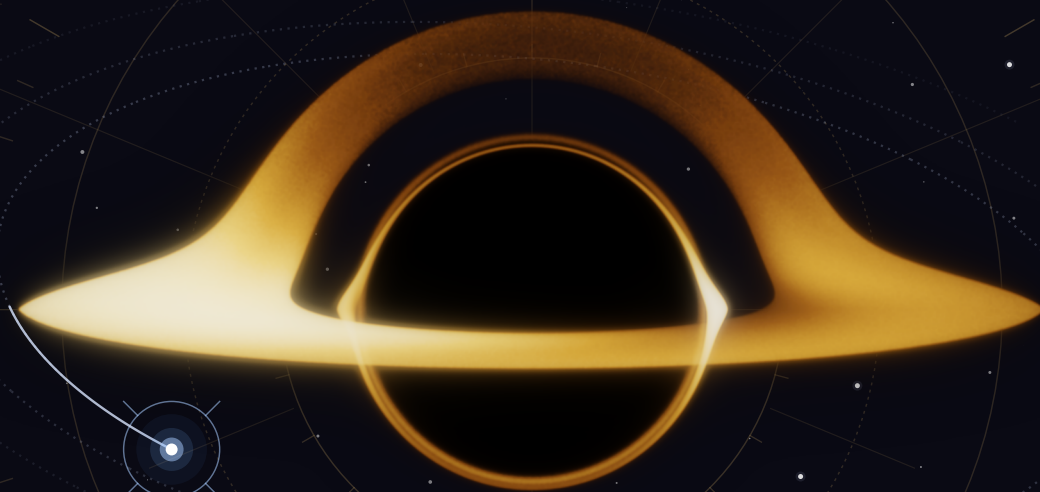


PHD THESIS



$$u^\mu \nabla_\mu u^\nu = f_{\text{self}}^\nu + f_{\text{env}}^\nu$$

Motion Through Matter in Curved Space

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MOTION THROUGH MATTER IN CURVED SPACE

PhD Thesis
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“Kilimanjaro is a pretty tricky climb you know, most of it’s up until you reach the very very top, and then it tends to slope away rather sharply.”

— MONTY PYTHON’S FLYING CIRCUS

ABSTRACT

Motion, in its most basic form, is the change in a body’s position or orientation over time. Rudimentary as it is, such a definition remained relevant for centuries, sufficient for Kepler to write down the laws of planetary motion and for Newton to derive the dynamics underlying them. Once lifted to the context of general relativity (GR) it encounters clear shortcomings, partly due to the absence of absolute position and time, but most acutely in the pathologies which quickly arise when one attempts to describe an extended body as a localised mass distribution along a worldline in GR.

In the Newtonian picture, this description is well-founded. The linearity of Newtonian gravity, together with the uniqueness theorem for solutions of Laplace’s equation, guarantees that the exterior gravitational field of any body is exactly and uniquely captured by its multipole expansion. Moreover, the effect of self-gravity on the centre-of-mass motion is precisely zero.

In general relativity this picture is fundamentally complicated. The field equations are nonlinear, and a point-particle source, a stress-energy tensor proportional to a delta function along a worldline, renders the metric singular precisely where one would evaluate the local self-forcing effects which arise in curved space. Resolving these pathologies through the notion of balance laws and singular-Regular splits, arriving at a rigorous notion of motion for compact objects in curved spacetime is the purpose of the self-force programme.

Over the decades, this programme has seen widespread success. Not only at the theoretical level, but also in the construction of accurate gravitational waveforms for the upcoming LISA mission. Which hopes to detect the inspiral of stellar-mass black holes on highly relativistic orbits around supermassive black holes. Such systems, known as extreme mass ratio inspirals (EMRIs), are natural test beds for probing the geometry and dynamics of the spacetime surrounding supermassive black holes. Most modelling to date in the context of self-force theory has focused on these systems in vacuum, with little in the way of first principles attempts to understand how the presence of additional extended matter fields complicates this picture. Yet a large fraction of supermassive black holes are expected to harbour dense matter environments in their vicinity. Such as those found in Active Galactic Nuclei (AGN), strongly motivating the modelling of environmental effects for EMRI sources. The primary goals of this thesis are twofold: first, to develop first-principles models of the matter response in the vicinity of EMRI and second,

to make explicit the relationship between forces on the secondary and integrated torques and perturbations associated with the matter field. Such notions are commonly understood in the Newtonian setting, but are not yet well understood in the context of bodies moving through matter in general relativity.

We begin Part [i](#) by laying the theoretical foundations already established in the literature and motivating the detection of asymmetric binary systems in astrophysical environments. We present the necessary background for understanding the geometry of spinning black hole spacetimes, describe perturbation theory on such geometries, and review known test-field environmental solutions. This part concludes with the first paper chapter, which considers the rectilinear motion of a Kerr black hole through a homogeneous medium.

Part [ii](#) then reviews the formal construction of point particles in general relativity through the method of matched asymptotic expansions, further developing the perturbative framework by which particles and matter can be consistently described within the theory. This part concludes with the second paper chapter, in which we perform the first self-consistent calculation of a matter response in a relativistic EMRI spacetime, in the context of a simple scalar field Boson cloud environment.

Part [iii](#) turns our focus to motion, first extending the previous scalar field calculation to a computation of the perturbations to a perfect fluid thin disc around a spinning black hole. This result is then used as a test bed for understanding torque balance through the disc, where we construct a relativistic quantity analogous to the Newtonian torque. We then show that in the case of this specific environment we can balance these torques against asymptotic fluxes, providing for the first time a handle on the construction of planetary torques in the context of relativistic EMRIs. The final main chapter then provides the key steps connecting these calculations to the motion of EMRIs, relating the torque quantities identified in the previous chapter directly to the evolution of the orbital elements, giving rise to a relativistic analogue of Newton’s third law for energy and angular momentum. We find an instantaneous form of the third law which importantly corrects for the acausal nature of the classical statement. Averaging over long timescales, we find that in the relativistic context a form of Newton’s third law holds, up to a correction term. This term is directly associated with the gravitational flux emitted from the interaction of the matter with the EMRI, driving the matter onto a new trajectory. Working to this end we hope to push the understanding of EMRIs through astrophysically relevant, dynamical environments.

DANSK RESUMÉ

Bevægelse i dens mest basale form er defineret ud fra ændringen af et legemes position eller orientering over tid. På trods af den simple beskrivelse har denne definition stået ved magt i århundreder, god nok til, at Kepler kunne formulere lovene for planetarisk bevægelse, og for Newton til at udlede den underliggende dynamik, der beskriver dem. Efter at løfte denne idé ind i en kontekst af den almene relativitetsteori støder vi dog på nogle problemer, overordnet set på grund af manglen på absolut position og tid, men mest akut i de patologier, der hurtigt opstår, når man forsøger at beskrive et udstrakt legeme som en lokaliseret massefordeling langs en worldline.

Fra det Newtonske perspektiv er denne beskrivelse velbegrundet. Shell-sætninger, entydighedsargumenter og lineariteten af Newtonsk tyngdekraft garanterer til sammen, at det ydre tyngdefelt fra ethvert legeme er præcist og entydigt bestemt af dets multipoludvikling. Det ydre tyngdefelt er derfor præcist beskrevet ved legemets multipolmomenter, givet en punktmasseidealiserings. Desuden er effekten af legemets selvgravitation på bevægelsen af dets massemidtpunkt præcis nul: ifølge Newtons tredje lov udligner de interne tyngdekrafter hinanden præcist, og legemets bevægelse bestemmes derfor udelukkende af det ydre tyngdefelt.

I den almene relativitetsteori er dette billede fundamentalt kompliceret. Feltligningerne er ikke-lineære, og en punktpartikelkilde, en stress-energy-tensor proportional med en deltafunktion langs en worldline, gør metrikken singulær præcis dér, hvor den er nødvendig. Selvgravitation vil generelt tvinge massive legemer til at afvige fra baggrundens geodætiske baner, og de kræfter, der er ansvarlige for dette, opstår præcis på denne worldline, hvor tyngdefeltet divergerer. At løse disse patologier og opnå en stringent beskrivelse af bevægelse for kompakte objekter i krumme rumtider er formålet med self-force-programmet.

Over de seneste årtier har dette program haft stor succes, ikke kun på det teoretiske plan, men også i udviklingen af præcise modeller for gravitationsbølger til den kommende LISA-mission, som har til formål at detektere inspiralet af sorte huller med stjernemasse på stærkt relativistiske baner omkring supermassive sorte huller. Sådanne systemer, kendt som extreme mass ratio inspirals (EMRIs), udgør naturlige laboratorier til at undersøge geometrien og dynamikken af rumtiden omkring supermassive sorte huller. Størstedelen af modelleringen inden for self-force-teorien har hidtil fokuseret på sådanne systemer i vakuum, mens der kun i begrænset omfang er udviklet modeller ud fra første principper, som beskriver, hvordan tilstedeværelsen af omgivende stof påvirker dette billede. En stor andel af supermassive sorte huller forventes imidlertid at være omgivet af tætte skyer af gas og andet stof, som eksempelvis dem der findes i Active Galactic Nuclei. Det

giver en stærk motivation for at beskrive sådanne miljøeffekter, som kunne være en mulig observationskilde for LISA. Formålet med denne afhandling er todelt. For det første udvikles modeller ud fra første principper for, hvordan det omgivende stof reagerer på et EMRI-inspiral. For det andet etableres en eksplicit sammenhæng mellem kræfterne på det sekundære legeme og det samlede drejningsmoment, som stoffet udøver. Disse størrelser beregnes ofte i den Newtonske teori, men deres indbyrdes sammenhæng er endnu ikke forstået for legemer, der bevæger sig i den almene relativitetsteori.

Vi indleder del [i](#) med at opbygge det teoretiske fundament, som allerede er etableret i litteraturen, samt at motivere studiet af asymmetriske binære systemer i astrofysiske miljøer. Vi gennemgår den nødvendige baggrund for at forstå geometrien af roterende sorte hullers rumtider, beskriver perturbationsteori på sådanne geometrier og giver en oversigt over kendte testfelt-løsninger i miljømæssige konfigurationer. Denne del afsluttes med det første artikelkapitel, som behandler den retlinede bevægelse af et Kerr-sort hul gennem et homogent medium.

Del [ii](#) gennemgår den formelle konstruktion af punktpartikler i den almene relativitetsteori ved hjælp af metoden *matched asymptotic expansions* og videreudvikler det perturbative rammeværk, hvormed partikler og stof konsistent kan beskrives inden for teorien. Denne del afsluttes med det andet artikelkapitel, hvor vi udfører den første selvkonsistente beregning af stofrespons i en relativistisk EMRI-rumtidskontekst, i tilknytning til et simpelt skalarfelt-bosonsky-miljø.

Del [iii](#) retter vores fokus mod bevægelse og udvider først den tidligere skalarfelt-beregning til en beregning af perturbationerne af en tynd, perfekt fluid-skive omkring et roterende sort hul. Resultatet anvendes herefter som et testområde til at forstå drejningsmomentbalance gennem skiven, hvor vi konstruerer en relativistisk størrelse analog til det Newtonske drejningsmoment. Vi viser derefter, at vi i dette specifikke miljø kan balancere disse drejningsmomenter mod asymptotiske fluxer, hvilket for første gang giver et redskab til konstruktionen af planetære drejningsmomenter i konteksten af relativistiske EMRI-systemer. Det afsluttende hovedkapitel giver dernæst de centrale trin, som forbinder disse beregninger med EMRI'ers bevægelse, idet de drejningsmomentstørrelser, der identificeres i det foregående kapitel, relateres direkte til udviklingen af de orbitale elementer og dermed giver anledning til en relativistisk analog til Newtons tredje lov for energi og impulsmoment. Vi finder en øjeblikkelig form af den tredje lov, som vigtigt korrigerer for den kausale problematik i den klassiske formulering. Ved tidlig gennemsnitdannelse finder vi, at en form for Newtons tredje lov gælder i den relativistiske kontekst, op til et korrektionsterm, som direkte kan associeres med den gravitationsflux, der udsendes fra vekselvirkningen mellem stoffet og EMRI-systemet, og som driver stoffet over i en ny bane. Dette giver den første konkrete forståelse af bevægelsen af kompakte objekter gennem stof i krum rumtid.

PUBLICATIONS

This thesis is based on and contains partial re-prints of the following publications and manuscripts in preparation:

1. C. Dyson, J. Redondo-Yuste, M. van de Meent, and V. Cardoso, “Relativistic aerodynamics of spinning black holes,” *Phys. Rev. D* **109**, 104038 (2024), [arXiv:2402.07981 \[gr-qc\]](#).
2. C. Dyson, T. F. M. Spieksma, R. Brito, M. van de Meent, and S. Dolan, “Environmental Effects in Extreme-Mass-Ratio Inspirals: Perturbations to the Environment in Kerr Spacetimes,” *Phys. Rev. Lett.* **134**, 211403 (2025), [arXiv:2501.09806 \[gr-qc\]](#).
3. C. Dyson and D. D’Orazio, “Spiral Density Waves and Torque Balance in the Kerr Geometry,” [arXiv:2601.19123 \[gr-qc\]](#).
4. C. Dyson, A. Pound, D. D’Orazio, M.v.d. Meent, L. Zwick, “A Relativistic Third Law for Asymmetric Binaries in Environments” (in prep).

Throughout the course of this program the following publications were also produced, however their contents and results do not appear explicitly in the following thesis:

5. C. Dyson and M. van de Meent, “Kerr-fully diving into the abyss: analytic solutions to plunging geodesics in Kerr,” *Class. Quantum Grav.* **40**, 195026 (2023), [arXiv:2302.03704 \[gr-qc\]](#).
6. C. Dyson and D. Pereñíguez, “Magnetic black holes: From Thomson dipoles to the Penrose process and cosmic censorship,” *Phys.Rev.D.* **108**, 084064 (2023), [arXiv:2306.157514 \[gr-qc\]](#).
7. J C.L. Chan, C. Dyson, M. Garcia, C. Redondo-Yuste and, L. Vujeva “Lensing and wave optics in the strong field of a black hole,” *Phys.Rev.D.* **112**, 064009 (2025), [arXiv:2502.14073 \[gr-qc\]](#).

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INTRODUCTION

Black holes (BHs), as described by Einstein’s theory of general relativity, are the simplest macroscopic objects in our Universe. They are characterised by three key parameters: mass, spin, and electric charge [1, 2]. Since we expect macroscopic astrophysical objects to be electrically neutral on average, this further restricts the space of astrophysically relevant isolated black holes to the Kerr family, fully described by two parameters: the mass (M) and spin (a).

Such simple objects were unheard of in astrophysics, and throughout the early twentieth century considerable scepticism surrounded their existence [3]. Indirect evidence for the existence of supermassive black holes (SMBHs), however, had been accumulating since the early 1960s, when the identification of quasi-stellar radio sources (quasars) as extraordinarily luminous extragalactic objects [4] led to the proposal that accretion onto a massive compact object was the only viable power source [5, 6], with quasars and their lower-luminosity counterparts, active galactic nuclei (AGN), later understood to be the active phase of what are now dormant SMBHs at galactic centres. By the early 1970s, however, evidence of their existence began to accrue from measurements of the X-ray binary system Cygnus X-1 [7, 8], where one of the constituent compact objects appeared to have a mass too large to be a neutron star. By the mid-to-late twentieth century, further concrete evidence began to build [6] for SMBHs. Leading to the Sgr A* programme [9, 10], which tracked stellar orbits at the centre of our Galaxy to determine the presence of a dark mass of order $\sim 10^6 M_\odot$ at the Galactic centre.

Such detections had brought the field of black hole physics to the forefront. However, both detections were in a sense indirect, owing to signals received from nearby objects orbiting these extreme objects rather than through direct emission. The first truly direct detections have only come in the past decade, with the first detection of gravitational waves in 2015 [11] emanating from a stellar mass binary black hole system. These waves, first predicted by Einstein [12], represent the emission of pure gravitational energy from the merger of two black holes, with up to $\sim 10\%$ of the system’s energy being radiated.

Detecting such signals from stellar-mass binary black hole systems is now commonplace, with over 300 such detections to date [13], using the global network of interferometry-based gravitational wave detectors: LIGO, Virgo, and KAGRA, collectively referred to as the LVK network. Such detections have driven our experimental understanding of GR over the past decade, yielding new constraints on modifications to GR [14] and precise measurements of the spectrum of relaxing black holes [15, 16]. Achieving these detections has required major breakthroughs

on two fronts: instrumentation, producing the most precise experiment in the history of humanity, capable of detecting changes in length of order $\sim 10^{-18}$ m; and theory, providing accurate models of the gravitational waveforms emitted during the highly non-linear collision of two black holes, which are then used to extract the true signal from the noisy detector data through the process of matched filtering.

Due to the scales involved in stellar mass black hole systems, we typically only observe the final seconds to milli-seconds of the binary black hole's lifetime with LVK. Additionally, at the frequencies at which these binaries are currently detected, their orbital timescale is much shorter than the dynamical timescale of any relevant surrounding environment, decoupling the binary dynamics from environmental effects. This has the advantage of making LVK sources *clean* to model and detect; however, it also comes at a cost: stellar-mass binaries do not interact significantly with their surrounding environment and are hence generally poor probes of relativistic matter fields in the vicinity of strong gravitational fields [17, 18]. Some environmental effects, such as Rømer delays from a nearby tertiary object, may be observable with next-generation detectors [19], however, the vast majority of environmental effects will not be observable with such systems.

At the same time, the success seen in measuring the properties of stellar-mass black hole binaries can be turned back towards systems involving supermassive black holes. These systems arise at the centres of almost all galaxies [20] and can operate almost entirely in isolation as quiescent objects, or in dense inflowing environments known as active galactic nuclei (AGN) [21–25]. Detecting gravitational waves from such systems is one of the primary science goals of upcoming space-based milli-hertz detectors, such as the Laser Interferometer Space Antenna (LISA) [26] and other prospective missions such as TianQin [27, 28] and Taiji [29]. Due to the larger scales of these systems (typically of order 10^5 – $10^8 M_\odot$), the associated gravitational waves are of much lower frequency, typically in the milli-Hz regime, which LISA is specifically designed to operate in.

These sources can be divided broadly into two classes: massive black hole binaries, where two supermassive black holes inspiral towards one another, typically arising during the merger of two galaxies [30, 31], and intermediate and extreme mass ratio inspirals (I/EMRIs) [32–34], whereby the much larger black hole at the centre of a galaxy captures a stellar-mass object, causing it to slowly inspiral. Such EMRI systems will be the focus of this thesis.

EMRIs are one of the key targets of the upcoming LISA mission [35]. At their core, an EMRI consists of a supermassive black hole, the primary, orbited by a much smaller stellar object, such as a stellar-mass black hole or neutron star, known as the secondary. Due to the mass disparity in EMRI systems, the signals which arise in the LISA band will persist for periods of order months to years. With accurate models we then expect to be able to precisely constrain the parameters of such systems, measuring the central black hole mass and spin to within five significant figures [36], gaining strong estimates of the orbital parameters and relativistic spin

magnitude of the secondary. Such measurements will place stringent bounds on the absorptive properties of the black hole horizon and constitute precision tests of the Kerr paradigm within general relativity [34, 35].

In the case of EMRIs arising in active galactic nuclei, where violent processes arise through magnetically supported turbulence and radiative emission, the secondary will also act as a precise probe of the surrounding matter in the vicinity of the central black hole, giving direct observational avenues towards testing the properties of accretion discs at galactic centres [37, 38]. Such detections carry important implications for probing binary formation rates, matter under extreme conditions, and black hole growth mechanisms. Additionally, multiple routes have been proposed through which supermassive black holes may accumulate extreme over-densities of dark matter from their surroundings [39–48], with implications for galactic structure and the feeding mechanisms governing SMBH growth.

Detection of EMRIs with LISA in such settings require not only major international collaboration to build and launch the detectors, currently scheduled for 2035, but also major theoretical advancements in order to extract signals from the data stream once LISA is operational. The primary theoretical framework for this in the context of EMRIs is the gravitational self-force formalism [49, 50], which models the dynamics of the EMRI system as an expansion in the mass ratio, achieving accurate waveform predictions for use in matched filtering. Achieving accurate calculations within the self-force formalism is a major long-term goal of the field [35, 51, 52]. However, such precision modelling has focused almost entirely on the vacuum case, relevant for modelling quiescent black holes in isolation.

Until recently, very little serious attempt has been made to consistently model the dynamics of EMRIs embedded in astrophysical environments, with most approaches relying on simplistic dynamical friction (DF) formulae [53, 54] far outside their realm of validity. Application of such approximations is not remotely sufficient even for Newtonian models of EMRIs, let alone the precision relativistic inspiral models required for LISA. The key issue is that such models fail to account for the fact that in scenarios where environmental effects are significant, the matter dynamics and their interaction with the orbit will generally give rise to distinct resonant interactions [55–57]. Alternatively, environments which are highly variable will also be dominated by turbulence and transient interactions [58], none of which is captured by DF models [17].

This thesis is concerned with developing the theoretical setup and formulation required to correctly describe the motion of EMRIs in matter, by lifting approaches already common in the Newtonian setting into general relativity, providing the foundational steps towards having realistic environmental-EMRI models for LISA. The thesis is set out as a monograph with four dedicated paper chapters developed throughout the course of my PhD.

Part i: Vacuum and Matter begins with a review of the basic properties of the Kerr geometry, primarily discussing its geodesic and algebraic structures. We then move

on to perturbation theory, focusing on vacuum perturbations to second order both at the level of the Einstein equation and through the Teukolsky equation. Next we discuss two simple black hole environments: a boson cloud in a pure $(\ell, m) = (1, 1)$ state and an axisymmetric thin disc. We conclude with the first paper chapter of the thesis, in which we consider the motion of a Kerr black hole through a homogeneous medium, serving as a starting point for our investigation into the dynamics of black holes in matter [59].

Part ii: Particles and Matter moves on to discussing the process of matched asymptotic expansions, through which a vacuum binary black hole system with unequal masses is re-expressed as an effective point-particle source moving around a central Kerr black hole. We then discuss how, through the methods of Dolan et al. [60], one can utilise the methods of vacuum perturbation theory with a specific choice of gauge to fully reconstruct the metric perturbation induced by such a point-particle source at leading order. With this we conclude the main review chapters and are well placed to proceed with the primary focus of this body of work. We continue by providing a concrete general formulation of the two-parameter framework, which assumes both the environment and secondary are subleading but independent at their respective leading orders with respect to the background Kerr geometry. This part concludes by applying the formalism to the scenario of an ultralight bosonic field in the vicinity of a supermassive black hole, determining the matter response and rate of asymptotic matter flux [61].

Part iii: Motion Through Matter connects the calculations of matter responses to the motion of the secondary. The first chapter, based on [62], takes the calculation previously carried out for bosonic fields and extends it to the case of a perfect fluid, providing the first relativistic fluid calculation of its kind. This chapter then begins the process of connecting the matter field response to the EMRI dynamics by constructing an explicit relation between integrated torques and asymptotic fluxes through the fluid. The second chapter covers the results of a manuscript in preparation, which details explicitly the connection between locally computed torques on the disc and the evolution of the orbital elements of the system, notably the energy and angular momentum, providing the relativistic analogue of Newton's third law. Having obtained a relativistic analogue of Newton's third law, this thesis then provides the first steps toward a calculation in its entirety: consistently computing the effect of the presence of an EMRI on a relativistic environment and relating that effect back onto the EMRI itself, providing a clear means toward the consistent treatment of EMRIs and environments in general relativity. We then conclude by recapping the key results of this body of work and laying out the future programme and route towards EMRI-disc models which properly encapsulate the realistic dynamics of complex strong-field environments such as AGN in the vicinity of Kerr black holes.

Throughout this thesis, we use natural units $G = c = \hbar = 1$, unless otherwise stated, and we work with a mostly plus metric signature.

Part I

VACUUM AND MATTER

KERR STRUCTURE

In the absence of macroscopic charge, the Kerr family describes the unique class of asymptotically flat, stationary, axisymmetric vacuum black-hole solutions. We begin the technical aspect of this thesis with the Einstein field equations, which describe the curvature of spacetime, and in vacuum take the full form

$$G_{\mu\nu}[g] := R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0. \quad (2.1)$$

Here $R := R^\mu{}_\mu = g^{\mu\nu}R_{\mu\nu}$ is the Ricci scalar, and the Ricci tensor is defined as the contraction $R_{\mu\nu} := R^\alpha{}_{\mu\alpha\nu}$ of the Riemann tensor,

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}, \quad (2.2)$$

with the Christoffel symbols

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}). \quad (2.3)$$

The full dynamical content of the spacetime is encoded in the metric $g_{\mu\nu}$, which carries ten tensor degrees of freedom. In addition the Einstein equations admit *diffeomorphism invariance*: the intrinsic geometry of the manifold $\mathcal{M}$ is independent of the coordinate chart used to describe it. This is properly understood as a diffeomorphism $\phi : \mathcal{M} \rightarrow \mathcal{M}$ physically maps the metric to a new tensor field,

$$g_{\mu\nu}(x) \longrightarrow (\phi^*g)_{\mu\nu}(x) = \frac{\partial\phi^\alpha}{\partial x^\mu} \frac{\partial\phi^\beta}{\partial x^\nu} g_{\alpha\beta}(\phi(x)), \quad (2.4)$$

and the Einstein equations are satisfied by ϕ^*g whenever they are satisfied by g .

Our approach throughout this thesis will be to describe the motion of bodies and matter in the vicinity of Kerr black holes [63]. In particular, we will assume that at any instant our dynamics will be dominated by the gravitational influence of a Kerr black hole. As this will be our basis, we begin the technical aspect of this thesis with the Kerr solution, which gives,

$$G_{\mu\nu}[g^{\text{Kerr}}] = 0. \quad (2.5)$$

The Kerr spacetime is described by the line element,

$$ds^2 = -\frac{\Delta}{\Sigma} \left(dt - a \sin^2 \theta d\phi \right)^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \frac{\sin^2 \theta}{\Sigma} \left(a dt - (r^2 + a^2) d\phi \right)^2, \quad (2.6)$$

written in Boyer-Lindquist coordinates [64], where,

$$\Sigma = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 - 2Mr + a^2. \quad (2.7)$$

In this form the coordinate singularity structure which indeed corresponds to the locations of both inner and outer black hole horizons is fixed by $\Delta = 0$, which occurs at $r_{\pm} := M \pm \sqrt{M^2 - a^2}$.

2.1 MOTION IN THE KERR GEOMETRY

Geodesic equations describe the natural paths of free fall along a manifold; the governing equations which define this motion are given by the standard geodesic equation,

$$u^\mu \nabla_\mu u^\nu = 0, \quad (2.8)$$

where $u^\mu = dx^\mu/d\lambda$ is the tangent vector to the worldline under an affine parametrisation λ and ∇_μ is the standard co-variant derivative. In the Kerr geometry these equations reduce to a simplified set of first-order ODEs admitting a rich geodesic structure, underpinned by the existence of four constants of motion arising from symmetries of the Kerr metric.

The first of these is the normalisation of timelike curves fixed by $u^\mu u_\mu = -1$. Next, the stationarity and axisymmetry of the Kerr geometry give rise to two Killing vectors, $(\partial_t)^\mu$ and $(\partial_\phi)^\mu$, whose contraction with the four-velocity u^μ yields the conserved specific energy and angular momentum:

$$\mathcal{E} := -g_{\mu\nu} u^\mu (\partial_t)^\nu, \quad \mathcal{L} := g_{\mu\nu} u^\mu (\partial_\phi)^\nu. \quad (2.9)$$

The fourth constant is the Carter constant [65], which does not arise from a Killing vector but rather from a rank-2 Killing tensor $\mathcal{K}_{\mu\nu}$, satisfying $\nabla_{(\alpha} \mathcal{K}_{\mu\nu)} = 0$, whose existence is a consequence of the type-D algebraic structure of the Kerr family [66]. The fundamental object underlying this structure is the principal conformal Killing–Yano tensor, an antisymmetric tensor defined by,

$$f_{\alpha\beta} = 2r n_{[\alpha} l_{\beta]} + 2ia \cos \theta \bar{m}_{[\alpha} m_{\beta]}, \quad (2.10)$$

where l^μ , n^μ , m^μ and $\bar{m}^\mu$ are the null vectors of the Kinnersley tetrad [67],

$$\begin{aligned} l^\mu &= \left[\frac{r^2 + a^2}{\Delta}, 1, 0, \frac{a}{\Delta} \right], & n^\mu &= \frac{1}{2\Sigma} [r^2 + a^2, -\Delta, 0, a], \\ m^\mu &= \frac{1}{\sqrt{2}\bar{\rho}} \left[ia \sin \theta, 0, 1, \frac{i}{\sin \theta} \right], \end{aligned} \quad (2.11)$$

and the bar operator simply denotes complex conjugation. The Killing tensor is then recovered as $\mathcal{K}_{\mu\nu} = f_{\mu\alpha} f_\nu{}^\alpha$, taking the explicit form

$$\mathcal{K}_{\mu\nu} = 2\Sigma l_{(\mu} n_{\nu)} + r^2 g_{\mu\nu}. \quad (2.12)$$

The Carter constant is then defined as:

$$Q := u^\mu \mathcal{K}_{\mu\nu} u^\nu - (\mathcal{L} - a\mathcal{E})^2. \quad (2.13)$$

These constants of motion $\{\mathcal{E}, \mathcal{L}, Q\}$ are sufficient to fully determine the equations of motion as a system of four first order ODEs. Introducing the substitution $z = \cos \theta$ and passing to Mino-Carter time λ [68], defined via:

$$d\tau = \Sigma d\lambda, \quad (2.14)$$

the equations of motion take the separated form [65]:

$$\left(\frac{dr}{d\lambda}\right)^2 = (\mathcal{E}(r^2 + a^2) - a\mathcal{L})^2 - \Delta(r^2 + (a\mathcal{E} - \mathcal{L})^2 + Q) =: V_r(r), \quad (2.15)$$

$$\left(\frac{dz}{d\lambda}\right)^2 = Q - z^2(a^2(1 - \mathcal{E}^2)(1 - z^2) + \mathcal{L}^2 + Q) =: V_z(z), \quad (2.16)$$

$$\frac{dt}{d\lambda} = \frac{r^2 + a^2}{\Delta} (\mathcal{E}(r^2 + a^2) - a\mathcal{L}) - a^2\mathcal{E}(1 - z^2) + a\mathcal{L}, \quad (2.17)$$

$$\frac{d\phi}{d\lambda} = \frac{a}{\Delta} (\mathcal{E}(r^2 + a^2) - a\mathcal{L}) + \frac{\mathcal{L}}{1 - z^2} - a\mathcal{E}. \quad (2.18)$$

The Mino time parametrisation causes the r and z sectors to decouple entirely, admitting a hierarchical solution strategy. One first solves for $r(\lambda)$ and $z(\lambda)$ independently, in which the coordinate time and azimuthal angle decompose additively as:

$$t(\lambda) = \Gamma\lambda + t^{(r)}(r(\lambda)) + t^{(z)}(z(\lambda)), \quad (2.19)$$

$$\phi(\lambda) = Y_\phi\lambda + \phi^{(r)}(r(\lambda)) + \phi^{(z)}(z(\lambda)), \quad (2.20)$$

where Γ and Y_ϕ are the frequencies of coordinate time t and ϕ with respect to Mino time λ , given by

$$\Gamma = Y_t^{(r)} + Y_t^{(z)} + a\mathcal{L}, \quad Y_\phi = Y_\phi^{(r)} + Y_\phi^{(z)} - a\mathcal{E}, \quad (2.21)$$

defined as in Eqs. 7 of [69].

In such a manner one can then obtain analytic solutions for bound, plunging, and scattering geodesics in the Kerr geometry. Remarkably, all classes of solution trajectories can be determined analytically in terms of elliptic integrals [69–72]. For bound timelike geodesics, V_r is a quartic admitting four real roots $r_1 \geq r_2 > r_3 \geq r_4$,

$$V_r(r) = - (1 - \mathcal{E}^2) (r - r_1) (r - r_2) (r - r_3) (r - r_4), \quad (2.22)$$

$$V_z(z) = a^2 (1 - \mathcal{E}^2) (z_1^2 - z^2) (z_2^2 - z^2), \quad (2.23)$$

where $r \in [r_2, r_1]$ defines the radial libration between periastron $r_2 = p/(1 + e)$ and apastron $r_1 = p/(1 - e)$. The polar coordinate $z = \cos \theta$ librates between $|z_1|$

and $-|z_1|$, with $|z_2| > |z_1|$. Through these definitions we can express the constants of motion in their quasi-Keplerian form,

$$p = \frac{2r_1 r_2}{r_1 + r_2}, \quad e = \frac{r_1 - r_2}{r_1 + r_2}, \quad I = \arccos(z_1). \quad (2.24)$$

giving the relativistic semi-latus rectum, eccentricity, and maximum inclination in Kerr, respectively [73]. To understand the phase structure of geodesics in the Kerr spacetime, one must also make contact with the various phase prescriptions that exist in the literature, such as the quasi-Keplerian [73] and Mino-frequency forms [69], in which the dynamics are re-expressed in terms of natural Mino-time frequencies,

$$q_r = Y_r \lambda + q_r^0, \quad q_z = Y_z \lambda + q_z^0, \quad (2.25)$$

with

$$Y_r = \frac{2\pi}{\Lambda_r}, \quad \Lambda_r := 2 \int_{r_2}^{r_1} \frac{dr}{\sqrt{V_r(r)}}, \quad (2.26)$$

$$Y_z = \frac{2\pi}{\Lambda_z}, \quad \Lambda_z := 4 \int_0^{z_+} \frac{dz}{\sqrt{V_z(z)}}. \quad (2.27)$$

These quantities can be calculated analytically in terms of elliptic integrals in the form

$$Y_r = \frac{\pi \sqrt{(1 - \mathcal{E})(r_1 - r_3)(r_2 - r_4)}}{2K(k_r)}, \quad k_r := \frac{r_1 - r_2}{r_1 - r_3} \frac{r_3 - r_4}{r_2 - r_4}, \quad (2.28)$$

$$Y_z = \frac{\pi \sqrt{a^2(1 - \mathcal{E})} z_1}{2K(k_z)}, \quad k_z := (z_2/z_1)^2, \quad (2.29)$$

and $K(\cdot)$ is the complete elliptic function of the first kind, $K(\cdot) = F(\pi/2, \cdot)$ [74]. In the Newtonian limit, one simply obtains that $Y_z \rightarrow 0$ and Y_r is related to the standard mean frequency. The remaining frequencies, Y_t and Y_ϕ are then fully determined. At the level of orbital parameters the system is then fully determined through the equations of motion, and

$$\frac{dp}{d\lambda} = 0, \quad \frac{de}{d\lambda} = 0, \quad \frac{dI}{d\lambda} = 0, \quad \frac{dq_r}{d\lambda} = Y_r, \quad \frac{dq_z}{d\lambda} = Y_z. \quad (2.30)$$

Moving forward, we will define the collection of conserved quantities describing the orbit as P_i and the collection of phases q_i , such that in Kerr the evolution equations become,

$$\frac{dP_i}{d\lambda} = 0, \quad \frac{dq_i}{d\lambda} = Y_i. \quad (2.31)$$

2.2 THE GEROCH–HELD–PENROSE FORMALISM

Beyond its geodesic structure, the Kerr geometry also admits the intrinsic symmetry of being Petrov type D. Spacetimes belonging to this class admit two repeated principal null directions. Taking the Weyl tensor to be the traceless component of the Riemann tensor,

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \left(g_{\mu[\rho} R_{\sigma]\nu} - g_{\nu[\rho} R_{\sigma]\mu} \right) + \frac{R}{3} g_{\mu[\rho} g_{\sigma]\nu}, \quad (2.32)$$

a null vector k^μ is a principal null direction if it satisfies the algebraic condition

$$k_{[\alpha} C_{\beta]\mu\nu[\rho} k_{\sigma]} k^\mu k^\nu = 0. \quad (2.33)$$

In such spacetimes we label these two null directions l^μ and n^μ , completing the tetrad with a complex transverse null vector m^μ . The four null vectors then satisfy the orthogonality relations

$$l^\mu l_\mu = n^\mu n_\mu = m^\mu m_\mu = \bar{m}^\mu \bar{m}_\mu = 0, \quad (2.34a)$$

$$l^\mu n_\mu = -m^\mu \bar{m}_\mu = 1, \quad (2.34b)$$

$$l^\mu m_\mu = l^\mu \bar{m}_\mu = n^\mu m_\mu = n^\mu \bar{m}_\mu = 0. \quad (2.34c)$$

Collecting the tetrad legs, one can build a local basis $e_i^\mu = \{l^\mu, n^\mu, m^\mu, \bar{m}^\mu\}$, with corresponding metric

$$\chi_{ij} = \chi^{ij} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (2.35)$$

The spacetime metric is then reconstructed from the tetrad as

$$g^{\mu\nu} = e_i^\mu e_j^\nu \chi^{ij} = -l^\mu n^\nu - n^\mu l^\nu + m^\mu \bar{m}^\nu + \bar{m}^\mu m^\nu. \quad (2.36)$$

The null tetrad carries a residual local Lorentz freedom, governed by the action of $\text{SO}(3,1) \cong \text{SL}(2, \mathbb{C})/\mathbb{Z}_2$ on the tetrad legs. The elements of this group fall into three distinct transformation classes. The first class consists of spin-boosts, which preserves both null directions via a boost by real $b > 0$ and a spatial rotation by angle θ ,

$$l^\mu \rightarrow b l^\mu, \quad n^\mu \rightarrow b^{-1} n^\mu, \quad m^\mu \rightarrow e^{i\theta} m^\mu. \quad (2.37a)$$

The remaining two are null rotations: one preserving the direction of n^μ and one preserving the direction of l^μ [75]. A null rotation about l^μ , parameterised by $d_1 \in \mathbb{C}$, preserves l^μ whilst transforming the remaining legs as

$$l^\mu \rightarrow l^\mu, \quad m^\mu \rightarrow m^\mu + d_1 l^\mu, \quad n^\mu \rightarrow n^\mu + \bar{d}_1 m^\mu + d_1 \bar{m}^\mu + |d_1|^2 l^\mu, \quad (2.38a)$$

and a null rotation about n^μ , parameterised by $d_2 \in \mathbb{C}$, preserves n^μ with

$$n^\mu \rightarrow n^\mu, \quad m^\mu \rightarrow m^\mu + d_2 n^\mu, \quad l^\mu \rightarrow l^\mu + \bar{d}_2 m^\mu + d_2 \bar{m}^\mu + |d_2|^2 n^\mu. \quad (2.39a)$$

Having defined a local tetrad basis, one can then project the curvature tensor into a collection of curvature scalars. The most fundamental such collection is the set of Weyl scalars,

$$\Psi_0 = -C_{\mu\nu\rho\sigma} l^\mu m^\nu l^\rho m^\sigma, \quad (2.40a)$$

$$\Psi_1 = -C_{\mu\nu\rho\sigma} l^\mu n^\nu l^\rho m^\sigma, \quad (2.40b)$$

$$\Psi_2 = -\frac{1}{2} C_{\mu\nu\rho\sigma} (l^\mu n^\nu l^\rho n^\sigma + l^\mu n^\nu m^\rho \bar{m}^\sigma), \quad (2.40c)$$

$$\Psi_3 = -C_{\mu\nu\rho\sigma} l^\mu n^\nu n^\rho \bar{m}^\sigma, \quad (2.40d)$$

$$\Psi_4 = -C_{\mu\nu\rho\sigma} n^\mu \bar{m}^\nu n^\rho \bar{m}^\sigma. \quad (2.40e)$$

Finally, we define the discrete *prime* symmetry, which exchanges the two null directions,

$$l^\mu \leftrightarrow n^\mu, \quad m^\mu \leftrightarrow \bar{m}^\mu. \quad (2.41)$$

Having described the key null tetrad structure, we can now project the tensorial Einstein equations (2.1) and the Bianchi identities onto a system of coupled scalar equations. In such a form one finds twelve independent spin coefficients $\kappa, \sigma, \rho, \tau, \beta, \epsilon, \kappa', \sigma', \rho', \tau', \beta', \epsilon'$, built from contractions of the tetrad legs with covariant derivatives of the metric, for example, $\kappa = -m^\mu l^\nu \nabla_\mu l_\nu$, alongside the derivative operators

$$D = l^\mu \nabla_\mu, \quad D' = n^\mu \nabla_\mu, \quad \delta = m^\mu \nabla_\mu, \quad \delta' = \bar{m}^\mu \nabla_\mu, \quad (2.42)$$

which together constitute the Newman–Penrose (NP) formalism [76].

One can carry this structure a step further through the Geroch–Held–Penrose (GHP) formalism [77], which embeds the inherent Lorentz co-variance of the system directly into the projected equations. Setting $\lambda = b e^{i\theta}$, the boost-rotation freedom (2.37) takes the form

$$l^\mu \rightarrow \lambda \bar{\lambda} l^\mu, \quad n^\mu \rightarrow (\lambda \bar{\lambda})^{-1} n^\mu, \quad m^\mu \rightarrow \lambda \bar{\lambda}^{-1} m^\mu. \quad (2.43)$$

A scalar ϕ is said to have *proper GHP type* (p, q) if it transforms as [77]

$$\phi \longrightarrow \lambda^p \bar{\lambda}^q \phi \quad (2.44)$$

under (2.43). The tetrad legs themselves carry types $(1, 1)$, $(-1, -1)$, $(1, -1)$, and $(-1, 1)$ for l_μ , n_μ , m_μ , and $\bar{m}_\mu$ respectively. The rules for combining typed objects are: only objects of the same type may be added; the product of a type- (p, q) and a type- (r, s) object has type $(p+r, q+s)$; and complex conjugation interchanges $p \leftrightarrow q$.

Of the twelve Newman–Penrose spin coefficients, only eight carry proper GHP type:

$$\begin{aligned} \kappa (3, 1), \quad \sigma (3, -1), \quad \rho (1, 1), \quad \tau (1, -1), \\ \kappa'(-3, -1), \quad \sigma'(-3, 1), \quad \rho'(-1, -1), \quad \tau'(-1, 1), \end{aligned} \quad (2.45)$$

while the remaining coefficients $\epsilon, \beta, \epsilon', \beta'$ transform with mixed spin weights. This is at odds with the central aim of the GHP formalism, which is to cast the Einstein equations entirely in terms of quantities well-defined under Lorentzian transformations. The remedy is to absorb these poorly-typed coefficients into modified derivative operators: acting on a type- (p, q) scalar ϕ , the four GHP-covariant derivatives are [77]

$$\mathfrak{p}\phi = (D - p\epsilon - q\bar{\epsilon})\phi, \quad \mathfrak{p}'\phi = (D' + p\epsilon' + q\bar{\epsilon}')\phi, \quad (2.46a)$$

$$\bar{\mathfrak{d}}\phi = (\delta - p\beta + q\bar{\beta}')\phi, \quad \bar{\mathfrak{d}}'\phi = (\bar{\delta} + p\beta' - q\bar{\beta})\phi. \quad (2.46b)$$

Through these projections and algebraic manipulation, together with application of the Bianchi identities, the Einstein equations reduce to a simplified form:

$$\bar{\mathfrak{d}}\rho - \bar{\mathfrak{d}}'\sigma = (\rho - \bar{\rho})\tau + (\bar{\rho}' - \rho')\kappa - \Psi_1, \quad (2.47a)$$

$$\mathfrak{p}\rho - \bar{\mathfrak{d}}'\kappa = \rho^2 + \sigma\bar{\sigma} - \bar{\kappa}\tau - \kappa\tau', \quad (2.47b)$$

$$\mathfrak{p}\sigma - \bar{\mathfrak{d}}\kappa = (\rho + \bar{\rho})\sigma - (\tau + \bar{\tau}')\kappa + \Psi_0, \quad (2.47c)$$

$$\mathfrak{p}\rho' - \bar{\mathfrak{d}}'\tau' = \rho'\bar{\rho} + \sigma\bar{\sigma} - \tau'\bar{\tau}' - \kappa\kappa' - \Psi_2. \quad (2.47d)$$

Each of these equations now yields an object of well-defined spin and boost weight, meaning each equation remains of proper type throughout. Finally, we note that while the choice of tetrad in the Kerr geometry admits a wide class of freedoms, a set of standard choices are often made in the literature. We now list several examples specialised to Boyer–Lindquist coordinates. The Kinnersley tetrad, as mentioned previously,

$$l_K^\mu = \left(\frac{r^2 + a^2}{\Delta}, 1, 0, \frac{a}{\Delta} \right), \quad (2.48a)$$

$$n_K^\mu = \frac{1}{2\Sigma} (r^2 + a^2, -\Delta, 0, a), \quad (2.48b)$$

$$m_K^\mu = \frac{1}{\sqrt{2}(r + ia\cos\theta)} \left(ia\sin\theta, 0, 1, \frac{i}{\sin\theta} \right), \quad (2.48c)$$

with $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$ in Kerr. This tetrad is particularly well suited to outgoing coordinate formulations, in which the principal null vectors take the simple form $l^\mu = (0, 1, 0, 0)$. In addition, the Hartle–Hawking tetrad takes the form

$$l_{HH}^\mu = \left(1, \frac{\Delta}{2(r^2 + a^2)}, 0, \frac{a}{r^2 + a^2} \right), \quad (2.49a)$$

$$n_{\text{HH}}^\mu = \left(0, -\frac{r^2 + a^2}{\Sigma}, 0, 0 \right), \quad (2.49\text{b})$$

$$m_{\text{HH}}^\mu = \frac{1}{\sqrt{2}(r + ia \cos \theta)} \left(ia \sin \theta, 0, 1, \frac{i}{\sin \theta} \right), \quad (2.49\text{c})$$

which is particularly well suited to working near the horizon, and in ingoing coordinates takes the simplified form $n^\mu = (0, -1, 0, 0)$. Finally, one can also work in the Carter tetrad, which is related to the Kinnersley tetrad under the Lorentzian boost $\lambda = \sqrt{\Sigma/\Delta}$.

Setting the Kerr geometry as our background, we now wish to develop for the reader the formalism required to describe small perturbations away from Kerr. While it will not be our focus to approach all aspects of this thesis with extreme mathematical rigour, it is worthwhile to briefly consider how perturbations to a spacetime can be made precise, with the first serious attempt to do so due to Stewart and Walker [75]. At its core, a perturbation to a spacetime manifold is a statement about comparing two distinct geometries, $(\mathcal{M}, g)$ and $(\mathcal{M}', g')$. The natural framework for doing so is to embed the continuously parameterised family of intermediate spacetimes $(\mathcal{M}_\epsilon, g_{\mu\nu}(\epsilon))$, with $\epsilon \in [0, 1]$, into a single five-dimensional manifold, where each $\mathcal{M}_\epsilon$ sits as a smooth four-dimensional submanifold, with $\epsilon = 0$ the background and $\epsilon = 1$ the perturbed spacetime. The ϵ -direction is rendered singular by $\nabla_\mu \epsilon = 0$, ensuring the submanifolds do not intersect, with their integral surfaces defined by $d\epsilon \neq 0$. The mapping between the background and a given perturbed slice is then furnished by a pushforward $\phi_\epsilon : \mathcal{M} \rightarrow \mathcal{M}_\epsilon$. The freedom in choosing such an identification introduces a large gauge dependence into our perturbative description. Gauge dependence on its own is not necessarily an issue, and knowing that such perturbative schemes can be made formal, we march ahead, directly mapping tensorial components to those of the background plus small deviations. Due to the nonlinear nature of the field equations, small perturbations do not truncate but instead source an infinite tower of corrections to the background solution, even in vacuum. The simplest means to derive these equations is to consider the functional variation of the Einstein operator itself, $G_{\mu\nu}[g] \rightarrow G_{\mu\nu}[g + \lambda h]$, which admits the expansion

$$G_{\mu\nu}[g + \lambda h] = G_{\mu\nu}[g] + \delta^{(1)}G_{\mu\nu}[h] + \delta^{(2)}G_{\mu\nu}[h, h] + \delta^{(3)}G_{\mu\nu}[h, h, h] + \dots, \quad (3.1)$$

where the n -th order variation is defined by

$$\delta^{(n)}G_{\mu\nu}[h] = \frac{1}{n!} \frac{d^n}{d\lambda^n} G_{\mu\nu}[g + \lambda h] \Big|_{\lambda=0}. \quad (3.2)$$

One then expands the perturbation as $h_{\mu\nu} = \epsilon h_{\mu\nu}^{(1)} + \epsilon^2 h_{\mu\nu}^{(2)} + \epsilon^3 h_{\mu\nu}^{(3)} + \dots$, which upon substitution yields the hierarchical system

$$\mathcal{O}(\epsilon) : \quad \delta^{(1)}G_{\mu\nu}[h^{(1)}] = 0, \quad (3.3)$$

$$\mathcal{O}(\epsilon^2) : \quad \delta^{(1)}G_{\mu\nu}[h^{(2)}] = -\delta^{(2)}G_{\mu\nu}[h^{(1)}, h^{(1)}], \quad (3.4)$$

$$\mathcal{O}(\epsilon^3) : \quad \delta^{(1)}G_{\mu\nu}[h^{(3)}] = -2\delta^{(2)}G_{(\mu\nu)}[h^{(1)}, h^{(2)}] - \delta^{(3)}G_{\mu\nu}[h^{(1)}, h^{(1)}, h^{(1)}]. \quad (3.5)$$

At each order the left-hand side is governed by the same linear operator $\delta^{(1)}G_{\mu\nu}$, while the right-hand side acts as an effective source built from solutions at lower orders, so that the full nonlinear problem is unfolded into a hierarchy of linearised equations sourced successively by lower-order data.

3.1 VACUUM TEUKOLSKY EQUATIONS

Even in the form above, which reduces the nonlinear system of 10 coupled partial differential equations to a hierarchy of linear problems, the linearised Einstein equations remain a complex task to solve. In the context of the background Kerr geometry, which is the primary point of interest in this thesis, the linearised equations do not decouple under a spheroidal harmonic decomposition, and hence, at minimum in the frequency domain, must be solved in the form $h_m(\omega, r, \theta)e^{im\phi - i\omega t}$ as a system of frequency-domain coupled *Fourier m-modes*, in which the Einstein equations take the form of two-dimensional coupled elliptic linear PDEs. Making use of the principal null directions and specialising to the GHP formalism discussed in the previous chapter, one can indeed decouple the perturbation equations. Specialising to a Petrov type-D background, all perturbed quantities are expanded as $\Psi_i \rightarrow \Psi_i^{(0)} + \epsilon\Psi_i^{(1)}$, which, using the Goldberg–Sachs theorem [78], gives the background conditions

$$\Psi_i^{(0)} = 0 \text{ for } i \in \{0, 1, 3, 4\}, \quad \kappa^{(0)} = \sigma^{(0)} = \kappa'^{(0)} = \sigma'^{(0)} = 0. \quad (3.6)$$

Retaining only $\mathcal{O}(\epsilon)$ terms, applying $\bar{\delta}'$ to the primed Bianchi identity and performing a number of additional manipulations as laid out in [75], one obtains the *vacuum Teukolsky equation* [79],

$$\mathcal{O}_{-2}[\Psi_4^{(1)}] = [(\mathfrak{p}' - 4\rho' - \bar{\rho}')(\mathfrak{p} - \rho) - (\bar{\delta}' - \bar{\tau} - 4\tau')(\bar{\delta} - \tau) - 3\Psi_2]\Psi_4^{(1)} = 0, \quad (3.7)$$

where the (-2) notes that are looking at the spin $s = -2$ rather than the spin $s = +2$ Teukolsky equation. One can also push this calculation to second order as was first laid out in Campanelli and Lousto [80] and later re-visited in Spiers et al. [81] to take the form,

$$\begin{aligned} \mathcal{O}_{-2}[\Psi_4^{(2)}] &= \mathcal{S}_{-2}^{(2)}[h^{(1)}, (e_i^\mu)^{(1)}] \\ &= \left[(\delta' - \bar{\tau} + \bar{\beta} - 3\beta' - 4\tau')(\delta + 4\beta - \tau)^{(1)} \right. \\ &\quad \left. - (D' + 3\epsilon' - \bar{\epsilon}' + 4\rho' + \bar{\rho}')(D + 4\epsilon - \rho)^{(1)} \right] \psi_4^{(1)} \\ &\quad - \left[(\delta' - \bar{\tau} + \bar{\beta} - 3\beta' - 4\tau')(\Delta + 4\bar{\rho}' + 2\bar{\epsilon}')^{(1)} \right. \\ &\quad \left. - (D' + 3\epsilon' - \bar{\epsilon}' + 4\rho' + \bar{\rho}')(\bar{\delta} + 4\bar{\tau}' + 2\bar{\beta}')^{(1)} \right] \psi_3^{(1)} \\ &\quad + 3 \left[(\delta' - \bar{\tau} + \bar{\beta} - 3\beta' - 4\tau')\bar{\kappa}'^{(1)} \right] \end{aligned}$$

$$\begin{aligned}
 & - (D' + 3\varepsilon' - \bar{\varepsilon}' + 4\rho' + \bar{\rho}') \bar{\sigma}'^{(1)} \Big] \psi_2^{(1)} \\
 & + 3 \Big[\left((\delta' - \bar{\tau} + \bar{\beta} - 3\beta' - 4\tau') - 3\pi \right)^{(1)} \bar{\kappa}'^{(1)} \\
 & - \left((D' + 3\varepsilon' - \bar{\varepsilon}' + 4\rho' + \bar{\rho}') - 3\bar{\rho}' \right)^{(1)} \bar{\sigma}'^{(1)} \Big], \tag{3.8}
 \end{aligned}$$

where the perturbations to all quantities must be carefully calculated with respect to both the background metric perturbations and the tetrad perturbations. Substituting the Kinnersley spin coefficients (2.45) and $\Psi_2^{(0)} = M\rho^3$ into the leading-order Teukolsky equation, one obtains the coordinate form of the operator for Ψ_4 in Boyer–Lindquist coordinates [82],

$$\begin{aligned}
 & \left[\frac{(r^2 + a^2)^2}{\Delta} - a^2 \sin^2 \theta \right] \frac{\partial^2 \Psi_4}{\partial t^2} + \frac{4Mar}{\Delta} \frac{\partial^2 \Psi_4}{\partial t \partial \phi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2 \theta} \right] \frac{\partial^2 \Psi_4}{\partial \phi^2} \\
 & - \Delta^2 \frac{\partial}{\partial r} \left(\Delta^{-1} \frac{\partial \Psi_4}{\partial r} \right) - \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Psi_4}{\partial \theta} \right) + 4 \left[\frac{a(r - M)}{\Delta} + \frac{i \cos \theta}{\sin^2 \theta} \right] \frac{\partial \Psi_4}{\partial \phi} \\
 & + 4 \left[\frac{M(r^2 - a^2)}{\Delta} - r + ia \cos \theta \right] \frac{\partial \Psi_4}{\partial t} + (4 \cot^2 \theta + 2) \Psi_4 = 0. \tag{3.9}
 \end{aligned}$$

Going further, one can write the ansatz

$$\Psi_4(t, r, \theta, \phi) = e^{-i\omega t} e^{im\phi} \Delta^{\frac{1}{2}} \mathcal{S}(\theta) \mathcal{R}(r), \tag{3.10}$$

for $\omega \in \mathbb{C}$ and $m \in \mathbb{Z}$, causing Eq. (3.9) to separate further into an angular equation,

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\mathcal{S}}{d\theta} \right) - \left[\frac{(m - 2 \cos \theta)^2}{\sin^2 \theta} - (a\omega)^2 \cos^2 \theta - 2a\omega \cos \theta \right] \mathcal{S} + \lambda \mathcal{S} = 0, \tag{3.11}$$

and a radial ODE,

$$\begin{aligned}
 & \Delta \frac{d^2 \mathcal{R}}{dr^2} + \frac{[\omega(r^2 + a^2) - am + 2i(r - M)]^2}{\Delta} \mathcal{R} \\
 & + \left(\frac{M^2 - a^2}{\Delta} - 8i\omega - \lambda - a^2 \omega^2 + 2am\omega \right) \mathcal{R} = 0, \tag{3.12}
 \end{aligned}$$

where λ is the angular separation constant.

In solving the vacuum Teukolsky equation $\mathcal{O}_s[\psi_s] = 0$, one can fully reconstruct the metric perturbation through the construction of a Hertz potential, which itself solves the adjoint equation. The method of reconstruction discussed here was first laid out by Chrzanowski, Cohen, and Kegeles [83–85], and is known as CCK reconstruction; it was later simplified through the method of adjoints by Wald [86]. The argument distills to the following: suppose $\mathcal{O}_s^\dagger[\Phi] = 0$, where $\dagger$ denotes the

adjoint operator. The sourced Teukolsky equation takes the form $\mathcal{O}_s[\psi_s] = \mathcal{S}_s[T]$, where $\mathcal{E}[h] = T$ are the linearised Einstein equations, given by

$$\mathcal{E}_{ab}[h] = -\frac{1}{2} \left[\nabla^c \nabla_c h_{ab} + \nabla_a \nabla_b h - 2 \nabla^c \nabla_{(a} h_{b)c} + g_{ab} \left(\nabla^c \nabla^d h_{cd} - \nabla^c \nabla_c h \right) \right] = T_{\alpha\beta}, \quad (3.13)$$

and $\psi_s = \mathcal{T}_s[h]$, defined through $\mathcal{T}_s$, the operation of extracting the Weyl curvature from the metric and contracting with the relevant tetrad legs, is given by

$$\mathcal{T}_{+2}[h] = \delta C_{abcd}[h] l^a m^b l^c m^d, \quad \mathcal{T}_{-2}[h] = \delta C_{abcd}[h] n^a \bar{m}^b n^c \bar{m}^d, \quad (3.14)$$

where $\delta C_{abcd}[h]$ is the linearised Weyl tensor, so that $\psi_{+2} = \Psi_0^{(1)}$ and $\psi_{-2} = \Psi_4^{(1)}$. One then obtains the operator identity

$$\mathcal{O}_s \mathcal{T}_s = \mathcal{S}_s \mathcal{E} \Rightarrow \mathcal{T}_s^\dagger \mathcal{O}_s^\dagger = \mathcal{E}^\dagger \mathcal{S}_s^\dagger. \quad (3.15)$$

Noting that the linearised Einstein operator is self-adjoint, $\mathcal{E} = \mathcal{E}^\dagger$, we immediately obtain $\mathcal{E}[\mathcal{S}_s^\dagger \Phi] = 0$, and hence $\mathcal{S}_s^\dagger \Phi$ is a solution to the vacuum Einstein equations. Importantly, Φ satisfies the equation for $\mathcal{O}_s^\dagger$, the adjoint of the Teukolsky operator, and is therefore distinct from the Teukolsky scalar. It is referred to instead as the *Hertz potential*, in keeping with the role it plays in the reconstruction procedure. An important note on the adjoint operator is in order: for the spin-weight s Teukolsky operator $\mathcal{O}_s$, one finds $\mathcal{O}_s^\dagger = \mathcal{O}_{-s}$, so that Φ carries spin weight $-s$. Crucially, however, this does not imply that the Hertz potential is given by the opposite-spin Weyl scalar; rather, the reconstruction of the corresponding Weyl scalar from the Hertz potential is mediated by the fourth-order differential relation

$$\psi_s = \mathcal{T}_s \mathcal{S}_s^\dagger[\Phi], \quad (3.16)$$

where the operator $\mathcal{T}_s \mathcal{S}_s^\dagger$ is not the identity. A key feature of the CCK reconstruction procedure is its implicit choice of gauge. In particular, depending on whether one reconstructs the metric from the Ψ_4 or Ψ_0 Weyl scalar and what choice of reconstruction operators one chooses, the metric will arise in either outgoing or ingoing radiation gauge (ORG/IRG), which, together with the tracelessness condition $h^\mu{}_\mu = 0$, requires orthogonality with respect to either the ingoing or outgoing null vector, $n^\mu h_{\mu\nu} = 0$ or $l^\mu h_{\mu\nu} = 0$, respectively.

Defining the Chandrasekhar radial and angular operators [87],

$$\mathcal{D} \equiv l^\mu \partial_\mu = \partial_r - \frac{iK}{\Delta}, \quad \mathcal{L}^\dagger \equiv \sqrt{2\rho} m^\mu \partial_\mu = \partial_\theta - Q, \quad (3.17)$$

$$\mathcal{D}^\dagger \equiv -\frac{2\Sigma}{\Delta} n^\mu \partial_\mu = \partial_r + \frac{iK}{\Delta}, \quad \mathcal{L} \equiv \sqrt{2\bar{\rho}} \bar{m}^\mu \partial_\mu = \partial_\theta + Q, \quad (3.18)$$

where $K \equiv \omega(r^2 + a^2) - am$ and $Q \equiv m \csc \theta - a\omega \sin \theta$, and their weighted forms,

$$\mathcal{D}_n \equiv \mathcal{D} + \frac{n\Delta'}{\Delta}, \quad \mathcal{L}_n \equiv \mathcal{L} + n \cot \theta, \quad (3.19)$$

one obtains the explicit metric perturbation satisfying $h_{\mu\nu}^{\text{IRG}} l^\nu = 0$,

$$\begin{aligned} h_{\text{IRG}}^{\mu\nu} = & -\frac{1}{2\rho^2} l^\mu l^\nu \left(\mathcal{L}_1^\dagger - \frac{2\rho_{,\theta}}{\rho} \right) \mathcal{L}_2^\dagger \Phi_- - m^\mu m^\nu \left(\mathcal{D} - \frac{2}{\rho} \right) \mathcal{D} \Phi_- \\ & + \frac{\sqrt{2}}{\rho} l^{(\mu} m^{\nu)} \left(\mathcal{D} \mathcal{L}_2^\dagger - \frac{1}{\rho} (\mathcal{L}_2^\dagger + \rho_{,\theta} \mathcal{D}) - \frac{1}{\bar{\rho}} (\mathcal{L}_2^\dagger + \bar{\rho}_{,\theta} \mathcal{D}) \right) \Phi_-, \end{aligned} \quad (3.20)$$

in terms of the Hertz potential Φ_- , where $\Delta^{-2}\Phi_-$ satisfies the $s = +2$ adjoint Teukolsky equation.

Having detailed the basic properties of motion, the algebraic structure, and vacuum perturbations in the Kerr geometry so far in Part i, we now turn to matter field models on the Kerr geometry. In particular, we consider massive complex scalar fields and their associated threshold solutions as boson clouds [39–48, 88], as well as self-consistently computed perfect fluids as thin accretion discs [17, 89–110]. Throughout this chapter, both matter fields are treated as test fields on the Kerr geometry, i.e. their own gravitational back-reaction is neglected. Although such an approximation is often made, it is rarely presented in its consistent formulation within black hole perturbation theory. Such a scheme is made formal by treating the matter field as containing subleading energy content relative to the background spacetime, allowing it to be incorporated as a perturbative source within the framework developed in Part i. We begin by making explicit how such a scheme is constructed.

Sections of this chapter are drawn directly from papers associated with this thesis. In particular, Section 4.2 constitutes material from a published article in *Phys. Rev. Lett.* 134 (2025) 21, 211403, co-authored with Thomas Spieksma, Richard Brito, Maarten van de Meent, and Sam Dolan, and significant sections of Section 4.3 are taken from an article currently under review, *Phys. Rev. D* (2026), co-authored with Daniel J. D’Orazio.

4.1 MATTER AS TEST FIELDS ON FIXED BACKGROUNDS

We wish to describe matter in curved spacetime, and consider the non-linear Einstein field equations coupled to matter as our starting point,

$$\begin{aligned} G_{\mu\nu}[g] &:= R_{\mu\nu}[g] - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu}^{\text{env}}[g, \{\psi\}], \\ \nabla^\mu T_{\mu\nu}^{\text{env}} &= 0. \end{aligned} \quad (4.1)$$

where g is the metric and $\{\psi\}$ is a collection of *scalar*, *vector* and/or *tensor* degrees of freedom describing the matter field in question, with $T_{\mu\nu}^{\text{env}}$ being its corresponding stress-energy tensor.

The full stress-energy tensor for a massive complex scalar field is given by,

$$T_{\mu\nu}^\Phi[g, \{\phi, \bar{\phi}\}] := \nabla_{(\mu}\phi\nabla_{\nu)}\bar{\phi} - \frac{1}{2}g_{\mu\nu}(\nabla^\alpha\phi\nabla_\alpha\bar{\phi} + \mu^2|\phi|^2) \quad (4.2)$$

where ϕ is the scalar field matter source minimally coupled to gravity and μ is its associated mass. Imposing the equations of motion then gives rise to the system of coupled equations,

$$\begin{aligned} G_{\mu\nu} [g] &= T_{\mu\nu}^\phi, \\ \square\phi - \mu^2\phi &= 0 \end{aligned} \quad (4.3)$$

where for now $\square$ is the d’alembertian operator on the full spacetime. From the fluid perspective, wherein this thesis we will always take the fluid stress-energy tensor to be truncated to zeroth order in gradients[11], gives,

$$T_{\mu\nu}^{\text{fl}}[g, \{\epsilon, p, u\}] := (\epsilon + p) u_\mu u_\nu + p g_{\mu\nu}, \quad (4.4)$$

with ϵ , p , and u^μ the fluid energy density, pressure, and four velocity fields respectively. Similarly, this then gives rise to the nonlinear coupled system of equations,

$$G_{\mu\nu} [g] = T_{\mu\nu}^{\text{fl}}, \quad (4.5)$$

$$(\epsilon + p) u^\mu \nabla_\mu u^\nu = -\nabla^\nu p - u^\nu \nabla_\mu ([\epsilon + p] u^\mu). \quad (4.6)$$

This system can be further simplified to the relativistic momentum and energy equations through the introduction of the projector tensor, $q^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$, to give the nonlinear coupled relativistic Euler equations,

$$G_{\mu\nu} [g] = T_{\mu\nu}^{\text{fl}}, \quad \nabla_\mu u^\mu = -\frac{u^\mu \nabla_\mu \epsilon}{(\epsilon + p)}, \quad u^\mu \nabla_\mu u^\nu = -\frac{q^{\mu\nu} \nabla_\mu p}{(\epsilon + p)}. \quad (4.7)$$

Working within the perturbation theory context motivates the general control of such stress-energy tensors by a small parameter λ such that,

$$G_{\mu\nu} [g] = \lambda^n T_{\mu\nu}^{\text{env}} [g, \{\psi^N\}] \quad (4.8)$$

where n is a number setting the overall power of λ at leading order, fixed by requiring the perturbative counting on constituent fields arises as integer quantities, and $N = \{n_i\}$ is a multi-index tracking the different leading-order scalings of the matter’s constituent fields.

In the absence of large overdensities, where the small controlling parameter maintains the “smallness” of the matter content, such a source and these assumptions directly imply the perturbative expansions,

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + \lambda^n h_{\mu\nu}^{(n)} + \dots, \quad (4.9)$$

$$\{\psi\} \rightarrow \lambda^N \{\psi^N\} + \lambda^{N+n} \{\psi^{N+n}\} + \dots, \quad (4.10)$$

on scales set by the background vacuum solution, where we adopt the notation,

$$\lambda^N \{\psi^N\} = \{\lambda^{n_1} \psi_1^{n_1}, \lambda^{n_2} \psi_2^{n_2}, \lambda^{n_3} \psi_3^{n_3}, \dots\}. \quad (4.11)$$

To make this concrete, consider the two matter models of interest in this section. For the scalar field we have,

$$\{\psi\} \rightarrow \{\lambda\phi^{(1)}, \lambda\bar{\phi}^{(1)}\}, \quad (4.12)$$

$$T^\phi \sim \phi\bar{\phi} \rightarrow \lambda^2\phi^{(1)}\bar{\phi}^{(1)}, \quad (4.13)$$

so that $N = \{1, 1\}$ and $n = 2$. For the perfect fluid we have an alternative prescription,

$$\{\psi\} \rightarrow \{\lambda\epsilon^{(1)}, \lambda p^{(1)}, u^{(0)}\}, \quad (4.14)$$

$$T^\text{fl} \sim (\epsilon + p)uu + pg \rightarrow \lambda \left((\epsilon^{(1)} + p^{(1)})u^{(0)}u^{(0)} + g^{(0)}p^{(1)} \right), \quad (4.15)$$

hence $N = \{1, 1, 0\}$ and $n = 1$. From this prescription it should be clear that each matter model requires its own slightly distinct treatment, but the general framework accommodates a wide class of matter fields within this language.

Taking this general perturbative prescription, our equations then cleanly separate into a system of hierarchical coupled equations,

$$\mathcal{O}(1) : G_{\mu\nu}[g] = 0, \quad (4.16)$$

$$\mathcal{O}(\lambda^N) : \nabla^\mu T_{\mu\nu}^{\text{env}}[g, \{\psi^N\}] = 0, \quad (4.17)$$

$$\mathcal{O}(\lambda^n) : \delta G_{\mu\nu}^{(1)}[h^{(n)}] = T_{\mu\nu}^{\text{env}}[g, \{\psi^N\}]. \quad (4.18)$$

$$(4.19)$$

where $\delta G_{\mu\nu}[\cdot]$ is as defined in sec. 9.2 and the source S built from lower-order fields will be clarified in later sections. Although rarely stated explicitly in this manner, the system in Eq. (7.16) forms the basis for any calculation of test field, boson cloud [112], dark matter spike [113], or accretion disk [92] models. At higher order, the final term also allows one to compute fluxes associated with $h_{\mu\nu}^{(n)}$ through the stress-energy tensor $T_{\mu\nu}^{\text{env}}$, as in [42]. This will form the basis of our approach to the dynamical friction force calculation of a Kerr black hole in linear motion through an asymptotically homogeneous medium in Chp. 5. Before proceeding to this chapter however we briefly motivate and provide an overview of the construction of the Boson Cloud and thin disk solutions which will be used in later chapters.

4.2 BOSON CLOUD

Through the process of *superradiance* [112, 114–116], massive bosonic fields can extract rotational energy from BHs, forming dense, macroscopic structures known as *boson clouds*. This process is most efficient when the Compton wavelength of the field is comparable to the gravitational radius of the BH, i.e., when the “gravitational coupling” $\alpha \equiv \mu M \sim \mathcal{O}(0.1)$ (with μ the boson mass). These clouds

can grow on astrophysical timescales [117, 118], reaching a quasi-stationary state known as the *superradiance threshold*, where the cloud has a characteristic frequency ω_c and the BH spin parameter is $a/M \approx 4m_b\alpha/(m_b^2 + 4\alpha^2)$ [119, 120]. Here, m_b is the azimuthal angular momentum of the scalar field. Throughout this work, we assume the cloud to be at this threshold, while residing in its dominant, dipolar ground state $|n_b\ell_b m_b\rangle = |211\rangle$.

The cloud-BH system, or *gravitational atom*, has gained significant attention as a target for future GW detectors. The reason is threefold: (i) the superradiance process is purely gravitational and does not require a pre-existing abundance of the scalar field; (ii) for astrophysical BHs, clouds form for bosons in the mass range $\mathcal{O}(10^{-20} - 10^{-10}) \text{ eV}/c^2$. Such ultralight particles are proposed solutions to the strong CP problem [121–123] and are plausible dark matter candidates [124–127]; (iii) studies on the Newtonian level have revealed a rich phenomenology when gravitational atoms are part of a binary system [101, 128–137], making them interesting candidates for testing fundamental physics with GWs.

For the purposes of modelling, we treat such a fundamental field as an agnostic additional scalar degree of freedom, minimally coupled to gravity. From the modelling perspective, boson clouds arise as bound-state solutions of the massive Klein–Gordon equation on the Kerr geometry,

$$\square\Phi - \mu^2\Phi = 0, \quad (4.20)$$

where $\mu = m_s/\hbar$ is the boson Compton wavenumber and m_s the boson mass.

Such solutions can be solved for directly as bound state solutions to Kerr black holes. However, modelling their initial formation is a separate matter. In the absence of other perturbing astrophysical forces, the scalar field exhibits exponential growth near the central black hole, leading to the formation of a macroscopic boson cloud. The mechanism driving this growth is superradiance [112], whereby the scalar field extracts energy from the spin of the central black hole. Solutions to this equation are generally decomposed in a spheroidal basis of the form,

$$\Phi = \sum_{\ell m} R_{\ell m}(r) S_{\ell m}(\theta, \phi) e^{-i\omega_c^{\ell m} t}, \quad (4.21)$$

where $S_{\ell m}$ are the spin-0 spheroidal eigenfunctions of the wave equation in Kerr. Waves satisfying $\omega_c^{\ell m} < m\Omega_H$ (where Ω_H is the angular velocity of the horizon) are amplified upon scattering in the ergoregion, extracting rotational energy from the black hole. When a bound state satisfies this condition, the mode grows exponentially as it is reflected off the potential barrier arising due to its mass, repeatedly interacting with the black hole inside the ergoregion.

Each mode $\omega_c^{\ell m}$ has its own characteristic growth rate, with the largest belonging to the ω_c^{11} mode. Consequently, such systems are often modelled using only a single dipolar $(\ell, m) = (1, 1)$ mode, yielding a profile of the form,

$$\Phi = R_{11}(r) S_{11}(\theta, \phi) e^{-i\omega_c t}, \quad (4.22)$$

where the indices on the frequency have been dropped.

Substituting the above ansatz into Eq. (4.20) gives rise to the radial massive scalar wave equation in the Kerr geometry. Multiple methods exist for solving this ordinary differential equation; one of the most prevalent is a modified Frobenius approach introduced in the context of gravitational physics by Leaver [138] and popularised in the context of boson clouds by Dolan [139]. The key refinement beyond the standard Frobenius method is to make the required boundary behaviour at each end of the domain explicit in the prefactors, before inserting the series expansion with undetermined coefficients for the field. This leads to the ansatz,

$$R(r) = (r - r_+)^{-i\sigma} (r - r_-)^{i\sigma + \chi - 1} e^{q r} \sum_{n=0}^{\infty} a_n \left(\frac{r - r_+}{r - r_-} \right)^n, \quad (4.23)$$

$$\sigma = \frac{2r_+(\omega - \omega_c)}{r_+ - r_-}, \quad q = \pm \sqrt{\mu_s^2 - \omega^2}, \quad h = \frac{\mu_s^2 - 2\omega^2}{q}.$$

where the coefficients a_n now satisfy a three-term recurrence relation

$$\alpha_0 a_1 + \beta_0 a_0 = 0, \quad (4.24)$$

$$\alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} = 0, \quad n > 0, \quad n \in \mathbb{N}, \quad (4.25)$$

with weightings α , β , and γ fixed by the radial KG equation. In particular, the inclusion of the exponential prefactor encodes the correct asymptotic behaviour at infinity, which the power series would not be able to reproduce on its own. This has the effect of ensuring convergence of the method throughout the entire domain, providing our first test-field environmental model on the Kerr geometry. Beyond the applications to fundamental physics laid out earlier in this section, such an environment also holds a clear attraction as one of the simplest self-consistent solutions for an environment in the Kerr geometry, governed by a simple scalar equation. We will later see that taking such a system as our initial setup serves as an excellent testbed for developing the methods required for more complicated environments.

4.3 THIN DISC

An additional, directly astrophysically motivated environment is that of a stationary, axisymmetric, locally isothermal thin disc on the Kerr geometry. We will now present the derivation of the background, following the approach of Abramowicz [140]. Parts of this section are taken directly from [62], which was co-authored with Daniel D’Orazio.

The form of thin disc profile we seek admits an additional simplifying parameter, a constant dimensionless aspect ratio $h \ll 1$ such that in Boyer-Lindquist coordinates $W(r) := 2H(r) = 2rh(r)$ is the width of the disk as a function of radius. This

additional small parameter allows us to consistently build a description of the Kerr geometry near the equatorial plane. Expanding in this parameter we then expand all fluid variables consistently to $\mathcal{O}(h^2)$ simplifying the equations of motion. The background perfect fluid equations on the Kerr geometry give the form,

$$(\epsilon + p) u^\mu \nabla_\mu u^\nu + q^{\mu\nu} \nabla_\mu p = 0. \quad (4.26)$$

where we take the flow $u^\mu = \alpha(r, \theta) \{1, 0, 0, \Omega(r, \theta)\}$ to be axi-symmetric in Boyer–Lindquist coordinates, with equation of state $p = c_s^2(r) \epsilon(r, \theta)$, fixing the disc to be poloidally isothermal but radially non-isothermal. Imposing hydrostatic equilibrium reduces Eq. (4.26) to

$$\Gamma_{\alpha\nu}^\mu u^\nu u_\mu = \frac{c_s^2}{1 + c_s^2} \frac{\partial_\alpha p}{p}, \quad (4.27)$$

with fluid variables scaling as $p \sim h^2$, $c_s \sim h$, $\delta\omega \sim h^2$, $\delta\alpha \sim h^2$ in analogy with the Newtonian case [141, 142], where $\Omega = \Omega^k(1 + \delta\omega)$ and $\alpha = \alpha^k(1 + \delta\alpha)$ denote the angular frequency and lapse perturbed away from their circular equatorial geodesic values.

Solving Eq. (4.27) self-consistently to $\mathcal{O}(z^2)$ yields the pressure profile

$$p(r, z) = p_0(r) e^{-z^2/2h^2}, \quad (4.28)$$

with the Boyer–Lindquist disc aspect ratio

$$h(r) \equiv c_s \sqrt{\frac{2ar^{3/2} - (3 - r)r^2}{(1 + c_s^2)(3a^2 - 4a\sqrt{r} + r^2)}}. \quad (4.29)$$

Fixing h to be constant determines the thermodynamic profile, giving

$$c_s = h \sqrt{\frac{4a\sqrt{r} - 3a^2 - r^2}{3a^2h^2 + (3 + h^2 - r)r^2 - 2a\sqrt{r}(2h^2 + r)}}, \quad (4.30)$$

and fixing the angular frequency perturbation via the radial component of Eq. (4.27),

$$\delta\omega = \frac{h^2(3a^2 - 4a\sqrt{r} + r^2) \partial_r p_0}{(2ar^{3/2} + (-3 + r)r^2) p_0}. \quad (4.31)$$

The sole remaining freedom is the equatorial energy density profile; for the remainder of this thesis we will arbitrarily choose the power-law

$$\epsilon = \epsilon_I \left(\frac{r}{r_I} \right)^{-2}, \quad (4.32)$$

where the subscript I denotes evaluation at the ISCO, hence providing the full solution to a perfect-fluid thin disc on the Kerr geometry to a controlled error of

$\mathcal{O}(\hbar^3)$. Such an environment is, as in the boson cloud case, particularly simple as a background; importantly, however, it is governed through a system of coupled non-linear PDEs as opposed to the single linear PDE found in the scalar case, making the calculation of the dynamics in such an environment a more arduous task. This does not, however, make the problem intractable, but instead represents an important development on top of the initial scalar model. We perform this development in Chp. 9.

Before building upon the theory developed so far and extending to the full problem of modelling EMRIs in environments, we first consider the response of matter to the presence of a black hole in rectilinear motion. This is the relativistic analogue of the classical dynamical friction problem studied by Chandrasekhar [53] and Ostriker [143], and is built upon the principles presented in the previous section, whereby test field solutions form the source of perturbation to the metric and can be used to extract analytic fluxes from the system. The fully relativistic treatment in this manner was first carried out by Vicente and Cardoso [42]. The following chapter, which constitutes the first of four *paper chapters* within this thesis, extends this picture to black holes of generic spin magnitude and alignment relative to the direction of motion. We explore this setup in two distinct regimes: that of a homogeneous corpuscular medium and that of a light scalar field, examining the differences between the two sets of results.

The classical dynamical friction force can be understood as follows. As a black hole moves through a medium, it attracts surrounding matter, creating a gravitational wake that in turn exerts an effective drag, slowing the black hole. This force has significant astrophysical consequences, ranging from the formation of supermassive black hole binaries to modifications in the phase evolution of compact binary mergers. In this chapter we explore the motion of *spinning* black holes through a medium. We find that the classical drag along the velocity direction is modified, and two novel forces appear: a rotational force, known in the context of fluid dynamics as the Magnus force, and a lift force orthogonal to the direction of motion. Within this chapter we also provide phenomenological fits to our numerical results which have seen recent application in the context of EMRIs [144].

The contents of this chapter primarily constitute the published article *Phys. Rev. D* **109**, 104038 (2024) [59], co-authored with Jaime Redondo-Yuste, Maarten van de Meent, and Vitor Cardoso.

5.1 PARTICLE-LIKE ENVIRONMENTS

We begin by studying the motion of a spinning BH in a medium of collision-less particles, with uniform density ρ , composed of particles with a mass scale m_p distributed on a cloud with radius $b_{\max}$ ¹. In the BH rest frame this problem is equivalent to scattering geodesics on that same BH background. We first obtain closed-form expressions for the transfer of momentum due to individual particles in geodesic motion. We numerically integrate the total contribution to the transfer of momentum due to each individual geodesic to compute the total force that the BH suffers. For this reason, although our approach is not fully analytical, it is sensitive to all the strong-field effects. Further analytical developments can be carried within a post-Minkowskian approach. Recently, expansions of the scattering and inclination angles were obtained up to order $\mathcal{O}(G_N^3 a^2)$, providing a potentially simpler integration scheme [146]. However, obtaining fully analytic formulas for the drag, Magnus and lift forces in a slow-spinning post-Minkowskian approach is out of the scope of this chapter.

We are interested both in scattering geodesics² and plunging geodesics (which contribute through accretion onto the BH) [72]. The separatrix between both cases occurs when the second radial turning point becomes complex, $r_2 = r_3$. Equations (2.15) allow us to describe equivalently the geodesic either knowing $\{p, e, x_{\text{inc}}^2\}$, or the orbital constants $\{\mathcal{E}, \mathcal{L}, Q\}$ (the radial and polar roots can be obtained analytically in terms of these, and vice-versa).

In addition to the orbital constants, in Boyer-Lindquist coordinates we need four initial phases $(q_t^0, q_r^0, q_z^0, q_\phi^0)$ to fully specify a geodesic [71]. Of these, we can always arrange q_t^0 and q_ϕ^0 to be zero, by applying a time translation and rotation to the Kerr background. We can also choose to always start our Mino time parameter at the pericenter of the trajectory, giving $q_r^0 = 0$. For generic bound geodesics, we could have used ergodicity of the orbit to also set $q_z^0 = 0$ without loss of generality. This, however, is not possible for scattering geodesics, since they complete in a finite amount of Mino time [147]. Different values of q_z^0 lead to physically different scattering orbits, and we will need to specify its value.

¹ It is well known that the drag force is (logarithmically) divergent when the cloud size is taken to infinity, $b_{\max} \rightarrow \infty$ [42, 145], so imposing such a cut-off is necessary from the computational point of view.

² These can be obtained via analytical continuation of the analytic solutions for bound geodesics obtained in [71] as explained in [147].

In order to compute the momentum flux imparted on the BH by an incoming particle, we make use of the scattering angle and the difference between the final and initial inclinations, which are [146, 147]

$$\begin{aligned}\cos \theta_{\text{in/out}} &= z_1 \text{sn} \left[K(k_z) \frac{2}{\pi} \left(q_z^0 \mp \frac{Y_z}{Y_r} q_r^S \right) \middle| k_z \right], \\ k_z &= a^2 \left(1 - \mathcal{E}^2 \right) \frac{z_1^2}{z_2^2}, \\ q_r^S &= \frac{\pi}{K(k_r)} F \left[\arcsin \sqrt{\frac{r_3 - r_1}{r_2 - r_1}} \middle| k_r \right], \\ k_r &= \frac{(r_1 - r_2)(r_3 - r_4)}{(r_1 - r_3)(r_2 - r_4)}.\end{aligned}\tag{5.1}$$

Here, $Y_{z(r)}$ are polar (radial) frequencies given by

$$\begin{aligned}Y_r &= \frac{\pi}{2K(k_r)} \sqrt{(1 - \mathcal{E}^2)(r_1 - r_3)(r_2 - r_4)}, \\ Y_z &= \frac{\pi z_2}{2K(k_r)},\end{aligned}\tag{5.2}$$

K is the complete elliptic integral of the first kind, sn the Jacobi elliptic sine function, and q_z^0 is the initial polar phase. Similarly, the scattering angle can be obtained as

$$\chi = 2q_r^S \frac{Y_z}{Y_r} - \phi_r(q_r^S) + \phi_z \left(q_z^0 + q_r^S \frac{Y_z}{Y_r} \right) - \phi_z \left(q_z^0 - q_r^S \frac{Y_z}{Y_r} \right),\tag{5.3}$$

where the phases $\phi_{r(z)}$ are defined in Eqs.(27)–(35) of [71], and for simplicity we set the initial radial phase to zero. We compute these geodesic quantities using the Black Hole Perturbation Toolkit [148].

We will now compute the force that is imparted to the BH due to the absorption of particles of the medium (plunging geodesics) and conservation of linear momentum, associated to the scattering of geodesics. We consider a homogeneous medium composed of point-particles of density ρ such that each individual mass $m_p \ll M$. The BH is moving with some velocity v in a direction which forms an angle β with its spin. For simplicity, we assume that the BH spin is oriented along the z axis, see Fig. 5.1 for an illustration of our setup. The only free parameters in this case are the velocity v of the BH and the angle β . In the BH frame, the motion of each particle can be mapped to either a scattering or a plunging geodesic. Each particle can be identified by the angle α that it forms with the plane defined by the BH spin, and a radial vector $b \in [0, b_{\text{max}}]$ which can be understood as an impact parameter. This is illustrated in Fig. 5.1. These variables provide a useful parametrization, from which we can define the particle's asymptotic four-velocity and impact vector as

$$\begin{aligned}u^\mu &= \gamma \left[1, 0, v \sin \beta, -v \cos \beta \right], \\ b^\mu &= b \left[0, \sin \alpha, -\cos \beta \cos \alpha, \sin \beta \cos \alpha \right].\end{aligned}\tag{5.4}$$

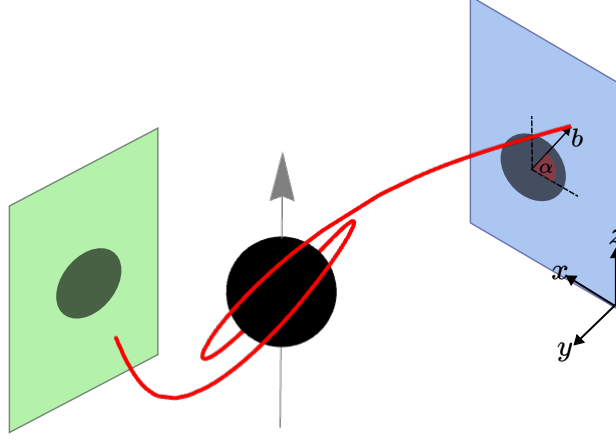


Figure 5.1: Diagram representing the motion of a geodesic (red) and the coordinate system used to describe it, with z pointing in the direction of the BH spin, and y such that the particle is in-coming along the (y, z) plane. The particle is identified by its impact parameter b and impact angle α , as shown in the blue impact plane. We will extract the momentum flux at the out-going plane, represented in green.

Above, $\gamma = (1 - v^2)^{-1/2}$ is the Lorentz factor. From these quantities, we can deduce the particle's constants of motion:

$$\begin{aligned} \mathcal{E} &= \gamma, \\ \mathcal{L} &= -\gamma b v \sin \alpha \sin \beta, \\ Q &= \frac{\gamma v^2}{4} \left[2a^2 - 3b^2 + (2a^2 - b^2) \cos(2\beta) - 2b^2 \cos(2\alpha) \sin^2(\beta) \right]. \end{aligned} \quad (5.5)$$

As discussed previously, these three quantities define radial and polar roots of the particle's trajectory, which can be obtained in closed form (see e.g. [147]). Once we know the roots, Eqs. (5.1)–(5.3) determine the initial and final inclination angles, as well as the scattering angle χ . The expression for the initial inclination angle is used to fix the initial phase q_z^0 such that $\theta_{\text{in}} = \beta$.

Plunging geodesics correspond to particles that get accreted by the BH, and therefore, the total momentum transfer due to plunging geodesics, δp_p^μ , by conservation of momentum, equals the initial momentum $m_p u_p^\mu$ of the plunging particles:

$$\frac{\delta p_p^\mu}{m_p} = u_p^\mu. \quad (5.6)$$

This implies that we need to distinguish between scattering and plunging geodesics. As discussed above, the separatrix is found when the second and third radial roots coincide $r_2 = r_3$. We solve numerically for this condition, building the separatrix

$b_{\text{crit}}(\alpha)$ in the impact plane for each configuration. Then, geodesics parametrized by some $\{b, \alpha\}$ such that $b \leq b_{\text{crit}}(\alpha)$ plunge, whereas in the opposite case, they correspond to scattering orbits.

In the scattering regime, the momentum transfer can be directly computed once we know the final inclination θ_{out} and the scattering angle χ . Indeed, conservation of momentum yields

$$\frac{\delta p_S^\mu}{m_p} = \gamma v \begin{pmatrix} 0 \\ \sin \theta_{\text{out}} \sin \chi \\ \sin \beta - \sin \theta_{\text{out}} \cos \chi \\ -\cos \beta - \cos \theta_{\text{out}} \end{pmatrix}, \quad (5.7)$$

where we remind the reader that we have chosen the initial phase such that $\theta_{\text{in}} = \beta$. We are not interested in the momentum transfer of each individual particle, but rather in the total momentum transfer due to the BH moving through a medium. We denote the total momentum transfer per unit mass by $\Delta \tilde{p}^\mu$. This is given by

$$\begin{aligned} m_p \Delta \tilde{p}^\mu &= \int_0^{2\pi} d\alpha \int_0^{b_{\text{crit}}(\alpha)} b db \delta p_P^\mu(b, \alpha) \\ &+ \int_0^{2\pi} d\alpha \int_{b_{\text{crit}}(\alpha)}^{b_{\text{max}}} b db \delta p_S^\mu(b, \alpha). \end{aligned} \quad (5.8)$$

In order to resolve accurately the second integral, the behavior of geodesics that end up scattering away, but whirl around the BH one or several times is crucial. In light of this we found it convenient to change the coordinates to

$$b \mapsto b_{\text{crit}}(\alpha) + [b_{\text{max}} - b_{\text{crit}}(\alpha)] e^{\tilde{b}}, \quad (5.9)$$

and then integrate numerically between $\tilde{b} \in [\tilde{b}_{\text{min}}, 0]$. We want to be able to resolve geodesics to sub-percent distance relative to the separatrix. Thus, we fix $\tilde{b}_{\text{min}} = -\log(10^{-\gamma_s} b_{\text{max}})$, where γ_s is the desired accuracy. For practical purposes we find that $\gamma_s = -3$ is enough to extract confidently the forces and achieve convergence. For convenience, we also change coordinates to resolve the plunge integral, defining $b = \tilde{b} b_{\text{crit}}(\alpha)$, so that the new variable is $\tilde{b} \in [0, 1]$. From the momentum flux, the forces, as defined in Fig. 5.2 can be computed directly as

$$F_D^{(P)} = \gamma v \rho M^2 (\Delta \tilde{p}^z \cos \beta - \Delta \tilde{p}^y \sin \beta), \quad (5.10)$$

$$F_M^{(P)} = \gamma v \rho M^2 \Delta \tilde{p}^x, \quad (5.11)$$

$$F_L^{(P)} = \gamma v \rho M^2 (\Delta \tilde{p}^y \cos \beta + \Delta \tilde{p}^z \sin \beta), \quad (5.12)$$

where the $\gamma v M^{-1}$ factor is the dilation factor of the co-moving volume that the BH spans in a unit time. Recovering the mass units from $\Delta \tilde{p}^\mu$ involves multiplying

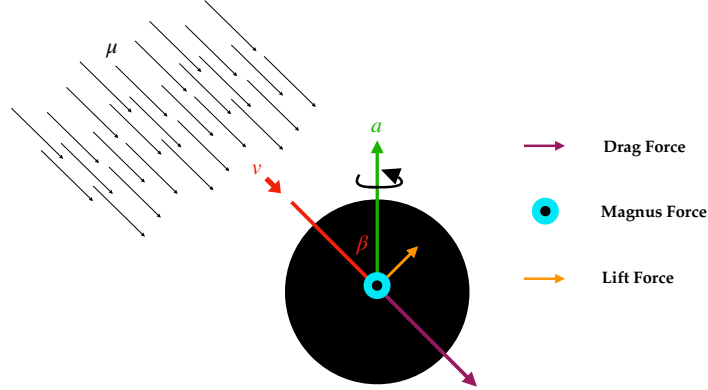


Figure 5.2: Schematic representation of the vector decomposition of the drag, Magnus, and lift forces relative to the velocity and spin directions. BH is viewed from side-on where in this frame the drag force is aligned with the velocity vector of the scalar field. The Magnus force in this representation is out of the plane and is given by the vector perpendicular to the plane spanned by the spin and velocity vectors (if the spin and velocity vectors are aligned then the Magnus force vanishes due to the symmetries of the system). The lift vector is in the direction perpendicular to the Magnus and drag vectors. Throughout this chapter, a positive sign for drag, Magnus and lift forces means they are aligned with the corresponding vectors in the figure.

everything by the mass scale of the individual environment particles $m_p = \rho M^3$, which combined with the above yields the overall dimensionless ρM^2 scaling factor. Our numerical results for forces will always be given in units of ρM^2 . Finally, $F_{D,M,L}$ denote the drag, Magnus and lift forces, respectively.

The drag force is the usual dynamical friction force, which is antiparallel to the motion of the BH. The novel forces that appear when considering spinning BHs moving at an angle with respect to their spin axis are the Magnus and the lift forces. The Magnus force is normal to the plane defined by the BH spin and velocity directions. The lift force, on the other hand, is orthogonal to both the BH velocity vector and the Magnus force. This is represented schematically in Fig. 5.2.

Already at this level we can observe that plunging geodesics, i.e., those associated to particles that fall into the BH, do *not* contribute to the Magnus or lift forces, resulting only in a net contribution to the drag due to accretion. Indeed, the force due to accretion is parallel to the BH velocity, therefore, its only contribution is in the usual dynamical friction [149].

5.2 WAVE-LIKE ENVIRONMENTS

We now wish to model the aerodynamic forces in a wave-like medium. We begin by considering a minimally coupled massive scalar field [150–152]

$$S = \int d^4x \sqrt{-g} \left(\frac{R}{8\pi} - \frac{1}{2} \nabla_\mu \Phi \nabla^\mu \Phi - \frac{1}{2} \mu^2 |\Phi|^2 \right), \quad (5.13)$$

where μ is the mass of the scalar field Φ , and ∇ denotes the covariant derivative. The equations of motion following from the action are

$$\begin{aligned} \square_g \Phi &= \mu^2 \Phi, \\ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} &= 8\pi T_{\mu\nu}, \end{aligned} \quad (5.14)$$

where $\square_g$ is the d'Alembert operator associated to the metric g , $R_{\mu\nu}$ is the Ricci tensor, and the stress–energy tensor of the scalar field is given by

$$T^{\mu\nu} = \nabla^{(\mu} \Phi \nabla^{\nu)} \Phi^* - \frac{g^{\mu\nu}}{2} (\nabla_\gamma \Phi \nabla^\gamma \Phi^* + \mu^2 |\Phi|^2). \quad (5.15)$$

In the regime where back reaction of the scalar field on the metric is subleading, we can take g_{ab} to be the Kerr metric. Then, substituting the solutions of Φ into Eq. (5.15) we can use the stress–energy tensor to obtain asymptotic quantities relating to forces.

We will approach this problem in the frequency domain, and we expand the scalar field in spheroidal harmonics as

$$\Phi = \sum_{\ell m} e^{-i\omega t} \phi_{\ell m}(r) S_{\ell m}(\theta, \varphi, \xi), \quad (5.16)$$

where $S_{\ell m}(\theta, \varphi, \xi)$ are spheroidal harmonics with angular numbers ℓm and spheroidicity $\xi = ik_\infty a$, with $J = aM$ being the angular momentum of the BH. Here we have defined the asymptotic wavenumber of the scalar field to be

$$k_\infty = \sqrt{\omega^2 - \mu^2}. \quad (5.17)$$

The radial equation governing $\phi_{\ell m}$ is [116]

$$\begin{aligned} \Delta \frac{d}{dr} \left[\Delta \frac{d\phi_{\ell m}}{dr} \right] + \left[\omega^2 (r^2 + a^2)^2 - 4aMm\omega r \right. \\ \left. + (ma)^2 - (\lambda_{\ell m} + \mu^2 (r^2 + a^2)) \Delta \right] \phi_{\ell m} = 0. \end{aligned} \quad (5.18)$$

Solutions of the previous equation that are regular asymptotically far away from the BH take the form

$$\phi_{\ell m} \sim \frac{I_{\ell m} e^{-ik_\infty r_*} + R_{\ell m} e^{ik_\infty r_*}}{\sqrt{r^2 + a^2}}, \quad (5.19)$$

with $r_* = r + \eta \log 2k_\infty r$, and $\eta \equiv M \left(\frac{\omega^2 + k_\infty^2}{k_\infty^2} \right)$. The asymptotic form of the field defines two independent solutions, $I_{\ell m}$ corresponding to in-going plane waves scattering towards the BH from past null infinity, $\mathcal{I}^-$, and $R_{\ell m}$ corresponding to out-going plane waves scattering off of the Kerr potential towards future null infinity, $\mathcal{I}^+$.

In order to solve Eq. (5.18) and calculate asymptotic forces on the Kerr BH, we first consider some fixed background scalar field given by

$$\Phi = e^{-i\mu t}. \quad (5.20)$$

This is a constant energy density solution to the Klein–Gordon equation in flat space, whose oscillation frequency is fixed by the mass of the field. It is also a asymptotic solution to (5.14) at large radii. By performing a Lorentz boost into the BH frame, the background field then takes the form

$$\Phi_{BC} = e^{-i\mu\gamma t} e^{-i\gamma\mu v r_*^{\text{out}} (\cos\beta \cos\theta + \sin\beta \sin\theta \sin\varphi)}, \quad (5.21)$$

where $r_*^{\text{out}} = r_*(r_{BC}^{\text{out}})$, and r_{BC}^{out} is fixed as the radius at which we evaluate the plane wave boundary condition. Here we have also defined v as the velocity of the BH, γ as the Lorentz factor and β as the inclination angle between the velocity vector and the spin–vector (which we fix to be the z -axis). The boost also redshifts the frequency of the field, fixing $\omega = \gamma\mu$, and $k_\infty = \gamma\mu v$.

We wish to set Eq. (5.21) as an outer boundary condition to the piece of the field incoming from $\mathcal{I}^-$ in Eq. (5.18). In doing so we must first decompose Eq. (5.21) in spheroidal harmonics. This is achieved through the following steps: we first rotate our coordinate system to a frame where $\beta = 0$. Next, we decompose the simplified form of Eq. (5.21) with $\beta = 0$ in spherical harmonics. After doing so we make use of the Wigner D-matrices to rotate back to the original frame with $\beta \neq 0$. Finally, we obtain the spheroidal modes by summing the spherical decomposition from Eq. (5.21) over spherical-spheroidal mixing coefficients. We will begin with the spatial boundary condition in the rotated frame

$$\tilde{\Phi}_{BC} = e^{-ik_\infty r_*^{\text{out}} \cos\theta}, \quad (5.22)$$

where the tilde denotes quantities defined in the rotated coordinate system. In this rotated system there is no φ dependence, meaning we will only obtain non-zero axi-symmetric modes, i.e., modes with $m = 0$.

Next, we project this expression onto spherical harmonics,

$$\begin{aligned} \tilde{W}_{\ell m} &= \int Y_{\ell m}(\theta, \varphi) \tilde{\Phi}_{BC} d\Omega \\ &= 2\pi \sqrt{\frac{2\ell+1}{4\pi}} \delta_{0m} \int_{-1}^1 \tilde{\Phi}_{BC} P_\ell(x) dx, \end{aligned} \quad (5.23)$$

with $x = \cos \theta$. Using Rodrigues' formula we obtain an expression that can be integrated by parts ℓ -times. All corresponding boundary terms vanish, leaving us with

$$\begin{aligned} \tilde{W}_{\ell m}(r) = & (ik_{\infty} r_{*}^{\text{out}})^{\ell} \frac{\sqrt{(2\ell+1)\pi}}{2^{\ell} \Gamma(\ell+1)} \delta_{m0} \\ & \times \int_{-1}^1 e^{-ik_{\infty} r_{*}^{\text{out}} x} [(x^2 - 1)^{\ell}] dx, \end{aligned} \quad (5.24)$$

which can be conveniently written in terms of Bessel functions as

$$\begin{aligned} \tilde{W}_{\ell m'} = & (-i)^{\ell} \pi \text{sgn}(k_{\infty} r_{*}^{\text{out}}) \\ & \times \sqrt{\frac{2(2\ell+1)}{k_{\infty} r_{*}^{\text{out}}}} J_{\ell+\frac{1}{2}}(k_{\infty} r_{*}^{\text{out}}) \delta_{m'0}. \end{aligned} \quad (5.25)$$

Having obtained the decomposition in a simplified rotated frame, we can then appeal to the Wigner-D matrices to transform the spherical harmonics between different angular frames, such that given $\tilde{W}_{\ell m'}$ in a certain frame, an equivalent $W_{\ell m}$ in a new rotated frame is given by

$$W_{\ell m} = \sum_{m'=-\ell}^{\ell} D_{mm'}^{\ell}(\alpha, \beta, \gamma) \tilde{W}_{\ell m'}, \quad (5.26)$$

where α , β and γ are the three Euler angles relating the two frames. In the context of our problem, we only require the $m' = 0$ matrix

$$D_{m0}^{\ell} = \sqrt{\frac{(\ell-m)!}{(\ell+m)!}} P_{\ell m}(\cos \beta) e^{-im\alpha}. \quad (5.27)$$

The Euler angle α corresponds to rotations along the direction of the spin of the BH, which is an axis of symmetry in our problem, thus we set $\alpha = 0$ without loss of generality, yielding

$$\begin{aligned} W_{\ell m} = & (-i)^{\ell+m} \pi \text{Sign}(k_{\infty} r_{*}^{\text{out}}) P_{\ell m}(\cos \beta) \\ & \times \sqrt{\frac{2(2\ell+1)(\ell-m)!}{k_{\infty} r_{*}^{\text{out}}(\ell+m)!}} J_{\ell+\frac{1}{2}}(k_{\infty} r_{*}^{\text{out}}). \end{aligned} \quad (5.28)$$

Here the m -mode dependence has now been reintroduced to our problem through the Wigner matrices and the full decomposition of the plane wave Eq. (5.21) is now given by

$$\Phi_{BC} = \sum_{\ell, m} W_{\ell m} Y_{\ell m}. \quad (5.29)$$

We want to isolate the in-going piece of the decomposition and set this as the boundary condition for our field. This can be achieved by separating the Bessel J function into constituent Hankel functions,

$$J_{\ell+\frac{1}{2}}(k_{\infty} r_{*}^{\text{out}}) = \frac{H_{\ell+\frac{1}{2}}^{(1)}(k_{\infty} r_{*}^{\text{out}}) + H_{\ell+\frac{1}{2}}^{(2)}(k_{\infty} r_{*}^{\text{out}})}{2}. \quad (5.30)$$

Using this we can isolate $W_{\ell m}$ into in-going and out-going pieces respectively as

$$W_{\ell m} = I_{\ell m}^{(p)} + R_{\ell m}^{(p)} \quad (5.31)$$

with

$$\begin{aligned} I_{\ell m}^{(p)} &= (-i)^{\ell+m} \pi \text{Sign}(k_{\infty} r_{*}^{\text{out}}) P_{\ell m}(\cos \beta) \\ &\quad \times \sqrt{\frac{(2\ell+1)(\ell-m)!}{2k_{\infty} r_{*}^{\text{out}}(\ell+m)!}} H_{\ell+\frac{1}{2}}^{(2)}(k_{\infty} r_{*}^{\text{out}}), \\ R_{\ell m}^{(p)} &= (-i)^{\ell+m} \pi \text{Sign}(k_{\infty} r_{*}^{\text{out}}) P_{\ell m}(\cos \beta) \\ &\quad \times \sqrt{\frac{(2\ell+1)(\ell-m)!}{2k_{\infty} r_{*}^{\text{out}}(\ell+m)!}} H_{\ell+\frac{1}{2}}^{(1)}(k_{\infty} r_{*}^{\text{out}}). \end{aligned} \quad (5.32)$$

Notably, from these expressions it can be shown that $\lim_{r \rightarrow \infty} I_{\ell m}^{(p)}$ is equivalent to Eq. (17) of [42].

The Klein–Gordon equation on a Kerr background does not separate in spherical harmonics, but in spheroidal harmonics. We can transform between both bases by using spherical–spheroidal mixing coefficients [153], which for a fixed spheroidicity ξ (as is the case here) we denote by $c_{\alpha \ell m}(\xi)$. Then, a spheroidal harmonic is decomposed into spherical harmonics as

$$S_{\ell m}(\theta, \varphi, \xi) = \sum_{\alpha \geq |m|} c_{\alpha \ell m}(\xi) Y_{\alpha m}(\theta, \varphi). \quad (5.33)$$

From this, we obtain the coefficients of the in-going plane wave component in terms of spheroidal harmonics as

$$J_{\ell m}^{(p)} = \sum_{\ell' \geq |m|} c_{\ell' \ell m}(ik_{\infty} a) I_{\ell' m}^{(p)}. \quad (5.34)$$

We begin implementing the boundary conditions by setting the purely in-going condition on the horizon. Firstly one should note that the radial Klein–Gordon equation (5.18) can be mapped to the confluent Heun equation [154–156]. Thus, solutions that are regular at the horizon, and in-going, can be written in terms of confluent Heun functions. We use this to set the inner boundary conditions at a finite radius, away from the horizon, $r_{\text{BC}}^{\text{in}} \approx 5M$ ³. Next, we integrate numerically the equation outwards up to large radii for each spheroidal mode. At large radii we may then assume the solution takes the form of Eq. (5.19). From the numerical solution, we can then extract the coefficients of the piece in-coming from past null infinity, given by $I_{\ell m}$.

³ Integrating numerically very close to the BH horizon introduces numerical instabilities. However, evaluating the special functions at very large distances is computationally very expensive. We found a fair trade-off between these two issues by setting the boundary conditions at $r_{\text{BC}}^{\text{in}} \approx 5M$.

We now want to fix that asymptotically the coefficient of the in-coming piece from past null infinity corresponds to the in-going component of a plane wave, given by $J_{\ell m}^{(p)}$ (5.34). In order to achieve this, it is sufficient to re-scale the solution by a factor $J_{\ell m}^{(p)} \sqrt{(r_{\text{BC}}^{\text{out}})^2 + a^2} / I_{\ell m}$, in virtue of the linearity of the problem. This then fixes the solution of the scalar field in the full spacetime.

Individually, Eqs. (5.32) are divergent subsequences of Eq. (5.29). However, one can see that by fixing $r_{\text{BC}}^{\text{out}}$ (choosing a cloud size) and analysing the asymptotic structure of the numerical solution, the $\phi_{\ell m}$ modes with $\ell > k_{\infty} r_{*}^{\text{out}}$ are exponentially suppressed, yielding a converging sum⁴. The field solution we obtain represents a plane wave scattered by the BH potential within some ball of radius r_{*}^{out} (outside this ball the field sharply decays). Having a setup that converges in ℓ such as this allows for a direct method of calculating finite drag forces without the need for any cutoff schemes like those employed in previous works [42, 46, 126, 157, 158].

The goal here, however, is to calculate spin-induced dynamical friction effects, which we expect to saturate and not depend on cloud size. Hence in practice, we take $r_{\text{BC}}^{\text{out}} = 10^7$, recovering the same boundary conditions as [42], and then sum to an appropriate number of ℓ -modes such that the forces have converged. An analysis of the convergence in ℓ of the forces can be found in Appendix A.1.

Having obtained the solution for the scalar field, we can now calculate momentum transfers between the field and the BH. Due to asymptotic flatness, we are provided with three translational Killing vectors ($\tilde{\zeta}_{\mu}^i = \delta_{\mu}^i$), from which we can define a rate of change of linear momentum,

$$F^i = \frac{dP^i}{dt} = - \lim_{r \rightarrow \infty} \int_{S_r} T^{ri} r^2 d\Omega, \quad (5.35)$$

where it is important to note that, in the large r limit,

$$\lim_{r \rightarrow \infty} r^2 T_{ri} \rightarrow \hat{x}^i r T_{rr}, \quad (5.36)$$

given $\hat{x}^i$ as the triad of cartesian unit vectors. The expressions $\hat{x}^i$ can be found analytically in terms of spherical harmonics, meaning the integral over the sphere becomes

$$\begin{aligned} F^x &= \sqrt{\frac{4\pi}{6}} \lim_{r \rightarrow \infty} \int_{S_r} (Y_{1-1} - Y_{11}) T_{rr} r^2 d\Omega, \\ F^y &= -\sqrt{\frac{4\pi}{6}} i \lim_{r \rightarrow \infty} \int_{S_r} (Y_{1-1} + Y_{11}) T_{rr} r^2 d\Omega, \\ F^z &= \sqrt{\frac{4\pi}{3}} \lim_{r \rightarrow \infty} \int_{S^2} Y_{10} T_{rr} r^2 d\Omega. \end{aligned} \quad (5.37)$$

⁴ This can be seen explicitly as in the $M \rightarrow 0$ limit, setting $I_{\ell m}^{(p)}$ as the in-going piece one recovers $W_{\ell m}$ as the full solution. Through the Bessel-J function, $W_{\ell m}$ has precisely this exponential convergence property.

At large radii we are now only required to calculate the (rr) component of the stress-energy tensor,

$$T_{rr} = \sum_{\ell, m} \sum_{\ell', m'} t_{\ell m}^{\ell' m'} S_{\ell m} S_{\ell' m'}^*, \quad (5.38)$$

$$t_{\ell m}^{\ell' m'} = \frac{1}{2} (\partial_r \phi_{\ell m} \partial_r \phi_{\ell' m'}^* + k_\infty^2 \phi_{\ell m} \phi_{\ell' m'}^*). \quad (5.39)$$

Expanding the spheroidal terms in Eq. (5.38) over spherical harmonics, one can then appeal to the standard expressions for the Wigner 3-j symbols [159] to compute the integrals in Eq. (5.37) analytically. Finally, we bring these integrals (including the sums over mixing coefficients) together with the numerical interpolants for $\phi_{\ell m}(r)$ to obtain radially oscillating functions of the forces. Analysing the asymptotic form of Eq. (5.19) it can be seen that, to leading order, the forces behave as

$$\lim_{r \rightarrow \infty} F = A + B e^{-2ik_\infty r_*} + C e^{2ik_\infty r_*} + \mathcal{O}\left(\frac{1}{r}\right), \quad (5.40)$$

where A, B , and C are coefficients to be determined. Including higher order terms in $\mathcal{O}(1/r)$ we perform a simple linear regression to extract the A coefficients. These are then the values of $\{F^x, F^y, F^z\}$ for a given point in $\{\mu, a, v, \beta\}$ parameter space. Finally, we apply a rotation to the cartesian force vectors to obtain the drag, Magnus and lift forces,

$$F_D^{(W)} = F^z \cos \beta - F^y \sin \beta, \quad (5.41)$$

$$F_M^{(W)} = F^x, \quad (5.42)$$

$$F_L^{(W)} = F^y \cos \beta + F^z \sin \beta. \quad (5.43)$$

5.3 RESULTS

We have evaluated the aerodynamic forces (5.10)–(5.12) and (5.41)–(5.43) semi-analytically as described above in a grid of Chebyshev points. The grid then covers the parameter space given by the velocity v , the dimensionless spin a/M , and the incidence angle β . From these points, we build a Chebyshev interpolant so that we can explore parameter space efficiently. Convergence of our results with the size of the cloud and details of the implementation of the interpolant (including the ranges in parameter space used to build them) are given in Appendix A.1. The data used to build the interpolants can be found in [59].

Our ultimate goal is to characterize the spin effects on dynamical friction. However, it is important to first check that our semi-analytical set-ups are consistent with previous results found for BHs with no spin, but in the fully relativistic regime [40, 42, 46]. For a particle-like medium, analytical formulas for the dynamical friction force (for Schwarzschild BHs) that allow for a direct comparison

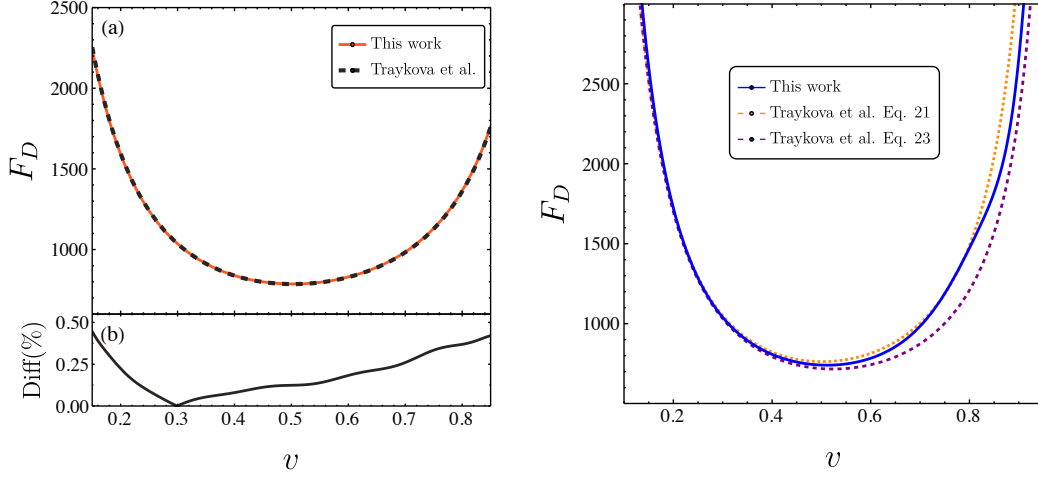


Figure 5.3: **Left:** Drag force (5.10) for particle-like environments, at zero spin and almost head-on $\beta = 0.1$ compared with the analytical predictions of Eq. (23) of [46] (top), and the percent-level relative difference between our method and [46], which is always sub-percent (bottom). **Right:** Drag force (5.41) for a wave-like medium, summed to $\ell_{\max} = 15$ with $\mu = 0.2$, $\beta = 0.001$ and $a = 0.0001M$. Eqs. (21) and (23) of [46] are evaluated with $R = 4000$; the particle and wave approximations bound our drag values over the majority of the domain.

with our results were given in Eq.(23) of [46]. In Fig. 5.3 we show that the relative difference between the drag force (5.10) that we obtain for non-rotating BHs and the analytical calculation is below 1%. The good accuracy of the comparison serves as a benchmark of our semi-analytical procedure discussed above.

For a wave-like, scalar-field environment, when the wavelength of the field is comparable to the size of the BH $\gamma\mu M \sim \mathcal{O}(1)$, the analytical work carried out for Schwarzschild BHs in Ref. [42, 46] is only an approximation. In particular, Ref. [46] found that the best match to numerical results was obtained by combining the scattering forces from the limit in which the scalar field is very light, $\gamma\mu M \ll 1$, with the accretion effects in the limit of particles (or very heavy scalars), $\gamma\mu M \gg 1$. In Fig. 5.3 we observe that our results for the drag force (5.41) (the blue line) are compatible and bracketed by the different approximations discussed in Ref. [46]. In particular, for this comparison we have set $\mu = 0.2$ (which is the mass that we cover parameter space with), giving $\gamma\mu \in (0.2, 0.45)$ on the range $v \in (0.1, 0.9)$. These values of $\gamma\mu$ lie outside the validity of the approximations of [42, 46]. In particular one should note the non-uniform behavior for our results near $v = 0.7$ in Fig. 5.3. For this choice of μ it is approximately at this velocity that the wavelength of the field in the BH frame becomes comparable to the length scale of the innermost stable circular orbit.

Finally, it is informative to observe the scalar field density on an equatorial slice, when the BH is moving in a direction orthogonal to its spin, as shown in Fig. 5.4.

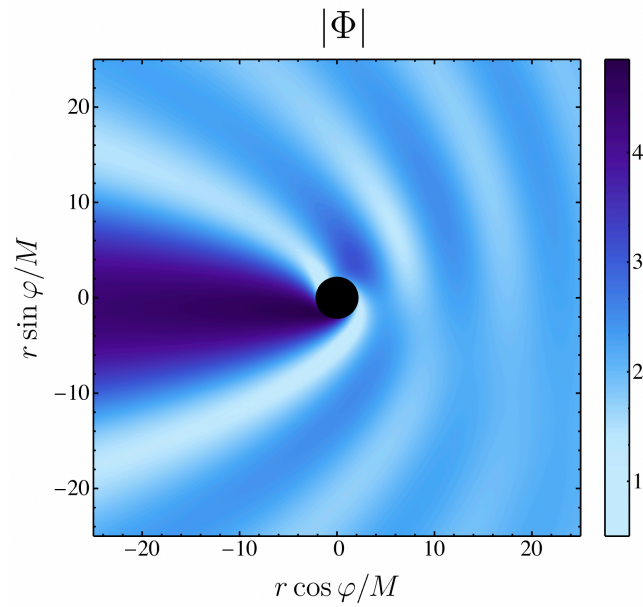


Figure 5.4: Equatorial slice of the magnitude of the scalar field $|\Phi|$, where $\mu M = 0.2$, over a region close to the BH. Here (r, φ) denote Boyer–Lindquist coordinates. The BH moves towards the right with $v = 0.8$, and its spin is along the axis pointing out of the plane, in an anti-clockwise direction, with magnitude $a = 0.9M$. We observe a wake of higher field density forming behind the BH, and clear wave-like patterns. Moreover one can notice that due to the spin of the BH, reflection along the $\varphi = \{0, \pi\}$ axis is no longer a symmetry of the problem.

For Schwarzschild BHs, we know that the field develops a cylindrically symmetric wake or an overdensity behind, that ultimately causes the drag force. Here we show for a rapidly spinning BH that a wake develops, and moreover that the field structure is asymmetric (c.f. Fig.1 in [42]). While the wake behind the BH gives a visual interpretation for the origin of the drag force, this asymmetry is directly responsible for the Magnus force.

5.3.1 Exploration of parameter space

Having compared our methods with previous results we now focus our attention on understanding the large parameter space covered by the Chebyshev interpolants. Our results are summarized in Figs. 5.5–5.7. It is important to note that although we provide a direct comparison of results between the particle and wave-like mediums there is no a priori reason to expect these results to coincide with one another. The limiting case in which we would expect agreement to occur is in the geometric optics regime as $\mu \rightarrow \infty$. This regime is not accessible to us with our semi-analytical approach due to slow convergence in the ℓ -mode sum as k_∞ becomes large.

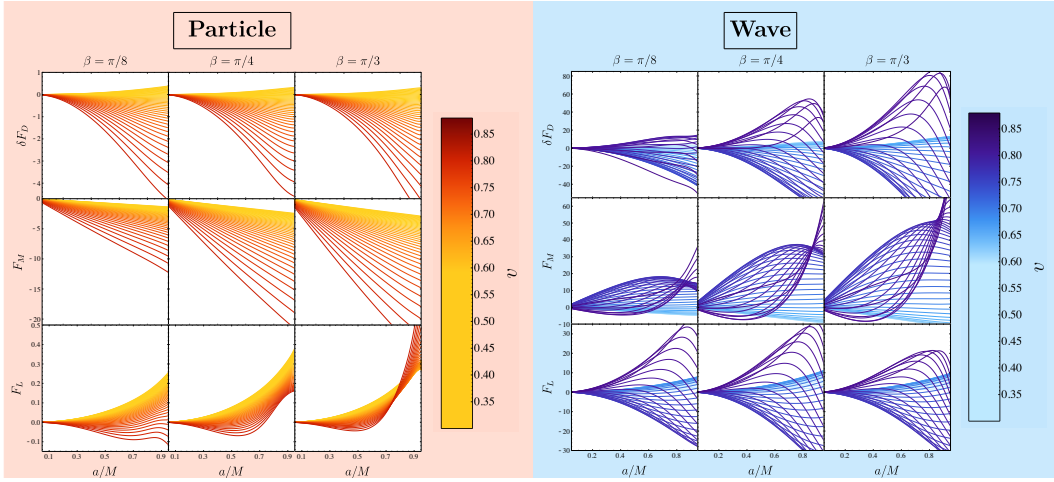


Figure 5.5: **Left:** The drag, Magnus and lift forces for different BH velocities and spin, in particle environments. Results are scaled by ρM^2 . We evaluate the forces at $\beta = \pi/8, \pi/4$, and $\pi/3$. In the low-velocity regime, the Magnus force is negative and there is a clear negative linear scaling with spin for all values of β tested. On the other hand, the drag and lift forces at small velocities are all positive and increase with spin. **Right:** Spin scaling for the variation in drag, Magnus, and lift forces for the wave setup. Here all quantities are evaluated equivalently to the particle case. Notice how in the low-velocity regime the spin scalings are the same as for the particle setup.

Secondly, we highlight that the overall variations of the drag with the spin and incidence angle are relatively small, and in particular, they are comparable to the other aerodynamic forces. For this reason, we decide to study carefully the variation in drag δF_D , defined as

$$\delta F_D(a, v, \beta) = F_D(a, v, \beta) - F_D(0, v, \beta). \quad (5.44)$$

As shown in Appendix A.1, this quantity does *not* depend on the cloud size, making it a more suitable quantity to track than the drag force itself, which grows logarithmically with the size of the cloud.

5.3.1.1 Dependence on spin

Figure 5.5 shows the dependence of the aerodynamic forces on the spin. The panel on the left (right) refers to a particle (scalar-field) environment. We focus on a scalar field with mass $\mu M = 0.2$, such that as the velocity increases we can study the transition from wave optics to the regime where the scalar field begins to probe the strong-field substructure of the spacetime.

Due to the logarithmic divergence in the drag force, the variation on drag is a small effect. In particular, the magnitude of the drag force for parameters of Fig. 5.5 is 3 orders of magnitude larger than δF_D . Thus the plot shows the small variations of the drag with the BH spin, with an approximately quadratic dependence, at low spins. We find that the Magnus force for particle-like environments is always negative. It is an *anti-Magnus* force, in reality, consistent with a post-Newtonian analysis in Ref. [149]. Our results disagree with the conclusions of Ref. [160] regarding the sign of the force, but we note a word of caution in a footnote of Ref. [160], suggesting that force estimates from fluxes may yield incorrect values. Evidently from the figure, the Magnus force grows linearly with the spin of the BH for a particle-like medium. This is also the case for a wave-like medium, but only for low velocities. Note also that for scalar-field environments, the Magnus force can be positive even at low spins. Since the Magnus force is ultimately a consequence of frame dragging, and the horizon frequency is linear in the BH spin, it is natural to expect that the Magnus force, at least in the low spin, low velocities regime, becomes linear in a/M .

Both the lift and the variation in the drag force, on the other hand, exhibit a quadratic scaling with the BH spin at low spins and low velocities. As can be seen from the construction of the aerodynamic forces in coordinates (5.37), the lift will “compete” with the drag force. Therefore we observe that they share some relevant features, such as the spin scaling. At large spins and velocity, the drag force can be enhanced or suppressed (relative to the non-spinning case).

Focusing on the high-velocity scenarios (represented by lines with a darker color), we observe that the same generic features as in the low velocity case persist for a particle-like medium, albeit with corrections that become more important for

higher spins (see e.g. the change of sign in F_L). On the other hand, for a wave-like medium, once the velocity becomes $v \sim 0.7$ the forces begin to oscillate, introducing new features that are not present in the weak field. These are a consequence of the wave-like nature of the scalar field, which at $v = 0.7$ has a wavelength $\lambda_C \sim 10M$, comparable to the length scale of the ISCO. We leave a detailed characterization of the strong field effects for future work.

5.3.1.2 Dependence on incidence angle

Fig. 5.5 also shows that the aerodynamic forces have an interesting structure depending on the incidence angle β . To study this further, we show the overall angular structure of the forces in Fig. 5.6. In order to better understand the structure of the β dependence we perform a re-scaling of the forces of the form,

$$\hat{F}_i(\beta) = \frac{F(a_i, v_0, \beta)}{\max_{\beta} F(a_{\max}, v_0, \beta)} + \frac{i}{c_0}, \quad (5.45)$$

where $v_0 \in \{0.325, 0.925\}$, i indexes the individual lines of different spin values in the subplots, and c_0 is some constant chosen to provide a clear separation between beta scalings for different spin values in each particular subplot. This arbitrary shift and scaling of the forces in a given subplot means the magnitude of individual lines on a single plot may be compared, however, we do not focus on the absolute values between different subplots, but instead on the angular pattern. It is evident that at low velocities the Magnus force has a clear $F_M \sim \sin \beta$ behavior, whereas $F_L \sim \sin(2\beta)$. The variation in the drag has a more complex structure. Worth remarking however is that the overall value in δF_D is much smaller at low velocities than it is at high velocities.

We find that the angular structure changes significantly at high spins and at high velocities: the lift force, in particular, develops an additional set of maxima and minima between which new roots arise. Interestingly, we have found these roots asymptote to $\pi/4$ and $3\pi/4$ in the limit $a, v \rightarrow 1$ (which is the point where the lift forces are maximised in the low-velocity limit). For the case where the wavelength of the medium becomes comparable with the size of the BH, the angular structure of the Magnus force and δF_D also change. Thus, the angle at which each of the aerodynamic forces is maximized depends sensitively on the BH velocity and spin.

5.3.1.3 Dependence on velocity

Finally, we characterize the dependence of the aerodynamic forces on the velocity in Fig. 5.7, where the forces are shown at an angle $\pi/3$. A cursory inspection of this figure confirms the qualitative agreement between the particle and wave setups. We observe the same characteristic sign change in the drag variation early in the velocity scaling. One also finds at low velocities a lift force that is positive for both

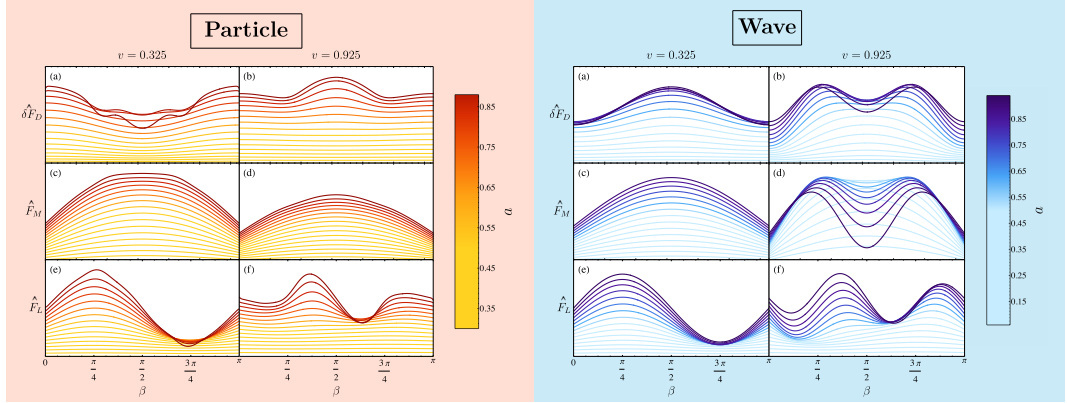


Figure 5.6: **Left:** dependence of forces for particle-like environments with incidence angle β for two different BH velocities, $v = 0.325$ and 0.925 . In the regions of parameter space tested, the Magnus force is always maximised when $\beta = \pi/2$ (i.e. when the spin and velocity vectors are orthogonal). Note also the development of additional maxima and minima that arise in the lift force. **Right:** same, for scalar-field environments, here with $\mu M = 0.2$. Features are similar to those of particle environments, but now the Magnus force also develops additional maxima and minima in the high velocity/spin regime.

setups. Additionally, at low velocities, we obtain an anti-Magnus force driving the BH in the opposite direction one would expect from the intuition provided by classical aerodynamics.

As velocity increases we also observe an order of magnitude increase in the forces for both the particle and wave mediums (except for the particle lift force). This can be thought of as a consequence of the BH interacting with an increasingly dense environment due to length contraction. In the wave set-up we find that as the velocity passes through $v \simeq 0.7$ there is a change in the sign of the forces. In simulating these scalings to higher velocities than is shown here ($v = 0.95$), it can be seen that this change in sign is characteristic of novel oscillatory effects that arise in the high-velocity limit, as the system transitions from the wave optics to geometric optics regimes. An in-depth analysis of this transition and the geometric optics limit of the scalar field is left as a task for future work.

5.3.2 Polynomial fits

For convenience, we provide polynomial fits to the aerodynamic forces in the weak field regime, which describe well our numerical results. The procedure developed here follows closely the construction of hyperfits for a ringdown model [161]. We provide fits for the range $a/M \in [0.1, 0.6]$ and $v \in [0.15, 0.5]$ (for a particle-like medium), and $a/M \in [0.04, 0.6]$, $v \in [0.05, 0.5]$ (for a wave-like medium with $\mu M = 0.2$). In order to do so, we first extract the expected scaling of the

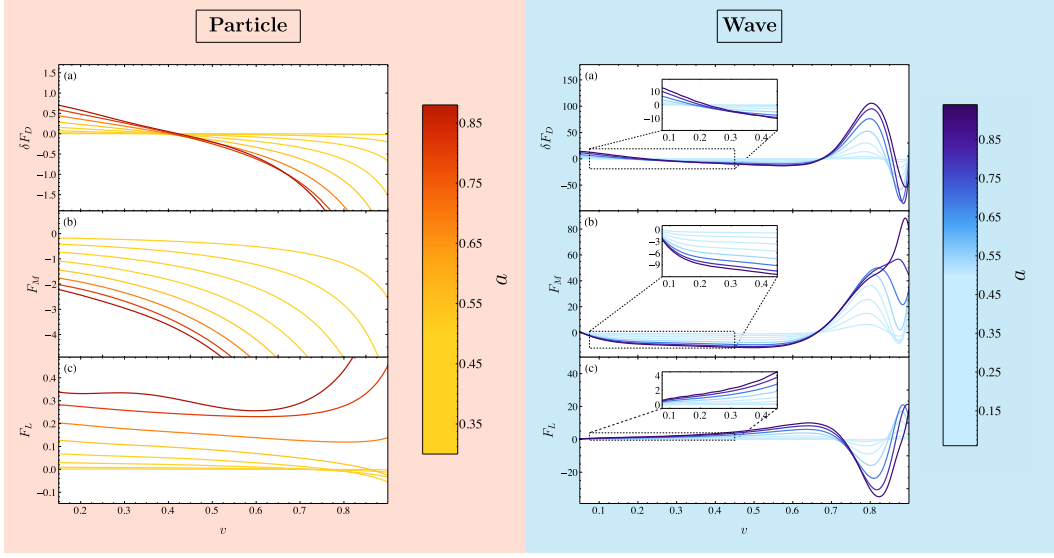


Figure 5.7: **Left:** Velocity dependence of the spin aerodynamic forces in the particle setup. Here we show the velocity scaling for a family of differing spin values (increasing in spin with darkness) evaluated at $\beta = \pi/3$. We find a transition from positive to negative variation in the drag at relatively low velocities with negative Magnus and positive lift forces. **Right:** Velocity scaling of the spin aerodynamic forces for the wave setup. Here all quantities are evaluated equivalently to the particle case, the insets run on the range $v \in (0.05, 0.4)$ and show explicitly in the low-velocity limit that the particle and wave cases have the same qualitative behavior. Returning to the full plot on the range $v \in (0.05, 0.9)$ one again sees a transition in the sign and an order of magnitude increase in the forces as the wavelength moves to scales relevant for probing the strong field structure.

forces with spin and angles, e.g., we define $\tilde{F}_M = F_M/(a \sin \beta)$, and similarly $\tilde{F}_L = F_L/(a^2 \sin(2\beta))$ and $\delta\tilde{F}_D = \delta F_D/a^2$. We perform a linear regression fit on the rescaled forces for a polynomial of degree N in the variables $\{a, \sqrt{v}, \cos \beta\}$. We fix the degree of the polynomial once the goodness of fit measure achieves a certain minimum value. We use as a goodness of fit the adjusted R squared measure, $\bar{\mathbf{R}}^2$, defined as

$$\bar{\mathbf{R}}^2 = 1 - \frac{n-1}{n-p} \mathbf{R}^2, \quad (5.46)$$

where n is the total number of points used for the fit (1024 for a particle medium, and 4000 for a wave medium), p is the number of parameters in the model, and $\mathbf{R}^2$ is the usual linear regression coefficient. We compute this for polynomials of different degree, and stop at the degree N for which $\bar{\mathbf{R}}^2 \geq 0.999$ ⁵.

⁵ Extracting the structure in the variations of the drag force proves to be more challenging, so we relax the accuracy requirement to $\bar{\mathbf{R}}^2 \geq 0.99$.

Finally, since a polynomial of, e.g. degree $N = 3$ in 3 variables contains $p = 20$ terms, we would like to reduce the number of terms included in the polynomial. To do this, we rank the contribution of each of the terms by computing the mismatch of the model once we do not include that given term. Then, we add one by one each term, in the order in which they were ranked, until the previous tolerance limit is achieved.

The fits obtained are given by⁶:

$$\begin{aligned}
 \delta F_D^{(P)} &= a^2 \left(1.11 - 3.26v + 0.74a - 1.07\sqrt{v}a - 0.59 \cos \beta + 1.38\sqrt{v} \cos \beta \right), \\
 F_M^{(P)} &= a \sin \beta \left(2.55 - 31.81\sqrt{v} + 69.81v - 59.15v^{3/2} - 0.12a\sqrt{v} \cos \beta \right), \\
 F_L^{(P)} &= a^2 \sin 2\beta \left(0.82 - 2.60\sqrt{v} + 5.06v - 3.79v^{3/2} \right. \\
 &\quad \left. + 0.28a + 1.28va - 0.26\sqrt{v}a \cos \beta \right. \\
 &\quad \left. + 0.11a \cos \beta - 1.22\sqrt{v}a - 0.02 \cos \beta + 0.04\sqrt{v} \cos \beta + 0.23\sqrt{v}a^2 \right), \quad (5.47) \\
 \delta F_D^{(W)} &= a^2 \left(-16.45 - 30.84\sqrt{v} + 316.40v - 296.10v^{3/2} \right. \\
 &\quad \left. + 5.44 \cos^2 \beta - 23.92\sqrt{v} \cos^2 \beta + 12.25\sqrt{v}a^2 - 9.16a^2 \right), \\
 F_M^{(W)} &= a \sin \beta \left(36.45 - 241.9\sqrt{v} + 415.8v - 248.6v^{3/2} \right), \\
 F_L^{(W)} &= a^2 \sin 2\beta \left(-1.023463 + 58.2938v - 165.9078v^{3/2} + 150.6706v^2 \right).
 \end{aligned}$$

For the Magnus force in particle environments, we can compare to the post-Newtonian based calculations carried out in Ref. [149]. The first correction, in the zero-velocity limit, in that case, would be ~ -2.5 . Evaluating our fit at the lowest velocity available $v = 0.15$, and low spin, returns $F_M \sim -2.73$, in close agreement with the results of [149] (we remind the reader that low velocities $v < 0.15$ were not available to construct the fit, and hence evaluating the fit provided above at exactly zero velocity would result in large extrapolation errors).

5.4 DISCUSSION

In previous sections we studied and characterized the way in which the dynamics of BHs moving in a nontrivial environment is modified due to BH spin. We can identify a key scenario where such aerodynamic forces might leave an aerodynamic imprint, Extreme mass ratio inspirals (EMRIs) which consist of binary systems formed by a

⁶ Here we reiterate that a positive δF_D refers to a variation that enhances the drag force causing the BH to slow down faster. Algebraic signs follow conventions in Fig. 5.2. A positive F_M is one in which the force is acting as in the standard classical fluid mechanics. A positive F_L force is one which is pushing the BH upwards along the positive z-axis in the coordinate we define in sec. 5.1.

solar mass BH and a supermassive BH. These are a promising source of gravitational waves in the LISA band [32], where they would emit long-lasting signals that can be modelled very accurately. The presence of a medium surrounding the central supermassive BH would change the orbital dynamics of the companion, resulting in a modified gravitational wave emission. We consider these separately.

5.4.1 EMRI aerodynamics

EMRIs are a very interesting prospect to learn about the geometry and the astrophysical environment very close to supermassive BHs [37]. The smaller object, typically with a mass a million times smaller than its companion, can orbit for years at distances smaller than 10 Schwarzschild radii of the central BH. The prospect of observing gravitational waves emitted from these systems with the space interferometer LISA motivates studying the impact of possible environments (such as gas, accretion disks or dark matter) on their emission patterns [162, 163].

In addition to gravitational wave emission, if there is matter in the vicinity of the central BH, it will create further dissipative channels that will affect the companion's orbital motion, whereby changing the trajectory would also modify the gravitational wave emission. Several studies in recent years have discussed that this is a potential opportunity to learn about said environments from the GW observations [101, 102, 164–169]. Here we consider whether the coupling between the spin of the secondary BH and the environment (through variations in the drag force, or the Magnus and lift forces, for instance), could have an impact.

The spin-dependent aerodynamic forces that we have discussed scale roughly as

$$F_X \equiv \mathcal{C} q^2 \frac{a}{M} f(v, a, \beta), \quad \mathcal{C} = \rho M_{\text{SMBH}}^2, \quad (5.48)$$

where $q = M/M_{\text{SMBH}}$ is the ratio between the mass of the secondary, M , and of the primary, M_{SMBH} , $f(v, a, \beta)$ is a dimensionless quantity of roughly order 1 (see the leading term in Eq. (5.47)), and $\mathcal{C}$ is the (also dimensionless) number that controls the overall strength of the interaction.

The spin-dependent aerodynamic forces on the smaller body therefore appear at the same formal order as the effective Mathisson–Papapetrou–Dixon (MPD) force [170–172] due to the secondary spin coupling to the background geometry. However, unlike the MPD force the spin-dependent aerodynamic forces can act dissipatively on the binary, secularly changing the constants of motion and appearing at leading order in the post-adiabatic expansion of the EMRI dynamics in terms of the mass ratio. Moreover, the dimensionless quantity that controls their effect is not necessarily very small ($\mathcal{C} \sim 10^{-6}$ for environments with the density of water, and $M_{\text{SMBH}} = 10^6$). This raises the prospect of the aerodynamic forces being the dominant effect due to the spin of the smaller object, potentially improving its detectability. A full assessment of the impact of the coupling between the spin

and the environment in EMRIs will, however, require an in-depth treatment of more realistic scenarios (including e.g. understanding the aerodynamic forces in generic orbits, and not in rectilinear motion). Furthermore, in any scenario where the spin-induced aerodynamic forces are significant, the regular component of the dynamical friction will be even bigger. A firm handle on the modelling uncertainties in the environment will therefore be paramount for any robust inference.

5.5 CONCLUSIONS

In this chapter, we have studied the impact of BH spin on dynamical friction in the fully relativistic regime. When the BH spin does not point along its direction of motion, cylindrical symmetry of the problem is broken, and two additional forces arise. These we dub Magnus and lift forces. Therefore the motion of a spinning BH through a medium will generically be curved out of its original plane of motion, as it decelerates. Besides the spin-induced aerodynamic forces, one would also expect the interaction of the spin with the environment to induce an effective torque on the spin itself [149]. We have not studied this effect in this chapter. Nonetheless, the introduced methods are amenable to such an analysis.

We have considered two different kinds of environments, representing very distinct physical regimes: on the one hand, a medium composed of collisionless massive particles, and on the other hand, an ultralight scalar field, in the regime where wave and strong field effects are most important. In the first case, we provide a semi-analytical calculation of these forces using the properties of scattering timelike geodesics on a Kerr background. In the second case, we have implemented analytically the boundary conditions corresponding to a plane wave scattering off a Kerr BH at an angle, and solved the relevant equations numerically. Our results agree with previous analytical (and also numerical) calculations in the fully relativistic regime in the case where the BH spin is aligned with its velocity [42, 46].

We have evaluated the forces on a grid of points in parameter space, which we share publicly in [59] and use to build rapid Chebyshev interpolants. From these we have learned the general features of the aerodynamic forces, including their spin scaling, angular structure, and dependence on the velocity of the BH. We also provide polynomial fits that are accurate in the regimes where the spin and the velocity of the BH are small or intermediate. Exploring in full detail the behaviour of the aerodynamic forces in the strong field regime, e.g., for very rapidly rotating BHs, is left for future study.

Finally, we have discussed one of the key astrophysical systems in which the spin dependence of the dynamical friction could be of relevance. Where we argue that for an EMRI system the aerodynamic forces could be the dominant consequence of the spin of the smaller object. Further exploration of the spin-dependent aerodynamic

forces in generic orbits, as well as their impact on gravitational wave emission, will be necessary to understand the full implications for EMRI observations.

In the process of carrying out our work on the spin aerodynamic forces, we became aware of a concurrent endeavour to study the gravitational Magnus effect from the perspective of Numerical Relativity [47]. This work makes use of the GRDzhadzha code-base [173–175] to calculate the Magnus force up to high spin and intermediate relativistic velocities (with the spin vector fixed to be perpendicular to the velocity). We see promising agreement between our results but leave a full comparative analysis as a task for future work.

Part II

PARTICLES AND MATTER

Having detailed the background theory surrounding the Kerr geometry, its perturbations, and associated stationary matter field configurations, and having explored our first proxy for the motion of black holes through matter, we now turn to the central goal of this thesis: constructing a self-consistent model for an asymmetric binary black hole system in the presence of matter in General Relativity. In *Part ii* we take that step, providing an overview of the means by which small bodies are represented as point particles in curved spacetime through the method of *matched asymptotic expansions* [176, 177]. The key goal here is to re-express the local properties of spacetime in terms of a collection of effective charges, equivalently expressing the local spacetime in terms of its multipolar decomposition.

In order to properly describe small bodies in vacuum we begin by seeking two perturbative solutions to the Einstein equations, one that is valid close to the secondary and another that is valid on the larger length scales associated with the primary. We match the two descriptions in an intermediate *buffer region*, where the two expansions overlap as seen in Fig. 6.1. We expand solutions both near and far away from the small body in terms of the small mass ratio ϵ :

$$\text{Outer expansion: } g_{\mu\nu} = g_{\mu\nu}^{\text{out}} + \sum_{n \geq 1} \epsilon^n h_{\alpha\beta}^{(n)}[\gamma(\epsilon)], \quad (6.1)$$

$$\text{Inner expansion: } g_{\mu\nu} = g_{\mu\nu}^{\text{in}}[\tilde{r}] + \sum_{n \geq 1} \epsilon^n H_{\alpha\beta}^{(n)}[\tilde{r}]. \quad (6.2)$$

Here $g_{\mu\nu}^{\text{out}}$ denotes the background spacetime on which the small body propagates, while $g_{\mu\nu}^{\text{in}}$ is the metric of the secondary object *in isolation*, that is, the geometry it would generate were it placed in an otherwise empty universe. The inner expansion is written in terms of the rescaled radial coordinate $\tilde{r}$ that zooms into the geometry of the small body and takes the form,

$$\tilde{r} = \frac{\mathcal{R}_{\text{out}}}{\mathcal{R}_{\text{in}}} r \quad (6.3)$$

where $\mathcal{R}_{\text{out}}$ and $\mathcal{R}_{\text{in}}$ are the characteristic length scales of the background.

To extract the matching conditions, we re-expand the outer solution at small radius and the inner solution at large radius, obtaining series valid in the buffer zone:

$$g_{\alpha\beta}^{\text{out}} = \sum_{p \geq 0} r^p g_{\alpha\beta}^{\text{out}(p)}, \quad h_{\alpha\beta}^{(n)} = \sum_{p \geq -n} r^p h_{\alpha\beta}^{(n;p)}, \quad (6.4)$$

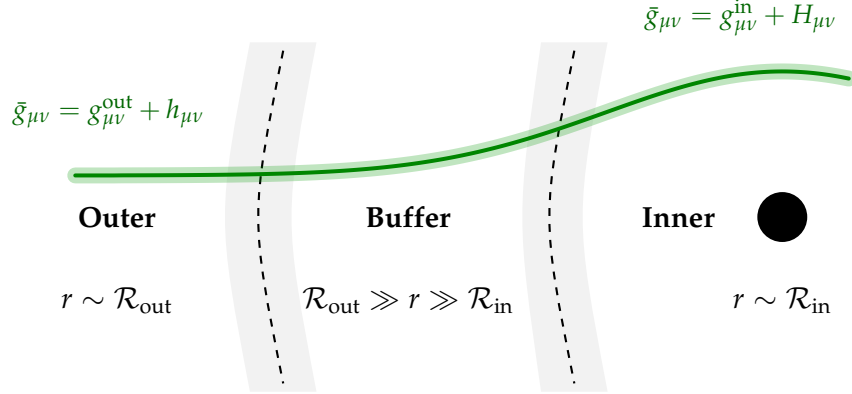


Figure 6.1: Diagrammatic representation of the distinct scaling regions and the location of the buffer region. The green curve schematically represents the full metric $\bar{g}_{\mu\nu}$ along the one-parameter family of solutions. In the outer region ($r \sim \mathcal{R}_{\text{out}}$) it is expanded as $\bar{g}_{\mu\nu} = g_{\mu\nu}^{\text{out}} + h_{\mu\nu}$ about the background, while in the inner region ($r \sim \mathcal{R}_{\text{in}}$) it is expanded as $\bar{g}_{\mu\nu} = g_{\mu\nu}^{\text{in}} + H_{\mu\nu}$ about the isolated secondary; the two descriptions are matched term by term in the intervening buffer zone.

$$g_{\alpha\beta}^{\text{in}} = \sum_{p \geq 0} \frac{1}{\tilde{r}^p} g_{\alpha\beta}^{\text{in}(p)}, \quad H_{\alpha\beta}^{(n)} = \sum_{p \geq -n} \frac{1}{\tilde{r}^p} H_{\alpha\beta}^{(n;p)}. \quad (6.5)$$

Requiring these two re-expansions to agree term by term in the buffer region constitutes the matching condition that ties the outer perturbations $h_{\alpha\beta}^{(n)}$ to the multipole structure of the secondary encoded in $g_{\alpha\beta}^{\text{in}}$. This takes the schematic form [176, 177]

$$g_{\mu\nu}^{\text{in}} \sim 1 + \frac{m}{\tilde{r}} + \frac{M_i n^i + \epsilon_{ijk} S^j n^k}{\tilde{r}^2} + \frac{M_{ij} n^i n^j + \epsilon_{ijk} S^j n^k n^q}{\tilde{r}^3} + \dots, \quad (6.6)$$

where m is the ADM mass, M_i the mass dipole, S^i the ADM angular momentum, and M_{ij} , S_{ij} the mass and current quadrupole moments of the secondary. The matching then fixes the most singular behaviour of the outer perturbations order by order: the $m/\tilde{r}$ term fixes the $1/r$ behaviour of the first-order field,

$$h_{\mu\nu}^{(1)} \sim \frac{m}{r} + \mathcal{O}(r^0), \quad (6.7)$$

while at second order the $1/\tilde{r}^2$ terms, involving the mass dipole and spin, fix

$$h_{\mu\nu}^{(2)} \sim \frac{m^2 + M_i n^i + \epsilon_{ijk} S^j n^k}{r^2} + \mathcal{O}(r^{-1}). \quad (6.8)$$

In the buffer zone the field equations reduce to a system of Poisson equations, naturally leading to a point-particle source of the form $\nabla^2(m/r) = -4\pi m \delta^{(3)}(r)$.

The key step which then leads to the strongly divergent behaviour in the field equations is that this buffer zone solution is extended back to the location of the secondary, replacing its true geometry with a multipolar decomposition. In taking this step one obtains a point-particle source in place of the secondary's geometry, which in the full spacetime takes the form

$$T_{(pp,1)}^{\mu\nu} := m \int_{\gamma} u^{\mu} u^{\nu} \frac{\delta^{(4)}(x^{\alpha} - z^{\alpha}(\tau))}{\sqrt{-g}} d\tau, \quad (6.9)$$

where $z^{\alpha}(\tau)$ parametrises the worldline γ [176]. Similarly, the point-particle source can be constructed at second order from the structure of Eq. (6.8). Even though this extension of the particle field down to the secondary is precisely what gives rise to the strong divergences which appear at higher order in self-force theory, having such rigorous control over the procedure also directly provides the means to remove the unphysical divergences at a later stage, through the construction of locally calculated singular fields and punctures, leading to well-defined *effective sources* at second order in perturbation theory [178, 179].

In practice all calculations at higher order make use of such an *effective source* prescription; however, non-regularised divergent sources can also be represented in a distributionally well-defined form at second order in the mass ratio through the construction of a highly regular gauge, which ameliorates the overly singular behavior of the second-order fields [178, 180]. This reformulation then allows one to explicitly construct the field equations in the distributional form,

$$\delta G_{\mu\nu}[h^{(1)}] = T_{\mu\nu}^{(pp,1)}, \quad (6.10)$$

$$\delta G_{\mu\nu}[h^{(2)}] = T_{\mu\nu}^{(pp,2)} - \delta^{(2)} G_{\mu\nu}[h^{(1)}, h^{(1)}]. \quad (6.11)$$

The existence of such a gauge has already proved to be a powerful method for establishing properties of the point-particle prescription at higher orders; however, in practice, calculating the gauge transformation for such systems has proved difficult.

Returning to the matter at hand, we wish to understand the motion of bodies through matter in GR; to do this we must first obtain the metric perturbation $h^{(1)}$ induced by the point-particle stress-energy tensor. In practice, the means by which progress has been made in this pursuit in the Kerr spacetime is through solving the linearised Teukolsky equation in the presence of a point-particle source. Solving the Teukolsky equation in this setting is standard and has been achieved for generic orbits in Kerr spacetime [71]. In reconstructing the metric perturbation, however, a problem persists in that the point-particle stress-energy tensor does not generally satisfy the traceless condition $g^{\mu\nu} T_{\mu\nu}^{(pp)} = 0$, making it incompatible with direct reconstruction from Ψ_4 . Such a gauge introduces distributional singularities along curves generated by n^{μ} . This problem can in principle be circumvented through the corrector tensor formalism of Green, Hollands, and Zimmerman (GHZ), but

in practice this has only been applied to effective-source calculations in the non-spinning case [181]; a review of available reconstruction approaches can be found in [60]. In Lorenz gauge, however, where $\nabla^\mu \bar{h}_{\mu\nu}^{(1)} = 0$, the singularity structure simplifies and localises to the position of the secondary in the form $\sim 1/r$, where r is a measure of distance from the secondary. Taking advantage of this structure is the approach we adopt in what follows.

6.1 PERTURBATIONS TO THE GEODESIC EQUATIONS

Much like perturbations to the field equations due to a particle source, one can ask how the trajectory of the smaller body differs when self-gravitational effects are taken into account. Through the method of matched asymptotic expansions, the leading-order singular-regular split [182] of the retarded field can be extended to at least second order in the mass ratio, i.e. $h_{\text{Reg}}^{(1/2)} = h_{\text{Ret}}^{(1/2)} - h_{\text{Sing}}^{(1/2)}$, such that the trajectory is a geodesic on the effective spacetime $g_{\mu\nu} = g_{\mu\nu} + \epsilon h_{\mu\nu}^{(1,\text{Reg})} + \epsilon^2 h_{\mu\nu}^{(2,\text{Reg})} + \dots$. Expanding the covariant derivative in this parameter yields the schematic expansion

$$\begin{aligned} u^\mu \nabla_\mu u^\nu &= u^\mu (\partial + \Gamma) u^\nu = 0 \\ &= u^\mu \left[\partial + \Gamma + \epsilon \delta\Gamma[h_{\text{Reg}}^{(1)}] + \epsilon^2 \left(\delta\Gamma[h_{\text{Reg}}^{(2)}] + \delta^{(2)}\Gamma[h_{\text{Reg}}^{(1)}] \right) + \dots \right] u^\nu = 0. \end{aligned} \quad (6.12)$$

Moving the perturbative terms to the right-hand side, one arrives at the *effectively forced* geodesic equations,

$$u^\mu \nabla_\mu u^\nu = \epsilon F_{(1)}^\nu + \epsilon^2 F_{(2)}^\nu + \dots \quad (6.13)$$

In this sense the worldline is geodesic on the effective spacetime, but appears forced when represented on the background. Notably, we wish to allow for a representation whereby we do not expand the velocity field u^μ . This is to guard against the fact that geodesics directly accumulate error as they integrate along the curve: if one were to also expand $u \rightarrow u^{(0)} + \epsilon u^{(1)} + \dots$, the solution would exhibit an $\mathcal{O}(1)$ error on a timescale $\sim 1/\epsilon$. Such approaches were proposed in the past for the vacuum self-force problem [183] but are not suitable for the long integration times required in EMRI modelling. Taking the form of the geodesic equation in Eq. (6.13) and transforming into the osculating geodesic form of Eq. (2.31), one obtains the forced osculating geodesic equations,

$$\frac{dP_i}{d\lambda} = \epsilon G^{(1)} + \epsilon^2 G^{(2)} + \dots, \quad (6.14)$$

$$\frac{dq_i}{d\lambda} = Y_i + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \dots. \quad (6.15)$$

Where in their current form, the functions $G^{(i)}$ and $f^{(i)}$ depend on both the slowly (P_i) and rapidly (q_i) evolving elements and are explicit functions of the associated metric perturbations. Naturally, solving these ODEs becomes slow as the force functions vary rapidly; the fast time dependence can, however, be removed through the method of Near-Identity Transformations (NITs) [184, 185], making such evolutions extremely computationally efficient.

Finally, there is an important point to note regarding the time-symmetry properties of the forcing functions arising in Eq. (6.13). This is made explicit by splitting the full retarded perturbation into its constituent pieces,

$$h_{\text{Ret}}^{(1)} = h_{\text{Sym}}^{(1)} + \frac{1}{2}h_{\text{Rad}}^{(1)}, \quad (6.16)$$

such that $h_{\text{Sym}}^{(1)}$ is symmetric under time reversal, leading to a conservative action on the orbit, whereas $h_{\text{Rad}}^{(1)}$ is time-antisymmetric, causing dissipative secular effects on the orbit. Such a splitting between dissipative and conservative forces is precisely what gives rise to the post-adiabatic counting in self-force theory, whereby contributions to the force at $\mathcal{O}(\varepsilon)$ that are conservative are pushed one order higher in the dynamics. This is clarified by the example of Φ , the total phase accumulated over an inspiral, for which the corrections take the form

$$\Phi = \frac{1}{\varepsilon} \left(\phi^{(0)}[h_{\text{Rad}}^{(1)}] + \varepsilon \phi^{(1)}[h_{\text{Ret}}^{(1)}, h_{\text{Rad}}^{(2)}] \right) + \dots \quad (6.17)$$

labeled as zeroth post-adiabatic (0PA – $\phi^{(0)}$) and first post-adiabatic (1PA – $\phi^{(1)}$) contributions to the dynamics. The statement of self-force order therefore refers to the order at which the forces in the geodesic equations are calculated, whereas post-adiabatic counting describes the order to which the phasing in the dynamics has been computed. From this expansion it becomes clear that in vacuum one needs only the radiative part of the second-order field to achieve sub-radian dephasing accuracy.

6.2 LORENZ GAUGE POINT PARTICLE METRIC RECONSTRUCTION IN THEORY

In this section we detail the approach developed by Dolan, Kavanagh, and Wardell [186] for obtaining the full retarded metric perturbation in the Kerr geometry for circular equatorial orbits. Rather than constructing the corrector piece to the radiation-gauge metric perturbation, they instead transform directly to Lorenz gauge. Exploiting the spatial compactness of the singularity structure in this gauge, they construct explicit jump conditions that impose continuity across the particle at the level of spheroidal modes, thereby recovering the missing distributional content without the need for a corrector tensor. The practical calculation, which we review and draw from throughout this section, can be found in [60]. Our aim here is to

give an intuitive description of the approach rather than revisit all technical aspects, which are already covered in detail in [60]. The approach proceeds as follows. We label the Lorenz-gauge perturbation $h_{\mu\nu}^L$, which satisfies

$$\nabla^\mu \bar{h}_{\mu\nu}^L = 0, \quad (6.18)$$

and the IRG perturbation $h_{\mu\nu}^{\text{IRG}}$, which satisfies

$$l^\mu h_{\mu\nu}^{\text{IRG}} = 0, \quad g^{\mu\nu} h_{\mu\nu}^{\text{IRG}} = 0. \quad (6.19)$$

Assuming one has access to the full IRG solution through the CCK reconstruction procedure laid out in chapter 3, the connection between the two gauges is given by the gauge vector ξ^μ via

$$h_L^{\mu\nu} = h_{\text{IRG}}^{\mu\nu} - 2\xi^{(\mu;\nu)}. \quad (6.20)$$

Taking the divergence of (6.20) and using the fact that $R_{\mu\nu} = 0$ in the vacuum Kerr spacetime, the transformation from radiation gauge to Lorenz gauge reduces to solving

$$\square \xi^\mu = \nabla_\nu h_{\text{IRG}}^{\mu\nu} \quad (6.21)$$

for the gauge vector ξ^μ . In vacuum, any particular solution of this equation would yield a valid Lorenz-gauge solution of the field equations. However, since our solutions will later be built from a point-particle source, additional homogeneous solutions of Eq. (6.21) will be required.

When working in a spheroidal basis, quantities on the sphere carry a definite spin weight. In the IRG the metric perturbation $h_{\mu\nu}^{\text{IRG}}$ is a purely spin-weight $s = -2$ quantity; however, the gauge vector must include spin-weight $s = -1$ and $s = 0$ contributions as well due to the source structure. Since the source is fixed entirely through the spin-2 fields, we decompose $\xi^\mu = \xi_{(s=2)}^\mu + \xi_{(s=1)}^\mu + \xi_{(s=0)}^\mu$, labelling by spin magnitude, with each piece satisfying

$$\square \xi_{(s=2)}^\mu = \nabla_\nu h_{\text{IRG}}^{\mu\nu}, \quad \nabla^\mu \xi_{(s=2)}^\mu = 0, \quad (6.22)$$

$$\square \xi_{(s=1)}^\mu = 0, \quad \nabla^\mu \xi_{(s=1)}^\mu = 0, \quad (6.23)$$

$$\square \xi_{(s=0)}^\mu = 0, \quad \nabla^\mu \xi_{(s=0)}^\mu = -\frac{1}{2}h^L, \quad (6.24)$$

where the spin-zero contribution is pure gauge, $h_{\mu\nu}^{L,(s=0)} = -2\xi_{(\mu;\nu)}^{(s=0)}$, and h^L denotes the trace $g^{\mu\nu} h_{\mu\nu}^L$. Building on the results of several works [186–188], these equations can be written on Fourier space, $h(t, x^i) \rightarrow h(\omega, x^i) e^{i\omega t}$, explicitly in terms of solutions to the spin-weighted Teukolsky equations as

$$\xi_{(s=2)}^\mu = -\sqrt{2}\bar{\rho}^2 \nabla_\nu \left(\rho \frac{l^{[\mu} m^{\nu]}}{3i\omega\bar{\rho}\Sigma} \mathcal{D}\mathcal{L}_2^+ \psi_- \right) - g^{\mu\nu} \partial_\nu \left(\frac{1}{24\omega^2} \mathcal{D}\mathcal{D}\mathcal{L}_1^+ \mathcal{L}_2^+ \psi_- \right), \quad (6.25)$$

$$\tilde{\xi}_{(s=1)}^\mu = -\frac{2}{i\omega\rho} \nabla_\nu \left(\phi_0 \bar{m}^{[\nu} n^{\mu]} - \phi_2 l^{[\nu} m^{\mu]} \right), \quad (6.26)$$

$$\tilde{\xi}_{(s=0)}^\mu = \frac{1}{2i\omega} f^{\mu\nu} \nabla_\nu h + 2\nabla^\mu \kappa, \quad (6.27)$$

where ψ_- is the Hertz potential satisfying the $s = -2$ Teukolsky equation, ϕ_0 the $s = +1$ Teukolsky equation, ϕ_2 the $s = -1$ Teukolsky equation [82], $f^{\mu\nu}$ is the conformal Killing–Yano tensor defined in Eq. (2.10), and κ satisfies the scalar wave equation $\square\kappa = -\frac{1}{2}h$, with the trace h solved for directly through the Einstein equations. With these explicit solutions in hand, one can directly calculate metric perturbations in the Lorenz gauge in terms of Teukolsky scalars. Such a representation is valid for homogeneous solutions; however, from the point-particle prescription the source will be supported at a single point in space. Due to the simplified structure induced by circular orbits, the corresponding stress-energy tensor takes the form,

$$\begin{aligned} T_{(pp,1)}^{\mu\nu} &= m_p \int_{-\infty}^{\infty} \frac{u^\mu u^\nu \delta^{(4)}(x^\alpha - z^\alpha(\tau))}{\sqrt{-g}} d\tau \\ &= \frac{m_p}{2\pi r_0^2 u^t} u^\mu u^\nu \sum_m \delta(r - r_0) \delta\left(\theta - \frac{\pi}{2}\right) e^{i(m\phi - \omega_m t)}. \end{aligned} \quad (6.28)$$

Such a form of source in Lorenz gauge then motivates the ansatz

$$h_{\mu\nu}^{(m)}(r, \theta) = \Theta(r - r_0) h_{\mu\nu}^{\text{up},(m)}(r, \theta) + \Theta(r_0 - r) h_{\mu\nu}^{\text{in},(m)}(r, \theta), \quad (6.29)$$

where $h_{\mu\nu}^{\text{up}}$ and $h_{\mu\nu}^{\text{in}}$ are vacuum solutions to the Einstein field equations and Θ is the Heaviside function. Such $h_{\mu\nu}^{\text{up/in}}$ can then be constructed in Lorenz gauge from the gauge transformations built from Teukolsky scalars as above, with physical boundary conditions. Doing so fixes the homogeneous solutions up to two overall constants, which are then determined by the source at the particle location. Substituting the ansatz (6.29) into the field equations and performing a local analysis around r_0 , one arrives at a series of jump conditions which fix these coefficients; as an example, the ll component takes the form

$$\lim_{\epsilon \rightarrow 0^+} \left[\partial_r h_{\ell m}^{\text{ll}}(r) \right]_{r_0-\epsilon}^{r_0+\epsilon} = -\frac{16\pi m_p}{u^t(r_0^2 - 2Mr_0 + a^2)} (u \cdot l)^2 Y_{\ell m}\left(\frac{\pi}{2}\right), \quad (6.30)$$

along with the continuity condition, as derived in [60]. These jump conditions fix the remaining freedom in (6.29), yielding the full Lorenz-gauge metric perturbation. Naturally, such a calculation involves several further complications: the reprojection onto different harmonic modes; the development of robust solution methods for the sourced wave equations, in particular for κ ; the treatment of the static modes, where the inversion relation introduces terms of the form $\sim 1/\omega$ that become ill-defined, requiring an alternative approach; and finally the incorporation of the completion

modes $h^{\delta M}$ and $h^{\delta a}$, tracking the perturbations to the spacetime's mass and spin respectively, which cannot be computed directly through Teukolsky reconstruction. All such additional subtleties are addressed and resolved for circular orbits in [60].

6.3 LORENZ GAUGE POINT PARTICLE METRIC RECONSTRUCTION IN PRACTICE

Having reviewed the theoretical means by which one can reconstruct Lorenz-gauge metric perturbations from Teukolsky variables [79], we now present some practical considerations that arise when the `KerrCircLorenz` Mathematica code [60] was employed to build metric data for the environment problems addressed later in this thesis. This section contains significant content from one of the *thesis papers* associated with this work, [62], co-authored with Daniel D'Orazio.

Notably, within the class of point-particle metric solutions, the Lorenz gauge is inherently asymptotically irregular. In order to obtain a solution that is regular in this gauge, Dolan et al. also remove the static monopole and dipole perturbations of the system, which arise due to the presence of the small body. This class of modified Lorenz-gauge solutions is commonly referred to as being within the Berndtson class of solutions [60, 189, 190]. We work within this class of solutions.

To numerically produce the required metric data, our practical requirements in later chapters will be the construction of sufficiently accurate Fourier m -modes of the point-particle metric perturbations denoted h_{ij}^m , with

$$h_{ij} = \sum_m h_{ij}^m e^{im\phi - i\omega_m t}, \quad (6.31)$$

where, in this context, tetrad components are denoted by Latin indices. To do so, we first construct the Kinnersley tetrad components of the Lorenz gauge solutions projected onto a spin-weighted spherical harmonic basis, as in [60]. Having obtained the spherical harmonic modes, we then construct the m -modes through,

$$h_{ij}^m(r, \theta) = \sum_{\ell=\max(m,s)}^{\ell_{\max}} h_{ij}^{\ell m}(r) {}_s Y_{\ell m}(\theta, \phi = 0), \quad (6.32)$$

typically taking $\ell_{\max} \sim 30$. The spin weight s is determined by the particular tetrad component being constructed. Having built the m -mode tetrad components, we numerically transform from the Kinnersley tetrad to Boyer–Lindquist coordinate components through the change of basis operation,

$$h_{\mu\nu}^{(m)} = \sum_{ij} \tilde{Z}_{\mu}^i \tilde{Z}_{\nu}^j h_{ij}^{(m)}, \quad (6.33)$$

with $\tilde{Z}_{\mu}^i = \{-n_{\mu}, -l_{\mu}, \bar{m}_{\mu}, m_{\mu}\}$, and where l , n , and m are the standard principal null directions of the Kinnersley tetrad in Kerr.

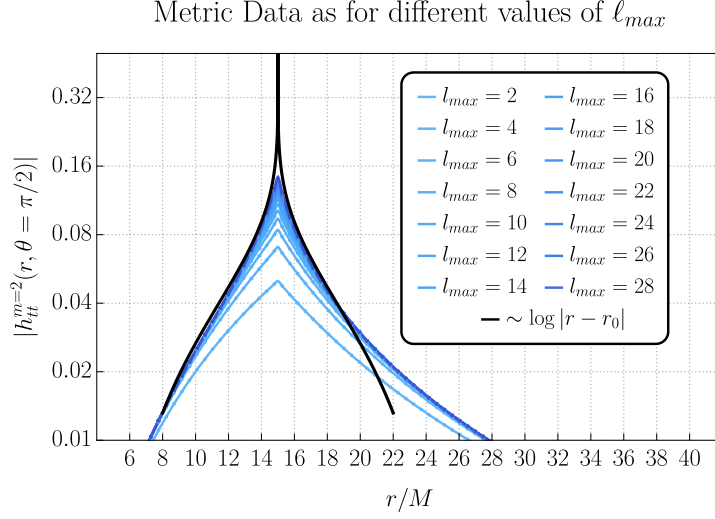


Figure 6.2: Equatorial slices of the metric perturbation for a single m -mode, constructed using varying numbers of input ℓ -modes. The lightest line corresponds to $\ell_{\max} = 2$, with successively darker lines representing $\ell_{\max} = 4, 6, \dots$ up to $\ell_{\max} = 28$. A characteristic logarithmic divergence is shown overlaid to illustrate the expected scaling of the source up to the softening length.

As stated previously, the Lorenz gauge solutions have a special singularity structure near the secondary, making them particularly suitable not only for this specific solution method but also for their subsequent use as forcing functions on environments. Due to the projection in the angular decomposition, Lorenz gauge solutions have different levels of regularity depending on whether one considers the four-dimensional, Fourier m -mode, or spherical-harmonic ℓ -mode solutions. The general singularity structure in each case is given by

$$\begin{aligned} h_{\mu\nu}^{4d}|_{r \rightarrow r_p} &\sim \frac{1}{|\mathbf{r} - \mathbf{r}_p|}, \\ h_{\mu\nu}^m|_{r \rightarrow r_p} &\sim \log |\mathbf{r} - \mathbf{r}_p|, \\ h_{\mu\nu}^{\ell m}|_{r \rightarrow r_p} &\sim |\mathbf{r} - \mathbf{r}_p|. \end{aligned}$$

Because we construct our m -modes from a finite sum of spherical-harmonic ℓ -modes, the resulting input m -mode metric data exhibits sharp but continuous behaviour across the particle's location, rather than the true logarithmic behaviour of the exact solution. This C^0 behaviour leads to a discontinuous but finite effective forcing function, which involves derivatives of the metric data. An example of this behaviour is shown in Fig. 6.2. Within $r_{\text{soft}} \sim r_p / \ell_{\max}$ of the particle, the metric perturbation exhibits a softened potential, while outside this distance from the particle, the field is properly converged and reproduces the correct logarithmic growth towards the secondary. Forthcoming work with the Multi-Scale Self-Force

Collaboration on the construction of the second-order source for nonlinear vacuum self-force calculations will present a method for regularising this field and consistently restoring the correct singular structure using the method of punctures [191, 192].

Having introduced the theory necessary to individually describe point particles and matter fields in curved space, we can proceed to describing their interactions in a fully relativistic manner. The analysis we provide here is general enough to also allow one to treat other stationary perturbations such as tides, as well as dynamical perturbations such as black hole ringdown. Throughout this section, however, we will be concerned with the case where the perturbation to the geometry is sourced by a secondary perturber.

Our starting point will be the assumption that, a priori, a matching procedure has been performed to separate an asymmetric binary system with a large-scale disparity into a background field with an associated massive particle in orbit. We have already covered the vacuum version of such a scenario in the previous sections of this thesis. Now, however, we assume that this has been done self-consistently in the presence of an associated matter field, i.e. a full matched asymptotic treatment has been performed self-consistently. Such a treatment will in general not be at all trivial, and will likely introduce boundary flux contributions due to mechanisms such as accretion onto the smaller body, as well as corrections to the associated mass and other charges derived from the multipole expansion due to the presence of bound matter in the Hill sphere of the smaller body. In this work we absorb all such pieces by defining an environmental correction to the point-particle stress-energy tensor. A more complete analysis of this structure and the physical relevance of such effects must be explored in future work.

The final assumption we will require in this section is that both the stress-energy of the environment and that of the particle are conserved independently. This would be at odds, for example, with the general expectation that boundary terms such as accretion onto the secondary may convert content from the environmental matter field into charge perturbations of the small body. In acknowledgement of these restrictions, we will nonetheless proceed, aware that more careful treatments of stress-energy conservation should provide a path towards incorporating such mechanisms.

7.1 THE RELATIVISTIC TWO-PARAMETER FRAMEWORK

We begin again by stating the problem in its covariant form to make contact with a fully nonlinear matter description. Our starting point is the full nonlinear Einstein equations, which we assume have already undergone matching to give the form,

$$G_{\mu\nu}[g] = \epsilon T_{\mu\nu}^{(p)}[g] + \lambda^n T_{\mu\nu}^{(\text{env})}[g, \{\Psi^N\}], \quad (7.1)$$

where $g_{\mu\nu}$ is the full spacetime metric and $\{\Psi\}$ denotes the collection of *scalar*, *vector*, and/or *tensor* degrees of freedom describing the environment's matter content. As before, the parameters ϵ and λ control the strength of the secondary's and the matter's gravitational fields respectively. The stress-energy $T^{(\text{env})}$ describes the matter content of the environment globally, whereas $T^{(p)}$ captures the near-field content of the secondary and its associated bound matter, re-expressed in terms of effective charges via the local matching procedure and a multi-polar decomposition. In general this term encompasses not only the vacuum content associated with the secondary, but also other contributions such as the mass shift due to matter locally bound within its Hill sphere. We further assume that the two stress-energy tensors are individually conserved,

$$\nabla^\mu T_{\mu\nu}^{(p)} = 0, \quad \nabla^\mu T_{\mu\nu}^{(\text{env})} = 0. \quad (7.2)$$

This condition will not hold in general: it neglects, for example, accretion onto the secondary, in which matter from $T^{(\text{env})}$ should be converted into changes in the effective charges of $T^{(p)}$. A proper treatment of this exchange will be of great importance in future work, with potential consequences for the development of physically motivated sink prescriptions in hydrodynamical simulations [97]. Nevertheless we will proceed with the current assumptions throughout this work.

At leading order we can consider the matter and secondary as sourcing two independent metric perturbations, $h^{(\text{env})}$ and $h^{(p)}$ respectively. This motivates one of the most interesting and striking regimes of validity of the two-parameter model, namely that where the metric perturbation due to the secondary stays small and is perturbatively well defined, but where the matter profile response is *nonlinear*. In this regime the matter stress-energy is then conserved on the perturbed spacetime,

$$\nabla_\mu^{g+\epsilon h^{(p)}} T_{\text{env}}^{\mu\nu}[g + \epsilon h^{(p)}, \{\Psi\}] = 0, \quad (7.3)$$

where we have assumed the self-gravity of the environment is negligible. Such an approach is internally consistent provided we do not venture to higher orders in perturbation theory in ϵ , where certain contributions would, in general, be partially resummed to all orders due to the nonlinear matter response. In this setting the equations of motion can be straightforwardly expanded in ϵ to obtain a nonlinear

equation for the matter field $\{\Psi\}$, forced by the presence of the metric perturbation $h^{(p)}$, given by,

$$\begin{aligned} \nabla_g^\mu T_{\mu\nu}^{\text{env}}[g, \{\Psi\}] &= -g^{\alpha\beta} \delta\Gamma_{\alpha\beta}^\gamma[h^{(p)}] T_{\gamma\nu}^{\text{env}}[g, \{\Psi\}] \\ &\quad + g^{\alpha\beta} \delta\Gamma_{\alpha\nu}^\gamma[h^{(p)}] T_{\gamma\beta}^{\text{env}}[g, \{\Psi\}] \\ &\quad - \nabla_g^\mu \delta^{(1,0)} T_{\mu\nu}^{\text{env}}[h^{(p)}, \{\Psi\}] \end{aligned} \quad (7.4)$$

where we have defined,

$$\delta^{(\kappa,m)} T_{\mu\nu}^{\text{env}}[h, \{\delta\Psi\}] := \frac{1}{k!m!} \frac{d}{dk} \frac{d}{dm} \left[T_{\mu\nu}^{\text{env}}[g + kh, \{\Psi\} + m\{\delta\Psi\}] \right]_{k=m=0}, \quad (7.5)$$

and

$$\Gamma_{\mu\nu}^\gamma[h] = \frac{1}{2} g^{\gamma\beta} (\nabla_\mu h_{\beta\nu} + \nabla_\nu h_{\beta\mu} - \nabla_\beta h_{\mu\nu}). \quad (7.6)$$

The formulation of Eq. (7.4) is precisely the starting point for calculating the nonlinear matter response to a fluid in curved space perturbed by a secondary potential.

If we instead enforce that both the metric and matter perturbations always remain small, we may develop a concrete systematic approach to higher orders in ϵ and λ . These assumptions directly imply the perturbative expansions,

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + \sum_{k+m \geq 0} \epsilon^k \lambda^m h_{\mu\nu}^{(k,m)}, \quad (7.7)$$

$$\{\psi\} \rightarrow \lambda^N \{\psi^N\} + \lambda^N \sum_{k+m \geq 0} \epsilon^k \lambda^m \{\psi^{(k,N+m)}\}, \quad (7.8)$$

on scales set by the background vacuum solution, where we have allowed the freedom for a dual expansion in the fields and adopt the multi-index notation,

$$\epsilon^k \lambda^m \lambda^N \{\psi^{(k,N+m)}\} = \epsilon^k \lambda^m \{\lambda^{n_1} \psi_1^{(k,n_1+m)}, \dots\}. \quad (7.9)$$

Taking this approach, the general contributions to the stress-energy tensors are then given by,

$$\lambda^n T_{(\text{env})}^{\mu\nu} = \lambda^n \sum_{k+m \geq 0} \epsilon^k \lambda^m T_{(\text{env},k,m+n)}^{\mu\nu}, \quad (7.10)$$

where the leading term $T_{(\text{env},0,n)}^{\mu\nu}$ describes the unperturbed environment on the background, while $T_{(\text{env},1,n)}^{\mu\nu}$ encodes the linear correction to the matter's stress-energy tensor due to the secondary's leading-order metric perturbation and the matter's corresponding response. We obtain for the orders mentioned above,

$$\mathcal{O}(\lambda^n) : T_{(\text{env},0,n)}^{\mu\nu} = T_{(\text{env})}^{\mu\nu}[g, \{\Psi^{(0,N)}\}] \quad (7.11)$$

$$\mathcal{O}(\epsilon\lambda^n) : T_{(\text{env},1,n)}^{\mu\nu} = \delta^{(1,0)} T_{(\text{env})}^{\mu\nu}[h^{(1,0)}, \{\Psi^N\}] + \delta^{(0,1)} T_{(\text{env})}^{\mu\nu}[g, \{\Psi^{(1,N)}\}], \quad (7.12)$$

Fixing the conservation of the stress-energy at each order, we then obtain the linearised equation

$$\begin{aligned} \nabla_g^\mu \delta^{(0,1)} T_{\mu\nu}^{\text{env}}[g, \{\Psi^{(1,N)}\}] &= -g^{\alpha\beta} \delta \Gamma_{\alpha\beta}^\gamma[h^{(p)}] T_{\gamma\nu}^{\text{env}}[g, \{\Psi^{(0,N)}\}] \\ &\quad + g^{\alpha\beta} \delta \Gamma_{\alpha\nu}^\gamma[h^{(p)}] T_{\gamma\beta}^{\text{env}}[g, \{\Psi^{(0,N)}\}] \\ &\quad - \nabla_g^\mu \delta^{(1,0)} T_{\mu\nu}^{\text{env}}[h^{(p)}, \{\Psi^{(0,N)}\}] \end{aligned} \quad (7.13)$$

governing the solution to the matter field perturbations $\{\Psi^{(1,N)}\}$. The secondary's stress-energy is then handled similarly in the same double series,

$$\epsilon T_{(p)}^{\mu\nu} = \epsilon \sum_{n+m \geq 0} \epsilon^n \lambda^m T_{(p,n+1,m)}^{\mu\nu}. \quad (7.14)$$

The leading term $T_{(p,1,0)}^{\mu\nu}$ is the standard geodesic point-particle stress-energy on the Kerr background, while $T_{(p,1,1)}^{\mu\nu}$ encodes the leading correction to the particle's stress-energy from the presence of the environment in its local zone. For compactness we define $T_{\mu\nu}^{(n,m)}$ as the total combination of all terms at order $\epsilon^n \lambda^m$,

$$T_{\mu\nu}^{(n,m)} := T_{\mu\nu}^{(p,n,m)} + T_{\mu\nu}^{(\text{env},n,m)}, \quad (7.15)$$

and similarly for metric perturbations, where naturally $T^{(p,0,1)} = T^{(\text{env},1,0)} = 0$. Having solved for the matter field equations, the metric perturbations are then sourced by

$$\mathcal{O}(1) : G_{\mu\nu}[g] = 0, \quad (7.16)$$

$$\mathcal{O}(\epsilon) : \delta G_{\mu\nu}^{(1)}[h^{(1,0)}] = T_{\mu\nu}^{(p,1,0)} \quad (7.17)$$

$$\mathcal{O}(\lambda^n) : \delta G_{\mu\nu}^{(1)}[h^{(0,n)}] = T_{\mu\nu}^{(\text{env},0,n)} \quad (7.18)$$

$$\mathcal{O}(\epsilon\lambda^n) : \delta G_{\mu\nu}^{(1)}[h^{(1,n)}] = T_{\mu\nu}^{(p,1,n)}[h^{(0,n)}] + T_{\mu\nu}^{(\text{env},1,n)}[h^{(1,0)}, \{\Psi^{(1,N)}\}]. \quad (7.19)$$

The forcing functions arise from imposing conservation of the particle stress-energy tensor on the full spacetime, $\nabla^\mu T_{\mu\nu}^{(\text{pp})} = 0$, which is equivalent to $u^\mu \hat{\nabla}_\mu u^\nu = 0$, where $\hat{\nabla}$ denotes the covariant derivative with respect to the full metric. Re-expanding in ϵ, λ and assuming no evolution in the mass of the secondary the functional form of the forcing is then given by,

$$\begin{aligned} u^\mu \nabla_\mu u^\nu &= \epsilon F^\nu[h^{(1,0)}] + \lambda^n F^\nu[h^{(0,n)}] + \epsilon\lambda^n \left(F^\nu[h^{(1,n)}] + Q^\nu[h^{(1,0)}, h^{(0,n)}] \right) \\ &\quad + \epsilon^2 \left(F^\nu[h^{(2,0)}] + Q^\nu[h^{(1,0)}, h^{(1,0)}] \right) + \dots \end{aligned} \quad (7.20)$$

with

$$F^\mu[h] = \frac{1}{2} (g^{\mu\nu} + u^\mu u^\nu) (h_{\sigma\lambda;\nu} - 2h_{\nu\sigma;\lambda}) u^\sigma u^\lambda, \quad (7.21)$$

and

$$Q^\mu[h^{(1)}, h^{(2)}] = -\frac{1}{2}(g^{\mu\nu} + u^\mu u^\nu) \left[h_\nu^{(1)\rho} \left(h_{\sigma\lambda;\rho}^{(2)} - 2h_{\rho\sigma;\lambda}^{(2)} \right) + h_\nu^{(2)\rho} \left(h_{\sigma\lambda;\rho}^{(1)} - 2h_{\rho\sigma;\lambda}^{(1)} \right) \right] u^\sigma u^\lambda, \quad (7.22)$$

where future studies understanding the subtleties of the singular-regular split giving rise to a well define form of Eq. (7.1) will be vital. We note that throughout this work we will generally avoid the careful local analysis required at the particle's location to express $h^{(1,0)}$ and $h^{(1,n)}$ in terms of their regularised counterparts, which is achieved through the construction of local puncture fields $h^{(p)}$ such that $h_{\text{Res}}^{(1,0)} = h_{\text{Ret}}^{(1,0)} - h_{\text{Punc}}^{(1,0)}$ and $h_{\text{Res}}^{(1,n)} = h_{\text{Ret}}^{(1,n)} - h_{\text{Punc}}^{(1,n)}$. Work along this route is currently being undertaken in collaboration with Samuel Upton in the context of a massive scalar field. In the action-angle formulation, this leads to corrections in the forces arising in the evolution of both actions and angles.

Notably, unlike in the vacuum self-force problem, where the forcing functions depend on $h^{(1,0)}$ and have a temporal structure set by the orbital periods (and are therefore inherently non-stationary), many astrophysical environments can be approximately modelled as stationary background states. This is the case for both the boson cloud and axisymmetric disc profiles we currently explore. In such a case the background effect of $h^{(0,n)}$ is purely conservative, and is therefore pushed to higher order in the post-adiabatic counting, entering at the same formal order as $h^{(1,n)}$, which, through its coupling to the EMRI orbit, carries both conservative and dissipative contributions. In such a scenario, assuming one is in the gravitational-wave-dominated regime of orbital evolution, one then obtains a dephasing structure of the form

$$\Phi = \frac{1}{\epsilon} \left(\phi^{(1,0)}[h^{(1,0)}] + \lambda^n \phi^{(1,n)}[h^{(1,0)}, h^{(0,n)}, h^{(1,n)}] + \epsilon \phi^{(2,0)}[h^{(1,0)}, h^{(2,0)}] \right) + \dots \quad (7.23)$$

The most striking and important aspect of this setup is that all environmental perturbations act independently of the vacuum first- and second-order self-force calculations, leaving an approach that is completely complementary to the current approaches laid out in the field in pushing vacuum calculation to second order in self-force theory.

Having established the two-parameter framework in application to a generic stress-energy tensor, we now turn to its first concrete application in the Kerr geometry: the gravitational atom, a Kerr black hole enveloped by a superradiant scalar cloud at the threshold configuration. The work we present here constitutes a key advance in the state of relativistic EMRI modelling in an astrophysical environment, representing the first calculation of a self-consistent matter response $\{\Psi^{(1,N)}\}$ of a massive scalar field due to an EMRI in the Kerr geometry [61]. This work builds on [193], which was the first clear example of such a calculation in a two-parameter framework for an extended matter field in curved space, taking the first step by computing the response around a Schwarzschild black hole. Our work here, beyond being the first such calculation in Kerr and a necessary step towards providing precision waveform models for boson clouds in the Kerr geometry, also acts as an ideal testing ground for later extension to more complex environmental models.

The results of this work reveal a rich wake structure in the perturbed scalar field of the boson cloud, detailing significant deviations from both Newtonian and Schwarzschild predictions across the LISA band, demonstrating that working in Kerr is not merely a refinement but a necessity. In an attempt to connect such matter calculations to the dynamics of the EMRI, we also calculate the energy and angular momentum flux of the matter field to infinity and into the horizon of the central Kerr black hole, in addition to the energy change of the matter field itself. The hope is that in some sense $\dot{E}_{\text{orbit}} = \dot{E}_{\text{Flux}} - \dot{E}_{\text{Cloud}}$; however, such a relation is only indicative at this point and does not allow for a concrete connection to the orbital dynamics. The focus of later chapters will be to make this connection to the orbital dynamics precise.

This chapter thus serves simultaneously as a proof of principle for the framework in the Kerr geometry and as a first step towards the more complex fluid environments we turn to next. The contents of this chapter primarily constitute those of a published article in *Phys. Rev. Lett.* 134 (2025) 21, 211403, co-authored with Thomas Spijksma, Richard Brito, Maarten van de Meent, and Sam Dolan. Importantly, some of this material also appeared in the PhD thesis of Thomas Spijksma at the University of Copenhagen.

8.1 SCALAR FIELDS

We begin by considering a gravitational atom perturbed by a point-particle on an equatorial, circular orbit in the Kerr geometry. The matter Lagrangian for a massive scalar is given by

$$\mathcal{L}^{\text{env}}[\Phi] = \nabla_\nu \Phi \nabla^\nu \Phi^* - \mu^2 |\Phi|^2, \quad (8.1)$$

yielding $n = 2$ and the equation of motion:

$$\mathcal{Q}[\Phi, g_{\mu\nu}] = \square \Phi - \mu^2 \Phi = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \partial^\mu \Phi) - \mu^2 \Phi = 0. \quad (8.2)$$

Following the previous section, we expand the scalar field as

$$\Phi = \lambda \phi^{(0,1)} + \epsilon \lambda \phi^{(1,1)} + \lambda^3 \phi^{(0,3)} + \dots. \quad (8.3)$$

We take the background solution $\lambda \phi^{(0,1)}$ to be that of a superradiant cloud, whose normalization is set such that its total mass is given by M_c . The characteristic length scale of the cloud is given by its Bohr radius $L^{\text{env}} = (\mu\alpha)^{-1} = \alpha^{-2}M$, with $\alpha = \mu M$, yielding, $\lambda = \alpha^3 \sqrt{M_c/M}$ and fixing our perturbative parameter. Next defining the variation of a general bi-function operator as,

$$\delta^{(n,m)} \mathcal{Q}[\tilde{\psi}, h_{\mu\nu}] = \frac{1}{n!m!} \frac{d^{n+m}}{d\kappa^n d\lambda^m} \left\{ \mathcal{Q}[\psi + \kappa \tilde{\psi}, g_{\mu\nu} + \lambda h_{\mu\nu}] \right\} \big|_{\lambda=0, \kappa=0}. \quad (8.4)$$

we can then expand the scalar equation of motion at leading dynamical order $\sim \mathcal{O}(\epsilon q)$ to obtain the expressions:

$$\delta^{(1,0)} \mathcal{Q}[\phi^{(1,1)}, g_{\mu\nu}] + \delta^{(0,1)} \mathcal{Q}[\phi^{(0,1)}, h_{\mu\nu}^{(1,0)}] = 0, \quad (8.5)$$

with

$$\delta^{(1,0)} \mathcal{Q}[\phi^{(1,1)}, g_{\mu\nu}] = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \partial^\mu \phi^{(1,1)}) - \mu^2 \phi^{(1,1)} = (\square - \mu^2) \phi^{(1,1)}, \text{ and} \quad (8.6)$$

$$\begin{aligned} \delta^{(1,1)} \mathcal{Q}[\phi^{(0,1)}, h_{\mu\nu}^{(1,0)}] = & -h_{(1,0)}^{\mu\nu} \nabla_\mu \nabla_\nu \phi^{(0,1)} - \frac{h^{(1,0)}}{2} \delta^{(1,0)} \mathcal{Q}[\phi^{(0,1)}, g_{\mu\nu}] \\ & - \left(\nabla_\mu \bar{h}_{(1,0)}^{\mu\nu} \right) \left(\nabla_\nu \phi^{(0,1)} \right). \end{aligned} \quad (8.7)$$

Here, $\bar{h}_{\mu\nu}$ is the trace-reversed metric perturbation and from now on, all covariant derivatives and d'Alembertian operators are defined with respect to the background metric. As $\phi^{(0,1)}$ is a test-field solution on Kerr, the second term on the right-hand side of Eq. (8.6) vanishes. Moreover, as we solve for the $\mathcal{O}(\epsilon)$ metric perturbation in Lorenz gauge, i.e., $\nabla_\mu \bar{h}_{(1,0)}^{\mu\nu} = 0$, the final term in Eq. (8.6) also vanishes, resulting in the expression:

$$\delta^{(1,1)} \mathcal{Q}[\phi^{(1,0)}, h_{\mu\nu}^{(1,0)}] = -h_{(1,0)}^{\mu\nu} \nabla_\mu \nabla_\nu \phi^{(0,1)}. \quad (8.8)$$

Hence Eq. (8.5) gives the leading-order *dynamical* perturbation from the secondary to the scalar field and is governed by

$$(\square - \mu^2) \phi^{(1,1)} = h_{(1,0)}^{\mu\nu} \nabla_\mu \nabla_\nu \phi^{(0,1)}. \quad (8.9)$$

The specialization to Lorenz gauge results in a source that diverges as $1/|\mathbf{r} - \mathbf{r}_p|$ at the position of the secondary. At the level of spheroidal harmonics, this leads to a source that is C^0 continuous, which makes it particularly well-suited for producing extended solutions across the entire domain.

8.2 NUMERICAL PROCEDURE AND VALIDATION

To solve Eq. (8.9), we proceed as follows. First, we construct the m -mode metric data $h_{\mu\nu}^{(1,0)}$ as laid out in Chapter 6 and the background field $\phi^{(0,1)}$ using Leaver's method [138, 139] as described in Chapter 4, both on a numerical two-dimensional (r, θ) grid. To calculate the derivatives of the scalar field, i.e., $\nabla_\mu \nabla_\nu \phi^{(0,1)}$, required for the source, we build a two-dimensional covariant derivative operator using the method of splines. As a consistency check, we contract this quantity with the background metric and confirm that $g_{\text{Kerr}}^{\mu\nu} \nabla_\mu \nabla_\nu \phi^{(0,1)} = \mu^2 \phi^{(0,1)}$ to within machine precision. In the Schwarzschild case, the absence of a stationary threshold configuration of the cloud leads to oscillatory behaviour at the horizon, where the field oscillates as $e^{-i\omega r_*}$. This contrasts with the Kerr case, where the field goes to a constant at the horizon (as $\omega \rightarrow \Omega_H$). To ensure the robustness of our Schwarzschild results, we recompute all quantities in ingoing Eddington-Finkelstein coordinates, which give regular derivatives at the horizon. We find that our results remain unchanged. With the m -mode tensors $h_{(0,1)}^{\mu\nu}$ and $\nabla_\mu \nabla_\nu \phi^{(1,0)}$ validated, we contract them to form the m -mode source term for Eq. (8.9). This source term is projected onto spheroidal harmonics (or spherical harmonics in the Schwarzschild case) using the same projection routines applied earlier.

Using the radial source functions, we solve the radial Klein-Gordon equation using a variation of parameters approach. We first build the “In” and “Up” solution numerically as those which solve the homogenous Klein-Gordon equation with boundary conditions given by

$$\begin{aligned} \lim_{r \rightarrow r_H} R_{\text{In}} &= e^{-i(\Omega_{m_g} + \omega_c - m\Omega_H)r^*}, \\ \lim_{r \rightarrow \infty} R_{\text{Up}} &= \frac{e^{ik_{m_g}r^*} r^{\frac{i\mu^2}{k_{m_g}}}}{\sqrt{a^2 + r^2}}, \end{aligned} \quad (8.10)$$

where $k_{m_g} = \sqrt{(\Omega_{m_g} + \omega_c)^2 - \mu^2}$ and ω_c is the frequency of the cloud. Taking the spheroidal projection of the source, $S_{\ell m}^{(1,1)}$, we directly solve for the scalar field perturbations as,

$$\phi_{\ell m}^{(1,1)}(r) = C_{\ell m}^{\text{Up}}(r)R_{\ell m}^{\text{Up}}(r) + C_{\ell m}^{\text{In}}(r)R_{\ell m}^{\text{In}}(r), \quad (8.11)$$

where

$$\begin{aligned} C_{\ell m}^{\text{Up}}(r) &= \int_{r_H}^r \frac{R_{\ell m}^{\text{In}} S_{\ell m}^{(1,1)}}{\mathcal{W}_0} dr', \\ C_{\ell m}^{\text{In}}(r) &= \int_r^\infty \frac{R_{\ell m}^{\text{Up}} S_{\ell m}^{(1,1)}}{\mathcal{W}_0} dr', \end{aligned} \quad (8.12)$$

and $\mathcal{W}_0$ is the constant Wronskian coefficient given by

$$\frac{\mathcal{W}_0}{\Delta} = R_{\ell m}^{\text{In}} \frac{dR_{\ell m}^{\text{Up}}}{dr} - R_{\ell m}^{\text{Up}} \frac{dR_{\ell m}^{\text{In}}}{dr}. \quad (8.13)$$

We explicitly verify that the behavior of the solution at both boundaries aligns with the boundary conditions of the homogeneous In and Up solutions. To cross-check our code, we insert our source data in a solver from an independent implementation [193], finding consistent results.

The solution of the Klein-Gordon equation (8.9) then gives $\phi_{\ell m}^{(1,1)}$, which we sum over ℓ and m to reconstruct the field profile, which is shown in e.g., Fig. 8.1. Defining $\phi_\ell^{(1,1)} = \sum_{m=-\ell}^\ell \phi_{\ell m}^{(1,1)}$ We find that the field perturbation is finite and continuous at the particle with the ℓ -modes of the perturbation constituting a convergent sequence decaying as ℓ^{-2} . The asymptotes of these extended solutions are then extracted to obtain the input of the flux formulae (A.13), generating the results shown in Fig. 8.2.

8.3 SCALAR FIELD WAKE

Having constructed our solution to the scalar field perturbation figure 8.1 details the wake profile of the matter field due to a secondary perturber on a prograde equatorial circular orbit. The system parameters are chosen deep in the relativistic regime, with orbital radius $r_p = 3.5M$, $a = 0.88M$ and $\alpha = 0.3$ (corresponding to a cloud whose density peaks at $r \sim 20M$). We solve Eq. (8.9) by decomposing in a spheroidal harmonic basis and find a full solution that constitutes a rich wake structure in the azimuthal and equatorial planes. As the secondary scatters matter through the transfer of angular momentum, a *low-density* trail is formed and a spiraling outwash causes matter flux to infinity.

We also analyze configurations at larger separations, where certain modes transition from radiative to bound configurations. At these radii, we observe complex

changes in the cloud morphology, including configurations where a low-density region forms in front of the secondary, with a *high-density* region in its wake. As we always stay in a regime where $\Omega_p < \omega_c$, this contrasts the picture presented in studies on linear motion of BHs in homogeneous media [40, 42, 46, 47, 88, 143, 194], where a trail of over-density is predicted to form in front of the secondary due to its relative velocity with respect to the background. Consequently, applying results from such studies to the binary case would yield incorrect conclusions.

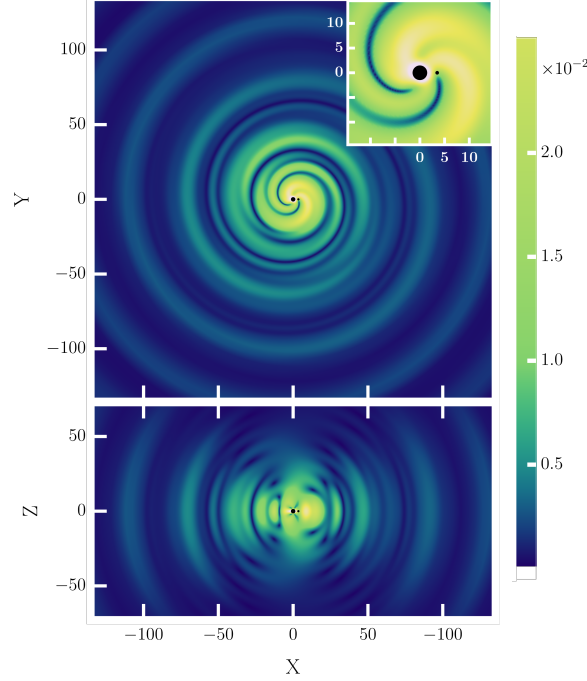


Figure 8.1: We show the absolute value of the perturbed scalar field $|\phi^{(1,1)}|$ for $\ell \geq 2$, taking $\alpha = 0.3$, $a = 0.88M$ and $r_p = 3.5M$. In the top panel, we show an equatorial slice of the field solution, in which the $\hat{Z}$ -axis is aligned with the BH spin. In the bottom panel, we show an azimuthal slice of the field, where the secondary moves “into the plane.”

8.4 RADIATIVE ENERGY LOSS

As the secondary orbits the central BH, its perturbation induces a transfer of energy and angular momentum to the scalar field and into GWs. It is clear from Eq. (8.3) that many dissipative and conservative effects will drive this change in orbital energy. For small mass ratios ϵ , the secondary evolves adiabatically, meaning that the energy dissipated over one orbit is much smaller than the total orbital energy. Consequently, the evolution of the secondary can be approximated by a sequence of geodesics [52, 162, 195–197]. As a first step towards determining how the secondary

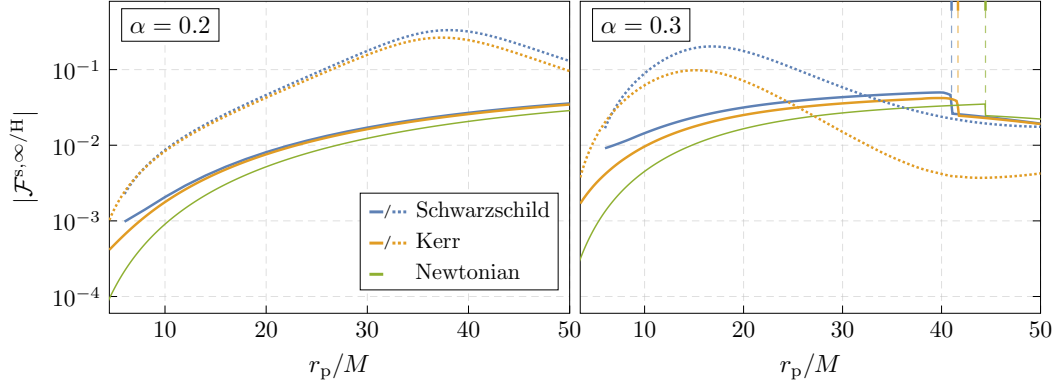


Figure 8.2: We show the total flux to infinity (solid lines) and through the horizon (dotted lines) for a prograde orbit with $\alpha = 0.2$ (left panel) and $\alpha = 0.3$ (right panel). Note that the horizon fluxes are negative on the entire radial domain. The sharp features in the flux to infinity in the right panel, computed using Eq. (8.15), are marked by vertical dashed lines. Note that the Schwarzschild results stop at the innermost stable circular orbit (ISCO) ($r_p = 6M$). We sum up to $\ell = 6$ (5) for the infinity (horizon) fluxes.

moves from one geodesic to the next, we consider the leading-order flux balance of the system:

$$\dot{E}_{\text{orb}} + \dot{M}_c = -\dot{E}^{\text{GW}, \infty} - \dot{E}^{\text{GW}, \text{H}} - \dot{E}^{\Phi, \infty} - \dot{E}^{\Phi, \text{H}}. \quad (8.14)$$

Defining the “scalar flux” of the environment as $\mathcal{F}^{s, \infty/\text{H}} \equiv \epsilon^{-2} \lambda^{-2} (\dot{E}^{\Phi, \infty/\text{H}} + \dot{M}_c^{\infty/\text{H}})$, as described in App. S.1, we calculate its emission to infinity and through the horizon from $\phi^{(1,1)}$, and apply Eq. (8.14) to find $\dot{E}_{\text{orb}}$. Importantly, this equation ignores conservative contributions from $h^{(0,2)}$ and $h^{(1,2)}$, which arise at the same order in the dynamics and does not truly connect to the motion of the small bodies through the geodesic equation attempts to do so will be constructed in later work. In Fig. 8.2, we show the scalar fluxes for $\alpha = 0.2$ and $\alpha = 0.3$, corresponding to highly spinning BHs with $a = 0.69M$ and $a = 0.88M$, respectively. For comparison, we also include the fluxes obtained in the Newtonian case [132–134]. The main features are: (i) as the orbital separation between the binary components grows, effects related to spin of the primary become small for the infinity flux, and Schwarzschild and Kerr become similar. Similarly, the green line, belonging to the Newtonian regime, converges toward the relativistic cases at large radii. We find the relative differences between Kerr and Schwarzschild is tens of percent, reaching 50% near the ISCO.

(ii) the horizon flux exceeds the infinity flux across most of the shown radial domain. It is dominated by the $(\ell, m) = (0, 0)$ mode and always negative, indicating that the binary’s orbit gains energy. This is due to a resonance between bound states of the cloud: the initial $|211\rangle$ state resonates with $|100\rangle$, which has lower energy and angular momentum. This surplus is fed back into the orbit, potentially giving rise to a *floating orbit* [131, 135, 137], where the binary’s evolution is slowed down

or even stalled for a period of time. For $\alpha = 0.3$, the $(\ell, m) = (2, 2)$ mode becomes significant around $r_p \sim 50M$, nearly overtaking the $(0, 0)$ contribution. Unlike the $(0, 0)$ mode, it produces a positive horizon flux, inducing a *sinking orbit*: a period of accelerated inspiral. Such resonances are key observables for probing the cloud's properties, yet they are also efficient at depleting the cloud itself [135, 137].

(iii) the right panel reveals a sharp feature in the flux to infinity, consistent with earlier studies [132–134, 193]. These arise when a new mode starts contributing to the flux, specifically, occurring when

$$r_p^{*,m} = \left(\frac{m - m_b}{\mu - \text{Re}[\omega_c]} - a \right)^{2/3} M. \quad (8.15)$$

Using Eq. (8.15) and calculating ω_c with Leaver's method [138, 139], we find that for $\alpha = 0.3$, $r_p^{*,2}/M = 41.01, 41.66, 44.44$ in the Schwarzschild, Kerr and Newtonian case, respectively, in precise agreement with Fig. 8.2 (vertical dashed lines). Notably, sharp features are absent in the $\alpha = 0.2$ case as they occur at larger radii (e.g. $r_p^{*,2} \sim 100M$). The physical origin of these features lies in the long-range nature of the gravitational potential, as detailed in App. D of [132]. Finally, close to $r_p^{*,m}$, the wavelength of the modes becomes extremely large, which requires the flux to be extracted far out. The small dip in the flux preceding the feature is thus merely a numerical artifact.

(iv) consistent with previous studies [132, 133, 193, 198], we observe that scalar fluxes tend to dominate over gravitational fluxes during the early inspiral stage. As the gravitational and scalar fluxes rely on independent perturbative parameters, a general comparison with the results in Fig. 8.2 should not be made.

8.5 DISCUSSION

In this chapter, we applied the two-parameter formalism to boson clouds and, for the first time, self-consistently compute the perturbation of an EMRI to the environment in the Kerr geometry. We have demonstrated the importance of performing these calculations in Kerr, by comparing them with fluxes in Schwarzschild. For less relativistic environments ($\alpha = 0.2$), where the density peaks at $\sim 50M$, we find relative differences of around 10%, increasing to 50% near the ISCO. In more relativistic environments ($\alpha = 0.3$), these differences are even more significant, attaining 30 – 100% throughout the region where EMRIs are expected to enter the LISA band.

Additionally, we solved for the field perturbations across the entire domain, revealing a rich wake structure induced by the secondary. Our results demonstrate how the morphology of the environment changes with the position of the secondary, emphasizing the intricate and rich dynamics of these systems, which are linked to striking observational signatures with future GW detectors [101, 109, 129, 131–137,

193, 198]. These results raise important questions about existing studies that use linear motion of BHs in a homogeneous medium as a proxy for dynamical friction in a binary inspiral (e.g. [199, 200]), showing that such approximations, or using Schwarzschild as a background is inadequate and will lead to significant errors. Instead, the correct approach in perturbation theory is to use the framework we have developed.

We expect this work to serve as the starting point for self-consistent modeling of EMRIs and environments in Kerr. There are several directions that warrant further exploration in the future. For instance, applying our framework to the Navier-Stokes system would provide a crucial step toward understanding EMRI dynamics in accretion disks in the fully relativistic regime. Another key challenge still lies in understanding the connection between the conservative and dissipative effects of all field perturbations and the evolution of the binary’s orbital parameters. Finally, we have yet to explore the slow-time contributions inherent to these systems. A two-timescale analysis will be necessary to understand all contributions to the evolution of the binaries’ orbital parameters to order $\mathcal{O}(\epsilon^2\lambda^2)$.

Part III

MOTION THROUGH MATTER

Having applied the two-parameter expansion to a boson cloud in the Kerr space-time, we have not only taken an important step towards accurate waveform modelling for EMRIs in gravitational atoms, but also provided the first proof-of-principle demonstration that such calculations can be carried out in the Kerr geometry with relative ease using the modern methods of self-force theory.

In the final *part iii* of the thesis, we build on these results, extending them to the fluid case and pushing beyond merely calculating the matter response and associated fluxes to begin developing a concrete handle on the EMRI dynamics in matter. In this next *paper chapter*, we construct, for the first time to the best of our knowledge, a measure of locally integrated torques on an extended matter field due to a secondary perturber in general relativity. Following Newton’s third law, the quantity governing the evolution of the orbital elements is the integrated work done on the matter by the gravitational field of the secondary; we show that in special environments lacking dissipation mechanisms these locally integrated torques can be strikingly balanced against asymptotic matter fluxes. The final chapter of this thesis will endeavour to make this connection explicit through the construction of a *Relativistic Third Law*.

This *chapter* begins by considering an environment of widespread interest: a thin fluid disc, whose EMRI interaction is commonly parameterised by ill-suited dynamical friction prescriptions [92], where the primary dissipative mechanism is entirely different from the linear dynamical friction picture and is instead set by resonant interactions between the EMRI orbit and differential disc flow, as first studied in the seminal works of Lin and Papaloizou [201] and Goldreich and Tremaine [56, 202]. Goldreich & Tremaine [56, 202] showed that angular momentum exchange in such discs is dominated by resonant interactions between the secondary and the disc. These interactions excite spiral density waves and underpin a large body of work on Newtonian “disc-satellite” interactions [e.g., 57, 203–213], and inducing “migration” of the perturbing body through the disc. The results of these calculations have been widely applied to the formation of planetary system architectures [e.g., 214], as well as a wide range of other applications including EMRI dynamics [e.g., 94, 95, 215].

Hirata [55, 216] extended some of these approaches to relativistic EMRIs using a Hamiltonian-based torque balance formalism. Such Hamiltonian approaches have recently been applied to studying torque exchange in collisionless discs [217–219]. However, relativistic studies have yet to perform a full fluid calculation self-consistently for EMRIs.

The goal of this work is to calculate the fluid perturbations and associated torque densities in an EMRI system in the spacetime of a spinning black hole, which we achieve by combining the two-parameter expansion framework [61, 193, 220–223] with self-force theory, Newtonian planetary migration methods, and the master enthalpy formalism of Ipser & Lindblom [224, 225]. We consider an adiabatic disc with a radial temperature gradient around a Kerr black hole, in the presence of a much less massive companion at strong relativistic separations of order 10 gravitational radii, restricting to the regime where the secondary mass is smaller than the thermal mass of the disc [226] and the linearised fluid prescription holds. We then demonstrate the balance between the local torque density and asymptotic fluxes in the fluid.

The contents of this chapter primarily constitute those of an article currently under review, *Phys. Rev. D* (2026), co-authored with Daniel J. D’Orazio.

9.1 SET-UP AND NON-LINEAR EQUATIONS OF MOTION

We begin with the Einstein equations coupled to matter,

$$G_{\mu\nu} = T_{\mu\nu}^{fl} \quad (9.1)$$

modelled as a perfect fluid,

$$T_{fl}^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + pg^{\mu\nu}, \quad (9.2)$$

with u^μ describing the four flow of the fluid, p is the pressure, and $\epsilon = \rho(c^2 + \epsilon)$ is the combination of rest-mass density and internal energy density of the fluid. The equations for conservation of energy-momentum are given by

$$\nabla_\mu T_{fl}^{\mu\nu} = 0, \quad (9.3)$$

taking the explicit form,

$$(\epsilon + p)A^\nu = -\nabla^\nu p - u^\nu \nabla_\mu ([\epsilon + p]u^\mu), \quad (9.4)$$

where $u^\mu \nabla_\mu u^\nu = A^\nu$ is the four acceleration of the fluid element. Due to the normalisation of the four-velocity $u^\mu u_\mu = -1$, we must satisfy the condition $A^\mu u_\mu = 0$. We define the projection tensor

$$q^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu. \quad (9.5)$$

Contraction with this tensor picks out elements of a four-vector orthogonal to the fluid flow and satisfies the conditions $q^{\mu\nu} = q^{\mu\alpha}q_\alpha^\nu$, $q_\alpha^\mu A^\alpha = A^\mu$, and $q_\alpha^\mu u^\alpha = 0$. Beginning from (9.4), we can contract with u_μ and q_μ^α respectively to obtain the relativistic energy and momentum conservation equations,

$$(\epsilon + p)\nabla_\mu u^\mu + u^\mu \nabla_\mu \epsilon = 0, \quad (9.6)$$

$$(\epsilon + p)u^\mu \nabla_\mu u^\nu + q^{\mu\nu} \nabla_\mu p = 0. \quad (9.7)$$

9.2 PERTURBATIVE SCHEME

In order to gain a handle on the problem, we elect to take a perturbative approach. In many astrophysical contexts, it is possible to perform expansions around some background geometry. Take a metric parametrically close to the Kerr spacetime [75], then we can expand our full metric in the form,

$$g_{\mu\nu} = g_{\mu\nu}^{\text{kerr}} + h_{\mu\nu}. \quad (9.8)$$

In the context we are considering here, the implicit assumption is that the effect on the geometry due to the small body and the matter field is perturbatively small

in comparison to the background. Here, $g_{\mu\nu}^{\text{kerr}}$ is the Kerr metric and, $h_{\mu\nu}$ contains the collection of all perturbative content. In order for this form of expansion to be valid, we must specify that the stress energy tensor of the isolated fluid in the absence of the secondary is also subleading and controlled by a small parameter [61, 193, 220–222]. Unlike in [61], we leave this parameter unspecified as an $\mathcal{O}(1)$ counting parameter λ . Next, we wish to introduce the effect of the secondary black hole, which can be developed formally through matched asymptotic expansions and the self-consistent formalism of Pound [177], however, for the purposes of this work we take the skeletonised approach which re-expresses the vacuum content of the smaller black hole as a point particle sourcing term to give,

$$G_{\mu\nu} = \lambda T_{\mu\nu}^{fl} + \epsilon T_{\mu\nu}^{pp}, \quad (9.9)$$

where ϵ is now a $\mathcal{O}(1)$ counting parameter associated with the mass ratio between the secondary and central black hole, $m_p/M \ll 1$, and

$$T_{pp}^{\mu\nu} = m_p \int \dot{x}^\mu(\tau) \dot{x}^\nu(\tau) \frac{\delta(x^\mu - x^\mu(\tau))}{\sqrt{-g}} d\tau, \quad (9.10)$$

where m_p is the secondaries mass, $\dot{x}^\mu$ its four-velocity and τ the proper time along the curve. Due to the nature of the fluid stress-energy tensor in Eq. (9.2), at leading order the fluid variables have the scalings, $\epsilon \sim \mathcal{O}(\lambda)$, $p \sim \mathcal{O}(\lambda)$, and $u^\mu \sim \mathcal{O}(1)$. A perturbative expansion provides the form of the metric perturbation,

$$h_{\mu\nu} = \sum_{|n|+|m|\geq 1} \epsilon^m \lambda^n h_{\mu\nu}^{(m,n)},$$

which allows us to expand Eqs. (9.6) and (9.7) and collect the $\mathcal{O}(\epsilon\lambda)$ expressions to give,

$$(\epsilon + p) \nabla_\mu u^\mu + u^\mu \nabla_\mu \epsilon = -\frac{1}{2} (\epsilon + p) u^\mu \nabla_\mu h^{(1,0)}, \quad (9.11)$$

$$(\epsilon + p) u^\mu \nabla_\mu u^\nu + q^{\mu\nu} \nabla_\mu p = q_\mu^\nu f_{(1,1)}^\mu. \quad (9.12)$$

where,

$$f_\mu^{(1,1)} = (\epsilon + p) u^\alpha u^\beta \left(\frac{h_{\alpha\beta;\mu}^{(1,0)}}{2} - h_{\mu\alpha;\beta}^{(1,0)} \right) + h_{\mu\nu}^{(1,0)} \partial^\nu p. \quad (9.13)$$

Here, all gradients and projectors are now defined with respect to the background Kerr metric and nonlinear fluid flow. After the expansion we have reprojected the second equation with the projector tensor to retain the orthogonal nature of the four-acceleration to the four-flow.

At this point, we have linearised the fluid equation in the metric degrees of freedom whilst keeping the fluid degrees of freedom non-linear. In principle, the equations at this point could be incorporated into a numerical relativistic hydrodynamics solver on a static metric [e.g., AthenaK 227]. For the purposes of this work, however, we now turn to a simplified, linearised fluid prescription.

s.1 Linearising the Fluid

In line with [220] and our previous motivation we expand the fluid variables as

$$\begin{aligned}\epsilon &= \sum_{n \geq 1, m \geq 0} \epsilon^m \lambda^n \epsilon^{(m,n)}, \\ p &= \sum_{n \geq 1, m \geq 0} \epsilon^m \lambda^n p^{(m,n)}, \\ u^\mu &= \sum_{n \geq 0, m \geq 0} \epsilon^m \lambda^n u_{(m,n)}^\mu.\end{aligned}$$

Here, $\epsilon^{(0,1)}$, $p^{(0,1)}$, and $u_{(0,0)}^\mu$ constitute the background fluid quantities on the Kerr geometry in the absence of the secondary BH and self-gravity. Having also introduced the expansion of our fluid variables, we are now in a position to linearise the fluid elements in Eq. (9.11) and obtain the $\mathcal{O}(\epsilon\lambda)$ component of the linearised Euler equations,

$$\left(\epsilon^{(1,1)} + p^{(1,1)}\right) \nabla_\mu u^\mu + (\epsilon + p) \nabla_\mu u_{(1,0)}^\mu + u^\mu \partial_\mu \epsilon^{(1,1)} \quad (9.14)$$

$$+ u_{(1,0)}^\mu \partial_\mu \epsilon = -\frac{1}{2}(\epsilon + p) u^\mu \nabla_\mu h^{(1,0)}, \quad (9.15)$$

$$\left(\epsilon^{(1,1)} + p^{(1,1)}\right) u^\mu \nabla_\mu u^\nu + (\epsilon + p) \left(u^\mu \nabla_\mu u_{(1,0)}^\nu + u_{(1,0)}^\mu \nabla_\mu u^\nu\right) \quad (9.16)$$

$$+ q^{\mu\nu} \nabla_\mu p^{(1,1)} + u^\nu u_{(1,0)}^\mu \nabla_\mu p = q_{\mu}^\nu f_{(1,1)}^\mu. \quad (9.17)$$

For simplicity of notation, any occurrence of a background fluid quantity, $\epsilon^{(0,1)}$, $p^{(0,1)}$, and $u_{(0,0)}^\mu$ has had the counting indices removed, all occurrences of the covariant derivatives and projectors are now related to the Kerr metric and the background fluid flow, and $h^{(1,0)} = g^{\mu\nu} h_{\mu\nu}^{(1,0)}$ is the trace of our metric perturbation.

9.3 MASTER ENTHALPY EQUATION

The use of a master enthalpy equation in studying disc dynamics in the presence of a secondary perturber has seen wide use in Newtonian studies. From the seminal works of Goldreich and Tremaine [56, 202] to more recent works understanding

the outflow conditions and extended density structure of the disc [209–211, 228–231]. In the relativistic context, seminal works, such as that of Ipser and Lindblom [224, 225], have derived relativistic forms of the master enthalpy equation for, non-barotropic, self-gravitating perfect fluid flows. In the following section, we connect these approaches by deriving the fully relativistic master enthalpy equation describing perturbations of a perfect fluid induced by an independently sourced metric perturbation (i.e. due to a small secondary perturber) in the Kerr spacetime. The derivation follows closely that of Ipser and Lindblom [224] with some minor deviations; due to this and the novelty of its usage in the context of a secondary perturber, we detail our version of the derivation in full.

As is often crucial in the derivation of such a master equation we will require some level of symmetry in our background flow in order for our linearised fluid equations to sufficiently simplify and become amenable to re-expressing as single scalar equation. In particular, in the case of a relativistic disc, we will require that our background flow follows the isometries of the Kerr geometry, i.e, that it is stationary and axi-symmetric and can be written as,

$$u_{(0,0)}^\mu = \alpha (t^\mu + \Omega \phi^\mu), \quad (9.18)$$

where in Boyer-Lindquist (BL) coordinates, t^μ and ϕ^μ represent the coordinate Killing vectors of the Kerr geometry, and α and Ω are generic functions of the radial and polar (r and θ) BL coordinates. With these restrictions on the background flow, at leading order the momentum equation simplifies to,

$$u_{(0,0)}^\mu \nabla_\mu u_{(0,0)}^\nu = -\frac{\nabla^\mu p}{\epsilon + p} \equiv -R^\mu. \quad (9.19)$$

To this point, our equations of motion are still underdetermined. In order to remedy this issue we must choose an equation of state in the co-moving Lagrangian frame of the fluid. We begin by denoting the leading order Lagrangian perturbations by Δp , $\Delta \epsilon$. One can then “close” the system by enforcing that local adiabatic perturbations to the fluid take the form,

$$\Delta p = c_s^2 \Delta \epsilon \quad (9.20)$$

By allowing c_s to be an arbitrary axi-symmetric, stationary function, and identifying it with the sound-speed given by the background relation, $p = c_s^2 \epsilon$, we fix our fluid to be in a $\gamma \rightarrow 1$ adiabatic state which as discussed in [210, 232] is distinct from the standard local isothermal approximation. To relate the Lagrangian and Eulerian perturbations of our fluid we write $\Delta p = p^{(1,1)} + \mathcal{L}_\xi p$ and $\Delta \epsilon = \epsilon^{(1,1)} + \mathcal{L}_\xi \epsilon$, where $\mathcal{L}_\xi$ is the Lie derivative along the Lagrangian displacement vector field ξ . For adiabatic perturbations, we can define the displacement vector between the perturbed and background flow through,

$$u_{(1,0)}^\nu = -(\mathcal{L}_\xi u)^\nu = u^\mu \partial_\mu \xi_{(1,0)}^\nu - \xi_{(1,0)}^\mu \partial_\mu u^\nu. \quad (9.21)$$

Using that perturbations are assumed to take the form $\sim e^{im\phi - i\omega t}$ and $\tilde{\omega} = m\Omega - \omega$, this can be reduced to the form,

$$u_{(1,0)}^\mu = \xi_{(1,0)}^\nu (i\tilde{\omega}\alpha\delta_\nu^\mu - \partial_\nu u^\mu). \quad (9.22)$$

Due to the assumed symmetries of the background, the matrix $\partial_\nu u^\mu$ is nilpotent, $\partial_\nu u^\mu \partial_\rho u^\nu = 0$. Hence, given $M^\mu_\nu = \delta^\mu_\nu - \partial_\nu u^\mu / (i\tilde{\omega}\alpha)$, we have $(M^{-1})^\mu_\nu = \delta^\mu_\nu + \partial_\nu u^\mu / (i\tilde{\omega}\alpha)$.

Specifying the form of u^μ we then obtain the equations,

$$\xi_{(1,0)}^\mu = \frac{1}{i\tilde{\omega}\alpha} \left(\delta_\nu^\mu + \frac{\partial_\nu u^\mu}{i\tilde{\omega}\alpha} \right) u_{(1,0)}^\nu. \quad (9.23)$$

Now that we have determined our displacement vector in terms of the fluid perturbations, we are in a position to relate our density perturbation to our pressure perturbation in the Eulerian frame:

$$\epsilon^{(1,1)} = \Delta\epsilon - \xi^\mu \partial_\mu \epsilon = \frac{1}{c_s^2} \left[p^{(1,1)} + \xi_{(1,0)}^\mu \partial_\mu p \right] - \xi_{(1,0)}^\mu \partial_\mu \epsilon. \quad (9.24)$$

Inputting the explicit form of Eq. (9.23), one finds the expression,

$$\epsilon^{(1,1)} = (\epsilon + p) \left[\frac{p^{(1,1)}}{(\epsilon + p)c_s^2} - \frac{B_\nu u_{(1,0)}^\nu}{i\tilde{\omega}\alpha} \right], \quad (9.25)$$

where,

$$B_\mu = \frac{1}{\epsilon + p} \left(\partial_\mu \epsilon - \frac{\partial_\mu p}{c_s^2} \right) = -\frac{\partial_\mu \log c_s^2}{1 + c_s^2}, \quad (9.26)$$

corresponds precisely to terms like Eq. 18 in [233], describing the thermal structure of the disc.

Having closed our system of equations, we are equipped to express our fluid perturbations in terms of a single master variable: an enthalpy-like perturbation $\mathfrak{h}^{(1,0)}$, defined through the identification,

$$p^{(1,1)} = (\epsilon + p)\mathfrak{h}^{(1,0)}, \quad (9.27)$$

leading to the expression,

$$\epsilon^{(1,1)} = (\epsilon + p) \left[\frac{\mathfrak{h}^{(1,0)}}{c_s^2} - \frac{B_\nu u_{(1,0)}^\nu}{i\tilde{\omega}\alpha} \right], \quad (9.28)$$

for the energy density perturbation. Our final task is to re-express the velocity perturbations in terms of the enthalpy scalar. We achieve this by substituting Eqs. (9.27) and (9.28) into Eq. (9.16). In doing so, it becomes apparent that the

assumed symmetries of our background flow allow the resulting equation to become purely algebraic in $u_{(1,0)}^\mu$, taking the form,

$$\left(\mathcal{Q}^{-1}\right)_\mu^\nu u_{(1,0)}^\mu = \frac{q_\mu^\nu f_{(1,1)}^\mu}{\epsilon + p} - q^{\nu\mu} \nabla_\mu \mathfrak{h}^{(1,0)} - \mathfrak{h}^{(1,0)} B^\nu, \quad (9.29)$$

where we have defined the operator,

$$\left(\mathcal{Q}^{-1}\right)_\mu^\nu = i\tilde{\omega}\alpha\delta_\mu^\nu + \Gamma_{\alpha\mu}^\nu u^\alpha + \nabla_\mu u^\nu + u^\nu R_\mu + \frac{R^\nu B_\mu}{i\alpha\tilde{\omega}}.$$

Inverting $\mathcal{Q}^{-1}$, we have now expressed all fluid quantities in terms of known expressions and the master enthalpy-like perturbation. Substituting Eqs. (9.27), (9.28), and (9.29) into the energy equation (9.14) yields a scalar equation expressing all perturbed fluid quantities purely in terms of the master variable $\mathfrak{h}^{(1,0)}$ and lower-order quantities, as in [61, 193].

The explicit form of the master equation is then given by,

$$c_s^2 \nabla_\mu \left(\mathcal{Q}_\alpha^\mu q^{\alpha\nu} \nabla_\nu \mathfrak{h}^{(1,0)} \right) + [R_\mu \mathcal{Q}_\alpha^\mu q^{\alpha\nu} + c_s^2 \mathcal{Q}_\alpha^\nu B^\alpha] \nabla_\nu \mathfrak{h}^{(1,0)} \quad (9.30)$$

$$- [i\alpha\tilde{\omega} - (c_s^2 \nabla_\mu + R_\mu) \mathcal{Q}_\alpha^\mu B^\alpha] \mathfrak{h}^{(1,0)} = \mathfrak{F}, \quad (9.31)$$

with source,

$$\mathfrak{F} = \frac{i\alpha\tilde{\omega}c_s^2}{2} h^{(1,0)} + [c_s^2 \nabla_\mu + R_\mu] \left(\frac{\mathcal{Q}_\nu^\mu q_\alpha^\nu f_{(1,1)}^\alpha}{\epsilon + p} \right), \quad (9.32)$$

and where $q_\nu^\mu f_{(1,1)}^\nu$ is the effective force of the metric perturbation on the fluid given by Eq. (9.13). Unlike in [224, 225], we have derived our equations in terms of the full fluid perturbation as opposed to its projected form $\hat{u}_{(1,0)}^\mu = q_\nu^\mu u_{(1,0)}^\nu$, allowing for explicit inversion of $\mathcal{Q}^{-1}$. This choice has the consequence of not automatically conserving the normalisation of our four-velocity ($u_\mu u^\mu = -1$) to order $\mathcal{O}(\epsilon^2)$. We remedy this by introducing a perturbative non-affine re-parameterisation of our four-velocity,

$$\tilde{u}^\mu(\lambda) = u^\mu(\chi + \epsilon f(\chi)) = u^\mu(\chi) - \epsilon f'(\chi) u^\mu(\chi), \quad (9.33)$$

where $\tilde{u}$ is the re-parameterised four-velocity. This freedom can be used to fix $\tilde{u}^\mu \tilde{u}_\mu = -1 + \mathcal{O}(\epsilon^2)$, which is straightforwardly achieved by setting,

$$f' = - \left(u_{(0,0)}^{(0,0)} u_{(1,0)}^\mu + \frac{1}{2} h_{\mu\nu}^{(1,0)} u_{(0,0)}^\mu u_{(0,0)}^\nu \right), \quad (9.34)$$

giving an altered description of our four-velocity,

$$\tilde{u}^\mu = u_{(0,0)}^\mu + \epsilon \left(u_{(1,0)}^\mu - f' u_{(0,0)}^\mu \right). \quad (9.35)$$

In our case, such a re-parameterisation does not affect the scalar fluid variables ϵ and p , due to their invariance under propagation along the background flow (i.e. $u_{(0,0)}^\mu \nabla_\mu p = 0$). Having reconstructed our perturbed four-velocity through the master enthalpy formalism, we therefore make the replacement,

$$u_{(1,0)}^\mu \rightarrow u_{(1,0)}^\mu - f' u_{(0,0)}^\mu. \quad (9.36)$$

9.4 OPERATOR PROJECTION AND ANALYSIS

Having obtained the master enthalpy equation (9.30) and all necessary background and metric contributions, we are now ready to solve it. In its current form, it is a second-order linear partial differential equation. Rather than solving it directly, we make use of the thin disc profile of our background solution to reduce the problem to a system of ordinary differential equations. To this end, we perform a vertical (height) averaging as in Potter [234] and adopt a modified local approximation (as described in Tanaka and Okada [212]) of the master enthalpy equation (9.30), denoting this combination of approaches the Projected Modified Local Approximation (PMLA). This procedure reduces the problem to a system of uncoupled radial ordinary differential equations, corresponding to solving the fully relativistic analogue of the dominant $n = 0$ mode identified in [212]. In this sense, we solve for the dominant cylindrical contribution to the three-dimensional fluid configuration rather than adopting a two-dimensional disc set-up *a priori*. We begin by taking the thin disc solution constructed in Chapter 4 and introducing the height-averaging procedure explicitly.

We follow closely the approaches in previous Newtonian works, where we consistently height-average our equations using the operator,

$$\langle \cdot \rangle = \int \frac{(\cdot) e^{-\frac{z^2}{2h^2}}}{\sqrt{2\pi}h} dz, \quad (9.37)$$

where $z = \cos \theta$ and h is fixed to be a constant and defined through in chapter 4. We simultaneously re-expand in the aspect ratio and retain all terms up to and including h^2 in the modified local approximation, i.e., the PMLA. The resulting equations can be seen as similar to governing the solution to the zeroth order term in a Hermite polynomial expansion of the polar degree of freedom. The notebooks providing the explicit form of these projected equations will be provided publicly in forthcoming work. In order to obtain a decoupled system of equations at this level we also take advantage of the up-down symmetry inherent in our background disc and equatorial orbits. This property prevents the existence of any terms of the form h or z in our master equation; problematic terms of the form $z \partial_z \mathfrak{h}$ or $\partial_z^2 \mathfrak{h}$ still arise however. We overcome the existence of such terms by integrating by parts until no further poloidal derivatives act on the master variable within the averaging

operation defined by Eq. (9.37). After averaging Eq. (9.30), appropriately reducing the derivatives through integration by parts, and re-expanding our equation to $\mathcal{O}(h^2)$, we arrive at a system of decoupled ordinary differential equations,

$$\mathcal{L}_m[\mathfrak{h}(r)] = \mathfrak{F}_m(r), \quad (9.38)$$

within the PMLA. We have reduced $\mathcal{L}_m[\mathfrak{h}(r)]$ and $\mathfrak{F}_m(r)$ such that the coefficient of the principal piece is moved into the sourcing function, leading to a form,

$$\mathfrak{h}_m''(r) + A_m(r)\mathfrak{h}_m'(r) + B_m(r)\mathfrak{h}_m(r) = \mathfrak{F}_m(r). \quad (9.39)$$

At the level of the operator, we retain the leading two orders in h , given by $A_m \sim A^{(0)}(r) + \mathcal{O}(h^2)$ and $B_m = B^{(-2)}(r)/h^2 + B^{(0)}(r) + \mathcal{O}(h^2)$.

We have now dropped the perturbative indices for readability. The term $B^{(0)}(r)$ contains all information regarding the pressure and thermodynamic structure of the background disc. All other terms are purely geometric, arising from the geodesic structure of the Kerr spacetime. We solve this system of equations using the standard variation-of-parameters approach, where we first build the Green's function for the system through its homogeneous solutions and then directly integrate against the source. Finally, we apply the same approximations to construct the source function, with the key caveat that, rather than appropriately height-averaging the full metric perturbation due to the secondary, we instead take directly the equatorial value of the metric, similarly to [234] and its m -mode perturbations. All other quantities in the source are consistently height-averaged.

In constructing solutions to Eq. (9.39), it is useful to first analyse the structure of the differential operator and its associated Wronskian. The Wronskian can be written as,

$$\mathcal{W}_m(r) = \mathcal{W}_m^p e^{-\int^r A_m(s) ds}. \quad (9.40)$$

Here $A_m(s)$ is as defined in Eq. (9.39), and $\mathcal{W}_p$ is an unspecified constant that will be fixed by boundary conditions. Notably, even given the complex structure of our master enthalpy equation, it is possible to find an analytic solution for the Wronskian, given by

$$\mathcal{W} = \mathcal{W}_p \frac{m^2 \left(\Omega_r^k - \Omega_{r_p}^k \right)^2 - \kappa_r^2}{(2a + (r-3)\sqrt{r}) (\Omega_r^k)^2 (\Omega_{r_p}^k)^2 \Delta_r} \sqrt{r}. \quad (9.41)$$

Here we have defined the radial epicycle frequency in Kerr by [235],

$$\kappa_r^2 = \frac{-3a^2\sqrt{r} + 8ar + (r-6)r^{3/2}}{r^{5/2} (a + r^{3/2})^2}. \quad (9.42)$$

The circular geodesic frequency in Kerr is now given by its purely equatorial form, $\Omega^k = 1/(a + r^{3/2})$, and the pressure and height corrections present have

been re-expanded and now arise as subleading terms, beyond second order in the expansion of $A_m(s)$.

We find that the Wronskian vanishes precisely at points where the difference in angular frequency between the secondary and a fluid element coincides with the fluid's radial epicycle frequency, leading to potential degeneracies in the solution space at these locations. These points are commonly known as Lindblad resonance locations and have been extensively studied in the literature [56, 202]. In nearly Keplerian discs, these resonances are well known to be the primary mechanism for the transfer of energy and angular momentum between the secondary and the fluid. For a given m -mode, each resonance occurs as a pair: one at an orbital radius interior to the secondary's orbit and one at an orbital radius exterior to it.

As shown in Fig. 9.1, for a given m -mode the region between these two locations (which act as turning points) contains a negative effective potential in equation (9.39). In this region, fluid perturbations exhibit exponential growth where waves are “excited”. Outside these turning points, the potential becomes positive, yielding wavelike solutions that propagate through the disc.

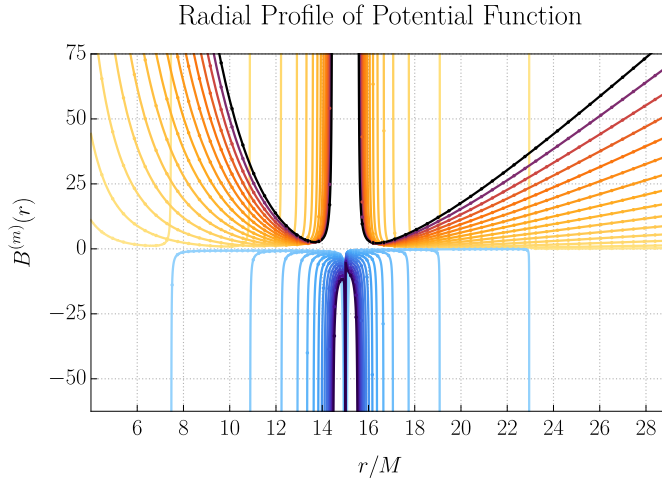


Figure 9.1: Plot exemplifying the dominating structure of the potential piece of the enthalpy wave equation for the fiducial parameter set, $r_p = 15$, $h = 0.1$, and $a = 0.6$, for a range of m -modes. Key to this plot is the identification of wave regions, i.e., when the potential is positive, as opposed to excitation regions where the potential is negative. Naturally, for higher m -modes, the excitation region contracts towards the location of the secondary. Here the lightest lines, with divergences further from the particle location, correspond to the $m = 1$ potential while subsequent darker lines, which diverge closer to the secondary, represent the potential of the $m = 2$, $m = 3 \dots, m = 15$ potential functions. The differential in colour schemes is simply to make clear the positions at which each equation transitions from a wavelike to an exponentially growing character.

Importantly, the PMLA does not eliminate the effect of pressure, which would normally shift the resonance locations; instead, it effectively re-expands these shifts and places them in the potential describing the wave zone, so that mode propagation can always occur at a finite distance from the secondary, resulting in convergent torques and mode sums. Nevertheless, for the purposes of singularity analysis, the divergence location is simplified [236, 237].

The positions of relativistic Lindblad resonances differ significantly from their Newtonian counterparts. In Fig. 9.2, we show the regions of exponential growth bounded by the resonant points. The region between Lindblad resonances is shown starting far from the central object for the $m = 1$ resonances, and darkening each time a new pair of resonances is crossed.

Notably, in the Newtonian case, the $m = 1$ resonance does not have an inner counterpart, since its location is fixed at $r = 0$. In the relativistic case, all spin values produce an inner resonance for that mode, which can subsequently contribute to the system's dynamics. Moving from Schwarzschild to Kerr, spin generally widens the separation between resonant locations (which we attribute to the decrease in total torque with spin, as found in [217, 218]). In addition, spin draws the inner resonance closer to the central black hole at a faster rate than it pushes the outer resonance outward.

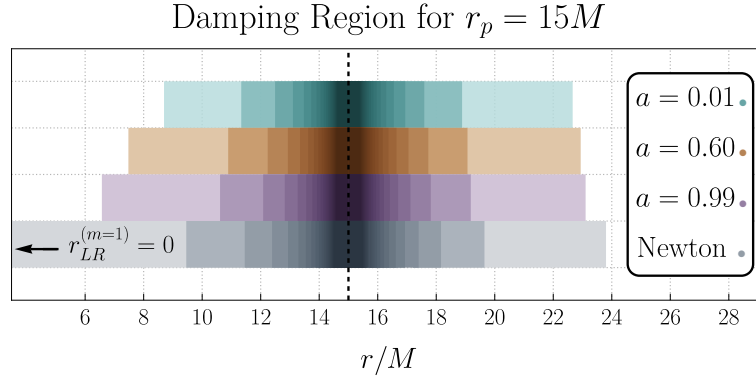


Figure 9.2: Illustration of the excitation regions arising between Lindblad resonances for a range of black hole spins at $r_p = 15M$, and for a reference Newtonian analogue. The outermost region, shown as the lightest (non-white) colour on each bar, corresponds to the area between the $m = 1$ inner and outer Lindblad resonances. Moving inward from the edges toward the secondary location, each successive darkening of the colour indicates crossing an additional resonant point. This colour progression provides a visual representation of how the resonance structure and excitation regions evolve with black hole spin compared to the Newtonian case.

An important but often overlooked aspect in relativistic disc-EMRI studies is the role of angular momentum transfer between the orbiting body and fluid elements at these resonant locations. In systems with EMRIs in near co-rotation with the

fluid it is these interactions, rather than classical dynamical friction, that dominate the system's evolution. Consequently, strong-field EMRI calculations that model disc effects through an approximated dynamical-friction force are almost always fundamentally inadequate.

In order to build the solutions of Eq. (9.39) we must first define the physical boundary conditions for our master equation $\mathcal{L}[\mathfrak{h}_m] = 0$. We fix ingoing and outgoing wave conditions at the ISCO and at large radius $r_{\max}$, respectively, thus enforcing that we do not have reflections from waves at the boundaries of our domain. Similarly to [210, 231, 238], we perform a second-order WKB approximation of the field at the boundaries, to give

$$\frac{d\mathfrak{h}_m^\pm}{dr} = (\pm ik_m - \frac{1}{2k_m} \frac{dk_m}{dr} - \frac{1}{2} A_m) \mathfrak{h}_m^\pm, \quad (9.43)$$

corresponding to outgoing (+) solutions at $r_{\max}$ and ingoing solutions (−) at the ISCO. IN this equation we have defined the wavenumber as,

$$k_m^2(r) = B_m - \frac{1}{2} \frac{dA_m}{dr} - \frac{1}{4} A_m^2. \quad (9.44)$$

Having specified the ingoing and outgoing boundary conditions, we are now ready to solve for the homogeneous solutions required for our variation-of-parameters approach. The homogeneous solutions however exhibit integrable divergences at co-rotation ($r = r_p$), which must be treated before proceeding.

The co-rotation resonance is another point of extensive interest in the planetary migration community, arising at the point where the orbital frequency of fluid elements matches the frequency of the secondary perturber. Performing a local expansion of our height-averaged equations, we find that our master equation at that point takes the following leading-order form,

$$\mathcal{L}[\mathfrak{h}]|_{r \rightarrow r_p} = \frac{d^2 \mathfrak{h}}{dr^2} - \frac{4\kappa_{r_p}^2}{9m^2 r_p \Delta_{r_p} (\Omega_{r_p}^k)^4 (r - r_p)^2} \mathfrak{h}. \quad (9.45)$$

where r_p is the location of the secondary. We find that our operator develops a regular-singular point at this location, making it ill-suited for numerical implementation. Notably, we do not find that an irregular-singular point arises at co-rotation, as in Tanaka and Okada [212]. Although the singularity here is a regular-singular point, numerical integration through it is still troublesome. We circumvent this issue by removing its singular behaviour analytically, solving numerically for the residual regularised field, and reinstating the singularity after the fact. Performing a local analysis of the operator $\mathcal{L}_m[\mathfrak{h}_m] = 0$ around the point $r = r_p$, we find two independent solutions, one regular and one singular. The singular solution takes the form,

$$S_m^p = (r - r_p)^{\frac{1}{2}} \left(1 - \sqrt{1 + \frac{16\kappa_{r_p}^2}{9m^2 r_p \Delta_{r_p} (\Omega_{r_p}^k)^4}} \right). \quad (9.46)$$

For fiducial parameters $a = 0.6M$, $r_p = 15M$, and $m = 2$, we find $S_m^p \sim (r - r_p)^{-0.08}$ giving rise to an integrable divergence. In order to remove this singularity, we substitute $\mathfrak{h}_m = S_m^p \psi_m$ into Eq. (9.39), obtaining a new equation $\hat{\mathcal{L}}[\psi_m] = 0$, which is now amenable to simple inbuilt ODE solvers in Mathematica.

9.5 ENTHALPY SOLUTIONS

Having appropriately rescaled the master equation through the introduction of the divergent factor in Eq. (9.46), and similarly rescaled the boundary conditions, we compute the homogeneous solutions for ψ_m corresponding to both ingoing and outgoing boundary conditions on a domain running from the ISCO (r_I) to some $r_{\max} \sim 100M$. We then analytically reinstate the correct singularity structure at co-rotation, yielding the full homogeneous ingoing ($\mathfrak{h}_m^+(r)$) and outgoing ($\mathfrak{h}_m^-(r)$) solutions across the computational domain. In Fig. 9.3, we present these solutions, with ingoing modes shown in *blue* and outgoing modes in *orange*. The absolute value of the ingoing solutions displays smooth behaviour within the domain interior to the particle (opaque blue), while it shows rapidly oscillatory behaviour in the exterior domain (translucent blue). The opposite pattern is observed for the outgoing solutions. As in Fig. 9.1, the lightest lines in each colour scheme correspond to the $m = 1$ mode, with successively darker lines representing $m = 2, 3, \dots$, up to $m_{\max} = 15$.

Having obtained our homogeneous solutions we build the full particular solution through the method of variation of parameters, whereby our solution is written in the form,

$$\mathfrak{h}_m(r) = C_m^+(r)\mathfrak{h}_m^+(r) + C_m^-(r)\mathfrak{h}_m^-(r), \quad (9.47)$$

with,

$$C_m^+(r) = \int_{r_I}^r \frac{\mathfrak{h}_m^-(s)\mathfrak{F}_m(s)}{\mathcal{W}_m^+(s)} ds, \quad (9.48)$$

$$C_m^-(r) = \int_r^{r_{\max}} \frac{\mathfrak{h}_m^+(s)\mathfrak{F}_m(s)}{\mathcal{W}_m^-(s)} ds. \quad (9.49)$$

The Wronskian is defined as in Eq. (9.40), and the associated Wronskian coefficient, $\mathcal{W}_p$, is fixed via

$$\mathcal{W}_p^\pm \equiv \mathfrak{h}^- \frac{d\mathfrak{h}^+}{dr} - \mathfrak{h}^+ \frac{d\mathfrak{h}^-}{dr} \Big|_{r=r_{\max}, r=r_I}. \quad (9.50)$$

In practice, we compute two distinct Wronskian coefficients: $\mathcal{W}_p^+$ evaluated at the outer boundary $r = r_{\max}$, and $\mathcal{W}_p^-$ evaluated at the inner boundary $r = r_I$. The introduction of separate Wronskian coefficients for the C^+ and C^- normalisation constants is motivated purely by numerical considerations. In particular, we find that the delicate cancellations required to reconstruct the physical fluid variables

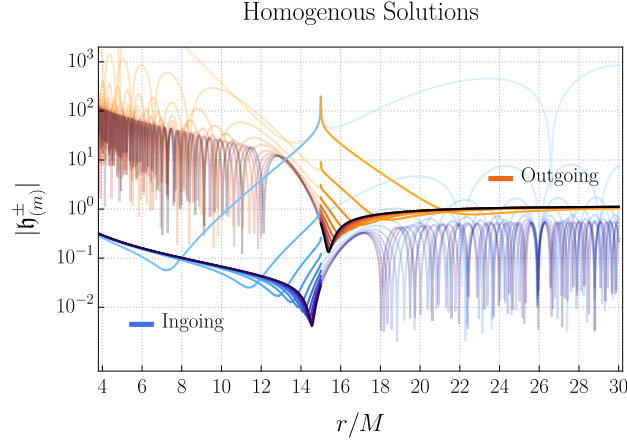


Figure 9.3: The independent m -mode solutions to the homogeneous enthalpy equations using the same lighter-to-darker colour scheme employed in previous figures to indicate increasing m -mode contributions. This example corresponds to the fiducial parameter set $r_p = 15$, $h = 0.1$, and $a = 0.6$. Solutions satisfying outgoing boundary conditions at the outer boundary are shown in *orange*, while those satisfying ingoing boundary conditions at the inner boundary are shown in *blue*. The ingoing solutions exhibit smooth behaviour within the domain interior to the particle (opaque blue) and rapidly oscillatory behaviour in the exterior domain (translucent blue). The use of differing opacities allows for easier identification of the uniform solution behaviour across the respective domains.

are significantly more stable when the Wronskian is fixed independently at each boundary.

We find that the integrals in Eqs. (9.48) diverge due to contributions at the inner and outer Lindblad resonances, which correspond to the zeros of the Wronskian in Eq. (9.41), where our solution space becomes degenerate.

Unlike in numerical schemes normally employed in the Newtonian literature [210, 211, 231, 239], the variation of parameters scheme causes these resonances to arise as $1/\delta r^2$ divergences in the integrands of Eqs. (??) (where δr is the distance from the divergent location). This arises because $\mathcal{W} \sim \delta r$ and $\mathcal{F} \sim 1/\delta r$ near the resonance, leading to a total divergence of the form, $1/\delta r^2$. Application of our method to the Newtonian analogue in [210] reveals precisely the same singularity structure. This combination of divergences in the variation of parameters framework leads to non-integrable contributions in our final solution. Examples of the $1/\delta r$ divergences which partially give rise to the non-integrable contributions can be seen explicitly in Fig. 9.4.

We alleviate these unphysical divergences through Hadamard regularisation [240]. The principle behind this regularisation procedure is basic: we remove the

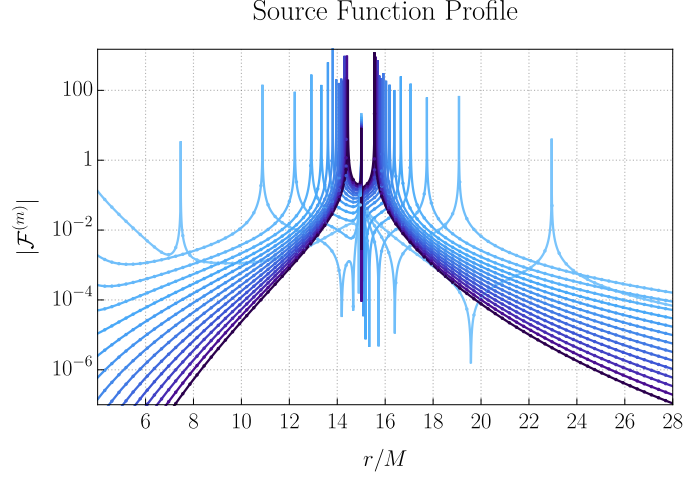


Figure 9.4: Plot exemplifying the sourcing function, Eq. (9.32), of the master enthalpy equation, with contributions from a high-accuracy interpolant of the effective forcing function, (9.13), contracted with the inversion matrix $\mathcal{Q}$ in the PMLA. The example is given for the fiducial parameter set $r_p = 15$, $h = 0.1$, and $a = 0.6$ for a range of m -modes. Notably, this plot explicitly shows the Lindblad resonant locations.

minimal content of the integrand that results in a finite integral. In practice, the approach is carried out as follows. Given an integral,

$$I(r) = \int_a^r \frac{F(x)}{(r_s - r)^2} dx, \quad (9.51)$$

with a singular part at a location r_s we define its partie finie contribution as

$$\text{p.f.}[I](r) = F(r_s) \left[\frac{1}{a - r_s} - \frac{1}{r - r_s} \right] + \int_a^r \frac{F(x) - F(r_s)}{(r_s - x)^2} dx \quad (9.52)$$

which explicitly removes the divergent contribution from the integrand with the characteristic form,

$$\int_{r_s - \epsilon}^{r_s + \epsilon} \frac{F(x)}{(r_s - x)^2} dx \sim \frac{F(r_s)}{\epsilon}. \quad (9.53)$$

Performing the same regularisation on the full solution at both Lindblad locations for every m -mode gives the full particular solutions to our master equation,

$$\mathfrak{h}_m(r) = \text{p.f.}[C_m^+](r) \mathfrak{h}_m^+(r) + \text{p.f.}[C_m^-](r) \mathfrak{h}_m^-(r). \quad (9.54)$$

We calculate the particular solutions to the master enthalpy equation up to an azimuthal number of $m_{\max} = 15$. Noting that throughout this work we never solve for the static $m = 0$ modes of our system.

Finally in this section we turn to reconstructing our fluid variables. We do so by first applying the PMLA to the velocity reconstruction operator given in Eq. (9.29).

In performing the reconstruction we find a delicate cancellation between terms linear in the enthalpy and terms linear in the metric perturbation. Schematically, the operator $\mathcal{Q}$ diverges as $\mathcal{Q} \sim 1/\delta r$ in the vicinity of Lindblad resonances. In contrast, the terms involving the enthalpy and metric perturbations remain finite. However, their combined contribution on the right hand side of Eq. (9.29) vanishes linearly, scaling as $\sim \delta r$, at the Lindblad resonances. This delicate cancellation acts to produce the correct smooth velocity profile of the fluid away from co-rotation. We find that this cancellation can be made numerically stable by fixing two separate Wronskian coefficients at either domain boundary, as in Eq. (9.50). Having calculated the four velocity of our fluid perturbations it is then straightforward to calculate the pressure and density perturbations through Eqs. (9.27) and (9.28).

9.6 SOLUTION PROPERTIES

Having derived the master equation for nonbarotropic linear perturbations to a perfect fluid in the Kerr geometry (§9.2, §9.3), and having developed a solution method for a thin, background disc perturbed by an orbiting secondary (9.4), we now explore the character of solutions, focusing on density wave morphology and angular momentum balance in the disc.

s.1 *Spiral Morphology*

In Fig. 9.5 we plot the perturbed disc energy density (analogue of Newtonian mass density), showing clear inner and outer spiral arms connecting at the location of the secondary, as expected from the Newtonian case.

While our solutions span the entire domain, for the purpose of plotting we have excised a region of width $\sim 1M$ around the secondary orbit, as the reconstruction of the 2D solution leads to divergences on the circle that intersects the co-rotation singularity. The inner white circle denotes the ISCO. To compare the phase structure of the relativistic and Newtonian cases, we overlay the analytical approximation (for $|r - r_p|/r_p \gtrsim h$) for the Newtonian spiral wavefront from [230] (white dashed line).

In the top panel, the agreement appears remarkable, with the outer Kerr spiral slightly more tightly wound (shifted to smaller radii) than the Newtonian spiral. Focusing on the inner disc, in the lower panel, reveals the detailed structure of the inner spiral arms and a rapid divergence between the Kerr and Newtonian predictions, with the Kerr arms again more tightly wound. The bottom panel also clearly demonstrates the emergence of a secondary spiral arm, arising at $r \lesssim 0.5r_p$ ¹. Miranda and Rafikov [210] have shown that the emergence of multiple arms in the

¹ Features in the higher-order spirals may not be fully resolved given the maximum m -mode considered here.

inner disc is a natural outcome of density wave propagation in differentially rotating discs, making their appearance in the Kerr geometry intriguing but unsurprising.

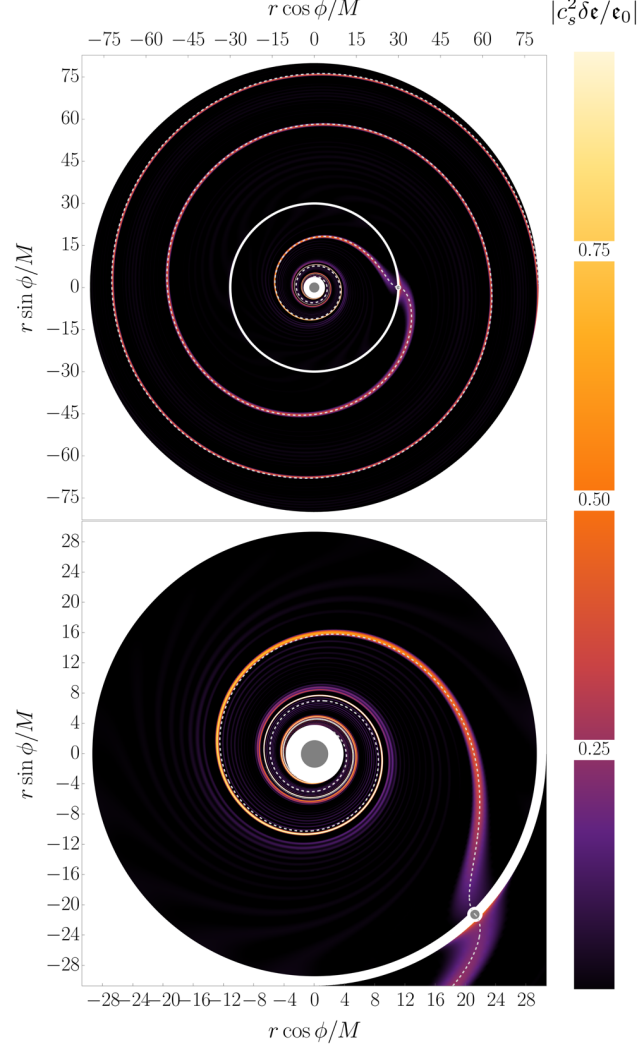


Figure 9.5: Extended structure of spiral density waves in the Kerr geometry for parameters $h = 0.1$, $a = 0.6M$, and $r_p = 30M$. *Top panel:* Large-scale structure of the disc with the analytic Newtonian spiral-arm prediction [230] overlaid. *Bottom panel:* Zoomed-in view of the inner disc, highlighting the detailed arm structure near the secondary and the emergence of both primary and secondary spiral arms near the central black hole. The discrepancy between the Kerr spiral arms and the analytic Newtonian prediction is clearly visible in the inner regions. The plotted quantity is chosen because it exhibits constant perturbation amplitude at large radius. We denote $\delta \epsilon = \epsilon_{(1,1)}$ and $\epsilon_{(1,0)} = \epsilon_0$ in the legends for clarity.

Fig. 9.6 explores the change in inner-spiral structure for different values of the central BH spin, at the same binary separation, $r_p = 30M$. Notably, the primary

effect of the spin is to allow the spirals to propagate more closely to the central black hole due to the contraction of the ISCO. This allows higher order spirals to become even more pronounced in the highest spin, right panel. Beyond this, the spiral arms for prograde discs tend to wind less tightly for higher spins. The strength of the spiral also decreases with increasing spin, implying that for a fixed orbital radius the integrated torques may weaken for higher spins. This is consistent with recent findings [217, 219].

Finally, we note that the close agreement with the Newtonian spiral structure is expected for an equatorial disc with weak pressure gradients, given the similarity in disc rotational frequencies in the Newtonian and Kerr cases $\Omega_N = r^{-3/2}$ while $\Omega_k = (r^{3/2} + a)^{-1}$. Further insight into how spiral wave morphologies depend on spin and disc thickness can be gained by analysing the spiral wavefronts. These can be accessed from the radial wave number in Eq. (9.44) [210, 230, 241], however, we leave further study of the rich spiral structure in the Kerr geometry for a future analysis, and now turn to angular momentum and torque balance in the disc.

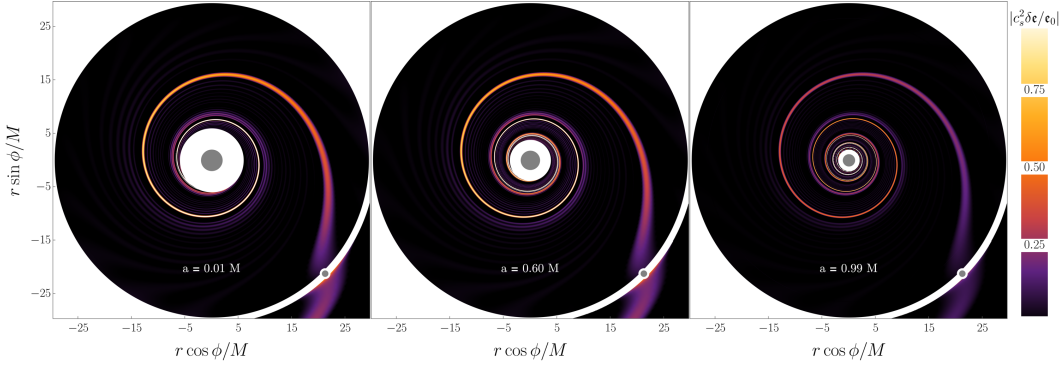


Figure 9.6: Equatorial-plane density profiles for a fixed disc aspect ratio $h = 0.1$ and secondary orbital radius $r_p = 30M$, shown for increasing values of the black hole spin parameter, $a = 0.01M$, $0.6M$, and $0.99M$ from left to right. These panels illustrate the systematic modification of the inner spiral arm structure with increasing black hole spin, highlighting the growing deviation from the low-spin morphology in the relativistic regime. The progression also demonstrates the robustness of our method at high spin values, enabling the resolution of spiral density waves deep in the Kerr potential where relativistic effects are strongest. The legends are the same as described in Fig. 9.5.

s.2 Torque Density and Advected Angular Momentum Balance

In black hole perturbation theory, a common approach to understanding the evolution of a secondary BH in the presence of matter is to first solve for the response of the matter fields and then treat the resulting perturbations as test fields on the background spacetime, exploiting the spacetime's Killing symmetries to

define energy and angular momentum fluxes through the horizon and to infinity [61, 167, 193, 220, 242]. These fluxes of energy and angular momentum are then often related directly back to the evolution of the secondary’s orbit.

This approach will encounter significant difficulties when applied to discs with viscous and heat dissipation and, more generally, to matter configurations where the interplay between the matter and secondary does not imprint directly on the fluxes at infinity. In many situations, the matter field and the secondary will exchange energy and angular momentum locally, but the corresponding fluxes measured at the horizon or at infinity will not be representative of the actual exchange governing the orbital evolution of the secondary. Consequently, the net fluxes through the horizon and at infinity will often fail to capture the physically relevant back reaction on the orbit.

By contrast, in Newtonian theory and in the hydrodynamical simulation literature, disc–secondary interactions are commonly quantified by computing the torque as a local quantity which is integrated to obtain the total force exerted by the perturber on the disc and vice versa by Newton’s third law.² In this section, we outline a pathway for describing an analogous construction of local torque densities within the relativistic framework of black hole perturbation theory, which we hope in future works may allow for a formal connection to the motion of EMRIs in matter. Although presented in the context of accretion discs, the approach we describe applies more generally to any relativistic prescription of matter in a perturbative binary black hole spacetime. Related ideas have appeared previously in the literature for describing the excitation of conserved quantities locally, for example in work associated with the GR-CHOMBO numerical relativity collaboration [244, 245]; however, while these methods move in a similar direction, they do not directly extend to an analogue of a local momentum–torque balance that we seek to construct.

Our goal here is therefore to formulate a balance equation that relates the local flux of momentum exchange through the matter to the locally applied force the EMRI applies to the matter, connecting the notion of fluxes to integrated matter torques, with the outlook that integrated matter torques are the physical quantities that we hope to allow for a formal balance with torques on the EMRI in future studies.

s.2.1 *General Torque Balance Equation*

We motivate a representation of the disc torques that connects directly to the Newtonian literature and takes the representative form described in App. A of [246]. To this end, we return to the fully non-linear Einstein equations. For circular orbits the primary means for understanding orbital evolution is through angular

² See for example [243], targeted towards intermediate mass ratio inspirals.

momentum exchange between the disc and the secondary. In this vein we begin with the equation,

$$\nabla_\mu \left(\xi_\phi^\nu T_\nu^\mu \right) = T_\nu^\mu \nabla_\mu \xi_\phi^\nu. \quad (9.55)$$

Here, $T^{\mu\nu}$ denotes the full fluid stress–energy tensor on the full spacetime manifold, assumed to satisfy the usual conservation equations, and $\xi_\phi^\mu = \{0, 0, 0, 1\}$ is the coordinate basis vector associated with the azimuthal direction. At this stage, ξ_ϕ^μ is not assumed to be a Killing vector.

This expression may be written equivalently as

$$\partial_\mu \left(\sqrt{-g} T_\phi^\mu \right) = \sqrt{-g} T_\nu^\mu \Gamma_{\mu\phi}^\nu, \quad (9.56)$$

where $\sqrt{-g}$ is the volume element of the full nonlinear spacetime and all other quantities also remain fully non-linear. We next expand this equation to $\mathcal{O}(\lambda\epsilon^2)$, the order at which we expect energy and angular momentum transfer to arise.

Before performing this expansion, however, we note that our objective is to obtain an expression for the orbit-averaged torque. Accordingly, we work with the orbit-averaged form of Eq. (9.56). The averaging operator for circular orbits is defined by,

$$\langle X \rangle = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} X dt.$$

Here, ω denotes the orbital frequency of the perturber and may be generalised beyond circular motion to include inclined, eccentric, and fully generic orbits as in [247]. At $\mathcal{O}(\lambda\epsilon^2)$, we are concerned only with quantities that are either quadratic in first-order perturbations or associated with the static ($m = 0$ for circular) modes of the second-order perturbations. Since our solutions are constructed in terms of azimuthal m -modes and for circular orbits, quadratic combinations of first-order quantities may be readily obtained by,

$$\langle XY \rangle = \text{Re} \left[\sum_m x_m \bar{y}_m \right], \quad (9.57)$$

where bar is the conjugation operator, $X = \sum x_m e^{im\phi} e^{-i\omega_m t}$ is a generic perturbative term, and similarly for Y . In the present work, we solve the field equations only to first order and therefore do not have access to the static second-order contributions that enter the leading-order angular momentum balance. We collectively denote these missing terms by $S_{m=0}^{(2)}$.

At this stage, we must also clarify a simplification employed throughout our prescription. Specifically, we have treated the background disc solution as time independent. Such an assumption is, in general, inconsistent for realistic disc configurations, particularly for adiabatic fluids or discs with heat dissipation or viscosity, where one must account for the secular evolution of the disc. A

more appropriate assumption is that the time derivatives of the background disc quantities are small. For example, if the energy density admits the expansion $\epsilon = \epsilon_{(0,1)} + \epsilon_{(1,1)}$, then we require

$$\frac{\partial \epsilon_{(0,1)}}{\partial t} \sim \mathcal{O}(\lambda \epsilon^2). \quad (9.58)$$

Alternatively, one may appeal to a multiscale expansion and reinstate the independence of the background quantities from the *fast* time (t) by introducing a new, independent *slow* time (T_D) variable governing the secular evolution of the disc, defined as $T_D = \epsilon^2 t$. In this framework,

$$\partial_t \epsilon_{(0,1)} \Rightarrow (\partial_t + \epsilon^2 \partial_{T_D}) \epsilon_{(0,1)} = \epsilon^2 \frac{\partial \epsilon_{(0,1)}}{\partial T_D}. \quad (9.59)$$

Such slow-time evolution of the background disc parameters is expected even in the example considered here, where the initial thermodynamic profile of our adiabatic fluid must evolve according to the energy equation, as discussed in [210].

Then, taking Eq. (9.56), including the new slow timescale, expanding to order $\mathcal{O}(\lambda \epsilon^2)$, and averaging over circular orbits we obtain the new balance equation given by,

$$\begin{aligned} \partial_{T_D} \left(\sqrt{-g} T_\phi^t \right) + \partial_i \left[\sqrt{-g} \left(\langle \delta^2 T_\phi^i \rangle + \langle h \delta T_\phi^i \rangle + \frac{1}{4} T_\phi^i \left(\langle h^2 \rangle - 2 \langle h_\beta^\alpha h_\alpha^\beta \rangle \right) \right) \right] \\ = \sqrt{-g} \left[\langle \delta T^{\alpha\beta} \nabla_\phi h_{\alpha\beta} \rangle - 2 T^{\alpha\beta} \left(\langle h_\alpha^\gamma \nabla_{[\gamma} h_{\beta]\phi} \rangle - \frac{1}{4} \langle h \nabla_\phi h_{\alpha\beta} \rangle \right) \right] + S_{m=0}^{(2)}, \end{aligned} \quad (9.60)$$

where all gradients and determinants are with respect to the background Kerr geometry, all occurrences of h refer to the metric perturbation and its trace, and $i \in \{r, \theta\}$. If we now impose in Eq. (9.60) our restrictions on the background flow, $u^r = u^\theta = 0$, then the explicit contributions to the balance equation become,

$$T_\phi^t = (\epsilon + p) u_{(0,0)}^t u_\phi^{(0,0)}, \quad (9.61)$$

$$\delta T_\phi^i = (\epsilon + p) u_{(1,0)}^i u_\phi^{(0,0)}, \quad (9.62)$$

$$\begin{aligned} \delta T^{\mu\nu} = & 2(\epsilon + p) u_{(1,0)}^{(\mu} u^{\nu)} \\ & + (\epsilon^{(1,1)} + p^{(1,1)}) u^\mu u^\nu \\ & + p^{(1,1)} g^{\mu\nu} - p h_{(1,0)}^{\mu\nu}, \end{aligned} \quad (9.63)$$

$$\begin{aligned} \delta^2 T^i_\phi = & (\epsilon + p) u^i_{(1,0)} \left(u^{(1,0)}_\phi + h^{(1,0)}_{\phi\mu} u^\mu \right) \\ & + \left(\epsilon^{(1,1)} + p^{(1,1)} \right) u^i_{(1,0)} u_\phi. \end{aligned} \quad (9.64)$$

By replacing ϕ with t in all of the above, one also recovers the energy balance equations, and although we derive this equation enforcing a Killing background flow and circular orbit frequencies, the result can be naturally extended to generic configurations.

In connection with the Newtonian literature and previous work in black hole perturbation theory, we define the advected angular momentum density as a subset of the terms quadratic in the variation of the stress–energy tensor, given by,

$$j^i_{\text{Adv}} = \sqrt{-g} \langle \delta^2 T \rangle^i_\phi - \dot{j}^i_M, \quad (9.65)$$

with

$$\dot{j}^i_M = \sqrt{-g} \left\langle \left(\epsilon^{(1,1)} \right) u^i_{(1,0)} \right\rangle U_\phi. \quad (9.66)$$

We choose this separation explicitly such that in the Newtonian limit,

$$j^r_{\text{Adv}} \rightarrow r^2 \rho_0 \langle \delta u^r \delta u^\phi \rangle, \quad (9.67)$$

$$\dot{j}^r_M \rightarrow r^2 \langle \delta \rho \delta u^r \rangle U^\phi. \quad (9.68)$$

We directly identify $\dot{j}^i_{\text{Adv}}$ with the relativistic analogue of the advected angular momentum flux through the disc [e.g., 56], and $\dot{j}^i_M$ with the angular momentum transport associated with the radial motion of perturbed fluid elements. Crucially, as emphasised in previous Newtonian analyses [248, 249], the term $\dot{j}^i_M$ is precisely the contribution expected to balance the otherwise unknown static second-order terms, which we collectively denote by $S^{(2)}_{m=0}$.³

Having accounted for the second-order stress–energy contributions, we now perform a canonical decomposition of the remaining terms in the torque balance equation. We separate the residual contributions into those that are linear in a single variation of the stress–energy tensor and a single metric perturbation, which we identify as the derivatives of the *matter torques*, $\partial T^\phi_{\text{Mat}}$, and those that depend only on the background fluid flow and are quadratic in the metric perturbations sourced by the secondary, which we interpret as derivatives of the *geometric torques*, $\partial T^\phi_{\text{Geo}}$. This decomposition yields,

$$\begin{aligned} \partial T^\phi_{\text{Mat}} = & \sqrt{-g} \left\langle \delta T^{\alpha\beta} \nabla_\phi h_{\alpha\beta} \right\rangle \\ & - \partial_i \left(\sqrt{-g} \left\langle h \delta T^\phi_i \right\rangle \right), \end{aligned} \quad (9.69)$$

³ We thank Callum Fairbairn for drawing our attention to these references.

and

$$\begin{aligned}\partial T_{\text{Geo}}^\phi = & -2\sqrt{-g}T^{\alpha\beta}\left\langle h_\alpha^\gamma\nabla_{[\gamma}h_{\beta]\phi}\right\rangle \\ & +\frac{1}{2}\sqrt{-g}T^{\alpha\beta}\left\langle h\nabla_\phi h_{\alpha\beta}\right\rangle \\ & -\frac{1}{4}\partial_i\left[\sqrt{-g}T_\phi^i\left(\left\langle h^2\right\rangle-2\left\langle h_\nu^\mu h_\mu^\nu\right\rangle\right)\right].\end{aligned}\tag{9.70}$$

This canonical separation of our equations gives rise to its collected form,

$$\begin{aligned}\partial_{T_D}\left(\sqrt{-g}T_\phi^t\right)+\partial_i\left(j_{\text{Adv}}^i+j_M^i\right)=\\ \partial T_{\text{Mat}}^\phi+\partial T_{\text{Geo}}^\phi+S_{m=0}^{(2)}.\end{aligned}\tag{9.71}$$

Similarly to our analysis of the angular momentum fluxes we now find that in the Newtonian limit,

$$\partial T_{\text{Mat}}^\phi\rightarrow r^2\left\langle\delta\rho\partial_\phi\Phi_p\right\rangle,\tag{9.72}$$

$$\partial T_{\text{Geo}}^\phi\rightarrow 0,\tag{9.73}$$

where Φ_p is the Newtonian potential of the smaller body, making a direct connection to previous works invoking torque-advection balance [e.g., 246]. This, along with the identification of the advected angular momentum, gives a direct relativistic analogue of the full advected angular momentum – torque density balance equation.

s.2.2 Application to the Relativistic Disc

Having obtained our solutions for the fluid variables, along with the input metric data and a prescribed background fluid, we apply the PMLA to our torque balance Eq. (9.60) to leading $\mathcal{O}(1)$ in the scale height expansion. This results in a purely radial, orbit-averaged equation. We present the results of applying the torque balance equation to our fluid solutions in Fig. 9.7.

In Fig. 9.7 and hereafter, we normalise our torques and torque density analogously to the Newtonian literature [e.g., 202, 250], with the exception that our torque is per unit volume, not area, and we here adjust for our initial choice of disc normalisation at the ISCO (Eq. (4.32)) by multiplying by a factor of $\epsilon_I(r_p/r_I)^{-2}$, to give,

$$J_0=\left(\frac{m_p}{M}\right)^2h^{-3}r_p^2\epsilon_Ir_I^2\Omega_p^2,\tag{9.74}$$

where variables with subscript I are background quantities evaluated on the ISCO. Figure 9.7 shows that, even with the PMLA, there is remarkable agreement between the radial profile of the orbit-averaged advected angular momentum flux density ($\partial_r j_{\text{Adv}}^r$), and our relativistic analogue of the disc excitation torque density

($\partial T_{\text{Mat}}^\phi$). Furthermore, the extended structure of the torque clearly exhibits the same qualitative behaviour as previously calculated torques in the Newtonian literature, including the primary trough and peak just inner to and outer from the secondary's radius, as well as the so-called negative torque density and torque wiggles at larger distances from the secondary (see [250, 251] for recent examples).

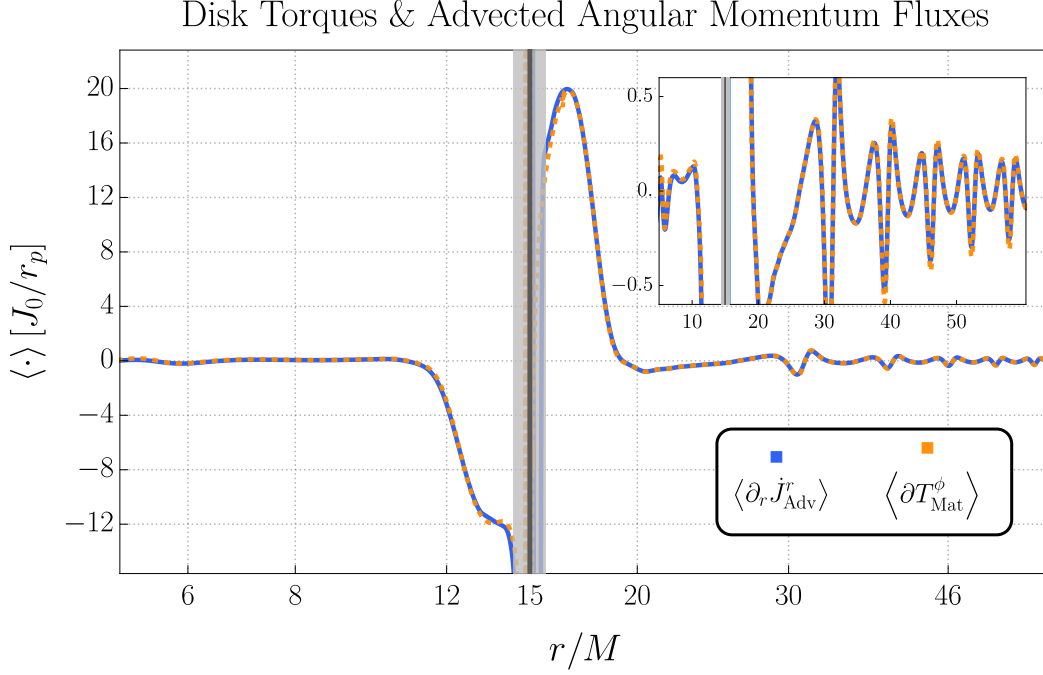


Figure 9.7: Different contributions to the torque density balance equation, constructed from the product of first-order metric and fluid perturbations, for the fiducial parameter set $h = 0.1$, $a = 0.99$, and $r_p = 15M$, summed up to an $m_{\text{max}} = 14$ with normalisation $[J_0/r_p]$. We mark the orbital radius with a dark black line surrounded by a shaded grey region, inside which we find poor convergence of our solutions. **Main Figure:** Global balance between the advected angular momentum and disc torque densities, showing the distinctive peak and trough on either side of the co-rotation point. **Inset:** Zoomed-in view of the same quantities, illustrating excellent agreement.

The $\partial_r \dot{J}_{\text{Adv}}^r$ contribution depends entirely on perturbed fluid quantities, while $\partial T_{\text{Mat}}^\phi$ depends on a cross product between the metric perturbation and perturbed fluid quantities. This comparison serves as a strong check on the validity of the approximations used throughout this work, the correctness of the numerical implementation, including the Hadamard regularisation procedure employed to remove the non-integrable divergences arising at Lindblad resonances, and the reconstruction of our fluid variables.

We also find that the new class of torques, which we denoted above as *geometric torques*, are subdominant, arising at around the 10^{-5} relative level to the *matter torques*. We also observed that the contribution in the $\dot{j}_M^r$ term grows away from the secondary. However, as previously noted in the literature [248, 249], and as can be seen by taking the orbit-averaged energy equation to second order, this growth in principle arises from the neglect of static second-order contributions. Specifically, we expect terms of the form $\epsilon_{(0,1)} u_\phi^{(0,0)} u_{(2,0),m=0}^r$ to balance this contribution. Finally, without careful calculation of the second-order static pieces or the slow-time change in our background variables (which also arises at second order) we cannot perform a full balance check. In any case, the strong agreement between $\partial_r \dot{j}_{\text{Adv}}^r$ and $\partial T_{\text{Mat}}^\phi$ suggests a clear route toward a torque balance framework for relativistic discs within black hole perturbation theory calculations.

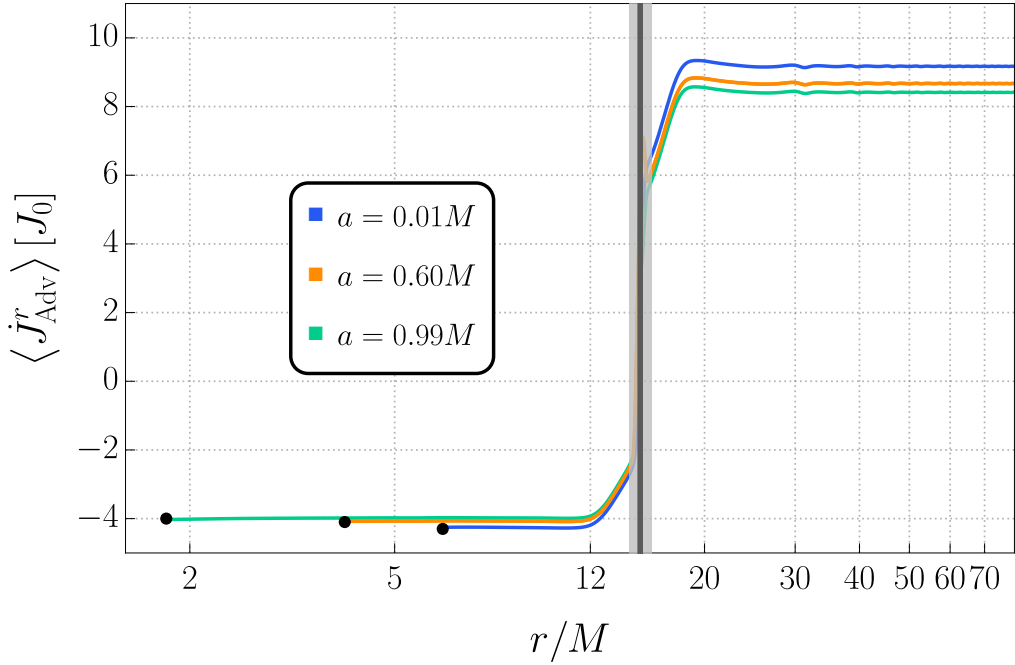


Figure 9.8: The total advected angular momentum for a range of spin values, with $h = 0.1$, $r_p = 15M$, and $m_{\text{max}} = 14$. The black dots in the inner regions correspond to the ISCO for different spins.

In the linear, adiabatic, Newtonian case, the torque density excited into the disc by the perturber is completely carried away by the advected angular momentum flux, hence allowing an association of the asymptotic angular momentum flux with the excitation Lindblad torque [210, 232]. Here we have found that a similar matching holds in the linear, adiabatic, relativistic case (Fig. 9.7), motivating that our angular momentum flux is an indicator of the total relativistic Lindblad torque.

In Fig. 9.8 we provide an example calculation of the advected fluxes for a range of spin values at fixed $r_p = 8M$. The values of $\langle \dot{J}_{\text{Adv}}^r \rangle$ at the ISCO and large r in Fig. 9.8 can be interpreted as the full one-sided inner and outer relativistic resonant Lindblad torques applied to the disc, where, in the case of the existence of a torque-balance law, the negative of their sum provides the differential torque applied to the secondary.

We identify two key results from Fig. 9.8. First, for all spin values considered, the outer angular momentum flux exceeds the inner flux. Extending the above analogy to the Newtonian case and identifying the integrated angular momentum flux with the resultant torque on the orbit, this implies that the net torque remains negative, indicating inward migration of the secondary at $r_p = 8M$. This contrasts with recent studies that report a torque reversal at this radius for lower spin values [217, 218]. Such reversal, however, appears to depend sensitively on the underlying disc properties. As a result, our findings are not inconsistent with previous work, but instead highlight the need for further investigation.

Second, and in agreement with recent relativistic studies, we find a decrease in the total torque with increasing black hole spin. Upcoming follow-up work will explore a wider region of parameter space in order to develop a more complete understanding of how our model differs from other recent relativistic treatments which treat the disc as a pressure-less collection of particles [217, 218].

Furthermore, this result points toward a path forward for the potential identification of a Newton’s third-law-type balance, from which one could hope to identify torques, $T_{\text{secondary}} = -T_{\text{Mat}}$. If such a connection exists, a fundamental advance in perturbative matter-field binary modelling could be achieved. Specifically, rather than calculating higher-order metric perturbations to determine the forces acting on a small body for orbital evolution, one could instead solve for the appropriate matter-field configuration of the binary, construct ∂T^ϕ for a given matter-field theory, and integrate over the volume to determine the binary evolution. We expect such a balance would take the schematic form,

$$\dot{J}_{\text{secondary}} \sim - \int \partial T_{\text{Mat}}^\phi dV. \quad (9.75)$$

Such an exploration will be of critical importance in future studies. A concrete connection between the force applied to the matter field and the force applied to the secondary will require a first-principles torque-balance derivation within self-force theory. However, even at the level of a heuristic implementation, the application of such a torque balance in this manner, we expect, would already represent a substantial improvement over the approximations presently employed in the literature.

9.7 CONCLUSION

In this chapter, we have presented the first fully relativistic fluid calculation in the binary potential of an extreme mass-ratio inspiral. Our approach leverages results from multiple fields, adapting solutions developed for self-gravitating relativistic stars and accretion discs and combining these with state-of-the-art self-force calculations, guided by current results in the planetary migration literature. Using these results, we derive a new form of master equation for the perturbations to a perfect fluid in the Kerr geometry, incorporating effective forces arising from independent metric perturbations, the source of which can be chosen generally. Here, we specialise to metric perturbations arising from a small secondary black hole on a circular orbit around a Kerr black hole, and to the case of an adiabatic fluid with adiabatic index approaching unity.

After constructing the relativistic background flow and the EMRI potential from self-force theory, we describe in detail the procedure for obtaining our height-averaged master equations, highlighting the subtleties encountered at different stages of calculating the enthalpy solutions. With the full fluid solutions reconstructed, we analyse the resulting spiral-arm structures and extended flow patterns, illustrating the deviations between the Kerr and Newtonian spiral-arm structures. We also compute the extended profiles for a range of spin values, exemplifying the contraction of the ISCO and the subtle increase in the winding rate of the inner arm with spin.

We then focus on the angular momentum balance, examining the interplay between advected angular momentum through the disc and the matter torques excited by the secondary potential. We derive a new, fully relativistic form of the torque balance equation, showing remarkable agreement between our definition of the advected flux densities and matter torque densities across the domain. This analysis also reveals new torque contributions, including what we have called geometric torques, and specific terms that require the calculation of static second-order fluid perturbations.

Our torque balance equation provides a crucial step toward understanding the interaction between orbital mechanics and disc physics. Instead of relying on traditional calculations of advected fluxes through the disc (as typically done in black hole perturbation theory), our approach relates these effects directly to local torques. In future work, this method could allow one to compute forces on the orbiting body without needing to perform complicated second-order self-force-type calculations. There are a number of immediate future directions in which to take this initial study. Firstly, we have computed solutions up to a maximum azimuthal Fourier mode of $m_{\max} = 15$, due to limitations arising from the computed modes of the metric perturbation. While this appears sufficient to capture basic properties of the disc morphology and torque balance, future work should test convergence with higher-order modes. Secondly, we do not calculate the contribution of the $m = 0$

mode to the fluid perturbations. Calculation of this term will be necessary in order to source the static second-order equations needed to perform a full torque balance.

Future developments of this framework should also include the explicit incorporation of the baryon number conservation equation, allowing a proper separation of rest-mass energy from internal energy. Such a split is essential for accurately tracking the thermodynamic evolution of the system over time, particularly when additional cooling effects are included.

Extensions to ideal magnetohydrodynamics (MHD) also appear feasible and promising, both at linear order within the framework laid out here, and numerically. Indeed, future extensions to linear theory could utilise the full nonlinear fluid equations, or their MHD extensions, coupled with the linear metric perturbations from the secondary, as derived in Eqs. (9.11)–(9.13).

Incorporating dissipative effects, such as viscosity, could be handled self-consistently through causal first-order fluid theories such as those by Bemfica, Disconzi, Noronha, and Kovtun (BDNK) [111, 252–254]. Such models, however, would involve background accretion and radial inflow, which would likely need to be addressed at the level of solving the full system of coupled fluid equations rather than through a reduced master equation. Progress in calculating master equations with BDNK has been made recently, however, in the context of spherically symmetric and low-spin viscous stars [255–258].

In this chapter, we have primarily focused on the formulation and methodology, providing several proof-of-principle calculations to demonstrate the robustness and potential of these results. Follow-up studies will explore the underlying physics and parameter space in greater detail, examining how relativistic effects modify the dynamics compared to the wealth of Newtonian studies already present in the literature, and determining the formal convergence criteria required to attain trustworthy results for use in waveform generation schemes for EMRIs in relativistic discs. These results represent the first steps toward fully relativistic fluid modelling around EMRIs and pave the way for more complex and astrophysically relevant models of strong-field EMRIs in active galactic nuclei.

A RELATIVISTIC THIRD LAW

Having applied the two-parameter expansion formalism to multiple environments in the Kerr geometry, and having connected with actual fluid calculations commonplace in the Newtonian literature, we move on to the final *paper chapter* of this thesis. In this final main chapter, we take the step foreshadowed in the introduction by making explicit the connection between the evolution of the orbital parameters of the EMRI and the locally integrated matter torques. As noted in previous chapters, the direct means to understand the force on the EMRI is through direct calculation of the field $h_{\mu\nu}^{(\text{env})}$. Computing such a contribution, however, would represent a significant leap beyond the current state of the art, requiring robust methods for calculating the metric perturbations induced by a time-dependent extended stress-energy tensor. Of course, such calculations are standard in numerical relativistic simulations; however, they come with excessive numerical cost and complications. The goal of this chapter is therefore to avoid such a calculation by relating direct forces on the EMRI to locally integrated torques on the matter field. Such a result would not only circumvent a large number of numerical calculations, but also provide a direct means for researchers expert in complex matter simulations of curved backgrounds to take their results and integrate a pre-calculated self-force source $\nabla h_{(p)}$ to directly determine the types of torques present on orbiting bodies in dynamical environments. Such a connection precisely entails a relativistic analogue of Newton's third law. Further, in the class of environments that do admit a torque-flux-balance relation, using the approach presented in the previous chapter, one can hope to rigorously construct a relation between EMRI forces, locally integrated torques, and asymptotic matter fluxes,

$$\underbrace{F[h^{(\text{env})}]|_{\gamma}}_{\text{EMRI force}} \sim \underbrace{\int T^{\text{env}} \nabla h_{(p)}}_{\text{locally integrated torques}} \sim \underbrace{\delta^2 T_{\text{env}}|_{\mathcal{H}^+/\mathcal{I}^+}}_{\text{asymptotic matter fluxes}}. \quad (10.1)$$

By the end of this chapter, we will have understood that for energy and angular momentum the first relation can be made precise.

This final *paper chapter* is laid out as follows. We begin by providing an explicit derivation of Newton's third law in the context of a point particle embedded in matter, and show how such a relation arises specifically through the structure of the Green's function of the Poisson operator. We then make explicit how such an approach can be directly lifted to the relativistic scenario, making use of the Green's function properties of the linearised Einstein operator in the Kerr geometry and considering the associated Killing symmetries of the spacetime. From the

construction of quasi-conserved charges on the spacelike hypersurfaces in Kerr, we derive an explicit third law in curved space which makes clear the causal requirements of such a balance. When averaged over long times, this recovers Newton's third law with a correction arising due to gravitational wave emission from the motion of the particle onto a new orbit.

We begin by providing such balance derivations at leading order, relevant for understanding interactions with turbulent non-stationary environments around EMRIs. With the subleading balance required for such calculation as provided in the previous chapter near on the horizon.

The contents of this chapter primarily constitute those of an article in preparation, co-authored with Adam Pound, Daniel J. D'Orazio, Maarten van de Meent, and Lorenz Zwick.

10.1 NEWTONIAN THIRD LAW

We begin by reviewing the Newtonian version of the calculation. We consider a point-particle secondary of mass m_p moving along a worldline γ parameterised by time t , with position $\mathbf{x}(\gamma)$, together with an environment whose mass density $\rho(\mathbf{x})$ is determined self-consistently in the binary potential. The Newtonian potential sourced by the secondary is

$$\Phi_p(\mathbf{x}) = -\frac{m_p}{|\mathbf{x} - \mathbf{x}(\gamma)|}, \quad (10.2)$$

evaluated at the field point $\mathbf{x}$ for the secondary located on γ . The potential sourced by the environment is

$$\Phi_{\text{env}}(\mathbf{x}) = -\int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x'. \quad (10.3)$$

The force on the secondary due to the environment's potential, evaluated on the worldline γ , is then

$$\mathbf{F}_p = -m_p \nabla \Phi_{\text{env}} \Big|_{\mathbf{x}=\mathbf{x}(\gamma)}, \quad (10.4)$$

while the total gravitational force on the environment due to the secondary's potential is

$$\mathbf{F}_{\text{env}} = -\int \rho(\mathbf{x}) \nabla \Phi_p(\mathbf{x}) d^3x. \quad (10.5)$$

The Newtonian analogue of circumventing the reconstruction of $h_{\mu\nu}^{(\text{env})}$ is to avoid computing Φ_{env} directly in order to determine $\mathbf{F}_p$, and instead relate it to $\mathbf{F}_{\text{env}}$. We do this by explicitly constructing Newton's third law in the context of point-particle-matter interactions, $\mathbf{F}_p = -\mathbf{F}_{\text{env}}$. Substituting Eq. (10.3) into Eq. (10.4),

$$\begin{aligned} \mathbf{F}_p &= -m_p \nabla_{\mathbf{x}} \left(-\int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x' \right) \Big|_{\mathbf{x}=\mathbf{x}(\gamma)} \\ &= \int \rho(\mathbf{x}') \nabla_{\mathbf{x}} \left(\frac{m_p}{|\mathbf{x} - \mathbf{x}'|} \right) \Big|_{\mathbf{x}=\mathbf{x}(\gamma)} d^3x'. \end{aligned} \quad (10.6)$$

Using the identity $\nabla_{\mathbf{x}} |\mathbf{x} - \mathbf{x}'|^{-1} = -\nabla_{\mathbf{x}'} |\mathbf{x} - \mathbf{x}'|^{-1}$, which follows from the fact that $|\mathbf{x} - \mathbf{x}'|^{-1}$ depends only on the difference $\mathbf{x} - \mathbf{x}'$ and allows one to transfer the gradient onto the dummy integration variable, Eq. (10.6) becomes

$$\mathbf{F}_p = \int \rho(\mathbf{x}') \nabla_{\mathbf{x}'} \Phi_p(\mathbf{x}') d^3x' = -\mathbf{F}_{\text{env}}, \quad (10.7)$$

where we used Eq. (10.5) in the last step. This is precisely Newton's third law: the force exerted on the secondary by the environment is equal and opposite to the integrated force exerted on the environment by the secondary. The key feature of this result is that $\mathbf{F}_p$ has been expressed entirely in terms of Φ_p , the

potential sourced by the secondary itself, with no reference to Φ_{env} . Importantly, it is precisely the structure of the Poisson operator's Green's function, $\nabla_{\mathbf{x}}|\mathbf{x} - \mathbf{x}'|^{-1}$, that permits this identification; it is this structure that we seek to generalise to the relativistic setting.

10.2 METRIC PERTURBATIONS AND THE RETARDED GREEN'S FUNCTION

To make contact with the above definitions, we begin with the relaxed form of the Einstein equations in Lorenz gauge, which at each order take the form

$$\mathcal{E}_{\alpha\beta}{}^{\mu\nu}\bar{h}_{\mu\nu}^{(n,m)} = -\frac{1}{2}(g_{\alpha}{}^{\mu}g_{\beta}{}^{\nu}\square + 2R_{\alpha}{}^{\mu}{}_{\beta}{}^{\nu})\bar{h}_{\mu\nu}^{(n,m)} = \mathcal{S}_{\alpha\beta}^{(n,m)}, \quad (10.8)$$

where $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}g_{\mu\nu}h^{\mu}{}_{\mu}$ is the trace-reversed metric perturbation and $\mathcal{S}_{\alpha\beta}^{(n,m)}$ is the full source term including both the content of the matter field stress energy tensor and the effective gravitational sources comprising all nonlinear contributions from lower-order perturbations. At leading perturbative orders the source $\mathcal{S}_{\alpha\beta}$ is given entirely by the leading order point particle and matter stress energy tensors in the form,

$$\mathcal{S}_{\alpha\beta} = 8\pi \left(\epsilon T_{\alpha\beta}^{(\text{p})} + \lambda T_{\alpha\beta}^{(\text{env})} \right) + \mathcal{O}(\epsilon^2, \epsilon\lambda, \lambda^2) \quad (10.9)$$

The metric perturbations at each order is then obtained by inverting via the retarded Green's function,

$$\bar{h}_{\alpha\beta}^{(n,m)} = \int G_{\alpha\beta\alpha'\beta'}^{\text{Ret}} \mathcal{S}_{\alpha'\beta'}^{(n,m)} dV'_0, \quad (10.10)$$

where $\mathcal{S}_{\alpha\beta}^{(n,m)}$ is the total source at order (n, m) is the total stress-energy at order (n, m) and dV'_0 is the covariant background volume element. Throughout this chapter we work with only the leading-order contributions to $T^{(\text{p})}$ and $T^{(\text{env})}$, connecting them correspondingly to metric perturbations $h^{(\text{p})}$ and $h^{(\text{env})}$ where as we are only counting to leading order we can set $n = 1$ in this section leaving the more precise higher-order treatment as work currently under progress.

A key tool in what follows will be the decomposition of the retarded Green's function into its time-symmetric ("Sym") and radiative ("Rad") parts,

$$G_{\alpha\beta\alpha'\beta'}^{\text{Ret}} = G_{\alpha\beta\alpha'\beta'}^{\text{Sym}} + G_{\alpha\beta\alpha'\beta'}^{\text{Rad}}, \quad (10.11)$$

and from this split we can also define the advanced Green's function by,

$$G_{\alpha\beta\alpha'\beta'}^{\text{Adv}} = G_{\alpha\beta\alpha'\beta'}^{\text{Sym}} - G_{\alpha\beta\alpha'\beta'}^{\text{Rad}}. \quad (10.12)$$

These constituent pieces are distinguished by their behaviour under exchange of the field and source points:

$$G_{\alpha\beta\alpha'\beta'}^{\text{Sym}}(x, x') = G_{\alpha'\beta'\alpha\beta}^{\text{Sym}}(x', x), \quad (10.13)$$

$$G_{\alpha\beta\alpha'\beta'}^{\text{Rad}}(x, x') = -G_{\alpha'\beta'\alpha\beta}^{\text{Rad}}(x', x). \quad (10.14)$$

In connection with the discussion of post-adiabatic counting earlier in the thesis, the symmetric part is time-even and carries no net flux to infinity; the radiative part is time-odd and encodes the outgoing radiation that drives the leading-order orbital evolution in vacuum. The importance of this split for the present work lies in how these pieces respond to Lie differentiation along a Killing vector ξ^α of the background. For any Green's function exhibiting the Killing symmetries of the Kerr geometry [49], one has the identity

$$\mathcal{L}_\xi G_{\alpha\beta\alpha'\beta'}(x, x') = -\mathcal{L}_{\xi'} G_{\alpha\beta\alpha'\beta'}(x, x'), \quad (10.15)$$

so that differentiation with respect to the field point is equivalent to differentiation with respect to the source point, up to a sign. For the symmetric and radiative pieces this gives

$$\mathcal{L}_\xi G_{\alpha\beta\alpha'\beta'}^{\text{Sym}}(x, x') = -\mathcal{L}_{\xi'} G_{\alpha'\beta'\alpha\beta}^{\text{Sym}}(x', x), \quad (10.16)$$

$$\mathcal{L}_\xi G_{\alpha\beta\alpha'\beta'}^{\text{Rad}}(x, x') = +\mathcal{L}_{\xi'} G_{\alpha'\beta'\alpha\beta}^{\text{Rad}}(x', x). \quad (10.17)$$

Such relations are strikingly similar to those required in the Newtonian setting to build the third law, suggesting that one should be able to straightforwardly construct a balance law for the energy and angular momentum associated with the orbit. Finally, we note the causal support properties that underpin the calculation: the retarded Green's function satisfies $G^{\text{Ret}}(x, x') = 0$ when $t < t'$ (the field point lies before the source), while the advanced Green's function satisfies $G^{\text{Adv}}(x, x') = 0$ when $t' < t$ (the source lies before the field point).

10.3 GLOBAL CONSERVATION LAWS

In Kerr spacetime, the stationarity and axisymmetry of the background give rise to two conserved Killing charges: the orbital energy and angular momentum, associated with the Killing vectors ∂_t and ∂_ϕ respectively. We now develop a balance law for these charges. As described earlier, the Carter constant $\mathcal{Q}$ is associated with a rank-two Killing tensor rather than a Killing vector, and therefore lies outside the present framework. Hence the work we present here will only allow for the inspiral of eccentric equatorial orbits in the Kerr geometry; further work will be required to incorporate inclined or generic orbits.

We begin by considering a spacetime with metric $g_{\alpha\beta}$ admitting a Killing vector ξ^α , alongside a perturbed metric $g_{\alpha\beta} = g_{\alpha\beta} + h_{\alpha\beta}$ sourced by a matter distribution with a stress-energy tensor $T^{\alpha\beta}$ satisfying $\nabla_\beta T^{\alpha\beta} = 0$, and a spacelike hypersurface Σ_t defined by $t = \text{constant}$. We provide a diagrammatic representation of the geometric setup in Fig. 10.1. Associated with these, we define a charge

$$Q_\xi(\Sigma_t) = \int_{\Sigma_t} T^\alpha_\beta \xi^\beta d\Sigma_\alpha, \quad (10.18)$$

where

$$d\Sigma_\alpha = n_\alpha d\Sigma, \quad d\Sigma = \sqrt{|\gamma|} d^3x, \quad (10.19)$$

is the future-directed surface element induced from $g_{\alpha\beta}$, γ is the determinant of the induced metric on Σ_t , and x^i are coordinates on Σ_t . The quantity

$$n_\alpha = -N\partial_\alpha t \quad (10.20)$$

is the future-directed unit normal to Σ_t , where

$$N = \frac{1}{\sqrt{-g^{\mu\nu}\partial_\mu t \partial_\nu t}} \quad (10.21)$$

is the lapse.

We are interested in the change in Q_ξ between two slices and particularly its rate of change. For concreteness, we choose a reference time t and initial and final times,

$$t_1 = t - T \quad \text{and} \quad t_2 = t + T, \quad (10.22)$$

for some $T > 0$. This choice allows us to discuss average rates of change in addition to total change. We consider the change

$$\Delta_T Q_\xi := Q_\xi(\Sigma_{t_2}) - Q_\xi(\Sigma_{t_1}) \quad (10.23)$$

between the charges on the two slices. Define a volume V_T with boundary $\partial V_T = \Sigma_{t_2} \cup \Sigma_{t_1} \cup \Gamma_T$, where Γ_T is a timelike boundary surface with two components: one near the bifurcation sphere $\mathcal{B}$ at the inner boundary, and one at large radius enclosing spatial infinity i^0 (see Fig. 10.1).

Applying Stokes' theorem, we find

$$\begin{aligned} \Delta_T Q_\xi &= \int_{V_T} \nabla_\alpha (T^{\alpha\beta} \xi_\beta) dV - \int_{\Gamma_T} T^\alpha_\beta \xi^\beta d\Sigma_\alpha \\ &= \frac{1}{2} \int_{V_T} T^{\alpha\beta} \mathcal{L}_\xi g_{\alpha\beta} dV - \int_{\Gamma_T} T^\alpha_\beta \xi^\beta d\Sigma_\alpha, \end{aligned} \quad (10.24)$$

where we have used $\nabla_\beta T^{\alpha\beta} = 0$, together with the identity $T^{\alpha\beta} \nabla_\alpha \xi_\beta = \frac{1}{2} T^{\alpha\beta} \mathcal{L}_\xi g_{\alpha\beta}$.

Here $g_{\alpha\beta}$ denotes the full spacetime metric; since ξ^α is Killing only with respect to the background $g_{\alpha\beta}$, one has $\mathcal{L}_\xi g_{\alpha\beta} = \mathcal{L}_\xi h_{\alpha\beta} \neq 0$ in general, and the volume term encodes the flux of energy/angular momentum sourced by the perturbation. The outward-directed surface element on Γ_T is $d\Sigma_\alpha = r_\alpha dS$, and $dV = N dt d\Sigma$ is the 4-dimensional volume element. We can now write the instantaneous rate of change as

$$\frac{dQ_\xi}{dt} = \frac{1}{2} \int_{\Sigma_t} T^{\alpha\beta} \mathcal{L}_\xi g_{\alpha\beta} N d\Sigma - \int_{\mathcal{B}_t} N T^{\alpha\beta} \xi_\beta r_\alpha d\Omega - \int_{i^0} N T^{\alpha\beta} \xi_\beta r_\alpha d\Omega, \quad (10.25)$$

where the boundary integrals are evaluated at the bifurcating 2-sphere $\mathcal{B}_t$ and at spatial infinity i^0 respectively.

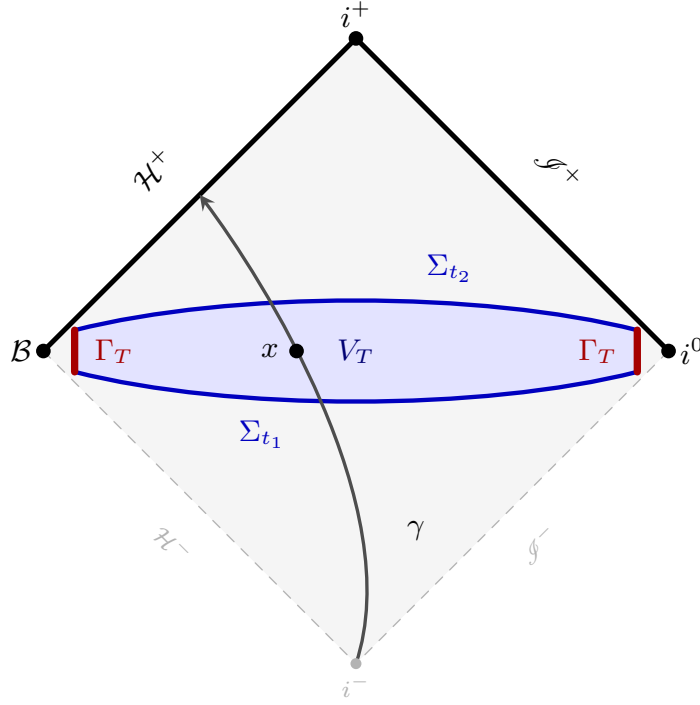


Figure 10.1: Penrose diagram of the black hole exterior. The constant- t hypersurfaces Σ_{t_1} and Σ_{t_2} run from the bifurcation sphere $\mathcal{B}$ to spatial infinity i^0 , bounding the volume V_T (shaded). The timelike boundary Γ_T connects the two slices at each end. A representative worldline γ of the small body travels from i^- , passes through V_T at x , and crosses $\mathcal{H}^+$.

10.4 LEADING ORDER BALANCE

Importantly, the leading-order results derived below are valid at $\mathcal{O}(\epsilon\lambda)$ and apply equally to the interaction with the background and to the nonlinear response equations. Starting from Eq. (10.25), the instantaneous rate of change of the particle's orbital charge receives a contribution at $\mathcal{O}(\epsilon\lambda)$ from the cross-coupling between the particle and matter stress-energies. Restricting attention to the component of the charge sourced by the particle stress-energy $T_{(p)}^{\alpha\beta}$, Eq. (10.25) reduces to

$$\frac{dQ_{\xi}^{(p)}}{dt} = \frac{1}{2} \int_{\Sigma_t} T_{(p)}^{\alpha\beta} \mathcal{L}_{\xi} h_{\alpha\beta} N d\Sigma, \quad (10.26)$$

where the boundary term now vanishes due to the point-particle stress-energy tensor having zero support on the boundaries. This contrasts with the charge associated with the matter stress-energy tensor, for which content accreting onto

the central black hole would accumulate on $\mathcal{B}$ and contribute a non-vanishing boundary flux.

Isolating the $\mathcal{O}(\epsilon\lambda)$ cross-term from the orbit and re-writing the metric perturbation $h^{(\text{env})}$ in terms of its Green's function representation, one obtains,

$$\frac{dQ_{\xi}^{(\text{p})}}{dt} = \frac{1}{2} \int_{\Sigma_t} T_{(\text{p})}^{\alpha\beta} \mathcal{L}_{\xi} h_{\alpha\beta}^{(\text{env})} d\Sigma = \frac{1}{2} \int_{-\infty}^{\infty} dt' \int_{\Sigma_{t'}} \int_{\Sigma_t} T_{(\text{p})}^{\alpha\beta} \mathcal{L}_{\xi} G_{\alpha\beta\alpha'\beta'}^{\text{Ret}} T_{(\text{env})}^{\alpha'\beta'} d\Sigma d\Sigma', \quad (10.27)$$

where through the geometric construction of our hypersurfaces we can write,

$$\int_V dV' = \int_{-\infty}^{\infty} dt' \int_{\Sigma_{t'}} d\Sigma' \quad (10.28)$$

We then decompose $G^{\text{Ret}} = G^{\text{Sym}} + G^{\text{Rad}}$ and use the adjoint Killing-derivative identities; the change in the orbital charge expands as

$$\begin{aligned} \frac{dQ_{\xi}^{(\text{p})}}{dt} &= \int_{\Sigma_t} d\Sigma \int_{V'} dV' T_{(p)}^{\alpha\beta} \mathcal{L}_{\xi} G_{\alpha\beta\alpha'\beta'}^{\text{Ret}} T_{(\text{env})}^{\alpha'\beta'} \\ &= - \int_{\Sigma_t} d\Sigma \int_{V'} dV' T_{(\text{env})}^{\alpha'\beta'} \mathcal{L}_{\xi'} G_{\alpha'\beta'\alpha\beta}^{\text{Adv}} T_{(p)}^{\alpha\beta}, \end{aligned}$$

where we used the Killing-derivative identity $\mathcal{L}_{\xi} G = -\mathcal{L}_{\xi'} G$ and the symmetry of $G^{\text{Ret/Adv}}$ to move the Lie derivative from the source point back to the field point. Next we wish to re-express the particle stress-energy tensor in terms of a metric perturbation as defined through integration with a Green's function of the linearised Einstein operator. We do this by defining a new effective source given by,

$$T_{(\text{p}|t^*)}^{\alpha\beta} := T_{(\text{p})}^{\alpha\beta}(t, \mathbf{x}) \delta(t - t^*),$$

which localises the point-particle source to the precise time at which we measure the rate of change of the orbital constants of motion. Subsequently we define the quantity

$$h_{\alpha'\beta'}^{(\text{p,Adv}|t)} = \int dV'' G_{\alpha'\beta'\alpha''\beta''}^{\text{Adv}} T_{(\text{p}|t)}^{\alpha''\beta''}, \quad (10.29)$$

where $h_{\alpha'\beta'}^{(\text{p,Adv}|t)}$ represents the corresponding advanced metric perturbation whose source is temporally localised precisely to the evaluation time of the rate of change of the conserved quantity; notably, this corresponds to a field with a non-conserved stress-energy tensor. This gives the final expression,

$$\frac{dQ_{\xi}^{(\text{p})}}{dt} = -\frac{1}{2} \int_{-\infty}^t dt' \int_{\Sigma_{t'}} T_{(\text{env})}^{\alpha'\beta'} \mathcal{L}_{\xi'} h_{\alpha'\beta'}^{(\text{p,Adv}|t)} d\Sigma'. \quad (10.30)$$

Resulting in the precise extension of Newton's Third Law to charges associated with approximate Killing symmetries in curved spacetime. We note that integrating the localised perturbation over all time recovers the full perturbation,

$$\int_{-\infty}^{\infty} h_{\alpha'\beta'}^{(\cdot|t)}(t', \mathbf{x}') dt = h_{\alpha'\beta'}^{(\cdot)}(t', \mathbf{x}'). \quad (10.31)$$

Importantly, $h_{\alpha\beta}^{(\text{p,Adv}|t)}$ is sourced by a non-conserved stress-energy tensor, the solution is purely vacuum except at a single point in spacetime. We expect to calculate such a perturbation to be computed as a vacuum solution to the relaxed Einstein equations, subject to non-conservative jump initial conditions; demonstrating this practically will be an important direction of future work.

We now briefly interpret Eq. (10.30) through the diagrammatic representation in Fig. 10.2. Newton's Third Law at its core is a statement that the force exerted on object A by object B is equal and opposite to the force exerted on B by A at that instant. In its naive form this is an instantaneous, acausal relation and breaks down even in flat space once retardation effects are accounted for.

In the standard picture the interaction occurs largely in the following way: the environment generates a gravitational field that propagates forward in time, at time t this results in a field sourced by all prior times $t' < t$ i.e. the environments retarded field. The derivative of this field evaluated on the worldline and contracted with the four velocity at the measurement time then gives the force on the EMRI. This is precisely the force appearing on the right-hand side of Eq. (10.27), and is illustrated by the red arrows in Fig. 10.2.

The balance law Eq. (10.30) is the relativistic, causal form of Newton's Third Law: it expresses the same rate of change from the perspective of the environment. To identify the reaction force in the environment's frame at a fixed instant t , one must integrate over the environment's entire past history $t' < t$, collecting every way in which the environment at earlier times will interact with the particle at time t . This is precisely what the localised advanced field $h^{(\text{p,Adv}|t)}$ encodes: in the time-reversed picture, it is the field emitted by the particle at time t and received by the environment at the earlier time t' . The right-hand side of Eq. (10.30) therefore integrates the environment's past stress-energy against this localised advanced field, as illustrated by the orange arrows in Fig. 10.2.

The localisation of the particle source to the single time t can also be understood from this picture. It is constructed such that the past instances of the environment, at all times t' , receive the advanced field from only a single future time slice, i.e. the slice on which we measure the force. Without this localisation, the integral would account for the full accumulated interaction between the environment and the particle along the entire worldline. Although it may seem strange, it is precisely this localisation that makes Eq. (10.30) the correct expression for $dQ_{\xi}^{(\text{p})}/dt$ at a given instant.

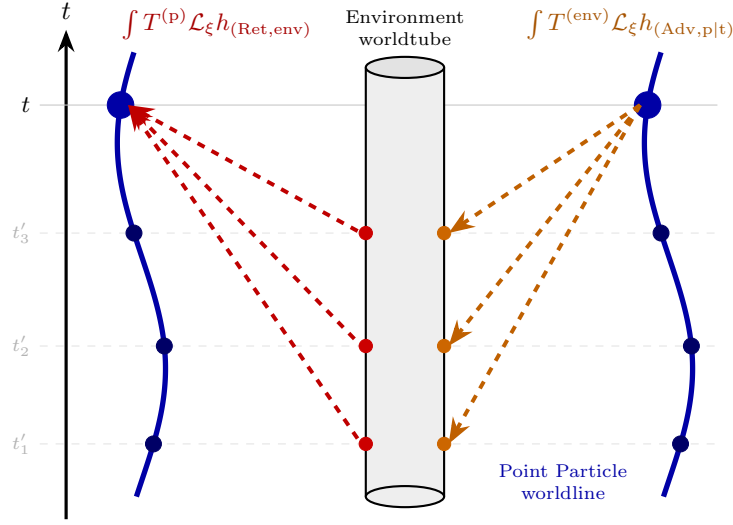


Figure 10.2: Visual representation of how the balance law takes the regular force on the particle from the environment's retarded field (left side of diagram) and relates it to the fluid perspective of how the environment will interact with the particle's potential in the future (right side of diagram), showing explicitly the causal form of Newton's third law in curved space from the view of distinct time slices.

Having derived the instantaneous force, we turn to the orbit-averaged rate of change of (10.30), which through the definition of the long time average,

$$\langle \cdot \rangle = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T dt \int_{\Sigma_t} d\Sigma' \quad (10.32)$$

simplifies to,

$$\left\langle \frac{dQ_\xi^{(p)}}{dt} \right\rangle = - \lim_{T \rightarrow \infty} \frac{1}{4T} \int_{V_T} T_{(env)}^{\alpha\beta} \mathcal{L}_\xi h_{\alpha\beta}^{Adv,(p)} dV. \quad (10.33)$$

Making use of the identity $h^{Adv} = h^{Ret} - 2h^{Rad}$, we arrive at the orbit-averaged relativistic third law:

$$\left\langle \frac{dQ_\xi^{(p)}}{dt} \right\rangle = - \left\langle \frac{d\hat{Q}_\xi^{(env)}}{dt} \right\rangle - \mathcal{F}_\xi^{env \times p}. \quad (10.34)$$

The third law cleanly splits into two quantities. The first is the integrated matter torques,

$$\left\langle \frac{d\hat{Q}_\xi^{(env)}}{dt} \right\rangle = \lim_{T \rightarrow \infty} \frac{\epsilon\lambda}{4T} \int_{V_T} T_{(env)}^{\alpha\beta} \mathcal{L}_\xi h_{\alpha\beta}^{Ret,(p)} dV, \quad (10.35)$$

which gives the integrated torque applied to the environment by the EMRI. We also obtain a term $\mathcal{F}_\xi^{\text{env}\times\text{p}}$, which collects all energy (or angular momentum) carried to infinity and down the horizon by gravitational waves sourced jointly by the secondary and the matter at $\mathcal{O}(\epsilon\lambda)$. Its form is given explicitly by,

$$\mathcal{F}_\xi^{\text{env}\times\text{p}} = \frac{\epsilon\lambda}{2T} \int_{V_T} \int_{V'_T} T_{(\text{p})}^{\alpha'\beta'} \mathcal{L}_\xi G_{\alpha\beta\alpha'\beta'}^{\text{Rad}} T_{(\text{env})}^{\alpha\beta}. \quad (10.36)$$

We have labelled Eq. (10.36) a flux term as it can be expressed in the standard form of an asymptotic flux by exploiting the factorised (mode-sum) representation of G^{Rad} , which decomposes into a product of ingoing and outgoing wave solutions of the Teukolsky equation.

This is the central result of the orbit-averaged calculation: the rate at which the secondary loses orbital energy (or angular momentum) to the environment is equal and opposite to the rate at which the environment gains that quantity, up to the gravitational-wave flux emitted at $\mathcal{O}(\epsilon\lambda)$. In scenarios where this flux is subdominant, one recovers Newton's third law in averaged form,

$$\left\langle \frac{dQ_\xi^{(\text{p})}}{dt} \right\rangle \approx - \left\langle \frac{d\hat{Q}_\xi^{(\text{env})}}{dt} \right\rangle. \quad (10.37)$$

Indeed, such a relation has here been derived at leading $\mathcal{O}(\epsilon\lambda)$ order, which in the case of an axisymmetric and stationary background would give zero. The relation required to understand the dynamics in both the case of the boson cloud and the thin disk presented previously thus requires the calculation to be pushed to $\mathcal{O}(\epsilon^2\lambda)$. Such a calculation is already well underway, with some details remaining to be ironed out; we omit the result from this thesis, directing the reader to the publication in preparation which we hope to release in the coming months, 2026.

Part IV

CONCLUSIONS

THE PATH FORWARD

We began this thesis motivated by the prospect of the upcoming LISA mission's ability to gain new insights into the behaviour of matter at the centres of galaxies in the strong-gravity regime. For decades, complex modelling of hydrodynamical and dark matter structures has been commonplace in the Newtonian literature. At the same time, precision vacuum EMRI modelling in the relativistic literature has seen enormous progress over the past decade, culminating in large part in the first second-order self-force calculation in the early 2020s [52]. Until recently, the same could not be said for the relativistic modelling of EMRIs in environments, with applications of overly simplistic force formulae remaining commonplace. This body of work sits at the onset of lifting relativistic EMRI modelling into the realm of complex astrophysical modelling, building toward more realistic EMRI–matter models and working toward the consistent treatment of their dynamics.

The scientific purpose of such developments is twofold: to create precision waveform models for relativistic EMRIs in environments and to gain a handle on the rich physics which can arise in dynamically coupled matter-binary systems in strong gravity. With recent advancements in black hole perturbation theory, public codes can now generate metric perturbation data for circular equatorial orbits in the Kerr geometry [60], along with progress laying out the route ahead for generic orbits [259, 260] the pre-requisite vacuum tools are available, setting the stage for advanced matter modelling around relativistic binaries.

11.1 REFLECTION

With the metric data in hand, complex astrophysical modelling of matter around relativistic binaries takes a tractable form: re-expressing the system as forced by the potential $h_{\mu\nu}^{(p)}$, one reduces the problem of solving fully nonlinear GR–matter systems coupled on dynamical backgrounds to that of a forced fluid on the fixed Kerr geometry. Having laid out the theory and background necessary to generate the metric data required for such simulations, this thesis works toward the aim of developing consistently calculated EMRI–environment systems.

The key outcomes of the work presented here can be distilled into three primary components: (i) the first application of such consistent matter–EMRI models in the Kerr geometry, with direct extension to a perfect fluid; (ii) within the matter equations, the connection of asymptotic fluxes with a relativistic form of the local integrated torque densities, making explicit the role of matter fluxes in such

systems; and (iii) the direct connection between the evolution of orbital elements and local torque densities. The results presented here also highlight a commonly misunderstood point: unlike in the vacuum context, the calculation of fluxes in a matter field alone is meaningless with regard to the evolution of a binary inspiral. A connection to the binary mechanics can only be made once one establishes the balance between local torques and asymptotic fluxes for a given matter model as in [62]. The meaningful quantity for evolving orbits is the torque, through the relativistic third law.

Our flux balance equations also clarify an additional point: the matter interaction terms carry associated gravitational flux corrections arising at the same perturbative order. After orbit averaging, these fluxes, which would arise from a modified Teukolsky-like setup [261], act as corrections to a Newtonian-type torque balance. Such an understanding only becomes clear after averaging, once the causal structure of the balance is smeared out. As explained in the previous chapter, any instantaneous balance will necessarily carry a complicated temporal structure, resulting in the need for the instantaneous particle perturbation.

The torque balance approach laid out here is, admittedly, one of pragmatism. One could always have directly taken the stress-energy tensor corresponding to a specific matter field, calculated the associated regularised metric perturbation from the linearised Einstein equation, and included the effective forcing term in the geodesic equation, evaluating the force along the worldline. Such an approach would be direct and would in principle yield all dissipative and conservative effects on the orbit, effects that we do not fully capture here. The prospect would be extremely valuable, but also extremely complex, requiring in general nonvacuum metric reconstruction of dynamical fields with non-compact support and development of the matched asymptotics in the context of matter. In addition, such an approach would require researchers fluent simultaneously in EMRI dynamics, relativistic matter simulations, and black hole perturbation theory, a combination of expertise that remains rare to say the least.

The approach presented here instead allows a much simpler route forward. A stress-energy tensor is computed on a fixed Kerr background, with or without an effective forcing function from a secondary object. The matter field solution is then integrated against a precomputed point-particle metric perturbation, immediately yielding the energy and angular momentum change in the orbit. The hope is that such an approach provides a direct route toward realistic astrophysical EMRI modelling in the near term, a prospect largely considered unfeasible four years ago, at the onset of this work.

11.2 OUTLOOK

The future ahead in this field is rich, with the models presented here representing a subset of simplified scenarios that make the point: modelling of relativistic EMRIs in complex environments is feasible and the route ahead is clear. True progress in this direction will require collaboration between the theoretical astrophysics and black hole perturbation theory communities. The years to come promise an era of realistic astrophysical modelling at the interface of astrophysics and self-force theory. Among the many exciting directions ahead are the following.

1. **Extension to Semi-Analytical Treatment of Discs:** In this work we present the first explicit calculation of an EMRI in a fluid environment which includes pressure. Notably, our fluid model remains extremely simple and does not feature a radial accretion flow, let alone explicit treatment of viscosity or radiation. Building upon the linearised framework developed here to incorporate such effects, viscosity, radiation transfer, and magnetic fields would constitute a natural and important extension towards more realistic disc models for LISA.
2. **Calculation of the Instantaneous Particle Perturbations:** A clear next step is the calculation of the instantaneous particle perturbation. Such a development will allow us to build the torque infrastructure required for the rapid evaluation of orbital torques, as described previously. It will further act as a necessary step towards the broader programme of environmental waveform modelling for LISA.
3. **Extensions to Generic Orbits:** EMRIs are expected to result from a wide range of formation channels, some of which are expected to place EMRIs in the LISA band whilst on highly eccentric orbits. Forthcoming work by Cunningham, Dolan, and Wardell [260] will deliver the extension of the Lorenz gauge circular orbit metric reconstruction to generic orbits in the Kerr geometry. Such a development will open the door to calculating EMRI–matter interactions on much more complex classes of trajectories.
4. **Matched Asymptotics with Matter:** One of the most visible gaps in the work presented here is the question of how the procedure of matched asymptotic expansions generalises to the coupled secondary–matter context. Work is currently under way to develop such a prescription for determining the *physical* local field near the particle, which would provide an exact analogue of the local self-force with environmental coupling. Such an endeavour is already under way in collaboration with Samuel Upton and will constitute a foundational step in understanding the motion of small bodies in general relativity coupled to matter.

5. **Connection with Hydro Simulations:** Building upon the previous direction, one of the clearest and most exciting next steps will be interfacing our formalism and results with realistic nonlinear hydrodynamical simulations. Such a route, in conjunction with the instantaneous torque balance equation, delivers a direct path towards understanding the complex causal interaction between dynamical environments and EMRIs. Such an extension promises deep insights into the behaviour of matter in strong gravity.
6. **Electromagnetic Counterparts:** EMRIs in AGN fluids and dark matter environments are not the only systems to which our formalism applies. Equally relevant are the interactions of EMRIs with tidal disruption event (TDE) transients, whereby a stellar object is disrupted near a supermassive black hole, forming a bright transient accretion disc. Quasi-periodic eruptions in these signals have been largely proposed to originate from stellar-mass compact object interactions with these discs. The formalism laid out here is equally applicable to describe such systems in the relativistic regime.

11.3 THE FINAL WORD

Through this thesis we have tackled the problem of motion through matter in curved space. Until recently, works on relativistic matter-EMRIs had failed to capture the complex matter response arising in the presence of an EMRI inspiral or to properly evolve the EMRI parameters through first-principles methods. We address both shortcomings, calculating for the first time the self-consistent matter response in the Kerr geometry for both bosonic fields and in the context of a true fluid prescription, a task often described as impractical in the field prior to the work presented here. In this work we have begun the development and the beginnings of a concrete understanding of EMRI-matter interaction through the construction of a relativistic extension to Newton's third law. With the foundations of EMRI-matter dynamics beginning to take shape, the road ahead is rich. Modelling more complex trajectories and environments now appears tractable, bringing EMRI modelling closer to true astrophysical realism. With LISA launching within the decade, the time has come when the development of such models is not only of theoretical interest but of the utmost importance, holding the promise of probing matter in the extreme and revealing dynamics richer than could have been imagined.

"This new learning amazes me, Sir Bedevere. Explain again how sheep's bladders may be employed to prevent earthquakes."

— MONTY PYTHON AND THE HOLY GRAIL

DECLARATION

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Conor Dyson, 17 Blegdamsvej, Copenhagen, Denmark, June 2026

Part V

APPENDIX

APPENDIX

A.1 CONVERGENCE OF MAGNUS RESULTS

In this Appendix we provide additional numerical evidence for the convergence of our results. This section is fully contained within the published article [59]

For the case of a particle-like medium, there are two factors to take into account: On the one hand, we numerically integrate over a disc in the impact plane. That numerical integration is performed on interpolants which use N^2 points. For each of those points the forces are computed analytically as described in the main text. Therefore, the only parameter controlling the numerical convergence is given by N . Fig. A.1 shows the aerodynamic forces for three different resolutions. The insets for the Magnus and Lift forces show that increasing the number of points achieves convergence.

Ultimately we want to extract the values of the forces for a cloud of a fixed size $b_{\max}$ where the quantities, $\delta F_D, F_M, F_L$ should *not* depend on the cloud size (since they appear due to the strong field region near the BH). Fig. A.1 shows that increasing $b_{\max} \geq 1000$ does not change the aerodynamic forces significantly, allowing us to safely perform the extraction at $b_{\max} = 1500$. The interpolant is constructed using the highest resolution available, given by $N = 64$. It uses $N_{\text{Cheb}} = 16^3$ points distributed in the ranges $a/M \in [0.05, 0.9]$, $\beta \in [0.05, 1.52]$ (reflection symmetry is later enforced, but has been checked separately), and $v \in [0.15, 0.925]$.

For the wave-like environment, there are also two convergence factors to consider. One being the accuracy requirements of the shooting method in numerically solving for the radial component of the scalar field. The other is the number of ℓ -modes summed to obtain the result. The accuracy of the numerical solver was found to be extremely stable once an explicit Runge-Kutta method was implemented and so was set with a precision goal of 16 and allowed 32 digits of arbitrary precision arithmetic. The more subtle factor in convergence however is the ℓ -mode sum. Following the same ethos as the cloud size in the convergence check for the particle case. We show in Fig. A.2 that for the case $\mu = 0.2$ we have rapid convergence in our measured forces as a function of $\ell_{\max}$ (which has an interpretation of cloud size here). In practice, we choose $\ell_{\max} = 15$ as an appropriately converged number of ℓ -modes for the calculation of the nodes in the Chebyshev interpolant. We have also additionally taken the time to check that at the most extreme edges of our Chebyshev grid (around $a = 0.9997, v = 0.95$) $\ell_{\max} = 15$ is also sufficient for convergence. The interpolant is constructed using 20 nodes in the spin and angular

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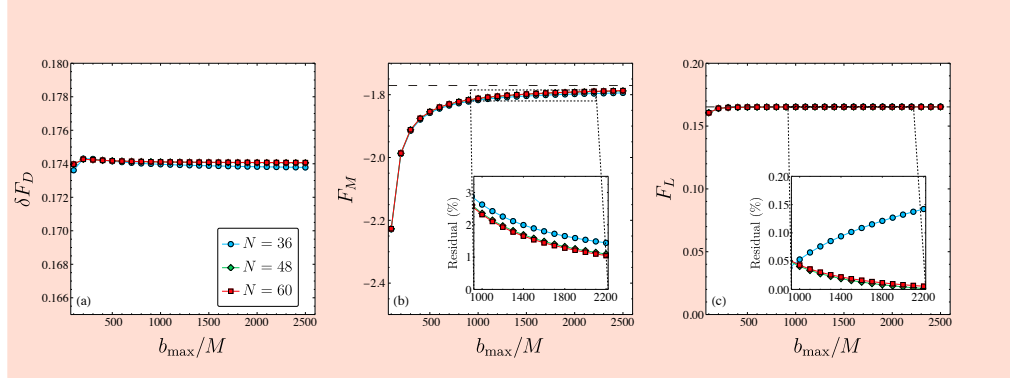


Figure A.1: **(a)**: Difference between the drag force evaluated at spin $a/M = 0.7$ and at very low spin, $a/M = 0.01$, keeping fixed the velocity $v = 0.3$ and the impact angle $\beta = \pi/4$, for different numerical resolutions (different colors, as shown in the label), as a function of the size of the cloud. We observe that this rapidly converges to a constant, showing that the spin-variation of the drag force is independent of the cloud size. **(b)**: Scaling of the Magnus force with the cloud size, for a BH spin $a/M = 0.7$, and the rest of the parameters equal to the previous panel. The inset panel shows the residual between the asymptotic value (obtaining by fitting the highest resolution available to a series of the type $A + c_1/b_{\max} + c_2/b_{\max}^2$) and the data points, in percent level. This shows that the higher resolutions converge towards zero, achieving an accuracy of around percent level for the Magnus. **(c)**: Same, but for the lift force.

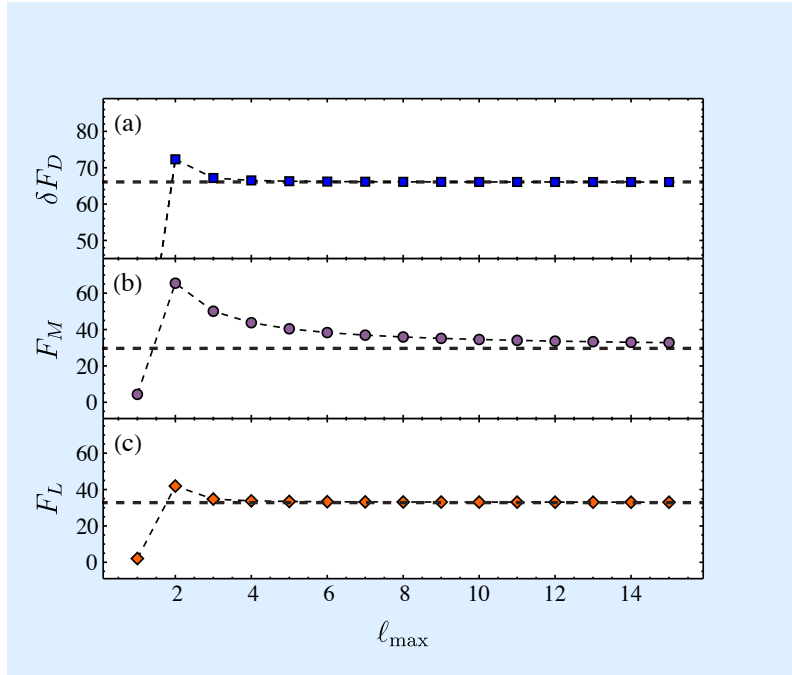


Figure A.2: Spin aerodynamic forces evaluated at spin $a/M = 0.95$, $v = 0.8$ and the impact angle $\beta = \pi/3$, as function of the $\ell_{\max}$. We observe that this variation in drag and lift forces rapidly converges to a constant, whereas the Magnus has a slightly slower convergence rate just in the case of the particle medium.

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directions, in the ranges $a/M \in [10^{-5}, 0.9997]$ and $\beta \in [10^{-5}, \pi - 10^{-5}]$, and 40 points in the velocity range given by $v \in [0.05, 0.95]$.

A.2 EMRIS IN BOSON CLOUDS APPENDIX

This section contains the appendices of the published article [61].

S.1 Fluxes

In addition to GWs, the secondary will induce emission of scalar waves, both to infinity and through the horizon of the primary. These will in turn determine how the secondary evolves. Here, we derive the explicit form of the flux formulae in the case of boson clouds, and compare our results with previous work in Schwarzschild [193]. In this section, we suppress much of the perturbative indexing in the interest of readability. However, the order of most quantities should be clear from context.

The (orbit-averaged) energy fluxes of the perturbed field to infinity and through the horizon can be calculated, respectively, as [82, 262]

$$\begin{aligned}\dot{E}^{\Phi, \infty} &= - \lim_{r \rightarrow +\infty} r^2 \int d\Omega T_{\mu r}^{\Phi} \xi_{(t)}^{\mu}, \\ \dot{E}^{\Phi, H} &= \lim_{r \rightarrow r_+} 2Mr_+ \int d\Omega T_{\mu\nu}^{\Phi} \xi_{(t)}^{\mu} l^{\nu},\end{aligned}\tag{A.1}$$

where

$$T_{\mu\nu}^{\Phi} = \nabla_{(\mu} \Phi \nabla_{\nu)} \Phi^* - \frac{g_{\mu\nu}}{2} (\nabla_{\delta} \Phi \nabla^{\delta} \Phi^* + \mu^2 |\Phi|^2),\tag{A.2}$$

and $d\Omega$ is the area element of the 2-sphere, $l^{\mu} = \partial/\partial t + \Omega_H \partial/\partial \varphi$ is a null vector normal to the horizon, with $\Omega_H = a/(2Mr_+)$ and $\xi_{(t)}^{\mu} \equiv \partial/\partial t$, $\xi_{(\varphi)}^{\mu} \equiv \partial/\partial \varphi$ are the Killing vectors of the Kerr metric. Analogous expressions for angular momentum fluxes are obtained by swapping $\xi_{(t)}^{\mu} \rightarrow \xi_{(\varphi)}^{\mu}$ in Eq. (A.1). We define the mass of the background boson cloud to be given by the volume integral on a spacelike slice of the time component of the stress-energy tensor. In particular, we define

$$M_c = \int_{r_H}^{\infty} \int_{S^2} T_t^t[\phi^{(1,0)}, g^{\text{Kerr}}] r^2 dr d\Omega.\tag{A.3}$$

We decompose $\phi^{(1,1)}$ in a spheroidal harmonics basis,

$$\phi^{(1,1)} = \sum_{\ell, m} \phi_{\ell m}^{(1,1)}(r) S_{\ell m}(\theta, \varphi, \gamma_{m_g}) e^{-i(\Omega_{m_g} + \omega_c)t},\tag{A.4}$$

where $\gamma_{m_g} = a \sqrt{(\Omega_{m_g} + \omega_c)^2 - \mu^2}$ and for circular orbits $\Omega_{m_g} = (m - m_b)\Omega_p \equiv m_g\Omega_p$. The above equations lead to the following expressions for the energy fluxes of individual modes:

$$\begin{aligned}\dot{E}_{\ell m}^{\Phi, \infty} &= \lim_{r \rightarrow +\infty} r^2 \left\{ 2 |\omega_c + \Omega_{m_g}| \right. \\ &\quad \times \operatorname{Re} \left[\sqrt{(\Omega_{m_g} + \omega_c)^2 - \mu^2} \right] |\phi_{\ell m}^{(1,1)}|^2 \left. \right\}, \\ \dot{E}_{\ell m}^{\Phi, H} &= \lim_{r \rightarrow r_+} 2Mr_+ \left\{ 2 (\omega_c + \Omega_{m_g}) \right. \\ &\quad \times (\omega_c + \Omega_{m_g} - m\Omega_H) |\phi_{\ell m}^{(1,1)}|^2 \left. \right\}.\end{aligned}\tag{A.5}$$

Here, we have set the frequency at the superradiant threshold, i.e., $\omega = \omega_c$ and we remind the reader that at leading-order $\phi_{\ell m}^{(1,1)} \propto 1/r$ when $r \rightarrow \infty$.

Similarly, the angular momentum fluxes are given by

$$\begin{aligned}\dot{L}_{\ell m}^{\Phi, \infty} &= \lim_{r \rightarrow +\infty} r^2 \left\{ 2ms_{m_g} \right. \\ &\quad \times \operatorname{Re} \left[\sqrt{(\Omega_{m_g} + \omega_c)^2 - \mu^2} \right] |\phi_{\ell m}^{(1,1)}|^2 \left. \right\}, \\ \dot{L}_{\ell m}^{\Phi, H} &= \lim_{r \rightarrow r_+} 2Mr_+ \left\{ 2m (\omega_c + \Omega_{m_g} - m\Omega_H) |\phi_{\ell m}^{(1,1)}|^2 \right\},\end{aligned}\tag{A.6}$$

where $s_{m_g} \equiv \operatorname{sgn}(\omega_c + \Omega_{m_g})$. While $\dot{E}^{\Phi, \infty/H}$ can be truly associated as a “scalar flux,” the scalar perturbations also affect the cloud, which in turn impacts the evolution of the secondary. Assuming an adiabatic evolution and conservation of energy and angular momentum, we thus have:

$$\begin{aligned}\dot{E}_{\text{orb}} + \dot{M}_c &= -\dot{E}^{\text{GW}, \infty} - \dot{E}^{\text{GW}, H} - \dot{E}^{\Phi, \infty} - \dot{E}^{\Phi, H}, \\ \dot{L}_{\text{orb}} + \dot{S}_c &= -\dot{L}^{\text{GW}, \infty} - \dot{L}^{\text{GW}, H} - \dot{L}^{\Phi, \infty} - \dot{L}^{\Phi, H},\end{aligned}\tag{A.7}$$

where S_c is the spin of the cloud. This balance equation thus allows one to evolve orbital parameters due to energy emission from the secondary and/or environment. Importantly, this formula excludes the effects of conservative energy transfer between the orbit and bound states of the cloud. As such, it should be used with some caution until all contributions up to order $\mathcal{O}(\epsilon^2 q^2)$ (e.g., $h^{(2,1)}$) are fully understood. Nevertheless, in the absence of a complete understanding, this equation can be used as a first step towards producing time-domain evolutions and generating relativistic waveforms for EMRIs with environments in Kerr. Some insights into how such conservative transfer might occur in a relativistic setting have been studied in [263].

To compute the rate at which the mass and spin of the cloud changes, we make use of the global $U(1)$ symmetry of the (complex) scalar field, whose conserved current implies the existence of a conserved Noether charge Q :

$$Q = \int_{\Sigma} d^3x \sqrt{-g} j_{\Phi}^0,\tag{A.8}$$

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where Σ is a space-like hypersurface and

$$j_\mu^\Phi = -i (\Phi^* \partial_\mu \Phi - \Phi \partial_\mu \Phi^*) . \quad (\text{A.9})$$

The mass and spin of the cloud are then related to the cloud's Noether charge, i.e., $M_c = \omega_c Q$, $S_c = m_b Q$, respectively. The rate of change of the scalar charge is

$$\begin{aligned} \dot{Q}^{\Phi,\infty} &= - \lim_{r \rightarrow +\infty} r^2 \int d\Omega j_r^\Phi , \\ \dot{Q}^{\Phi,H} &= \lim_{r \rightarrow r_+} 2Mr_+ \int d\Omega j_\mu^\Phi l^\mu , \end{aligned} \quad (\text{A.10})$$

leading to:

$$\begin{aligned} \dot{Q}_{\ell m}^{\Phi,\infty} &= - \lim_{r \rightarrow +\infty} r^2 \left\{ 2s_{m_g} \right. \\ &\quad \times \text{Re} \left[\sqrt{(\Omega_{m_g} + \omega_c)^2 - \mu^2} \right] |\phi_{\ell m}^{(1,1)}|^2 \Big\} , \\ \dot{Q}_{\ell m}^{\Phi,H} &= - \lim_{r \rightarrow r_+} 2Mr_+ \left\{ 2(\omega_c + \Omega_{m_g} - m\Omega_H) |\phi_{\ell m}^{(1,1)}|^2 \right\} . \end{aligned} \quad (\text{A.11})$$

Through the Noether charge, we can then define the scalar energy and angular momentum “power” as

$$\begin{aligned} \dot{E}^{s,\infty/H} &= \dot{E}^{\Phi,\infty/H} + \omega_c \dot{Q}^{\Phi,\infty/H} , \\ \dot{L}^{s,\infty/H} &= \dot{L}^{\Phi,\infty/H} + m_b \dot{Q}^{\Phi,\infty/H} . \end{aligned} \quad (\text{A.12})$$

Plugging in Eqs. (A.5), (A.6) and (A.11) in the balance equation (A.7), we then find the explicit terms shown in Fig. 8.2:

$$\begin{aligned} \dot{E}_{\ell m}^{s,\infty} &= \lim_{r \rightarrow +\infty} r^2 \left\{ 2\Omega_{m_g} s_{m_g} \right. \\ &\quad \times \text{Re} \left[\sqrt{(\Omega_{m_g} + \omega_c)^2 - \mu^2} \right] |\phi_{\ell m}^{(1,1)}|^2 \Big\} , \\ \dot{E}_{\ell m}^{s,H} &= \lim_{r \rightarrow r_+} 2Mr_+ \left\{ 2\Omega_{m_g} (\omega_c + \Omega_{m_g} - m\Omega_H) |\phi_{\ell m}^{(1,1)}|^2 \right\} . \end{aligned} \quad (\text{A.13})$$

Finally, for the angular momentum, we have

$$\begin{aligned} \dot{L}_{\ell m}^{s,\infty} &= \lim_{r \rightarrow +\infty} r^2 \left\{ 2m_g s_{m_g} \right. \\ &\quad \times \text{Re} \left[\sqrt{(\Omega_{m_g} + \omega_c)^2 - \mu^2} \right] |\phi_{\ell m}^{(1,1)}|^2 \Big\} , \\ \dot{L}_{\ell m}^{s,H} &= \lim_{r \rightarrow r_+} 2Mr_+ \left\{ 2m_g (\omega_c + \Omega_{m_g} - m\Omega_H) |\phi_{\ell m}^{(1,1)}|^2 \right\} , \end{aligned} \quad (\text{A.14})$$

which satisfy

$$\dot{E}^{s,\infty/H} = \Omega_p \dot{L}^{s,\infty/H} . \quad (\text{A.15})$$

Consequently, the backreaction from the leading-order scalar fluxes onto the secondary evolves circular orbits into circular orbits, justifying the quasi-circular limit studied in this work. Because of this relation (A.15), it is also sufficient to look at the energy fluxes only, as is done in the main text.

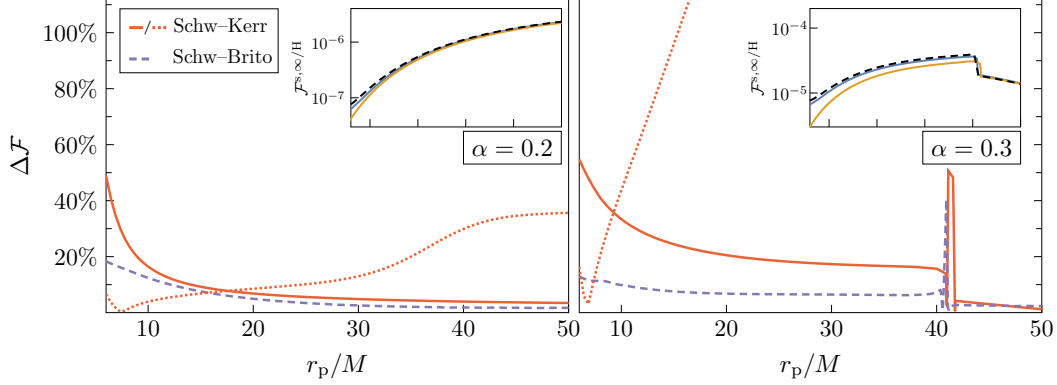


Figure A.3: We show the relative error $\Delta\mathcal{F}$ in the total flux to infinity (solid lines) and through the horizon (dotted lines), taking our Schwarzschild data as a reference. We consider a prograde orbit and $\alpha = 0.2$ (left panel) or $\alpha = 0.3$ (right panel). The comparison is between our Kerr and Schwarzschild (red lines) and between our Schwarzschild and the Schwarzschild from an earlier work [193] (purple line), which used a different gauge. In the inset, we show the actual flux $\mathcal{F}^{s,\infty}$ in Schwarzschild/Kerr from our data (blue/orange solid) compared to the one from [193] (black dashed). The horizontal axis is the same as in the main plot.

S.1.1 Comparisons

We shortly detail the derivation of the Newtonian fluxes [132–134] shown in the main text (Fig. 8.2) and compare the fluxes between Kerr and Schwarzschild, as well as between our Schwarzschild results and those from an earlier work [193].

The study of dynamical friction in gravitational atoms in the Newtonian regime has been referred to as *ionization* [132], due to analogy with atomic physics (see [134] for a thorough comparison between “classic” dynamical friction and ionization). Below, we summarize the ionization process and refer to [132] for details.

In the language of quantum mechanics, ionization describes the transfer of the cloud from its bound state $|n_b \ell_b m_b\rangle$, to any unbound state $|k \ell m\rangle$, where k represents the wavenumber. This process is governed by the coupling strength between these states, defined by the matrix element:

$$\eta = \langle k \ell m | V | n_b \ell_b m_b \rangle, \quad (\text{A.16})$$

where V is the gravitational perturbation of the secondary, which is expressed as a multipole expansion of the Newtonian potential.

The “ionization power” is then found by summing over all the unbound states. On circular, equatorial orbits, it is given by

$$P_{\text{ion}} = \frac{M_c}{\mu} \sum_{\ell, m} \Omega_{m_g} |\eta(k_*)|^2 \Theta(k_*^2), \quad (\text{A.17})$$

where $k_* = -\mu\alpha^2/(2n_b^2) \pm \Omega_{m_g}$ and Θ is the Heaviside step function. This quantity is equivalent to the energy flux to infinity, $P_{\text{ion}} \equiv \dot{E}^{s,\infty}$.

In Schwarzschild, massive scalar fields still settle on quasi-bound states, even though spin is essential for superradiance to occur. However, these states are decaying ($M\text{Im}[\omega_c] < 0$) due to absorption at the horizon, preventing the cloud from achieving a stationary configuration. In the Schwarzschild case, following [193], we explicitly ignore this, setting $M\text{Im}[\omega_c] = 0$.

In the main text (Fig. 8.2), we compare fluxes in the Newtonian, Schwarzschild and Kerr regimes. Here, we focus on comparing in detail two cases: Kerr versus Schwarzschild, and “our” Schwarzschild results versus those from an earlier work [193] which used a different method and gauge. The relative differences in the fluxes at infinity and at the horizon are shown in Fig. A.3, using our Schwarzschild results as the reference.

As expected, at large radii, the Schwarzschild and Kerr flux become similar, with relative differences dropping to a few percent at $r_p = 50M$. Closer to the ISCO, the differences become larger, emphasizing the necessity of performing calculations in Kerr and not in Schwarzschild. When comparing our Schwarzschild results to those in [193], we find relative differences up to 20%, due to gauge ambiguities, as discussed in the main text. In addition, mode-by-mode comparisons reveal notable differences between our results and [193], particularly in the quadrupole.

In the relativistic case, we sum up to $\ell = 6$, ensuring that the flux increment remains below 1% across the considered radial domain. In the Newtonian case instead, the computational cost is much lower, allowing us to easily sum up to $\ell = 10$, which pushes the flux increment below 0.01%. At larger radii than we are showing in this work, more modes might be required to obtain accurate results, which poses a computational challenge in the relativistic regime. Therefore, to compute the total flux across a large radial domain, e.g., for waveform modeling in packages like FEW [195–197], a smooth interpolation between both approaches might be required.

Importantly, the scalar flux we calculate does not describe the complete flux at $\mathcal{O}(\epsilon^2 q^2)$. This is due to the fact that there are two additional gravitational contributions to the energy fluxes which we do not consider. One is due to the expansion of the Einstein operator, $\delta^2 G[h^{(1,2)}, h^{(1,0)}]$ which arises as an additional term in the integrand of $\mathcal{F}_{(2,2)}^{s,\infty/H}$, which we do not calculate. Additionally, there is a term coming from the $\mathcal{O}(\epsilon^2)$ correction to the orbital frequency ($\Omega_p^{(2,0)}$) from the background matter field. Thus, to calculate the full gauge invariant flux at $\mathcal{O}(\epsilon^2 q^2)$ one requires the full expression:

$$\mathcal{F}_{(2,2)}^{\infty/H} \equiv \mathcal{F}_{(2,2)}^{s,\infty/H} + \frac{d\mathcal{F}_{(0,2)}^{\text{GW},\infty/H}}{d\Omega_p^{\text{Kerr}}} \Omega_p^{(2,0)}, \quad (\text{A.18})$$

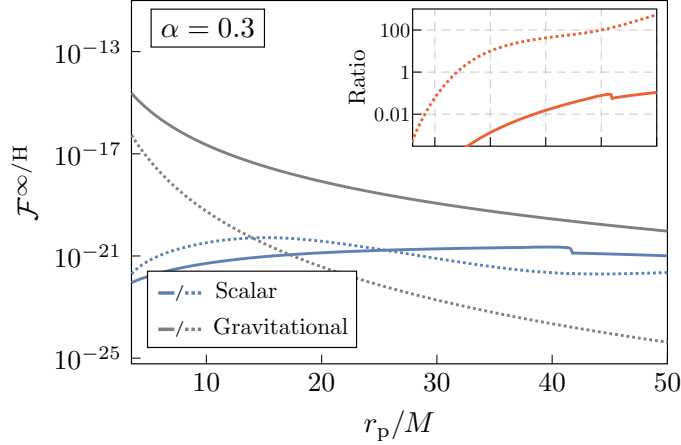


Figure A.4: We show the fluxes to infinity and the horizon for the scalar and gravitational case, including the correct perturbative prefactors, considering $q = 10^{-6}$, $\epsilon^2 = 0.1\alpha^3$, $\alpha = 0.3$. In the inset, we show the ratio between the two, i.e., $\mathcal{F}_s/\mathcal{F}_{\text{GW}}$. The horizontal axis is the same as in the main plot.

where $\mathcal{F}_{(2,2)}^{s,\infty/H}$ should be corrected to include the term arising from the expansion of the Einstein operator and $\mathcal{F}_{(0,2)}^{\text{GW},\infty/H}$ is the first order vacuum flux due to the secondary. Moreover, Ω_p^{Kerr} is the frequency in Kerr for a given orbital radius. The calculation of the $h^{(2,0)}$ and $h^{(2,1)}$ metric perturbations is necessary to calculate these additional terms and is still an open problem.

s.2 Scalar vs GW flux

Previous works have shown that scalar fluxes can dominate over gravitational fluxes at large radii. While we cannot directly compare the two quantities, as they come in at different perturbative orders, i.e., q and ϵ are independent, we can consider an example system. Since both quantities scale as q^2 , this factor cancels out in their ratio. In Fig. A.4, we show a case for which the cloud has obtained a maximum mass $\eta = M_c/M = 0.1$ [119, 120, 264], with a typical EMRI mass ratio $q = 10^{-6}$ and $\alpha = 0.3$. The results show that scalar horizon fluxes quickly dominate over the gravitational horizon fluxes, while the fluxes to infinity will probably overtake at larger radii.

Finally, we note that the horizon flux is less relevant in spherically symmetric structures that were studied before [193, 198]. A possible reason is that for spherical structures the $(\ell, m) = (0, 0)$ mode does not contribute to $\dot{E}^{s,H}$, unlike for dipolar clouds. Since there is no angular barrier for $(\ell, m) = (0, 0)$ modes, those are more “easily” absorbed at the horizon.

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