial R. \tag{17b}$$

The matching condition for Φ at the boundary $\chi = \chi_0$, $R = R_{sf}$ is that the derivative of Φ, with respect to proper distance normal to the boundary, is continuous and that Φ itself is continuous. (This can be shown by using Gauss's theorem on a slab-like volume including the boundary.) It should be noticed that the system of Eqs. (16) and (17) and the matching condition in no way single out $r = 2M$ as a special surface. Viewing the local problem in these mathematical terms, we should be very surprised if a singularity develops there.

To solve the dynamical problem we also need to know the motion of the scalar charge carriers. This motion is freely specifiable since we are ignoring the forces due to the scalar field. A natural choice is to have each dust particle in the star carry a fixed charge; this fixes the time dependence of j at any one value of χ, θ, ϕ:

$$j(\tau) \propto \rho(\tau) \propto a^{-3}(\eta).$$

The form of $j(\tau, \chi, \theta, \phi)$ depends how the charge per particle varies from point to point in the star.

We choose the radial dependence so that j vanishes smoothly at the surface, and we choose the angular dependence to be a spherical harmonic, e.g.,

$$j(\tau, \chi, \theta, \phi) = \epsilon[a_0/a^3(\eta)](1-\chi^2/\chi_0^2)Y_l^m(\theta, \phi). \tag{18}$$

Here ϵ is an expansion parameter which is chosen small enough so that we can ignore scalar-field forces and stress-energy. The coupling constant κ in (2) is taken to be $(2M)^2$ so that Φ is dimensionless.

It remains only to specify Φ and its time derivative initially. Outside the star we take Φ to be static, that is, a solution of (3) for $\Phi(R)$. These solutions characteristically go as $1/R^{l+1}$ at large R. In the interior, where there is not so natural a choice for Φ, we choose it such that $\Phi_{,\tau}$ and $\Phi_{,\tau\tau}$ vanish inside. The initial interior field will then be a superposition of the particle solution – depending on the scalar charge distribution – and the homogeneous solution of (16):

$$\Phi_{\text{interior}} = \Phi_{\text{particular}} + A\Phi_{\text{homogeneous}} . \tag{19a}$$

The exterior field will be

$$\Phi_{\text{exterior}} = B\Phi_{\text{static}} , \tag{19b}$$

and the constants A and B must be determined so as to make the initial data satisfy the junction con-

TABLE I. Details of the local calculation.

Feature of the problem	Region (see Fig. 1)	Equations
1. Geometry	Interior D	Friedmann line element. See Eq. (10).
	Exterior E, F	Schwarzschild geometry described in comoving coordinates. See Eq. (14).
2. Scalar charge density	Interior D	(i) $l=0$, $\kappa j = -[a_0/a^3(\eta)](\sqrt{2}\ 105/8)\cos\chi\cos^2 2\chi$. [This choice of j gives a simple particular solution for Φ in Eq. (16), at $\eta=0$.] (ii) $l=3$, $\kappa j = -[a_0/a^3(\eta)](1-\chi^2/\chi_0^2)Y_3^0(\theta,\phi)$.
	Exterior E, F	$j=0$.
3. Equations for the evolution of Φ	Exterior E, F	$[a^2(\eta)\Phi_{,\eta}]_{,\eta} - (\sin^2\chi\Phi_{,\chi})_{,\chi}/\sin^2\chi + l(l+1)\Phi/\sin^2\chi = \begin{cases} (\sqrt{2}\cdot 105/8)[a_0/a(\eta)]\cos\chi\cos^2 2\chi & \text{for } l=0, \\ [a^0/a(\eta)](1-\chi^2/\chi_0^2)Y_3^0(\theta,\phi) & \text{for } l=3. \end{cases}$
	Dynamic exterior E	Equation (16), with $j=0$.
	Static exterior F	Φ remains static; it is a function of r only and is given by the initial conditions in feature 5 of this table.
4. Junction conditions of the star's surface	Stellar surface C	Φ is continuous and $\vec{n}\cdot\vec{\nabla}\Phi$ is continuous, i.e., $\Phi_{,\chi}/a(\eta) = \Phi_{,R}/\sqrt{2}(\partial r/\partial R)$. (Here $\vec{n}$ is a unit normal to the world line of the stellar surface.)
5. Initial conditions	Initial hypersurface (exterior) A	The initial field is static outside the star ($\Phi_{,t}=0$); hence Φ is given by (i) $l=0$, $\Phi = B_0\ln(1-2M/R)$, $B_0=1$, (ii) $l=3$, $\Phi = B_3\Phi_{\text{stat}}$, where Φ_{stat} is a solution to Eq. (3) and $\Phi_{\text{stat}} \to 1/R^4$ at large R. $B_3 = -0.0144$.
	Initial hypersurface (interior) B	Φ is a solution to the equations in 3 of this table at $\eta=0$. We choose Φ such that $\Phi_{,\eta}=0$ and $\Phi_{,\eta\eta}=0$ initially. Then (i) $l=0$, $\Phi = A_0+\Phi_p$, where $\Phi_p = \sqrt{2}/8\cos\chi(-11+\sin^2 2\chi)$, $A_0 = 1-\ln 2$, (ii) $l=3$, $\Phi = A_3\Phi_h + \Phi_p$, where Φ_p is the particular solution of $(\sin^2\chi\Phi_{,\chi})_{,\chi}/\sin^2\chi - 12\Phi/\sin^2\chi = Y_3^0(\Phi,\phi)(1-\chi^2/\chi_0^2)$ which goes as $\frac{1}{6}\chi^2 Y_3^0$ near $\chi=0$; Φ_h is the homogeneous solution that goes as $\chi^3 Y_3^0$ near $\chi=0$. $A_3 \approx 0.394$.
6. Results for Φ field	D, E, F	See Fig. 2 for numerical results.
7. Numerical constants		χ_0 is chosen as $\frac{1}{4}\pi$ for convenience. The star collapses from initial radius $r=4M$, so that $a_0 = 4\sqrt{2}M$. The surface passes through its event horizon at $\eta = \frac{1}{2}\pi$, $\tau/2M = \sqrt{2}(1+\frac{1}{2}\pi) \approx 3.64$.

ditions at the stellar surface.

The details of the calculation of the evolution of Φ are summarized in Table I and the results of numerical computations are shown in Fig. 2.

In both the $l=0$ and $l=3$ case, the field increases as the star grows smaller, but in neither case is there any strong local effect to distinguish the point at which the star's surface crosses the event horizon. At this moment of crossing (at $\eta=\frac{1}{2}\pi$) both Φ and its derivatives (in comoving coordinates) are neither zero nor infinity. *The scalar perturbations do remain small.*

III. EVOLUTION OF THE SCALAR FIELD

A. The Curvature Potential; Characteristic Data

Having established that no catastrophic local phenomenon will interrupt the collapse, and having calculated data on the stellar surface, we now turn our attention to the evolution of the exterior field. In particular, we are interested in seeing how the asymptotic field at $t\to\infty$ avoids the singularity of a static solution.

The comoving coordinates which are so useful in studying the local problem are poorly suited to the radiation problem. To understand the nature of the radiation in the exterior we must use a reference system related to the static nature of the background – so our system must be stationary with respect to the Schwarzschild system. The line element is then (1) or if we use the r^* coordinate defined in (4), the line element is

$$ds^2=(1-2M/r)(dt^2-dr^{*2})-r^2(d\theta^2+\sin^2\theta\, d\phi^2), \tag{20}$$

where $r(r^*)$ is a solution of the implicit Eq. (4). Note that the null radial lines in the geometry are $dr^*=\pm dt$.

Assuming that Φ has the angular dependence of a spherical harmonic, the equation governing its evolution is

$$\begin{aligned}\Phi_{;\nu}{}^{;\nu}&=(1-2M/r)^{-1}(\Phi_{,tt}-\Phi_{,r^*r^*})\\&\quad-2r^{-1}\Phi_{,r^*}+l(l+1)r^{-2}\Phi\\&=0.\end{aligned} \tag{21}$$

If we introduce a modified field variable $\Psi=r\Phi$, then (21) takes the very simple form

$$\Psi_{,tt}-\Psi_{,r^*r^*}+F_l^{sc}(r^*)\Psi=0, \tag{22a}$$

with

$$F_l^{sc}(r^*)=(1-2M/r)[2M/r^3+l(l+1)/r^2]. \tag{22b}$$

The function $F_l^{sc}(r^*)$ which is very important to our analysis, is peaked strongly around small absolute values of r^*. (See Fig. 3.) Its asymptotic forms are

$$F_l^{sc}(r^*)\approx\begin{cases}\dfrac{l(l+1)}{r^{*2}}+\dfrac{4Ml(l+1)\ln(r^*/2M)}{r^{*2}} & \text{if } l\neq 0\\ 2M/r^{*3} & \text{if } l=0\end{cases}\Bigg\}\; r^*\gg M \\ \qquad\quad \dfrac{l(l+1)+1}{(2M)^2}\exp[(r^*/2M)+1], \quad r^*\ll -M. \tag{23}$$

[Here the constant in (4) has been chosen such that $r^*=0$ at $r=4M$.] The shape of $F_l^{sc}(r^*)$ is shown in Fig. 3.

The wave Eq. (22) can give us a simple picture of the nature of the radiation problem. If $F_l^{sc}(r^*)$ were simply the centrifugal barrier, $l(l+1)/r^{*2}$ in the region $r^*>0$, then waves of the scalar field would propagate freely; there would be no gravitational effects on them. The existence of the region $r^*<0$ and the fact that $F_l^{sc}(r^*)$ is *not* $l(l+1)/r^{*2}$ for $r^*>0$ are due to the curvature of space-time. To isolate curvature effects from the effects due to spherical coordinates (i.e., the centrifugal barrier), we can subtract $l(l+1)/r^{*2}$ from $F_l^{sc}(r^*)$ for r^* greater than, say, $20M$. The part of $F_l^{sc}(r^*)$ which remains and which is due to curvature we shall call the *curvature potential*.

The useful and interesting property of the curvature potential is that it is a very localized barrier to scalar waves. It can be thought of as a barrier between a flat-space close zone adjacent to the stellar surface source, and a flat-space distant zone where we are most interested in the manifestations of this radiation.[25]

In this picture of the problem we will find the coordinates u and v, advanced and retarded time, to be useful. They are related to t and r^* by a $45°$ rotation:

$$u=t-r^*, \quad v=t+r^*. \tag{24}$$

They are important in that they are null coordinates; if it were not for the scattering by the potential, information would propagate along u, v coordinate lines without distortion.

The specification of the problem is complete when we give the initial conditions that Ψ is static at $\tau = 0$ (and hence on the "first ray"), and when we put in the values of Ψ and its normal derivative on the stellar surface, from Sec. II. The problem is presented in this form in Fig. 4 and Table II.

Consider the initial Cauchy data on the surface of the star. In Sec. II we saw that Ψ and $\partial\Psi/\partial R$ are well-behaved functions of comoving coordinates from the onset of collapse to the passage through the event horizon. The fact that the variation of Ψ is bounded on a curve of finite length in comoving coordinates means that its variations on the curve of infinite length in Fig. 4 must be very small, asymptotically zero in fact at $u \to \infty$. Mathematically we can show that Ψ approaches its final value according to

$$\Psi \to a + b\exp(-u/4M)$$
$$\text{as } t \to \infty \text{ (and } u \to \infty) \quad (a, b \text{ constants}). \tag{25}$$

This effect is in fact just the ordinary time-dilation phenomenon between a falling frame and a static one and does not depend in any way on the surface falling in on a geodesic.

These asymptotic properties can be established most easily by using Kruskal coordinates[26]

$$U = -4Me^{-u/4M}, \quad V = 4Me^{v/4M}. \tag{26}$$

Since U and V are well-behaved coordinates at $r = 2M$, the partial derivatives $\partial\Psi/\partial U$ and $\partial\Psi/\partial V$ should be finite. This implies that $\partial\Psi/\partial u$ must fall off sharply at the event horizon because

$$\frac{\partial\Psi}{\partial u} = \frac{\partial\Psi}{\partial U}\exp(-u/4M), \tag{27}$$

and $u \to \infty$ at $r = 2M$. The advanced time v is finite at $r = 2M$ (for an ingoing world line) so that $\partial\Psi/\partial v$ is finite. If we picture the path of the surface through space-time as $v(u)$ or $V(U)$, then near $r = 2M$ we have

$$\frac{dv}{du} = \frac{dV}{dU}\frac{dv}{dV}\Big/\frac{du}{dU} \sim \exp(-u/4M). \tag{28}$$

The world line of the surface, therefore, appears *in our coordinates* to be almost an ingoing null line, and we conclude that

$$\frac{d\Psi}{du} = \frac{\partial\Psi}{\partial v}\frac{dv}{du} + \frac{\partial\Psi}{\partial u} \sim \exp(-u/4M), \tag{29}$$

for large u. We shall see that the evolution of Ψ at large t depends only on the asymptotic, large u, behavior of Ψ on the surface.

For convenience, we will specify data on an in-

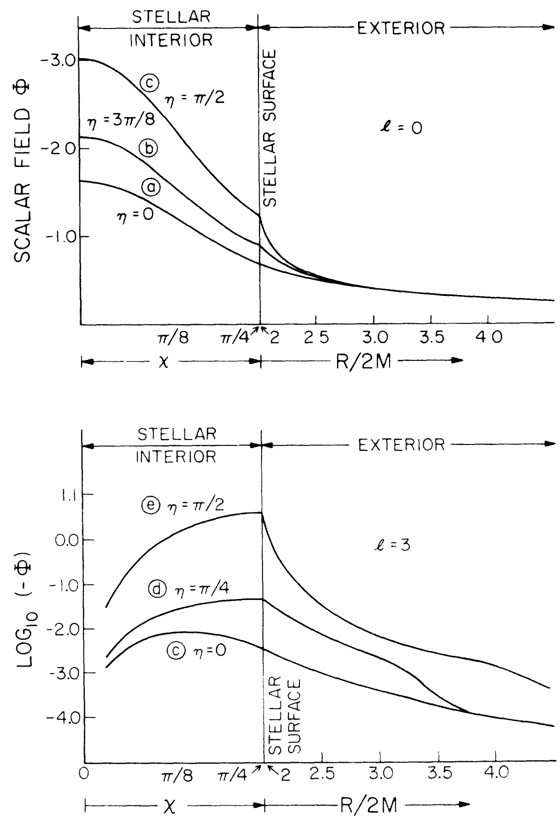

FIG. 2. The results of computer integrations for the evolution of Φ for $l=0$ and $l=3$. In the $l=3$ case the great variation of Φ from $\eta=0$ to $\eta=\frac{1}{2}\pi$ necessitates plotting the logarithm of Φ. In both the $l=0$ and $l=3$ plots, the scale of coordinates has been chosen to make the curves for $\eta=0$ appear smooth. For later times the radial derivatives appear discontinuous at the stellar surface. This is wholly due to the change in time of the radial coordinates. The derivative of Φ with respect to proper radial distance is always continuous. For curves a and c: $\eta=0$, $\tau=0$, and the values of Φ are the initial, static values. For curve b: $\eta=\frac{3}{8}\pi$, $\tau/2M=2.973$, and Φ is static for $R/2M>4.344$. For curve d: $\eta=\frac{1}{4}\pi$, $\tau/2M=2.111$, and Φ is static for $R/2M>3.635$. For curves c and e: $\eta=\frac{1}{2}\pi$, $\tau/2M=3.636$, and Φ is static for $R/2M>4.900$. For a description of initial data and further details, see Table I and the text.

going null line rather than on the stellar surface. If the v distance between these two curves is δv at some u, then the error in $\Psi_{,u}$ is approximately

$$\delta(\Psi_{,u}) \approx (\Psi_{,u})_{,v}\delta v = -\tfrac{1}{4}F_l^{sc}(r^*)\Psi\,\delta v$$
$$\propto \exp(-u/4M)\delta v. \tag{30}$$

So using the null line is justifiable in that it does not change the nature of the asymptotic data. (See Fig. 4.)

Summarizing then, we have reduced the physical problem to a mathematical problem in wave prop-

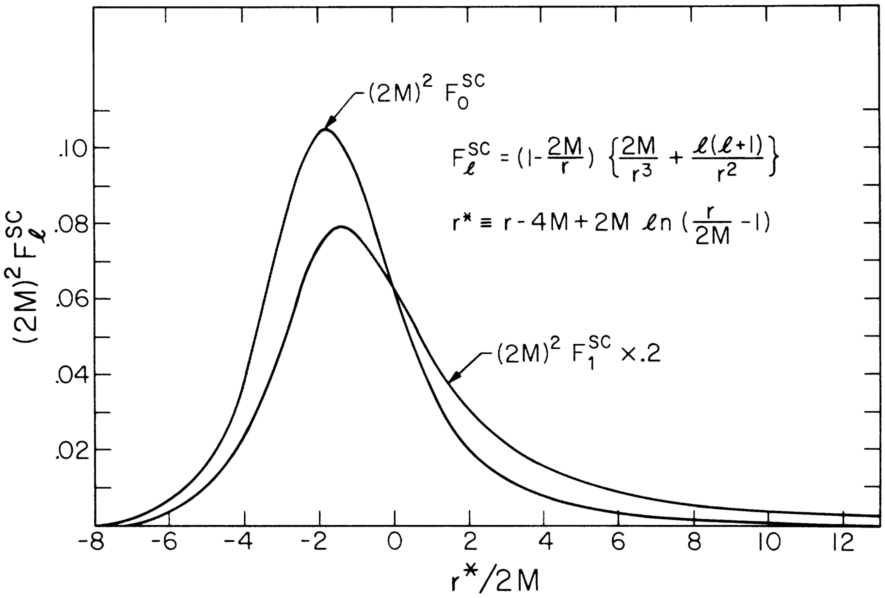

FIG. 3. The appearance of the peak of $F_l^{sc}(r^*)$ for $l=0$ and $l=1$. Here the constant in Eq. (4) has been chosen so that $r^*=r-4M+2M\ln(r/2M-1)$, and $r^*=0$ at $r=4M$, which is the radius from which the star starts to collapse in the calculations of Sec. III. Note that F_l^{sc} is sharply peaked in the neighborhood of $r^*=0$. In fact, the peak occurs at $r=\frac{8}{3}M$ for $l=0$ and for $l\neq 0$ at $r_{\rm peak}=2M\{3(L-1)+[9(L-1)^2+32L]^{1/2}\}/4L$ where $L\equiv l(l+1)$. For $l=1$, $r_{\rm peak}=2.88M$; for $l=2$, $r_{\rm peak}=2.95M$; for $l=3$, $r_{\rm peak}=2.97M$; for $l\to\infty$, $r_{\rm peak}\to 3M$.

agation, with data given on two characteristics[27]: the "first ray" $u=0$, and the "stellar surface" $v=v_0$. The partial differential equation is (21) or equivalently

$$\Psi_{,uv}+\tfrac{1}{4}F_l^{sc}(r^*)\Psi=0. \tag{31}$$

The form of the characteristic data is

$$\Psi(u,v=v_0)\to a+b\exp(-u/4M)\quad\text{at } u\gg M \tag{32a}$$

and

$$\Psi(u=0,v)\to \text{static solution}\to r^{*\,-l}\to v^{-l}\quad\text{at } v\gg M. \tag{32b}$$

B. An Idealized Potential

Before going on to look closely at the manner in which the fields evolve, it is interesting to look at a very idealized analog to our wave equation.

$$\Psi_{,tt}-\Psi_{,r^*r^*}+F_l(r^*)\Psi=0, \tag{33a}$$

where

$$F_l(r^*)=\begin{cases} l(l+1)/r^{*2} & \text{for } r^*\geq 1\\ 0 & \text{for } r^*<1.\end{cases} \tag{33b}$$

The input data as before will be on characteristics: an exponentially damped falloff at $v=v_0$, and first-ray data corresponding to an initially static solution. The extent to which we have eliminated some important physics with this idealization will become apparent presently.

Rather than dealing with a general l we shall specialize to $l=1$; the following calculation can be done in the same manner for any l. It is interesting that even in this simple model equation, we have the "paradox." The static solutions are

$$\Psi=\begin{cases} c_1+c_2r^*, & r^*<1\\ C_1/r^*+C_2r^{*2}, & r^*\geq 1.\end{cases} \tag{34}$$

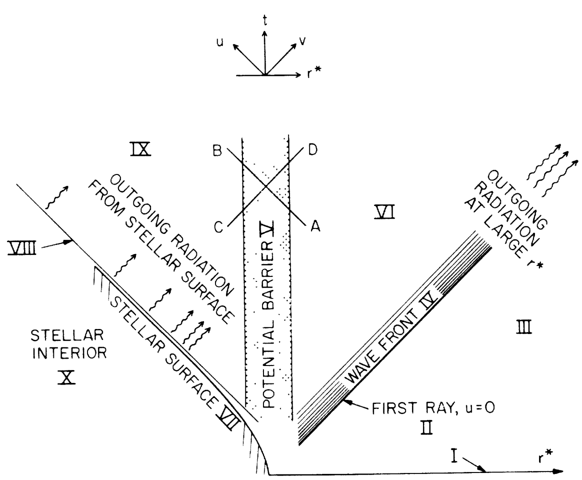

FIG. 4. The "radiation problem" pictured in r, t or u,v coordinates. For explanations and descriptions of features of this diagram, see Table II.

TABLE II. Regions of the radiation problem.

Region (see Fig. 4)	Description
I	The initial Cauchy hypersurface $t=0$, outside the star. On this hypersurface Φ is chosen to be static. For $l=0$, $\Phi=\text{const}\times\ln(1-2M/R)$. For $l\neq 0$, Φ is a solution to Eq. (3).
II	The first ray, $u=0$. This first outgoing scalar ray carries information to the exterior that the star has begun to collapse.
III	The static region. This region has not yet received information that the star has begun to collapse. See region F of Fig. 1.
IV	The wave front. Most of the high-frequency radiation from the stellar source moves on outgoing null lines, is affected only slightly by the potential, and is contained in a wave front of extent $\Delta u \sim M$.
V	The potential barrier. This region near $r^*\approx 0$ is the domain in which $F_l(r^*)$ is large. (See Fig. 3.)
VI	The distant wave zone. This is the space-time region far from the star and subsequent to the first ray. It is where scalar (and other) radiation would be detected by antennas.
VII	The world line of the surface of the star. The data for Φ, and its derivative normal to the surface, on region VII are a result of the computations of Sec. II B. (See also region C of Fig. 1.)
VIII	The "stellar surface" $v=v_0$. This is a null line approximating the world line of the stellar surface. [See Eq. (30).]
IX	The near wave region. The vacuum exterior near the stellar surface. The field here obeys $\Psi_{,tt}-\Psi_{,r^*r^*}\approx 0$.
X	The stellar interior. The dynamics of this region affects the star's exterior only via the data it creates on the stellar surface VII.

We cannot match the two good solutions (with the usual conditions that Ψ and $\Psi_{,r^*}$ are continuous), so there can be no static solution that is well behaved at both $r^*=+\infty$ and $r^*=-\infty$.

We get perhaps the clearest picture of the nature of the paradox if we regard this problem as a purely mathematical problem in the propagation of waves, in one dimension, under the influence of a rather strange potential. Prior to the first ray, a distant observer sees a static field, the source of which is the charge in the star, or equivalently the field on the stellar surface. At $u\to\infty$ the stellar surface field again becomes static, and nonzero, so we might expect the distant observer to see a static nonvanishing field. This is impossible without singularities. In a sense then, this idealization is a reduction of the essence of the paradox to its simplest terms.

The advantage of the idealization is clear; the solutions in the two regions $r^*\geq 1$ and $r^*<1$ can be written in very convenient forms depending on four arbitrary functions:

$$\Psi=\begin{cases}\dfrac{df(v)}{dv}-\dfrac{f(v)}{r^*}+\dfrac{dg(u)}{du}+\dfrac{g(u)}{r^*}, & r^*\geq 1\\ \alpha(u)+\beta(v), & r^*<1.\end{cases} \tag{35}$$

For further convenience we redefine, for now, u and v as

$$v=t+r^*-1,$$
$$u=t-r^*+1, \tag{36}$$

so that $u=v$ when $r^*=1$, and we use characteristic boundaries at $u=0$ and $v=0$.

At $u=0$ we choose the condition $\Psi=1/r^*$, while at $v=0$ we choose

$$\Psi=1+(2k)^{-1}-(2k)^{-1}e^{-ku}. \tag{37}$$

(The constants are chosen so that Ψ and $\Psi_{,u}$ are continuous at $t=0$.) The solution to (33) with this input is

$$\Psi=-(2k)^{-1}e^{-ku}+(A/4k)e^{-kv}+kA^{1/2}e^{-v/2}\cos(\tfrac{1}{2}v-\phi)\quad\text{for } r^*\leq 1, \tag{38a}$$

$$\Psi=-\tfrac{1}{2}kAe^{-ku}-kA^{1/2}e^{-u/2}\sin(\tfrac{1}{2}u-\phi)+r^{*-1}[\tfrac{1}{2}Ae^{-ku}+\sqrt{2}\,kA^{1/2}e^{-u/2}\cos(\tfrac{1}{2}u-\phi-\tfrac{1}{4}\pi)]\quad\text{for } r^*\geq 1, \tag{38b}$$

where

$$A = 1/(k^2 - k + \tfrac{1}{2}), \tag{39a}$$

$$\tan\phi = 1/(2k-1), \quad 0 \leq \phi \leq \tfrac{3}{4}\pi. \tag{39b}$$

The terms in Ψ that go as $\exp(-ku)$ represent the outgoing waves from the "stellar surface"; the $\exp(-kv)$ term represents reflected waves. The coefficients

$$T = \tfrac{1}{2}kA/(2k)^{-1} = k^2/(k^2 - k + \tfrac{1}{2}),$$

$$R = (A/4k)/(2k)^{-1} = \tfrac{1}{2}/(k^2 - k + \tfrac{1}{2})$$

indicate the strength of the radiation transmitted through and reflected from, respectively, the potential peak near $r^* = 1$. In the limit that the input waves have very high frequency ($k \to \infty$), the waves are transmitted completely with no reflection. But *in the low-frequency limit* ($k \to 0$) *they are completely reflected and there is no transmission.* The exponentials with frequencies $\omega = -\frac{1}{2} \pm \frac{1}{2}i$ are transients which are characteristic of the potential, and which enable the conditions at $r^* = 1$ to be satisfied.[28]

The crucial thing to notice is that the solution falls off exponentially in time everywhere – i.e., at any r^* – thus avoiding the catastrophe of a static asymptotic solution. The paradox was founded on a belief that the value of Ψ at $v = 0$ and large u would penetrate the potential and show up at large r^*. We now see that the potential acts as a very effective barrier against zero-frequency waves from the "surface of the star." It is this potential barrier that causes the paradox (there would be well-behaved static solutions if not for the potential) and that resolves it.

Although it seems likely that this vanishing of the field is really the essence of the resolution of our paradox, we shall go on to look into the details of the real problem. We shall in particular be concerned with the question: Does the solution to (22) vanish asymptotically at large t and, if so, how fast? We shall see that for more realistic potentials than those of the idealized example, the solution does not fall off exponentially but rather develops a power-law falloff asymptotically, which dominates the exponential falloff of the previous example. Nevertheless, the solution does fall off at large t.

The power-law tail is caused by scattering of the radiation off the anomalous curvature part of the potential – i.e., from the fact that the real potentials do not have the convenient forms $l(l+1)/r^{*2}$, but have higher-order terms at large r^* also. The convenient forms of (35) correspond to unimpeded ingoing and outgoing waves. When we add the other curvature-induced parts of $F_l(r^*)$ we scatter these waves, effectively slowing the dilution of the field. Another viewpoint on this comes from the study of the spreading effect of potentials by Kundt and Newman.[29] In effect, they show that there is a zero measure of potential functions which give a nice separation of ingoing and outgoing waves as in (35). It seems that an exponential falloff of Ψ (in the case of exponential falloff of surface data) is associated with these nonspreading potentials; our potentials – the anomalous parts of which come from the curvature of space-time – will not be in this exalted class and we must expect scattering and other, slower, falloffs.

[Heuristically, we may argue for a nonexponential falloff as follows: In the Kundt-Newman formalism we may formally write a solution for any potential in a form like (35). For the spreading potentials, however, an infinite number of derivatives of $f(v)$ and $g(u)$ are required. This gives rise to an infinite number of transient frequencies. (In our idealized case we had only $\omega = -\frac{1}{2} \pm \frac{1}{2}i$.) The sum of an infinite number of transient terms may be viewed as the Fourier integral of a function other than an exponential.]

We shall now investigate the solution for the case of the actual potential and we will concern ourselves chiefly with the asymptotic solution (large u, large t). The simplicity of the monopole case makes it particularly suitable to a detailed discussion and to numerical solution.

C. Monopole Fields

Since the scalar monopole can be radiated just as well as higher multipoles, there is no reason to expect its asymptotic solution to differ *qualitatively* from that of multipoles with $l > 0$. The great advantage in considering the monopole case is that the $l(l+1)/r^2$ centrifugal-barrier term vanishes and we can think of the total potential as localized near $r^* = 0$.

If $F_0^{sc}(r^*)$ vanished everywhere – this would be the idealized potential for $l = 0$ – then the solution to our wave equation, with the characteristic data of Eq. (25), would simply be

$$\Psi(u, v) = a + b\exp(-u/4M), \tag{40}$$

representing free propagation outward of the data on the stellar surface. Although this cannot be the total solution, this should be the behavior of the high-frequency components of the radiation ($e^{i\omega t}$ with $\omega \gg [\max(F_0^{sc})]^{1/2} \sim 1/2M$). This phenomenon has appeared in our simple example and is a well-known occurrence in quantum mechanics where an energetic wave train is little affected by a potential barrier of much lower energy. Equation (40) is then a first approximation to the behavior of the solution. Inasmuch as it predicts a concentration of the waves near the first ray $u=0$, this solution represents a *wave front* which will be the dominant solution near $u=0$. The exact form of this wave front depends greatly on the details of the collapse; the crucial point here is that the wave front is exponentially damped.[30]

It is obvious that it is the low frequencies which are really involved in the paradox and in its resolution. These low frequencies make the greatest contribution (e.g., to a Fourier integral) at large times and so may very well lead to a modified asymptotic solution.

For now we assume that F_0^{sc} is absolutely localized[31] in some region $|r^*|<\beta M$. (It should be clear that the exponential tail of the potential at $r^*\to-\infty$ is ignorable. Later we must also justify ignoring the effects of the M/r^{*3} tail on the evolution of the asymptotic solution.) So now we have

$$\Psi_{,uv}=0 \quad \text{(in regions VI, IX of Fig. 4).} \tag{41}$$

From (41) and the data on the $v=v_0$ characteristic boundary (32a) it follows that the solution for Ψ at large u in region IX is

$$\Psi = b\exp(-u/4M)+f(v), \tag{42}$$

where $f(v)$ is an, as yet, unspecified function. On the other characteristic boundary $u=0$, we have

$$\Psi(u=0,v)=\frac{r}{2M}\ln(1-2M/r)$$

$$=-1-\frac{2M}{v}+O([\ln(v/2M)]/v^2)\,. \tag{43a}$$

According to (7), Ψ in region VI must be

$$\Psi=-1-\frac{2M}{v}+\cdots+g(u), \tag{43b}$$

where $g(u)$ is a function we must determine along with $f(v)$. Notice that $f(v)$ and $g(u)$ represent waves which propagate away from the potential in regions IX and VI, respectively.

Now let us assume that the solution in the region $t\gg r^*$ is not an exponential in time – but rather something slower like a power law. (This will be justified in the results.) The solution in this region then can be written as

$$\Psi=\psi(t)\phi(r^*)+\text{terms which fall off faster in time than } \psi(t) \text{ for } t\gg|r^*|, \tag{44}$$

$$\psi_{,t}/\psi\ll 1.$$

For convenience let us use the symbols $\mathcal{U}, \mathcal{V}$ for $\partial\Psi/\partial u$ and $\partial\Psi/\partial v$, respectively. If A is a point in region VI and B is a point in region IX with the same v coordinate (see Fig. 4), then by (31) and (44)

$$\mathcal{V}_B-\mathcal{V}_A\approx+\tfrac{1}{2}\psi(t)\int_A^B F_0^{sc}(r^*)\phi(r^*)dr^*. \tag{45}$$

Now $\mathcal{U}=0$ (modulo an exponential falloff) in region IX so that there

$$\mathcal{V}\equiv\Psi_{,v}=\Psi_{,t}\,, \tag{46}$$

and also in region IX, for $t\gg r^*$,

$$\mathcal{V}(B)=\psi_{,v|B}=\dot\psi(t)\phi(r^*). \tag{47}$$

Now if $\mathcal{V}(B)$ falls off in time as $\dot\psi(t)$ and hence faster than $\psi(t)$, it must be that $\mathcal{V}(A)$ cancels the integral in (45) – i.e., in region VI $\psi(t)$ must fall off like $\mathcal{V}$ for $t\gg r^*$. Furthermore, $\mathcal{U}_{,v}=\mathcal{V}_{,u}$ implies that

$$\mathcal{U}(D)-\mathcal{U}(C)\approx\mathcal{V}(B)-\mathcal{V}(A)\approx-\mathcal{V}(A)+O(\mathcal{V}'(A)) \tag{48}$$

at large t. Since $\mathcal{U}(C)=0$ we conclude that $\mathcal{U}(D)\approx-\mathcal{V}(A)$, or the incoming and outgoing parts of the tail are equal in magnitude for $t\gg r^*$. This almost total reflection of the ingoing waves is another manifestation of the impenetrability of the barrier to low-frequency waves.

Now from (43b) we see that in region VI, $\mathcal{V}$ must be $2M/v^2$ so

(i) $g(t)$ must fall off as $1/t^2$ for $t\gg r^*$.

(ii) From (48) and (43b), in region VI,

$$\Psi=2M\left(\frac{1}{u}-\frac{1}{v}\right)+\frac{\gamma}{v^2}\quad(\gamma \text{ some constant}). \tag{49}$$

(iii) In region IX, $\mathcal{V}$ must fall off as $1/t^3$ for $t\gg r^*$ so that

$$\mathcal{V}\sim O(1/v^3),$$
$$\psi\sim O(1/v^2). \tag{50}$$

In (49) we see that at any r^* if $t\gg r^*$, then $\Psi=(4Mr^*+\gamma)/t^2$, that is, Ψ falls off as $1/t^2$, and from (50) we see that this must be true for region IX also. Thus, *a sufficiently long time after the wave front passes in region VI, or after the surface passes in region IX, the solution will fall off in time as* $1/t^2$.

Before going on to discuss the meaning and implication of these results, we must justify having ignored the $2M/r^{*3}$ tail of the potential. It is clear that ignoring this tail in our analysis of the evolution amounts to assuming that in region VI, $\mathcal{V}$ is transported unchanged (on a line of constant v) from the first ray to the edge of the potential barrier. We can calculate how much $\mathcal{V}$ will change on this path for our solution, due to the $2M/r^{*3}$ tail of the potential:

$\delta\mathcal{V}\equiv$ (change in $\mathcal{V}$ in region I, on a line of constant v, due to the tail of F_0: $\delta F_0\sim 1/r^{*3}$)

$$=-\tfrac{1}{4}\int_{u=0}^{r^*\approx\beta M} du\,\delta F_0\Psi$$

$$= -\frac{M}{2}\int_{u=0}^{u=v-\beta M} du\,\frac{1}{r^{*3}}\times\begin{cases}\Psi\propto 1 & \text{near } u=0\\ \Psi\propto M(1/u-1/v) & \text{for } u\gg M.\end{cases} \tag{51}$$

We can divide the integral into two parts: (i) the contribution at small u due to the wave front and (ii) the contribution, mostly near the barrier, due to the tail $2M(1/u-1/v)+\gamma/v^2$ which we have calculated for Ψ. The first contribution is of order M/v^3 and we need give it no further consideration. The contribution of the tail is of order $M/\beta v^2$ and thus falls off at the same rate as $\mathfrak{V}$, but we can make this error as small as we wish merely by making β sufficiently large.

The problem of monopole radiation just discussed was attacked numerically with an IBM 360/75 computer. The *exact* problem was investigated. That is, the exact potential $F_0^{\text{sc}}(r^*)$ as given in (22b) was used as well as the exact first-ray solution $\Psi=(r/2M)\ln(1-2M/r)$. The results were in perfect agreement with the arguments presented above. Specifically, Ψ was found to have a large wave front at small u which gave way to an asymptotic solution for $t\gg r^*$ that did in fact go precisely as $2M(1/u-1/v)+\gamma/v^2$ in region VI. In region IX the solution was found to be very accurately independent of u, and to go as $\Psi=\text{const}/v^2$. Furthermore, the program kept track of $\mathfrak{U}$ and $\mathfrak{V}$ in region VI; from the first ray to quite small values of r^* it was found that $\delta\mathfrak{V}$ as defined in (51) does fall off as $1/v^2$ but is always much smaller than $\mathfrak{V}$ for $r^*>20M$ or so. Results of these computations are presented in Figs. 5, 6, and 7.

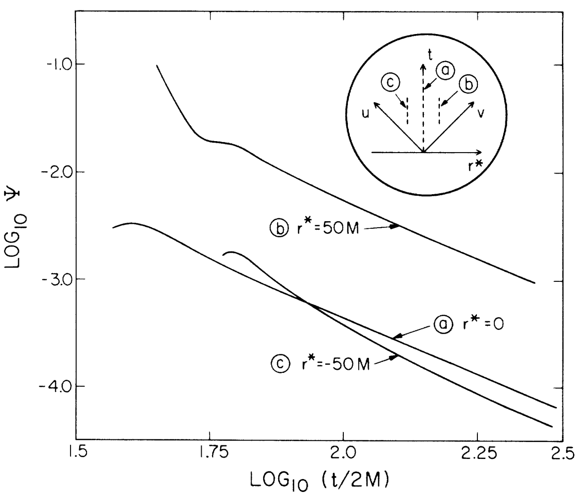

FIG. 5. The results of computer integrations of the asymptotic falloff of Ψ for a coordinate stationary observer in the case $l=0$. The slopes of the logΨ vs log$t$ curves all approach a slope of -2 at large t, verifying the t^{-2} falloff derived in the text. The computations for these curves use $R_{\text{surf}}(t=0)=4M$, and $\Psi=(r/2M)\ln(1-2M/r)$ on the first ray. The "surface" data at $v=0$ were taken to be $a+b\exp(-u/4M)$, with a and b chosen so that Ψ on the "surface" and on the first ray matches smoothly at $t=0$. (As in Fig. 3, r^* is defined as zero at $r=4M$.) The dashed lines in the circled insert depict the points for which Ψ is plotted in the three curves.

D. Multipoles of General l

We shall now discuss the asymptotic evolution of multipoles of general l for two initial conditions. (i) A static perturbation field exists outside the star prior to the onset of collapse. (ii) There is no initial perturbation field outside the star, but one develops during the collapse. These calculations are only outlined here; details are left to the Appendix.

As in the monopole case we can correctly analyze the large-time dynamics using a localized idealization of the potential

$$-4\Psi_{,uv}=\begin{cases}\Psi\, l(l+1)/r^{*2}, & r^*>\beta M\\ 0, & r^*<-\beta M\\ F_l\Psi, & -\beta M\leqslant r^*\leqslant\beta M.\end{cases} \tag{52}$$

[This is justified by the calculation in the Appendix.] With this simplification we can write the solution for Ψ, in the regions $|r^*|>\beta M$, in terms of

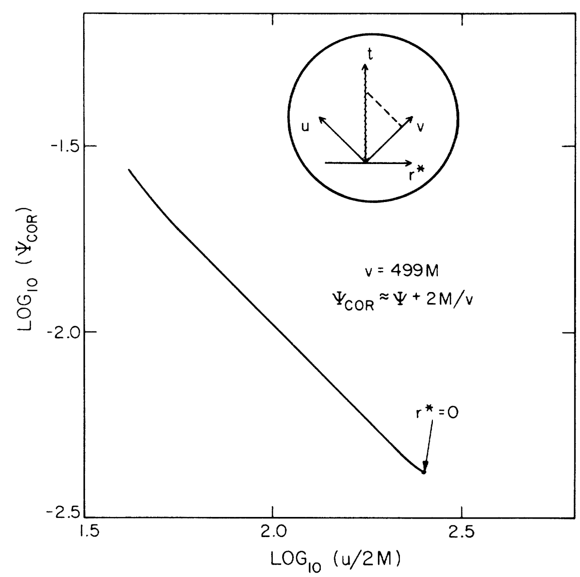

FIG. 6. The results of computer integrations for the behavior of Ψ in region VI of Fig. 4, along a line of constant v. The "corrected" value of Ψ is defined: $\Psi_{\text{COR}}(u)\equiv\Psi(u)+[1-\Psi(u=0)]\approx\Psi(u)+2M/v$. According to the analysis in the text Ψ_{COR} should be approximately $2M/u$ except very near the wave front or the potential barrier. The computer results verify this. The dashed line in the circled insert depicts the points for which values of Ψ_{COR} are plotted here. Note that $\Psi_{\text{COR}}\simeq 2M/u$ even for $r^*=0$. For further discussion, see the text [especially Eq. (49)].

four unknown functions [a generalization of (35)]:

$$\Psi = \alpha(u) + \gamma(v) \tag{53}$$

for $r^* < -\beta M$ and

$$\Psi = f^{(l)}(v) - A_l^1 f^{(l-1)}(v)/r^* + \cdots + (-1)^l A_l^l f(v)/r^{*l} + g^{(l)}(u) + A_l^1 g^{(l-1)}(u)/r^* + \cdots + A_l^l g(u)/r^{*l}, \tag{54}$$

where

$$A_l^p = (l+p)!/[2^p p!(l-p)!] \tag{55}$$

for $r^* < \beta M$.

From the arguments preceding (44) we have that

$$\Psi = \psi(t)\phi_{\text{static}}(r^*) \tag{56}$$

in the asymptotic region $t \gg r^*$. By expanding (54) for $t \gg r^*$ and comparing the result with (56) we find the functional relations

$$g(t) = (-1)^{l+1} f(t) + [\text{terms that fall off at least as fast as } f^{(2l+1)}(t), \text{ at large } t] \tag{57}$$

and

$$\psi(t) = \text{const} \times f^{(2l+1)}(t) + [\text{terms that fall off faster at large } t]. \tag{58}$$

A large-v expansion of Ψ near the wave front, carried out in the Appendix, shows that for general l

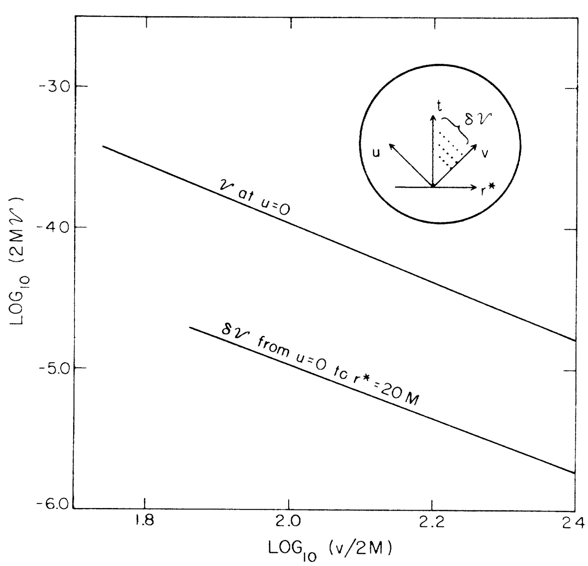

FIG. 7. The justification for localizing the curvature potential in the monopole case. The computer integrations show that $\mathcal{V} \equiv \Psi_{,v}$ on the first ray falls off as v^{-2}. Also plotted here is $\delta\mathcal{V}$: the change in $\mathcal{V}$, on a line of constant v, between the first ray and $r^* = 20M$ (i.e., the change in $\mathcal{V}$ along one of the dotted lines in the circled insert). The plot of $\delta\mathcal{V}$ as a function of v (i.e., as a function of which dotted line in the insert is used) shows that $\delta\mathcal{V} \propto 1/v^2$. Though $\delta\mathcal{V}$ falls off at the same rate as $\mathcal{V}$, it is only 10% as large. For further discussion, see the text [especially Eq. (51)].

$$f(v) \approx \text{const}/v \tag{59}$$

if a static l pole is initially present, and

$$f(v) \approx \text{const}/v^2 \tag{60}$$

if there is no initial perturbation field. [These are the two most interesting cases, but the form of $f(v)$ for any initial condition can easily be calculated.] From (58) it then follows that *(i) an initial static multipole perturbation will die out as* $t^{-(2l+2)}$ *at large time and (ii) a multipole perturbation which develops during the collapse will die out as* $t^{-(2l+3)}$ *at large time.*

Let us now summarize the physics of the evolution of scalar-field multipoles.

(i) Near the first ray (i.e., at small u) the solution is dominated by a wave front: outgoing waves from the stellar surface that have passed through the potential barrier. These primary waves fall off exponentially in u since the variation of Ψ on the stellar surface is exponentially damped.

(ii) The wave front of primary waves is backscattered by the tail of the potential and the "input" to the post-wave-front region is the ingoing radiation caused by this backscattering. This ingoing radiation has the form, near the wave front, $\Psi \sim v^{-(l+1)}$ if the field was initially static, and $\Psi \sim v^{-(l+2)}$ if there was no initial field.

(iii) From (57) we see that the ingoing radiation from the wave front is almost totally reflected by the potential barrier near $r^* = 0$.

(iv) In region IX (see Fig. 4) the outgoing radiation from the stellar source dies out exponentially. At large t the solution is dominated by the ingoing radiation, from the wave front, that does manage to penetrate the potential barrier. This radiation falls off as $t^{-(2l+2)}$ or $t^{-(2l+3)}$ depending on initial conditions.

(v) In region VI, for $t \gg r^*$, in- and outgoing waves interfere destructively, leaving an uncanceled field which falls off as $t^{-(2l+2)}$ or $t^{-(2l+3)}$.

(vi) Though we have started the collapse from a very relativistic static configuration, it is easy to see that our conclusions are independent of this. If the collapse starts from a radius $\gg M$, then the primary waves of (i) dominate for a longer time, but the qualitative evolution after the primary waves have passed is unchanged.

E. A Picture of the Decay of Ψ

In general, invariance of a problem under some group of transformations leads to a conserved quantity. For our radiation problem, the background space is independent of time and we can derive an energylike conserved quantity for the scalar field. This quantity can help us picture the decay of the field. The wave equation

$$\Psi_{,tt} - \Psi_{,r^*r^*} + F_l(r^*)\Psi = 0$$

leads us to define

$$\mathcal{H} \equiv \tfrac{1}{2}(\Psi_{,t})^2 + \tfrac{1}{2}(\Psi_{,r^*})^2 + \tfrac{1}{2}F_l\Psi^2$$
$$= (\Psi_{,u})^2 + (\Psi_{,v})^2 + \tfrac{1}{2}F_l\Psi^2 \qquad (61)$$

and

$$\mathcal{S} \equiv \Psi_{,t}\Psi_{,r^*} = (\Psi_{,v})^2 - (\Psi_{,u})^2, \qquad (62)$$

so that the equation of motion takes the form

$$\frac{\partial \mathcal{H}}{\partial t} = \frac{\partial \mathcal{S}}{\partial r^*}\,. \qquad (63)$$

We can interpret $\mathcal{H}$ as being like an energy density, $\mathcal{S}$ a sort of energy flux, and (63) as a divergence equation. We take comfort in the fact that $\mathcal{H}$ is positive definite.

Let us apply this to the radiation problem with boundary values given on the null lines $u=0$ and $v=0$. On a spacelike hypersurface of constant t the total "energy" in the wave zone is

$$H \equiv \int_{r^*=-t}^{r^*=+t} dr^*\mathcal{H}. \qquad (64)$$

This total "energy" can only be changed by "energy" flowing across the boundaries

$$\frac{dH}{dt} = \int_{r^*=-t}^{r^*=+t} dr^*\,\frac{\partial \mathcal{H}}{\partial t} + \mathcal{H}|_{r^*=+t} + \mathcal{H}|_{r^*=-t}$$
$$= \int_{r^*=-t}^{r^*=+t} dr^*\,\frac{\partial \mathcal{S}}{\partial r^*} + \mathcal{H}|_{r^*=+t} + \mathcal{H}|_{r^*=-t}$$
$$= [\mathcal{H}+\mathcal{S}]_{r^*=+t} + [\mathcal{H}-\mathcal{S}]_{r^*=-t}$$
$$= [2\Psi_{,v}{}^2 + \tfrac{1}{2}F_l\Psi^2]_{u=0} + [2\Psi_{,u}{}^2 + \tfrac{1}{2}F_l\Psi^2]_{v=0}\,. \qquad (65)$$

The two terms in the last line represent, respectively, the "energy" flowing across the first ray and across the stellar surface. If we consider what the asymptotic contributions at large t are, we find that the second term is exponentially small and hence negligible. The first term at most gives a contribution that falls off as t^{-2} so that $dH/dt \propto t^{-2}$ or less, which means that $H \propto a - b/t$ at large t.

Now we notice that $\mathcal{H}$ is a positive definite quantity and

$$H \gtrsim \tfrac{1}{2}\int F_l\Psi^2 dr^* \propto t(\overline{\Psi})^2, \qquad (66)$$

where Ψ is some sort of average value of Ψ on the hypersurface. This tells us that this average value of Ψ must fall off essentially as $t^{-1/2}$ or faster since H is essentially constant at large t.

If the information at $t=0$ were dispersed by a very strong potential uniformly through the future light cone, Ψ would fall off everywhere as $t^{-1/2}$. This is the familiar case of the so-called diffraction of waves studied by Lewis[32] and others. On the other hand, if there were no backscattering the waves would not spread at all so the integral for H would have a nonvanishing contribution in a spatial region independent of t, and Ψ would have a constant value on an ingoing or an outgoing characteristic.

For our problem neither limit applies. In a sense the high frequencies propagates on the characteristics and the low frequencies tend to spread, but the correct asymptotic solution demands a deeper analysis. While arguments based on the conserved flow cannot tell us just what sort of asymptotic solution will develop in the presence of our curvature potential, they do help in picturing the physics of the situation. One way of interpreting this picture of "energy" flow is to say the "final" value of Ψ on the surface of the star, as the surface crosses its gravitational radius, is ineffective in stopping the decay of the field.

In Paper II we shall deal with a complex field satisfying an equation like (22); the only thing that must be changed to accommodate the complex field is that we define $\mathcal{H}$ and $\mathcal{S}$ as the real quantities

$$\mathcal{H} \equiv \tfrac{1}{2}|\Psi_{,t}|^2 + \tfrac{1}{2}|\Psi_{,r^*}|^2 + \tfrac{1}{2}F_l|\Psi|^2 \qquad (67)$$

and

$$\mathcal{S} = \tfrac{1}{2}(\Psi_{,t}\overline{\Psi}_{,r^*} + \overline{\Psi}_{,t}\Psi_{,r^*}), \qquad (68)$$

where the bar over the Ψ denotes the complex conjugate.

IV. GRAVITATIONAL PERTURBATIONS

The study of the scalar field is more than a plausible analog; from the mathematics of the previous sections we can directly infer the dynamics of gravitational perturbations. In Paper II a unified view of all integer-spin massless field perturbations will be given with the aid of the null-tetrad formalism of Newman and Penrose.[33] Here we shall describe the physical nature of the falloff of gravitational perturbations. To be concise, we shall usually refer specifically to the evolution of an l-pole perturbation field which is initially static.

Although the mathematical description of gravitational perturbations is not greatly more difficult than that for other perturbations, the physical interpretation is complicated by gauge arbitrariness. Gravitational perturbations (e.g., perturbations in the Riemann tensor) are unavoidably mixed with perturbations in the background geometry. In physical terms, to give a value for a gravitational perturbation we must specify how it would be measured. Nevertheless, the physical nature of the falloff of the perturbations is fairly clear.

The description of gravitational perturbations used here is essentially that of Regge and Wheeler (RW).[17,34] This involves the use of vector and tensor spherical harmonics to separate the angular variables, and a convenient choice of gauge. In this RW gauge two functions of radius and time describe the odd-parity perturbations and three functions suffice to describe the even ones.

A. Odd Parity

We are not concerned with multipoles of $l<2$. Such multipoles for the spin-2 gravitational field are *nonradiatable*. Specifically there can be no $l=0$ odd-parity perturbation, and the $l=1$ multipole has been fully investigated. As Vishveshwara[21] and independently Campolattaro and Thorne[35] have shown, the odd-parity dipole perturbation must be stationary (a consequence of the field equations) and corresponds to a small angular momentum in the star.

For quadrupole and higher-multipole perturbations, Regge and Wheeler[17] found that the field equations lead to a wave equation similar to (22):

$$Q_{,tt}-Q_{,r^*r^*}+F_l^{\text{op}}(r^*)Q=0. \tag{69a}$$

Here the curvature potential for odd-parity gravitational waves is

$$F_l^{\text{op}}(r^*)=(1-2M/r)[l(l+1)/r^2-6M/r^3]. \tag{69b}$$

The RW metric perturbations h_0 and h_1 can be derived from Q according to

$$h_1=rQ(1-2M/r)^{-1}, \tag{70a}$$

$$h_{0,t}=(rQ)_{,r^*}. \tag{70b}$$

The formal similarity of (22) and (69) is striking but to continue the analogy between the odd-parity gravitational perturbations and scalar perturbations we must ask whether Q, as a measure of the gravitational perturbations, is free from pathological coordinate effects. We shall see that it is not; Q vanishes at the event horizon even though locally measured perturbations are finite there.

Let us define

$$q(r,t)=\int_\infty^t Q(r,T)dT. \tag{71}$$

In Paper II it is proven that q is measurable in the following sense: It is a linear combination of the components of the Riemann tensor referred to the orthonormal tetrad of a falling observer, and the coefficients in this linear combination are finite at $r=2M$. This implies that on the stellar surface q and its proper time derivative are finite at the event horizon. Since $Q=q_{,t}$ then due to time-dilation effects Q on the stellar surface will vanish as $(1-2M/r)$, when the surface crosses the event horizon at $t=\infty$.

If we now integrate (69a) over the time variable from ∞ to t, we find[36] that q must satisfy the same equation as Q,

$$q_{,tt}-q_{,r^*r^*}+F_l^{\text{op}}(r^*)q=0. \tag{72}$$

The behavior of q on the stellar surface follows from the measurable nature of q. Since q and its proper time derivative are finite, the argument of (25) to (30) implies that for $u\gg M$ on the stellar surface,

$$q=q_0+q_1\exp(-u/4M). \tag{73}$$

The initial-value problem for q also requires data on a line $u=$constant. If we choose the star and field outside it to be momentarily at rest, then q on the first ray signalling the onset of collapse must be the static solution of (72) which is well behaved at spatial infinity.

The structure of the initial-value problems for Ψ and for q are then almost identical. The only difference is in the details of the potentials, but the calculation in the Appendix makes it clear that it is only the dominant asymptotic terms in the potential, at $r^*=+\infty$ and $r^*=-\infty$, which are important to the large-time behavior of the solution. The analysis and results of Sec. III therefore apply immediately to q. The asymptotic evolution of q (for $l\geq 2$) is precisely the same as that of Ψ. *In particular, at a fixed r, q falls off as $t^{-(2l+2)}$ if the perturbation was initially static.*

The evolution in time of the RW functions Q and h_1 can be found easily from (70). They fall off for large time at constant r, as

$$h_1\sim Q\sim t^{-(2l+3)}. \tag{74}$$

Equation (70b) implies

$$h_{0,t}=(rq_{,t})_{,r^*} \tag{75}$$

so that

$$h_0=(rq)_{,r^*}+b(r). \tag{76}$$

But $b(r)$ must be zero or h_0 would be nonzero at large t and there would be a physical singularity[21] at $r=2M$ or $r=\infty$. At large t then

$$h_0\sim t^{-(2l+2)}.$$

B. Even Parity

As in the odd-parity case, the properties of the *nonradiatable* $l<2$ even-parity multipoles are well known. (i) By Birkhoff's theorem an $l=0$ perturbation can only be a small static change in the mass. (ii) Even-parity dipole perturbations correspond to a coordinate displacement of the origin. Such displacements have no physical meaning and can always be annihilated by a gauge transformation.[35] To analyze the $l \geq 2$ radiatable multipoles we need a wave equation like (22) or (72). Fortunately, Zerilli[18] has recently supplied such an equation. Zerilli's equation is in the context of the RW formalism and the RW gauge. Thus, we describe the even-parity perturbations by three functions: H, H_1, and K in the RW notation. Zerilli assumes perturbations to have $\exp(-ikt)$ time dependence. This does not suit our purposes here so while Zerilli replaces H_1 by $R \equiv H_1/k$, we define[37] it as

$$R(r,t) \equiv i\int_t^\infty H_1(r,T)dT. \tag{77}$$

Following Zerilli we define certain linear combinations of R and K:

$$\tilde{K} = a_1 K + a_2 R, \tag{78a}$$

$$\tilde{R} = a_3 K + a_4 R, \tag{78b}$$

where

$$a_1 = r^2/(\Lambda r + 3M), \tag{79a}$$

$$a_2 = \frac{-i(r-2M)}{\Lambda r + 3M}, \tag{79b}$$

$$a_3 = \frac{-\Lambda r^2 + 3\Lambda M r + 3M^2}{(\Lambda r + 3M)^2}, \tag{79c}$$

$$a_4 = \frac{i(r-2M)[\Lambda(\Lambda+1)r^2 + 3\Lambda M r + 6M^2]}{r^2(\Lambda r + 3M)^2}, \tag{79d}$$

$$\Lambda = \tfrac{1}{2}(l-1)(l+2). \tag{79e}$$

(The third function, H, can be found from $\tilde{R}$ and $\tilde{K}$ by means of the field equations.[18]) With the definitions of (78) and (79), we can put the field equations in a very simple form:

$$\tilde{K}_{,r^*} = \tilde{R}, \tag{80a}$$

$$\tilde{R}_{,r^*} = F_l^{ep}(r^*)\tilde{K} + \tilde{K}_{,tt}, \tag{80b}$$

with

$$F_l^{ep}(r^*) = \left(1 - \frac{2M}{r}\right) \times \frac{2\Lambda^2(\Lambda+1)r^3 + 6\Lambda^2 M r^2 + 18\Lambda M^2 r + 18M^3}{r^3(\Lambda r + 3M)^2}. \tag{80c}$$

We can now combine (80a) and (80b) to get Zerilli's effective-potential equation,

$$\tilde{K}_{,tt} - \tilde{K}_{,r^*r^*} + F_l^{ep}(r^*)\tilde{K} = 0, \tag{81}$$

an equation of the same form as (22) and (72).

In Paper II it is demonstrated that $\tilde{K}$ describes the even-parity perturbations with no pathological coordinate effects. That is, if locally measured gravitational perturbations are finite on the stellar surface during the passage through the event horizon, then $\tilde{K}$ and its proper-time derivative on the stellar surface are also finite. From the argument of (25) to (30) it follows that for $u \gg M$ on the stellar surface,

$$\tilde{K} = K_0 + K_1[\exp(-u/4M)]. \tag{82}$$

As we did for scalar and odd-parity waves, we may start the star and $\tilde{K}$ field from a momentarily static situation. The remaining input is then a static solution to (81) on the "first ray", u = constant. The initial-value problems for Ψ, q, and $\tilde{K}$ are now quite similar. Furthermore, from (80c) $F_l^{ep}(r^*)$ has the same asymptotic behavior, at $r^* = +\infty$ and at $r^* = -\infty$, as the potentials in (22) and (72), so that we may apply the results of Sec. III to $\tilde{K}$. In particular an initially static $\tilde{K}$ will fall off at large t as $t^{-(2l+2)}$. From (80) we see that $\tilde{R}$ has this same large-time behavior[38] and therefore by (78), K and R also die out as $t^{-(2l+2)}$. Using the field equations[17] we can show that H therefore dies out at this same rate. Since H_1 is a time derivative of R, it must fall off faster, as $t^{-(2l+3)}$. With these results and those for odd-parity perturbations, we conclude: *Initially static gravitational multipole perturbations vanish at large time as* $t^{-(2l+2)}$, *and it is this vanishing of the perturbations that resolves the paradox of the singularities.*

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APPENDIX

We shall consider here the evolution of Ψ at late times, for $l > 0$. The wave equation to be used,

$$\Psi_{,tt} - \Psi_{,r^*r^*} + \left(1 - \frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2} + R(r)\right]\Psi = 0, \tag{A1a}$$

$$R(r) = R_1 \frac{2M}{r^3} + R_2 \frac{(2M)^2}{r^4} + \cdots, \tag{A1b}$$

is general enough to encompass the wave equations of (22), (72), and (81).

If $\Psi = 0$ before the first ray $u = u_0$, then the solution after the first ray can be written as an expansion for $u \ll r$:

$$\Psi = \sum_{p=0}^{l} A_l^p r^{-p} G^{(l-p)}(u) + \sum_{p=0}^{\infty} B_l^p(r) G^{(l-p-1)}(u), \tag{A2}$$

in which $G(u)$ and its first l derivatives vanish on the first ray,

$$G(u_0)=G'(u_0)=\cdots=G^{(l)}(u_0)=0. \tag{A3}$$

The negative-order derivatives, to be interpreted as integrals

$$G^{(-1)}(u)\equiv\int_{u_0}^{u}G(\tilde u)d\tilde u, \tag{A4}$$

$$G^{(-2)}(u)\equiv\int_{u_0}^{u}G^{(-1)}(\tilde u)d\tilde u,\ldots, \tag{A5}$$

also vanish on the first ray. In (A2) the coefficients A_l^p are those given in (55) [solution to flat space-time wave equation], and the functions $B_l^p(r)$ [curvature-induced corrections] have yet to be calculated. The first sum in (A2) represents the primary waves in the wave front; the second sum represents waves backscattered by the anomalous part of the curvature potential.

By putting (A2) into the wave equation (A1) we can derive the recursion relation

$$2B_l^{p\prime}=\left[\left(1-\frac{2M}{r}\right)B_l^{p-1\,\prime}\right]'-B_l^{p-1}\left[\frac{l(l+1)}{r^2}+R(r)\right]$$
$$-A_l^p r^{-(p+3)}[p(p+2)2M+r^3R(r)]. \tag{A6}$$

(Prime denotes d/dr.) From this recursion relation B_l^p can be found as a power series

$$B_l^p=a_l^p/r^{p+2}+b_l^p/r^{p+3}+\cdots. \tag{A7}$$

Some expansion coefficients which follow from (A6) and (A7), and which will be important later, are

$$a_l^{l-1}=\frac{2MA_l^l[(l-1)(l+1)+R_1]}{2(l+1)}, \tag{A8a}$$

$$a_l^l=\frac{A_l^l\,2M(2l+1)}{2(l+2)}. \tag{A8b}$$

If there were no tail of the potential at large r (i.e., if it consisted of a centrifugal barrier only, as in flat space-time), Ψ could be made to vanish at late time by "turning off the source," i.e., by specifying $G(u)=0$ for $u>u_1>u_0$. But the negative-order derivatives and the nonzero values of the coefficients $B_l^p(r)$ when space-time is curved, show that backscattered waves persist, even after the source is turned off.

In general we do not expect $G(u)$ to go strictly to zero at some finite u, but we do know that $G(u)$ and its (positive-order) derivatives fall off as $\exp(-u/4M)$, becoming negligible for $u>u_1\gg M$. After the passage of the primary waves, when $u>u_1$, the dominant term at large r is

$$B_l^l(r)G^{(-1)}(u)\approx a_l^lG^{(-1)}(u)/r^{l+2}. \tag{A9}$$

This can be thought of as the dominant backscatter of the primary waves,[39] if there is no initial static multipole.

If there is an initial static multipole of magnitude μ, then to (A2) we must add the static solution of (A1):

$$\Psi_{\text{static}}=\mu\left\{r^{-l}+r^{-(l+1)}\frac{2M[l(l+2)+R_1]}{2(l+1)}+O(M^2/r^{l+2})\right\}. \tag{A10}$$

When the source of the primary waves is "turned off" (i.e., they become exponentially small) at $u=u_1$, then $G(u)$ must be constant (i.e., it has exponentially small variation) for $u>u_1$, in order to cancel Ψ_{static}; in fact we must have

$$A_l^lG(u_1)=-\mu. \tag{A11}$$

Thus after the primary waves are "turned off" the dominant backscatter at large r is

$$\frac{a_l^{l-1}G(u_1)+\mu 2M[l(l+2)+R_1]/2(l+1)}{r^{l+1}}$$
$$=\frac{2M\mu[(2l+1)/2(l+1)]}{r^{l+1}}. \tag{A12}$$

It is significant that (A12), the dominant backscatter in the case of an initially static perturbation, and (A9), the backscatter in the case of no initial perturbation, are both independent of the $R(r)$ term in the potential. Mathematically, it is the relation of r to r^* which is critical in determining the dominant backscatter; potential terms of order M/r^3 only influence backscatter to a higher order.

Next we shall calculate the asymptotic evolution of Ψ, as outlined in Sec. IIID. For $r\gg M$ the ingoing and outgoing waves can be written [see (54)]

$$\Psi_{\text{in}}=\left[\sum_{k=0}^{l}\frac{A_l^k(-1)^kf^{(l-k)}(v)}{r^{*k}}\right]$$
$$\times[1+O(2Mr^{*-1}\ln(r^*/2M))], \tag{A13a}$$

$$\Psi_{\text{out}}=\left[\sum_{k=0}^{l}\frac{A_l^kg^{(l-k)}(u)}{r^{*k}}\right][1+O(2Mr^{*-1}\ln(r^*/2M))]. \tag{A13b}$$

We have seen that $\Psi\to\psi(t)\phi_{\text{static}}(r^*)$, asymptotically for $t\gg r^*$, so that for $t\gg r^*\gg M$,

$$\Psi=\psi(t)r^{*\,l+1}[1+O(2Mr^{*-1}\ln(r^*/2M))]$$
$$+\text{terms which fall off faster in } t \text{ than } \psi(t), \tag{A14a}$$

and for $t\gg -r^*\gg M$,

$\Psi = \text{const}\times\psi(t) + \text{terms which fall off faster in } r^* \text{ or in } t.$ (A14b)

[The constant in (A14b) can be found by numerically solving the static wave equation.] We can now expand (A13) for $t \gg r^*$ and write Ψ as a superposition of ingoing and outgoing waves:

$$\Psi = \sum_{n=-l}^{\infty} K_n^l r^{*n}[f^{(l+n)}(t) + (-1)^n g^{(l+n)}(t)] \times [1 + O(2Mr^{*-1}\ln(r^*/2M))], \qquad \text{(A15)}$$

where

$$\left.\begin{aligned} &K_n^l = 0 \quad \text{if } l-n \text{ is odd},\\ &K_n^l = (\tfrac{1}{2})^l(-1)^{(l-n)/2}\frac{(l-n)!}{[\frac{1}{2}(l-n)]![\frac{1}{2}(l+n)]!} \quad \text{if } l-n \text{ is even},\\ &K_{l+1}^l = 1/(2l+1)!!\,. \end{aligned}\right\} \text{ for } -l \le n \le l. \qquad \text{(A16)}$$

By comparing (A14) and (A15) we can conclude that

$$\psi(t) = \left(\frac{d}{dt}\right)^{2l+1}\frac{f(t) + (-1)^{l+1}g(t)}{(2l+1)!!} \qquad \text{(A17)}$$

and that $f(t) \approx (-1)^{l+1}g(t)$ or, more precisely,

$$g(t) = (-1)^{l+1}f(t) + O((2M)^{2l+1}f^{(2l+1)}(t)), \qquad \text{(A18)}$$

so that

$$\psi(t) = 2f^{(2l+1)}(t)/(2l+1)!!\,. \qquad \text{(A19)}$$

From (A9) and (A12) we know the behavior of Ψ as a function of v on a line of constant $u = u_1$, where $v \gg u_1 \gg M$,

$$\Psi_{\text{in}} = a_l^l G^{(-1)}(u_1)[\tfrac{1}{2}v]^{-(l+2)}[1 + O(2Mv^{-1}\ln(v/2M))]$$

if no initial l pole (A20a)

$$= 2M\mu\frac{(2l+1)}{2(l+1)}(\tfrac{1}{2}v)^{-(l+1)}[1 + O(2Mv^{-1}\ln(v/2M))]$$

if initial static l pole of magnitude μ. (A20b)

Now we can compare (A20) with (A13a) in the region $r^* \gg u$ to find that

$$f(v) = (-1)^l 2MG^{(-1)}(u_1)/v^2, \qquad \text{(A21a)}$$

$$\psi(t) = \frac{2(-1)^{l+1}(2l+2)!}{(2l+1)!!}\frac{2M}{t^{2l+3}}\int_{u_0}^{u_1}G(u)\,du, \qquad \text{(A21b)}$$

if there is no initial static multipole, and

$$f(v) = (-2)^l 2M\mu[l!/(2l)!]/v,$$

$$\psi(t) = \frac{(-2)^{l+1}(2l+1)l!}{(2l+1)!!}\frac{2M\mu}{t^{2l+2}},$$

if there is an initial static multipole of magnitude μ. Notice that these asymptotic solutions do not depend on the potential term $R(r)$ since the initial backscatter does not depend on it. The asymptotic solutions for (22b), (69b), and (80c) are identical.

Another approach to this calculation, using Laplace transforms, will be published elsewhere by Thorne.[40] Yet another method has recently been devised by Fackerell.[41]

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[19]Static, here and throughout, means independent of the time coordinate geared to the timelike Killing field, i.e., Schwarzschild time.

[20]Actually the solutions are confluent hypergeometric functions.

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[22]It is, of course, possible that Φ couples to the Riemann tensor (just as the electromagnetic vector potential couples to the Ricci tensor). It is difficult to see how this coupling could be achieved without introducing higher derivatives or violating the equivalence principle. Such questions are further discussed in Paper II.

[23]The description used here is primarily due to D. L. Beckedorff and C. W. Misner, 1962 (unpublished report) and D. L. Beckedorff, A. B. senior thesis, Princeton University, 1962 (unpublished). See also the reference in footnote 26.

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[25]T. Regge and J. A. Wheeler (see footnote 17) also used the concept of an effective potential in their analysis of the stability of the Schwarzschild geometry against odd-parity gravitational perturbations.

[26]Our notation for u, v and for U, V is that of C. W. Misner, in *Astrophysics and General Relativity, Vol. 1,* edited by M. Chrétien, S. Deser, and J. Goldstein (Gordon and Breach, New York, 1969).

[27]Characteristic data require only the specification of Ψ, not of its normal derivatives.

[28]For a discussion of the significance of such transients to the gravitational radiation spectrum from a black hole see W. H. Press (unpublished report).

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[30]This exponentially damped wave front is the total solution proposed by Patashinski and Harkov. (See Ref. 11.)

[31]We could get the same results using a δ-function potential and Fourier integrals. This would be less instructive and harder to justify.

[32]R. W. Lewis, in *Asymptotic Solutions of Differential Equations and Their Application,* edited by C. Wilcox (Wiley, New York, 1964).

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[36]In deriving (72) we must use the fact $\partial Q/\partial t \to 0$ as $t \to \infty$. This follows from the argument (see the discussion in Sec. I or the Ref. 4) that the field must be static at large t.

[37]The choice of ∞ as the upper limit on the integral allows us to conclude from the field equations that $(d/dr)[iR(1-2M/r)]=H+K$. This is necessary in Zerilli's derivation.

[38]In region IX, $\tilde{K} \approx \gamma(v)$, so that (80a) would seem to imply that $\tilde{R}=\tilde{K}\sim t^{-(2l+3)}$, and this would be incompatible with (80b). Throughout Sec. III we have assumed $F_l=0$ in region IX, so that we have ignored terms like $\psi(t)\exp(r^*/2M)$. This term makes a negligible difference to Ψ, q, or $\tilde{K}$ in region IX, but it gives the dominant asymptotic time behavior for $\Psi_{,r^*}$, $q_{,r^*}$, and $\tilde{K}_{,r^*}$.

[39]The interpretation of these integral terms as backscatter is reasonable because they depend on data spread out over a section of the past light cone. Outgoing waves depend only on data at a fixed u.

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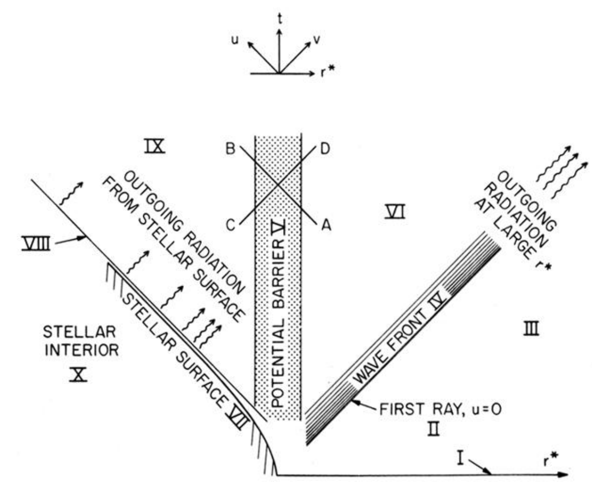

FIG. 4. The "radiation problem" pictured in r, t or u, v coordinates. For explanations and descriptions of features of this diagram, see Table II.