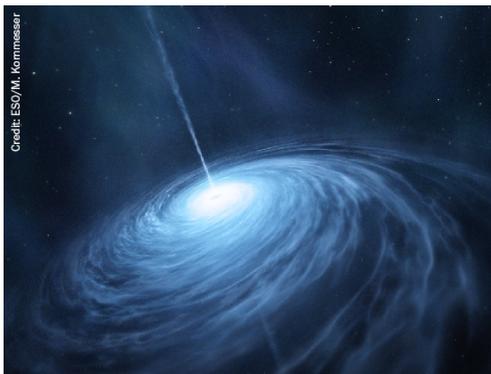


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# Quantum instability of the Cauchy horizon in Reissner–Nordström–deSitter spacetime

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## Abstract

In classical general relativity, the values of fields on spacetime are uniquely determined by their values at an initial time within the domain of dependence of this initial data surface. However, it may occur that the spacetime under consideration extends beyond this domain of dependence, and fields, therefore, are not entirely determined by their initial data. This occurs, for example, in the well-known (maximally) extended Reissner–Nordström or Reissner–Nordström–deSitter (RNdS) spacetimes. The boundary of the region determined by the initial data is called the ‘Cauchy horizon.’ It is located inside the black hole in these spacetimes. The strong cosmic censorship conjecture asserts that the Cauchy horizon does not, in fact, exist in practice because the slightest perturbation (of the metric itself or the matter fields) will become singular there in a sufficiently catastrophic way that solutions cannot be extended beyond the Cauchy horizon. Thus, if strong cosmic censorship holds, the Cauchy horizon will be converted into a ‘final singularity,’ and determinism will hold. Recently, however, it has been found that, classically this is not the case in RNdS spacetimes in a certain range of mass, charge, and cosmological constant. In this paper, we consider a quantum scalar field in RNdS spacetime and show that quantum theory comes to the rescue of strong cosmic censorship. We find that for any state that is nonsingular (i.e., Hadamard) within the domain of dependence, the expected stress-tensor blows up with affine parameter,  $V$ , along a radial null geodesic transverse to the Cauchy horizon as  $T_{VV} \sim C/V^2$



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with  $C$  independent of the state and  $C \neq 0$  generically in RNdS spacetimes. This divergence is stronger than in the classical theory and should be sufficient to convert the Cauchy horizon into a singularity through which the spacetime cannot be extended as a (weak) solution of the semiclassical Einstein equation. This behavior is expected to be quite general, although it is possible to have  $C = 0$  in certain special cases, such as the BTZ black hole.

Keywords: strong cosmic censorship, quantum field theory on curved spacetime, Reissner-Nordström-deSitter spacetime

(Some figures may appear in colour only in the online journal)

## 1. Introduction

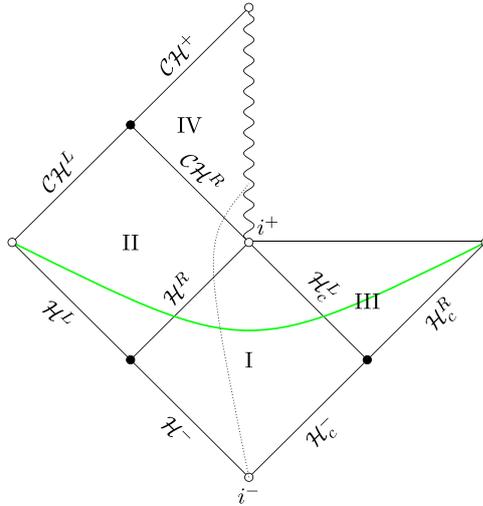
The Reissner–Nordström–deSitter (RNdS) spacetime describes a charged, static and spherically symmetric ‘eternal’ black hole in deSitter spacetime. It is an exact solution to the Einstein–Maxwell field equations with a positive cosmological constant  $\Lambda$ . The part of the maximally extended spacetime relevant for the discussions in this paper is drawn in figure 1, see for example [1, 2] or section 2 for detailed discussion.

The regions II and IV are inside the black hole, the regions I, III outside, and the green line is a spacelike surface,  $\Sigma$ . By the usual properties of solutions to hyperbolic field equations such as the covariant Klein–Gordon wave equation,

$$(\square - \mu^2)\Phi = 0, \tag{1}$$

if we prescribe initial data for  $\Phi$  on such a surface, then  $\Phi$  will be determined uniquely in the ‘domain of dependence’ of  $\Sigma$ , which in the case at hand is the union of regions I, II, III. However,  $\Phi$  is not determined uniquely beyond—i.e., in particular, in region IV—representing a breakdown of determinism. The horizon separating region IV from the rest is called the ‘Cauchy horizon’. A similar situation occurs for the more general Kerr–Newman–dS black holes (not discussed in this paper) and for other fields such as the linearized gravitational field.

The maximally extended RNdS spacetime also extends through the surface labeled as  $\mathcal{CH}^L$  in figure 1 to a region isometric to region IV. The surface  $\mathcal{CH}^L$  also comprises part of the full Cauchy horizon of the maximally extended RNdS spacetime. However, an eternal static black hole is physically unrealistic. For a black hole produced by the collapse of a charged, spherically symmetric body, the region exterior to the body would be described by the RNdS spacetime, but the region inside the body would be replaced by a suitable ‘interior metric.’ The dotted line in figure 1 represents a typical trajectory of the surface of a collapsing body. The region of RNdS spacetime to the right of this line is thus physically relevant for the gravitational collapse spacetime. However, the region to the left of this line would be ‘covered up’ by the collapsing body and is not physically relevant. The key point is that the singularity in the region beyond  $\mathcal{CH}^L$  will always be ‘covered up’ [3]. Indeed, all of  $\mathcal{CH}^L$  and part of  $\mathcal{CH}^R$  may be ‘covered up,’ as is the case for the trajectory shown in figure 1. But even if  $\mathcal{CH}^L$  is not fully covered up, it will no longer play the role of a Cauchy horizon in a gravitational collapse spacetime. On the other hand, a portion of  $\mathcal{CH}^R$  ‘near’ the event horizon will never be covered up and will correspond to a portion of the Cauchy horizon of the gravitational collapse spacetime. In this paper, we will consider the behavior of a quantum scalar field as one approaches  $\mathcal{CH}^R$  in the extended RNdS spacetime shown in figure 1. Our results will be applicable to the portion of the gravitational collapse spacetime outside of the collapsing body.

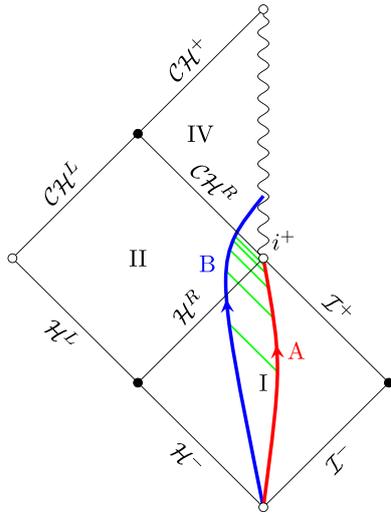


**Figure 1.** Sketch of partially extended RNdS. The wiggled line indicates the curvature singularity and the double line conformal infinity. Filled circles correspond to bifurcation surfaces, whereas open circles indicate singular points and/or points at infinity. Also indicated, as a green line, is a Cauchy surface for the partially extended RNdS, up to the region IV beyond the Cauchy horizon  $\mathcal{CH}^R$ . To the r.h.s. of the dotted curve is the outer region of a shell in a typical collapse spacetime.

It was argued many years ago by Penrose [4] that dynamical fields such as  $\Phi$  would have a very bad behavior at the Cauchy horizon  $\mathcal{CH}^R$ , thereby converting the Cauchy horizon to a singularity. The conjecture that the maximal Cauchy evolution of suitable initial data generically yields an inextendible spacetime is known as *strong cosmic censorship* (sCC). If sCC holds in general relativity, then there would be no breakdown of determinism. Penrose’s original argument<sup>3</sup> for the instability of  $\mathcal{CH}^R$ , phrased for Reissner–Nordström (RN) spacetime where  $\Lambda = 0$ , is that a timelike observer staying in the external region, such as A in figure 2, will reach future infinity  $i^+$  at infinite proper time, while an infalling observer such as B in figure 2 will reach the Cauchy horizon in finite proper time. Thus, if A sends periodic (according to her time) light signals into the black hole, illustrated by the green rays in figure 2, these will arrive more and more frequently at B according to his time. Similarly, source free solutions  $\Phi$  that oscillate only moderately near  $\mathcal{I}^+$  will oscillate extremely rapidly near  $\mathcal{CH}^R$ , i.e., there will be an infinite ‘blueshift effect’ as one approaches  $\mathcal{CH}^R$ . Such a rapidly oscillating field would have an infinite transversal derivative at  $\mathcal{CH}^R$ , thus resulting in a singular stress tensor. In the full theory, one might expect this behavior to be sufficiently singular to render the Einstein equations ill defined at  $\mathcal{CH}^R$ , thus ‘solving’ the problem of indeterminism—or rather relegating it to the domain of quantum gravity taking over near this singularity.

On the other hand, as observed also some time ago [6–8], in the presence of a positive cosmological constant, the amplitude of a dynamical field  $\Phi$  is expected to decay exponentially in region I (and on  $\mathcal{H}^R$ ) due to a similar ‘red-shift effect’ related to the cosmological expansion. In principle, this decay could counterbalance the blue-shift effect, thus leading only to a mild—if any—singularity of the stress tensor at the Cauchy horizon, implying a potential violation of sCC. The earlier investigations of [6–8] remained to a certain extent

<sup>3</sup> See [5] for a closely related argument formulated in terms of ‘mass-inflation’.



**Figure 2.** Illustration of the blueshift effect in Reissner–Nordström spacetime.

inconclusive, partly because the precise conclusions depend on the regularity of the initial data for  $\Phi$  (see e.g. [9] for a detailed discussion and [2] who advocate less regular initial data). More recently, the decay of fields near the Cauchy horizon was analyzed in a mathematically rigorous manner both in RN and RNdS spacetime, see [1, 2, 10–21]. Concretely, it was proven by Hintz and Vasy [1] (partly building on earlier work by [22–30]) that, starting from smooth initial data e.g. on  $\Sigma$ , the solution  $\Phi$  has Sobolev regularity  $H^{1/2+\beta-\epsilon}$  for any small  $\epsilon > 0$  locally near  $\mathcal{CH}^R$ . Here the parameter governing regularity at the Cauchy horizon is

$$\beta = \frac{\alpha}{\kappa_-}. \tag{2}$$

In this expression,  $\kappa_-$  is the surface gravity of the Cauchy horizon and  $\alpha$  the spectral gap, i.e., the infimum

$$\alpha = \inf\{-\text{Im}(\omega_{n\ell m})\}, \tag{3}$$

with  $\omega_{n\ell m}$  the so-called quasi-normal frequencies [31] of region I. Together with their results on ‘conormal regularity’ at the Cauchy horizon, this implies via the Sobolev embedding theorem that the field  $\Phi$  is Hölder-continuous ( $C^{0,\beta-\epsilon}$ ) across the Cauchy horizon  $\mathcal{CH}^R$ . Taking the field  $\Phi$  as a proxy for a dynamical metric in the full Einstein–Maxwell equations, this would correspond to a rather weak singularity in the sense that an observer crossing  $\mathcal{CH}^R$  would not be crushed/stretched by an infinite amount<sup>4</sup>.

Taking derivatives, the Sobolev regularity translates into a behavior not more singular than

$$T_{VV} \sim D|V|^{-2+2\beta} \quad \text{as } V \rightarrow 0^- \tag{4}$$

for the classical stress-energy component  $T_{VV} = (\partial_V \Phi)^2$  near the Cauchy horizon  $\mathcal{CH}^R$  at  $V = 0$ , where  $V$  denotes a Kruskal-type null coordinate. (Here  $D$  depends on the solution  $\Phi$ .) This result is roughly consistent with the results of [8]. The spectral gap,  $\alpha$ , which vanishes

<sup>4</sup> A similar conclusion for RN, where  $\beta = 0$ , was reached a long time ago in the context of the Israel–Poisson mass inflation scenario [5, 32] and later in a mathematically rigorous fashion by [13].

for  $\Lambda = 0$ , has recently been analyzed in detail by Cardoso *et al* [33, 34] for  $\mu = 0$  in this context. They have given strong numerical evidence that there are black hole parameters for which, e.g.  $\beta > \frac{1}{2}$ . This would correspond to a singularity that is sufficiently weak that a solution could be extended in different ways as a weak solution across the Cauchy horizon; see e.g. [13] for a more detailed discussion of such matters. Thus, it seems that, from the classical viewpoint, RNdS black holes can violate sCC for certain ranges of the black hole parameters.

In this work, we therefore ask whether quantum effects can change this classical picture in an essential way. For this, we consider a free, real Klein–Gordon quantum field equation (1) propagating in RNdS. First, we set up a class of quantum states that are regular, in a natural sense, near the initial data surface such as  $\Sigma$  in figure 1—technically, they are of ‘Hadamard type’. This condition is necessary and sufficient in order to have a smoothly varying, and in particular finite, expectation value of any ‘composite field’ (i.e. a suitably renormalized [35] monomial in covariant derivatives of  $\Phi$ ) such as the stress tensor in the regions I, II, III, but not necessarily at the Cauchy horizon  $\mathcal{CH}^R$ . In this sense, Hadamard states are the quantum analog of classical solutions that are smooth on an initial data surface  $\Sigma$ . They are thus the natural class of states to work with because—at least in our opinion—the statement of sCC becomes obscure if we work with states having singularities already to begin with<sup>5</sup>, i.e. states which fail to be Hadamard near  $\Sigma$  already.

Our main result is that the expectation value of the stress tensor operator component  $T_{VV}$  in such a state  $\Psi$  generically diverges as

$$\langle T_{VV} \rangle_{\Psi} \sim C|V|^{-2} + t_{VV} \quad \text{as } V \rightarrow 0^- \quad (5)$$

in terms of the Kruskal type coordinate  $V$  adapted to the Cauchy horizon (see section 2 for details), where  $C$  is some function that, in the limit  $V \rightarrow 0^-$ , goes to a constant, depending only on the black hole parameters  $M, Q, \Lambda$ , but *not* on the state  $\Psi$ . In the following, we will identify  $C$  with its limit and treat it as a constant, but it should be kept in mind that there also may be subleading divergences associated with the term  $C|V|^{-2}$  in (5). By contrast,  $t_{VV}$  is a piece depending on the state  $\Psi$ , diverging, however, no more strongly than the stress tensor of a classical solution with smooth initial data, so  $t_{VV}$  roughly behaves as (4), i.e.  $t_{VV} \sim |V|^{-2+2\beta}$ . Our investigations, which partly rely on numerical calculations, indicate that  $C \neq 0$  for generic values of the black hole parameters, consistent with recent results of [36] for the case of RN, who have also determined the constant  $C$  in that case; see also [37] for related results. We expect a similar behavior in Kerr–Newman–dS.<sup>6</sup>

The proof of equation (5) can be given quite easily for the case of a massless scalar field in 2 spacetime dimensions. In this case, the classical divergence (4) of the stress tensor component  $T_{VV}$  at  $\mathcal{CH}^R$  becomes simply  $T_{VV} \sim D|V|^{-2+2\frac{\kappa_c}{\kappa_-}}$ . As we shall show in section 3, the behavior of the quantum stress tensor is determined by the trace anomaly and the continuity equation alone. We find that equation (5) holds, with  $C \sim \kappa_c^2 - \kappa_-^2$ . Our arguments are technically rather similar to considerations made a long time ago by [38–40], although there are some minor differences in how we set up the states. By contrast, our arguments for equation (5) in 4

<sup>5</sup> Our arguments rest on the assumption of regularity across the cosmological horizon, or the weaker condition that  $T_{vv}$  vanishes on the cosmological horizon, with  $v$  the ‘Killing parameter’. This condition has a natural limit in RN, that of absence of radiation infalling from spatial infinity. It seems natural to discuss the behavior at the Cauchy horizon in terms of such asymptotic conditions, as without asymptotic conditions also the classical consideration of strong cosmic censorship becomes meaningless, as discussed for example in the Introduction to [13]: the domain of dependence of a compact achronal set in any globally hyperbolic spacetime has a Cauchy horizon, across which it is of course smoothly extendible.

<sup>6</sup> The methods of our paper also apply to this more general case in principle.

dimensions are considerably more involved. Our analysis in 4 dimensions involves arguments from microlocal analysis (to estimate  $t_{VV}$ ) and a mode scattering computation in regions I, II (to obtain a mode sum expression for  $C$ , which is then evaluated numerically). The scattering coefficients arising in the mode sum are evaluated only in the conformally coupled case in which  $\mu^2 = \frac{1}{6}R = \frac{2}{3}\Lambda$ , and the numerical evaluation is done only in this case as well. Nevertheless, the fact that we find  $C \neq 0$  in this case provides strong evidence that  $C \neq 0$  generically.<sup>7</sup>

That the divergence of the state dependent term  $t_{VV}$  in (5) is not worse than that of the stress tensor of a classical solution is proven in subsection 5.1 for the range  $\beta > \frac{1}{2}$ . This is not a serious restriction, as this is precisely the range in which sCC is potentially violated classically, see e.g. [13]. It is remarkable that, provided  $C \neq 0$ , the strength of the singularity is state independent, so that the blow-up at the Cauchy horizon is not restricted to ‘generic initial data’ as in classical formulations of sCC. Restoring units, one checks that  $C$  is of  $O(\hbar)$ , so that the leading divergence (5) is caused entirely by quantum effects.

The divergence (5) is sufficiently strong that, if backreaction of the quantum stress tensor on the metric were considered via the semiclassical Einstein equation, it would stretch/crush observers on geodesics crossing the Cauchy horizon into region IV. With a stress-energy tensor of the form (5), the Ricci-tensor should behave as  $R_{\lambda\lambda} := R_{\mu\nu}\dot{\gamma}^\mu\dot{\gamma}^\nu \sim C\lambda^{-2}$  for a null geodesic  $\gamma$  crossing the Cauchy horizon at the value  $\lambda = 0$  of its affine parameter. In contrast with the behavior found in the classical theory by [5, 32] and [1, 2, 10–21], this would correspond to a ‘strong’ rather than a ‘weak’ singularity, e.g. in the sense of [42, 43]. In more detail, let  $Z_1, Z_2$  be two Jacobi fields normal to the null geodesic  $\gamma$ . Denote by  $n$  a null vector field along  $\gamma$  such that  $n_\mu\dot{\gamma}^\mu = 1$  and  $0 = Z_1^\mu n_\mu = Z_2^\mu n_\mu$ . We may use these to define an area element  $\epsilon = \frac{1}{2}Z_1 \wedge Z_2$  at each point of  $\gamma$ . Furthermore, the area of this element relative to that induced by the metric is  $A$  where  $6n_{[\mu}\dot{\gamma}_{\nu}\epsilon_{\alpha\beta]} = A\eta_{\mu\nu\alpha\beta}$ . Writing  $|A| = x^2$  we find that the geodesic deviation equation implies in the usual way that

$$\frac{d}{d\lambda}(x^2\sigma^A_B) = -x^2C^A_{\lambda B\lambda}, \quad \frac{d^2}{d\lambda^2}x = -\frac{1}{2}(2\sigma^A_B\sigma^B_A + R_{\lambda\lambda})x, \quad (6)$$

with  $\sigma^A_B$ ,  $A, B = 1, 2$ , being the components of the shear tensor in a parallelly propagated frame and  $C^A_{\lambda B\lambda}$  being the components of the Weyl tensor (see, e.g., [44] for details). Since  $R_{\lambda\lambda} \sim C\lambda^{-2}$ , the functions  $x, \sigma^A_B$  must clearly behave badly as  $\lambda \rightarrow 0$  unless  $C = 0$ , which as we have argued, should not be the generic case. In fact, if  $C > 0$ , we immediately learn from the second equation that  $x \rightarrow 0$  as the Cauchy horizon is reached, which corresponds to an infinite amount of crushing. (A similar conclusion can be reached for time-like geodesics.) On the other hand, when  $C < 0$ , to avoid an infinite stretching (i.e.  $x \rightarrow \infty$  as  $\lambda \rightarrow 0$ ), one eigenvalue of  $\sigma^A_B$  should diverge as  $\lambda^{-1}$ . Then by the first equation, if  $x$  stays away from zero as  $\lambda \rightarrow 0$ , at least one eigenvalue of  $C^A_{\lambda B\lambda}$  must diverge as  $\lambda^{-2}$ . Then, by the Jacobi equation for  $Z_i^A$ , at least one of the Jacobi fields must go to infinity, thus resulting in an infinite stretching. This kind of behavior presumably would convert the Cauchy horizon into a singularity through which the spacetime could not be extended as a (weak) solution of the semiclassical Einstein equation. If so, quantum effects can be said to rescue sCC, at least in the case of RNdS.

This paper is organized as follows: in section 2, we introduce RNdS spacetime and some standard coordinate systems. In section 3, we analyze the two-dimensional case. In

<sup>7</sup> As we shall discuss in section 6, this divergent behavior differs significantly from the results found for the BTZ-black hole spacetimes by [41], since  $C = 0$  in that case.

section 4, we recall the basic setup for linear quantum field theory, including the construction of Hadamard states by prescribing data on suitable null surfaces ('null quantization'). We construct the Unruh state in RNdS spacetime and show it is Hadamard in the union of regions I, II, and III. In section 5, we treat the four-dimensional case. This requires (i) showing that the difference in the stress-energy of the Unruh state and an arbitrary Hadamard state behaves as in the classical case and (ii) performing a mode sum analysis to estimate the behavior of the Unruh state stress-energy. Numerical evaluation of the mode sum is required to argue that generically,  $C \neq 0$  in (5). In section 6 we compare our results to the case of the BTZ black hole recently analyzed by [41], where, in fact,  $C = 0$ . In section 7, we draw our conclusions.

## 2. RNdS spacetime and coordinate systems

The metric of RNdS spacetime is

$$g = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\Omega^2, \quad (7)$$

where  $d\Omega^2$  is the line element of the unit 2-sphere, and where

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2, \quad (8)$$

with  $M > 0$ ,  $Q$  and  $\Lambda > 0$  having the interpretation of the black hole mass, charge and the cosmological constant, respectively. For physically reasonable<sup>8</sup> parameters, assumed throughout, this function has three positive roots  $r_- < r_+ < r_c$ , corresponding to the Cauchy, the event, and the cosmological horizon, respectively. In terms of these, one may express  $f$  as

$$f(r) = \frac{(r - r_-)(r - r_+)(r_c - r)(r - r_o)}{r^2(r_o^2 - r_-r_c - r_+r_- - r_+r_c)} \quad (9)$$

with

$$r_o = -(r_c + r_+ + r_-) \quad (10)$$

being the the fourth, negative, root. The quantities

$$\kappa_X = \frac{1}{2}|f'(r_X)|, \quad X \in \{-, +, c, o\} \quad (11)$$

are the corresponding surface gravities. One has  $\kappa_- \geq \kappa_+$ , with equality in the extremal case  $r_- = r_+$ . Far from extremality, we have  $\kappa_-, \kappa_+ > \kappa_c$ , but in the extremal limit  $r_- \rightarrow r_+$ , one has  $\kappa_-, \kappa_+ \rightarrow 0$  while  $\kappa_c$  stays finite. We will refer to the parameter range where  $\kappa_- < \kappa_c$  as the near extremal regime.

We frequently use the tortoise coordinate  $r_*$ , defined up to an integration constant by  $dr_* = f(r)^{-1}dr$ . For later convenience<sup>9</sup>, we choose the integration constant in the interior region such that, near  $r = r_+$ ,

$$r_* = \frac{1}{2\kappa_+} \log |r - r_+| + D + O(r - r_+), \quad (12)$$

<sup>8</sup>I.e., solutions representing a black hole hidden from the outside by an event horizon, as opposed to a representing a naked singularity visible from the outside.

<sup>9</sup>This becomes relevant when matching the phases of the various mode functions across  $r = r_+$  in section 5.2.



one can extend  $V_-$  (and the metric) analytically from  $(-\infty, 0)$  to  $(-\infty, \infty)$  over the Cauchy horizon  $\mathcal{CH}^R$ . Figure 3 gives a sketch of the geometry and indicates the coordinates and their range. It is sometimes useful to also consider a  $u$  coordinate in the interior region II, where it can be defined as

$$u = -\frac{1}{\kappa_+} \log U. \quad (19)$$

It diverges to  $+\infty$  on the event horizon  $\mathcal{H}^R$  and to  $-\infty$  on  $\mathcal{CH}^L$ , cf also figure 3. Due to (12), this is consistent with (14), also in region II. Finally, a coordinate adapted to  $\mathcal{CH}^L$ , i.e., that allows for an extension through  $\mathcal{CH}^L$ , can be defined as

$$U_- = e^{\kappa_- u} = U^{-\frac{\kappa_-}{\kappa_+}}. \quad (20)$$

For our arguments in sections 4.2.2 and 5.1, we will need estimates on the asymptotic behavior of classical solutions near  $i^+$  (or analogously at  $i^-$ ). This has been analyzed in the present context by Hintz and Vasy [1] and is best expressed resolving  $i^+$  ( $i^-$ ) by a blow up procedure also introduced in [1]. We first concentrate on region I where  $r_+ \leq r \leq r_c$ . In this region we can define, following [1], a new time coordinate (basically the Kerr-star coordinate) by

$$dt_* = dt \pm [f(r)^{-1} + c(r)]dr. \quad (21)$$

Here  $c(r)$  is a smooth function on the interval  $[r_+, r_c]$ , up to the boundary, such that

$$c(r) \begin{cases} = -f(r)^{-1} & r \in (r_+ + \delta, r_c - \delta), \\ \in (-2f(r)^{-1}, 0) & r \in [r_+, r_+ + \delta/2) \cup (r_c - \delta/2, r_c]. \end{cases} \quad (22)$$

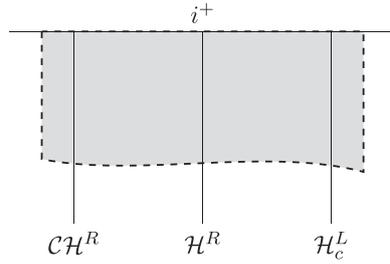
Note that near  $r_+$  and  $r_c$ , the function  $c(r)$  is not fully prescribed, but only restricted to a ( $r$  dependent) interval. The  $+$  sign in (21) is chosen near  $r_+$  and the  $-$  sign at  $r_c$ . For  $r \in (r_+ + \delta, r_c - \delta)$ ,  $dt_*$  coincides with  $dt$ , and the modifications near  $r_+, r_c$  ensure that  $t_*$  can be extended as a time-like coordinate to  $r < r_+, r > r_c$ . In particular, in the coordinate system  $(t_*, r, \Omega)$  thus set up for  $r \in [r_+, r_c]$ , the metric takes the form (7) for  $r \in (r_+ + \delta, r_c - \delta)$ , and

$$g = -f(r)dt_*^2 \pm 2[1 + c(r)f(r)]drdt_* - c(r)[2 + c(r)f(r)]dr^2 + r^2d\Omega^2 \quad (23)$$

for  $r \in [r_+, r_+ + \delta/2) \cup (r_c - \delta/2, r_c]$ . The coordinate transformations has achieved that the ‘point’  $i^+$  has been blown up to the corridor  $(r_+, r_c)_r \times S_\Omega^2$  which is represented by the lower part of the boundary at the coordinate location  $\tau = 0$  of  $\tau = e^{-t_*}$  of a rectangle  $(t_0, \infty)_{t_*} \times (r_+, r_c)_r \times S_\Omega^2$  representing the region I near  $i^+$ . A similar procedure can be carried through with the regions II, III, IV; see section 2.1 of [1]. The blow up of  $i^+$  is shown in figure 4. The vertical lines in the domain  $\mathcal{R}$  represent (parts of)  $\mathcal{CH}^R$ ,  $\mathcal{H}^R$ ,  $\mathcal{H}_c^L$ , respectively, and the horizontal boundary at  $\tau = 0$  represents a blow up of  $i^+$ .<sup>10</sup>

As discussed in the Introduction, only parts of the partially extended RNdS would be present in a more realistic spacetime representing a collapsing spherical object. However, even in the presence of a collapsing star, a portion of  $\mathcal{CH}^R$  near  $i^+$  is always part of the spacetime, and our arguments would apply to that part. To make our analysis simple, we will ignore the collapsing star, which is inessential to this problem, and work with the portions I, II, III, IV of RNdS spacetime as described above.

<sup>10</sup>In [1], also a further artificial horizon beyond the Cauchy horizon is introduced. We refer to [1] for details.



**Figure 4.** Sketch of the ‘blow-up’ of  $i^+$  and of a sequence of Cauchy surfaces as described in the text. The shaded region is called  $\mathcal{R}$ .

### 3. Quantum stress-energy in two-dimensional RNdS spacetime

In this section, we consider a massless, real scalar field  $\Phi$  satisfying

$$\square\Phi = 0 \quad (24)$$

on the two-dimensional spacetime obtained from (7) by suppressing the angular coordinates, i.e.,

$$g = -f(r)dt^2 + f(r)^{-1}dr^2, \quad (25)$$

with  $f$  given by equation (8).

#### 3.1. Behavior of the classical and quantum stress-energy tensor near $\mathcal{CH}^R$

We first consider the behavior of the stress-energy component  $T_{vv}$  near  $\mathcal{CH}^R$  for a *classical* field (24). This can be determined immediately from the facts that (i) the stress-energy tensor is conserved,  $\nabla^\nu T_{\mu\nu} = 0$ , and (ii) the classical stress-energy tensor is traceless,  $T^\mu{}_\mu = 0$ . Thus, in the null coordinates  $u, v$  of (14), we have  $T_{uv} = 0$ . The  $\mu = v$  component of the continuity equation  $\nabla^\nu T_{\mu\nu} = 0$  then yields

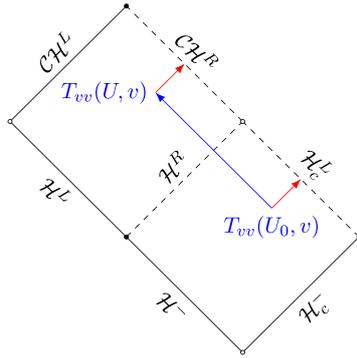
$$\partial_u T_{vv} = -f\partial_v(f^{-1}T_{uv}) = 0. \quad (26)$$

Thus,  $T_{vv}$  is constant along any ‘left moving’ null geodesic, such as the blue line in figure 5, i.e., we have  $T_{vv}(U, v) = T_{vv}(U_0, v)$ . We now take the limit as  $v \rightarrow \infty$  with  $U_0 < 0$  and  $U > 0$ , so that  $(U, v)$  approaches  $\mathcal{CH}^R$  and  $(U_0, v)$  approaches  $\mathcal{H}_c^L$  (see figure 5). We transform to regular coordinates  $V = V_-$  and  $V = V_c$  near  $\mathcal{CH}^R$  and  $\mathcal{H}_c^L$ , respectively, (see equations (17) and (18)). We obtain

$$T_{V_-V_-} = \frac{\kappa_c^2}{\kappa_-^2}(-V_-)^{2\frac{\kappa_c}{\kappa_-}-2} T_{V_cV_c}. \quad (27)$$

If the classical solution is smooth on  $\mathcal{H}_c^L$ , then (27) shows that if  $T_{V_cV_c} \neq 0$  on  $\mathcal{H}_c^L$  and if  $\kappa_c < \kappa_-$ , then  $T_{V_-V_-}$  blows up on  $\mathcal{CH}^R$ . However, in all cases, if  $T_{V_cV_c}$  has a finite limit on  $\mathcal{H}_c^L$ , then  $V_-^2 T_{V_-V_-} \rightarrow 0$  on  $\mathcal{CH}^R$ .

We now consider the corresponding behavior of the quantum stress-energy tensor near  $\mathcal{CH}^R$ . The only difference in our analysis from the classical case is that, as is well-known



**Figure 5.** The blue arrow indicates the integration path and the red arrows the limit  $v \rightarrow \infty$  in the discussion of the 2d case.

(see e.g. [38] for a thorough discussion) in any conformally invariant field theory in 2 dimensions, we have a trace anomaly of the form

$$g^{\mu\nu}T_{\mu\nu} = aR, \tag{28}$$

where  $a = 1/24\pi$  for the case of a massless scalar field. For the metric (25), we have  $R = -f''$ , where the prime denotes derivative w.r.t.  $r$ , so

$$T_{uv} = \frac{a}{4}f f''. \tag{29}$$

Therefore, the  $\mu = v$  component of the continuity equation now reads

$$\partial_u T_{vv} = -f \partial_v (f^{-1} T_{uv}) = -\frac{a}{8} f^2 f'''. \tag{30}$$

This can be integrated to yield

$$T_{vv}(U, v) = \frac{a}{8} [2f f'' - (f')^2]_{r(U_0, v)}^{r(U, v)} + T_{vv}(U_0, v). \tag{31}$$

We choose  $U_0 < 0$  and  $U > 0$  and take the limit as  $v \rightarrow \infty$ . Since

$$\lim_{v \rightarrow \infty} r(U, v) = \begin{cases} r_c & U < 0, \\ r_- & U > 0. \end{cases} \tag{32}$$

we obtain

$$\lim_{v \rightarrow \infty} T_{vv}(U, v) = \lim_{v \rightarrow \infty} T_{vv}(U_0, v) - \frac{a}{2} (\kappa_-^2 - \kappa_c^2). \tag{33}$$

This differs from the classical case by the step of  $-a(\kappa_-^2 - \kappa_c^2)/2$  that occurs at  $U = 0$ .

When we transform (33) to regular coordinates  $V = V_-$  and  $V = V_c$  near  $\mathcal{CH}^R$  and  $\mathcal{H}_c^L$ , respectively, the first term will give rise to the behavior (27) found in the classical case. However, the second term will give rise to singular behavior of the form

$$T_{V_- V_-} \sim \frac{a}{2} \frac{\kappa_c^2 - \kappa_-^2}{\kappa_-^2} \frac{1}{V_-^2}. \tag{34}$$

Unless  $\kappa_c = \kappa_-$ , this behavior is more singular than the classical behavior. Furthermore, in the quantum case, the  $V_-^{-2}$  divergence of  $T_{V_-V_-}$  at the Cauchy horizon  $\mathcal{CH}^R$  is present whenever  $\kappa_c \neq \kappa_-$  even if  $T_{V_cV_c}$  vanishes on the cosmological horizon  $\mathcal{H}_c^L$ .

Similar arguments for the RN case were previously given in [39]. Apart from not assuming stationarity, the main difference of our argument is that we impose the ‘initial condition’ of nonsingularity at  $\mathcal{H}_c^L$  instead of the event horizon, as in [39], which seems physically much better motivated.

### 3.2. The Unruh state in two-dimensional RNdS

The main shortcoming of the analysis of the previous subsection is that it does not generalize to higher dimensions, as the trace anomaly does not give sufficient information to determine  $T_{vv}$ . To make an argument, one has to invoke the—very plausible—absence of numerical coincidence of a cancellation between a geometric (from the trace anomaly) and a state-dependent (from the tangential pressure) contribution to the integral of  $\partial_u T_{vv}$  [39].<sup>11</sup>

In order to make contact with the analysis that will be given in the following sections for the four-dimensional case, it is useful to analyze the properties of the Unruh state, defined in the usual way (see e.g. [46] or (53) below) by the positive frequency mode solutions  $\square\psi_k^{U,\text{in/up}} = 0$  given by

$$\psi_k^{U,\text{in}} = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2k}} e^{-ikV_c}, \quad \psi_k^{U,\text{up}} = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2k}} e^{-ikU}, \quad (35)$$

We write the metric in the form

$$g = GdUdV_c \quad (36)$$

The expectation value of the stress tensor in the state defined by these modes is given by [47]

$$\langle T_{V_cV_c} \rangle_U = -\frac{1}{12\pi} G^{\frac{1}{2}} \partial_{V_c}^2 G^{-\frac{1}{2}}, \quad \langle T_{UU} \rangle_U = -\frac{1}{12\pi} G^{\frac{1}{2}} \partial_U^2 G^{-\frac{1}{2}}. \quad (37)$$

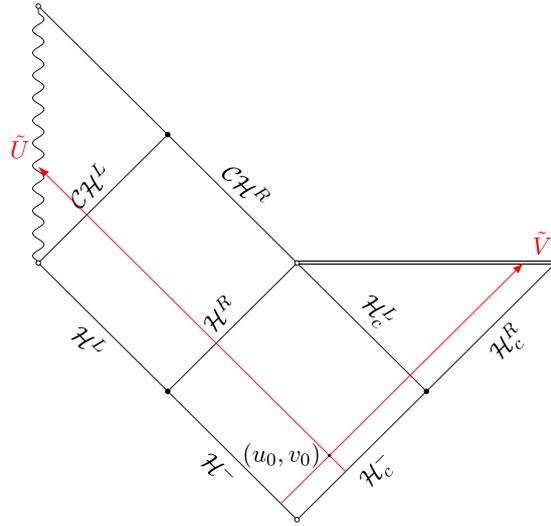
One easily verifies that  $\langle T_{V_cV_c} \rangle_U$  is finite at the cosmological horizon  $\mathcal{H}_c^L$  and that near the Cauchy horizon  $\mathcal{CH}^R$

$$\langle T_{V_-V_-} \rangle_U = -\frac{1}{48\pi} \left( 1 - \frac{\kappa_c^2}{\kappa_-^2} \right) V_-^{-2}, \quad (38)$$

consistent with (34) and the conformal anomaly coefficient  $a = \frac{1}{24\pi}$  of the scalar field.

We remark that in the stationary state defined by the modes (35), also  $\langle T_{UU} \rangle_U$  diverges as  $U^{-2}$  at  $\mathcal{CH}^L$  in adapted coordinates  $U = U_-$ . However, following [40], it is not difficult to see that it is possible to choose a nonstationary state that is regular on  $\mathcal{CH}^L$ . To do so, choose a point  $(u_0, v_0)$  in the exterior region of RNdS and define  $\tilde{U}(u)$  and  $\tilde{V}(v)$ , respectively, as the affine parameter of ingoing and outgoing null geodesics starting at  $(u_0, v_0)$ , cf figure 6. Defining the state by positive frequency mode solutions  $(4\pi k)^{-\frac{1}{2}} e^{-ik\tilde{U}}$ ,  $(4\pi k)^{-\frac{1}{2}} e^{-ik\tilde{V}}$ , one finds that  $\langle T_{UU} \rangle$  is finite on  $\mathcal{CH}^L$ , but  $\langle T_{V_-V_-} \rangle$  is still singular at  $\mathcal{CH}^R$ . The state is Hadamard also in an extension beyond  $\mathcal{CH}^L$ . See figure 6 for a sketch of the setup and the domain in which

<sup>11</sup> Another argument for a divergence at the Cauchy horizon that does not invoke absence of numerical coincidences was put forward in [45]: under the assumptions of spherical symmetry, stationarity, and a non-vanishing total flux of radiation, the stress tensor diverges either at  $\mathcal{CH}^R$  or  $\mathcal{CH}^L$ . However, as discussed in the Introduction, a divergence at  $\mathcal{CH}^L$  is irrelevant in general.



**Figure 6.** The domain in which the Markovic–Unruh state is Hadamard, cf the description in the text.

the state is Hadamard. However, since regularity at  $\mathcal{CH}^L$  is inessential for us, but stationarity is quite convenient for computational purposes, we shall work with the stationary Unruh state defined by the modes (35).

Note that a single ingoing Unruh-mode  $\psi_k^{U,\text{in}}$  has a stress tensor at the Cauchy horizon  $\mathcal{CH}^R$

$$T_{V_- V_-}[\psi_k^{U,\text{in}}] = \partial_{V_-} \psi_k^{U,\text{in}} \partial_{V_-} \overline{\psi_k^{U,\text{in}}} = \frac{1}{4\pi} k \frac{\kappa_c^2}{\kappa_-^2} V_-^{2\frac{\kappa_c}{\kappa_-} - 2}, \quad (39)$$

with a divergence which, of course, is the same as the one found in (27) for the classical stress tensor and is weaker than that found for the full quantum stress tensor. The divergence (38) of the quantum stress tensor is thus a UV effect, i.e. it stems from integration over modes of arbitrarily high frequency. This can be seen more explicitly by—instead of renormalizing the stress tensor via Hadamard point-splitting, i.e. using (37)—computing the difference of the expectation value of  $T_{V_- V_-}$  evaluated in the state defined by the modes (35) and a ‘comparison’ state which is Hadamard across the Cauchy horizon  $\mathcal{CH}^R$  (so that the latter has  $T_{V_- V_-}$  finite at  $\mathcal{CH}^R$ ). Such a state can be defined by using, instead of the modes  $\psi_k^{U,\text{in}}$ , the modes  $\square\psi_k^{C,\text{out}} = 0$ , with

$$\psi_k^{C,\text{out}} = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2k}} e^{-ikV} \quad (40)$$

where now  $V = V_-$ .

This difference in expectation values can be easily computed directly, with a point-split procedure. Having in mind our later discussion of the four-dimensional case, we perform this calculation differently. Restricting to the in-modes, we consider the ‘Boulware’ mode solutions (just as a computational device, not changing the definition of the state!)

$$\psi_\omega^{\text{in,I}}(V_c) = \begin{cases} (2\pi)^{-\frac{1}{2}} (2\omega)^{-\frac{1}{2}} e^{-i\omega v} & V_c < 0 \\ 0 & V_c > 0 \end{cases}, \quad (41a)$$

$$\psi_{\omega}^{\text{in,III}}(V_c) = \begin{cases} 0 & V_c < 0 \\ (2\pi)^{-\frac{1}{2}}(2\omega)^{-\frac{1}{2}}e^{-i\omega v} & V_c > 0 \end{cases}, \quad (41b)$$

where for  $V_c > 0$ , we defined

$$v = \kappa_c^{-1} \log V_c. \quad (42)$$

Here I stands for the original exterior region and III for the region beyond the cosmological horizon  $\mathcal{H}_c^L$ , where  $V_c > 0$ , cf figure 1. As usual [38], the formal mode integral over the Unruh modes  $\psi_k^{\text{U,in}}$  can also be expressed as

$$\langle T_{vv} \rangle_{\text{U}} = \int d\omega \coth(\pi\omega\kappa_c^{-1}) T_{vv}[\psi_{\omega}^{\text{in,I}}] \quad (43)$$

with

$$T_{vv}[\psi_{\omega}^{\text{in,I}}] = \partial_v \psi_{\omega}^{\text{in,I}} \partial_v \overline{\psi_{\omega}^{\text{in,I}}} = \frac{1}{4\pi} \omega \quad (44)$$

the evaluation of  $T_{vv}$  in the solution  $\psi_{\omega}^{\text{in,I}}$ . For the difference between the expectation value of  $T_{vv}$  evaluated in the state defined by the modes  $\psi_k^{\text{U,in}}$  and the modes  $\psi_k^{\text{C,out}}$  defining the comparison state  $\langle \cdot \rangle_{\text{C}}$  (the latter being Hadamard across  $\mathcal{CH}^R$ ), one thus obtains

$$\langle T_{vv} \rangle_{\text{U}} - \langle T_{vv} \rangle_{\text{C}} = \frac{1}{4\pi} \int_0^{\infty} d\omega \omega \left( \coth \frac{\pi\omega}{\kappa_c} - \coth \frac{\pi\omega}{\kappa_-} \right) = \frac{1}{48\pi} (\kappa_c^2 - \kappa_-^2), \quad (45)$$

which coincides with the result (33), for the anomaly coefficient  $a = (24\pi)^{-1}$  of the scalar field. We note that in terms of the modes  $\psi_{\omega}^{\text{in,I}}$ , the stress tensor for a single mode *does* give the correct prediction for the degree of singularity of the stress tensor at the Cauchy horizon, cf (44). There is no contradiction to the previous discussion in terms of the  $\psi_k^{\text{U,in}}$  modes, as a single  $\psi_{\omega}^{\text{in,I}}$  mode contains arbitrarily high frequencies w.r.t.  $V_c$ . The point is that for a single  $\psi_{\omega}^{\text{in,I}}$  mode the stress tensor is divergent both at the cosmological and the Cauchy horizon, but at the cosmological horizon the Boltzmann weights of these modes are exactly such that their contributions cancel upon renormalization. At the Cauchy horizon, on the other hand, the Boltzmann weights are then incommensurate, unless  $\kappa_c = \kappa_-$ , so that a finite value of  $T_{vv}$  (and, thus, a singular stress-energy tensor) at the Cauchy horizon is obtained.

#### 4. Construction of the Unruh state in four-dimensional RNdS spacetime

The analysis just given for the two-dimensional case is not adequate to treat the four-dimensional case because (a) the trace anomaly and continuity equation do not give enough information to determine the behavior of the stress energy tensor near the Cauchy horizon and (b) in the two-dimensional case the field is an exact linear superposition of left- and right-movers, whereas the ‘backscattering of modes’ occurs in 4 dimensions. In order to make our arguments in 4 dimensions with the necessary clarity and precision, we introduce in this section some notions and results from QFT in curved spacetime. The main result of this section is the construction of the Unruh state as a stationary Hadamard state in regions I, II, and III of figure 1. The reader not interested in the general discussion of QFT in curved spacetime or in the details of the Unruh state construction may skip to section 5.

#### 4.1. Quantum fields and Hadamard states

Usually, one thinks of observables, such as the (smeared) stress tensor, as operators on a Hilbert space. However, properly speaking, it is problematical to fix from the outset a fixed Hilbert space representation in which the states live as vectors or statistical operators (density matrices). This is because (a) frequently states must be considered that are not expressible as density matrices on the fixed Hilbert space, and (b) the usual Hilbert space quantization is tied closely to the existence of a Cauchy surface (to define canonical fields), which is not available in our situation.

Therefore, it is formally much more convenient—and essentially requires no work (!)—to set up the quantum theory in an algebraic manner for the purpose of the discussion. Recall that a spacetime  $(\mathcal{N}, g)$  is called ‘globally hyperbolic’ if it has a Cauchy surface  $\Sigma$ , i.e. a spacelike or null codimension-one surface such that every inextendible causal curve hits  $\Sigma$  precisely once, see [44] for details. In such a situation, the Klein–Gordon wave operator  $\square - \mu^2$  possesses a unique retarded and advanced propagator, called  $E^\pm$ . We can think about the propagators as integral operators  $E^\pm : C_0^\infty(\mathcal{N}) \rightarrow C^\infty(\mathcal{N})$ , which are uniquely determined by the properties

$$(\square - \mu^2)E^\pm = 1, \quad \text{supp}(E^\pm f) = J^\pm(\text{supp} f), \quad (46)$$

where ‘supp’ is the support of a function  $f$  (the closure of the set where it is not zero), and where  $J^\pm(\mathcal{O})$  are the causal future/past of a set  $\mathcal{O}$ , see e.g. [44] for the detailed definition of these notions. The existence and uniqueness property of  $E^\pm$  express the fact that the initial value problem for  $(\square - \mu^2)\Phi = 0$  has a unique solution, i.e. they express determinism. The ‘commutator function’ is defined as  $E = E^+ - E^-$ , and it is often identified with a distributional kernel  $E(x_1, x_2)$  on  $\mathcal{N} \times \mathcal{N}$  (this identification implicitly depends on the choice  $d\eta$  of the integration element, which we take to be that given by the metric  $g$ ). Associated with  $(\mathcal{N}, g)$ , we then define an algebra,  $\mathfrak{A}(\mathcal{N}, g)$ , whose generators are ‘smeared’ field observables  $\Phi(f)$ , their formal adjoints<sup>12</sup>  $\Phi(f)^*$  and an identity 1, where  $f$  is a complex-valued  $C^\infty$ -function on  $\mathcal{N}$  with compact support, and whose relations are:

- (A1) For any  $f_1, f_2$  and any complex numbers  $c_1, c_2$ , we have  $\Phi(c_1 f_1 + c_2 f_2) = c_1 \Phi(f_1) + c_2 \Phi(f_2)$ .
- (A2) For any  $f$  we have  $\Phi[(\square - \mu^2)f] = 0$ .
- (A3) For any  $f$  we have  $\Phi(f)^* = \Phi(\bar{f})$ .
- (A4) For any  $f_1, f_2$ , we have  $\Phi(f_1)\Phi(f_2) - \Phi(f_2)\Phi(f_1) = iE(f_1, f_2)1$ .

Item (A1) essentially says that  $\Phi(f)$  is an operator-valued distribution. By analogy with the conventions in distribution theory, we write

$$\Phi(f) = \int_{\mathcal{N}} \Phi(x)f(x)d\eta(x) \quad (47)$$

as if  $\Phi(x)$  was a function, and work from now with the formal point-like object  $\Phi(x)$ . Item (A2) says that  $(\square - \mu^2)\Phi(x) = 0$  in the sense of distributions and item (A3) that  $\Phi(x) = \Phi(x)^*$  in the sense of distributions, i.e. we have a hermitian field. Item (A4) gives the usual commutation relation in covariant form  $[\Phi(x_1), \Phi(x_2)] = iE(x_1, x_2)1$ , with  $E(x_1, x_2)$  the distributional kernel of  $E$ .

<sup>12</sup> Here, the  $*$  is an anti-linear operation satisfying the usual rules for the adjoint, namely  $A^*B^* = (BA)^*$ .

In RNdS, see figure 1, the regions I, II, III separately are globally hyperbolic, as is their union. However, region IV is not globally hyperbolic. Since we somehow want to talk about the field in this region, we must find a way to generalize this construction. This can be done in the following cheap way. Let  $\mathcal{M}$  be a non globally hyperbolic spacetime such as the the union  $I \cup II \cup III \cup IV$  depicted in figure 1. We consider all globally hyperbolic subregions  $\mathcal{N} \subset \mathcal{M}$  whose causal structure coincides with that of  $\mathcal{M}$ , i.e.  $J_{\mathcal{N}}^{\pm}(p) = J_{\mathcal{M}}^{\pm}(p) \cap \mathcal{N}$  for all  $p \in \mathcal{N}$ , where  $J_{\mathcal{N}}^{\pm}(p)$  denotes the causal future/past of  $p$  in  $\mathcal{N}$ .<sup>13</sup> For two such globally hyperbolic subregions  $\mathcal{N}_1$  and  $\mathcal{N}_2$  such that  $\mathcal{N}_1 \subset \mathcal{N}_2$ , one then has  $\mathfrak{A}(\mathcal{N}_1) \subset \mathfrak{A}(\mathcal{N}_2)$ . In a first step, let us define a (too large) algebra  $\bigvee_{\mathcal{N} \subset \mathcal{M}} \mathfrak{A}(\mathcal{N}, g)$  *freely* generated by the algebras  $\mathfrak{A}(\mathcal{N}, g)$  of such globally hyperbolic subregions of  $\mathcal{M}$  (i.e. no relations imposed between generators of different  $\mathfrak{A}(\mathcal{N}_i, g)$ ). Then we divide out the relation  $\Phi_{\mathcal{N}_1}(f) = \Phi_{\mathcal{N}_2}(f)$  for all  $f$ ,  $\mathcal{N}_1, \mathcal{N}_2$  such that  $\text{supp } f \subset \mathcal{N}_1, \mathcal{N}_2$ . This construction gives a satisfactory way of defining the observables in a spacetime like RNdS which is not globally hyperbolic, by imposing all relations that can be obtained in globally hyperbolic subregions<sup>14</sup>.

Of course, to do physics, we not only need an algebra but also states. A state in this framework is simply a positive, normalized, linear functional on  $\mathfrak{A}(\mathcal{M}, g)$ , where ‘positive’ means a functional  $\mathfrak{A}(\mathcal{M}, g) \ni A \mapsto \langle A \rangle_{\Psi} \in \mathbb{C}$  such that  $\langle A^*A \rangle_{\Psi} \geq 0$  for all  $A$ , and where ‘normalized’ means  $\langle 1 \rangle_{\Psi} = 1$ . Since  $\mathfrak{A}(\mathcal{M}, g)$  is presented in terms of generators and relations, a state is given once we know its correlation functions  $\langle \Phi(x_1) \cdots \Phi(x_n) \rangle_{\Psi}$ . Among all states, we will focus on ‘Gaussian states’, which are determined uniquely in terms of their 2-point correlation (‘Wightman’-) function  $\langle \Phi(x_1)\Phi(x_2) \rangle_{\Psi}$ , see e.g. [35] for details. From the definitions, we must have

- (S1) (Commutator)  $\langle \Phi(x_1)\Phi(x_2) \rangle_{\Psi} - \langle \Phi(x_2)\Phi(x_1) \rangle_{\Psi} = iE(x_1, x_2)$  whenever  $x_1, x_2$  are contained in some globally hyperbolic portion  $\mathcal{N} \subset \mathcal{M}$ .
- (S2) (Wave equation)  $(\square - \mu^2)_{x_1} \langle \Phi(x_1)\Phi(x_2) \rangle_{\Psi} = (\square - \mu^2)_{x_2} \langle \Phi(x_1)\Phi(x_2) \rangle_{\Psi} = 0$ .
- (S3) (Positive type)  $\int \langle \Phi(x_1)\Phi(x_2) \rangle_{\Psi} f(x_1) \overline{f}(x_2) d\eta(x_1) d\eta(x_2) \geq 0$  for any smooth, compactly supported function  $f$  on  $\mathcal{M}$ .

Conversely, any distribution  $\langle \Phi(x_1)\Phi(x_2) \rangle_{\Psi}$  with these properties defines a Gaussian state. One may always represent the field algebra  $\mathfrak{A}(\mathcal{M}, g)$  on some Hilbert space such that the expectation functional  $\langle \cdot \rangle_{\Psi}$  corresponds to some ‘vacuum’ vector in that Hilbert space, although for a generic state the terminology ‘vacuum’ has no physical meaning. At any rate, it will be fully sufficient for this paper to work with the expectation functionals.

While the above three conditions characterize states in general, to obtain physically reasonable states (in a sense explained below), one should impose more stringent conditions on the short-distance behavior of the 2-point function  $\langle \Phi(x_1)\Phi(x_2) \rangle_{\Psi}$ . One such condition is the ‘Hadamard condition’. To state this condition, one introduces the notion of a convex normal neighborhood,  $\mathcal{O}$ , in the total spacetime  $\mathcal{M}$ , which is a globally hyperbolic sub-spacetime  $\mathcal{O} \subset \mathcal{M}$  such that any pair of points  $x_1, x_2$  from  $\mathcal{O}$  can be connected by a unique geodesic. In such a neighborhood, we can define uniquely the signed squared geodesic distance  $\sigma(x_1, x_2)$  for  $x_1, x_2$  from  $\mathcal{O}$ , and we can, non-uniquely, define a time function  $T(x)$ . E.g., in RNdS,  $T(x) = t$  in region I, or  $T(x) = r$  in region II.

<sup>13</sup> The issue that is circumvented with this restriction was pointed out in [48].

<sup>14</sup> Imposing ‘boundary conditions’ at the singularity in region IV would in effect render the equation hyperbolic on all of  $\mathcal{M}$ . In this case an algebra  $\mathfrak{A}(\mathcal{M})$  could be defined directly. It would correspond to a quotient of the algebra defined previously by additional relations.

**Definition 4.1.** (Hadamard condition, see, e.g. [49]). For  $(x_1, x_2) \in \mathcal{O} \times \mathcal{O}$ , where  $\mathcal{O}$  is any convex normal globally hyperbolic neighborhood, the 2-point function has the general form

$$\langle \Phi(x_1)\Phi(x_2) \rangle_\Psi = \frac{1}{4\pi^2} \left( \frac{U(x_1, x_2)}{\sigma + i0T} + V(x_1, x_2) \log(\sigma + i0T) + W_\Psi(x_1, x_2) \right). \quad (48)$$

Here  $U = \Delta^{1/2}$  is the square root of the VanVleck determinant [50],  $T = T(x_1) - T(x_2)$ ,  $V$  is determined by the Hadamard–deWitt transport equations [51] and is thereby, as  $U, \sigma$ , entirely determined by the local geometry within  $\mathcal{O}$ . The state dependence is contained in  $W_\Psi$ , which is required to be a  $C^\infty$  function on  $\mathcal{O} \times \mathcal{O}$ .

Roughly speaking, the Hadamard condition states that the singular part of the 2-point function  $\langle \Phi(x_1)\Phi(x_2) \rangle_\Psi$  is entirely determined by the local geometry. In any Hadamard state, one can define the  $n$ -point correlation functions of ‘renormalized composite fields’. Such fields are classically given by polynomials of the field  $\Phi$  and its covariant derivatives  $\nabla_{\mu_1} \dots \nabla_{\mu_k} \Phi$  such as the stress tensor<sup>15</sup>

$$T_{\mu\nu} = \nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{2} g_{\mu\nu} (\nabla^\sigma \Phi \nabla_\sigma \Phi + \mu^2 \Phi^2). \quad (49)$$

At the quantum level, these are defined (non-uniquely), as operator valued distributions in some larger algebra containing  $\mathfrak{A}(\mathcal{M}, g)$  [35, 52]. For us, it is only important how their 1-point function is defined. This is easiest to explain if there are no derivatives and if the polynomial is quadratic, i.e. for  $\Phi^2$ . Then one defines

$$\langle \Phi^2(x) \rangle_\Psi = \lim_{x_1, x_2 \rightarrow x} W_\Psi(x_1, x_2). \quad (50)$$

As the singular part of (48)—which is effectively subtracted—is covariant, this ‘point-split renormalization’ prescription<sup>16</sup> is covariant—in a sense it is the ‘same’ on all spacetimes, see [35, 54] for the precise meaning of this statement. The 1-point functions  $\langle \nabla_{\mu_1} \dots \nabla_{\mu_k} \Phi \nabla_{\nu_1} \dots \nabla_{\nu_l} \Phi(x) \rangle_\Psi$  would be defined in the same way, except that we take the derivatives before the coincidence limit. There are certain ambiguities in the definition of the composite quantum fields  $\nabla_{\mu_1} \dots \nabla_{\mu_k} \Phi \nabla_{\nu_1} \dots \nabla_{\nu_l} \Phi(x)$ , in the sense that the above point-split scheme is not the only one which is ‘the same on all spacetimes’. These ambiguities are fully characterized in a mathematically precise manner in [35]. They would correspond in the case of  $\Phi^2$  to adding a constant multiple of the scalar curvature and the mass,  $\Phi^2 \rightarrow \Phi^2 + c_1 R + c_2 \mu^2$ , and in the case of more general quadratic composite fields, to adding linear combinations of higher curvature invariants of the correct ‘dimension’. In the case of  $T_{\mu\nu}$  this freedom must be used to ensure that  $\nabla^\mu T_{\mu\nu} = 0$ . These ambiguities, while important in general, are however of no consequence for our investigation here, since they only give corrections to the expectation values that are smooth functions on spacetime, whereas we are interested in their (potential) divergent part as we approach a Cauchy horizon.

For us, the following properties of Hadamard states will be relevant:

(H1) For any Hadamard state,  $\langle \nabla_{\mu_1} \dots \nabla_{\mu_k} \Phi \nabla_{\nu_1} \dots \nabla_{\nu_l} \Phi(x) \rangle_\Psi$  is a  $C^\infty$  function of  $x$  on  $\mathcal{M}$ .

<sup>15</sup> There is an ambiguity in the definition of the classical stress tensor, as one could interpret the mass term also as a curvature coupling.

<sup>16</sup> The ‘point-split’ prescription is equivalent to saying that  $\Phi^2$  is the first non-trivial, i.e. aside from the identity, operator in the OPE, see [53].

(H2) For any pair  $\Psi, \Psi'$  of Hadamard states,  $\langle \Phi(x_1)\Phi(x_2) \rangle_\Psi - \langle \Phi(x_1)\Phi(x_2) \rangle_{\Psi'}$  is a  $C^\infty$  function on  $\mathcal{N} \times \mathcal{N}$  on any globally hyperbolic portion  $\mathcal{N}$  of  $\mathcal{M}$ .

(H3) For any pair  $\Psi, \Psi'$  of Hadamard states and  $x \in \mathcal{M}$ , we have

$$\langle \Phi^2(x) \rangle_\Psi - \langle \Phi^2(x) \rangle_{\Psi'} = \lim_{x_1, x_2 \rightarrow x} (\langle \Phi(x_1)\Phi(x_2) \rangle_\Psi - \langle \Phi(x_1)\Phi(x_2) \rangle_{\Psi'}), \quad (51)$$

with a similar formula for general quadratic composite fields  $\nabla_{\mu_1} \dots \nabla_{\mu_k} \Phi \nabla_{\nu_1} \dots \nabla_{\nu_l} \Phi(x)$ .

Properties (H1) and (H3) are immediate consequences of the definition since  $W$  is locally smooth. Property (H2) is remarkable, because the Hadamard form only refers to an arbitrarily small neighborhood  $\mathcal{O}$ , while (H2) makes a statement about arbitrarily large globally hyperbolic subsets of  $\mathcal{M}$ , such as the union of regions I, II, III in RNdS. Its proof combines nontrivially the commutator, field equation, and positive type property of Hadamard states [55]. It can only be given using ‘microlocal’ techniques, which we will briefly describe in the next subsection. States which are not Hadamard typically have infinite fluctuations, e.g.  $\langle \Phi^2(f_1)\Phi^2(f_2) \rangle_\Psi = \infty$ , of smeared composite fields, even if their expectation value should be finite  $\langle \Phi^2(f) \rangle_\Psi < \infty$  (which need not be the case either, see e.g. [56]). Furthermore, interacting quantum field theories (e.g. adding a  $\Phi^4$ -interaction) require Hadamard states [52, 57]. Reasonable states in flat spacetime such as the vacuum, thermal, finite particle number, steady states are Hadamard, and if a state is Hadamard near any Cauchy surface  $\Sigma$  in a globally hyperbolic spacetime, i.e. ‘initially’, then it remains Hadamard [58].

Together, these properties strongly suggest that Hadamard states are the reasonable class to consider as ‘regular states’ (analogous classically to  $C^\infty$ -functions) on globally hyperbolic spacetimes. In this paper, we will study in a sense whether a state which is initially Hadamard (in regions I, II, III) remains Hadamard across the Cauchy horizon  $\mathcal{CH}^R$  of RNdS.

#### 4.2. Hadamard states from null surfaces

Having defined Hadamard states, we must ask whether (a) such states exist, at least on globally hyperbolic spacetimes, and (b) how to construct/characterize concretely Hadamard states on given spacetime representing given physical setups. (a) has been established by several rigorous methods, see [59–63]. In particular, given any one Hadamard state  $\Psi$ , we may go to a representation of the field algebra by operators on a Hilbert space, in which the state is represented by a vector. Then, applying any product  $\Phi(f_1) \dots \Phi(f_N)$  to this vector, we get a new vector giving a new expectation functional. It can be shown that this is again a Hadamard state. Thus, in globally hyperbolic spacetimes, there is an abundance of Hadamard states. (b) In the present context, it is particularly natural to define particular Hadamard states with a concrete interpretation by ‘prescribing positive frequency modes’ on suitable null surfaces. This idea goes back already to the beginnings of the subject [64]. The Hadamard property of such a state was first established by [62] in the specific context of null-cones in curved space, and later independently by [65] in the case of the ‘Unruh state’ in Schwarzschild.

**4.2.1. ‘Local vacuum states’ defined on null-cones.** To motivate the constructions in RNdS, we first consider the case of a massless scalar field on Minkowski spacetime. There, we have of course the global vacuum state with the usual Wightman 2-point function  $\langle \Phi(x_1)\Phi(x_2) \rangle_0 = \frac{1}{4\pi} [(x_1 - x_2)^2 + i0(x_1^0 - x_2^0)]^{-1}$ . If we restrict attention to the causal future  $J^+(0)$  of the origin 0, we can alternatively write this state in terms of modes with characteristic initial data on the

null boundary,  $\dot{J}^+(0)$ , of this cone. For this, we let  $\psi_{\omega\ell m}$  be the mode solutions  $\square\psi_{\omega\ell m} = 0$  such that

$$\psi_{\omega\ell m}(x) = (2\pi)^{-\frac{3}{2}}(2\omega)^{-\frac{1}{2}}Y_{\ell m}(\theta, \phi)r^{-1}e^{-i\omega r} \quad \text{on } \dot{J}^+(0). \tag{52}$$

Here,  $r$  is viewed as an affine parameter along the null generators of  $\dot{J}^+(0)$ . As shown in [62], the vacuum 2-point function can be written inside the future cone as

$$\langle \Phi(x_1)\Phi(x_2) \rangle_0 = \int_0^\infty \sum_{\ell, m} \overline{\psi_{\omega\ell m}(x_1)}\psi_{\omega\ell m}(x_2)d\omega. \tag{53}$$

This formula can be rewritten as follows. Let  $\Phi(f) = \int \Phi(x)f(x)d\eta(x)$  be the smeared field, and let  $f_1, f_2$  be  $C^\infty$ -functions supported inside the future light cone  $J^+(0)$ . Let  $E = E^+ - E^-$  be the commutator function (retarded minus advanced fundamental solution) and let  $F_1 = E^- f_1|_{j^+(0)}$ , with a similar formula for  $F_2$ . Then we can write alternatively, with  $\Omega = (\theta, \phi)$  and  $d^2\Omega = d\theta^2 + \sin^2\theta d\phi^2$ ,

$$\langle \Phi(f_1)\Phi(f_2) \rangle_0 = -\frac{1}{\pi} \int_0^\infty \int_0^\infty \int_{S^2} \frac{r_1 r_2 F_1(r_1, \Omega) F_2(r_2, \Omega)}{(r_1 - r_2 - i0)^2} d^2\Omega dr_1 dr_2. \tag{54}$$

As argued in [62], and later independently in [66], one can generalize these constructions to locally define states associated with lightcones in curved spacetime  $(\mathcal{M}, g)$  for a conformally coupled field  $(\square - \frac{1}{6}R)\Phi = 0$ , in the following manner.

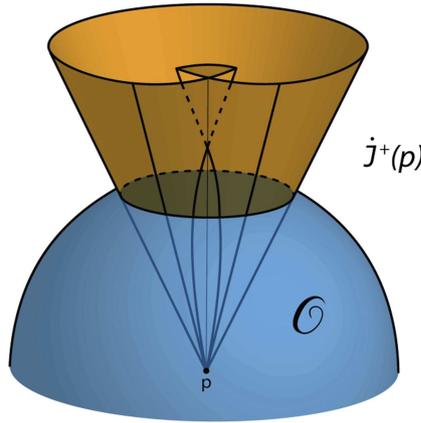
First, we pick a reference point  $p \in \mathcal{M}$ , and we let  $J^+(p)$  be its causal future. The boundary  $\dot{J}^+(p)$  will in general not have the structure of an embedded hypersurface far away from  $p$  due to caustics, but defining  $\dot{J}^+(p) \cap \mathcal{O}$ , with  $\mathcal{O}$  a convex normal neighborhood of  $p$  with smooth boundary,  $\dot{J}^+(p) \cap \mathcal{O}$  is diffeomorphic to a future lightcone in Minkowski space cut off by a spacelike plane, cf Figure 7. We can introduce coordinates  $(r, \Omega)$  on  $\dot{J}^+(p) \cap \mathcal{O}$  analogous to the above as follows: choose a smooth assignment  $S^2 \ni \Omega \mapsto l(\Omega) \in T_p\mathcal{M}$  such that each null ray in  $T_p\mathcal{M}$  is intersected exactly once by this mapping. Then  $(r, \Omega) \mapsto \exp_p(r l(\Omega))$  is a diffeomorphism between a neighborhood of the origin in  $\mathbb{R}_+ \times S^2$  and  $\dot{J}^+(p) \cap \mathcal{O}$ . Then, we can define<sup>17</sup> a local vacuum state by

$$\langle \Phi(f_1)\Phi(f_2) \rangle_p = -\frac{1}{\pi} \int_0^\infty \int_0^\infty \int_{S^2} \frac{r_1 r_2 \left(\Delta^{-\frac{1}{2}} F_1\right)(r_1, \Omega) \left(\Delta^{-\frac{1}{2}} F_2\right)(r_2, \Omega)}{(r_1 - r_2 - i0)^2} d^2\Omega dr_1 dr_2, \tag{55}$$

for testfunctions  $f_1, f_2$  supported in  $J^+(p) \cap \mathcal{O}$ , where now  $F_1 = E f_1|_{j^+(p) \cap \mathcal{O}}$ , and similarly for  $F_2$ . The presence of the VanVleck determinant  $\Delta(p, \cdot)$ , which is equal to 1 in Minkowski spacetime, ensures that the definition is independent of the arbitrary choice of coordinates  $(r, \Omega)$  [62].

By construction, the 2-point function (55) defines a state within  $J^+(p) \cap \mathcal{O}$  because it can be seen to satisfy the commutator, field equation, and positivity requirements [62]. Furthermore, it can be viewed as a ‘local vacuum state’, in the sense [67] that its construction only depends on the local geometry within  $J^+(p) \cap \mathcal{O}$ , but not on arbitrary choices of coordinates. But is it also Hadamard inside  $J^+(p) \cap \mathcal{O}$ ? The answer is yes [62], but the proof cannot be obtained simply by checking the definition. Instead, one must use the methods of microlocal analysis. Since

<sup>17</sup> A minor subtlety is to prove that the integrals converge at the tip of the lightcone,  $p$ , see [62].



**Figure 7.** Sketch of the of the subset  $J^+(p) \cap \mathcal{O}$  of the light cone on which one can define a state by (55).

we will use a similar argument below for RNdS spacetime, we here present the derivation. For details on distribution theory and functional analysis, we refer to [68].

If  $\varphi$  is a smooth function on  $\mathbb{R}^n$ , then its Fourier transform  $\hat{\varphi}(k)$  decays faster than any inverse power  $|k|^{-n}$  as  $|k| \rightarrow \infty$ . If  $\varphi$  is distributional, we say that a non-zero  $k \in \mathbb{R}^n$  is a regular direction at  $x$  if there exists a smooth cutoff function  $\chi$  of compact support such that  $\chi(x) \neq 0$  and an open cone  $\Gamma$  containing  $k$  such that for all  $N > 0$  there is a constant  $C_N$  for which

$$|\widehat{\chi\varphi}(p)| \leq C_N(1 + |p|)^{-N} \quad \forall p \in \Gamma. \tag{56}$$

The ‘wave-front set’,  $\text{WF}_x(\varphi)$  of  $\varphi$  at point  $x$  is, loosely speaking, the set of directions  $k$  such that  $\chi\varphi(x)$  fails to have a rapidly decreasing Fourier transform. More precisely, it is the complement of the regular directions at  $x$ , with  $k = 0$  by definition never in  $\text{WF}_x(\varphi)$ . It can be seen that under a diffeomorphism  $\psi$  of a neighborhood of  $x$ ,  $\text{WF}_x(\varphi \circ \psi) = \psi^*\text{WF}_{\psi(x)}(\varphi)$ . Thus, on an  $n$ -dimensional manifold  $\mathcal{X}$ ,  $\text{WF}_x(\varphi)$  should be viewed as a dilation invariant subset of  $T_x^*\mathcal{X} \setminus o$ , where  $o$  denotes the zero section. One also sets  $\text{WF}(\varphi) = \bigcup_{x \in \mathcal{X}} \text{WF}_x(\varphi) \subset T^*\mathcal{X} \setminus o$ . In the present context, we take  $\mathcal{X} = \mathcal{N} \times \mathcal{N}$ , where  $\mathcal{N}$  is some globally hyperbolic subset of a spacetime  $\mathcal{M}$ .

By an important result of Radzikowski [55], a state is Hadamard if and only if its 2-point correlation function  $G_\Psi(x_1, x_2) = \langle \Phi(x_1)\Phi(x_2) \rangle_\Psi$  satisfies

$$\text{WF}'(G_\Psi) = \mathcal{C}^+, \tag{57}$$

where

$$\mathcal{C}^\pm := \{(x_1, \xi_1, x_2, \xi_2) \in T^*(\mathcal{N} \times \mathcal{N}) \setminus o \mid (x_1, \xi_1) \sim (x_2, \xi_2), \pm \xi_1 \triangleright 0\}. \tag{58}$$

Here, the notation means  $\xi_1 \in T_{x_1}^*\mathcal{N}$  etc and  $(x_1, \xi_1) \sim (x_2, \xi_2)$  means that  $x_1$  and  $x_2$  can be joined by a null geodesic  $\gamma$  such that  $\xi_1, \xi_2$  (viewed as vectors using the metric  $g$ ) are tangent to  $\gamma$ .  $\xi \triangleright 0$  means that that the corresponding vector is future-pointing. If  $K(x_1, x_2)$  is a distributional kernel, the primed wave front set is defined as  $\text{WF}'(K) = \{(x_1, \xi_1, x_2, \xi_2) \mid (x_1, \xi_1, x_2, -\xi_2) \in \text{WF}(K)\}$ . With this characterization one can show:

**Theorem 4.2.** *The local vacuum state (55) is Hadamard in  $J^+(p) \cap \mathcal{O}$ .*

**Proof.** The full proof is given in [62]. Apart from a minor subtlety arising from tip of the lightcone at  $p(r=0)$ , it uses only straightforward techniques from microlocal analysis. The ingredients are as follows.

- The wave front set of the Schwartz kernel  $K_E$  of the commutator function satisfies  $\text{WF}'(K_E) = \mathcal{C}^+ \cup \mathcal{C}^-$ ; this is a restatement of the ‘propagation of singularities theorem’ [69].
- The wave front set of the distribution kernel  $K_A(r_1, \Omega_1, r_2, \Omega_2) = \delta(\Omega_1, \Omega_2)/(r_1 - r_2 - i0)^2$  satisfies  $\text{WF}'(K_A) = \{(\Omega, r, \xi_\Omega, \xi_r; \Omega, r, \xi_\Omega, \xi_r) | \xi_r > 0\}$  away from the tip  $r=0$  by direct computation. So, microlocally, the corresponding operator  $A$  ‘removes the negative frequencies’.
- The kernel  $K_R$  of the restriction operator  $RF = F|_{J^+(p) \cap \mathcal{O}}$  from  $C^\infty$ -functions on  $\mathcal{O}$  to  $C^\infty$ -functions on  $J^+(p) \cap \mathcal{O}$ , parameterized by  $(r, \Omega)$  via a diffeomorphism  $\psi : (0, r_0) \times S^2 \rightarrow J^+(p) \cap \mathcal{O} \setminus \{p\}$  away from the tip satisfies:

$$\text{WF}'(K_R) \subset \{(r, \Omega, \xi_r, \xi_\Omega; y, \eta) | y = \psi(r, \Omega), (d\psi)^t(r, \Omega)\eta = (\xi_r, \xi_\Omega) \quad \text{if } r > 0\} \quad (59)$$

by the restriction theorem (theorem 8.2.4 of [68]).

Looking at the definition of the local vacuum state, we see that the 2-point function is a composition of the kernels of  $E, A, R$ . These results are then combined with a standard result about the composition of distributional kernels. Let  $B : \mathcal{D}(\mathcal{X}_1) \rightarrow \mathcal{D}'(\mathcal{X}_2)$  and  $A : \mathcal{D}(\mathcal{X}_2) \rightarrow \mathcal{D}'(\mathcal{X}_3)$  be linear continuous maps. By the Schwartz Kernel theorem these correspond to distribution kernels  $K_B \in \mathcal{D}'(\mathcal{X}_2 \times \mathcal{X}_1)$  and  $K_A \in \mathcal{D}'(\mathcal{X}_3 \times \mathcal{X}_2)$ . If  $K \in \mathcal{D}'(\mathcal{X}_1 \times \mathcal{X}_2)$ , one defines

$$\text{WF}(K)_{\mathcal{X}_2} := \{(x_2, \xi_2) | (x_1, 0; x_2, \xi_2) \in \text{WF}(K)\}. \quad (60)$$

Then one has (theorem 8.2.14 of [68]): if  $K_B$  has proper support, and

$$\text{WF}'(K_A)_{\mathcal{X}_2} \cap \text{WF}'(K_B)_{\mathcal{X}_2} = \emptyset \quad (61)$$

then the composition  $A \circ B$  is well defined and

$$\begin{aligned} \text{WF}'(K_{A \circ B}) \subset & \text{WF}'(K_A) \circ \text{WF}'(K_B) \\ & \cup ((\mathcal{X}_1 \times \{0\}) \times \text{WF}(K_B)_{\mathcal{X}_3}) \cup (\text{WF}(K_A)_{\mathcal{X}_1} \times (\mathcal{X}_3 \times \{0\})), \end{aligned} \quad (62)$$

where

$$\begin{aligned} \text{WF}'(K_A) \circ \text{WF}'(K_B) := & \{(x_3, \xi_3, x_1, \xi_1) | \text{there exist } \xi_2 \neq 0 \quad \text{and} \quad x_2 \text{ s.t.} \\ & \times (x_2, \xi_2, x_1, \xi_1) \in \text{WF}'(K_B) \quad \text{and} \quad (x_3, \xi_3, x_2, \xi_2) \in \text{WF}'(K_A)\}. \end{aligned} \quad (63)$$

Combining this information, one then finds that  $\text{WF}'(G_p)$  is contained in the set  $\mathcal{C}^+$ . The tip of the lightcone in effect plays no role for this argument, as backward null geodesics starting in the interior of  $J^+(p) \cap \mathcal{O}$  never reach it<sup>18</sup>. The theorem is proven if we can show that both

<sup>18</sup>The tip of the lightcone only needs to be examined a little more carefully for the proof that the commutator property holds, and plays a role similar to  $i^-$  in the subsequent construction in RNdS.

sets are in fact equal. To see this, introduce  $G_p^t(f_1, f_2) = G_p(f_2, f_1)$ . Then, by the definition of the wave front set,

$$\text{WF}(G_p^t) = \text{WF}(G_p)^t := \{(x_1, \xi_1, x_2, \xi_2) | (x_2, \xi_2, x_1, \xi_1) \in \text{WF}(G_p)\},$$

from which it follows that  $\text{WF}'(G_p^t) \subset \mathcal{C}^-$ . Therefore, by the commutator property of the 2-point function,

$$\mathcal{C}^+ \cup \mathcal{C}^- = \text{WF}'(E) = \text{WF}'(G_p - G_p^t) \subset \text{WF}'(G_p) \cup \text{WF}'(G_p^t) \subset \mathcal{C}^+ \cup \mathcal{C}^-,$$

and the above inclusions must in fact be equalities. Since  $\mathcal{C}^+ \cap \mathcal{C}^- = \emptyset$ , this means that in fact  $\text{WF}'(G_p) = \mathcal{C}^+$ , which is the statement of the theorem.  $\square$

**4.2.2. The ‘Unruh’ and a ‘comparison’ state in RNdS.** We now define two states by a similar construction in RNdS which play a role for the subsequent argument.

**Unruh state:** the first is an analogue of the ‘Unruh state’ [70] in Schwarzschild. We first define (see figure 1)

$$\mathcal{H}_c = \mathcal{H}_c^- \cup \mathcal{H}_c^R, \quad \mathcal{H} = \mathcal{H}^- \cup \mathcal{H}^L, \tag{64}$$

so  $\mathcal{H}_c \cup \mathcal{H}$  is formally, i.e. apart from the ‘point’  $i^-$  at infinity, a characteristic surface for the union  $\mathcal{N}$  of the regions I, II, III, where the Unruh state will be defined. Heuristically, the Unruh state is specified as in (53) by the ‘positive frequency modes ( $k \geq 0$ )’  $\psi_{klm}^{\text{U, in/up}}$ , which are solutions ( $\square - \mu^2$ ) $\psi_{klm}^{\text{U, in/up}} = 0$  defined on  $\mathcal{N}$  in terms of their initial data on  $\mathcal{H}_c \cup \mathcal{H}$ :<sup>19</sup>

$$\psi_{klm}^{\text{U, in}}(x) = \begin{cases} (2\pi)^{-\frac{3}{2}}(2k)^{-\frac{1}{2}} Y_{lm}(\theta, \phi) r_c^{-1} e^{-ikV} & \text{on } \mathcal{H}_c, \\ 0 & \text{on } \mathcal{H}, \end{cases} \tag{65a}$$

$$\psi_{klm}^{\text{U, up}}(x) = \begin{cases} 0 & \text{on } \mathcal{H}_c, \\ (2\pi)^{-\frac{3}{2}}(2k)^{-\frac{1}{2}} Y_{lm}(\theta, \phi) r_+^{-1} e^{-ikU} & \text{on } \mathcal{H}, \end{cases} \tag{65b}$$

where  $V = V_c$  is the Kruskal type (affine) coordinate adapted to the cosmological horizon, and where  $U$  is the Kruskal type (affine) coordinate adapted to the event horizon; see section 2.

In order to see that the Unruh state is well-defined, and to understand some of its basic properties, we formally rewrite the definition (53) (with the modes  $\psi_{klm}^{\text{U, in/up}}$ ) in a similar way as in (55), noting that the VanVleck determinant vanishes on a Killing horizon. Let  $f_1, f_2 \in C_0^\infty(\mathcal{N})$ , and define for  $X \in \{+, c\}$  the restrictions of the corresponding solutions,  $F_1^X = E^- f_1|_{\mathcal{H}_X}, F_2^X = E^- f_2|_{\mathcal{H}_X}$ . Then the precise definition of the Unruh state is:

**Definition 4.3.** The Unruh state is defined by the 2-point function on  $\mathcal{N} \times \mathcal{N}$  given by:

$$\begin{aligned} \langle \Phi(f_1)\Phi(f_2) \rangle_U = & -\frac{1}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{S^2} \frac{F_1^+(U_1, \Omega) F_2^+(U_2, \Omega)}{(U_1 - U_2 - i0)^2} d^2\Omega dU_1 dU_2 \\ & -\frac{1}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{S^2} \frac{F_1^c(V_1, \Omega) F_2^c(V_2, \Omega)}{(V_1 - V_2 - i0)^2} d^2\Omega dV_1 dV_2 \end{aligned} \tag{66}$$

<sup>19</sup>This is to be understood in the sense of wave packets, cf section 5.2 for how to translate this into asymptotic data of the corresponding radial functions.

where null-generators of the event and cosmological horizon are parameterized by  $U$  resp.  $V$ , and  $d^2\Omega$  is the induced integration element on the spheres of constant  $V$  resp.  $U$ .

This definition is actually still formal as it stands, because the convergence of the integrals over  $U$  and  $V$  has not been ensured. Also, one needs to show the commutator, field equation, and positivity properties (S1)–(S3), and one would also like to show that the Unruh state is Hadamard in the union  $\mathcal{N}$  of the regions I, II, III where it is defined. These properties have been proven in the case of Schwarzschild spacetime ( $\Lambda = Q = 0$ ) by [65]. A similar proof can be given in the present context, taking into account the features of the dynamical evolution special to RNdS.

First, the positivity property (S3) is obvious—assuming (66) is well defined—because  $-(x - i0)^{-2}$  is a kernel of positive type (its Fourier transform is non-negative). The field equation (2) is also obvious, because the commutator function  $E$  is a bi-solution. For a Cauchy surface  $\Sigma$  as in figure 1,  $E$  can be written in the form [71]:

$$E(f_1, f_2) = \int_{\Sigma} n^\mu (F_1 \nabla_\mu F_2 - F_2 \nabla_\mu F_1) d\eta_{\Sigma}, \tag{67}$$

with  $F_1 = Ef_1, F_2 = Ef_2$ . Since  $(\square - \mu^2)F_1 = (\square - \mu^2)F_2 = 0$ , Gauss' theorem shows that the expression is independent of the chosen Cauchy surface for  $\mathcal{N}$ . Now, if we could formally deform  $\Sigma$  to the pair of null surfaces  $\mathcal{H}_c \cup \mathcal{H}$ , then we would have, formally

$$E(f_1, f_2) = \int_{\mathbb{R}} \int_{S^2} (F_1^+ \partial_U F_2^+ - F_2^+ \partial_U F_1^+) d^2\Omega dU + \int_{\mathbb{R}} \int_{S^2} (F_1^c \partial_V F_2^c - F_2^c \partial_V F_1^c) d^2\Omega dV. \tag{68}$$

Although this argument is not rigorous because it ignores a potential contribution from  $i^-$ , let us proceed for the moment and assume that the integrals in (66) converge in a suitable sense. Then using the formula  $\text{Im}(x - i0)^{-2} = -\pi\delta'(x)$ , on both terms, one formally gets

$$\begin{aligned} \langle \Phi(f_1)\Phi(f_2) \rangle_U - \langle \Phi(f_2)\Phi(f_1) \rangle_U &= 2i \text{Im} \langle \Phi(f_1)\Phi(f_2) \rangle_U \\ &= i \int_{\mathbb{R}} \int_{S^2} (F_1^+ \partial_U F_2^+ - F_2^+ \partial_U F_1^+) d^2\Omega dU \\ &\quad + i \int_{\mathbb{R}} \int_{S^2} (F_1^c \partial_V F_2^c - F_2^c \partial_V F_1^c) d^2\Omega dV. \end{aligned} \tag{69}$$

Combining with the previous equation, this would show the commutator property (S1).

In order to make this argument rigorous, we should first check (68), and for that, we need the asymptotic behavior of  $F_1, F_2$  near  $i^-$ . This asymptotic behavior has been analyzed by Hintz and Vasy [1] and is best expressed resolving  $i^-$  by a blow up procedure sketched in section 2. The results of [1] express the regularity of the forward solution  $F = E^+f$  near  $i^+$  in terms of certain ‘variable order’ Sobolev spaces. By time reflection symmetry, these analogously hold for  $F = E^-f$  near  $i^-$ . The variable orders of (fractional) differentiability express the different regularity properties of the solution near the various horizons, i.e. the different locations of  $r$ , see appendix A of [1] for the precise definitions. The variable forward order functions  $s(r)$  considered by these authors are such that  $s(r)$  is constant for  $r < r_- + \epsilon$  and  $r > r_- + 2\epsilon$ , with

$$s(r) \begin{cases} < 1/2 + \alpha/\kappa_- & \text{for } r < r_- + \epsilon \\ \geq 1/2 + \max\{\alpha/\kappa_c, \alpha/\kappa_-\} & \text{for } r > r_- + 2\epsilon \end{cases} \tag{70}$$

and  $s'(r) \geq 0$ . The parameter  $\alpha$  of main interest is the spectral gap, but forward order functions can in principle be considered for any  $\alpha \in \mathbb{R}$ . Based on a forward order function, a certain Sobolev space  $H_{b,+}^s(\mathcal{R})$  of functions supported toward the future of the Cauchy surface  $\Sigma$ , is then defined, see appendix B of [1] for details, and figure 4 for the definition of the domain  $\mathcal{R}$ . Further, [1] define the spaces  $H_{b,+}^{s,\alpha}(\mathcal{R}) = \tau^\alpha H_{b,+}^s(\mathcal{R})$  of functions with an exponential decay toward  $t_* \rightarrow \infty$ , i.e. toward  $i^+$ , see section 2 for the definition of  $t_* = -\log \tau$ . Their main result can be phrased as follows:

**Theorem 4.4.** *If<sup>20</sup>  $\mu^2 > 0$  and  $f \in C_0^\infty(\mathcal{N})$ , then  $F = E^+ f \in H_{b,+}^{s,\alpha}(\mathcal{R})$  for any forward order function, and furthermore  $\|F\|_{H_{b,+}^{s,\alpha}(\mathcal{R})} \lesssim \|f\|_{C^m(\mathcal{N})}$  for some  $m$  depending on  $s$ .*

**Proof.** The proof of this theorem is essentially contained in that of proposition 2.17 of [1]. Here we only somewhat pedantically go through the norm estimates provided by [1]. [1] first modify the wave operator to  $P = \square - \mu^2 - iQ$ , where  $Q$  is a suitable formally self-adjoint pseudo-differential operator having its support in the region  $r < r_-$  inside the black hole behind the Cauchy horizon  $\mathcal{CH}^L$ . The construction of  $Q$  is roughly made in such a way that singularities propagating into the interior of the black hole get ‘absorbed’ by  $Q$ . Then they show (theorem 2.13) that with a suitable choice of  $-\epsilon < \tilde{\alpha} < 0$  and of  $Q$ , the map  $P$  is a Fredholm operator between their weighted ‘forward’ Sobolev spaces  $H_{b,+}^{\tilde{s},\tilde{\alpha}}(\mathcal{R}) \rightarrow H_{b,+}^{\tilde{s}-1,\tilde{\alpha}}(\mathcal{R})$ , with  $\tilde{s}$  a forward order function for  $\tilde{\alpha}$ , and with  $\mathcal{R}$  a domain of the shape indicated in figure 4. More precisely,  $\mathcal{R}$  should be extended to the left beyond the Cauchy horizon  $r < r_-$  by effectively modifying the metric function  $f(r)$  such that the singularity at  $r = 0$  gets replaced with the origin of polar coordinates or another horizon. By the standard theory of Fredholm operators, the inverse  $P^{-1}$  is defined as a bounded operator on a subspace of  $H_{b,+}^{\tilde{s}-1,\tilde{\alpha}}(\mathcal{R})$  of finite co-dimension. In fact, it is shown in [1], lemma 2.15 that the projector onto this subspace may be chosen as  $Rf = f - \sum_{i=1}^N \phi_i v_i(f)$ , with  $\phi_i$  smooth and supported in  $r < r_-$  and  $v_i$  distributions. Thus, if  $f \in C_0^\infty(\mathcal{M})$ , it follows that the forward solution of  $PF = f$  has

$$\|F\|_{H_{b,+}^{\tilde{s},\tilde{\alpha}}(\mathcal{R})} \lesssim \|f\|_{C^m(\mathcal{M})} \tag{71}$$

for a sufficiently large  $m$  depending on the choice of  $\tilde{s}$ . Then, by the propagation of singularities theorem, proposition 2.9 of [1], one has

$$\|F\|_{H_{b,+}^{s,\alpha}(\mathcal{R})} \lesssim \|F\|_{H_{b,+}^{\tilde{s},\tilde{\alpha}}(\mathcal{R})} + \|PF\|_{H_{b,+}^{\tilde{s},\tilde{\alpha}}(\mathcal{R})} \lesssim \|f\|_{C^m(\mathcal{M})}, \tag{72}$$

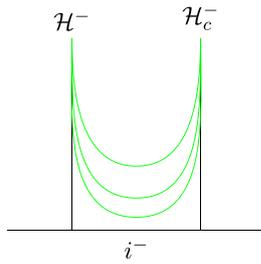
with  $s \leq \tilde{s} + 1$  a forward order function for any weight  $\alpha > 0$  such that  $\alpha/\kappa_- < 1$ . By [1], section 2 and [26], lemma 3.1 and remark 3.4, (essentially a Fourier–Laplace-transform argument in the variable  $t_* = -\log \tau$ ),  $F$  can be developed in an asymptotic expansion

$$F(x, t_*) = \sum_j t_*^{m_j} e^{\sigma_j t_*} a_j(x) + F'(x, t_*) \tag{73}$$

with  $-\alpha < \sigma_j < -\tilde{\alpha}$  the resonances of the Fourier–Laplace-transformed operator  $\widehat{P}(\sigma)$ ,  $m_j$  their multiplicity, and with a norm bound

$$\|F'\|_{H_{b,+}^{s,\alpha}(\mathcal{R})} \lesssim \|f\|_{C^m(\mathcal{M})}. \tag{74}$$

<sup>20</sup>The restriction to  $\mu^2 > 0$  is made here simply to avoid complications arising from constant solutions in the case  $\mu^2 = 0$ .



**Figure 8.** Sketch of the ‘blow-up’ of  $i^-$  and of a sequence of Cauchy surfaces as described in the text.

By [1], lemma 2.14,  $F' = F$  in the region  $r > r_-$  if  $\alpha$  is chosen so that there are no quasi-normal frequencies for the scattering problem in region I with  $0 \leq \text{Im}(\omega_{nlm}) > \alpha$ —in fact they argue that any resonance of  $\hat{P}(\sigma)$  not equal to such a quasi-normal mode<sup>21</sup> has  $a_j(x) = 0$  for  $r > r_-$ .

In the case at hand, we have  $\mu^2 > 0$ . It follows that there can be no such quasi-normal frequencies on the real line (unlike for the massless wave operator considered by [1], who have to consider separately the constant mode), as one can see by a simple argument involving the Wronskian of the radial equation, cf (104) below, for real  $\omega$ . Thus, we may put  $\alpha$  to be the value (3).  $\square$

These results will now be used in order to investigate the properties of the Unruh state. As already mentioned, theorem 4.4 analogously holds for the behavior of the backward solution  $F = E^-f$  near  $i^-$ . We are interested in its behavior in region I. But, as no lower bound is imposed on the forward order function  $s(r)$  in the region  $r_+ \leq r \leq r_c$ , using Sobolev embedding, there follows a bound

$$|\partial^N F(t_*, r, \Omega)| \lesssim e^{\alpha t_*} \|f\|_{C^m(\mathcal{N})} \quad \text{for } r_+ \leq r \leq r_c, \tag{75}$$

where  $\partial^N$  is a derivative in  $(t_*, r, \Omega)$  of order  $N > 0$  and  $m$  depends only on  $N$ . By choosing in (67) a sequence of Cauchy surfaces as depicted in figure 8 moving downwards toward the horizontal boundary, it immediately follows by writing (67) out in the coordinates  $(t_*, r, \Omega)$  that in the limit only the contributions from the two vertical boundaries  $\mathcal{H}^-$ ,  $\mathcal{H}_c^-$  contribute near  $i^-$ , but not the interpolating surface depicted in green. Hence, in this limit, (68) holds true.

We may also see immediately that the integrals in (66) converge when  $f_1, f_2$  are functions in  $C_0^\infty(\mathcal{N})$ , making the Unruh state actually well defined. For this, we note that  $e^{\alpha t_*} \sim |U|^{-\alpha/\kappa_+}$  for  $U \rightarrow -\infty$  on  $\mathcal{H}$  and  $e^{\alpha t_*} \sim |V|^{-\alpha/\kappa_c}$  for  $V \rightarrow -\infty$  on  $\mathcal{H}_c$ . In combination with (75), this easily implies convergence of the integrals (69). We therefore conclude:

**Proposition 4.5.** *The Unruh state is well-defined on the algebra  $\mathcal{A}(\mathcal{N})$ ,  $\mathcal{N} = \text{I} \cup \text{II} \cup \text{III}$  generated by  $\Phi(f)$  with  $f \in C_0^\infty(\mathcal{N})$ .*

We now verify that the Unruh 2-point function  $G_U(x_1, x_2) = \langle \Phi(x_1)\Phi(x_2) \rangle_U$  is Hadamard in the union of regions I, II, III (but not necessarily across  $\mathcal{CH}^R$ !). Let us begin by recalling basic results from the analysis of partial differential equations. For any distribution  $\varphi$  on  $\mathcal{X}$  and (pseudo-) differential operator  $A$ , it is known that

$$\text{WF}(\varphi) \subset \text{WF}(A\varphi) \cup \text{char}(A), \tag{76}$$

<sup>21</sup> Note that  $e^{\sigma t_*} a(x)$  satisfies QNM boundary conditions for  $\sigma < 0$  due to the definition of  $t_*$  in section 2.

where  $\text{char}(A) = \{(x, \xi) \in T^*\mathcal{X} \setminus o \mid a(x, \xi) = 0\}$ , and where  $a(x, \xi)$  is the principal symbol of  $A$ , see theorem. 8.3.1 of [68]. Furthermore, the propagation of singularities theorem [69] states that if the differential operator  $A$  has real principal symbol  $a$  and if  $A\varphi$  is smooth then  $\text{WF}(\varphi)$  is contained in the intersection of  $\text{char}(A)$  and the Hamiltonian orbits of the symbol  $a(x, \xi)$  on ‘phase space’  $T^*\mathcal{X} \setminus o$ . In the case of  $A = \square - \mu^2$ , the principal symbol is  $g^{\mu\nu}(x)\xi_\mu\xi_\nu$ , and the Hamiltonian orbits in the characteristic set  $\text{char}(\square - \mu^2)$  are precisely the null geodesics with initial condition  $(x, \xi)$ . We denote one such an orbit, called bicharacteristic, as  $\mathcal{B}(x, \xi)$ . Based on this definition, we distinguish three cases:

(1) Neither the bicharacteristic  $\mathcal{B}(x_1, \xi_1)$  nor  $\mathcal{B}(x_2, \xi_2)$  enters region I: in this case, the claim follows by exactly the same argument as given in the previous section for lightcones, because every backward null geodesic emanating from these regions will eventually hit  $\mathcal{H} \cup \mathcal{H}_c$ .

(2) The bicharacteristics  $\mathcal{B}(x_1, \xi_1)$  and  $\mathcal{B}(x_2, \xi_2)$  enter region I: by propagation of singularities, it suffices to consider  $x_1, x_2 \in \text{I}$ . In region I, past-directed null geodesics trapped on the photon ring (see e.g. [44]), will not come from  $\mathcal{H} \cup \mathcal{H}_c$  but begin their lives in the ‘point’  $i^-$ . Thus, the type of argument made in the case of lightcones—where this cannot occur—does not work, and one needs another argument worked out in detail in the case of Schwarzschild by [65]. The argument is based on the fact that the Unruh state is static, i.e. invariant under translations in the coordinate  $t$ . A simplified version of the argument which also works in the present case runs as follows.

First, for an arbitrary but fixed open region  $\mathcal{O}$  whose closure is contained in region I (i.e. not touching the horizons  $\mathcal{H} \cup \mathcal{H}_c$ ), we introduce the maps,  $X \in \{c, +\}$ ,

$$K^X : C_0^\infty(\mathcal{O}) \rightarrow L^2(\mathbb{R}_\omega \times S_\Omega^2), \quad K^X f(\omega, \Omega) = r_X \left( \frac{\omega e^{2\pi r_X \omega}}{\sinh(2\pi r_X \omega)} \right)^{\frac{1}{2}} \int_{\mathbb{R}} F^X(s, \Omega) e^{i\omega s} ds \tag{77}$$

where  $F^+(u, \Omega) = Ef|_{\mathcal{H}^-}(u, \Omega)$  and  $F^c(v, \Omega) = Ef|_{\mathcal{H}_c^-}(v, \Omega)$ . Next, we let  $Kf := K^+f \oplus K^c f \in L^2 \oplus L^2$ , and then we can write the Unruh 2-point function as [49]

$$\langle \Phi(f_1)\Phi(f_2) \rangle_U = \langle Kf_1, Kf_2 \rangle_{L^2 \oplus L^2}, \tag{78}$$

which follows from (66) by Fourier transformation and the relations  $U = -e^{-\kappa_+ u}$ ,  $V = -e^{-\kappa_c v}$ . The decay and regularity results, theorem 4.4, together with the Sobolev embedding theorem and the relation between  $t_*$  and  $u$  resp.  $v$  on  $\mathcal{H}^-$  resp.  $\mathcal{H}_c^-$  imply that  $|\partial_s^N F^X(s, \Omega)| \leq C_N e^{-\alpha|s|}$ , where the constant  $C_N$  is controlled by some  $C^m$  norm of  $f$  by (75). As a consequence,  $K = K^+ \oplus K^c$  is shown to be a distribution in  $\mathcal{O}$  with values in the Hilbert space  $L^2 \oplus L^2$ : since  $|\partial_s^N F^X(s, \Omega)| \lesssim \|f\|_{C^m(\mathcal{O})} e^{-\alpha|s|}$  for all  $f \in C_0^\infty(\mathcal{O})$ , we find  $\|Kf\|_{L^2 \oplus L^2} \lesssim \|f\|_{C^m(\mathcal{O})}$ , which is the continuity required from a distribution.

Next, invariance of  $E$  under translations of  $t$  in the coordinates  $(t, r_*, \Omega)$  and the exponentially decreasing prefactor of order  $O(e^{2\pi r_X \omega})$  in (77) for  $\omega \rightarrow -\infty$  imply that the distribution  $K$  has a holomorphic extension to the strip  $\{t + is \mid s \in (0, 2\pi r_+)\}$  in the  $t$ -coordinate. Moreover it is the boundary value of this holomorphic extension, in the sense of distributions, as  $s \rightarrow 0^+$ , for all test-functions from  $C_0^\infty(\mathcal{O})$ . By a very slight modification of the proof of theorem 3.1.14 of [68] we then get, for some  $m$ , the inequality

$$\|K(t + is, r, \Omega)\|_{L^2 \oplus L^2} \lesssim s^{-m} \quad \text{for } (t, r, \Omega) \in \mathcal{O} \text{ and } s \in (0, \pi r_+). \tag{79}$$

In view of theorem 8.4.8 of [68], we then further get

$$\text{WF}(K)|_{\mathcal{O}} \subset \{(x, \xi) \in T^*\mathcal{O} \setminus \mathcal{O} \mid \langle \xi, \partial_t \rangle > 0\}. \tag{80}$$

Combining (78) and the rules for the wave front set under the composition of distributions, we finally get

$$\text{WF}'(G_U)|_{\mathcal{O} \times \mathcal{O}} \subset \{(x_1, \xi_1, x_2, \xi_2) \in T^*(\mathcal{O} \times \mathcal{O}) \setminus \mathcal{O} \mid \langle \xi_i, \partial_t \rangle > 0\}. \tag{81}$$

Next, using the relationship between  $t$  and  $u, v$ , it follows immediately from (77), (78) that the 2-point function  $G_U$  is invariant under translation of  $t$  in both arguments. In infinitesimal form,  $(\partial_{t_1} + \partial_{t_2})G_U = 0$ . From (76), it follows that in fact

$$\text{WF}'(G_U)|_{\mathcal{O} \times \mathcal{O}} \subset \{(x_1, \xi_1, x_2, \xi_2) \in T^*(\mathcal{O} \times \mathcal{O}) \setminus \mathcal{O} \mid \langle \xi_1, \partial_t \rangle = \langle \xi_2, \partial_t \rangle > 0\}. \tag{82}$$

If we assume that  $\mathcal{O}$  is the entire (open) region I, viewed as a globally hyperbolic spacetime in its own right with Cauchy-surface  $\Sigma_{\mathcal{O}}$ , then it follows from the propagation of singularities theorem that

$$\text{WF}'(G_U)|_{\mathcal{O} \times \mathcal{O}} \subset \{\mathcal{B}(x_1, \xi_1) \times \mathcal{B}(x_2, \xi_2) \setminus \mathcal{O} \mid x_i \in \Sigma_{\mathcal{O}}, g(\xi_i, \xi_i) = 0\}. \tag{83}$$

We now distinguish the cases (a)  $\mathcal{B}(x_1, \xi_1) = \mathcal{B}(x_2, \xi_2)$  and (b)  $\mathcal{B}(x_1, \xi_1) \neq \mathcal{B}(x_2, \xi_2)$ . In case (a), we obtain, from (82) and (83), that  $(x_1, \xi_1, x_2, \xi_2) \notin \text{WF}'(G_U)$ , unless  $\xi_1 \sim \xi_2$  and  $\xi_1 \triangleright 0$ . In case (b), we may, without loss of generality, assume that  $x_1, x_2 \in \Sigma_{\mathcal{O}}$  are spacelike separated. We may then choose real-valued cutoff functions  $\chi_1, \chi_2$  supported in a coordinate chart near  $x_1, x_2$  such that  $\chi_i(x_i) \neq 0$ , with  $\text{supp}(\chi_1)$  remaining spacelike to  $\text{supp}(\chi_2)$ . Since  $G_U$  is a distribution of positive type, it follows in view of the Cauchy–Schwarz inequality that

$$|G_U(\chi_1 e_{-k_1}, \chi_2 e_{k_2})| \leq |G_U(\chi_1 e_{-k_1}, \chi_1 e_{k_1})|^{\frac{1}{2}} |G_U(\chi_2 e_{-k_2}, \chi_2 e_{k_2})|^{\frac{1}{2}}, \tag{84}$$

where  $e_k(x) := e^{-ikx}$ . Now let

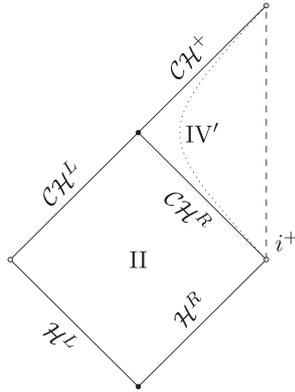
$$\mathcal{V}_i^{\pm} = \{(x, \xi) \in T^*\mathcal{O} \setminus \mathcal{O} \mid x \in \text{supp}\chi_i, g(\xi, \xi) = 0, \langle \xi, \partial_t \rangle \gtrless 0\}. \tag{85}$$

It follows from the wave front conditions (82), (83) applied to the right side of (84) that the left side of (84) is decaying rapidly in  $(k_1, k_2)$  (i.e. is bounded by  $\leq C_N(1 + |k_1| + |k_2|)^{-N}$  for any  $N$ ) in any conic neighborhood not intersecting  $\mathcal{V}_1^+ \times \mathcal{V}_2^+$ . However, since the arguments are spacelike, the commutator property of 2-point functions also gives

$$G_U(\chi_1 e_{-k_1}, \chi_2 e_{k_2}) = G_U(\chi_2 e_{k_2}, \chi_1 e_{-k_1}). \tag{86}$$

Applying the same reasoning to  $G_U(\chi_2 e_{k_2}, \chi_1 e_{-k_1})$ , it follows that the left side of (84) is decaying as fast in  $(k_1, k_2)$  in any conic neighborhood not intersecting  $\mathcal{V}_1^- \times \mathcal{V}_2^-$ . For  $\text{supp}\chi_{1/2}$  small enough,  $\mathcal{V}_1^+ \times \mathcal{V}_2^+ \cap \mathcal{V}_1^- \times \mathcal{V}_2^- = \emptyset$ , so that  $(x_1, \xi_1, x_2, \xi_2) \notin \text{WF}'(G_U)$  in case (b). Combining this information, we have

$$\begin{aligned} \text{WF}'(G_U)|_{\mathcal{O} \times \mathcal{O}} \subset \{(x_1, \xi_1, x_2, \xi_2) \in T^*(\mathcal{O} \times \mathcal{O}) \setminus \mathcal{O} \mid x_1, x_2 \text{ connected by null geodesic } \gamma, \\ \xi_1 = c_1 g(\cdot, \dot{\gamma}_1), \xi_2 = c_2 g(\cdot, \dot{\gamma}_2), \langle \xi_1, \partial_t \rangle = \langle \xi_2, \partial_t \rangle > 0\}. \end{aligned} \tag{87}$$



**Figure 9.** Sketch of the spacetime in which the comparison state is Hadamard. The dashed line indicates the origin of coordinates. The region beyond the dotted line is region where  $f(r)$  is modified.

The set on the right hand side is easily seen to be equal to  $\mathcal{C}^+$ , thus we learn  $\text{WF}'(G_U)|_{\mathcal{O} \times \mathcal{O}} \subset \mathcal{C}^+$ , and we actually get equality by the commutator argument used at the end of the proof of theorem 4.2.

(3) *One bicharacteristic, say  $\mathcal{B}(x_1, \xi_1)$ , will enter I, but  $\mathcal{B}(x_2, \xi_2)$  not:* we can argue as in subcase (b) above, i.e. using that the right hand side of the inequality (84) is decaying rapidly in  $(k_1, k_2)$ , due to (1), (2).

**Comparison state:** for the sake of the discussion in the next section, it is also convenient to have another state that is stationary and that is manifestly Hadamard in a two-sided open neighborhood of the Cauchy horizon  $\mathcal{CH}^R$ , see figure 1. This state is not of particular physical interest and introduced purely in order to make our arguments with greater ease. In order to define it, we modify the metric function  $f(r)$  beyond the Cauchy horizon for  $r < r_- - \delta$  with a small  $\delta > 0$  in such a way that it smoothly interpolates between  $f(r)$  and the constant function 1 in a neighborhood of  $r = 0$ . We call the new function  $f_*(r)$ . Evidently, it defines the RNdS metric in regions I, II, III and an open neighborhood of  $\mathcal{CH}^R$ , but replaces the singularity in region IV with a smooth origin of polar coordinates  $r = 0$ , see region IV' in figure 9.

In this—made-up unphysical—spacetime, we now define a vacuum state called  $\langle \cdot \rangle_{\mathcal{C}}$  for regions II, IV' by the ‘positive frequency modes ( $k \geq 0$ )’  $\psi_{k\ell m}^{\mathcal{C}, \text{out}}$ , which are solutions  $(\square - \mu^2)\psi_{k\ell m}^{\mathcal{C}, \text{out}} = 0$  defined on the union of II, IV' in terms of their final data on  $\mathcal{CH} = \mathcal{CH}^+ \cup \mathcal{CH}^L$ :

$$\psi_{k\ell m}^{\mathcal{C}, \text{out}}(x) = (2\pi)^{-\frac{3}{2}}(2k)^{-\frac{1}{2}}Y_{\ell m}(\theta, \phi)r_c^{-1}e^{-ikV} \quad \text{on } \mathcal{CH}, \tag{88}$$

where  $V = V_-$  is the Kruskal type (affine) coordinate adapted to this Cauchy horizon, see figure 9. Again, it can be shown by exactly the same methods as in the lightcone case that these modes define via (53) a Hadamard 2-point function  $\langle \Phi(x_1)\Phi(x_2) \rangle_{\mathcal{C}}$  in regions II, IV', which we call the ‘comparison state’.

We have shown:

**Theorem 4.6.** *The Unruh state has a 2-point function of Hadamard form in the union of region I, II, III. The comparison state is Hadamard in the union of region II and an open (two-sided) neighborhood of the Cauchy horizon  $\mathcal{CH}^R$ .*

## 5. Behavior of the quantum stress tensor at the Cauchy horizon in 4 dimensions

In this section, we will prove our main result, (5), concerning the behavior of the stress-energy tensor near the Cauchy horizon,  $\mathcal{CH}^R$ , of four-dimensional RNdS spacetime for an arbitrary state  $\Psi$  that is Hadamard in the union of regions I, II, and III. We will proceed by writing  $\langle T_{\mu\nu} \rangle_\Psi$  as

$$\langle T_{\mu\nu} \rangle_\Psi = \underbrace{(\langle T_{\mu\nu} \rangle_\Psi - \langle T_{\mu\nu} \rangle_U)}_{=t_{\mu\nu} \text{ as in prop. 5.1}} + \underbrace{(\langle T_{\mu\nu} \rangle_U - \langle T_{\mu\nu} \rangle_C)}_{\text{calculated in sec. 5.4}} + \underbrace{\langle T_{\mu\nu} \rangle_C}_{\text{smooth at } \mathcal{CH}^R}. \quad (89)$$

In subsection 5.1, we will show that the first term behaves like the stress-energy tensor of a classical field. Thus, it has behavior not more singular than (4). The last term is smooth by construction of the state  $\langle \cdot \rangle_C$ . Most of our effort will be spent in calculating the middle term, which will be done via a mode expansion that will be carried out in subsections 5.2–5.4.

In the two-dimensional case, we found that the limiting behavior of  $\langle T_{vv} \rangle_U - \langle T_{vv} \rangle_C$  on  $\mathcal{CH}^R$  was given by (45). Since  $\langle T_{\mu\nu} \rangle_C$  is smooth on  $\mathcal{CH}^R$ , this implies that  $\langle T_{vv} \rangle_U \rightarrow (\kappa_c^2 - \kappa_-^2)/48\pi$  on  $\mathcal{CH}^R$ . Thus  $\langle T_{vv} \rangle_U$  goes to a finite limit on  $\mathcal{CH}^R$ , which is nonvanishing unless  $\kappa_c = \kappa_-$ . As discussed in section 3, for  $\kappa_c \neq \kappa_-$ , this implies that  $\langle T_{V_-V_-} \rangle_U$  blows up on  $\mathcal{CH}^R$  as  $1/V_-^2$ , a worse singularity than classical behavior. In 4 dimensions, the logical possibilities are that (i)  $\langle T_{vv} \rangle_U$  does not approach a limit at  $\mathcal{CH}^R$ , (ii)  $\langle T_{vv} \rangle_U$  approaches a nonzero limit at  $\mathcal{CH}^R$ , or (iii)  $\langle T_{vv} \rangle_U \rightarrow 0$  on  $\mathcal{CH}^R$ . In case (ii), we again would have that  $\langle T_{V_-V_-} \rangle_U$  blows up on  $\mathcal{CH}^R$  as  $1/V_-^2$ . In case (i) the behavior of  $\langle T_{V_-V_-} \rangle_U$  would be more singular than a  $1/V_-^2$  blow up. Only in case (iii) would the quantum stress-energy possibly behave in a manner similar to the classical case.

In subsection 5.2, we will obtain a mode sum expression for  $\langle T_{vv} \rangle_U - \langle T_{vv} \rangle_C$  in terms of scattering coefficients (see (127)). The scattering coefficients are calculated in subsection 5.3 for the conformally coupled case,  $\mu^2 = \frac{1}{6}R = \frac{2}{3}\Lambda$ . The mode sum is then evaluated numerically in subsection 5.4. We find that the mode sum converges to a nonzero value, i.e., case (ii) occurs and the behavior of the stress-energy tensor is given by (5) with  $C \neq 0$ . Although our evaluation of  $C$  is done only in the conformally coupled case  $\mu^2 = \frac{2}{3}\Lambda$  and at particular black hole parameters, we would expect  $C$  to vary analytically with  $\mu$  and the black hole parameters, in which case we would have  $C \neq 0$  generically.

### 5.1. Reduction to the Unruh state

In this subsection, we show that  $\langle T_{\mu\nu} \rangle_\Psi - \langle T_{\mu\nu} \rangle_U$  has the same behavior near  $\mathcal{CH}^R$  as the stress-energy tensor of a classical field. Let  $\langle \cdot \rangle_\Psi$  be an arbitrary state that is Hadamard near the Cauchy surface  $\Sigma$  as drawn in figure 1. By the propagation of singularities [58], it remains Hadamard throughout regions I, II, III, but it is not defined (at least not uniquely) in region IV, just as if we were to specify initial data for the classical wave equation on  $\Sigma$ . Since the state is Hadamard in I, II, III, the expectation value  $\langle T_{\mu\nu} \rangle_\Psi$  is finite and smooth in I, II, III, but may diverge<sup>22</sup> as we move toward the Cauchy horizon,  $\mathcal{CH}^R$ . We would like to see how it diverges, if at all. The first step is to relate the behavior of  $\langle T_{\mu\nu} \rangle_\Psi$  to behavior of the corresponding quantity  $\langle T_{\mu\nu} \rangle_U$  for the Unruh state, which we have a better way of calculating.

<sup>22</sup>The coordinate indices of course should refer to a coordinate system regular there such as  $(U, V, \theta, \varphi)$ , since we are not interested in artificial singularities created by a bad choice of coordinates.

Since we are interested in the local behavior, we pick a point  $p$  on the Cauchy horizon  $\mathcal{CH}^R$  contained in a coordinate chart,  $(x^\mu) = (V, y^i)$ , where  $V = V_-$  is the Kruskal-type coordinate and  $y^i, i = 1, 2, 3$  are local coordinates parameterizing  $\mathcal{CH}^R$ . So,  $V = 0$  locally defines  $\mathcal{CH}^R$ , and the intersection of this chart with region II locally corresponds to  $V < 0$ . Let  $\mathcal{U}$  be a small open neighborhood contained in this coordinate chart. Then the following proposition basically follows from theorem 4.4.

**Proposition 5.1.** *Let  $\Psi, \Psi'$  be Hadamard states near the Cauchy surface  $\Sigma$  as drawn in figure 1, and assume  $\beta > 1/2$ , see (2). Then:*

- each component of  $t_{\mu\nu} := \langle T_{\mu\nu} \rangle_\Psi - \langle T_{\mu\nu} \rangle_{\Psi'}$ , is smooth in  $y$ , and viewed as a function of  $V < 0$ , is locally in  $L^p(\mathbb{R}_-)$ , where  $1/p = 2 - 2\beta + 2\epsilon$  for arbitrarily small  $\epsilon > 0$  (thus  $p > 1$ ), with norm uniformly bounded in  $y$  within  $\mathcal{U}$ .
- viewed as a function of  $V$ ,  $t_{\mu\nu}$  has  $s = \beta - 1/2 - \epsilon$  derivatives locally in  $L^p(\mathbb{R}_-)$  for  $1/p = 3/2 - \beta + \epsilon$  and arbitrarily small  $\epsilon$ .

**Remarks:** (1) For a classical field with smooth initial data on  $\Sigma$ , the stress tensor is a function known to have the same regularity as  $t_{\mu\nu}$ , so the result is that the difference between the quantum stress tensors in two states that are initially Hadamard behaves basically like the classical stress tensor near the Cauchy horizon. (2) The restriction to  $\beta > 1/2$  is of technical nature. For  $\beta \leq 1/2$ , already a classical solution  $\phi$  would generically fail to be of Sobolev class  $H^1$  near  $\mathcal{CH}^R$ , thus making it impossible to extend the solution through  $\mathcal{CH}^R$  already at the classical level, see [13] for a discussion of such matters. Thus,  $\beta > 1/2$  is the interesting case for the discussion of sCC.

**Proof.** Let  $\mathcal{N} = \text{I} \cup \text{II} \cup \text{III}$ . Since  $\mathcal{N}$  is globally hyperbolic and since both states are Hadamard near  $\Sigma$ , it follows by propagation of singularities that the difference between the corresponding two point functions  $W(x_1, x_2) = \langle \Phi(x_1)\Phi(x_2) \rangle_\Psi - \langle \Phi(x_1)\Phi(x_2) \rangle_{\Psi'}$  is in  $C^\infty(\mathcal{N} \times \mathcal{N})$ . It has been shown in [72], lemma 3.7 that given a causal normal neighborhood  $\mathcal{N}$  of  $\Sigma$  (see [49] for the definition), and any open subset  $\mathcal{O}$  with compact closure contained in  $\mathcal{N}$ , there exists a smooth function  $B$  with support in  $\mathcal{O} \times \mathcal{O}$  such that

$$W(f_1, f_2) = \int_{\mathcal{O} \times \mathcal{O}} (Ef_1)(x_1)(Ef_2)(x_2)B(x_1, x_2)d\eta(x_1)d\eta(x_2), \tag{90}$$

for any testfunctions  $f_1, f_2$  with support in the domain of dependence  $D(\mathcal{O})$ . By making  $\mathcal{O}$  sufficiently large, we may assume that the neighborhood  $\mathcal{U}$  of interest near the Cauchy horizon is contained in  $D(\mathcal{O})$ . Furthermore, by the algebraic relations (A3), (A4) and the positivity property (S3) of states, we have  $B(x_1, x_2) = \overline{B(x_1, x_2)} = B(x_2, x_1)$ .

Let  $m$  be a fixed natural number. Using the result [72], appendix B, and the symmetry properties of  $B$ , one can see that there exist  $C_0^m$  functions  $\{b_j\}$  on  $\mathcal{O}$  such that

$$B(x_1, x_2) = \sum_j c_j \overline{b_j(x_1)} b_j(x_2), \quad \sum_j \|b_j\|_{C^m(\mathcal{O})}^2 < \infty, \quad c_j \in \{\pm 1\}. \tag{91}$$

Now let  $\psi_j = E^+ b_j$ , with  $E^+$  the retarded fundamental solution. Then  $(\square - \mu^2)\psi_j = b_j$ , and furthermore,

$$W(x_1, x_2) = \sum_j c_j \overline{\psi_j(x_1)} \psi_j(x_2) \quad \text{for } x_1, x_2 \in \mathcal{U}. \tag{92}$$

To estimate  $\psi_j$ , we can apply the results of [1] already mentioned in theorem 4.4. First, we define new local coordinates  $(z, y^i)$  in the following way. First we locally parameterize the

orbit  $S_p^2$  of  $p \in \mathcal{CH}^R$  by angles  $\Omega$ . Then we transport these coordinates away from  $\mathcal{CH}^R$  along  $\partial/\partial V$  and let  $z$  be the parameter along this curve. Finally, we transport the coordinates  $(z, \Omega)$  along the Killing vector field  $\partial/\partial t$ , and then  $(y^i) = (t, \Omega)$ .

If we apply an arbitrary product  $\prod_{j=1}^N X^{(a_j)}$  of the vector fields  $X^{(a)}$ ,  $a = 1, 2, 3, 4$  generating the symmetry group  $\mathbb{R} \times SO(3)$  of RNdS to the defining equation  $(\square - \mu^2)\psi_j = b_j$  (with retarded initial conditions) and commute these through  $\square - \mu^2$ , the function  $\prod_{a=1}^N X^{(a)}\psi_j$  is seen to satisfy an equation of the same form and initial conditions with a new source. By theorem 4.4 and the construction of  $(z, y^i)$ , this gives

$$\|\partial_y^N \psi_j\|_{H^{1/2+\beta-\epsilon}(\mathcal{U})}^2 \lesssim \|b_j\|_{C^m(\mathcal{O})}^2 \tag{93}$$

for an implicit constant independent of  $j$  and a sufficiently large  $m$  depending on  $N$ . We can also use our expression for the difference of the expected stress tensor in two Hadamard states, see (51). This shows in combination with (92) that

$$t_{\mu\nu}(x) = \sum_j c_j \left\{ \partial_\mu \overline{\psi_j(x)} \partial_\nu \psi_j(x) - \frac{1}{2} g_{\mu\nu} (\partial_\gamma \overline{\psi_j(x)} \partial^\gamma \psi_j(x) + \mu^2 \overline{\psi_j(x)} \psi_j(x)) \right\}. \tag{94}$$

A standard argument based on the Fourier transform in  $y$ , Cauchy’s inequality and Parseval’s theorem shows that for  $\varphi$  supported in the coordinate patch covered by  $(z, y^i)$ ,

$$\begin{aligned} |\varphi(z, y)| &= \int d^3 \xi \tilde{\varphi}(z, \xi) e^{iy\xi} \\ &\leq \left( \int d^3 \xi (1 + \xi^2)^{-N/2} \right)^{1/2} \left( \int d^3 \xi (1 + \xi^2)^{N/2} |\tilde{\varphi}(z, \xi)|^2 \right)^{1/2} \\ &\lesssim \left\| (-\partial_y^2 + 1)^{N/2} \varphi(z, -) \right\|_{L^2(\mathbb{R}_y^3)}, \end{aligned} \tag{95}$$

taking  $N > 3$ . In combination with the Sobolev embedding theorem,  $p$  as in the statement of the proposition,

$$\|\varphi(-, y)\|_{L^{2p}(\mathbb{R}_-)} \lesssim \|(-\partial_z^2 + 1)^{(\beta-1/2-\epsilon)/2} \varphi(-, y)\|_{L^2(\mathbb{R}_-)} \lesssim \|(-\partial_y^2 + 1)^{N/2} \varphi\|_{H^{\beta-1/2-\epsilon}(\mathcal{U})}, \tag{96}$$

uniformly in  $y$  within  $\mathcal{U}$ . We now take  $\varphi = \partial_\mu \psi_j$  or  $\varphi = \psi_j$  and combine this with (93), thereby obtaining

$$\|\partial_\mu \psi_j(-, y)\|_{L^{2p}(\mathbb{R}_-)} \lesssim \|b_j\|_{C^m(\mathcal{O})}. \tag{97}$$

Therefore, using the Cauchy–Schwarz inequality,

$$\begin{aligned} \|t_{\mu\nu}(-, y)\|_{L^p(\mathbb{R}_-)} &\lesssim \sum_j \left( \sum_{\gamma=1}^4 \|\partial_\gamma \psi_j(-, y)\|_{L^{2p}(\mathbb{R}_-)} + \|\psi_j(-, y)\|_{L^{2p}(\mathbb{R}_-)} \right)^2 \\ &\lesssim \sum_j \|b_j\|_{C^m(\mathcal{O})}^2 < \infty. \end{aligned} \tag{98}$$

By considering  $\varphi = \partial_\mu \partial_y^M \psi_j$ , we can similarly obtain a corresponding bound for  $\|\partial_y^M t_{\mu\nu}(-, y)\|_{L^p(\mathbb{R}_-)}$ . Transforming from the coordinates  $(z, y^i)$  to  $(V, y^i)$  gives the first statement.

To prove the second result, we use that (95) also implies

$$\|D_z^{s/2} \varphi(-, y)\|_{L^{2p}(\mathbb{R}_-)} \lesssim \|(-\partial_y^2 + 1)^{N/2} \varphi\|_{H^s(\mathcal{U})} \tag{99}$$

for  $(-\partial_z^2 + 1)^{s/2} = D_z^s$ ,  $s$  and  $p$  as indicated in the second statement, and  $\varphi$  of compact support in  $\mathcal{U}$ . With  $\varphi = \partial_\mu \psi_j$ , we get

$$\|D_z^{s/2} \partial_\mu \psi_j(-, y)\|_{L^{2p}(\mathbb{R}_-)} \lesssim \|\partial_y^N \psi_j\|_{H^{s+1}(\mathcal{U})}^2, \tag{100}$$

for  $s < \beta - 1/2$ . We have classical commutator estimates of the type [73], theorem A.8,

$$\|D^s(f_1 f_2) - D^s f_1 f_2 - f_1 D^s f_2\|_{L^p(\mathbb{R}^n)} \lesssim \|D^{s'} f_1\|_{L^{p'}(\mathbb{R}^n)} \|D^{s''} f_2\|_{L^{p''}(\mathbb{R}^n)} \tag{101}$$

for  $0 < s = s' + s''$ ,  $s', s'' > 0$ ,  $0 < 1/p = 1/p' + 1/p'' < 1$ . These basically allow us to distribute the fractional derivatives  $D_z^s$  on a product  $\partial_\mu \psi_j \partial_\nu \psi_j$  such as in  $t_{\mu\nu}$  up to an error term which is controlled in a lower Sobolev norm using  $\|\partial_y^N \psi_j\|_{H^{s+1}(\mathcal{U})}^2 \lesssim \|b_j\|_{C^m(\mathcal{O})}$ . The second statement then follows taking  $s' = s'' = s/2$ ,  $p' = p'' = 2p$ . This concludes the proof of the proposition.  $\square$

The proposition says that for the purposes of calculating  $\langle T_{\mu\nu} \rangle_\Psi$  near the Cauchy horizon in a state  $\Psi$  which starts out as a Hadamard state near the initial time surface  $\Sigma$ , we may work with the Unruh state  $\langle T_{\mu\nu} \rangle_U$  because, as we have shown in thm. 4.6, it is also a Hadamard state on the initial time surface  $\Sigma$ . In doing this, we must accept an error  $t_{\mu\nu}$  which has the regularity described in proposition 5.1 at the Cauchy horizon, and which, as we have hinted, is the behavior of the classical stress tensor. This is acceptable since, as we will show, the behavior of  $\langle T_{\mu\nu} \rangle_U$  is more singular than this at the Cauchy horizon  $\mathcal{CH}^R$ .

We turn, now to the calculation of the middle term in (89), namely  $\langle T_{\mu\nu} \rangle_U - \langle T_{\mu\nu} \rangle_C$ . By stationarity of the Unruh and the comparison state, it suffices to perform the computation in a neighborhood of  $\mathcal{CH}^L$ . The calculation is outlined in subsection 5.2, with the details of the mode calculations relegated to subsection 5.3.

### 5.2. Outline of mode calculation for Unruh state

The mode calculation is a straightforward effective one-dimensional scattering construction. It proceeds in two steps:

- We solve a scattering problem in region I, with asymptotic conditions determined by our choice of initial state on the surfaces  $\mathcal{H}^-$ ,  $\mathcal{H}_C^-$ .
- We subsequently solve a scattering problem in region II, with asymptotic conditions determined by our choice of initial state on  $\mathcal{H}^L$  and by the data from the previous step on  $\mathcal{H}^R$ .

Because similar constructions have been described in the literature previously for the case of RN [37], we will be brief and focus on the novel aspects. Actually, the calculation is not more difficult for the operators  $(\partial_V^N \Phi)^2$ , which are equal to  $T_{VV}$  for  $N = 1$ , using the obvious generalization of (51) to the operators  $(\partial_V^N \Phi)^2$ .

Since  $\partial v / \partial V = -(\kappa_- V)^{-1}$  with  $V = V_-$  the outgoing Kruskal-type coordinate regular in a neighborhood of  $\mathcal{CH}^R$ , we can work instead with the variable  $v$ , so we are interested in

$\langle(\partial_v^N \Phi)^2\rangle_U - \langle(\partial_v^N \Phi)^2\rangle_C$ . As in [37], for computational purposes only (not changing the state!), it is useful to consider the ‘Boulware’ modes<sup>23</sup>, defined by the wave equation and asymptotic data<sup>24</sup>

$$\psi_{\omega\ell m}^{\text{in,I}} = \begin{cases} (2\pi)^{-\frac{3}{2}}(2|\omega|)^{-\frac{1}{2}}Y_{\ell m}(\theta, \phi)r_c^{-1}\Theta(-V)e^{-i\omega v} & \text{on } \mathcal{H}_c, \\ 0 & \text{on } \mathcal{H}, \end{cases} \quad (102a)$$

$$\psi_{\omega\ell m}^{\text{up,I}} = \begin{cases} 0 & \text{on } \mathcal{H}_c \\ (2\pi)^{-\frac{3}{2}}(2|\omega|)^{-\frac{1}{2}}Y_{\ell m}(\theta, \phi)r_+^{-1}\Theta(-U)e^{-i\omega u} & \text{on } \mathcal{H}, \end{cases} \quad (102b)$$

$$\psi_{\omega\ell m}^{\text{up,II}} = \begin{cases} 0 & \text{on } \mathcal{H}_c \\ (2\pi)^{-\frac{3}{2}}(2|\omega|)^{-\frac{1}{2}}Y_{\ell m}(\theta, \phi)r_+^{-1}\Theta(U)e^{-i\omega u} & \text{on } \mathcal{H}, \end{cases} \quad (102c)$$

$$\psi_{\omega\ell m}^{\text{out,II}} = \begin{cases} (2\pi)^{-\frac{3}{2}}(2|\omega|)^{-\frac{1}{2}}Y_{\ell m}(\theta, \phi)r_-^{-1}\Theta(-V)e^{-i\omega v} & \text{on } \mathcal{CH}^L \\ 0 & \text{on } \mathcal{CH}^+, \end{cases} \quad (102d)$$

with  $\Theta$  the step function, and where we used the notation (64). Furthermore,  $V = V_{c/-}$  in the first/last equation. For later convenience, we defined these modes also for  $\omega < 0$ . We omitted the modes  $\psi^{\text{in,III}}$  and  $\psi^{\text{out,IV}}$ , which are not relevant for the calculations that we are going to perform. In the next section 5.3, we shall find the exact solutions to the above characteristic initial value problems in mode form, i.e. in the general form

$$\psi_{\omega\ell m}(t, r, \theta, \phi) = (2\pi)^{-\frac{3}{2}}(2|\omega|)^{-\frac{1}{2}}Y_{\ell m}(\theta, \phi)r^{-1}R_{\omega\ell}(r)e^{-i\omega t}, \quad (103)$$

where the radial function  $R$  has to solve

$$-\partial_{r_*}\partial_{r_*}R_{\omega\ell} + (V_\ell - \omega^2)R_{\omega\ell} = 0, \quad (104)$$

with a smooth effective potential

$$V_\ell = f \left( \frac{\ell(\ell+1)}{r^2} + \frac{f'}{r} + \mu^2 \right). \quad (105)$$

The particular solutions corresponding to various asymptotic conditions are denoted by superscripts, such as in  $\psi_{\omega\ell m}^{\text{in,I}}$ , with corresponding radial function  $R_{\omega\ell}^{\text{in,I}}$ , etc. For example,  $R_{\omega\ell}^{\text{in,I}}$  should satisfy the asymptotic condition

$$R_{\omega\ell}^{\text{in,I}}(r) = \begin{cases} e^{-i\omega r_*} + \mathcal{D}_{\omega\ell}^{\text{in,I}}e^{i\omega r_*} & r_* \rightarrow \infty, \\ \mathcal{D}_{\omega\ell}^{\text{in,I}}e^{-i\omega r_*} & r_* \rightarrow -\infty. \end{cases} \quad (106)$$

Since the effective potential is smooth and, when expressed in terms of  $r_*$ , decays, with all its derivatives, faster than any power, we must have [74], proposition 5.2.9,

<sup>23</sup> These are analogous to (41) in 2 dimensions.

<sup>24</sup> This has to be understood in the sense of wave packets, cf (106) for how to translate this into asymptotic conditions for a scattering problem.

$$\mathcal{R}_{\omega\ell}^{\text{in,I}} = \begin{cases} O(\omega^{-\infty}) & \omega \rightarrow \infty, \\ 1 + O(\ell^{-\infty}) & \ell \rightarrow \infty, \end{cases} \quad |\mathcal{T}_{\omega\ell}^{\text{in,I}}|^2 = \begin{cases} 1 + O(\omega^{-\infty}) & \omega \rightarrow \infty, \\ O(\ell^{-\infty}) & \ell \rightarrow \infty. \end{cases} \quad (107)$$

where  $O(\omega^{-\infty})$  indicates a decay faster than any power. This provides asymptotic data on  $\mathcal{H}^R$  and  $\mathcal{H}^L$  (the latter vanish), which can be transported to  $\mathcal{CH}^L$  by again solving (104), now with the asymptotic condition

$$R_{\omega\ell}^{\text{in,II}}(r) = \mathcal{T}_{\omega\ell}^{\text{in,I}} \begin{cases} e^{-i\omega r_*} & r_* \rightarrow -\infty, \\ \mathcal{T}_{\omega\ell}^{\text{in,II}} e^{-i\omega r_*} + \mathcal{R}_{\omega\ell}^{\text{in,II}} e^{i\omega r_*} & r_* \rightarrow \infty. \end{cases} \quad (108)$$

Note that the new coefficients fulfill the relation

$$|\mathcal{T}_{\omega\ell}^{\text{in,II}}|^2 - |\mathcal{R}_{\omega\ell}^{\text{in,II}}|^2 = 1, \quad (109)$$

which in particular means that they need not be bounded. Nevertheless, we again have

$$\mathcal{R}_{\omega\ell}^{\text{in,II}} = O(\omega^{-\infty}), \quad |\mathcal{T}_{\omega\ell}^{\text{in,II}}|^2 = 1 + O(\omega^{-\infty}). \quad (110)$$

We thus obtain, for the restriction of the ‘Boulware’ in-mode to  $\mathcal{CH}^L$ ,

$$\psi_{\omega\ell m}^{\text{in,I}}|_{\mathcal{CH}^L} = \mathcal{T}_{\omega\ell}^{\text{in,II}} \mathcal{T}_{\omega\ell}^{\text{in,I}} \psi_{\omega\ell m}^{\text{out,II}}. \quad (111)$$

For the up-modes  $\psi_{\omega\ell m}^{\text{up,I}}$ , we proceed similarly, i.e., we first scatter them to  $\mathcal{H}^R$  by solving again (104), now with the asymptotic condition

$$R_{\omega\ell}^{\text{up,I}}(r) = \begin{cases} e^{i\omega r_*} + \mathcal{R}_{\omega\ell}^{\text{up,I}} e^{-i\omega r_*} & r_* \rightarrow -\infty, \\ \mathcal{T}_{\omega\ell}^{\text{up,I}} e^{i\omega r_*} & r_* \rightarrow \infty. \end{cases} \quad (112)$$

One then scatters to  $\mathcal{CH}^L$  as for the in-modes and obtains

$$\psi_{\omega\ell m}^{\text{up,I}}|_{\mathcal{CH}^L} = \mathcal{T}_{\omega\ell}^{\text{in,II}} \mathcal{R}_{\omega\ell}^{\text{up,I}} \psi_{\omega\ell m}^{\text{out,II}}, \quad (113)$$

For the up-modes  $\psi_{\omega\ell m}^{\text{up,II}}$ , only a single scattering is necessary. The radial function  $R_{\omega\ell}^{\text{up,II}}$  is defined by the asymptotic condition

$$R_{\omega\ell}^{\text{up,II}}(r) = \begin{cases} e^{i\omega r_*} & r_* \rightarrow -\infty, \\ \mathcal{T}_{\omega\ell}^{\text{up,II}} e^{i\omega r_*} + \mathcal{R}_{\omega\ell}^{\text{up,II}} e^{-i\omega r_*} & r_* \rightarrow \infty. \end{cases} \quad (114)$$

This gives

$$\psi_{\omega\ell m}^{\text{up,II}}|_{\mathcal{CH}^L} = \mathcal{R}_{\omega\ell}^{\text{up,II}} \psi_{\omega\ell m}^{\text{out,II}}. \quad (115)$$

Comparing (114) with (108) yields

$$\mathcal{R}_{\omega\ell}^{\text{up,II}} = \overline{\mathcal{R}_{\omega\ell}^{\text{in,II}}}, \quad \mathcal{T}_{\omega\ell}^{\text{up,II}} = \overline{\mathcal{T}_{\omega\ell}^{\text{in,II}}}. \quad (116)$$

The symmetrized two-point function, i.e., the Hadamard function, of the Unruh state defined by the modes (65) can be expressed in terms of the Boulware modes (102) as, see [75] for a similar calculation (but note the change of sign of  $u$  in the interior region w.r.t. that reference),

$$\begin{aligned} \langle \{\Phi(x_1), \Phi(x_2)\} \rangle_U = & \sum_{\ell m} \int_0^\infty d\omega \left[ \coth \frac{\pi\omega}{\kappa_c} \left\{ \psi_{\omega\ell m}^{\text{in,I}}(x_1), \overline{\psi_{\omega\ell m}^{\text{in,I}}(x_2)} \right\} \right. \\ & + 2\text{csch} \frac{\pi\omega}{\kappa_+} \text{Re} \left\{ \psi_{\omega\ell m}^{\text{up,I}}(x_1), \overline{\psi_{\omega\ell m}^{\text{up,II}}(x_2)} + \coth \frac{\pi\omega}{\kappa_+} \left( \left\{ \psi_{\omega\ell m}^{\text{up,I}}(x_1), \overline{\psi_{\omega\ell m}^{\text{up,I}}(x_2)} \right\} \right) \right\} \\ & \left. + \left\{ \psi_{-\omega\ell m}^{\text{up,II}}(x_1), \overline{\psi_{-\omega\ell m}^{\text{up,II}}(x_2)} \right\} \right], \end{aligned} \quad (117)$$

where we omitted the contribution from  $\psi_{\omega\ell m}^{\text{in,III}}$ , which is absent in  $I \cup II$ , and used the notation

$$\{A(x_1), B(x_2)\} = \frac{1}{2} (A(x_1)B(x_2) + A(x_2)B(x_1)). \quad (118)$$

We now turn to the comparison state. The specification of this state in terms of modes on a null hypersurface has the property that, after selecting an angular momentum component,

$$\Phi_{\ell m}(U, V) = r \int \overline{Y_{\ell m}(\Omega)} \Phi(U, V, \Omega) d^2\Omega, \quad (119)$$

and taking at least one derivative w.r.t.  $V = V_-$ , one can evaluate the two-point function on the null hypersurface, i.e.,

$$\lim_{U \rightarrow \infty} \langle \partial_V^n \Phi_{\ell_1 m_1}^*(U, V_1) \partial_{V_2}^n \Phi_{\ell_2 m_2}(U, V_2) \rangle_C = \delta_{\ell_1 \ell_2} \delta_{m_1 m_2} \frac{1}{16\pi^3} \int_0^\infty dk k^{2n-1} e^{-ik(V_1 - V_2 - i0)}. \quad (120)$$

When both points are on  $\mathcal{CH}^L$ , one may reexpress this as, see [75] for example,

$$\lim_{U \rightarrow \infty} \langle \partial_v^n \Phi_{\ell_1 m_1}^*(U, v_1) \partial_v^n \Phi_{\ell_2 m_2}(U, v_2) \rangle_C = \delta_{\ell_1 \ell_2} \delta_{m_1 m_2} \frac{1}{16\pi^3} \int_0^\infty d\omega \omega^{2n-1} \coth \frac{\pi\omega}{\kappa_-} e^{-i\omega(v_1 - v_2 - i0)}. \quad (121)$$

With (111), (113), (115) and (117), we thus obtain for the difference of the expectation value in the Unruh and the comparison state, in a coinciding point limit,

$$\begin{aligned} & \lim_{U \rightarrow \infty, v_1 \rightarrow v_2} \left[ \langle \partial_v^n \Phi_{\ell_1 m_1}^*(U, v_1) \partial_v^n \Phi_{\ell_2 m_2}(U, v_2) \rangle_U - \langle \partial_v^n \Phi_{\ell_1 m_1}^*(U, v_1) \partial_v^n \Phi_{\ell_2 m_2}(U, v_2) \rangle_C \right] \\ & = \delta_{\ell_1 \ell_2} \delta_{m_1 m_2} \frac{1}{16\pi^3} \int_0^\infty d\omega \omega^{2n-1} n_{\ell_1}(\omega), \end{aligned} \quad (122)$$

where

$$\begin{aligned} n_\ell(\omega) = & |\mathcal{T}_{\omega\ell}^{\text{in,I}}|^2 |\mathcal{T}_{\omega\ell}^{\text{in,II}}|^2 \coth \frac{\pi\omega}{\kappa_c} + \left( |\mathcal{R}_{\omega\ell}^{\text{up,I}}|^2 |\mathcal{T}_{\omega\ell}^{\text{in,II}}|^2 + |\mathcal{R}_{\omega\ell}^{\text{up,II}}|^2 \right) \coth \frac{\pi\omega}{\kappa_+} \\ & + 2\text{csch} \frac{\pi\omega}{\kappa_+} \text{Re}(\mathcal{R}_{\omega\ell}^{\text{up,I}} \mathcal{T}_{\omega\ell}^{\text{in,II}} \overline{\mathcal{R}_{\omega\ell}^{\text{up,II}}}) - \coth \frac{\pi\omega}{\kappa_-}. \end{aligned} \quad (123)$$

From the asymptotic behavior of the transmission and reflection coefficients, it follows that (123) falls off faster than any power in  $\omega$  for  $|\omega| \rightarrow \infty$ , so that the integral (122) is UV finite for all  $n \geq 1$ . However, one may worry that there is an IR divergence for  $n = 1$ , due to the fact that generically  $\mathcal{T}_{\omega\ell}^{\text{in,II}}$  and hence, by (109), also  $\mathcal{T}_{\omega\ell}^{\text{in,I}}$  have a simple pole at  $\omega = 0$ , [76],

proposition 6.2, so that the second and the third term on the r.h.s. are individually IR divergent for  $n = 1$  (in the first term,  $|\mathcal{T}_{\omega\ell}^{\text{in,I}}|^2$  vanishes rapidly as  $\omega \rightarrow 0$ ).<sup>25</sup> However, the potential IR divergences due to these terms cancel: we first notice that  $\mathcal{R}_{\omega\ell}^{\text{up,I}} = -1 + iC_\ell\omega + O(\omega^2)$  as  $\omega \rightarrow 0$ , where  $C_\ell \in \mathbb{R}$ , see [37]. The statement then follows from (116) and the fact, derivable from inspection of the proof of [76], proposition 6.2, that, in case of a divergent  $\mathcal{T}_{\omega\ell}^{\text{in,II}}$ ,

$$\mathcal{T}_{\omega\ell}^{\text{in,II}} = \frac{iC'_\ell}{\omega} + O(\omega^0), \quad \mathcal{R}_{\omega\ell}^{\text{in,II}} = -\frac{iC'_\ell}{\omega} + O(\omega^0), \tag{124}$$

where  $C'_\ell \in \mathbb{R}$ .<sup>26</sup> It is then straightforward to see that the potential IR divergences in (122) cancel.

We also note that due to  $|\mathcal{T}_{\omega\ell}^{\text{in,I}}|^2 = |\mathcal{T}_{\omega\ell}^{\text{up,I}}|^2$ , it suffices to determine  $\mathcal{R}_{\omega\ell}^{\text{up,I}}$ ,  $\mathcal{T}_{\omega\ell}^{\text{up,II}}$ , and  $\mathcal{R}_{\omega\ell}^{\text{up,II}}$  in order to evaluate (123).

If the integral on the r.h.s. of (122) gives a non-zero value for any  $n$ , then the Unruh state is singular at  $\mathcal{CH}^R$ : by stationarity and continuity, the values on  $\mathcal{CH}^R$  and  $\mathcal{CH}^L$  coincide. Converting the resulting tensor to adapted coordinates  $V = V_-$  instead of  $v$ , one obtains

$$\langle \partial_V^n \Phi_{\ell m}^* \partial_V^n \Phi_{\ell m} \rangle_U - \langle \partial_V^n \Phi_{\ell m}^* \partial_V^n \Phi_{\ell m} \rangle_C \sim c_{n\ell} |V|^{-2n}, \tag{125}$$

where  $c_{n\ell}$  is  $\kappa_-^{2n}$  times the value of the integral on the right hand side of (122). That the integral vanishes for all  $\ell$  and all  $n \geq 1$  seems quite improbable, and this expectation is confirmed in section 5.4. Because the comparison state C is Hadamard in an open neighborhood on  $\mathcal{CH}^R$  by theorem 4.6, any singular behavior must be due to the Unruh state U and not the comparison state C. Therefore, in view of property (H1) of subsection 4.1, we conclude:

**Conclusion 1:** unless the quantity on the right side of (122) vanishes for all  $\ell$  and all  $n \geq 1$ , the Unruh state cannot be a Hadamard state in a two sided open neighborhood of  $\mathcal{CH}^R$  (of course it is Hadamard away from  $\mathcal{CH}^R$  by theorem 4.6).

While for the angular momentum components  $\Phi_{\ell m}$  the limits  $U \rightarrow \infty$  and  $v_1 \rightarrow v_2$  on the left hand side of (122) can be controlled and shown to converge to the right hand side, this is not straightforward for  $\Phi$ , required for the evaluation of the difference of expectation values of local Wick powers such as the stress tensor  $T_{vv}$  on  $\mathcal{CH}^L$ . Noting that we can write

$$\partial_v \Phi(x_1) \partial_v \Phi(x_2) = \frac{1}{r_1 r_2} \sum_{\ell_1, \ell_2} \sum_{m_1, m_2} Y_{\ell_1 m_1}^*(\Omega_1) Y_{\ell_2 m_2}(\Omega_2) \partial_v \Phi_{\ell_1 m_2}^*(U_1, V_1) \partial_v \Phi_{\ell_2 m_2}(U_2, V_2), \tag{126}$$

and assuming that we may interchange the limits  $U \rightarrow \infty$  and  $v_1 \rightarrow v_2$  with the summations, we obtain

$$\langle T_{vv} \rangle_U - \langle T_{vv} \rangle_C = \frac{1}{16\pi^3 r_-^2} \sum_{\ell} \frac{2\ell + 1}{4\pi} \int_0^\infty d\omega \omega n_\ell(\omega), \tag{127}$$

with  $n_\ell(\omega)$  as in (123). As discussed above, the integral on the right hand side converges for all  $\ell$ . However, the convergence of the sum over  $\ell$  is not obvious. Numerical evidence, see below, indicates that it does converge, which we take as an indication that the above interchange of

<sup>25</sup> The nature of this pole can also be seen explicitly from the expressions given for the scattering coefficients in section 5.3.

<sup>26</sup> This follows from the fact that the coefficients  $A, B$  in (6.5) of [76] are real.

limits is justified. In view of the regularity of  $C$  on  $\mathcal{CH}^R$ , the relationship between  $v$  and  $V_-$ , and proposition 5.1, we then also conclude:

**Conclusion 2:** if the sum on right side of (127) converges to a nonzero value, then for any initially a Hadamard state  $\Psi$  in a neighborhood of the Cauchy surface  $\Sigma$  as in figure 1 and for any RNdS spacetime with  $\beta > 1/2$  (see (2)), then the expected stress tensor behaves in region II near  $\mathcal{CH}^R$  as

$$\langle T_{VV} \rangle_{\Psi} \sim C|V|^{-2} + t_{VV}. \tag{128}$$

Here  $C$  is given by  $\kappa_-^2$  times the right side of (127) and only depends on the parameters of the black hole but not on  $\Psi$ .  $t_{VV}$ , which depends on  $\Psi$ , has the regularity proposition 5.1, i.e.  $t_{VV} \in L^p$  as a function of  $V$  locally near  $\mathcal{CH}^R$  with  $1/p = 2 - 2\beta + 2\epsilon$  and with  $\beta - 1/2 - \epsilon$  fractional derivatives w.r.t.  $V$  in  $L^{1/(3/2-\beta+\epsilon)}$ , where  $\epsilon > 0$  is arbitrarily small.

Based on the results of the next section 5.3, we shall numerically evaluate the right sides of (123) and (127) in the case of the conformally coupled scalar field and find them to be nonvanishing. In particular, this provides evidence that (128) holds with  $C \neq 0$  for generic values of the black hole parameters.

**Remark:** these results should be compared with the expression (45) in the two-dimensional case. In particular, in (123) one explicitly sees the effect of backscattering in 4 dimensions, as the up modes also contribute. Analogous expressions for the evaluation of  $T_{vv}$  in the Unruh state on RN (without the subtraction of the contribution of the comparison state) can be found in [36].

### 5.3. Computation of the scattering coefficients

We next describe how to determine the scattering coefficients  $\mathcal{R}^{\text{up},\text{I}}$ ,  $\mathcal{T}^{\text{in},\text{II}}$ ,  $\mathcal{R}^{\text{in},\text{II}}$  needed for the evaluation of (123). When expressed in terms of  $r$ , the radial equation (104) has regular singular points at  $r = r_-, r_+, r_c, r_o, \infty$  (recall (10) for the definition of  $r_o$ ). The transformation

$$x = \frac{(r_- - r_o)(r - r_+)}{(r_- - r_+)(r - r_o)}, \tag{129}$$

maps  $r_-, r_+, r_c, r_o, \infty$  to

$$x_- = 1, \quad x_+ = 0, \quad x_c = \frac{(r_- - r_o)(r_c - r_+)}{(r_- - r_+)(r_c - r_o)}, \quad x_o = \infty, \quad x_{\infty} = \frac{r_- - r_o}{r_- - r_+} \tag{130}$$

which are the singular points of the radial equation written in terms of  $x$ . In the conformally coupled case,  $\mu^2 = \frac{1}{6}R = \frac{2}{3}\Lambda$ , it is possible to factor out the singularity at  $x_{\infty}$  by a suitable transformation [77] of the radial equation<sup>27</sup>. On account of this simplification, we restrict consideration to the conformally coupled case from this point on.

We define

$$b_+ = +i\frac{\omega}{2\kappa_+}, \quad b_- = -i\frac{\omega}{2\kappa_-}, \quad b_c = -i\frac{\omega}{2\kappa_c}, \quad b_o = +i\frac{\omega}{2\kappa_o}, \tag{131}$$

<sup>27</sup> One can in principle also treat the case of different values of  $\mu$  by a variant of the method below using results by [78].

and make the ansatz

$$(r^{-1}R)(r(x)) = (\pm x)^{\sigma_+ b_+} (1-x)^{\sigma_- b_-} \left(\frac{x-x_c}{1-x_c}\right)^{\sigma_c b_c} \frac{x-x_\infty}{1-x_\infty} H(x), \tag{132}$$

where  $\sigma_X = \pm 1$  and the sign in the first factor is chosen below depending on whether one considers solutions in I or II. It is then found [77] that  $H(x)$  satisfies Heun’s equation [79] with the singular points  $x_-, x_+, x_c, \infty$ . The parameters of this equation can be written in terms of these and  $x_\infty$ .

It is known that solutions to Heun’s equation which are analytic in a domain of the complex plane containing two of the four singular points can be constructed as series of various special functions, see [78, 80, 81] for the origins of this method. For us, it will be useful to follow a variant of this method where the special functions are hypergeometric functions, see [82], chapter 15. Of particular interest for us are the domains containing  $\{x_+, x_-\}$  respectively  $\{x_c, x_\infty\}$ . In the first case one can define solutions fulfilling ‘in’ or ‘up’ boundary conditions at  $r_+$  and compute the scattering coefficients  $\mathcal{S}^{\text{up,II}}, \mathcal{R}^{\text{up,II}}$ . In the second case, one can define solutions fulfilling specified boundary conditions at  $r_c$ . As the domains overlap, one can compute the scattering coefficients  $\mathcal{R}^{\text{up,I}}$ .

The solution to the radial equation (104) with ‘up’ boundary condition at  $r_+$  in region II can be written, up to normalization, as

$$r^{-1}R_{\omega, \ell, \nu}^{\text{up,II}}(r) = x^{b_+} (1-x)^{b_-} \left(\frac{x-x_c}{1-x_c}\right)^{b_c} \frac{x-x_\infty}{1-x_\infty} \times \sum_{n=-\infty}^{\infty} a_n^\nu F(-n-\nu+b_++b_-, n+\nu+b_++b_-+1; 1+2b_+; x), \tag{133}$$

with  $F = {}_2F_1$  the Gauss hypergeometric function. The so-called characteristic exponent  $\nu$  is, *a priori*, an undetermined parameter unrelated to the parameters appearing in the radial—resp. Heun equation. The ansatz can be shown [77] to yield a formal solution for *any*  $\nu$ , provided the coefficients  $a_n^\nu$  solve the three term recursion relation

$$\alpha_n^\nu a_{n+1}^\nu + \beta_n^\nu a_n^\nu + \gamma_n^\nu a_{n-1}^\nu = 0 \tag{134}$$

with initial condition

$$a_0^\nu = 1 \tag{135}$$

and coefficients

$$\alpha_n^\nu = -\frac{(n+\nu-b_+-b_-+1)(n+\nu-b_c+b_o+1)(n+\nu-b_c-b_o+1)(n+\nu-b_++b_-+1)}{2(n+\nu+1)(2n+2\nu+3)}, \tag{136a}$$

$$\begin{aligned} \beta_n^\nu &= \frac{(b_++b_-)(b_+-b_-)(b_c-b_o)(b_c+b_o)}{2(n+\nu)(n+\nu+1)} + \left(\frac{1}{2}-x_c\right)(n+\nu)(n+\nu+1) \\ &+ \frac{1}{2}((2b_c+1)(b_+-b_-) - (2b_-+1)(b_++b_-) + (b_++b_-+b_c+1)^2 - b_o^2) \\ &+ x_c((b_++b_-)^2 + b_++b_-) + \nu, \end{aligned} \tag{136b}$$

$$\gamma_n^\nu = -\frac{(n + \nu + b_+ + b_-)(n + \nu + b_c - b_o)(n + \nu + b_c + b_o)(n + \nu + b_+ - b_-)}{2(n + \nu)(2n + 2\nu - 1)}, \tag{136c}$$

where

$$v = 2\omega^2 \frac{r_+^3(r_+r_- - 2r_-r_c + r_+r_c)(r_o^2 - r_+r_- - r_+r_c - r_-r_c)^2}{(r_+ - r_-)^3(r_+ - r_c)^2(r_+ - r_o)(r_c - r_o)} - 2 \frac{r_o^2}{(r_+ - r_-)(r_c - r_o)} - x_\infty - (1 + x_c)b_+ - x_cb_- - b_c - 2b_+(x_cb_- + b_c) - \ell(\ell + 1) \frac{r_o^2 - r_+r_- - r_+r_c - r_-r_c}{(r_+ - r_-)(r_c - r_o)}. \tag{137}$$

However, for generic values of the characteristic exponent  $\nu$ , the solution is only formal in the sense that the series in (133) fails to converge. To get a convergent (bilateral) series,  $\nu$  has to be determined such that the recursion relation (134) has a solution which is ‘minimal’ in the sense of [83], both for the ascending  $n \rightarrow +\infty$  and for the descending  $n \rightarrow -\infty$  series. By the general theory of three-term relations, a solution which is minimal in both directions can be determined as follows. Using one of the algorithms for finding the minimal solution to a three-term recursion relation described in [83], one determines

$$\rho_n^\nu = \frac{a_n^\nu}{a_{n-1}^\nu}, \quad \lambda_n^\nu = \frac{a_n^\nu}{a_{n+1}^\nu}, \tag{138}$$

the former by recursion starting at  $n = +\infty$ , the latter by recursion starting at  $n = -\infty$  (for numerical investigations, at some integer of large modulus). The critical exponent  $\nu$  is then determined by solving the transcendental equation

$$\rho_n^\nu \lambda_{n-1}^\nu = 1. \tag{139}$$

Choosing this minimal solution, it follows by theorem 2.2 of [83] and the well-known asymptotics of  $F = {}_2F_1$  that the series for (133) converges in some ellipse in the complex plane with focal points  $\{x_+, x_-\}$  [77].

It is obvious from the series (133) that the critical exponent is only determined up to an integer. Furthermore, one can show [77] that if  $\nu$  is a solution to (139), then so is  $-\nu - 1$ , with the corresponding coefficients related by

$$a_{-n}^{-\nu-1} = a_n^\nu. \tag{140}$$

Although the solution of the transcendental equation (139) does not seem possible in closed form, one can in practice get solutions as power series expansions in suitably small parameters. For example, one can obtain the following (not necessarily convergent) asymptotic expansion for large  $r_c$  (near RN) and small  $\omega$ ,

$$\nu \sim \ell \left[ 1 + \sum_{m,n \geq 0} \nu_{n,m} \left(\frac{\omega}{\ell}\right)^n \left(\frac{1}{r_c}\right)^m \right], \tag{141}$$

where the prime indicates that  $(m, n) = (0, 0)$  is excluded. The coefficients depend on  $\ell$  and  $r_-, r_+$ , remain bounded for  $\ell \rightarrow \infty$ , and can be found by direct substitution into the defining

equation for  $\nu$ . The lowest order in  $n + m$  coefficients turn out to be

$$\nu_{2,0} = \frac{\ell(11r_-^2 + 14r_-r_+ + 11r_+^2) - \ell^2(\ell + 1)(15r_-^2 + 18r_-r_+ + 15r_+^2)}{2(2\ell + 1)(4\ell(\ell + 1) - 3)}, \tag{142a}$$

$$\nu_{0,2} = \frac{(15\ell^4 + 30\ell^3 + 9\ell^2 - 6\ell - 4)(r_-^2 + r_+^2) + (18\ell^4 + 36\ell^3 + 10\ell^2 - 8\ell - 4)r_-r_+}{2\ell(2\ell + 1)(4\ell(\ell + 1) - 3)}, \tag{142b}$$

$$\nu_{0,3} = \frac{-(15\ell^4 + 30\ell^3 + 9\ell^2 - 6\ell - 4)(r_-^2 + r_+^2)^2 - (33\ell^4 + 66\ell^3 + 19\ell^2 - 14\ell - 8)r_-r_+(r_- + r_+)}{2\ell(8\ell^3 + 12\ell^2 - 2\ell - 3)}. \tag{142c}$$

To determine the normalization of the solution (133), we note that, by  $F(a, b; c; 0) = 1$ , it behaves as

$$r^{-1}R_{\omega\ell,\nu}^{\text{up,II}}(x) \sim e^{i\omega(r_*+D')} \left(\frac{-x_c}{1-x_c}\right)^{b_c} \frac{1-x_\infty}{1-x_\infty} \sum_{n=-\infty}^{\infty} a_n^\nu \tag{143}$$

near  $r_+$ , with a constant  $D'$ , related to the integration constant  $D$  in (12). To determine the asymptotic behavior at the Cauchy horizon, we use [82], 15.3.6,

$$\begin{aligned} F(a, b; c; x) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a, b; a+b-c+1; 1-x) \\ &+ (1-x)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} F(c-a, c-b; c-a-b+1; 1-x). \end{aligned} \tag{144}$$

It follows that near  $r_-$ , the solution (133) behaves as

$$\begin{aligned} r^{-1}R_{\omega\ell,\nu}^{\text{up,II}} &\sim e^{i\omega(r_*+D'')} \sum_{n=-\infty}^{\infty} a_n^\nu \frac{\Gamma(1+2b_+)\Gamma(-2b_-)}{\Gamma(b_+ - b_- + n + \nu + 1)\Gamma(b_+ - b_- - n - \nu)} \\ &+ e^{-i\omega(r_*+D'')} \sum_{n=-\infty}^{\infty} a_n^\nu \frac{\Gamma(1+2b_+)\Gamma(2b_-)}{\Gamma(b_+ + b_- - n - \nu)\Gamma(b_+ + b_- + n + \nu + 1)}, \end{aligned} \tag{145}$$

with another integration constant  $D''$ . From this, one can read off

$$\begin{aligned} \mathcal{R}_{\omega\ell}^{\text{up,II}} &= e^{-i\omega(D'-D'')} \frac{r_-}{r_+} \left(\frac{1-x_c}{-x_c}\right)^{b_c} \frac{1-x_\infty}{-x_\infty} \Gamma(1+2b_+)\Gamma(-2b_-) \\ &\times \frac{\sum_{n=-\infty}^{\infty} a_n^\nu [\Gamma(b_+ - b_- + n + \nu + 1)\Gamma(b_+ - b_- - n - \nu)]^{-1}}{\sum_{n=-\infty}^{\infty} a_n^\nu}, \end{aligned} \tag{146a}$$

$$\begin{aligned} \mathcal{R}_{\omega\ell}^{\text{up,II}} &= e^{-i\omega(D'+D'')} \frac{r_-}{r_+} \left(\frac{1-x_c}{-x_c}\right)^{b_c} \frac{1-x_\infty}{-x_\infty} \Gamma(1+2b_+)\Gamma(2b_-) \\ &\times \frac{\sum_{n=-\infty}^{\infty} a_n^\nu [\Gamma(b_+ + b_- - n - \nu)\Gamma(b_+ + b_- + n + \nu + 1)]^{-1}}{\sum_{n=-\infty}^{\infty} a_n^\nu}. \end{aligned} \tag{146b}$$

From the factors  $\Gamma(\pm 2b_-)$ , one easily verifies the asymptotic behavior (124). Note that in both expressions the fraction of the sums converges to

$$-\frac{1}{\pi} \sin \pi \nu \frac{\sum_{n=-\infty}^{\infty} (-1)^n a_n^\nu}{\sum_{n=-\infty}^{\infty} a_n^\nu} \tag{147}$$

in the limit as  $\omega \rightarrow 0$ , so that the divergence at  $\omega \rightarrow 0$  cannot be enhanced but possibly canceled if  $\nu \rightarrow \mathbb{Z}$  for  $\omega \rightarrow 0$ , which, as can be seen from (141), happens in the RN limit, consistent with results by [76].

In order to determine the coefficient  $\mathcal{R}^{\text{up},\text{I}}$ , we first define the up mode  $R_{\omega\ell,\nu}^{\text{up},\text{I}}$  in region I by replacing in (133) the first factor by  $(-x)^{b_+}$ . Then, by [82], 15.3.8, and (140), one can write

$$R_{\omega\ell,\nu}^{\text{up},\text{I}}(x) = R_{\omega\ell,\nu}^{+, \text{I}}(z) + R_{\omega\ell,-\nu-1}^{+, \text{I}}(z), \tag{148}$$

where

$$\begin{aligned} r^{-1} R_{\omega\ell,\nu}^{+, \text{I}}(z) &= (z-1)^{b_+} z^{b_-} \left(1 - \frac{z}{z_c}\right)^{b_c} \left(1 - \frac{z}{z_\infty}\right) \Gamma(1+2b_+) \\ &\times \sum_{n=-\infty}^{\infty} a_n^\nu \frac{\Gamma(2n+2\nu+1)}{\Gamma(n+\nu+b_++b_-+1)\Gamma(n+\nu+b_+-b_-+1)} z^{n+\nu-b_+-b_-} \\ &\times F(b_++b_--n-\nu, b_+-b_--n-\nu; -2n-2\nu; 1/z). \end{aligned} \tag{149}$$

Here  $z = 1 - x$ , and analogously for  $z_c, z_\infty$ . By (144), the behavior near the event horizon is given by

$$\begin{aligned} r^{-1} R_{\omega\ell,\nu}^{+, \text{I}} &\sim e^{i\omega(r_*+D''')} \left(\frac{-x_c}{1-x_c}\right)^{b_c} \frac{-x_\infty}{1-x_\infty} \frac{\sin[\pi(\nu+b_++b_-)] \sin[\pi(\nu+b_+-b_-)]}{\sin 2\pi b_+ \sin 2\pi \nu} \\ &\times \sum_{n=-\infty}^{\infty} a_n^\nu + e^{-i\omega(r_*+D''')} \left(\frac{-x_c}{1-x_c}\right)^{b_c} \frac{-x_\infty}{1-x_\infty} \Gamma(1+2b_+) \Gamma(2b_+) \Gamma(1+2\nu) \Gamma(-2\nu) \\ &\times \sum_{n=-\infty}^{\infty} a_n^\nu [\Gamma(1+n+\nu+b_++b_-) \Gamma(1+n+\nu+b_+-b_-) \\ &\times \Gamma(-n-\nu+b_++b_-) \Gamma(-n-\nu+b_+-b_-)]^{-1}, \end{aligned} \tag{150}$$

with some integration constant  $D'''$ , where we also used

$$\Gamma(z)\Gamma(1-z) = \pi \csc \pi z. \tag{151}$$

An analogous solution in region I which is regular at the cosmological horizon  $r_c$  is given by

$$\begin{aligned} r^{-1} R_{\omega\ell,\nu}^{c,\text{I}}(z) &= (z-1)^{b_+} z^{b_-} \left(1 - \frac{z}{z_c}\right)^{b_c} \left(1 - \frac{z}{z_\infty}\right) \frac{1}{\Gamma(b_++b_--b_c-b_o+1)} \\ &\times \sum_{n=-\infty}^{\infty} a_n^\nu \frac{\Gamma(n+\nu-b_c+b_o+1)\Gamma(n+\nu-b_c-b_o+1)}{\Gamma(2n+2\nu+2)} \left(\frac{z}{z_c}\right)^{n+\nu-b_+-b_-} \\ &\times F(n+\nu+b_+-b_-+1, n+\nu+b_++b_-+1; 2n+2\nu+2; z/z_c). \end{aligned} \tag{152}$$

It follows from the coefficients in our three-term relation, by theorem 2.2 of [83] and the well-known asymptotics of  $F = {}_2F_1$  that the series for (133) converges in some ellipse in the complex  $z_c/z$  plane with focal points  $\{0, 1\}$ , containing in particular  $\{x_c, x_\infty\}$  in the complex  $x$  plane. By comparison of coefficients one then finds that [77]

$$R_{\omega\ell,\nu}^{+,I} = K_{\omega\ell}^\nu R_{\omega\ell,\nu}^{c,I} \tag{153}$$

with

$$\begin{aligned} K_{\omega\ell}^\nu &= z_c^{p+\nu-b_+-b_-} \\ &\times \frac{\Gamma(1+2b_+)\Gamma(b_+ + b_- - b_c - b_o + 1)}{\Gamma(p + \nu - b_+ - b_- + 1)\Gamma(p + \nu - b_+ + b_- + 1)\Gamma(p + \nu + b_c - b_o + 1)\Gamma(p + \nu + b_c + b_o + 1)} \\ &\times \sum_{n=p}^{\infty} (-1)^{n-p} a_n^\nu \frac{\Gamma(n + \nu - b_+ - b_- + 1)\Gamma(n + \nu - b_+ + b_- + 1)\Gamma(p + n + 2\nu + 1)}{\Gamma(n + \nu + b_+ + b_- + 1)\Gamma(n + \nu + b_+ - b_- + 1)(n - p)!} \\ &\times \left( \sum_{n=-\infty}^p a_n^\nu \frac{\Gamma(n + \nu - b_c + b_o + 1)\Gamma(n + \nu - b_c - b_o + 1)}{\Gamma(n + \nu + b_c - b_o + 1)\Gamma(n + \nu + b_c + b_o + 1)\Gamma(p + n + 2\nu + 2)(p - n)!} \right)^{-1}. \end{aligned} \tag{154}$$

Here  $p$  is an arbitrary integer, on which the result does not depend (keeping this parameter provides a check for the subsequent formulas). The solutions  $R_{\omega\ell,\nu}^{c,I}$  and  $R_{\omega\ell,-\nu-1}^{c,I}$  are linearly independent. Using (140), (144), and the identity (151), one can show that the linear combination

$$\begin{aligned} R_{\omega\ell,\nu}^{\text{out},I} &= \sin[\pi(b_c + b_o - \nu)] \sin[\pi(b_c - b_o - \nu)] R_{\omega\ell,\nu}^{c,I} \\ &\quad - \sin[\pi(b_c + b_o + \nu)] \sin[\pi(b_c - b_o + \nu)] R_{\omega\ell,-\nu-1}^{c,I} \end{aligned} \tag{155}$$

is given by

$$\begin{aligned} r^{-1} R_{\omega\ell,\nu}^{\text{out},I}(z) &= (z - 1)^{b_+} z^{b_-} \left(1 - \frac{z}{z_c}\right)^{b_c} \left(1 - \frac{z}{z_\infty}\right) \frac{1}{\Gamma(b_+ + b_- - b_c - b_o + 1)} \frac{\pi \sin 2\pi\nu}{\Gamma(1 + 2b_c)} \\ &\quad \times \sum_{n=-\infty}^{\infty} a_n^\nu \left(\frac{z}{z_c}\right)^{n+\nu-b_+-b_-} \\ &\quad \times F(n + \nu + b_c - b_o + 1, n + \nu + b_c + b_o + 1; 1 + 2b_c; 1 - z/z_c). \end{aligned} \tag{156}$$

Its asymptotic form near  $r = r_c$  is given by  $R_{\omega\ell,\nu}^{\text{out},I} \sim D_\omega e^{i\omega r_*}$  with some irrelevant factor  $D_\omega$ . Thus, it represents an outgoing solution near the cosmological horizon. Using (153), we may reexpress  $R_{\omega\ell,\nu}^{\text{out},I}$  as

$$R_{\omega\ell,\nu}^{\text{out},I} = \sin[\pi(b_c + b_o - \nu)] \sin[\pi(b_c - b_o - \nu)] (K_{\omega\ell}^\nu)^{-1} R_{\omega\ell,\nu}^{+,I} - \{\nu \leftrightarrow -\nu - 1\}. \tag{157}$$

From (150), we can thus read off

$$\begin{aligned}
 \mathcal{R}_{\omega\ell}^{\text{up,I}} &= -e^{-2i\omega D'} \pi \sin(2\pi b_+) \Gamma(1 + 2b_+) \Gamma(2b_-) \\
 &\times \left[ \sin[\pi(b_c + b_o - \nu)] \sin[\pi(b_c - b_o - \nu)] (K_{\omega\ell}^\nu)^{-1} + \{\nu \leftrightarrow -\nu - 1\} \right] \\
 &\times \sum_{n=-\infty}^{\infty} a_n^\nu \left[ \Gamma(1 + n + \nu + b_+ + b_-) \Gamma(1 + n + \nu + b_+ - b_-) \Gamma(b_+ + b_- - n - \nu) \right. \\
 &\times \Gamma(b_+ - b_- - n - \nu) \left. \right]^{-1} \left[ \sin[\pi(b_c + b_o - \nu)] \sin[\pi(b_c - b_o - \nu)] \right. \\
 &\times \sin[\pi(b_+ + b_- + \nu)] \sin[\pi(b_+ - b_- + \nu)] (K_{\omega\ell}^\nu)^{-1} \\
 &\left. + \{\nu \leftrightarrow \nu - 1\} \right]^{-1} \left[ \sum_{n=-\infty}^{\infty} a_n^\nu \right]^{-1}. \tag{158}
 \end{aligned}$$

Regarding the integration constants  $D'$ ,  $D''$ , one easily checks that these cancel in the next-to-last term in (123), which is the only term sensitive to phases.

#### 5.4. Numerical results

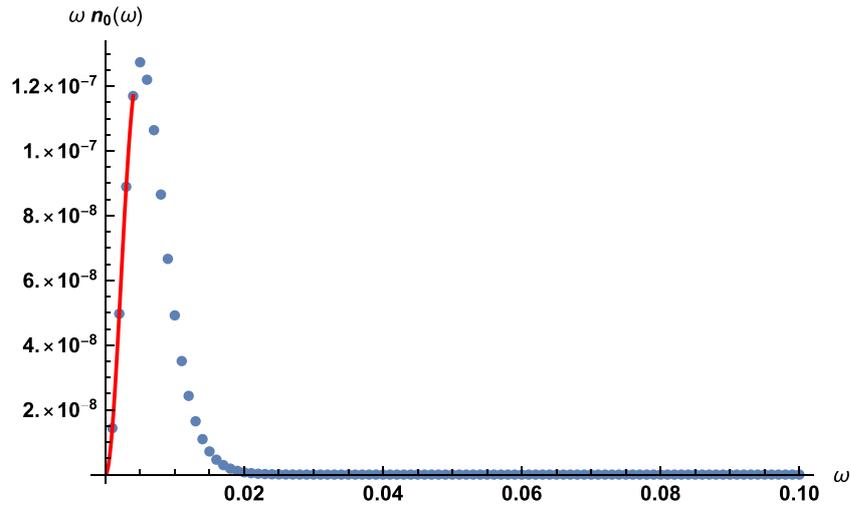
From the discussion in the previous sections, two open questions remain: (1) Is the ‘density of states’  $n_\ell(\omega)$  as given in (123) non-vanishing<sup>28</sup>, with generically non-vanishing integrals of the form (122)? (2) Does the sum (127) over angular momenta converge and give a generically non-vanishing result, corresponding to a nonzero  $C$  in (128), resp. (5)? Without performing an exhaustive parameter scan, we provide numerical evidence that the answer to both questions is affirmative for the case of a conformally coupled scalar field.

As the near extremal case seems to be the most interesting one, due to large violations of sCC in the classical case, we pick an example from that regime. Figure 10 shows  $\omega n_0(\omega)$  for the parameters  $r_c = 100$ ,  $r_+ = 2$ ,  $r_- = 1.95$ , corresponding to  $\kappa_c = 0.0094$ ,  $\kappa_+ = 0.0062$ ,  $\kappa_- = 0.0066$ . Shown in the figure are results obtained with the method based on Heun functions described in section 5.3, combined with results obtained by direct numerical integration of the radial equation, which is the method employed in [36] for the RN case. The consideration of the results obtained by direct integration of the radial equation not only provides a consistency check for the results obtained with the Heun function method, but is at present also necessary as the search for a solution of (139), i.e. for the critical exponent  $\nu$ , becomes difficult for large frequencies  $\omega$  for which the asymptotic expansion (141) does not provide a useful estimate for the starting point of a numerical search algorithm<sup>29</sup>. On the other hand, the Heun method seems to have advantages for  $\omega \rightarrow 0$  where some scattering coefficients diverge.

At any rate, clearly  $\omega n_0(\omega)$  is not only non-vanishing, but also has a non-vanishing integral, answering the first of the above questions. As for the second question, one finds that  $\omega n_1(\omega)$  is already of order  $10^{-15}$ , i.e. suppressed by eight orders of magnitude. The contributions of higher angular momenta are already drowned in numerical noise. The same behavior, i.e. non-vanishing integrands  $\omega n_\ell(\omega)$  with non-vanishing integrals and a rapid decrease in angular momentum were also found for the other parameters  $r_X$  at which we numerically evaluated  $\omega n_\ell(\omega)$  (both in the near-extremal and the ‘normal’ regime  $\kappa_- > \kappa_c$ ). We thus conclude that the coefficient  $C$  in (5) is indeed generically non-vanishing.

<sup>28</sup> As discussed in the next section, the analog of  $n_\ell(\omega)$  in the case of the BTZ black hole vanishes identically.

<sup>29</sup> However, we do not doubt that with more effort the range of applicability of the Heun function method can be extended to greater frequencies.



**Figure 10.** Numerical results for  $\omega n_0(\omega)$  for the parameters  $r_c = 100$ ,  $r_+ = 2$ ,  $r_- = 1.95$ . In red the results from the Heun function method, in blue results obtained by direct numerical integration of the radial equation.

## 6. Comparison with the BTZ black hole

In [41] Dias *et al* have studied the stress tensor of a scalar field obeying  $(\square - \mu^2)\Phi = 0$  in the three-dimensional BTZ black hole spacetime [84, 85]. In local coordinates it has the metric

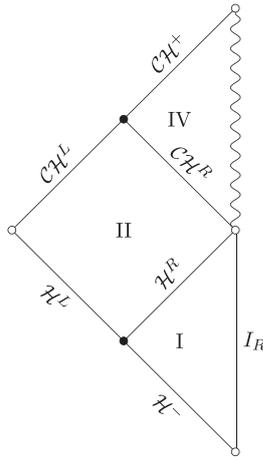
$$g = -f dt^2 + f^{-1} dr^2 + r^2(d\phi - \Omega dt)^2, \quad (159)$$

with

$$f = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2}, \quad \Omega = \frac{r_+ r_-}{r^2}, \quad (160)$$

and Penrose diagram as in figure 11. (For simplicity, we have set the AdS radius  $L = 1$ ; we refer to [41] for details on how to construct the diagram.) Even though the BTZ black hole is a solution with a negative cosmological constant, these authors showed that the regularity of the classical stress tensor  $T_{\mu\nu}$  is governed by a parameter  $\beta$  similar to (2). However, in contrast with RNdS, they showed that  $\alpha$  is not the spectral gap (3) of all quasi-normal modes in region I, but only of the ‘counter-rotating’ ones. As a consequence, it can be seen [41] that  $\beta \rightarrow \infty$  in the BTZ case as the spacetime approaches extremality  $r_- \rightarrow r_+$ . Hence, classically, the stress tensor becomes as regular as we wish across the Cauchy horizon, implying an even stronger violation of sCC than in the case of RNdS at the classical level. The authors [41] have also considered the expected quantum stress tensor in the Hartle–Hawking state and have presented arguments that it similarly becomes arbitrarily regular at the Cauchy horizon when we approach extremality, in marked contrast to our results in RNdS.

As [41] have pointed out, these results are in effect due to a cancellation that appears to be special to the BTZ case. Nevertheless, since the behavior of the quantum stress-energy tensor near  $CH^R$  for the BTZ black hole found by [41] is very different from the behavior we have found in the RNdS case, it is instructive to understand how this difference arises from the perspective of the type of mode calculation given in the previous section. Therefore, we will



**Figure 11.** Sketch of the relevant part of the BTZ spacetime. Here the double line indicates the asymptotic timelike boundary.

now analyze the BTZ case by the methods of the previous section. The upshot of the analysis that we will now give is that the constant  $C$  appearing in (5) is zero in the BTZ case, consistent with the findings of [41].

Following [41], we introduce the variable

$$z = \frac{r^2 - r_-^2}{r_+^2 - r_-^2}, \tag{161}$$

and write generic mode solutions as

$$\psi_{\omega m}(t, z, \phi) = e^{-i\omega t} e^{im\phi} R_{\omega m}(z). \tag{162}$$

Solutions are given in terms of hypergeometric functions involving the parameters

$$a = \frac{1}{2} \left( \Delta - i \frac{\omega - m\Omega_-}{\kappa_-} - i \frac{\omega - m\Omega_+}{\kappa_+} \right), \tag{163a}$$

$$b = \frac{1}{2} \left( 2 - \Delta - i \frac{\omega - m\Omega_-}{\kappa_-} - i \frac{\omega - m\Omega_+}{\kappa_+} \right), \tag{163b}$$

$$c = 1 - i \frac{\omega - m\Omega_-}{\kappa_-}. \tag{163c}$$

Here  $\Delta = 1 + \sqrt{1 + \mu^2}$ , and the surface gravities  $\kappa_{\pm}$  and the angular velocities  $\Omega_{\pm}$  of event and Cauchy horizon, which can be expressed in terms of the horizon radii  $r_{\pm}$  as

$$\kappa_{\pm} = \frac{r_+^2 - r_-^2}{r_{\pm}}, \quad \Omega_{\pm} = \frac{r_{\mp}}{r_{\pm}} = \frac{\kappa_{\pm}}{\kappa_{\mp}}. \tag{164}$$

The solutions with special behavior near the Cauchy horizon  $r_-$  are given by

$$R_{\omega m}^{\text{out},-}(z) = z^{-\frac{1}{2}(1-c)} (1-z)^{\frac{1}{2}(a+b-c)} F(a, b; c; z), \tag{165a}$$

$$R_{\omega m}^{\text{in},-}(z) = z^{+\frac{1}{2}(1-c)}(1-z)^{\frac{1}{2}(a+b-c)}F(a-c+1, b-c+1; 2-c; z). \quad (165b)$$

The solutions with special behavior near the event horizon  $r_+$  are given by

$$R_{\omega m}^{\text{out},+}(z) = z^{-\frac{1}{2}(1-c)}|1-z|^{-\frac{1}{2}(a+b-c)}F(c-b, c-a; -a-b+c+1; 1-z), \quad (166a)$$

$$R_{\omega m}^{\text{in},+}(z) = z^{-\frac{1}{2}(1-c)}|1-z|^{+\frac{1}{2}(a+b-c)}F(a, b; a+b-c+1; 1-z). \quad (166b)$$

Finally, the solution with the fast decay at infinity is given by

$$R_{\omega m}^{0,\infty}(z) = z^{-\frac{1}{2}(2a-c+1)}(z-1)^{\frac{1}{2}(a+b-c)}F(a, a-c+1; a-b+1; 1/z). \quad (167)$$

From the linear transformation formulas for hypergeometric functions, it follows that

$$R_{\omega m}^{\text{out},+} = \mathcal{A}_{\omega m}R_{\omega m}^{\text{out},-} + \mathcal{B}_{\omega m}R_{\omega m}^{\text{in},-}, \quad (168a)$$

$$R_{\omega m}^{\text{in},+} = \tilde{\mathcal{A}}_{\omega m}R_{\omega m}^{\text{in},-} + \tilde{\mathcal{B}}_{\omega m}R_{\omega m}^{\text{out},-}, \quad (168b)$$

$$\tilde{\mathcal{T}}_{\omega m}R_{\omega m}^{0,\infty} = R_{\omega m}^{\text{out},+} + \tilde{\mathcal{R}}_{\omega m}R_{\omega m}^{\text{in},+}, \quad (168c)$$

where

$$\mathcal{A}_{\omega m} = \frac{\Gamma(1-c)\Gamma(1-a-b+c)}{\Gamma(1-a)\Gamma(1-b)}, \quad \mathcal{B}_{\omega m} = \frac{\Gamma(c-1)\Gamma(1-a-b+c)}{\Gamma(c-a)\Gamma(c-b)}, \quad (169a)$$

$$\tilde{\mathcal{A}}_{\omega m} = \frac{\Gamma(c-1)\Gamma(a+b-c+1)}{\Gamma(a)\Gamma(b)}, \quad \tilde{\mathcal{B}}_{\omega m} = \frac{\Gamma(1-c)\Gamma(a+b-c+1)}{\Gamma(a-c+1)\Gamma(b-c+1)}, \quad (169b)$$

$$\tilde{\mathcal{T}}_{\omega m} = \frac{\Gamma(a)\Gamma(a-c+1)}{\Gamma(a-b+1)\Gamma(a+b-c)}, \quad \tilde{\mathcal{R}}_{\omega m} = \frac{\Gamma(a)\Gamma(a-c+1)\Gamma(c-a-b)}{\Gamma(1-b)\Gamma(c-b)\Gamma(a+b-c)}. \quad (169c)$$

We also recall the definition

$$r_* = \frac{1}{2\kappa_+} \log \left| \frac{r-r_+}{r+r_+} \left( \frac{r+r_-}{r-r_-} \right)^{\Omega_+} \right| \quad (170)$$

of the tortoise coordinate and introduce the corotating angles

$$\phi_{\pm} = \phi - \Omega_{\pm}t. \quad (171)$$

As before, we use boundary data to define the relevant states. To define the comparison state we use the modes with boundary data

$$\psi_{\omega m}^{\text{out}} = \begin{cases} (2\pi)^{-1}(2|\omega|)^{-\frac{1}{2}}r_-^{-\frac{1}{2}}e^{im\phi_-}e^{-i\omega v} & \text{on } \mathcal{CH}^L, \\ 0 & \text{on } \mathcal{CH}^R. \end{cases} \quad (172)$$

One easily checks that

$$\psi_{\omega m}^{\text{in},-} = (2\pi)(2|\omega - m\Omega_-|)^{\frac{1}{2}} r_-^{\frac{1}{2}} \left(\frac{4r_-}{\kappa_-}\right)^{i\frac{\omega - m\Omega_-}{2\kappa_-}} \left(\frac{r_+ - r_-}{r_+ + r_-}\right)^{i\frac{\omega - m\Omega_-}{2\kappa_+}} \psi_{\omega - m\Omega_-, m}^{\text{out}}. \tag{173}$$

To define the Hartle–Hawking (HH) state (an analog of our Unruh state), we use the modes with boundary data

$$\psi_{\omega m}^{\text{up},\text{I}} = \begin{cases} (2\pi)^{-1}(2|\omega|)^{-\frac{1}{2}} r_+^{-\frac{1}{2}} e^{im\phi_+} e^{-i\omega u} & \text{on } \mathcal{H}^-, \\ O(z^{-\Delta/2}) & \text{at } \infty, \end{cases} \tag{174a}$$

$$\psi_{\omega m}^{\text{up},\text{II}} = \begin{cases} (2\pi)^{-1}(2|\omega|)^{-\frac{1}{2}} r_+^{-\frac{1}{2}} e^{im\phi_+} e^{-i\omega u} & \text{on } \mathcal{H}^L, \\ 0 & \text{on } \mathcal{H}^R. \end{cases} \tag{174b}$$

In I and II, respectively, these are related to the mode solutions introduced above by

$$\psi_{\omega m}^{\text{out},+} + \tilde{\mathcal{R}}_{\omega m} \psi_{\omega m}^{\text{in},+} = (2\pi)(2|\omega - m\Omega_+|)^{\frac{1}{2}} r_+^{\frac{1}{2}} \left(\frac{4r_+}{\kappa_+}\right)^{+i\frac{\omega - m\Omega_+}{2\kappa_+}} \left(\frac{r_+ - r_-}{r_+ + r_-}\right)^{+i\frac{\omega - m\Omega_+}{2\kappa_-}} \psi_{\omega - m\Omega_+, m}^{\text{up},\text{I}}, \tag{175a}$$

$$\psi_{\omega m}^{\text{out},+} = (2\pi)(2|\omega - m\Omega_+|)^{\frac{1}{2}} r_+^{\frac{1}{2}} \left(\frac{4r_+}{\kappa_+}\right)^{+i\frac{\omega - m\Omega_+}{2\kappa_+}} \left(\frac{r_+ - r_-}{r_+ + r_-}\right)^{+i\frac{\omega - m\Omega_+}{2\kappa_-}} \psi_{\omega - m\Omega_+, m}^{\text{up},\text{II}}. \tag{175b}$$

Similar to [75], and analogously to (117), the symmetrized two-point function in the Hartle–Hawking state can be expressed as

$$\begin{aligned} \langle \{\Phi(x_1), \Phi(x_2)\} \rangle_{\text{HH}} &= \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega \left[ \frac{\text{sgn } \omega}{\sinh \frac{\pi\omega}{\kappa_+}} \text{Re} \left\{ \psi_{\omega, m}^{\text{up},\text{I}}(x_1), \overline{\psi_{\omega, m}^{\text{up},\text{II}}(x_2)} \right\} \right. \\ &\quad \left. + \frac{\text{sgn } \omega}{1 - e^{-\frac{2\pi\omega}{\kappa_+}}} \left( \left\{ \psi_{\omega, m}^{\text{up},\text{I}}(x_1), \overline{\psi_{\omega, m}^{\text{up},\text{I}}(x_2)} \right\} + \left\{ \psi_{-\omega, m}^{\text{up},\text{II}}(x_1), \overline{\psi_{-\omega, m}^{\text{up},\text{II}}(x_2)} \right\} \right) \right] \tag{176} \end{aligned}$$

Restricting to region II, using the above relations and neglecting the  $\psi^{\text{out},-}$  modes, which are irrelevant for the restriction to  $\mathcal{CH}^L$  that we want to perform in the end, one obtains

$$\begin{aligned} \langle \{\Phi(x_1), \Phi(x_2)\} \rangle_{\text{HH}} &= \frac{r_-}{r_+} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega \frac{|\omega|}{|\omega - m\Omega_+ + m\Omega_-|} \\ &\quad \times \left[ \frac{\text{sgn}(\omega - m\Omega_+ + m\Omega_-)}{1 - e^{-\frac{2\pi(\omega - m\Omega_+ + m\Omega_-)}{\kappa_+}}} |\tilde{\mathcal{A}}_{\omega + m\Omega_-, m}|^2 \left\{ \psi_{\omega, m}^{\text{out}}(x_1), \overline{\psi_{\omega, m}^{\text{out}}(x_2)} \right\} \right] \end{aligned}$$

$$\begin{aligned}
 & + \frac{\operatorname{sgn}(\omega - m\Omega_+ + m\Omega_-)}{1 - e^{-\frac{2\pi(\omega - m\Omega_+ + m\Omega_-)}{\kappa_+}}} |\mathcal{B}_{\omega+m\Omega_-,m}|^2 \left\{ \psi_{-\omega,m}^{\text{out}}(x_1), \overline{\psi_{-\omega,m}^{\text{out}}(x_2)} \right\} \\
 & + \frac{\operatorname{sgn}(\omega - m\Omega_+ + m\Omega_-)}{\sinh \frac{\pi(\omega - m\Omega_+ + m\Omega_-)}{\kappa_+}} \\
 & \times \operatorname{Re} \left\{ \tilde{\mathcal{H}}_{\omega+m\Omega_-,m} \tilde{\mathcal{A}}_{\omega+m\Omega_-,m} \psi_{\omega,m}^{\text{out}}(x_1), \overline{\mathcal{B}_{\omega+m\Omega_-,m} \psi_{\omega,m}^{\text{out}}(x_2)} \right\}.
 \end{aligned} \tag{177}$$

For the difference of the Hartle–Hawking and the comparison state, differentiated  $n$  times w.r.t.  $v$  and restricted to  $\mathcal{CH}^L$ , one thus obtains formally, after symmetrization under the simultaneous transformation  $\omega \rightarrow -\omega, m \rightarrow -m$ ,

$$\left[ \langle \partial_v^n \Phi(x_1) \partial_v \Phi(x_2) \rangle_{\text{HH}} - \langle \partial_v^n \Phi(x_1) \partial_v \Phi(x_2) \rangle_C \right]_{x_1 \rightarrow x_2 \rightarrow \mathcal{CH}^L} = \frac{1}{8\pi^2 r_-^2} \sum_{m=-\infty}^{\infty} \int_0^{\infty} d\omega \omega^{2n-1} n_m(\omega), \tag{178}$$

where

$$\begin{aligned}
 n_m(\omega) = & \frac{r_-}{r_+} \frac{\omega}{\omega - m\Omega_+ + m\Omega_-} \left[ \coth \frac{\pi(\omega - m\Omega_+ + m\Omega_-)}{\kappa_+} \left( |\tilde{\mathcal{A}}_{\omega+m\Omega_-,m}|^2 + |\mathcal{B}_{\omega+m\Omega_-,m}|^2 \right) \right. \\
 & \left. + 2\operatorname{csch} \frac{\pi(\omega - m\Omega_+ + m\Omega_-)}{\kappa_+} \operatorname{Re} \left( \tilde{\mathcal{H}}_{\omega+m\Omega_-,m} \tilde{\mathcal{A}}_{\omega+m\Omega_-,m} \overline{\mathcal{B}_{\omega+m\Omega_-,m}} \right) \right] - \coth \frac{\pi\omega}{\kappa_-}.
 \end{aligned} \tag{179}$$

Equation (179) is the analog of (122). In the present case, however, one can use standard identities for products of  $\Gamma$  functions and trigonometric identities to show that  $n_m(\omega)$  identically vanishes! This means in particular that  $\langle T_{vv} \rangle_{\text{HH}} - \langle T_{vv} \rangle_C \propto \sum_m \int d\omega \omega n_m(\omega)$  vanishes at  $\mathcal{CH}^L$ , and since both states are stationary, i.e. invariant under the flow of  $\partial_t$ , the same must be true at  $\mathcal{CH}^R$ . Hence, the constant  $C$  appearing in (5) is zero in the BTZ case, whereas we have presented evidence that  $C \neq 0$  generically in the case of RNdS. Thus, our findings are mathematically consistent with those of [41]. We believe that the vanishing of  $n_m(\omega)$  is a very special property of the BTZ black hole spacetime.

### 7. Conclusions

In this work, we have analyzed the expected stress tensor of a real, linear Klein–Gordon quantum field near the Cauchy horizon inside the RNdS black hole. We have presented arguments that, due entirely to quantum effects, the  $VV$ -component of the stress tensor diverges as  $V^{-2}$  near the Cauchy horizon (defined by  $V = 0$ ) with a coefficient that does not depend on the initial state, provided only that the state is regular (Hadamard) near an initial Cauchy surface extending beyond the cosmological horizon. Numerical evidence indicates that this coefficient is not zero at least for the black hole parameter values which we have considered. Since the coefficient would at any rate be expected to be basically an analytic function of the black hole parameters, it should be zero only on a set of measure zero of finely tuned parameters.

Although we have studied in this paper the case of RNdS spacetime, our methods should apply with relatively minor modifications to the the Kerr–Newman–dS black holes, and we strongly expect the same conclusions. More generally, we expect the behavior (5) to hold generically in eternal black hole spacetime whose Cauchy horizon is a bifurcate Killing horizon.

Our analysis does not preclude the existence of spacetimes for which  $C = 0$  and for which the behavior of the quantum stress-energy is less singular than (5). Examples of such spacetimes are the two-dimensional RNdS spacetime with  $\kappa_c = \kappa_-$  and the three-dimensional BTZ black hole spacetime analyzed by [41]. Nevertheless, we shall argue elsewhere that even for such spacetimes in which the quantum stress-energy is not (badly) singular as one approaches the Cauchy horizon, a Hadamard state can not be extended beyond the Cauchy horizon, which typically manifests itself in the divergence of some local Wick power at the horizon. Thus, it appears that quantum field effects enforce strong cosmic censorship—or, at least, move it to a regime where full quantum gravity will be needed to determine to what extent it holds.

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## References

- [1] Hintz P and Vasy A 2017 Analysis of linear waves near the Cauchy horizon of cosmological black holes *J. Math. Phys.* **58** 081509
- [2] Dafermos M and Shlapentokh-Rothman Y 2018 Rough initial data and the strength of the blue-shift instability on cosmological black holes with  $\Lambda > 0$   $\{x=[ ]\}$  *Class. Quantum Grav.* **35** 195010
- [3] Boulware D G 1973 Naked Singularities, Thin Shells, and the Reissner-Nordström Metric *Phys. Rev. D* **8** 2363
- [4] Penrose R 1974 Gravitational collapse *Gravitational Radiation and Gravitational Collapse* ed C DeWitt-Morette (Berlin: Springer) pp 82–91
- [5] Poisson E and Israel W 1990 Internal structure of black holes *Phys. Rev. D* **41** 1796
- [6] Mellor F and Moss I 1990 Stability of Black Holes in De Sitter Space *Phys. Rev. D* **41** 403
- [7] Mellor F and Moss I 1992 A reassessment of the stability of the Cauchy horizon in de Sitter Space *Class. Quantum Grav.* **9** L43
- [8] Brady P R, Moss I G and Myers R C 1998 Cosmic censorship: As strong as ever *Phys. Rev. Lett.* **80** 3432
- [9] Dias O J C, Reall H S and Santos J E 2018 Strong cosmic censorship: taking the rough with the smooth *J. High Energy Phys.* **JHEP10(2016)001**
- [10] McNamara J M 1978 Behaviour of scalar perturbations of a reissner-nordström black hole inside the event horizon *Proc. R. Soc. A* **364** 121

- [11] Dafermos M 2003 Stability and instability of the Cauchy horizon for the spherically symmetric Einstein-Maxwell-scalar field equations *Ann. Math.* **875**
- [12] Dafermos M 2005 The Interior of charged black holes and the problem of uniqueness in general relativity *Commun. Pure Appl. Math.* **58** 0445
- [13] Dafermos M and Luk J 2017 The interior of dynamical vacuum black holes I: The  $C^0$ -stability of the Kerr Cauchy horizon (arXiv:1710.01722)
- [14] Dafermos M 2014 Black holes without spacelike singularities *Commun. Math. Phys.* **332** 729
- [15] Sbierski J 2014 On the initial value problem in general relativity and wave propagation in black-hole spacetimes *PhD Thesis* University of Cambridge
- [16] Luk J and Oh S J 2017 Proof of linear instability of the Reissner-Nordström Cauchy horizon under scalar perturbations *Duke Math. J.* **166** 437
- [17] Costa J L *et al* 2015 On the global uniqueness for the Einstein-Maxwell-scalar field system with a cosmological constant: I. Well posedness and breakdown criterion *Class. Quantum Grav.* **32** 015017
- [18] Costa J L *et al* 2015 On the global uniqueness for the Einstein-Maxwell-scalar field system with a cosmological constant. Part 2: Structure of the solutions and stability of the Cauchy horizon *Commun. Math. Phys.* **339** 903
- [19] Costa J L *et al* 2014 On the global uniqueness for the Einstein-Maxwell-scalar field system with a cosmological constant. Part 3: Mass inflation and extendibility of the solutions (arXiv:1406.7261)
- [20] Franzen A T 2016 Boundedness of Massless Scalar Waves on Reissner-Nordström Interior Backgrounds *Commun. Math. Phys.* **343** 601
- [21] Costa J L and Franzen A T 2017 Bounded energy waves on the black hole interior of Reissner-Nordström-de Sitter *Annales Henri Poincaré* **18** 3371
- [22] Barreto A S and Zworski M 1997 Distribution of resonances for spherical black holes *Math. Res. Lett.* **4** 103
- [23] Bony J F and Häfner D 2008 Decay and non-decay of the local energy for the wave equation on the de Sitter-Schwarzschild metric *Commun. Math. Phys.* **282** 697
- [24] Dyatlov S 2011 Quasi-normal modes and exponential energy decay for the Kerr-de Sitter black hole *Commun. Math. Phys.* **306** 119
- [25] Wunsch J and Zworski M 2011 Resolvent estimates for normally hyperbolic trapped sets *Ann. Henri Poincaré* **12** 1349
- [26] Vasy A 2013 Microlocal analysis of asymptotically hyperbolic and Kerr-de Sitter spaces (with an appendix by Semyon Dyatlov) *Invent. Math.* **194** 381
- [27] Dyatlov S 2016 Spectral gaps for normally hyperbolic trapping *Annales Inst. Fourier* **66** 55
- [28] Dyatlov S 2015 Resonance projectors and asymptotics for  $r$ -normally hyperbolic trapped sets *J. Am. Math. Soc.* **28** 311
- [29] Nonnenmacher S and Zworski M 2015 Decay of correlations for normally hyperbolic trapping *Invent. Math.* **200** 345
- [30] Hintz P and Vasy A 2018 Asymptotics for the wave equation on differential forms on Kerr-de Sitter space *J. Diff. Geom.* **110** 221
- [31] Nollert H P 1999 Quasinormal modes: the characteristic ‘sound’ of black holes and neutron stars *Class. Quantum Grav.* **16** R159
- [32] Ori A 1991 Inner structure of a charged black hole: An exact mass-inflation solution *Phys. Rev. Lett.* **67** 789
- [33] Cardoso V *et al* 2018 Quasinormal modes and Strong Cosmic Censorship *Phys. Rev. Lett.* **120** 031103
- [34] Luna R *et al* 2019 Strong Cosmic Censorship: the nonlinear story *Phys. Rev. D* **99** 064014
- [35] Hollands S and Wald R M 2001 Local Wick polynomials and time ordered products of quantum fields in curved space-time *Commun. Math. Phys.* **223** 289
- [36] Zilberman N, Levi A and Ori A 2019 Quantum fluxes at the inner horizon of a spherical charged black hole (arXiv:1906.11303)
- [37] Sela O 2018 Quantum effects near the Cauchy horizon of a Reissner-Nordström black hole *Phys. Rev. D* **98** 024025
- [38] Christensen S M and Fulling S A 1977 Trace Anomalies and the Hawking Effect *Phys. Rev. D* **15** 2088
- [39] Birrell N D and Davies P C W 1978 On falling through a black hole into another universe *Nature* **272** 35

- [40] Marković D and Unruh W G 1991 Vacuum for a massless scalar field outside a collapsing body in de Sitter space-time *Phys. Rev. D* **43** 332
- [41] Dias O J C, Reall H S and Santos J E 2019 The BTZ black hole violates strong cosmic censorship (arXiv:1906.08265)
- [42] Tipler F J 1977 Singularities in conformally flat spacetimes *Phys. Lett. A* **64** 8
- [43] Clarke C J S 1993 *The Analysis of Space-Time Singularities* (Cambridge: Cambridge University Press)
- [44] Wald R M 1984 *General Relativity* (Chicago, IL: University of Chicago Press)
- [45] Hiscock W A 1980 Quantum mechanical instability of the Kerr-Newman black hole interior *Phys. Rev. D* **21** 2057
- [46] Wald R M 1994 *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics* (Chicago, IL: The University of Chicago Press)
- [47] Davies P C W, Fulling S A and Unruh W G 1976 Energy Momentum Tensor Near an Evaporating Black Hole *Phys. Rev. D* **13** 2720
- [48] Kay B S 1992 The Principle of locality and quantum field theory on (nonglobally hyperbolic) curved space-times *Rev. Math. Phys.* **4** 167
- [49] Kay B S and Wald R M 1991 Theorems on the Uniqueness and Thermal Properties of Stationary, Nonsingular, Quasifree States on Space-Times with a Bifurcate Killing Horizon *Phys. Rep.* **207** 49
- [50] Poisson E 2004 The Motion of point particles in curved space-time *Living Rev. Rel.* **7** 6
- [51] DeWitt B S and Brehme R W 1960 Radiation damping in a gravitational field *Ann. Phys., Lpz.* **9** 220
- [52] Hollands S and Wald R M 2002 Existence of local covariant time ordered products of quantum fields in curved space-time *Commun. Math. Phys.* **231** 309
- [53] Hollands S and Wald R M 2010 Axiomatic quantum field theory in curved spacetime *Commun. Math. Phys.* **293** 85
- [54] Brunetti R, Fredenhagen K and Verch R 2003 The Generally covariant locality principle: A New paradigm for local quantum field theory *Commun. Math. Phys.* **237** 31
- [55] Radzikowski M 1996 Micro-local approach to the Hadamard condition in quantum field theory on curved space-time *Commun. Math. Phys.* **179** 529
- [56] Brunetti R, Fredenhagen K and Hollands S 2005 A Remark on alpha vacua for quantum field theories on de Sitter space *J. High Energy Phys.* **JHEP05(2016)063**
- [57] Brunetti R and Fredenhagen K 2000 Microlocal analysis and interacting quantum field theories: Renormalization on physical backgrounds *Commun. Math. Phys.* **208** 623
- [58] Radzikowski M J and Verch R 1996 A Local to global singularity theorem for quantum field theory on curved space-time *Commun. Math. Phys.* **180** 1
- [59] Fulling S, Narcowich F and Wald R M 1981 Singularity structure of the two point function in quantum field theory in curved space-time. II *Ann. Phys., Lpz.* **136** 243
- [60] Junker W and Schrohe E 2002 Adiabatic vacuum states on general space-time manifolds: Definition, construction, and physical properties *Ann Henri Poincaré* **3** 1113
- [61] Gérard C and Wrochna M 2014 Construction of Hadamard states by pseudo-differential calculus *Commun. Math. Phys.* **325** 713
- [62] Hollands S 2000 Aspects of Quantum Field Theory on Curved Spacetime *PhD Thesis* University of York
- [63] Dappiaggi C, Moretti V and Pinamonti N 2017 *Hadamard States from Light-like Hypersurfaces* (Berlin: Springer) (arXiv:1706.09666)
- [64] Hawking S W 1975 Particle Creation by Black Holes *Commun. Math. Phys.* **43** 199
- [65] Dappiaggi C, Moretti V and Pinamonti N 2011 Rigorous construction and Hadamard property of the Unruh state in Schwarzschild spacetime *Adv. Theor. Math. Phys.* **15** 355
- [66] Dappiaggi C, Pinamonti N and Poppmann M 2011 Local causal structures, Hadamard states and the principle of local covariance in quantum field theory *Commun. Math. Phys.* **304** 459
- [67] Kay B S 2001 Application of linear hyperbolic PDE to linear quantum fields in curved space-times: Especially black holes, time machines and a new semilocal vacuum concept (arXiv:gr-qc/0103056)
- [68] Hörmander L 2003 *The Analysis of Linear Partial Differential Operators. I* (Berlin: Springer)
- [69] Duistermaat J J and Hörmander L 1972 Fourier integral operators. ii *Acta Math.* **128** 183
- [70] Unruh W G 1976 Notes on black hole evaporation *Phys. Rev. D* **14** 870
- [71] Dimock J 1980 Algebras of local observables on a manifold *Commun. Math. Phys.* **77** 219
- [72] Verch R 1994 Local definiteness, primarity and quasiequivalence of quasifree Hadamard quantum states in curved space-time *Commun. Math. Phys.* **160** 507

- [73] Kenig C E, Ponce G and Vega L 1993 Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle *Commun. Pure Appl. Math.* **46** 527
- [74] Yafaev D 2010 *Mathematical Scattering Theory: Analytic Theory* (Providence, RI: American Mathematical Society)
- [75] Lanir A *et al* 2018 Two-point function of a quantum scalar field in the interior region of a Reissner-Nordstrom black hole *Phys. Rev. D* **97** 024033
- [76] Kehle C and Shlapentokh-Rothman Y 2019 A Scattering Theory for Linear Waves on the Interior of Reissner-Nordström Black Holes *Annales Henri Poincaré* **20** 1583
- [77] Suzuki H, Takasugi E and Umetsu H 1999 Analytic solutions of Teukolsky equation in Kerr-de Sitter and Kerr-Newman-de Sitter geometries *Prog. Theor. Phys.* **102** 253
- [78] Schmidt D 1979 Die Lösung der linearen Differentialgleichung 2. Ordnung um zwei einfache Singularitäten durch Reihen nach hypergeometrischen Funktionen *J. Reine Angew. Math.* **309** 127
- [79] Ronveaux A 1995 *Heun's Differential Equation* (Oxford: Oxford University Press)
- [80] Leaver E W 1986 Solutions to a generalized spheroidal wave equation: Teukolsky's equations in general relativity, and the two-center problem in molecular quantum mechanics *J. Math. Phys.* **27** 1238
- [81] Leaver E W 1986 Spectral decomposition of the perturbation response of the Schwarzschild geometry *Phys. Rev. D* **34** 384
- [82] Abramowitz M and Stegun I A 1972 *Handbook of Mathematical Functions* (New York: Dover)
- [83] Gautschi W 1967 Computational aspects of three-term recurrence relations *SIAM Rev.* **9** 24
- [84] Banados M, Teitelboim C and Zanelli J 1992 The Black hole in three-dimensional space-time *Phys. Rev. Lett.* **69** 1849
- [85] Banados M *et al* 1993 Geometry of the (2+1) black hole *Phys. Rev. D* **48** 1506  
Banados M *et al* 2013 Geometry of the (2+1) black hole *Phys. Rev. D* **88** 069902 (erratum)