THE ASTROPHYSICAL JOURNAL, 170:L105–L108, 1971 December 15

LONG WAVE TRAINS OF GRAVITATIONAL WAVES FROM A VIBRATING BLACK HOLE

WILLIAM H. PRESS
California Institute of Technology
Received 1971 October 12

ABSTRACT

The vibrations of a black hole of mass M, perturbed from spherical symmetry, have been studied numerically. Initial perturbations of high spherical-harmonic index ($l \gg 1$) which contain Fourier components of long wavelength ($2\pi M \gtrsim \lambda \gg 2\pi M/l$) produce long-lasting vibrations. The vibrational energy is radiated away gradually as a long, nearly sinusoidal wave train of gravitational radiation with angular frequency $\omega \approx (27)^{-1/2}\, l/M$.

A Schwarzschild black hole, perturbed from spherical symmetry, will radiate gravitational waves to restore sphericity. This fact follows from the recent work of Price (1971), which applied generally to perturbations of any integer-spin, zero-rest-mass field, including gravity. The exact dynamics of this process, for gravitational perturbations, is governed by equations due to Zerilli (1970*a*, *b*) (even-parity case) and to Regge and Wheeler (1957) (odd-parity).

A priori, one might expect the black hole to divest itself of the unwanted perturbations in a single large belch, a burst of radiation of duration $\sim M$, the hole's mass or gravitational radius (units with $G = c = 1$). This Letter reports numerical computations which exhibit a totally different behavior: Initial perturbations of multipolarity $l \gg 1$ which contain Fourier components of wavelength $2\pi M/l \ll \lambda \lesssim 2\pi M$ are radiated only gradually, yielding a long and nearly sinusoidal wave train of gravitational radiation. The characteristic angular frequency ω of the wave train depends on the mass of the black hole and on the multipolarity of the perturbation, but is otherwise independent of the form of the initial perturbation: $\omega \approx (27)^{-1/2}\, l/M$. Loosely speaking, the black hole vibrates around spherical symmetry in a quasi-normal mode, and the mode is slowly damped by gravitational radiation.

The Zerilli and Regge-Wheeler equations governing black-hole perturbations have the form

$$\varphi^l{}_{,tt} - \varphi^l{}_{,r^*r^*} + V^l(r^*)\varphi^l = 0\,. \tag{1}$$

Here φ^l is a scalar quantity which describes the l-pole components of the gravitational radiation. (The components of the metric tensor are obtained by applying particular differential operators to φ^l; see Price 1971 or Thorne 1971.) The radial coordinate r^* is defined in terms of the Schwarzschild coordinate r by

$$r^* = r + 2M \ln\left(\frac{r}{2M} - 1\right). \tag{2}$$

Thus $r = 2M$ corresponds to $r^* = -\infty$, and $r = +\infty$ to $r^* = +\infty$. $V^l(r^*) \equiv \hat{V}^l(r)$ is the so-called curvature potential,

$$\hat{V}^l(r) = \begin{cases} \left(1 - \dfrac{2M}{r}\right) \dfrac{(2\Lambda^2(\Lambda + 1)r^3 + 6\Lambda^2 M r^2 + 18\Lambda M^2 r + 18M^3)}{r^3(\Lambda r + 3M)^2} & \text{even-parity (Zerilli)} \\[2ex] \left(1 - \dfrac{2M}{r}\right)\left(\dfrac{l(l+1)}{r^2} - \dfrac{6M}{r^3}\right) & \text{odd-parity (Regge-Wheeler)}\,, \end{cases} \tag{3}$$

where $\Lambda = \frac{1}{2}(l-1)(l+2)$. Asymptotically for large l, the even- and odd-parity potentials become identical.

To study black-hole vibrations, we choose a set of initial conditions at time $t = 0$: $\varphi^l(r^*, t = 0)$ and $\varphi^l{}_{,t}(r^*, t = 0)$ for $-\infty < r^* < +\infty$. We then solve equation (1) numerically to determine the subsequent evolution. Solutions have been computed from a variety of initial conditions, and for various values of l. It is immediately clear that initial conditions containing predominantly short-wavelength Fourier components (e.g., a narrow peak or a high-frequency sine wave) are uninteresting: the potential term in equation (1) has only a slight dispersive influence, so the perturbation is radiated outward and inward with essentially its original profile (i.e., this case *does* yield a single belch). This expected behavior has been verified numerically.

Initial perturbations of greater interest are broad, "thick" ones which contain long-wavelength Fourier components; these cannot propagate as free waves in the region of the potential. Two such initial conditions are shown in Figure 1. The curvature potential $V^l(r^*)$ is indicated by the crosshatched curve. In general $V^l(r^*)$ is peaked at about $r^* = 2M$ and drops off exponentially in the inward direction ($r^* \to -\infty$), and as r^{*-2} in the outward direction ($r^* \to \infty$). In these examples, the initial time derivative of the perturbations is chosen zero.

The subsequent evolution of the perturbations, as computed numerically, can be described and understood as follows. At any given r^*, the perturbation initially oscillates (no propagation leftward or rightward!) with angular frequency approximately $[V^l(r^*)]^{1/2}$. Since $V^l(r^*)$ varies with r^*, the oscillations soon become out of phase from point to point, and the initially smooth perturbation builds up components of ever shortening wavelength. When wavelengths as short as the critical value $2\pi[V^l(r^*)]^{-1/2}$ have developed, the perturbations begin to propagate as free waves out of the region of the potential. For small l (say 2 or 3 or 4), the potential is low and the free propagation is almost immediate (single belch); but for large l the shortening process is gradual, and

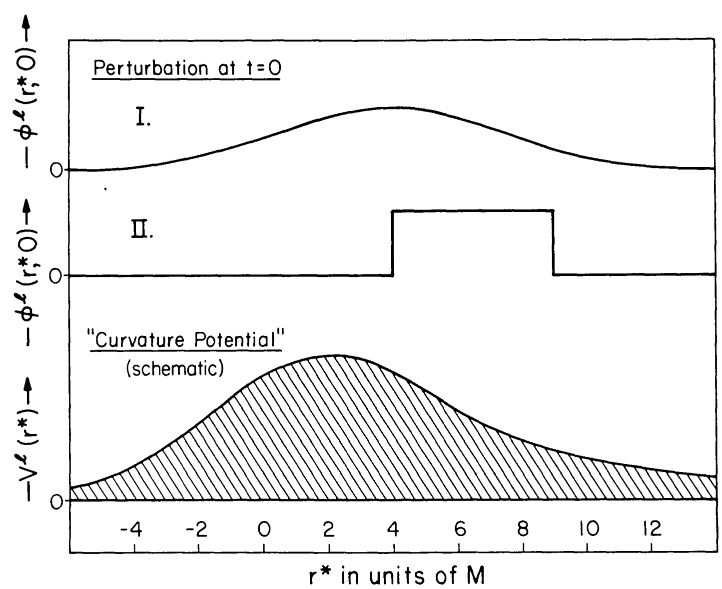

FIG. 1.—Two interesting perturbations of a Schwarzschild black hole. The initial radial "wave forms" of the perturbations are shown in I and II. Their initial time derivatives are assumed zero, and their angular dependence is a spherical harmonic of order l. The perturbations are spread out broadly over the region of strong "curvature potential" $V^l(r^*)$, so spacetime curvature prevents them from propagating until they have developed a wave form containing wavelengths shorter than the characteristic length $2\pi[V^l(r^*)]^{-1/2}$ (see text for discussion).

long wave trains are emitted, of characteristic angular frequency

$$\omega \approx [V^l(r^*)]_{\max}{}^{1/2} \approx \frac{l}{(27)^{1/2}M}\,. \tag{4}$$

Figure 2 shows the profile of the propagating gravitational wave trains at large t for the two inital conditions of Figure 1 and the two multipolarities $l = 20$ and $l = 40$. The estimate of equation (4) is seen to be approximately correct. The length of the wave train depends somewhat on the precise initial conditions chosen, but seems to be rather independent of l. These characteristics are typical of our numerical results in general; but we are able to give no analytic estimate for the train length.

How much of the perturbation radiates down the hole instead of off to infinity? A simple rule of thumb summarizes all our numerical calculations: The quantity

$$\mathfrak{E} = |\varphi^l{}_{,t}|^2 + |\varphi^l{}_{,r^*}|^2 + V^l(r^*)|\varphi^l|^2 \tag{5}$$

is a mathematical energy density which is exactly conserved by the evolution of equation (1). Measured in terms of $\mathfrak{E}$, that long-wavelength energy initially located outside the potential maximum is typically radiated outward; that energy initially inside the potential maximum goes down the hole. Thus, the offset of initial conditions I and II from the potential maximum results in most of the energy radiating outward ($\sim$80 or 90 percent).

We emphasize that the phenomenon here exhibited, the "free oscillation of a black

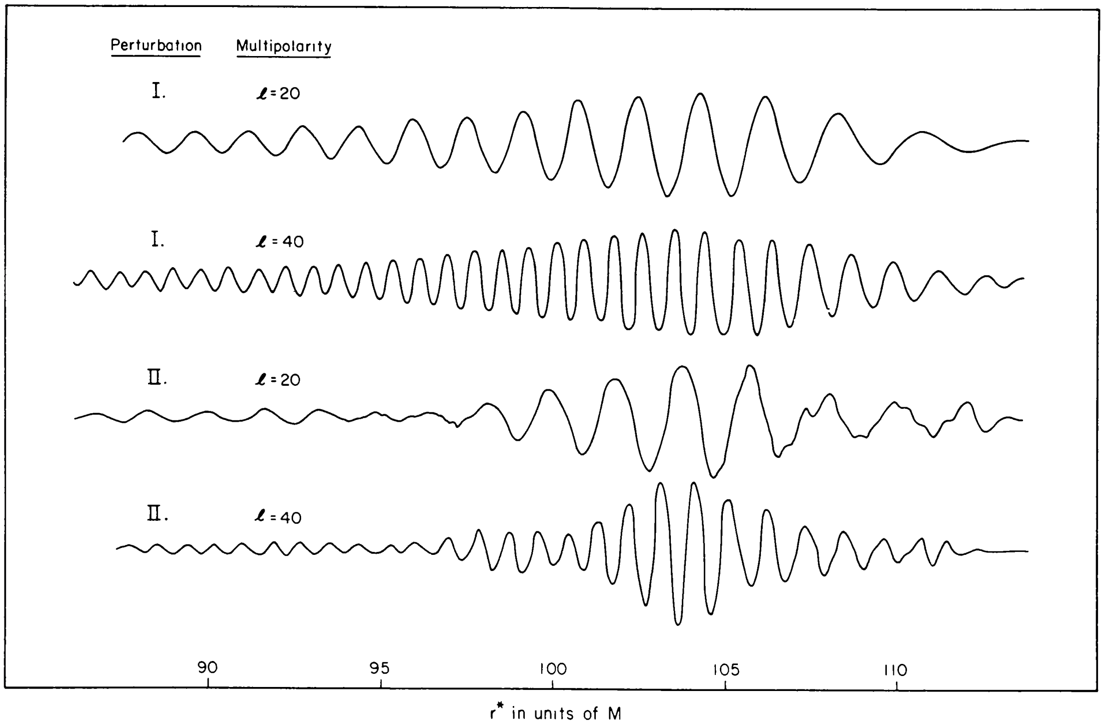

FIG. 2.—Wave forms $\varphi^l(r^*, t)$ of gravitational radiation at large r^* and fixed t, as produced by the initial perturbations I and II of Fig. 1 for $l = 20$ and $l = 40$. The waves are propagating rightward, away from the region of strong "curvature potential," where they originated. Since "wavelength shortening" in the region of the potential proceeds gradually, the waves have the form of a long sinusoidal train of angular frequency $\omega \simeq [V^l(r^*)_{\max}]^{1/2} \simeq (27)^{-1/2}\, l/M$. This frequency can be interpreted as the "vibration" frequency of the black hole (see text).

hole," is distinct from the curvature-potential effect studied by Price (1971) and Fackerell (1971) in which the potential acts as a high-pass filter of gravitational radiation.

The free oscillations of a bell are initiated by a mechanical blow; the Earth's free oscillations are excited by large earthquakes. What processes can induce a black hole to oscillate, i.e., can supply the initial perturbation which we have supplied by fiat in our numerical calculations? Recent calculations by Davis *et al.* (1971) show that vibrational modes are excited—though weakly for high l—by a test particle falling radially into a black hole. (In fact, the entire calculated spectrum can be understood qualitatively as a superposition of such vibrations.) Whether high-l vibrations can be excited *preferentially* by some other pattern of infalling matter is a problem—presently unsolved—of considerable astrophysical relevance. In some cases symmetry considerations can at least inhibit low multipole radiation. For example, the turbulent influx of matter into a black hole might produce perturbations whose dominant multipolarity is determined by the size L of the turbulent cell $l \sim 2\pi M/L \gg 1$.

The essential point of this Letter is that a black hole can be a dynamical entity rather than merely an arena for dynamics. This new point of view suggests new directions of research: How does the rotation of a black hole affect its vibrations? *Are* black-hole vibrations excited significantly by natural astrophysical processes? Might they play a significant role as sources of gravitational radiation?

If Weber's (1969, 1970*a*, *b*) observed gravitational radiation is verified and found to have a highly oscillatory wave form (indicating vibrations of large l as a possible source), black-hole vibrations will become a strong candidate for explaining the observations. Vibration is a mechanism by which "short"-wavelength gravitational radiation can be emitted by a black hole of large mass, so there is no limit in principle on the mass of a black hole which radiates at the frequency of Weber's detection apparatus.

I am pleased to thank Dr. Richard Price and Professor Kip S. Thorne for their invaluable assistance and encouragement. Dr. Remo Ruffini provided helpful suggestions. I thank the Fannie and John Hertz Foundation for their support. This work was supported in part by the National Science Foundation [GP-27304 and GP-28027].

REFERENCES

Davis, M., Ruffini, R., Press, W., and Price, R. 1971, *Phys. Rev. Letters*, **27**, 1466.
Fackerell, E. D. 1971, *Ap. J.*, **166**, 197.
Price, R. H. 1971, paper submitted to *Phys. Rev.*
Regge, T., and Wheeler, J. A. 1957, *Phys. Rev.*, **108**, 1063.
Thorne, K. S. 1971, *Nonspherical Gravitational Collapse—A Short Review* (in press).
Weber, J. 1969, *Phys. Rev. Letters*, **22**, 1320.
———. 1970*a*, *ibid.*, **24**, 276.
———. 1970*b*, *ibid.*, **25**, 180.
Zerilli, F. J. 1970*a*, *Phys. Rev. Letters*, **24**, 737.
———. 1970*b*, *Phys. Rev.*, **D2**, 2141.