Gravitational Field of a Particle Falling in a Schwarzschild Geometry Analyzed in Tensor Harmonics*†

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(Received 5 January 1970; revised manuscript received 4 August 1970)

We are concerned with the pulse of gravitational radiation given off when a star falls into a "black hole" near the center of our galaxy. We look at the problem of a small particle falling in a Schwarzschild background ("black hole") and examine its spectrum in the high-frequency limit. In formulating the problem it is essential to pose the correct boundary condition: gravitational radiation not only escaping to infinity but also disappearing down the hole. We have examined the problem in the approximation of linear perturbations from a Schwarzschild background geometry, utilizing the decomposition into the tensor spherical harmonics given by Regge and Wheeler (1957) and by Mathews (1962). The falling particle contributes a δ-function source term (geodesic motion in the background Schwarzschild geometry) which is also decomposed into tensor harmonics, each of which "drives" the corresponding perturbation harmonic. The power spectrum radiated in infinity is given in the high-frequency approximation in terms of the traceless transverse tensor harmonics called "electric" and "magnetic" by Mathews.

I. INTRODUCTION

It was pointed out by Dyson[1] that a pulse of gravitational radiation will result from the capture of a star by a black hole (that is, a collapsed star). We consider the problem of a particle of mass m_0 falling along a geodesic of a Schwarzschild geometry produced by a larger mass m. The particle emits gravitational radiation as it falls until it is absorbed through the Schwarzschild surface at $2m$. The question of boundary conditions is interesting here. In a Euclidean topology we would require outgoing waves at infinity and regularity at the origin. In the Schwarzschild case there is no origin. However, the Schwarzschild surface at $2m$ has the property that future timelike or null trajectories pass through it only toward the interior region. Hence a natural boundary condition to replace regularity at the origin is to require that there are only ingoing waves at the Schwarzschild surface, that is, nothing coming out of the black hole.

Zel'dovich and Novikov[2] have considered the problem of the radiation of gravitational waves by bodies moving in the field of a collapsing star. They base their calculations on the formula, given by Landau and Lifshitz,[3] for the gravitational power radiated in terms of the third time derivative of the quadrupole moment of the system. Unfortunately, such considerations can only be valid for bodies which move at distances large compared to the Schwarzschild radius of the central body. But a substantial part of the radiation comes from the region r near $2m$. It is for this reason that we consider the field produced by the falling particle as a perturbation on the background Schwarzschild geometry so that we are not restricted to large distances from $2m$. The source term $T_{\mu\nu}$ is given by an integral over the world line of the particle, the integrand containing a four-dimensional invariant δ function. The source term is then guaranteed to be divergence-free if the world line is a geodesic in the background geometry.

Because of the spherical symmetry of the Schwarzschild field, the field equations for the perturbation $h_{\mu\nu}$ are in the form of a rotationally invariant differential operator on $h_{\mu\nu}$ set equal to the source term $T_{\mu\nu}$. We use this rotational invariance to separate the angular variables in the field equations. The usefulness of scalar harmonics Y_{LM} in separating, for example, Laplace's equation lies in the fact that they transform under a particular irreducible representation of the rotation group. Thus a rotationally invariant operator on Y_{LM} gives a quantity which transforms under the same irreducible representation and hence is a linear combination of Y_{LM} of the same order L. When dealing with tensor fields, we use tensor harmonics which transform under a particular representation of the rotation group.

In Appendix G we discuss the solutions of the $L=0$ and $L=1$ equations. There is no $L=0$ odd-parity-type (magnetic) harmonic. By suitable gauge transformations, it is possible to solve explicitly the partial differential equations in r and t for the $L=0$ and $L=1$ even-parity-type (electric) harmonics and the $L=1$ magnetic harmonics. The $L=0$ electric equations give the expected result. Let $R(t)$, $\Theta(t)$, and $\Phi(t)$ denote the Schwarzschild coordinates of the falling particle at Schwarzschild time t. Then inside the sphere $r<R(t)$ the perturbation from the background is zero, while outside it simply represents an augmentation of the Schwarzschild mass by $m_0\gamma_0$, the mass-energy of the falling particle. The $L=1$ magnetic equations give as a solution zero perturbation for $r<R(t)$, but give an $h_{0\phi}$ term outside of $R(t)$ which goes as $(\sin^2\theta/r)$ and which, according to the criterion given in Landau and Lifshitz,[3] represents a metric with angular momentum m_0a, where

* Based on the author's Ph.D. thesis, Princeton University, 1969 (unpublished).
† Work supported in part by NSF Grant No. GP 7769 and NSF Graduate Fellowship.
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[1] F. Dyson (private communication).
[2] Ya. Zel'dovich and I. Novikov, Dokl. Akad. Nauk SSSR 155, 1033 (1964) [Soviet Phys. Doklady 9, 246 (1964)].
[3] L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley, Reading, Mass., 1962).

$m_0 a$ is the conserved angular momentum of the falling particle. The $L=1$ electric equations also give zero inside $R(t)$ and a nonzero $h_{\mu\nu}$ outside $R(t)$ which can be removed by a gauge transformation which is interpretable by a distant observer as a shift of the origin of the coordinate system. When the particle is far from $2m$, that is, $R(t)$ is large, the shift looks like a transformation to the center-of-momentum system where the particles orbit each other with distances from the center of momentum which are in inverse proportion to their relativistic masses.

For $L\geq 2$ we cannot solve the equations explicitly. We look at the Fourier transform in Schwarzschild time t of the equations which results in ordinary differential equations in r. In the high-frequency limit we obtain asymptotic expansions for the solutions of the homogeneous equations. The angular dependence is, of sourse, given by the tensor harmonics; by a gauge transformation to a gauge in which the leading terms of the expansion are traceless and divergenceless, the fields are expressed in terms of the transverse traceless electric and magnetic harmonics. We can write the expression for the radiated energy using the Landau-Lifshitz pseudotensor for r large compared to $2m$. Isaacson[4] has shown that this is a suitable expression everywhere, in the high-frequency limit, provided it is suitably averaged. Since we are dealing with Fourier transforms, this averaging amounts to taking field amplitudes times their complex conjugates. The stress tensor becomes singular at $r=2m$. If we transform to Kruskal coordinates, we observe that for outgoing waves the singularity is at $2m$ but $t=+\infty$, while for ingoing waves it is singular at $2m$ but at $t=-\infty$. This behavior agrees with Trautman's[5] result for the propagation of a discontinuity in the Riemann tensor in a Schwarzschild field. If we look at the leading terms of the correction to the Riemann tensor for our solution, we see that it has the same type of behavior as Trautman's discontinuity, and that the Riemann tensor is type N or radiation type in the Petrov-Pirani classification.

Using a Green's function formed from the high-frequency-limit solutions of the homogeneous equations, we obtain amplitudes for the ingoing (at $r=2m$) and outgoing (at $r=\infty$) radiation for a particle falling radially into the black hole. We use what is basically the saddle-point procedure to evaluate asymptotically the integral expressions for these amplitudes. We give in Appendix J a rough estimate for the power spectrum, which goes as a power of the frequency for low frequencies, reaches a peak at $\omega\sim 3/16\pi m$, and falls off exponentially for high frequencies. We estimate the total energy radiated to be $(1/625)$ (m_0^2/m) times a factor of order 1.

[4] R. Isaacson, Phys. Rev. **166**, 1263 (1968); **166**, 1272 (1968).
[5] A. Trautman, Lectures on General Relativity, lecture notes, Kings College, London, 1958 (unpublished).

II. WAVE EQUATIONS FOR RADIATION TREATED AS PERTURBATION

We write the metric in the form

$$g_{\alpha\beta}=(g_{\alpha\beta})_{\text{Schwarzschild}}+h_{\alpha\beta}\,,$$

where

$$(g_{\alpha\beta})_{\text{Schwarzschild}}dx^\alpha dx^\beta=-(1-2m/r)dt^2 +(1-2m/r)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta\, d\varphi^2)\,.$$

The complete description in space and time of the gravitational wave generated by the infalling particle is provided by giving the perturbation in the metric, the symmetric tensor $\mathbf{h}=\{h_{\alpha\beta}\}$, as a function of r, θ, ϕ, and t. This tensor is expanded in tensor harmonic functions of angle $\mathbf{a}_{LM}{}^{(0)}(\theta,\phi)$, $\mathbf{a}_{LM}{}^{(1)}(\theta,\phi)$, $\mathbf{a}_{LM}$, $\mathbf{b}_{LM}{}^{(0)}$, $\mathbf{b}_{LM}$, $\mathbf{c}_{LM}{}^0$, $\mathbf{c}_{LM}$, $\mathbf{d}_{LM}$, $\mathbf{g}_{LM}$, and $\mathbf{f}_{LM}$. These ten harmonics are (a) complete over the space of symmetric tensor fields on a two-sphere and (b) orthonormal. They have been treated elsewhere[6–9] and are listed for convenience in Appendix A. Three of these harmonics are labeled "magnetic" and the other seven are labeled "electric." There is some ambiguity in the terminology in the literature. See Appendix B. The general first-order small perturbation in the geometry (Appendix C) is given by the following prescription: (a) Take each row in turn in Table I; (b) multiply each factor in the first column (tensor harmonic; function of θ and ϕ: details in Appendix A) by the factor in the second column ("coefficient in expansion in harmonics"; function of r and t); (c) sum all ten such products ("totalized part of metric perturbation of harmonic indes L, M"); and (d) sum over all L and M. This analysis presumes that one has a way to get the ten radial factors in this expansion directly as functions of r and t. For this purpose it proves simplest to express these factors as Fourier integrals, as, for example,

$$h_{0LM}(r,t)=(2\pi)^{-1/2}\int_{-\infty}^{+\infty}h_{0LM}(\omega,r)e^{-i\omega t}d\omega\,,$$

$$h_{0LM}(\omega,r)=(2\pi)^{-1/2}\int_{-\infty}^{+\infty}h_{0LM}(r,t)e^{i\omega t}dt\,,$$

with similar expressions for the other nine coefficients.

We have chosen the form of the coefficient functions to agree with the notation of Regge and Wheeler.[6] They go on to specialize the gauge (choice of four coordinates; see Appendix D for methods and options) so as to annul the four radial factors $h_{2LM}(r,t)$, $h_{0LM}{}^{(m)}(r,t)$, $h_{1LM}{}^{(m)}(r,t)$, and $G_{LM}(r,t)$ (reduction from 3 to 2 in number of radial factors in "magnetic" part of perturbation; and from 7 to 4 in number of radial factors in "electric" part). This choice of gauge simplifies the differ-

[6] T. Regge and J. A. Wheeler, Phys. Rev. **108**, 1063 (1957).
[7] J. Mathews, J. Soc. Ind. Appl. Math. **10**, 768 (1962).
[8] F. Zerilli, J. Math. Phys. **11**, 2203 (1970).
[9] J. Stachel, Nature **220**, 779 (1968).

TABLE I. Components in expression of perturbation of metric in terms of tensor harmonics. The first three terms describe the "magnetic" part of the perturbation, and the last seven terms describe the "electric" part. The coefficients in this expansion ("radial factors") are listed as functions of r and t, but for the Fourier analysis (see text) the same formulas apply except that now the typical factor, for example, h_{0LM}, is to be understood as a function of ω and r. The third column shows the special values of these factors (whether functions of r, t, or of ω, r) in Regge-Wheeler (RW) gauge. The fourth column applies only in the (ω,r) representation; it gives the radial factors in radiation gauge, insofar as they have been expressed explicitly (details in Appendix I) in terms of the radial factors in RW gauge, as the latter may be evaluated from the differential equations given in the text. The outgoing radiation is governed exclusively by the third and by the next to the last row in the table. The tensor harmonics that come into these two terms are the transverse traceless harmonics $\mathbf{d}_{LM}(\theta,\varphi)$ and $\mathbf{f}_{LM}(\theta,\varphi)$, identical up to a sign with the harmonics $\mathbf{T}$ of Mathews.

Tensor harmonic	Coefficient	Specialization of coefficient to RW gauge	Value of coefficient for case of radiation gauge expressed of coefficients for RW gauge
$\mathbf{c}_{LM}^{(0)}(\theta,\varphi)$	$(-1/r)[2L(L+1)]^{1/2}h_{0LM}(r,t)$	same	zero
$\mathbf{c}_{LM}(\theta,\varphi)$	$(i/r)[2L(L+1)]^{1/2}h_{1LM}(r,t)$	same	$h_{1LM}^{(\mathrm{rad})}(\omega,r)=-(L-1)(L+2)(\omega r)^{-2}$ $\times(1-2m/r)h_{1LM}(\omega,r)$ (neglect at large ωr)
$-\mathbf{T}_{LM}^{(e)}=\mathbf{d}_{LM}(\theta,\varphi)$	$(\frac{1}{2}r^{-2})[2L(L+1)(L-1)(L+2)]^{1/2}h_{2LM}(r,t)$	zero	$h_{2LM}^{(\mathrm{rad})}(\omega,r)=(2i/\omega)h_{0LM}(\omega,r)$
$\mathbf{a}_{LM}^{(0)}(\theta,\varphi)$	$(1-2m/r)H_{0LM}(r,t)$	same	order of $1/\omega^2r^2$ at large ωr
$\mathbf{a}_{LM}^{(1)}(\theta,\varphi)$	$-2^{1/2}iH_{1LM}(r,t)$	same	order of $1/\omega^2r^2$ at large ωr
$\mathbf{a}_{LM}(\theta,\varphi)$	$(1-2m/r)^{-1}H_{2LM}(r,t)$	same	order of $1/\omega^2r^2$ at large ωr
$\mathbf{b}_{LM}^{(0)}(\theta,\varphi)$	$-(i/r)[2L(L+1)]^{1/2}h_{0LM}^{(m)}(r,t)$	zero	zero
$\mathbf{b}_{LM}(\theta,\varphi)$	$(1/r)[2L(L+1)]^{1/2}h_{1LM}^{(m)}(r,t)$	zero	order of $1/\omega^2r^2$ at large ωr
$\mathbf{T}_{LM}^{(m)}=\mathbf{f}_{LM}(\theta,\varphi)$	$[\frac{1}{2}L(L+1)(L-1)(L+2)]^{1/2}G_{LM}(r,t)$	zero	$[\frac{1}{2}L(L+1)(L-1)(L+2)]^{1/2}(1/i\omega r)K_{LM}(\omega,r)$ at large ωr
$\mathbf{g}_{LM}(\theta,\varphi)$	$\sqrt{2}K_{LM}(r,t)-2^{-1/2}L(L+1)G_{LM}(r,t)$	only K term	order of $1/\omega^3r^3$ at large ωr

ential equations. However, it has the feature that the perturbation in the metric increases with distance from the center of attraction in the "electric" part of $\mathbf{h}$; and in the "magnetic" part it keeps an unchanging order of magnitude. By contrast, for the calculation of the radiation one needs a gauge in which the magnitude of the perturbation falls off as $1/r$. The quantities needed in radiation gauge are expressed in Table I in terms of the radial factors in the Regge-Wheeler (RW) gauge because integrations seem easiest to do in the RW gauge. Only for the odd waves is the gauge transformation given explicitly; for the even waves the gauge transformation is spelled out only asymptotically in the limit of large ωr, the limit relevant for radiation (details of gauge transformations in Appendix D).

The perturbation in the geometry is "driven" by the source term in Einstein's field equations, 8π times the tensor of stress and energy, which tensor here—for a test particle moving on a goedesic—takes the form

$$T^{\mu\nu}=m_0\int_{-\infty}^{\infty}\delta^{(4)}(x-z(\tau))\frac{dz^\mu}{d\tau}\frac{dz^\nu}{d\tau}d\tau$$
$$=m_0\frac{dT}{d\tau}\frac{dz^\mu}{dt}\frac{dz^\nu}{dt}\frac{\delta(r-R(t))}{r^2}\delta^{(2)}(\Omega-\Omega(t)),\quad(1)$$

where the notation is as follows: $\delta^{(4)}$ is the invariant δ function, defined by

$$\iiiint\delta^{(4)}(x)(-g)^{1/2}d^4x=1,$$

τ is the proper time along the world line

$$z^\mu=z^\mu(\tau)=(T(\tau),R(\tau),\Theta(\tau),\Phi(\tau)),$$

Ω is an abbreviation for (θ,ϕ), and $\delta^{(2)}(\Omega)=\delta(\cos\theta)\delta(\phi)$. The stress-energy tensor is expressed in spherical harmonics in Appendix E and the procedure is given for evaluating its Fourier transform.

Appendix F gives the ordinary differential equations for the radial factors in the (ω,r) representation, RW gauge. In the odd case, three coupled equations for two radial factors are given; in the even case, where we have specialized to the problem of a mass falling straight in (and where in the absence of this specialization we would have had seven coupled equations for four radial factors), we have six interdependent equations for three radial factors.

An exact solution for the $L=0$ and $L=1$ terms is given in Appendix G. These terms describe the changes in mass, velocity, and angular momentum of the center of attraction produced by the arriving particle.

The harmonics of order $L=2$, 3, ... describe the radiation. Asymptotically for large r in radiation gauge the perturbation $\mathbf{h}$ in the metric is the sum of two simple terms, each involving only one transverse traceless tensor harmonic:

$$\mathbf{h}^{(m)}(r,\theta,\varphi,t)=(2\pi)^{-1/2}\int_{-\infty}^{+\infty}\mathbf{h}^{(m)}(\omega,r,\theta,\varphi)e^{-i\omega t}d\omega$$
$$\sim(2\pi)^{-1/2}\left[\int_{-\infty}^{+\infty}\frac{2i}{\omega}h_{0LM}(\omega,r)e^{-i\omega t}d\omega\right]\mathbf{d}_{LM}(\theta,\varphi)$$

and

$$\mathbf{h}^{(e)}(r,\theta,\varphi,t)\sim(2\pi)^{-1/2}[\tfrac{1}{2}L(L+1)(L-1)(L+2)]^{1/2}$$
$$\times\left[\int_{-\infty}^{+\infty}\frac{1}{i\omega r}K_{LM}(\omega,r)e^{-i\omega t}d\omega\right]\mathbf{f}_{LM}(\theta,\varphi).$$

Asymptotically for large r, we find that radial factors have the form

$$h_{0LM}(\omega,r)\sim -rA_{LM}^{(m)}(\omega)\exp i\omega r^* \qquad (2)$$

and

$$K_{LM}(\omega,r)\sim A_{LM}^{(e)}(\omega)\exp i\omega r^* \qquad (3)$$

for magnetic and electric waves, respectively, where

$$r^*=r+2m\ln(r/2m-1). \qquad (4)$$

The coefficients $A_{LM}(\omega)$ in these asymptotic expressions are found by integrating the radial wave equation. In terms of these coefficients, we can calculate intensity as a function of angle and as a function either of time or of frequency (case of pulse radiation; see Appendix H for case of multiply periodic motion). Thus we have

$$-\frac{d^2E}{d\omega d\Omega}=\frac{1}{32\pi}\sum_{LML'M'}[L(L+1)(L-1)(L+2)\times L'(L'+1)(L'-1)(L'+2)]^{1/2}\times\{A_{L'M'}^{(e)*}(\omega)A_{LM}^{(e)}(\omega)\mathbf{T}_{L'M'}^{(e)*}:\mathbf{T}_{LM}^{(e)}+A_{L'M'}^{(m)*}(\omega)A_{LM}^{(m)}(\omega)\mathbf{T}_{L'M'}^{(m)*}:\mathbf{T}_{LM}^{(m)}\} \qquad (5)$$

and

$$-\frac{dE}{d\omega}=\frac{1}{32\pi}\sum_{LM}L(L+1)(L-1)(L+2)\times[|A_{LM}^{(e)}(\omega)|^2+|A_{LM}^{(m)}(\omega)|^2]. \qquad (6)$$

To determine the distribution of the energy in time rather than in frequency, we must form the Fourier integrals for electric and magnetic waves, and construct the stress-energy tensor from these time-dependent fields. In a more extended account,[10] a treatment is also given for the amount of gravitational energy "going down the black hole."

The part of the radiation of more direct physical interest, that goes to great distances, is evidently determined completely by the coefficients $A_{LM}^{(m)}(\omega)$ and $A_{LM}^{(e)}(\omega)$ ("amplitudes of magnetic and electric waves"). We find these coefficients by solving the wave equations in the two cases, driven by the specified sources, and comparing the asymptotic behavior of the so obtained solutions with the asymptotic expressions (2) and (3).

Appendix I gives the analysis in question for the special case of a particle falling straight into the center of attraction. There is no magnetic term, and all the electric amplitudes also vanish, except for those with $M=0$, for which we find in the limit of high frequencies

$$A_L^{(\text{out})}\sim -4m_0(L+\tfrac{1}{2})^{1/2}e^{-4\pi m\omega}[\tfrac{1}{3}\sqrt{2}e^{5\pi i/8}\times\Gamma(\tfrac{3}{4})(m\omega)^{-3/4}+\tfrac{1}{12}\pi(m\omega)^{-1}+\cdots]. \qquad (7)$$

From this result we find the radiation emitted in the Lth mode to be

$$-(dE/d\omega)_L\sim(1/32\pi)L(L+1)(L-1)(L+2)\times|A_L^{(\text{out})}(\omega)|^2 \qquad (8)$$

at high frequencies. Appendix J discusses the qualitative behavior expected for this contribution at lower frequencies, the total output, and its comparison with the results of Zel'dovich and Novikov[2] based on the formulas cited by Landau and Lifshitz.[3]

For a fuller treatment it will be necessary to solve the radial equations for frequencies where an expansion asymptotic in ω is not appropriate. For this purpose it is possible to use directly the coupled systems of equations given in Appendix F. However, one gains insight into the structure of the equations by transforming from several dependent variables (coupled radial factors) to a single function. It satisfies a second-order wave equation with an effective potential $V_L(r)$ that lends itself to ready visualization. It has a peak ("barrier summit") at an r value of the order $r\sim 3m$ and goes to zero both at $r=\infty$ and as r approaches $2m$. "Large ω" in the asymptotic expansion previously employed means $\omega^2\gg V_{\text{peak}}$. When ω^2 is comparable to or less than V_{peak}, standard JWKB or numerical methods are appropriate. The details of the wave equation complete this paper.

For magnetic waves the new radial factor is $R_{LM}^{(m)}(\omega,r)$. In terms of it, the two old radial factors (RW gauge) are

$$h_{1LM}=r^2R_{LM}^{(m)}/(r-2m) \qquad (9)$$

and

$$h_{0LM}=-\frac{i}{\omega}\frac{d}{dr^*}(rR_{LM}^{(m)})-\frac{8\pi r(r-2m)}{\omega[\frac{1}{2}L(L+1)(L-1)(L+2)]^{1/2}}D_{LM}(\omega,r), \qquad (10)$$

where $D_{LM}(\omega,r)$ is the source term listed in Appendix E. The new radial factor obeys the wave equation [in terms of the new variable $r^*=r+2m\ln(r/2m-1)$]

$$\frac{d^2R_{LM}^{(m)}}{dr^{*2}}+[\omega^2-V_L^{(m)}(r)]R_{LM}^{(m)}=-\frac{8\pi i}{[\frac{1}{2}L(L+1)(L-1)(L+2)]^{1/2}}\frac{r-2m}{r^2}\times\left\{\frac{d}{dr}[r(r-2m)D_{LM}]+2(r-2m)D_{LM}+(r-2m)[(L-1)(L+2)]^{1/2}Q_{LM}\right\}, \qquad (11)$$

where

$$V_L^{(m)}(r)=\left(1-\frac{2m}{r}\right)\left(\frac{L(L+1)}{r^2}-\frac{6m}{r^3}\right) \qquad (12)$$

For electric waves the new radial factor is

[10] F. Zerilli, Ph.D. thesis, Princeton University, 1969 (unpublished).

$R_{LM}{}^{(e)}(\omega,r)$. In terms of it, the old radial factors are

$$K_{LM}=\frac{\lambda(\lambda+1)r^2+3\lambda mr+6m^2}{r^2(\lambda r+3m)}R_{LM}{}^{(e)}+\frac{r-2m}{r}\frac{dR_{LM}{}^{(e)}}{dr}, \quad (13)$$

$$H_{1LM}=-i\omega\frac{\lambda r^2-3\lambda mr-3m}{(r-2m)(\lambda r+3m)}R_{LM}{}^{(e)}-i\omega r\frac{dR_{LM}{}^{(e)}}{dr}, \quad (14)$$

$$H_{0LM}=\frac{\lambda r(r-2m)-\omega^2r^4+m(r-3m)}{(r-2m)(\lambda r+3m)}K_{LM}+\frac{m(\lambda+1)-\omega^2r^3}{i\omega r(\lambda r+3m)}H_{1LM}-\tilde{B}_{LM}, \quad (15)$$

$$H_{2LM}=H_{0LM}+16\pi r^2\times[\tfrac{1}{2}L(L+1)(L-1)(L+2)]^{-1/2}F_{LM}, \quad (16)$$

where

$$\lambda=\tfrac{1}{2}(L-1)(L+2),$$

and

$$\tilde{B}_{LM}=\frac{8\pi r^2(r-2m)}{\lambda r+3m}\{A_{LM}+[\tfrac{1}{2}L(L+1)]^{-1/2}B_{LM}\}-\frac{4\pi\sqrt{2}}{\lambda r+3m}\frac{mr}{\omega}A_{LM}{}^{(1)}. \quad (17)$$

The new radial factor obeys the wave equation

$$\frac{d^2R_{LM}{}^{(e)}}{dr^{*2}}+[\omega^2-V_L{}^{(e)}(r)]R_{LM}{}^{(e)}=S_{LM}, \quad (18)$$

where the source term is

$$S_{LM}=-i\frac{r-2m}{r}\frac{d}{dr}\left[\frac{(r-2m)^2}{r(\lambda r+3m)}\left(\frac{ir^2}{r-2m}\tilde{C}_{1LM}+\tilde{C}_{2LM}\right)\right]+i\frac{(r-2m)^2}{r(\lambda r+3m)^2}\left[\frac{\lambda(\lambda+1)r^2+3\lambda mr+6m^2}{r^2}\tilde{C}_{2LM}+i\frac{\lambda r^2-3\lambda mr-3m^2}{r-2m}\tilde{C}_{1LM}\right],$$

where

$$V_L{}^{(e)}(r)=\left(1-\frac{2m}{r}\right)\times\frac{2\lambda^2(\lambda+1)r^3+6\lambda^2mr^2+18\lambda m^2r+18m^3}{r^3(\lambda r+3m)^2}. \quad (19)$$

The quantities $\tilde{C}_{1LM}$ and $\tilde{C}_{2LM}$ are combinations of source coefficients:

$$\tilde{C}_{1LM}=-\frac{8\pi}{\sqrt{2}\omega}A_{LM}{}^{(1)}-\frac{1}{r}\tilde{B}_{LM}, \quad (20)$$

$$\tilde{C}_{2LM}=\frac{8\pi r^2}{i\omega}\frac{[\tfrac{1}{2}L(L+1)]^{-1/2}}{r-2m}B_{LM}{}^{(0)}-\frac{ir}{r-2m}\tilde{B}_{LM}. \quad (21)$$

ACKNOWLEDGMENT

I am grateful to Professor John A. Wheeler for proposing the problem and for many suggestions and much encouragement.

APPENDIX A: TENSOR HARMONICS

If $\mathbf{T}$ is a symmetric second-rank covariant tensor, it can be expanded in tensor harmonics[6–8] as follows:

$$\mathbf{T}=\sum_{LM} A_{LM}{}^{(0)}\mathbf{a}_{LM}{}^{(0)}+A_{LM}{}^{(1)}\mathbf{a}_{LM}{}^{(1)}+A_{LM}\mathbf{a}_{LM}+B_{LM}{}^{(0)}\mathbf{b}_{LM}{}^{(0)}+B_{LM}\mathbf{b}_{LM}+Q_{LM}{}^{(0)}\mathbf{c}_{LM}{}^{(0)}+Q_{LM}\mathbf{c}_{LM}+G_{LM}\mathbf{g}_{LM}+D_{LM}\mathbf{d}_{LM}+F_{LM}\mathbf{f}_{LM}, \quad (A1)$$

where

$$\mathbf{a}_{LM}{}^{(0)}=\begin{bmatrix} Y_{LM} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}, \quad (A2a)$$

$$\mathbf{a}_{LM}{}^{(1)}=(i/\sqrt{2})\begin{bmatrix} 0 & Y_{LM} & 0 & 0\\ Y_{LM} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}, \quad (A2b)$$

$$\mathbf{a}_{LM}=\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & Y_{LM} & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}, \quad (A2c)$$

$$\mathbf{b}_{LM}{}^{(0)}=ir[2L(L+1)]^{-1/2}\times\begin{bmatrix} 0 & 0 & (\partial/\partial\theta)Y_{LM} & (\partial/\partial\phi)Y_{LM}\\ 0 & 0 & 0 & 0\\ * & 0 & 0 & 0\\ * & 0 & 0 & 0\end{bmatrix}, \quad (A2d)$$

$$\mathbf{b}_{LM}=r[2L(L+1)]^{-1/2}\times\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & (\partial/\partial\theta)Y_{LM} & (\partial/\partial\phi)Y_{LM}\\ 0 & * & 0 & 0\\ 0 & * & 0 & 0\end{bmatrix}, \quad (A2e)$$

$$\mathbf{c}_{LM}{}^{(0)}=r[2L(L+1)]^{-1/2}\times\begin{bmatrix} 0 & 0 & (1/\sin\theta)(\partial/\partial\phi)Y_{LM} & -\sin\theta(\partial/\partial\theta)Y_{LM}\\ 0 & 0 & 0 & 0\\ * & 0 & 0 & 0\\ * & 0 & 0 & 0\end{bmatrix}, \quad (A2f)$$

$$\mathbf{c}_{LM}=ir[2L(L+1)]^{-1/2}\times\begin{bmatrix}0&0&0&0\\0&0&(1/\sin\theta)(\partial/\partial\phi)Y_{LM}&-\sin\theta(\partial/\partial\theta)Y_{LM}\\0&*&0&0\\0&*&0&0\end{bmatrix},\tag{A2g}$$

$$\mathbf{d}_{LM}=-ir^2[2L(L+1)(L-1)(L+2)]^{-1/2}\times\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&-(1/\sin\theta)X_{LM}&\sin\theta\,W_{LM}\\0&0&*&\sin\theta\,X_{LM}\end{bmatrix},\tag{A2h}$$

$$\mathbf{g}_{LM}=(r^2/\sqrt{2})\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&\sin^2\theta\end{bmatrix}Y_{LM},\tag{A2i}$$

$$\mathbf{f}_{LM}=r^2[2L(L+1)(L-1)(L+2)]^{-1/2}\times\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&W_{LM}&X_{LM}\\0&0&*&\sin^2\theta\,W_{LM}\end{bmatrix},\tag{A2j}$$

$$X_{LM}=2\frac{\partial}{\partial\phi}\left(\frac{\partial}{\partial\theta}-\cot\theta\right)Y_{LM},$$

$$W_{LM}=\left(\frac{\partial^2}{\partial\theta^2}-\cot\theta\frac{\partial}{\partial\theta}-\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)Y_{LM}.$$

The symbol $*$ denotes components derived from the symmetry of the tensors. The above set of tensors is orthonormal in the inner product

$$(T,S)\equiv\int\int T^*{:}Sd\Omega,$$

where

$$T{:}S\equiv\eta^{\mu\lambda}\eta^{\nu\kappa}T_{\mu\nu}S_{\lambda\kappa}$$

and $\eta_{\mu\nu}$ is the Minkowski metric. Thus $A_{LM}^{(0)}=(\mathbf{a}_{LM}^{(0)},\mathbf{T})$, $B_{LM}^{(0)}=(\mathbf{b}_{LM}^{(0)},\mathbf{T})$, etc.

In place of $\mathbf{f}_{LM}$, Regge and Wheeler[6] use the harmonic (up to a normalization factor)

$$\mathbf{e}_{LM}=[2L(L+1)-2]^{-1/2}\{[(L-1)(L+2)]^{1/2}\mathbf{f}_{LM}-[L(L+1)]^{1/2}\mathbf{g}_{LM}\}.$$

Two of the harmonics in Eqs. (A2) are the transverse traceless electric and magnetic harmonics given by Mathews[7]:

$$\mathbf{d}_{LM}=-\mathbf{T}_{LM}^{(e)},\quad \mathbf{f}_{LM}=\mathbf{T}_{LM}^{(m)}.$$

Since our background metric is spherically symmetric, Eqs. (2.4) are in the form $Q[\mathbf{h}]=-16\pi\mathbf{T}$, where Q is a rotationally invariant operator. We have denoted the metric perturbation tensor by $\mathbf{h}$ and the source tensor by $\mathbf{T}$. Thus we can separate the angular variables θ, ϕ by expanding $\mathbf{h}$ and $\mathbf{T}$ in tensor harmonics.

APPENDIX B: EVEN-ODD CONVENTION

Table II is a correlation of the even-odd convention as used here and elsewhere.[11]

TABLE II. Correlation between terminology used here for the two types of harmonics and the terminology used elsewhere in the literature.

Listed in Table I	First three	Last seven
Parity	$(-1)^{L+1}$	$(-1)^L$
Number of such tensorial harmonics	3	7
Regge and Wheeler (Ref. 6)	Odd parity	Even parity
Mathews (Ref. 7); Zerilli (Ref. 8)	Electric	Magnetic
Thorne and Campolattaro (Ref. 11)	Odd or magnetic	Even or electric
Present paper	Magnetic	Electric
Emitted when mass falls straight in to Schwarzschild geometry?	No	Yes
Name employed for vector harmonic of same parity in case of electromagnetic radiation[a]	Magnetic	Electric
Emitted when charge falls straight towards copper sphere?	No	Yes

[a] Employing a definition of "magnetic" and "electric" such that "a vector potential A equal to one of these harmonics yields electromagnetic fields of precisely these types (magnetic or electric)."

APPENDIX C: PERTURBATIONS ON SCHWARZSCHILD METRIC

In this appendix we discuss the equations for linear perturbations from a background geometry that enables us to treat the gravitational interaction of two systems in an approximation which is good if the difference between the geometry of the base system (Schwarzschild geometry) and that of the interacting systems (mass falling in Schwarzschild geometry) is small. We will not discuss the question of what constitutes a "small" perturbation since this would be most adequately treated in the theory of "superspace," the set whose elements are three-dimensional Riemannian manifolds.[12] It will suffice to assume the norm in a particular coordinate system to be $\|g_{\mu\nu}\|=\sup\sum_{\mu\nu}|g_{\mu\nu}(x)|^2$, and we will consider a perturbation $\lambda h_{\mu\nu}$ small if there is a gauge transformation to an admissible coordinate system in which $|\,\|g_{\mu\nu}+\lambda h_{\mu\nu}\|-\|g_{\mu\nu}\|\,|\rightarrow 0$ as $\lambda\rightarrow 0$.

Now let us consider linear perturbations[13] on a metric.

[11] K. Thorne and A. Campolattaro, Astrophys. J. **149**, 591 (1967).

[12] J. A. Wheeler, "Superspace and the Nature of Quantum Geometrodynamics," in *Battelle Rencontres* (Benjamin, New York, 1968); A. E. Fischer, Ph.D. thesis, Princeton University, 1969 (unpublished).

[13] We use units in which $G=c=1$. We follow the convention proposed by C. W. Misner, K. S. Thorne, and J. A. Wheeler in "An Open Letter to Relativity Theorists," 1968 (unpublished): Latin indices run from 1 to 3 and indicate spacelike coordinates; Greek indices run from 0 to 2. The metric has signature +2 (spacelike convention). The connection coefficients and Riemann tensor are $\Gamma^\lambda{}_{\mu\nu}=\frac{1}{2}g^{\lambda\sigma}(g_{\sigma\mu,\nu}+g_{\sigma\nu,\mu}-g_{\mu\nu,\sigma})$ and $R^\alpha{}_{\lambda\mu\nu}=\Gamma^\alpha{}_{\lambda\nu,\mu}-\Gamma^\alpha{}_{\lambda\mu,\nu}+\Gamma^\alpha{}_{\sigma\mu}\Gamma^\sigma{}_{\lambda\nu}-\Gamma^\alpha{}_{\sigma\nu}\Gamma^\sigma{}_{\lambda\mu}$. The contracted Riemann tensor is $R_{\mu\nu}=R^\alpha{}_{\mu\alpha\nu}$ and the Einstein equations are $G_{\mu\nu}(g_{\alpha\beta})\equiv R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=8\pi T_{\mu\nu}$.

Let $g_{\mu\nu}$ be a solution of the Einstein equations, and let $\tilde{g}_{\mu\nu}=g_{\mu\nu}+h_{\mu\nu}$. Then $G_{\mu\nu}(\tilde{g}_{\alpha\beta})=8\pi\tilde{T}_{\mu\nu}$, and $h_{\mu\nu}$ must satisfy

$$\Delta G_{\mu\nu}(g_{\alpha\beta},h_{\alpha\beta})\equiv G_{\mu\nu}(g_{\alpha\beta}+h_{\alpha\beta})-G_{\mu\nu}(g_{\alpha\beta})=8\pi(\tilde{T}_{\mu\nu}-T_{\mu\nu})\equiv 8\pi\Delta T_{\mu\nu}.$$

If $T_{\mu\nu}$ is "small," we can assume that $h_{\mu\nu}$ is "small." Now

$$\Delta G_{\mu\nu}(g_{\alpha\beta},h_{\alpha\beta})=\delta G_{\mu\nu}(g_{\alpha\beta})(h_{\rho\sigma})+O[h_{\rho\sigma}{}^2],$$

where $\delta G_{\mu\nu}(g_{\alpha\beta})$ is a linear operator on $h_{\rho\sigma}$ [$\delta G_{\mu\nu}(g_{\alpha\beta})$ is the derivative of the mapping which takes the tensor $g_{\mu\nu}$ into the tensor $G_{\mu\nu}$]. If $h_{\mu\nu}$ is small, then this linear operator is a good approximation to $\Delta G_{\mu\nu}(g_{\alpha\beta},h_{\alpha\beta})$. We thus consider solutions of

$$\delta G_{\mu\nu}(g_{\alpha\beta})(h_{\rho\sigma})=+8\pi\Delta T_{\mu\nu}. \quad \text{(C1)}$$

The expression for $\delta G_{\mu\nu}(g_{\alpha\beta})(h_{\rho\sigma})$ is straightforward to calculate and has been given in several places.[14] Let the semicolon denote covariant differentiation in the base metric $g_{\mu\nu}$. Then

$$-\delta R_{\mu\nu}(g_{\alpha\beta})(h_{\rho\sigma})=\tfrac{1}{2}(h_{\mu\nu;\alpha}{}^{;\alpha}-h_{\mu\alpha;\nu}{}^{;\alpha}-h_{\nu\alpha;\mu}{}^{;\alpha}+h^{\alpha}{}_{\alpha;\mu;\nu}) \quad \text{(C2)}$$

and Eq. (C1) becomes

$$\delta R_{\mu\nu}(g_{\alpha\beta})(h_{\rho\sigma})-\tfrac{1}{2}g_{\mu\nu}(h_{\alpha\lambda}{}^{;\alpha;\lambda}-h_{\alpha}{}^{\alpha}{}_{;\lambda}{}^{;\lambda})-\tfrac{1}{2}h_{\mu\nu}R+\tfrac{1}{2}g_{\mu\nu}h_{\alpha\beta}R^{\alpha\beta}=+8\pi\Delta T_{\mu\nu}.$$

Now by commuting covariant derivatives and denoting $h_{\mu\alpha}{}^{;\alpha}\equiv f_{\mu}$, we write Eq. (C1) as

$$-[h_{\mu\nu;\alpha}{}^{;\alpha}-(f_{\mu;\nu}+f_{\nu;\mu})+2R^{\rho}{}_{\mu}{}^{\alpha}{}_{\nu}h_{\rho\alpha}+h^{\alpha}{}_{\alpha;\mu;\nu}-R^{\rho}{}_{\nu}h_{\mu\rho}-R^{\rho}{}_{\mu}h_{\nu\rho}]-g_{\mu\nu}(f_{\lambda}{}^{;\lambda}-h^{\alpha}{}_{\alpha;\lambda}{}^{;\lambda})-h_{\mu\nu}R+g_{\mu\nu}h_{\alpha\beta}R^{\alpha\beta}=+16\pi\Delta T_{\mu\nu}. \quad \text{(C3)}$$

The part in square brackets is $2\delta R_{\mu\nu}$. Finally, if the background is Ricci flat, $R_{\mu\nu}=0$, then

$$[h_{\mu\nu;\alpha}{}^{;\alpha}-(f_{\mu;\nu}+f_{\nu;\mu})+2R^{\rho}{}_{\mu}{}^{\alpha}{}_{\nu}h_{\rho\alpha}+h^{\alpha}{}_{\alpha;\mu;\nu}]+g_{\mu\nu}(f_{\alpha}{}^{;\alpha}-h^{\alpha}{}_{\alpha;\lambda}{}^{;\lambda})=-16\pi\Delta T_{\mu\nu}. \quad \text{(C4)}$$

Consider now "gauge" transformations. If we make a coordinate transformation $x'^{\mu}=x^{\mu}+\xi^{\mu}$, then the transformed tensor field is $\tilde{g}'_{\mu\nu}=\tilde{g}_{\mu\nu}-\xi_{\mu;\nu}-\xi_{\nu;\mu}$ keeping terms to first order in ξ_{μ}. We can assume $\tilde{g}_{\mu\nu}'=g_{\mu\nu}+h'_{\mu\nu}$, then $h'_{\mu\nu}=h_{\mu\nu}-\xi_{\mu;\nu}-\xi_{\nu;\mu}$. We note[4] that if $R_{\mu\nu}(g_{\alpha\beta})=0$, it is consistent with (C4) to choose a gauge in which $h_{\mu\nu}{}^{;\nu}=0$ and $h_{\mu}{}^{\mu}=0$.

We will now restrict our attention to a background geometry which has spherical symmetry[15] and whose metric in a local coordinate system can be written

$$-e^{\nu}dt^2+e^{\lambda}dr^2+r^2(d\theta^2+\sin^2\theta\, d\phi^2). \quad \text{(C5)}$$

In particular, for the Schwarzschild geometry, $e^{\nu}=1-2m/r$ and $e^{\lambda}=e^{-\nu}$. In this case the coordinate system is singular at $r=2m$. However, we know that the manifold is smooth at $2m$ since there are coordinate systems which cover the region near $r=2m$ which are nonsingular and in which the metric tensor is analytic. In fact, the maximal analytic extension of this manifold is covered by one coordinate patch, that of Kruskal.[16] The singularity at $r=0$, however, is real and is a three-dimensional manifold which is the boundary of the four-dimensional space-time.[17] All the physics dealt with in this paper takes place in the region from $r=2m$ to $r=\infty$.

Expanding the perturbation $h_{\mu\nu}$ and the source tensor $\Delta T_{\mu\nu}$ in tensor harmonics, the field equations (C4) become, in the case of magnetic-parity harmonics,

$$\frac{\partial^2 h_0}{\partial r^2}-\frac{\partial^2 h_1}{\partial r\partial t}-\frac{2}{r}\frac{\partial h_1}{\partial t}+\left[\frac{4m}{r^2}-\frac{L(L+1)}{r}\right]\frac{h_0}{r-2m}=-8\pi r[\tfrac{1}{2}L(L+1)]^{-1/2}Q_{LM}{}^{(0)}, \quad \text{(C6a)}$$

$$\frac{\partial^2 h_1}{\partial t^2}-\frac{\partial^2 h_0}{\partial r\partial t}+\frac{2}{r}\frac{\partial h_0}{\partial t}+(L-1)(L+2)(r-2m)\frac{h_1}{r^3}=8\pi i(r-2m)[\tfrac{1}{2}L(L+1)]^{-1/2}Q_{LM}, \quad \text{(C6b)}$$

$$\left(1-\frac{2m}{r}\right)\frac{\partial h_1}{\partial r}-\left(1-\frac{2m}{r}\right)^{-1}\frac{\partial h_0}{\partial t}+\frac{2m}{r^2}h_1=8\pi i r^2[\tfrac{1}{2}L(L+1)(L-1)(L+2)]^{-1/2}D_{LM}. \quad \text{(C6c)}$$

For the electric harmonics, we have

$$\left(1-\frac{2m}{r}\right)^2\frac{\partial^2 K}{\partial r^2}+\left(1-\frac{2m}{r}\right)\left(3-\frac{5m}{r}\right)\frac{1}{r}\frac{\partial K}{\partial r}-\left(1-\frac{2m}{r}\right)^2\frac{1}{r}\frac{\partial H_2}{\partial r}-\left(1-\frac{2m}{r}\right)\frac{1}{r^2}(H_2-K)-\left(1-\frac{2m}{r}\right)\frac{1}{2r^2}L(L+1)(H_2+K)=8\pi A_{LM}{}^{(0)}, \quad \text{(C7a)}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial K}{\partial r}+\frac{1}{r}(K-H_2)-\frac{m}{r(r-2m)}K\right)-\frac{L(L+1)}{2r^2}H_1=\frac{8\pi i}{\sqrt{2}}A_{LM}{}^{(1)}, \quad \text{(C7b)}$$

$$\left(1-\frac{2m}{r}\right)^{-2}\frac{\partial^2 K}{\partial t^2}-\frac{1-m/r}{r-2m}\frac{\partial K}{\partial r}-\frac{2}{r-2m}\frac{\partial H_1}{\partial t}+\frac{1}{r}\frac{\partial H_0}{\partial r}+\frac{1}{r(r-2m)}(H_2-K)+\frac{L(L+1)}{2r(r-2m)}(K-H_0)=8\pi A_{LM}, \quad \text{(C7c)}$$

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[14] C. Lanczos, Z. Physik **31**, 112 (1925); E. Lifshitz, J. Phys. USSR **10**, 116 (1946); P. C. Peters, Phys. Rev. **146**, 938 (1966).

[15] For a discussion of spherically symmetric space-times, see J. P. Vajk, Ph.D. thesis, Princeton University, 1968 (unpublished).

[16] M. D. Kruskal, Phys. Rev. **119**, 1743 (1960); for a discussion of the Kruskal picture, see R. Fuller and J. A. Wheeler, *ibid.* **128**, 919 (1962).

[17] R. Geroch, J. Math. Phys. **9**, 450 (1968).

$$\frac{\partial}{\partial r}\left[\left(1-\frac{2m}{r}\right)H_1\right]-\frac{\partial}{\partial t}(H_2+K)$$
$$=-8\pi i r[\tfrac{1}{2}L(L+1)]^{-1/2}B_{LM}{}^{(0)}, \quad \text{(C7d)}$$

$$-\frac{\partial H_1}{\partial t}+\left(1-\frac{2m}{r}\right)\frac{\partial}{\partial r}(H_0-K)$$
$$+\frac{2m}{r^2}H_0+\frac{1-m/r}{r}(H_2-H_0)$$
$$=-8\pi(r-2m)[\tfrac{1}{2}L(L+1)]^{-1/2}B_{LM}, \quad \text{(C7e)}$$

$$-\left(1-\frac{2m}{r}\right)^{-1}\frac{\partial^2 K}{\partial t^2}+\left(1-\frac{2m}{r}\right)\frac{\partial^2 K}{\partial r^2}+\left(1-\frac{m}{r}\right)\frac{2}{r}\frac{\partial K}{\partial r}$$
$$-\left(1-\frac{2m}{r}\right)^{-1}\frac{\partial^2 H_2}{\partial t^2}+\frac{2\partial^2 H_1}{\partial r\partial t}-\left(1-\frac{2m}{r}\right)\frac{\partial^2 H_0}{\partial r^2}$$
$$+\frac{2}{r-2m}\left(1-\frac{m}{r}\right)\frac{\partial H_1}{\partial t}-\frac{1}{r}\left(1-\frac{m}{r}\right)\frac{\partial H_2}{\partial r}$$
$$-\frac{1}{r}\left(1+\frac{m}{r}\right)\frac{\partial H_0}{\partial r}-\frac{1}{2r^2}L(L+1)(H_2-H_0)$$
$$=-8\pi\sqrt{2}G_{LM}, \quad \text{(C7f)}$$

$$\tfrac{1}{2}(H_0-H_2)$$
$$=-8\pi r^2[\tfrac{1}{2}L(L+1)(L-1)(L+2)]^{-1/2}F_{LM}. \quad \text{(C7g)}$$

These equations are written in the RW gauge. The terms on the right-hand side are the "radial factors" in the expansion of the source in tensor harmonics (see Appendix A). We obtain, for each (L,M), three magnetic-parity equations for two unknown functions h_0 and h_1, and seven electric-parity equations for four unknown functions H_0, H_1, H_2, and K. Consistency is assured for the vacuum equations since the Einstein equations satisfy the Bianchi identities; in the case where we have a nonzero source term, this implies that the divergence of the source stress tensor $\mathbf{T}$ must be zero. It may be verified that the divergence of the stress tensor vanishes if and only if the source particle follows a geodesic.

APPENDIX D: GAUGE TRANSFORMATION

The general perturbation is expressed by

$$\mathbf{h}=\sum_{LM}[\mathbf{h}_{LM}{}^{(e)}+\mathbf{h}_{LM}{}^{(m)}], \quad \text{(D1)}$$

where the magnetic $[(-1)^{L+1}$ parity$]$ terms are

$$\mathbf{h}_{LM}{}^{(m)}=(i/r)[2L(L+1)]^{1/2}[ih_{0LM}(r,t)\mathbf{c}_{LM}{}^{(0)}(\theta,\phi)+h_{1LM}(r,t)\mathbf{c}_{LM}(\theta,\phi)-(i/2r^2)\times[(L-1)(L+2)]^{1/2}h_{2LM}(r,t)\mathbf{d}_{LM}(\theta,\phi)] \quad \text{(D2a)}$$

and the electric $[(-1)^L$ parity$]$ terms are

$$\mathbf{h}_{LM}{}^{(e)}=(1-2m/r)H_{0LM}\mathbf{a}_{LM}{}^{(0)}-\sqrt{2}iH_{1LM}\mathbf{a}_{LM}{}^{(1)}+(1-2m/r)^{-1}H_{2LM}\mathbf{a}_{LM}-(1/r)[2L(L+1)]^{1/2}\times(ih_{0LM}{}^{(m)}\mathbf{b}_{LM}{}^{(0)}-h_{1LM}{}^{(m)}\mathbf{b}_{LM})+[\tfrac{1}{2}L(L+1)(L-1)(L+2)]^{1/2}G_{LM}\mathbf{f}_{LM}+\sqrt{2}[K_{LM}-\tfrac{1}{2}L(L+1)G_{LM}]\mathbf{g}_{LM}. \quad \text{(D2b)}$$

We have chosen the form of the coefficient functions to agree with the notation of Regge and Wheeler.[6]

We can simplify (D2a) and (D2b) by using the freedom to choose a particular gauge. Since we can choose four linearly independent vector fields ξ, we can eliminate four of the ten coefficient functions (see Table I) in (D2). Let ∇ denote the covariant derivative in the Schwarzschild geometry. Then the gauge-transformed perturbation is

$$\mathbf{h}'=\mathbf{h}-2[\nabla\xi]_s,$$

where the subscript s denotes symmetrization.

There is one vector harmonic of order (LM) and parity $(-1)^{L+1}$, viz., $(0,\mathbf{L}Y_{LM})$. Thus let

$$\xi_{LM}{}^{(m)}=(i/r)\Lambda_{LM}(r,t)(0,\mathbf{L}Y_{LM}(\theta,\phi)).$$

Then

$$2[\nabla\xi_{LM}{}^{(m)}]_s=(i/r)[2L(L+1)]^{1/2}\{i(\partial\Lambda_{LM}/\partial t)\mathbf{c}_{LM}{}^{(0)}+r^2(\partial/\partial r)(\Lambda_{LM}/r^2)\mathbf{c}_{LM}+(1/r)\times[(L-1)(L+2)]^{1/2}\Lambda_{LM}\mathbf{d}_{LM}\}. \quad \text{(D3)}$$

Regge and Wheeler's choice of gauge makes h_{2LM} zero. Thus

$$h_{LM}{}^{(m)}=(i/r)[2L(L+1)]^{1/2}(ih_{0LM}\mathbf{c}_{LM}{}^{(0)}+h_{1LM}\mathbf{c}_{LM}).$$

There are three vector harmonics of degree (LM) and parity $(-1)^L$; they are e_tY_{LM}, e_rY_{LM}, and $(0,\nabla Y_{LM})$ (e_t and e_r are unit vectors along t and r). Thus let

$$\xi_{LM}{}^{(e)}=M_0(r,t)Y_{LM}e_t+M_1(r,t)Y_{LM}e_r+M_2(r,t)(0,\nabla Y_{LM}).$$

Then verify that

$$2[\nabla\xi_{LM}{}^{(e)}]_s=2\left(\frac{\partial M_0}{\partial t}-\frac{m}{r^3}(r-2m)M_1\right)\mathbf{a}_{LM}{}^{(0)}-\sqrt{2}i\left[\frac{\partial M_1}{\partial t}+\frac{\partial M_0}{\partial r}-\frac{2m}{r(r-2m)}M_0\right]\mathbf{a}_{LM}{}^{(1)}$$
$$+2\left[\frac{\partial M_1}{\partial r}+\frac{m}{r(r-2m)}M_1\right]\mathbf{a}_{LM}-\frac{i}{r}[2L(L+1)]^{1/2}\left(\frac{\partial M_2}{\partial t}+M_0\right)\mathbf{b}_{LM}{}^{(0)}+\frac{1}{r}[2L(L+1)]^{1/2}\left(\frac{\partial M_2}{\partial r}-\frac{2}{r}M_2+M_1\right)\mathbf{b}_{LM}$$
$$+\frac{\sqrt{2}}{r^2}[2(r-2m)M_1-L(L+1)M_2]\mathbf{g}_{LM}+\frac{1}{r^2}[2L(L+1)(L-1)(L+2)]^{1/2}M_2\mathbf{f}_{LM}. \quad \text{(D4)}$$

We can choose the three functions M_{0LM}, M_{1LM}, and M_{2LM} to eliminate three of the coefficients in (D2b). Regge and Wheeler choose the gauge so that

$$\mathbf{h}_{LM}^{(e)}=(1-2m/r)H_{0LM}\mathbf{a}_{LM}^{(0)}-\sqrt{2}iH_{1LM}\mathbf{a}_{LM}^{(1)}+(1-2m/r)^{-1}H_{2LM}\mathbf{a}_{LM}+\sqrt{2}K_{LM}\mathbf{g}_{LM}.$$

The preceding considerations hold in general for $L\geq 2$. For $L=0$ and $L=1$, the situation is somewhat simpler since there are fewer independent harmonics. For $L=0$, it is clear that $\mathbf{h}_{00}^{(m)}\equiv 0$ since $\mathbf{c}_{LM}^{(0)}$, $\mathbf{c}_{LM}$, and $\mathbf{d}_{LM}$ are zero for $L=0$, while

$$\mathbf{h}_{00}^{(e)}=(1-2m/r)H_{0}\mathbf{a}_{00}^{(0)}-\sqrt{2}iH_{1}\mathbf{a}_{00}^{(1)}+(1-2m/r)^{-1}\mathbf{a}_{00}+\sqrt{2}K\mathbf{g}_{00}, \quad \text{(D5)}$$

since $\mathbf{b}_{LM}^{(0)}$, $\mathbf{b}_{LM}$, and $\mathbf{f}_{LM}$ are zero for $L=0$. For $L=1$, since $\mathbf{d}_{1M}\equiv 0$ and $\mathbf{f}_{1M}\equiv 0$, we have

$$\mathbf{h}_{1M}^{(m)}=(2i/r)(ih_0\mathbf{c}_{1M}^{(0)}+h_1\mathbf{c}_{1M}), \quad \text{(D6)}$$

while

$$\mathbf{h}_{1M}^{(e)}=(1-2m/r)H_0\mathbf{a}_{1M}^{(0)}-\sqrt{2}iH_1\mathbf{a}_{1M}^{(1)}+(1-2m/r)^{-1}\mathbf{a}_{1M}-(2i/r)h_0^{(m)}\mathbf{b}_{1M}^{(0)}+(2/r)h_1^{(m)}\mathbf{b}_{1M}+\sqrt{2}K\mathbf{g}_{1M}. \quad \text{(D7)}$$

Thus our gauge transformations allow us to eliminate one of the functions h_0 or h_1 for the magnetic harmonics and allow us to eliminate three of the six (instead of seven for $L\geq 2$) functions for the electric harmonics.

APPENDIX E: STRESS-ENERGY TENSOR EXPRESSED IN TENSOR HARMONICS

The following prescription gives the stress-energy tensor expressed in terms of tensor harmonics: (a) Take each row in turn in Table III; (b) multiply the factor in the first column ("coefficient in expansion in harmonics"; a function of r and t) by the factor in the second column (tensor harmonic; function of θ and ϕ); (c) sum all ten such products ("totalized part of stress energy tensor of index L, M"); and (d) sum over all L and M.

For the Fourier transforms of these radial factors in the expansion of the source, we multiply by $(2\pi)^{-1/2}$ and by $e^{i\omega t}dt$ and integrate from $-\infty$ to $+\infty$. The analysis is simplest when $r=R(t)$ is a monotonic function of time. Then we write $dt=dR/(dR/dt)$. The function $\delta(r-R(t))$ integrates out immediately. The net

TABLE III. Components in expression of stress-energy tensor of test particle in terms of tensor harmonics. The symbol γ is an abbreviation for the quantity $\gamma=dT(\tau)/d\tau$. The first five and the last two terms drive the electric part of the perturbation in the geometry; the remaining three terms drive the magnetic part. The arrangement of the terms is chosen to make more readily apparent the similarities in form between one coefficient and another. In the table Y_{LM} denotes the usual normalized spherical harmonic, X_{LM} and W_{LM} are functions derived from Y_{LM} as listed at the end of Appendix A, and * denotes complex conjugate.

Description	Dependence of "driving term" on r and t	Tensor harmonic (Appendix A)
Electric	$A_{LM}(r,t)=m_0\gamma\left(\frac{dR}{dt}\right)^2(r-2m)^{-2}\delta(r-R(t))Y_{LM}^*(\Omega(t))$	$\mathbf{a}_{LM}(\theta,\phi)$
Electric	$A_{LM}^{(0)}=m_0\gamma\left(1-\frac{2m}{r}\right)^2r^{-2}\delta(r-R(t))Y_{LM}^*(\Omega(t))$	$\mathbf{a}_{LM}^{(0)}(\theta,\phi)$
Electric	$A_{LM}^{(1)}=\sqrt{2}im_0\gamma\frac{dR}{dt}r^{-2}\delta(r-R(t))Y_{LM}^*(\Omega(t))$	$\mathbf{a}_{LM}^{(1)}(\theta,\phi)$
Electric	$B_{LM}^{(0)}=[\frac{1}{2}L(L+1)]^{-1/2}im_0\gamma\left(1-\frac{2m}{r}\right)r^{-1}\delta(r-R(t))dY_{LM}^*(\Omega(t))/dt$	$\mathbf{b}_{LM}^{(0)}(\theta,\phi)$
Electric	$B_{LM}=[\frac{1}{2}L(L+1)]^{-1/2}m_0\gamma(r-2m)^{-1}\frac{dR}{dt}\delta(r-R(t))dY_{LM}^*(\Omega(t))/dt$	$\mathbf{b}_{LM}(\theta,\phi)$
Magnetic	$Q_{LM}^{(0)}=[\frac{1}{2}L(L+1)]^{-1/2}m_0\gamma\left(1-\frac{2m}{r}\right)r^{-1}\delta(r-R(t)) \times\left[\frac{1}{\sin\Theta}\frac{\partial Y_{LM}^*}{\partial\Phi}\frac{d\Theta}{dt}-\sin\Theta\frac{\partial Y_{LM}^*}{\partial\Theta}\frac{d\Phi}{dt}\right]$	$\mathbf{c}_{LM}^{(0)}(\theta,\phi)$
Magnetic	$Q_{LM}=[\frac{1}{2}L(L+1)]^{-1/2}im_0\gamma(r-2m)^{-1}\delta(r-R(t))\frac{dR}{dt} \times\left[\frac{1}{\sin\Theta}\frac{\partial Y_{LM}^*}{\partial\Phi}\frac{d\Theta}{dt}-\sin\Theta\frac{\partial Y_{LM}^*}{\partial\Theta}\frac{d\Phi}{dt}\right]$	$\mathbf{c}_{LM}(\theta,\phi)$
Magnetic, transverse, traceless	$D_{LM}=-[\frac{1}{2}L(L+1)(L-1)(L+2)]^{-1/2}im_0\gamma\delta(r-R(t))\left\{\frac{1}{2}\left[\left(\frac{d\Theta}{dt}\right)^2-(\sin\Theta^2)\left(\frac{d\Phi}{dt}\right)^2\right] \times\frac{1}{\sin\Theta}X_{LM}^*[\Omega(t)]-\sin\Theta\frac{d\Phi}{dt}\frac{d\Theta}{dt}W_{LM}^*[\Omega(t)]\right\}$	$\mathbf{d}_{LM}(\theta,\phi)$
Electric, transverse, traceless	$F_{LM}=[\frac{1}{2}L(L+1)(L-1)(L+2)]^{-1/2}m_0\gamma\delta(r-R(t))\left\{\frac{d\Theta}{dt}\frac{d\Phi}{dt}X_{LM}^*[\Omega(t)] +\frac{1}{2}\left[\left(\frac{d\Theta}{dt}\right)^2-(\sin\Theta)^2\left(\frac{d\Phi}{dt}\right)^2\right]W_{LM}^*[\Omega(t)]\right\}$	$\mathbf{f}_{LM}(\theta,\phi)$
Electric	$G_{LM}=\frac{m_0\gamma}{\sqrt{2}}\delta(r-R(t))\left[\left(\frac{d\Theta}{dt}\right)^2+(\sin\Theta)^2\left(\frac{d\Phi}{dt}\right)^2\right]Y_{LM}^*[\Omega(t)]$	$\mathbf{g}_{LM}(\theta,\phi)$

result is to transform each radial factor in the table in the following three respects: (a) Multiply by $(2\pi)^{-1/2}$; (b) replace $e^{i\omega t}$ by $e^{i\omega T(r)}$, where $T(r)$ is the function inverse to $R(t)$; and (c) replace the δ function by $1/(dR/dt)$. For example, in the case when the particle falls from $r=\infty$ to $2m$ starting at $t=-\infty$ with zero velocity, we have

$$A_L^{(0)}(\omega,r)=(m_0/2\pi)[(L+\tfrac{1}{2})(r/2m)]^{1/2}(1/r^2)e^{i\omega T(r)},$$
$$A_L^{(1)}(\omega,r)=-i(m_0/2\pi)(2L+1)^{1/2}(1/r^2)\times(1-2m/r)^{-1}e^{i\omega T(r)}, \quad \text{(E1)}$$
$$A_L(\omega,r)=(m_0/2\pi)[(L+\tfrac{1}{2})(2m/r)]^{1/2}\times(r-2m)^{-2}e^{i\omega T(r)}.$$

APPENDIX F: FOURIER-TRANSFORMED FIELD EQUATIONS FOR RADIAL FACTORS IN METRIC PERTURBATION

We write the Fourier transform of the field equations listed in Appendix C. The magnetic equations are

$$\omega^2 h_{1LM}-\frac{i\omega dh_{0LM}}{dr}+\frac{2i\omega h_{0LM}}{r}-(r-2m)(L-1)(L+2)\frac{h_{1LM}}{r^3}=-8\pi i[\tfrac{1}{2}L(L+1)]^{-1/2}(r-2m)Q_{LM}(\omega,r), \quad \text{(F1a)}$$

$$(r-2m)\left(\frac{d^2h_{0LM}}{dr^2}+\frac{i\omega dh_{1LM}}{dr}+\frac{2i\omega h_{1LM}}{r}\right)+\left[\frac{4m}{r^2}-\frac{L(L+1)}{r}\right]h_{0LM}=-8\pi[\tfrac{1}{2}L(L+1)]^{-1/2}\times r^2Q_{LM}^{(0)}(\omega,r), \quad \text{(F1b)}$$

$$-(r-2m)\frac{dh_{1LM}}{dr}-\frac{i\omega r^2h_{0LM}}{r-2m}-\frac{2mh_{1LM}}{r}=8\pi i[\tfrac{1}{2}L(L+1)(L-1)(L+2)]^{-1/2}r^3D_{LM}(\omega,r). \quad \text{(F1c)}$$

The electric parity equations are

$$\left(1-\frac{2m}{r}\right)^2\frac{d^2K}{dr^2}+\frac{1}{r}\left(1-\frac{2m}{r}\right)\left(3-\frac{5m}{r}\right)\frac{dK}{dr}-\frac{1}{r}\left(1-\frac{2m}{r}\right)^2\frac{dH_2}{dr}-\frac{1}{r^2}\left(1-\frac{2m}{r}\right)(H_2-K)-L(L+1)\frac{1}{2r^2}\left(1-\frac{2m}{r}\right)(H_2+K)=8\pi A_{LM}^{(0)}(\omega,r), \quad \text{(F2a)}$$

$$-\omega^2\left(1-\frac{2m}{r}\right)^{-2}K+L(L+1)\frac{1}{2r^2}\left(1-\frac{2m}{r}\right)^{-1}(K-H_0)+\frac{1}{r(r-2m)}(H_2-K)+\frac{2i\omega}{r}\left(1-\frac{2m}{r}\right)^{-1}H_1-\frac{1}{r}\left(1-\frac{m}{r}\right)\left(1-\frac{2m}{r}\right)^{-1}\frac{dK}{dr}+\frac{1}{r}\frac{dH_0}{dr}=8\pi A_{LM}(\omega,r), \quad \text{(F2b)}$$

$$-i\omega\left[\frac{dK}{dr}+\frac{1}{r}(K-H_2)-\frac{m}{r^2}\left(1-\frac{2m}{r}\right)^{-1}K\right]-L(L+1)\left(\frac{1}{2r^2}\right)H_1=4\pi\sqrt{2}iA_{LM}^{(1)}(\omega,r), \quad \text{(F2c)}$$

$$\frac{d}{dr}\left[\left(1-\frac{2m}{r}\right)H_1\right]+i\omega(H_2+K)=-8\pi i[\tfrac{1}{2}L(L+1)]^{-1/2}rB_{LM}^{(0)}(\omega,r), \quad \text{(F2d)}$$

$$i\omega H_1+\frac{2m}{r^2}H_0+\left(1-\frac{2m}{r}\right)\frac{d}{dr}(H_0-K)+\frac{1}{r}\left(1-\frac{m}{r}\right)(H_2-H_0)=-8\pi[\tfrac{1}{2}L(L+1)]^{-1/2}\times(r-2m)B_{LM}(\omega,r), \quad \text{(F2e)}$$

$$r^2\omega^2\left(1-\frac{2m}{r}\right)^{-1}(H_2+K)+r(r-2m)\frac{d^2}{dr^2}(K-H_0)-\frac{2i\omega r^2dH_1}{dr}-2i\omega r\left(1-\frac{m}{r}\right)\left(1-\frac{2m}{r}\right)^{-1}H_1+2(r-m)\frac{dK}{dr}-r\left(1-\frac{m}{r}\right)\frac{dH_2}{dr}-r\left(1+\frac{m}{r}\right)\frac{dH_0}{dr}-\tfrac{1}{2}L(L+1)(H_2-H_0)=-\frac{16\pi r^2}{\sqrt{2}}G_{LM}(\omega,r), \quad \text{(F2f)}$$

$$\tfrac{1}{2}(H_{0LM}-H_{2LM})=-16\pi[2L(L+1)(L-1)(L+2)]^{-1/2}\times r^2F_{LM}(\omega,r). \quad \text{(F2g)}$$

These equations were first given by Regge and Wheeler for the case of vacuum perturbations. There were some minor errors in the magnetic-parity equations given by Regge and Wheeler. A corrected version has been given by Vishveshwara.[18] The equations given here differ from those of Regge-Wheeler-Vishveshwara only in that we consider variations in $G_{\mu\nu}$ rather than $R_{\mu\nu}$. The sources terms are zero in (F2d)–(F2f) in the case for motion without angular momentum along the z

[18] C. V. Vishveshwara, Ph.D. thesis, University of Maryland, 1968 (unpublished).

axis. Also in this case, the solution of the magnetic-parity equations which satisfies the boundary conditions is the zero solution. Thus we will only have to consider the electric equations for the case of the particle falling straight in.

For the magnetic-parity waves we have three equations. Equation (F1b) is a consequence of (F1a) and (F1c), provided that the source term satisfies the divergence condition. Thus we have a system of two first-order linear equations. The two first-order equations (F1a) and (F1c) can be expressed as a simple second-order Schrödinger-type equation (see text).[6]

For the electric-parity waves we have six equations and three unknown functions. Three of the first-order equations are sufficient to determine a solution provided the divergence conditions on the source term are satisfied. Further, as noted by Regge and Wheeler,[6] we obtain an algebraic equation relating the three unknown functions. Let us take (F2c)–(F2e) as our basic equations and solve them for the first derivatives of K, H_2, and H_1 as follows:

$$\frac{dK}{dr}+\frac{1}{r}\left(1-\frac{3m}{r}\right)\left(1-\frac{2m}{r}\right)^{-1}K-\frac{1}{r}H_2+L(L+1)\frac{1}{2i\omega r^2}H_1=-\frac{4\pi\sqrt{2}}{\omega}A_{LM}{}^{(1)},\quad\text{(F3a)}$$

$$\begin{aligned}\frac{dH_2}{dr}&+\frac{1}{r}\left(1-\frac{3m}{r}\right)\left(1-\frac{2m}{r}\right)^{-1}K-\frac{1}{r}\left(1-\frac{4m}{r}\right)\left(1-\frac{2m}{r}\right)^{-1}H_2\\&+\left[i\omega\left(1-\frac{2m}{r}\right)^{-1}+L(L+1)\frac{1}{2i\omega r^2}\right]H_1\\&=-\frac{4\pi\sqrt{2}}{\omega}A_{LM}{}^{(1)}-8\pi r[\tfrac{1}{2}L(L+1)]^{-1/2}B_{LM}\\&\quad+16\pi[\tfrac{1}{2}L(L+1)(L-1)(L+2)]^{-1/2}\times\left[\frac{d}{dr}(r^2F_{LM})-\frac{r-3m}{r(-r2m)}F_{LM}\right],\quad\text{(F3b)}\end{aligned}$$

$$\frac{dH_1}{dr}+i\omega\left(1-\frac{2m}{r}\right)^{-1}(K+H_2)+\frac{2m}{r^2}\left(1-\frac{2m}{r}\right)^{-1}H_1=-\frac{8\pi ir^2}{r-2m}[\tfrac{1}{2}L(L+1)]^{-1/2}B_{LM}{}^{(0)}.\quad\text{(F3c)}$$

If we substitute these in (F2a), we obtain a relation between the source terms which is the divergence condition. If we substitute (F3) into (F2b), we obtain

$$\begin{aligned}F(r)&=16\pi r(r-2m)A_{LM}-\frac{8\pi\sqrt{2}m}{\omega}A_{LM}{}^{(1)}+\frac{16\pi r(r-2m)}{[\frac{1}{2}L(L+1)]^{1/2}}B_{LM}-16\pi[(L-1)(L+2)+6m/r]\\&\quad\times[\tfrac{1}{2}L(L+1)(L-1)(L+2)]^{-1/2}r^2F_{LM},\quad\text{(F4)}\end{aligned}$$

where

$$\begin{aligned}F(r)&=-\left[\frac{6m}{r}+(L-1)(L+2)\right]H_2+\left[(L-1)(L+2)-2\omega^2r^2\left(1-\frac{2m}{r}\right)^{-1}+\frac{2m}{r}\left(1-\frac{3m}{r}\right)\left(1-\frac{2m}{r}\right)^{-1}\right]K\\&\quad+\left[2i\omega r+L(L+1)\frac{m}{i\omega r^2}\right]H_1.\end{aligned}$$

A straightforward way of showing the consistency of (F3) and (F4) is to eliminate one of the functions H, H_1, and K from two of the equations (F3) and use the divergence condition to show that the third equation is satisfied identically. Thus we reduce our original system to a system of two first-order equations for two unknown functions. We can proceed to write this as a single second-order equation,[19] given in the text [Eq. (18)].

APPENDIX G: SOLUTIONS OF $L=0$, 1 EQUATIONS

The fact that for $L=0$ and $L=1$ there are fewer independent tensor harmonics makes it possible for us to give exact solutions of the perturbation equations in these cases. We have noted previously that the $L=0$ magnetic-parity harmonic is identically zero. Thus we are left with the $L=1$ magnetic equations and the $L=0$ and $L=1$ electric equations. We will, for these three cases, give the solutions to the homogeneous equations and the solutions for the case where there is a source term which is produced by a point mass m_0 falling on a geodesic of the background.

A. Monopole (Mass) Perturbation

Let us consider, first, the $L=0$ electric-parity equations. The general form of the perturbation is given by Eq. (D5). Making a gauge transformation $\xi^{(e)}=M_0Y_{00}e_t+M_1Y_{00}e_r$, we can choose $M_0(r,t)$ and $M_1(r,t)$ so that $H_1(r,t)=K(r,t)=0$. Since $\mathbf{b}_{00}{}^{(0)}\equiv\mathbf{b}_{00}\equiv\mathbf{f}_{00}\equiv0$, the only trivial magnetic equations for $L=0$ are Eqs. (C7a)–(C7c) and (C7f). Equation (C7f) is satisfied identically by a solution of the other three provided that the source term satisfies the divergence condition. In the source-free case, Eqs. (C7a) and (C7b) give

$$H_2=2(4\pi)^{1/2}c/(r-2m),$$

[19] F. Zerilli, Phys. Rev. Letters **24**, 737 (1970).

where c is a constant, while (C7c) gives

$$H_0=H_2+f(t),$$

where f is an arbitrary function of time. A gauge transformation of the form $\xi^{(e)}=M_0 e_t Y_{00}$ allows us to eliminate $f(t)$. From Eq. (D4) we see that if

$$M_0(r,t)=\tfrac{1}{2}(1-2m/r)\int f(t)dt,$$

then in the new gauge we have

$$H_0=H_2=2(4\pi)^{1/2}c/(r-2m), \quad \text{(G1)}$$

and so

$$\mathbf{h}_{00}{}^{(e)}=(16\pi)^{1/2}cr^{-1}\mathbf{a}_{00}{}^{(0)}+(1-2m/r)^{-2}+(16\pi)^{1/2}cr^{-1}\mathbf{a}_{00}, \quad \text{(G2)}$$

and up to the linear approximation in which we are dealing, this is simply

$$\tilde{g}_{00}=g_{00}+h_{00}=1-2m/r+2c/r,$$
$$\tilde{g}_{11}=g_{11}+h_{11}=(1-2m/r+2c/r)^{-1}. \quad \text{(G3)}$$

Thus we obtain the result which we would expect: The $L=0$ perturbation, being spherically symmetric, represents only a change in the Schwarzschild mass, a result required by Birkhoff's theorem. Regge and Wheeler[6] stated this result by explicitly assuming the solutions to be time independent.

Now let us consider the case where there is a point mass m_0 falling along the geodesics of the Schwarzschild geometry. From Eq. (C7) we have

$$(\partial/\partial r)[(r-2m)H_2]=-(16\pi)^{1/2}m_0\gamma(1-2m/r)\times\delta(r-R(t)),$$

and integrating we obtain

$$(r-2m)H_2=0, \quad r<R(t)$$
$$=-(16\pi)^{1/2}m_0[1-2m/R(t)]dT/ds, \quad r>R(t).$$

But $[1-2m/R(t)]dT/ds=\gamma_0$, and γ_0 is a constant of the motion which is an energy parameter; for example, $\gamma_0=1$ if the particle falls from infinity starting with zero velocity $dR/ds=0$ at $r=\infty$. Thus

$$H_2(r,t)=0, \quad r<R(t)$$
$$=-2(4\pi)^{1/2}m_0\gamma_0/(r-2m), \quad r>R(t). \quad \text{(G4)}$$

From Eq. (C7c) we obtain

$$H_0(r,t)=0, \quad r<R(t)$$
$$=-2(4\pi)^{1/2}m_0\gamma_0/(r-2m)+f(t), \quad r>R(t) \quad \text{(G5)}$$

where

$$f(t)=2(4\pi)^{1/2}m_0\gamma\frac{(dR/dt)^2}{[1-2m/R(t)]^2R(t)}.$$

As in the vacuum case, $f(t)$ is eliminated by a gauge transformation and, referring to Eqs. (G1)–(G3), we see that the solution is that of a Schwarzschild field of mass m inside the two-sphere of radius $R(t)$ and is a Schwarzschild field of mass $m+m_0\gamma_0$ outside the sphere of radius $R(t)$.

B. Magnetic Dipole (Angular Momentum) Perturbation

Now we discuss the $L=1$ magnetic-parity equations. Equation (D6) gives the general form of the perturbation. We perform a gauge transformation $\xi_{1M}{}^{(m)}=(i/r)\Lambda_{1M}(r,t)(0,\mathbf{L}Y_{1M})$, which makes $h_0(r,t)=0$. Then

$$\mathbf{h}_{1M}{}^{(m)}=(2i/r)h_1(r,t)\mathbf{c}_{1M}.$$

Since $\mathbf{d}_{1M}\equiv 0$, the only nontrivial electric equations for $L=1$ are (C6a) and (C6b). Thus, in the source-free case, (C6a) gives $h_1(r,t)=f(r)t+g(r)$, where f and g are arbitrary functions of r. Then (C6b) gives $f'(r)+(2/r)f(r)=0$, whose solution is $f(r)=3c/r^2$ where c is a constant. The function $g(r)$ is entirely arbitrary. However, $g(r)$ can be eliminated by a guage transformation; at the same time we will transform to a gauge in which the perturbation is easily interpretable. We choose the gauge function

$$\Lambda(r,t)=-\frac{ct}{r}+r^2\int g(r)r^{-2}dr.$$

Then in the new guage, $h_1=0$ while $h_0=-\partial\Lambda/\partial t=c/r$. Thus

$$\mathbf{h}_{1M}{}^{(m)}=(2ic/r^2)\mathbf{c}_{1M}{}^{(0)}. \quad \text{(G6)}$$

This metric has only 0θ and 0ϕ components. We will see that this perturbation can be interpreted as adding angular momentum to the background metric. If we transform this perturbation to Kruskal coordinates, we find that it is singular at $2m$. However, Vishveshwara[18] has shown that a suitable gauge transformation brings it to a form which is regular everywhere in Kruskal coordinates.

Landau and Lifshitz[3] show that, for a weak field, the coefficients of the $1/r^2$ terms of the $dtdx^i$ components of the metric tensor are related to the angular momentum tensor (in a Cartesian frame at infinity):

$$h_{0i}=\sum_{j=1}^{3}\frac{2M_{ij}n_j}{r^2}, \quad i=1,2,3 \quad \text{(G7)}$$

where n_i are the components of e_r in Cartesian coordinates: $n_x=\sin\theta\ \sin\phi$, $n_y=\sin\theta\cos\phi$, $n_z=\cos\theta$. Transforming (G7) to spherical coordinates, we obtain

$$h_{0r}=0,$$
$$h_{0\theta}=(2/r)(l_x\sin\phi-l_y\cos\phi), \quad \text{(G8)}$$
$$h_{0\phi}=(2/r)[-l_z\sin^2\theta+(l_x\cos\phi+l_y\sin\phi)\sin\theta\cos\theta],$$

where $l_x=M_{yz}$, $l_y=M_{zx}$, and $l_z=M_{xy}$. We now note that (G8) can be written as a sum of the tensor harmonics $\mathbf{c}_{1M}{}^{(0)}$ as follows:

$$\mathbf{h}=(4i/r^2)(\tfrac{4}{3}\pi)^{1/2}(l_{+1}\mathbf{c}_{11}{}^{(0)}+l_z\mathbf{c}_{10}{}^{(0)}+l_{-1}\mathbf{c}_{-1,1}{}^{(0)}), \quad \text{(G9)}$$

where $l_{\pm 1}=\mp\frac{1}{2}\sqrt{2}(l_x\pm il_y)$. We see that the solution (G6) gives a perturbation which has angular momentum.

Let us now proceed to the case where there is a source term due to a particle orbiting in the Schwarzschild field. Since the motion lies in a plane, assume that $\Theta(t)=\frac{1}{2}\pi$ and then from (C6b) we obtain

$$\frac{1}{r^2}\frac{\partial^2}{\partial t\partial r}(r^2h_1)=8\pi m_0\left(\frac{3}{4\pi}\right)^{1/2}\delta(r-R(t))\frac{a}{R^2},$$

where $a=R^2d\Phi/ds$ is the z component of angular momentum per unit mass of the orbiting particle. Since $\Theta=\frac{1}{2}\pi$, the z component is the only nonzero component (and the only nonzero source terms Q_{1M} and $Q_{1M}^{(0)}$ have the index $M=0$). Integrating with respect to r, we obtain

$$\begin{aligned} h_1(r,t)&=0\,, \quad r<R(t)\\ &=8\pi(3/4\pi)^{1/2}m_0at/r^2\,, \quad r>R(t)\,. \end{aligned} \tag{G10}$$

As in the homogeneous case, if we perform a gauge transformation with

$$\begin{aligned} \Lambda(r,t)&=0\,, \quad r<R(t)\\ &=-2m_0a(3/4\pi)^{-1/2}t/r\,, \quad r>R(t) \end{aligned}$$

we obtain, in the new gauge,

$$\begin{aligned} \mathbf{h}^{(m)}&=0\,, \quad r<R(t)\\ &=(2i/r)(4\pi/3)^{1/2}(2m_0a/r)\mathbf{c}_{10}^{(0)}\,, \quad r>R(t)\,. \end{aligned} \tag{G11}$$

Referring to Eq. (G9), we see that this perturbation is zero inside the sphere $r<R(t)$ while it has, for $r>R(t)$, angular momentum

$$l_z=m_0a\,,$$

which is precisely the conserved angular momentum of the orbiting particle.

C. Electric Dipole (Coordinate) Perturbation

Proceeding now to the last of the nonradiative cases, we discuss the $L=1$ magnetic-parity equations. Equation (D7) gives the general form of the perturbation. Using our gauge freedom, we can make $K_{1M}=0$. Since $\mathbf{f}_{1M}\equiv 0$, there are six field equations: (C7a)–(C7f). Three of the equations determine a solution, and consistency is assured by the divergence condition on the coefficients $A_{1M}^{(0)}$, $A_{1M}^{(1)}$, etc. Looking at the homogeneous case, we see by substitution that any solution which satisfies (C7a) and (C7b) automatically satisfies (C7d). Also (C7e) and (C7d) imply (C7f). To show this, differentiate (C7e) with respect to r and (C7d) with respect to t. Finally, from (C7e) and (C7c) we obtain

$$3mH_0=mH_2-r^2\partial H_1/\partial t\,. \tag{G12}$$

Thus the system is reduced to (C7a)–(C7c) and (G12). But it is easily verified that any solution of (C7a), (C7b), and (G12) identically satisfies (C7). In the homogeneous case, the solution of (C7a) is

$$H_2(r,t)=f(t)/(r-2m)^2\,,$$

where f is an arbitrary function of t. Then (C7b) and (G12) give, for H_0 and H_1,

$$\begin{aligned} H_1(r,t)&=-rf'(t)/(r-2m)^2\,,\\ H_0(r,t)&=f(t)/3(r-2m)^2+r^3f''(t)/3m(r-2m)^2\,. \end{aligned}$$

This will also be the form of the solution to the inhomogeneous equations, where the function $f(t)$ will be determined by the source term.

We will now see that this perturbation can be removed entirely by a guage transformation which we will attempt to interpret as a translation to the c.m. system of the two bodies. Thus we find a vector field $\xi_1^{(e)}=M_0e_tY_{1M}+M_1e_rY_{1M}+M_2\nabla Y_{1M}$ such that

$$2[\nabla\xi_{1M}^{(e)}]_s=\mathbf{h}_{1M}^{(e)}\,.$$

In fact, let

$$\begin{aligned} M_2&=r^2g(t)/(r-2m)\,,\\ M_1&=r^2g(t)/(r-2m)^2\,,\\ M_0&=-r^2g'(t)/(r-2m)\,, \end{aligned}$$

where $g(t)=-f(t)/6m$. Thus the $L=1$ magnetic perturbations are strictly removable by a gauge transformation.

Now we give an interpretation of this gauge transformation. In the limit of large r ($r\gg 2m$), we can write

$$\xi_{1M}^{(e)}\sim -g_M'(t)re_tY_{1M}+g_M(t)e_rY_{1M}+g_M(t)r\nabla Y_{1M}\,. \tag{G13}$$

Just for the purposes of the following discussion, let ∇ denote the operator whose covariant components are $(-\partial/\partial t,\ \partial/\partial x,\ \partial/\partial y,\ \partial/\partial z)$ in a Cartesian coordinate system; and since we consider r large, we will raise and lower indices with the Minkowski metric $(+1, -1, -1, -1)$. Then (G13) may be written compactly

$$\xi_{1M}^{(e)}\sim\nabla[g_M(t)rY_{1M}(\theta,\phi)]\,.$$

But $rY_{1M}=(3/4\pi)^{1/2}x_M$, where $x_0=z$, $x_{\pm1}=\mp\frac{1}{2}\sqrt{2}\times(x\pm iy)$. For example, in contravariant components,

$$\xi_{10}^{(e)}\sim -(3/4\pi)^{1/2}(g_0'(t)z,0,0,g_0(t))\,.$$

Thus we identify this with a displacement along the z axis by $\rho_0(t)=(3/4\pi)^{1/2}g_0(t)$. That is, $x'^\mu=x^\mu+\xi^\mu$ or

$$\begin{aligned} t'&=t-\rho_0'(t)z\,, \quad x'=x\,,\\ y'&=y\,, \quad z'-z=\rho_0(t)\,. \end{aligned}$$

This is a Lorentz transformation along the z axis; $\rho_0'(t)$ is the velocity v. The factor of $(1-v^2)^{1/2}$ does not appear here since we are dealing with linear perturbations and this factor is of quadratic rather than linear order. Similarly, $\xi_{11}^{(e)}+\xi_{1,-1}^{(e)}$ represents a translation in the (x,y) plane by $(\rho_x(t),\rho_y(t))$, where $\rho_x=-(3/8\pi)^{1/2}\times(g_{+1}-g_{-1})$ and $\rho_y=-i(3/8\pi)^{1/2}(g_{+1}+g_{-1})$. Thus we identify the gauge transformation as the analog, in the

Schwarzschild geometry, of a Lorentz transformation in flat space since this is what it looks like to the distant observer.

Now let us consider the inhomogeneous equations. The solution of Eq. (C7a) is

$$H_{2M}(r,t)=0\,,\quad r<R(t)$$
$$=f_M(t)/(r-2m)^2\,,\quad r>R(t)$$

where

$$f_M(t)=-8m\pi_0\gamma(t)Y_{1M}^*(\Omega(t))[R(t)-2m]^2/R(t)\,.$$

Now from (C7b) we obtain

$$H_{1M}(r,t)=0\,,\quad t<R(r)$$
$$=-rf_M'(t)/(r-2m)^2\,,\quad r>R(t)\,.$$

Note that the source terms contribute only a δ function on the surface $r=R(t)$ to H_{1M}. The same is true for H_0, and using (G12), we obtain

$$H_{0M}(r,t)=0\,,\quad r<R(t)$$
$$=[f_M(t)+(r^3/m)f_M''(t)]/3(r-2m)^2\,,\quad r>R(t)\,.$$

Thus the solution is of the same form as that for the homogeneous case, and $f_M(t)$ is determined by the source term. Now define $\rho(t)$ by

$$\rho(t)=(m_0/m)\gamma(t)[R(t)-2m]^2/R(t)\,.$$

Then $f_M(t)=-8\pi m\rho(t)Y_{1M}^*(\Omega(t))$, and the gauge transformation which eliminates this perturbation is (covariant components in spherical coordinates)

$$\begin{aligned}\xi_0&=-\frac{\partial}{\partial t}[r^2(r-2m)^{-1}\rho(t)\sigma(t,\theta,\phi)]\,,\\ \xi_1&=\left[\left(1-\frac{2m}{r}\right)^{-2}\rho(t)\sigma(t,\theta,\phi)\right]\\ \xi_2&=\frac{\partial}{\partial\theta}\left[r\left(1-\frac{2m}{r}\right)^{-1}\rho(t)\sigma(t,\theta,\phi)\right]\,,\\ \xi_3&=\frac{\partial}{\partial\phi}\left[r\left(1-\frac{2m}{r}\right)^{-1}\rho(t)\sigma(t,\theta,\phi)\right]\,,\end{aligned}\qquad\text{(G14)}$$

where $\sigma(t,\theta,\phi)=\frac{4}{3}\pi\sum_M Y_{1M}^*(\Omega(t))Y_{1M}(\theta,\phi)$. But $r\rho(t)\times\sigma(t,\theta,\phi)=\mathbf{x}\cdot\boldsymbol{\xi}(t)$, where $\boldsymbol{\xi}(t)=(\rho(t),\Theta(t),\Phi(t))$. Thus for r large compared with $2m$, we can write

$$\xi\sim\nabla[\boldsymbol{\xi}(t)\cdot\mathbf{x}]\,,$$

where ∇ is defined as above in the discussion after Eq. (G13). We see that this represents the Lorentz transformation

$$t'=t-\mathbf{x}\cdot\boldsymbol{\xi}'(t)\,,$$
$$\mathbf{x}'=\mathbf{x}-\boldsymbol{\xi}(t)\,,$$

since, in contravariant components,

$$\nabla[\boldsymbol{\xi}(t)\cdot\mathbf{x}]=-[\boldsymbol{\xi}'(t)\cdot\mathbf{x},\boldsymbol{\xi}(t)]\,.$$

Thus we identify the gauge transformation as an "analog" Lorentz transformation to a moving frame given by the displacement

$$\boldsymbol{\xi}(t)=\{(m_0/m)\gamma(t)[R(t)-2m]^2/R(t),\,\Theta(t),\,\Phi(t)\}\,.$$

When the falling particle m_0 is far from the central mass m, that is, $R(t)\gg 2m$, then

$$\boldsymbol{\xi}(t)\sim[m_0(/m)\gamma(t)R(t),\Theta(t),\Phi(t)]\,,$$

which is the usual classical mechanical transformation to the center-of-momentum frame of the two bodies. Thus we are tempted to interpret the gauge transformation (G14) as the analog of a "transformation to the center-of-momentum system."

APPENDIX H: RADIATION IN MULTIPLY PERIODIC MOTION

The present investigation is concerned with the pulse of gravitational radiation given out when a small mass m_0 plunges into the black hole associated with a much larger mass m. In this connection we analyze the mechanical quantities descriptive of the motion of m_0, and analyze the gravitational radiation itself into Fourier integrals. The results of such an analysis for radiation emitted in a hyperbolic orbit are already known not to be at all complicated when the departure of the space from flatness is so slight that the geometry can be idealized as nearly Lorentzian. In that limit, the familiar textbook formula for the rate of radiation,

$$-\frac{dE}{dt}=\frac{1}{45}\frac{d^3Q^{pq}(t)}{dt^3}\frac{d^3Q^{pq}(t)}{dt^3}\,,$$

in terms of the quadrupole moment

$$Q^{pq}(t)=\sum_{\text{all masses}}m(3x^px^q-\delta^{pq}x^sx^s)\,,$$

lends itself readily to Fourier analysis:

$$Q^{pq}(t)=(2\pi)^{-1/2}\int_{-\infty}^{+\infty}Q^{pq}(\omega)e^{-i\omega t}d\omega\,,$$

$$Q^{pq}(\omega)=(2\pi)^{-1/2}\int_{-\infty}^{+\infty}Q^{pq}(t)e^{i\omega t}dt\,.$$

The amount of energy emitted per unit interval of frequency ν $(=\omega/2\pi)$, integrated over the entire time of the pulse, is

$$-dE/d\nu=(4\pi/45)Q^{pq*}(\omega)Q^{pq}(\omega)\,.$$

This expression reduces at low frequencies (ω small compared to the reciprocal of the time required for the orbit to undergo the major part of its change in direction) to[20]

$$-\frac{dE}{d\nu}=\frac{2}{45}\sum_{p,q}\Big[\sum_{\text{masses}} m(3\dot{x}^p\dot{x}^q-\delta^{pq}\dot{x}^s\dot{x}^s)\big|_{\text{before}}^{\text{after}}\Big]^2,$$

where the dot means differentiation with respect to t. We expect the formulas given in the present text to reduce to these simple expressions in the appropriate limit. However, the present analysis should by no means be considered to be confined to the case of aperiodic motion and pulse radiation. On the contrary, we have in atomic physics (see, for example, Born's *Atommechanik*) an example of the analysis of multiply periodic motion, and of the radiation given out in such motion, which can be taken over practically without change to the present problem. The energy of the system is expressed as a function of three action variables,

$$E=E(J_r,J_\theta,J_\varphi),$$

of which the last two measure total angular momentum and angular momentum about the z axis. The circular frequencies associated with the three modes of motion are

$$\omega_r=\partial E/\partial J_r,\quad \omega_\theta=\partial E/\partial J_\theta,\quad \omega_\varphi=\partial E/\partial J_\varphi.$$

In the limit of nearly flat space, where Cartesian coordinates are appropriate, one has

$$x=\sum_{n_r,n_\theta,n_\varphi} X_{n_r n_\theta n_\varphi}\exp[-i(n_r\omega_r+n_\theta\omega_\theta+n_\varphi\omega_\varphi)t],$$

and similar Fourier series for the other two coordinates and for the third time rate of change of the quadrupole moment as well. The rate of emission of quadrupole gravitational radiation at the circular frequency $\omega=n_r\omega_r+n_\theta\omega_\theta+n_\varphi\omega_\varphi$ is

$$\left(-\frac{dE}{dt}\right)_{\omega=n_\mu\omega_\mu}=\frac{2}{45}\left(\frac{d^3Q^{pq*}}{dt^3}\right)_{n_r,n_\theta,n_\varphi}\left(\frac{d^3Q^{pq}}{dt^3}\right)_{n_r,n_\theta,n_\varphi}.$$

The generalization of this formula to the Schwarzschild geometry allows itself in principle to be read out of the present work, once the determination of the appropriate Fourier coefficients in the multiply periodic motion has been carried through.

APPENDIX I: AMPLITUDES FOR SPECIAL CASE OF PARTICLE FALLING STRAIGHT IN

We do a standard asymptotic expansion in the parameter ω for large ω. This simultaneously gives us an asymptotic expansion for large r. First consider the magnetic equations, in particular, the homogeneous case.

[20] R. Ruffini and J. A. Wheeler, "Cosmology from Space Platforms," European Space Research Organization report (unpublished).

Let $\epsilon=+1$ for outgoing waves and $\epsilon=-1$ for incoming waves; then we may write

$$h_1\sim r(1-2m/r)^{-1}[1-(\epsilon/i\omega r)(\lambda+1-3m/2r)+O(1/\omega^2)]\exp(\epsilon i\omega r^*),\quad \text{(I1)}$$
$$h_0\sim -\epsilon r[1-(\epsilon/i\omega r)(\lambda+m/r)+O(1/\omega^2)]\exp(\epsilon i\omega r^*).$$

Now we notice that for large r these functions go as $O(r)$. If we transform the perturbation to a Cartesian coordinate system, this implies that the perturbation goes as $O(1)$. We can easily see this by looking at Eq. (D2a).

Now in order to apply the usual pseudotensor criteria for energy and momentum radiated, we must have a perturbation that goes as $O(1/r)$ in order that the space be asymptotically flat. We remedy the difficulty by a gauge transformation. This gauge transformation does several things. It makes the new $h_0^{(N)}=0$ and the new $h_1^{(N)}=O(1/\omega r)$ and thus these functions do not contribute in the radiation field since $h_1=O(1/\omega r)$ implies that the perturbation is $O(1/\omega r^2)$. At the same time, it introduces a nonzero $h_2^{(N)}$ in the new gauge which is $O(r)$ and hence, as can be seen by looking at Eq. (D2a), the perturbation is then $O(1/r)$, which enables us to use the pseudotensor to calculate the energy or momentum radiated. Note that the angular dependence of the radiation term is given by Mathews's electric harmonic and that in this gauge the perturbation is divergenceless and traceless to $O(1/r)$. The form of the gauge transformation which does all this is easily found. From Eq. (D3) we see that setting $\Lambda_{LM}=-(1/i\omega)h_{0LM}$ makes $h_0^{(N)}$ zero. Also, using Eq. (F1a), we see that

$$h_{1LM}^{(N)}=-(2\lambda/\omega^2r^2)(1-2m/r)h_{1ML},$$

which is $O(1/r)$. Finally,

$$h_{2LM}^{(N)}=2\Lambda_{LM}=-(2/i\omega)h_{0LM}.$$

We will then denote the canonical solution of the homogeneous magnetic equations by $\mathbf{h}_{LM}^{(m)}(\omega,r,\phi)$ and

$$\mathbf{h}_{LM}^{(m)(\epsilon)}\sim(\epsilon/2i\omega r)\exp(\epsilon i\omega r^*)\times[2L(L+1)(L-1)(L+2)]^{1/2}\times\mathbf{T}_{LM}^{(e)}(\theta,\phi)+O(1/\omega^2r^2).\quad \text{(I2)}$$

Gauge transformations of a similar nature have been discussed by Edelstein[21] and by Price and Thorne.[22] By looking at (I1), we see that the second term of the asymptotic expansion is less than the first term when $2m\omega>\lambda+1$. Thus we can expect the asymptotic expansion to be a good approximation for $4m\omega>L(L+1)$. This means that the approximation is good if ω^2 is above the peak of the effective potential $V(r)$ given by Regge and Wheeler.

Now let us turn to the magnetic-parity equations. Again we look at the homogeneous equations.

[21] L. Edelstein, Ph.D. thesis, University of Maryland, 1969 (unpublished).

[22] R. Price and K. Thorne, Astrophys. J. 155, 163 (1969).

Letting $\epsilon=+1$ denote outgoing waves and $\epsilon=-1$ denote ingoing waves, we have the result

$$K^{(\epsilon)}\sim\left\{1+\epsilon\left(\frac{3m}{2i\omega r^2}\right)+\frac{1}{\omega^2}\left[\frac{\lambda(\lambda+1)}{2r^2}+\frac{m(2\lambda-1)}{2r^3}+\frac{15m^2}{8r^4}\right]+O(1/\omega^3)\right\}\exp(\epsilon i\omega r^*),$$

$$H^{(\epsilon)}\sim-\epsilon H_1{}^{(\epsilon)}+O(1/\omega^3),\quad\text{(I3)}$$

$$H_1{}^{(\epsilon)}\sim-i\omega r\left(1-\frac{2m}{r}\right)^{-1}\left\{1-\frac{\epsilon}{i\omega}\left(\frac{\lambda}{r}+\frac{3m}{2r^2}\right)-\frac{1}{\omega^2}\left[\frac{\lambda(\lambda+1)}{2r^2}+\frac{m(\lambda+1)}{2r^3}-\frac{3m^2}{8r^4}\right]+O(1/\omega^3)\right\}\times\exp(\epsilon i\omega r^*).$$

The perturbation given by (I3) is $O(r)$ as $r\to\infty$, as can be seen by looking at Eq. (D2b). Again we must find a suitable gauge transformation so that the perturbation is $O(1/r)$ as $r\to\infty$. This means that in the new gauge all the coefficient functions in (D2b) must be $O(1/r)$ except for $h_0{}^{(e)}$ and $h_1{}^{(e)}$, which must be $O(1)$ in order for $\mathbf{h}_{LM}{}^{(e)}\sim O(1/r)$.

We will be able to find a gauge transformation which makes $h_0{}^{(e)}=0$ and which makes the perturbation divergenceless, traceless, and proportional to Mathews's transverse traceless magnetic harmonic, at least up to the order of the radiation terms. Most important of all, it gives the proper asymptotic dependence for the perturbation as $r\to\infty$. From (D2b) and (D4) we obtain seven equations relating M_0, M_1, M_2, H, H_1, and K to the transformed coefficient functions [denoted with a superscript (N)]. Since K is $O(1)$ and we want $K^{(N)}$ to be $O(1/r)$, we must require

$$M_1=-\tfrac{1}{2}r(1-2m/r)^{-1}[1+\mu_1(r)/\omega+\mu_2(r)/\omega^2]\times\exp(\epsilon i\omega r^*),$$

where $\mu_1(r)=O(1/r)$, $\mu_2(r)=O(1/r^2)$ are still free to be chosen. The requirement $H_0{}^{(N)}=H_2{}^{(N)}$ fixes M_0 in terms of M_1, and it follows also that $H_1{}^{(N)}=-\epsilon H^{(N)}+O(1/\omega^3)$. The condition $h_0{}^{(e)(N)}\equiv0$ fixes $M_2=(1/i\omega)M_1$. We can then choose $\mu_1(r)$ and $\mu_2(r)$ so that $H^{(N)}$ and $H_1{}^{(N)}$ are $O(1/\omega^2)$ [and hence also $O(1/r^2)$]. Then $G^{(N)}$ and $h_1{}^{(e)(N)}$ are fixed by the remaining relations: $K^{(N)}-\tfrac{1}{2}L(L+1)G^{(N)}=O(1/\omega^3r^3)$, $(1/r)h_1{}^{(e)(N)}=O(1/\omega^2r^2)$, and finally

$$K^{(N)(\epsilon)}=\frac{\epsilon}{2i\omega r}L(L+1)\left[1-\frac{\epsilon(2\lambda r+m)}{2i\omega r^2}+O(1/\omega^2r^2)\right]\times\exp(\epsilon i\omega r^*).\quad\text{(I4)}$$

Thus

$$\mathbf{h}_{LM}{}^{(e)}=\frac{\epsilon}{2i\omega r}\exp(\epsilon i\omega r^*)\times[2L(L+1)(L-1)(L+2)]^{1/2}\mathbf{T}_{LM}{}^{(m)}+O(1/\omega^2r^2).\quad\text{(I5)}$$

The trace of $\mathbf{h}_{LM}{}^{(e)}$ is $O(1/\omega^3r^3)$ and $\mathrm{div}(\mathbf{h}_{LM}{}^{(e)})=O(1/\omega^2r^2)$.

Using these results, we construct a high-frequency-limit Green's function, and, applying the boundary condition of outgoing waves at $r=\infty$ and ingoing waves at $r=2m$, we obtain amplitudes for the ingoing and outgoing radiation fields as integrals. One of these integrals is evaluated in an asymptotic expansion in ω by the method of saddle points in a beautiful procedure given by van der Waerden.[23] The other integral is evaluated in an asymptotic expansion using a theorem given by Copson.[24] First we discuss the stress tensor for radiation and some pecularities of the wave solutions of the homogeneous equations.

Isaacson[4] shows that in the high-frequency limit (wavelength of the radiation short compared with the curvature of the background space) for small perturbations a suitable stress tensor for gravitational radiation is

$$T_{\mu\nu}=(32\pi)^{-1}\{h_{\rho\sigma;\mu}h^{\rho\sigma}{}_{;\nu}\}_{\mathrm{av}}+O(h^3),\quad\text{(I6)}$$

where $\{\ \}_{\mathrm{av}}$ denotes an average over several wavelengths of the radiation. The expression (I6) is valid in the gauge where $\mathrm{tr}(\mathbf{h})$ and $\mathrm{div}(\mathbf{h})$ are zero. The result (I6) is also equivalent to using the Landau-Lifshitz[3] pseudotensor in the limit of large r. Since we are dealing with the Fourier transforms of the fields, averaging corresponds to taking the field amplitudes times their complex conjugates; that is,

$$T_{\mu\nu}=(32\pi)^{-1}h^*{}_{\rho\sigma;\mu}h^{\rho\sigma}{}_{;\nu}.$$

We are interested in the energy density and flux given by the components of $T_{\mu\nu}$ with μ and ν equal to zero or one. If we keep only the leading terms in ω, we can replace the covariant derivative with respect to t by $-i\omega$ and the covariant derivative with respect to r by $\epsilon i\omega(1-2m/r)^{-1}$. Now let

$$h(\omega,r,\theta,\phi)=\sum_{LM}\{A_{LM}{}^{(e)}(\omega)\mathbf{h}_{LM}{}^{(e)}(\omega,r,\theta,\phi)+A_{LM}{}^{(m)}(\omega)\mathbf{h}_{LM}{}^{(m)}(\omega,r,\theta,\phi)\},$$

where $\mathbf{h}_{LM}{}^{(e)}$ and $\mathbf{h}_{LM}{}^{(m)}$ are solutions which, in the absence of sources, are given asymptotically by (I5) and (I2). Let us consider the energy flux for large r. Then the power per unit solid angle per unit frequency

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[23] B. L. van der Waerden, Appl. Sci. Res. **B2**, 33 (1960); also discussed by H. A. Lauwerier, *Asymptotic Expansions* (Mathematisch Centrum, Amsterdam, 1966).

[24] E. T. Copson, *Asymptotic Expansions* (Cambridge U. P., New York, 1965), p. 21.

for a particular electric or magnetic LM multipole is

$$\frac{dS_{LM}(\omega,\Omega)}{d\Omega}=\epsilon(32\pi)^{-1}L(L+1)(L-1)(L+2)\times|A_{LM}(\omega)|^2\mathbf{T}_{LM}^*:\mathbf{T}_{LM}, \quad (17)$$

while the total power per unit frequency is

$$\lim_{r\to\infty} 2\,\mathrm{Re}\left\{r^2\int\int T_{10}d\Omega\right\}$$

or

$$S(\omega)=\epsilon(32\pi)^{-1}\sum_{LM}L(L+1)(L-1)(L+2)\times\{|A_{LM}^{(e)}(\omega)|^2+A_{LM}^{(m)}(\omega)|^2\}. \quad (I8)$$

By appealing to conservation of energy,[25] we can say that this expression also gives the energy flux through the Schwarzschild surface $r=2m$; that is, to find the power radiated into the $2m$ surface by some source, we take the ingoing wave solution of the homogeneous equations which has the same amplitude as that of the solution with the source term. Then we can calculate the power flowing inward by looking at the amplitude of the homogeneous solutions at large r where energy has a well-defined meaning and where a well-defined method of calculating the energy exists. Price and Thorne[22] discuss the polarization of waves described by tensor harmonics and also discuss the linear and angular momentum carried by these waves.

Let us look at the 00, 01, and 11 components of the stress tensor $T_{\mu\nu}$ for a particular harmonic. Let

$$U_{LM}(\theta,\phi)=(64\pi)^{-1}L(L+1)(L-1)(L+2)\times\mathbf{T}_{LM}^{(m)*}:\mathbf{T}_{LM}^{(m)}.$$

Then

$$T_{00}=r^{-2}U_{LM}(\theta,\phi),$$
$$T_{01}=-\epsilon r^{-2}(1-2m/r)^{-1}U_{LM}(\theta,\phi),$$
$$T_{11}=(r-2m)^{-2}U_{LM}(\theta,\phi).$$

If we transform this tensor to Kruskal coordinates, we obtain

$$T^K{}_{00}=16m^2r^{-2}(u-\epsilon v)^{-2}U_{LM}(\theta,\phi),$$
$$T^K{}_{01}=-\epsilon T^K{}_{00},\quad T^K{}_{11}=T^K{}_{00}.$$

Thus we see that the stress tensor is singular along $u=v$ (that is, $r=2m$, $t=\infty$) for outgoing waves ($\epsilon=+1$) while the stress tensor is singular along $u=-v$ (that is, $r=2m$, $t=-\infty$) for ingoing waves ($\epsilon=-1$). This singularity is a manifestation of the fact that the perturbation $h_{\mu\nu}$ is singular in just this manner at $r=2m$. This shows up in the higher-order terms which we ignored in our asymptotic expansions, that is, in the functions $H^{(N)}$ and $H_1^{(N)}$. Doroshkevich, Zel'dovich, and Novikov[26] arrive at a similar result and argue, therefore, that, in gravitational collapse, the higher moments of the gravitational field must be attenuated with collapse.

Vishveshwara[18] has shown that the solutions of the time-independent perturbation equations for $L\geq 2$ cannot be nonsingular at $r=2m$. He points out that these singularities pose no problem: It is possible to build wave packets which stay bounded away from $u=-v$ for ingoing waves or $u=+v$ for outgoing waves, and hence no singularity appears. For example, one way of producing a packet bounded in the manner described is to hold a particle at constant distance r_2 and then release it at a certain time, let it fall for a while (during this time it radiates), and then keep it at constant distance r_1 less than r_2. It has also been pointed out[27] that because of the sinusoidal behavior of the perturbations, energy pours out (say, for the outgoing waves) forever toward $r=\infty$ at a uniform rate, and thus in the Kruskal picture there must be an infinite amount of radiation in the region $0<u-v<\epsilon m$ for any $\epsilon>0$. In any case we will show that the singular behavior of the perturbations is not unexpected. Trautman[5] has examined the propagation of a discontinuity in the Riemann tensor for a Schwarzschild geometry. The result, if expressed in an orthonormal tetrad basis along the t, r, θ, ϕ directions, is that

$$\Delta R_{(\mu)(\lambda)(\alpha)(\beta)}(r)=\Delta R_{(\mu)(\lambda)(\alpha)(\beta)}(r_0)(r_0-2m)/(r-2m).$$

This is radiationlike, that is, $O(1/r)$, for large r, but is singular at $r=2m$. Now compare the leading term of the Riemann tensor for our asymptotic solutions with this result. The first-order correction to the Riemann tensor is

$$R_{\alpha\beta\gamma\delta}{}^{(1)}=-\tfrac{1}{2}\{h_{\alpha\gamma,\beta\delta}+h_{\beta\delta,\alpha\gamma}-h_{\beta\gamma,\alpha\delta}-h_{\alpha\delta,\beta\gamma}\}.$$

If we keep only the leading terms in ω [let us consider the electric solution (I5)] and if we transform $R_{\mu\nu\alpha\beta}$ to the orthonormal tetrad components and use the correspondence

$$\begin{Bmatrix}A\\ \alpha\beta\end{Bmatrix}=\begin{Bmatrix}1&2&3&4&5&6\\ 23&31&12&01&02&03\end{Bmatrix},$$

then we can write $R_{(\alpha)(\beta)(\gamma)(\delta)}$ as the 6×6 matrix:

$$\|R_{AB}\|\sim-\frac{1}{2}\frac{\epsilon i\omega}{r-2m}\exp(\epsilon i\omega r^*)\times\begin{pmatrix}0&0&0&0&0&0\\0&-\sigma&-\tau&0&\epsilon\tau&-\epsilon\sigma\\0&-\tau&\sigma&0&-\epsilon\sigma&-\epsilon\tau\\0&0&0&0&0&0\\0&\epsilon\tau&-\epsilon\sigma&0&\sigma&\tau\\0&-\epsilon\sigma&-\epsilon\tau&0&\tau&-\sigma\end{pmatrix}. \quad (I9)$$

[25] C. W. Misner (private communication); this argument is due to L. Edelstein.

[26] A. Doroshkevich, Ya. Zel'dovich, and I. Novikov, Zh. Eksperim. i Teor. Fiz. **49**, 170 (1965) [Soviet Phys. JETP **22**, 122 (1966)].

[27] K. Thorne (private communication).

We recognize precisely the same $1/(r-2m)$ behavior that occurred in the shock-front propagation, and we also see that this tensor is type N in the Petrov-Pirani classification[28] and corresponds to a gravitational wave propagating in the (ϵ) direction.

Denote the outgoing and ingoing wave solutions of the homogeneous electric equation by ψ_{out} and ψ_{in}, where ψ is the column vector

$$\psi=\begin{pmatrix} R_{LM}{}^{(e)} \\ dR_{LM}{}^{(e)}/dr^* \end{pmatrix}.$$

Then let $\Psi(r)=(\psi_{\text{out}}|\psi_{\text{in}})$, where the notation means the matrix whose columns are ψ_{out} and ψ_{in}, respectively. Then the required solution to the inhomogeneous system is

$$\psi(r)=\int_{2m}^{r}\Psi(r)\Psi^{-1}(\rho)s(\rho)d\rho+\psi_{\text{hom}}(r),$$

where

$$s(\rho)=\begin{pmatrix} 0 \\ S_{LM} \end{pmatrix}$$

and where $\psi_{\text{hom}}(r)$ is a solution of the homogeneous system chosen so that $\psi(r)$ satisfies the ingoing wave condition at $2m$ and the outgoing wave condition at ∞. Now denote

$$\Psi^{-1}(\rho)s(\rho)=\begin{pmatrix} c_1(\rho) \\ c_2(\rho) \end{pmatrix},$$

where

$$s(\rho)=\begin{pmatrix} 0 \\ S_{LM}(\rho) \end{pmatrix},$$

S_{LM} being the source term given in Eq. (18). Then

$$\psi(r)=\left\{\int_{2m}^{r}c_1(\rho)d\rho\right\}\psi_{\text{out}}(r)+\left\{\int_{2m}^{r}c_2(\rho)d\rho\right\}\psi_{\text{in}}(r)+\psi_{\text{hom}}(r).$$

Thus to ensure outgoing waves for large r and ingoing waves for r near $2m$ we choose

$$\psi_{\text{hom}}=-\left\{\int_{2m}^{\infty}c_2(\rho)d\rho\right\}\psi_{\text{in}}(r)$$

and then, up to terms of $O(1/\omega)$, we have the result

$$\psi(r)=A_L{}^{(\text{out})}(\omega,r)\psi_{\text{out}}(r)+A_L{}^{(\text{in})}(\omega,r)\psi_{\text{in}}(r), \quad \text{(I10)}$$

where

$$A_L{}^{(\text{out})}(\omega,r)=-2m_0(L+\tfrac{1}{2})^{1/2}\int_{2m}^{r}\exp\{i\omega[T(\rho)-\rho^*]\}\times\tilde{c}_1(\rho)(\lambda\rho+3m)^{-1}(1-2m/\rho)^{-1}d\rho,$$

[28] F. Pirani, Phys. Rev. **105**, 1089 (1957).

$$A_L{}^{(\text{in})}(\omega,r)=2m_0(L+\tfrac{1}{2})^{1/2}\int_{r}^{\infty}\exp\{i\omega[T(\rho)+\rho^*]\}\times\tilde{c}_2(\rho)(\lambda\rho+3m)^{-1}d\rho,$$

and

$$\tilde{c}_1(\rho)=[1+(2m/\rho)^{1/2}]\{1+(1/i\omega\rho)[m/2\rho+(\lambda\rho+\sqrt{2}m^{1/2}\rho^{1/2}+m)(\rho+3m/\lambda)^{-1}]\},$$

$$\tilde{c}_2(\rho)=[1+(2m/\rho)^{1/2}]^{-1}\{-1+(1/i\omega\rho)[m/2\rho+(\lambda\rho-\sqrt{2}m^{1/2}\rho^{1/2}+m)(\rho+3m/\lambda)^{-1}]\},$$

$$T(\rho)=-4m(\rho/2m)^{1/2}-\tfrac{4}{3}m(\rho/2m)^{3/2}-\ln[(\rho/2m)^{1/2}-1]+\ln[(\rho/2m)^{1/2}+1].$$

These are, in the high-frequency limit, the amplitudes for the outgoing and ingoing waves. We have

$$A_L{}^{(\text{out})}(\omega)=\lim_{r\to\infty}A_L{}^{(\text{out})}(\omega,r) \quad \text{(I11)}$$

and

$$A_L{}^{(\text{in})}(\omega)=\lim_{r\to 2m}A_L{}^{(\text{in})}(\omega,r). \quad \text{(I12)}$$

Equations (I7) and (I8) then give us the power radiated in a unit frequency interval. In principle one can calculate the field $\psi(r)$ to any order in $1/\omega$ and thus obtain integrals for the amplitudes $A_L{}^{(\text{out})}$ and $A_L{}^{(\text{in})}$. These integrals can also be expanded in an asymptotic expansion. Let us first consider $A_L{}^{(\text{out})}$. We note that the integral is not absolutely convergent, the integrand going like $1/\rho$ times an oscillatory factor for $\rho\to+\infty$ while going to a constant times an oscillatory factor for $\rho^*\to-\infty$ (that is, $\rho\to 2m$). This behavior is indicative of a δ function in ω for $\omega=0$. Since our approximation is valid for large ω, we will ignore this contribution, and this will be done in a natural manner in the procedure to be discussed. Let us make the transformation $x^2=\rho/(2m)$ followed by $e^y+1=x$ and let $k=2m\omega$. Then (I11) becomes

$$A_L{}^{(\text{out})}(\omega)=-2m_0(L+\tfrac{1}{2})^{1/2}I(k,\lambda),$$

where

$$I(k,\lambda)=\int_{-\infty}^{\infty}e^{ikf(y)}g(y,k,\lambda)dy,$$

$$f(y)=-\tfrac{2}{3}x^3-x^2-2x-2y,$$

$$g(y,k,\lambda)=4x^2(2\lambda x^2+3)^{-1}\{1+(1/ikx^2)\times[\tfrac{1}{4}x^{-2}+\lambda(2\lambda x^2+3)^{-1}(2\lambda x^2+2x+1)]+O(1/k^2)\}.$$

Thus we have an integral which is in a suitable form for asymptotic expansion by the saddle-point method. Now $f'(y)=0$ implies $e^{y_0}=-1$, which implies $y_{0n}=(2n+1)i\pi$ for any integer n. Thus the saddle points are at $(2n+1)i\pi$, where n is an integer. We see that a singularity of $g(y,k,\lambda)$ coincides with the saddle points. However, van der Waerden[23] has shown that we can still use the saddle-point method. Using van der Waerden's method, we make the transformation $w=-if(y)$

[assume $\omega>0$; for $\omega<0$, $A(\omega)=A^*(-\omega)$]. Then

$$I(k,\lambda)=\int_{-i\infty}^{+i\infty} e^{-kw}g(y(w),k,\lambda)(dy/dw)dw.$$

The saddle points in the y plane become branch points in the w plane. These branch points are at $w_n=2(2n+1)\pi$ for n an integer. Now make branch cuts along the real axis between the branch points, and deform the contour $C'=(-i\infty, i\infty)$ into the contour C by pushing it to the right as far as possible without passing any branch points. The contour C comes in from $+\infty$ below the positive real axis, goes around $w_0=2\pi$, and goes back out to $+\infty$ above the positive real axis. Clearly C is reached from C' by going downhill on the real part of e^{-kw}. This at most eliminates contributions to the integral from the infinite parts of the contour which, as we noted above, we expect to give a δ-function type of behavior near $\omega=0$. In an asymptotic expansion of the integral the most important contribution comes from the left-most branch point $w_0=2\pi$, the other points giving exponentially smaller terms (that is, asymptotic to zero compared with the contributions from $w_0=2\pi$). We obtain for the outgoing wave amplitude

$$A_L^{(\text{out})}\sim -4m_0(L+\tfrac{1}{2})^{1/2}e^{-4\pi m\omega}\{\tfrac{1}{3}\sqrt{2}e^{5\pi i/8}\Gamma(\tfrac{3}{4})(m\omega)^{-3/4} +\tfrac{1}{12}\pi(m\omega)^{-1}+\cdots\}. \quad \text{(I13)}$$

The dominant feature of this amplitude is that it decreases exponentially with increasing frequency.

Now let us turn to the asymptotic evaluation of $A_L^{(\text{in})}(\omega)$ given in Eq. (I12). Here the integrand is well behaved near $2m$ (that is, as $\rho^*\to -\infty$) but goes as $1/\rho$ for $\rho\to\infty$ with an oscillatory factor whose exponent is

$$T(\rho)+\rho^*=-4m(\rho/2m)^{1/2}-\tfrac{4}{3}m(\rho/2m)^{3/2}+\rho +4m\ln[1+(\rho/2m)^{1/2}].$$

The method of evaluation is again an example of the method of steepest descents, but in this case it is the end point of the contour which is most important. To evaluate the integral, we use a theorem of Copson[24] on asymptotic expansions which is just the analog, for integrals along the imaginary axis, of the result that the asymptotic behavior of the Laplace transform of a function depends on the behavior of the function near the origin. Let us go back to (I12) and make the substitution $x^2=\rho/(2m)$; then

$$A_L^{(\text{in})}(\omega)=2m_0(L+\tfrac{1}{2})^{1/2}\int_1^\infty e^{ikf(x)}g(x,k,\lambda)dx,$$

where

$$f(x)=-2x-\tfrac{2}{3}x^3+x^2+2\ln(1+x),$$

$$g(x)=4x^2(2\lambda x^2+3)^{-1}(1+x)^{-1}\{-1+(1/ikx^2)\times[\tfrac{1}{4}x^{-2}+\lambda(2\lambda x^2+3)^{-1}(2\lambda x^2-2x+1)]+O(1/k^2)\}.$$

We evaluate the terms to $O(1/k^2)$ and obtain

$$A_L^{(\text{in})}(\omega)=-2m_0(L+\tfrac{1}{2})^{1/2}ie^{2mi\beta\omega}(2\lambda+3)^{-1}(m\omega)^{-1}\times[1+\tfrac{1}{2}i(\lambda+5/4)(m\omega)^{-1}+O(1/\omega^2)], \quad \text{(I14)}$$

where $\beta=5/3-2\ln 2$.

APPENDIX J: SOME QUALITATIVE CONSIDERATIONS

Although the low-frequency part of the spectrum has not been adequately described in the preceding calculations, we will give here some estimates of a very qualitative nature which have been suggested in a conversation with John Wheeler.

We have seen [Eq. (I13)] that the amplitude $A_L^{(\text{out})}(\omega)$ for outgoing waves at ∞ goes like $2m_0^{1/2}\times(m\omega)^{-3/4}e^{-4\pi m\omega}$. From Eq. (I8), the power per unit frequency is then

$$S_L(\omega)\sim 0.01kL^4|A_L(\omega)|^2\sim k0.04m_0^2L^5\times(m\omega)^{-3/2}e^{-8\pi m\omega}, \quad \text{(J1)}$$

where k is a numerical factor of order 1. We have also seen that the asymptotic approximation is good if $8m\omega\gtrsim L^2$. Thus, for a fixed frequency ω, we expect that the power as a function of the degree L first goes up as L^5, reaches a peak, and starts falling off rapidly with increasing L at the "barrier" $8m\omega\sim L_B^2$. Thus the power per unit ω, summed over all L's, is approximately

$$S(\omega)\sim\int_0^{L_B} S_L(\omega)dL$$

or

$$S(\omega)\sim 4m_0^2(m\omega)^{3/2}e^{-8\pi m\omega}k. \quad \text{(J2)}$$

As a function of ω, this looks like a power law for small ω, reaches a peak at $\omega=3/16\pi m$. and decreases exponentially thereafter. Further, we may integrate this power spectrum over ω and obtain an estimate for the total energy radiated. Thus we obtain

$$E=\int_0^\infty S(\omega)d\omega\sim 4(8\pi)^{5/2}\Gamma(\tfrac{5}{2})(m_0^2/m)k \sim k0.0016m_0^2/m. \quad \text{(J3)}$$

Compare this with a calculation using the linearized theory of Landau and Lifshitz[3]; they give

$$-\frac{dE}{dt}=\frac{1}{45}\sum_{i,j=1}^{3}\left(\frac{d^3D_{ij}}{dt^3}\right)^2,$$

where

$$D_{ij}=\iiint\rho(x)(3x_ix_j-\delta_{ij}r^2)d^3x.$$

For the case of radial motion starting at $r=\infty$ with zero velocity, we have

$$t^{1/3}\simeq -\tfrac{2}{3}(2mr)^{1/2}$$

for $r \gg 2m$. Thus

$$dE/dt = -(1/30)(m_0/m)^2(2m/r)^5 = -(1/30) \times (m_0/m)^2(2m/3t)^{10/3}.$$

Integrating this expression, we obtain the energy radiated in falling from ∞ to r:

$$E(r) = (1/70)(m_0/m)^2(2m/3)^{10/3}t^{-7/3} = (1/105)(m_0^2/m)(2m/r)^{7/2}. \quad \text{(J4)}$$

Thus the energy radiated in falling to ten Schwarzschild radii (it is reasonable to expect that the linearized approximation is fairly good up to this ponit) is

$$\Delta E(20m) \cong (1/330000)(m_0^2/m),$$

which is less than 0.3% of the total radiation given by (J3). If we use (J4) to calculate the radiation up to four Schwarzschild radii, we obtain

$$\Delta E(8m) \cong (1/14000)(m_0^2/m),$$

which is less than 6% of the total given by (J3). Thus all indications are that a substaintal portion of the radiation comes from the part of the particle's trajectory which is between one and four Schwarzschild radii.

APPENDIX K: TIME-INDEPENDENT PERTURBATIONS FOR $L \geq 2$

The case where the perturbation is assumed time independent, $\partial h/\partial t = 0$, has been discussed by Regge and Wheeler[6] and by Vishveshwara.[18] We present here the solutions of the electric time-independent equations which can be given in terms of hypergeometric functions (for the homogeneous equations).

Setting the terms in (C7) which contain derivatives with respect to time equal to zero, we obtain the following equations:

$$H_1 = 0, \quad \text{(K1)}$$

$$\frac{d}{dr}(H-K) + 2mr^{-2}\left(1-\frac{2m}{r}\right)^{-1} H = 0, \quad \text{(K2)}$$

$$\left(1-\frac{m}{r}\right)\left(1-\frac{2m}{r}\right)^{-1}\frac{dK}{dr} - \frac{dH}{dr} - \tfrac{1}{2}(L-1)(L+2)(r-2m)^{-1}(K-H) = 0. \quad \text{(K3)}$$

From these equations we obtain

$$\frac{d^2H}{dr^2} + \frac{d}{dr}\left[\frac{2}{r}\left(1-\frac{m}{r}\right)\left(1-\frac{2m}{r}\right)^{-1} H\right] - (L-1)(L+2)r^{-1}(r-2m)^{-1}H = 0, \quad \text{(K4)}$$

or, letting $M = r(r-2m)H$ and $x = r/2m$, we have

$$x(1-x)\frac{d^2M}{dx^2} + (2x-1)\frac{dM}{dx} + (L-1)(L+2)M = 0. \quad \text{(K5)}$$

This is a form of the hypergeometric equation, and a particular solution is[29]

$$M(x) + x^2F(L+1, -L; 3; x). \quad \text{(K6)}$$

This is a polynomial in r of degree $L+2$ and goes to ∞ as $r \to \infty$. The other solution of the equation is

$$M = r^{-L+1}F(L+1, L-1; 2L+2; 2m/r). \quad \text{(K7)}$$

This solution, however, goes as $(r-2m)^{-L}$ as $r \to 2m$. Vishveshwara interprets these results as showing that there cannot exist any time-independent perturbations (for $L \geq 2$) on the Schwarzschild metric. Similar results hold for the electric-parity equations. We give the expressions for the hypergeometric functions in the above solutions:

$$F(-L, L+1; 3; z) = \sum_{n=0}^{L} \frac{\Gamma(L+n)\Gamma(L+n+1)\Gamma(3)}{\Gamma(L)\Gamma(L+1)\Gamma(n+3)} \frac{z^n}{n!}. \quad \text{(K8)}$$

The polynomial (K8) is the Jacobi polynomial

$$[\tfrac{1}{2}(L+1)(L+2)]^{-1}P_L^{(2,-2)}(1-2z). \quad \text{(K9)}$$

Also,[30]

$$F(L+1, L-1; 2L+2; 1/z) = -\frac{(2L+1)!}{(L!)^2(L+2)!(L-2)!} \times \frac{d^{L-2}}{dt^{L-2}}\left\{(1-t)^L \frac{d^{L+2}}{dt^{L+2}}\left[\frac{\ln(1-t)}{t}\right]\right\}_{t=1/z}. \quad \text{(K10)}$$

[29] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. Stegun (Dover, New York, 1965), Chap. 15, p. 562.

[30] *Higher Trancendental Functions*, edited by A. Erdélyi *et al.* (McGraw-Hill, New York, 1953), Vol. I, Chap. 2.