



A Proof of Friedman’s Ergosphere Instability for Scalar Waves

Georgios Moschidis 

Department of Mathematics, Princeton University, Fine Hall, Washington Road, Princeton, NJ 08544, USA.
E-mail: gm6@math.princeton.edu

Received: 29 November 2016 / Accepted: 23 August 2017

Published online: 4 November 2017 – © Springer-Verlag GmbH Germany 2017

Abstract: Let (\mathcal{M}^{3+1}, g) be a real analytic, stationary and asymptotically flat spacetime with a non-empty ergoregion \mathcal{E} and no future event horizon \mathcal{H}^+ . In Friedman (Commun Math Phys 63(3):243–255, 1978), Friedman observed that, on such spacetimes, there exist solutions φ to the wave equation $\square_g \varphi = 0$ such that their local energy does not decay to 0 as time increases. In addition, Friedman provided a heuristic argument that the energy of such solutions actually grows to $+\infty$. In this paper, we provide a rigorous proof of Friedman’s instability. Our setting is, in fact, more general. We consider smooth spacetimes (\mathcal{M}^{d+1}, g) , for any $d \geq 2$, not necessarily globally real analytic. We impose only a unique continuation condition for the wave equation across the boundary $\partial\mathcal{E}$ of \mathcal{E} on a small neighborhood of a point $p \in \partial\mathcal{E}$. This condition always holds if (\mathcal{M}, g) is analytic in that neighborhood of p , but it can also be inferred in the case when (\mathcal{M}, g) possesses a second Killing field Φ such that the span of Φ and the stationary Killing field T is timelike on $\partial\mathcal{E}$. We also allow the spacetimes (\mathcal{M}, g) under consideration to possess a (possibly empty) future event horizon \mathcal{H}^+ , such that, however, $\mathcal{H}^+ \cap \mathcal{E} = \emptyset$ (excluding, thus, the Kerr exterior family). As an application of our theorem, we infer an instability result for the acoustical wave equation on the hydrodynamic vortex, a phenomenon first investigated numerically by Oliveira et al. in (Phys Rev D 89(12):124008, 2014). Furthermore, as a side benefit of our proof, we provide a derivation, based entirely on the vector field method, of a Carleman-type estimate on the exterior of the ergoregion for a general class of stationary and asymptotically flat spacetimes. Applications of this estimate include a Morawetz-type bound for solutions φ of $\square_g \varphi = 0$ with frequency support bounded away from $\omega = 0$ and $\omega = \pm\infty$.

Contents

1. Introduction	438
2. Statement of the Main Results	442
2.1 Assumptions on the spacetimes under consideration	442

2.2	The main results	446
2.3	Discussion on Assumption A1	447
2.4	Examples of spacetimes satisfying Assumptions G1–G3 and A1	448
2.5	A remark on the Carleman-type estimates in the proof of Theorem 2.1	451
3.	Notational Conventions and Hardy Inequalities	451
3.1	Constants and dependence on parameters	452
3.2	Inequality symbols	452
3.3	Some special subsets of \mathcal{M}	452
3.4	Notations on metrics, connections and integration	453
3.5	Coordinate charts on $\mathcal{M} \setminus \mathcal{H}^-$	453
3.6	Notations for derivatives on \mathbb{S}^{d-1}	454
3.7	The $O_k(\cdot)$ notation	455
3.8	Vector field multipliers and currents	455
3.9	Hardy-type inequalities	455
4.	Proof of Theorem 2.1	456
4.1	Construction of initial data on Σ with negative higher order energy	457
4.2	Decay outside the extended ergoregion	459
4.3	Limiting behaviour for solutions of (1.2)	459
4.4	Finishing the proof	461
5.	Frequency Decomposition	465
5.1	Weighted energy estimates for ψ	465
5.2	Frequency cut-off	467
5.3	Bounds for the frequency-decomposed components	469
6.	A Carleman-Type Estimate Outside the Extended Ergoregion	480
6.1	Parameters and cut-off functions in the proof of Proposition 6.1	483
6.2	Construction of the auxiliary functions w_R, \tilde{w}_R	483
6.3	The seed functions f, \tilde{f}, h and \tilde{h}	489
6.4	The integration-by-parts scheme	490
6.5	Estimates for the zeroth order term	494
6.6	Estimates for the first order terms	496
6.7	Proof of Proposition 6.1	499
6.8	Proof of Corollary 6.1	507
6.9	Proof of Proposition 6.1 in the case of Dirchlet or Neumann boundary conditions	509
7.	Proof of Proposition 4.1	512
8.	Proof of Corollary 2.1	516
9.	Aside: Discussion on Friedman’s Heuristic Argument	517
9.1	Friedman’s argument	517
9.2	Comparison with the proof of Theorem 2.1	518
	References	519

1. Introduction

In the field of general relativity, stationary and asymptotically flat spacetimes (\mathcal{M}, g) arise naturally as models of the asymptotic state of isolated self-gravitating systems. In this context, questions on the stability properties of such spacetimes as solutions to the initial value problem for the Einstein equations

$$Ric_{\mu\nu}(g) - \frac{1}{2}R(g)g_{\mu\nu} = 8\pi T_{\mu\nu} \quad (1.1)$$

(where $T_{\mu\nu}$ is the stress-energy tensor associated to the matter fields, with $T_{\mu\nu} = 0$ in the vacuum case) are of particular importance, being directly related to the physical relevance of the spacetimes themselves.

The stability of Minkowski spacetime (\mathbb{R}^{3+1}, η) as a solution to the vacuum Einstein equations was established in the monumental work of Christodoulou–Klainerman [8]. Until today, Minkowski spacetime is the only stationary and asymptotically flat vacuum spacetime that is known to be non-linearly stable. A more complicated example of a family of stationary and asymptotically flat spacetimes expected to be stable are the subextremal Kerr exterior spacetimes $(\mathcal{M}_{M,a}, g_{M,a})$, with mass M and angular momentum a satisfying $0 \leq |a| < M$ (for a detailed formulation of the Kerr stability conjecture, see [10]). While the non-linear stability of the family $(\mathcal{M}_{M,a}, g_{M,a})$ has not been established so far, the *linear* stability of the Schwarzschild exterior (i.e., $(\mathcal{M}_{M,a}, g_{M,a})$ for $a = 0$) was recently obtained by Dafermos–Holzegel–Rodnianski (see [10]).

Owing to the fact that the wave equation

$$\square_g \varphi = 0 \tag{1.2}$$

can be viewed as a simple model of the linearised vacuum Einstein equations (1.1) around $(\mathcal{M}_{M,a}, g_{M,a})$, the stability properties of equation (1.2) in the case $0 \leq |a| < M$ had been extensively studied in the years preceding [10], culminating in the proof of polynomial decay estimates for solutions φ to (1.2) on $(\mathcal{M}_{M,a}, g_{M,a})$ in the full subextremal case $0 \leq |a| < M$ in [17, 36]. For earlier results in the Schwarzschild case $a = 0$ and the very slowly rotating case $|a| \ll M$, see [3, 4, 11–13, 25] and [2, 14–16, 38] respectively.

One important aspect of the geometry of $(\mathcal{M}_{M,a}, g_{M,a})$ in the case $a \neq 0$ is the existence of an *ergoregion* (or “ergosphere”) \mathcal{E} ; recall that $\mathcal{E} \subset \mathcal{M}_{M,a}$ is defined as

$$\mathcal{E} \doteq \overline{\{p \in \mathcal{M}_{M,a} \mid g(T_p, T_p) > 0\}}, \tag{1.3}$$

where T is the stationary Killing vector field on $(\mathcal{M}_{M,a}, g_{M,a})$. The fact that \mathcal{E} is non-empty when $a \neq 0$ gives rise to the phenomenon of *superradiance* for solutions to (1.2) on $(\mathcal{M}_{M,a}, g_{M,a})$, $a \neq 0$: there exist solutions φ to (1.2) such that their T -energy flux through future null infinity \mathcal{I}^+ is greater than their T -energy flux initially. In general, superradiance poses a serious difficulty in obtaining stability results for Eq. (1.2). In the case of $(\mathcal{M}_{M,a}, g_{M,a})$, superradiance does not eventually render equation (1.2) unstable, owing, partly, to the presence of the future *event horizon* \mathcal{H}^+ , allowing for part of the energy of solutions of (1.2) to “leave” the black hole exterior. Notice, however, that superradiance-related mode instabilities do appear on $(\mathcal{M}_{M,a}, g_{M,a})$ for the Klein–Gordon equation (see [35]), or even for the wave equation with a (well-chosen) short-range non-negative potential (see [32]).

Stationary and asymptotically flat spacetimes (\mathcal{M}, g) with a non-empty ergoregion \mathcal{E} but lacking a future event horizon \mathcal{H}^+ appear in the literature as models for rapidly rotating self-gravitating objects, for instance, as models of self-gravitating dense rotating fluids (see [5]). In [20], Friedman studied the instability properties of equation (1.2) on such spacetimes, making the following observation: There exist smooth solutions φ to (1.2) with negative T -energy flux initially, i.e.,

$$\int_{\Sigma} J_{\mu}^T(\varphi) n_{\Sigma}^{\mu} < 0 \tag{1.4}$$

on a Cauchy hypersurface Σ of (\mathcal{M}, g) (see Sect. 3 for our notations on vector field currents), and, in view of the conservation of the T -energy flux, the absence of a future

event horizon \mathcal{H}^+ and the non-negativity of $J_\mu^T(\cdot)n_\Sigma^\mu$ outside \mathcal{E} , any such function φ satisfies for all $\tau \geq 0$:

$$\int_{\Sigma_\tau \cap \mathcal{E}} J_\mu^T(\varphi)n_{\Sigma_\tau}^\mu \leq \int_\Sigma J_\mu^T(\varphi)n_\Sigma^\mu < 0 \tag{1.5}$$

(where Σ_τ denotes the image of Σ under the flow of T for time τ). Therefore, the local energy of φ cannot decay to 0 with time.

Based on the above observation, Friedman provided a heuristic argument suggesting that, under the additional assumption that the spacetime (\mathcal{M}, g) is real analytic, any such solution φ satisfies

$$\limsup_{\tau \rightarrow +\infty} \int_{\Sigma_\tau} J_\mu^N(\varphi)n_{\Sigma_\tau}^\mu = +\infty \tag{1.6}$$

for a globally timelike T -invariant vector field N . In view of the aforementioned connection between equation (1.2) and the Einstein equations (1.1), Friedman suggested that such spacetimes cannot appear as the final state of the evolution of a self-gravitating system (see [20] for more details).¹ For a numerical investigation of Friedman’s instability, see [6, 9, 39].

In this paper, we will provide a rigorous proof of Friedman’s instability for equation (1.2). Our proof will in fact not require that (\mathcal{M}, g) is real analytic, but we will assume, instead, a substantially weaker unique continuation condition for equation (1.2) through a subset of the boundary $\partial\mathcal{E}_{ext}$ of the “extended” ergoregion \mathcal{E}_{ext} , where we define \mathcal{E}_{ext} to be equal to the union of the ergoregion \mathcal{E} with the connected components of $\mathcal{M} \setminus \mathcal{E}$ which intersect neither \mathcal{H}^+ nor the asymptotically flat region of \mathcal{M} .² Note that, in the case when $\mathcal{M} \setminus \mathcal{E}$ is connected, \mathcal{E}_{ext} coincides with \mathcal{E} . In particular, we will establish the following result:

Theorem 1.1. *Let (\mathcal{M}^{d+1}, g) , $d \geq 2$, be a smooth, globally hyperbolic, stationary and asymptotically flat spacetime with a non-empty ergoregion \mathcal{E} and a future event horizon \mathcal{H}^+ which is either empty or satisfies $\mathcal{E} \cap \mathcal{H}^+ = \emptyset$. Assume, in addition, that the following unique continuation condition through the boundary $\partial\mathcal{E}_{ext}$ of the “extended” ergoregion \mathcal{E}_{ext} holds:*

Unique continuation condition: *There exists a point $p \in \partial\mathcal{E}_{ext}$ and an open neighborhood \mathcal{U} of p in \mathcal{M} such that, for any solution φ to Eq. (1.2) on \mathcal{M} with $\varphi \equiv 0$ on $\mathcal{M} \setminus \mathcal{E}_{ext}$, we have $\varphi = 0$ also on $\mathcal{E}_{ext} \cap \mathcal{U}$.*

Then, there exists a smooth solution φ to (1.2) with compactly supported initial data on a Cauchy hypersurface Σ of (\mathcal{M}, g) , such that

$$\limsup_{\tau \rightarrow +\infty} \int_{\Sigma_\tau} J_\mu^N(\varphi)n^\mu = +\infty, \tag{1.7}$$

where T is the stationary Killing field of (\mathcal{M}, g) , N is a globally timelike and T -invariant vector field on \mathcal{M} , coinciding with T in the asymptotically flat region of \mathcal{M} , and Σ_τ is the image of Σ under the flow of T for time τ .

¹ In general, it is expected that rotating, self-gravitating compressible fluids satisfying the Einstein–Euler system suffer (at the linearised level) from the so-called Chandrasekhar–Friedman–Schutz instability, see [7, 22]. In this case, the associated instability mechanism does not depend on the existence of an ergoregion. However, in the presence of an ergoregion, the linearised Einstein–Euler system is also expected to exhibit an instability similar to Friedman’s instability for Eq. (1.2), associated to the so-called *w-modes* (see [21, 26, 27]). At the non-linear level, it is believed that the effect of the CFS instability will generally dominate the effect of the *w-mode* instability in an astrophysical setting (see [26]).

² Notice that $\partial\mathcal{E}_{ext} \subset \partial\mathcal{E}$.

Remark. Note that the assumption $\mathcal{E} \cap \mathcal{H}^+ = \emptyset$ excludes the Kerr exterior family with angular momentum $a \neq 0$.

For a more detailed statement of Theorem 1.1 and the assumptions on the spacetimes under consideration, see Sect. 2. For a comparison between the heuristics of Friedman in [20] and the methods and results of this paper, see Sect. 9.

Let us remark that, in the detailed statement of Theorem 1.1, we will introduce an additional restriction on the class of spacetimes (\mathcal{M}, g) under consideration, namely the condition that every connected component of $\mathcal{M} \setminus \mathcal{E}$ intersecting \mathcal{H}^+ also intersects the asymptotically flat region of (\mathcal{M}, g) . However, our proof of Theorem 1.1 can be adapted to the case when this condition does not hold.

We should also remark that the unique continuation condition through an open subset of $\partial \mathcal{E}_{ext}$, appearing in the statement of Theorem 1.1, is always satisfied in the case when (\mathcal{M}, g) possesses an axisymmetric Killing field Φ such that the span of T, Φ on $\partial \mathcal{E}_{ext}$ contains a timelike direction, or in the case when the spacetime (\mathcal{M}, g) is real analytic in an open subset $\mathcal{U} \subset \mathcal{M}$ such that $\mathcal{U} \cap \partial \mathcal{E}_{ext} \neq \emptyset$; see the discussion in Sect. 2.3. It would be natural to expect that this condition can be completely removed from the statement of Theorem 1.1, but we have not succeeded so far in doing so.

The proof of Theorem 1.1, presented in Sect. 4, proceeds by contradiction. In particular, assuming that every smooth solution φ of equation (1.2) on (\mathcal{M}, g) with compactly supported initial data satisfies

$$\limsup_{\tau \rightarrow +\infty} \int_{\Sigma_\tau} J_\mu^N(\varphi) n^\mu < +\infty, \tag{1.8}$$

it is shown that φ decays in time on $\mathcal{M} \setminus \mathcal{E}$. This fact is then shown to lead to a contradiction after a suitable choice of the initial data for φ , combined with the unique continuation assumption of Theorem 1.1. See Sect. 4 for more details. The decay of φ on $\mathcal{M} \setminus \mathcal{E}$ is established through some suitable Carleman-type estimates, derived in Sect. 6. These estimates could have been obtained by methods similar to the ones implemented in [30], but we chose instead to provide an alternative proof, based entirely on the method of first order multipliers for Eq. (1.2). For more details on this, see Sect. 2.5.

The instability mechanism proposed by Friedman is of interest not only in general relativity, but also in all areas of mathematical physics where stationary and asymptotically flat Lorentzian manifolds (\mathcal{M}, g) , and the associated wave equation (1.2), arise. For instance, in the field of fluid mechanics, the steady flow of a (locally) irrotational, inviscid and barotropic fluid on an open subset \mathcal{V} of \mathbb{R}^3 gives rise to a stationary Lorentzian metric g on $\mathcal{M} = \mathbb{R} \times \mathcal{V}$, the so called *acoustical* metric, and the wave equation (1.2) associated to g governs the evolution of small perturbations of the flow. In [33], the authors investigate numerically the Friedman instability for the acoustic wave equation on the hydrodynamic vortex $(\mathbb{R} \times \mathcal{V}_{hyd,\delta}, g_{hyd})$, where $\mathcal{V}_{hyd,\delta} = \mathbb{R}^3 \setminus \{\bar{r} \leq \delta\}$ for some $\delta \ll 1$ (in the cylindrical (\bar{r}, ϑ, z) coordinate system) and

$$g_{hyd} = -\left(1 - \frac{C^2}{\bar{r}^2}\right) dt^2 + d\bar{r}^2 - 2C dt d\vartheta + \bar{r}^2 d\vartheta^2 + dz^2, \tag{1.9}$$

with suitable boundary conditions imposed for (1.2) at $\bar{r} = \delta$. Note that the quotient of $(\mathbb{R} \times \mathcal{V}_{hyd}, g_{hyd})$ by the group of translations in the z direction is asymptotically flat, possesses a non-empty ergoregion $\mathcal{E} = \{\delta < \bar{r} \leq C\}$ (corresponding to the region where the fluid velocity exceeds the speed of sound) and has no event horizon.

As a straightforward application of Theorem 1.1, we will establish a Friedman-type instability for the acoustical wave equation on the hydronic vortex:

Corollary 1.1. *For any $\delta < 1$, there exist smooth and z -invariant solutions φ_D, φ_N to the acoustical wave equation (1.2) on $(\mathbb{R} \times \mathcal{V}_{\text{hyd},\delta}, g_{\text{hyd}})$, satisfying Dirichlet and Neumann boundary conditions, respectively, on $\{\bar{r} = \delta\}$, with smooth initial data at time $t = 0$ which are compactly supported when restricted on $\{z = 0\}$, such that (in the $(t, \bar{r}, \vartheta, z)$ coordinate chart on $\mathbb{R} \times \mathcal{V}_{\text{hyd},\delta}$):*

$$\limsup_{\tau \rightarrow +\infty} \int_{\{t=\tau\} \cap \{z=0\} \cap \{\bar{r} \geq \delta\}} (|\partial_t \varphi_D|^2 + |\nabla_{\mathbb{R}^3} \varphi_D|^2) \bar{r} d\bar{r} d\vartheta = +\infty \tag{1.10}$$

and

$$\limsup_{\tau \rightarrow +\infty} \int_{\{t=\tau\} \cap \{z=0\} \cap \{\bar{r} \geq \delta\}} (|\partial_t \varphi_N|^2 + |\nabla_{\mathbb{R}^3} \varphi_N|^2) \bar{r} d\bar{r} d\vartheta = +\infty. \tag{1.11}$$

For a more detailed statement of Corollary 1.1, see Sect. 2.2.

Remark. While Corollary 1.1 provides an instability statement for both Dirichlet and Neumann boundary conditions for φ on $\{\bar{r} = \delta\}$, only the Neumann condition is relevant for the case when $\{\bar{r} = \delta\}$ represents a physical boundary for the fluid flow, since in this case the fluid velocity is only allowed to be tangential to $\{\bar{r} = \delta\}$.

2. Statement of the Main Results

In this section, we will outline in detail the assumptions on the spacetimes (\mathcal{M}, g) under consideration, and we will state the main results of this paper.

2.1. Assumptions on the spacetimes under consideration. Let (\mathcal{M}^{d+1}, g) , $d \geq 2$, be a smooth, globally hyperbolic Lorentzian manifold with piecewise smooth boundary $\partial\mathcal{M}$ (allowed to be empty). Before stating our main results, we will need to introduce a number of assumptions on the structure of (\mathcal{M}, g) . In Sect. 2.4, we will present some explicit examples of spacetimes (\mathcal{M}, g) satisfying all the assumptions that will be introduced in this section.

2.1.1. Assumption G1 (Asymptotic flatness and stationarity). We will assume that (\mathcal{M}, g) satisfies the following conditions:

- There exists a Killing field T on (\mathcal{M}, g) with complete orbits which is tangential to $\partial\mathcal{M}$, as well as a smooth Cauchy hypersurface $\tilde{\Sigma} \subset \mathcal{M}$, such that $T|_{\tilde{\Sigma}}$ is everywhere transversal to $\tilde{\Sigma} \setminus \partial\mathcal{M}$ and timelike outside a compact subset of $\tilde{\Sigma}$.
- The triad $(\tilde{\Sigma}, g_{\tilde{\Sigma}}, k_{\tilde{\Sigma}})$, where $g_{\tilde{\Sigma}}$ is the induced (Riemannian) metric on $\tilde{\Sigma}$ and $k_{\tilde{\Sigma}}$ its second fundamental form, defines an asymptotically flat Riemannian manifold (possibly with boundary $\tilde{\Sigma} \cap \partial\mathcal{M}$), with a finite number of asymptotically flat ends (possibly more than one); see also the definition in Section 2.1.1 of [30]. Let \mathcal{I}_{as} be the asymptotically flat region of \mathcal{M} (see [30] for the relevant definition). Expressed in a polar coordinate chart of the form (t, r, σ) in each connected component of \mathcal{I}_{as} , g has the following form:

$$g = -\left(1 + O_4(r^{-1})\right) dt^2 + \left(1 + O_4(r^{-1})\right) dr^2 + r^2 \cdot \left(g_{\mathbb{S}^{d-1}} + O_4^{\mathbb{S}^{d-1}}(r^{-1})\right) + O_4(1) dt d\sigma, \tag{2.1}$$

where $O_4^{\mathbb{S}^{d-1}}(\rho^{-1})$ is a symmetric $(0, 2)$ -tensor field on the coordinate sphere $\{r = \rho\} \simeq \mathbb{S}^{d-1}$ with $O_4(\rho^{-1})$ asymptotics as $\rho \rightarrow +\infty$. See Sect. 3.7 for the $O_k(\cdot)$, $O_k^{\mathbb{S}^{d-1}}(\cdot)$ notation and Sect. 3.6 for the σ notation on the angular variables of a polar coordinate chart.

- Let $\mathcal{H} = \partial(J^+(\mathcal{I}_{as}) \cap J^-(\mathcal{I}_{as}))$ be the horizon of \mathcal{M} , split as $\mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^-$, with $\mathcal{H}^+ = \overline{J^+(\mathcal{I}_{as})} \cap \partial J^-(\mathcal{I}_{as})$ and $\mathcal{H}^- = \overline{J^-(\mathcal{I}_{as})} \cap \partial J^+(\mathcal{I}_{as})$. Then \mathcal{H} coincides with $\partial\mathcal{M}$, and \mathcal{H}^+ and \mathcal{H}^- are smooth null hypersurfaces with smooth boundary $\mathcal{H}^+ \cap \mathcal{H}^-$, with $T \neq 0$ on $\mathcal{H} \setminus \mathcal{H}^-$ (the case $\mathcal{H}^+ = \emptyset$ or $\mathcal{H}^- = \emptyset$ is also trivially included in this condition).

See Assumption 1 in Section 2.1.1 of [30] for a detailed statement of these conditions and their related geometric constructions, as well as their implications on the geometry of \mathcal{M} . Notice that the domain of outer communications of the asymptotically flat region \mathcal{I}_{as} of \mathcal{M} is the whole of $\mathcal{M} \setminus \mathcal{H}$. In view of the remarks in Section 2.1.1 of [30], \mathcal{H}^+ and \mathcal{H}^- are invariant under the flow of the stationary Killing field T .

Let Σ be a spacelike hypersurface intersecting \mathcal{H}^+ transversally (if $\mathcal{H}^+ \neq \emptyset$) and satisfying $\Sigma \cap \mathcal{H}^- = \emptyset$, such that Σ coincides with $\tilde{\Sigma}$ outside a small neighborhood of \mathcal{H}^+ . In view of the remarks in Section 2.1.1 of [30], our assumption on the structure of (\mathcal{M}, g) implies that $\Sigma \cap \mathcal{H}^+$ is compact. Notice that, in case $\mathcal{H}^+ \neq \emptyset$, Σ will not be a Cauchy hypersurface of \mathcal{M} .

We will also fix a smooth spacelike hyperboloidal hypersurface $\mathcal{S} \subset \mathcal{M}$ terminating at future null infinity \mathcal{I}^+ as in Section 2.1.1 of [30], such that $\mathcal{S}|_{\{r \leq R_1\}} \equiv \Sigma|_{\{r \leq R_1\}}$ for some fixed constant $R_1 \gg 1$ (Fig. 1).

Remark. In view of the remarks of Section 2.1.1 of [30], the causal future sets $J^+(\Sigma)$, $J^+(\mathcal{S})$ of Σ , \mathcal{S} , respectively, in \mathcal{M} coincide with the future domains of depen-

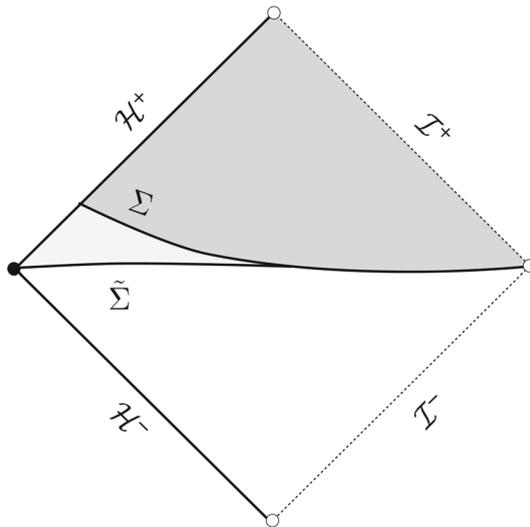


Fig. 1. The subextremal Kerr exterior spacetime $(\mathcal{M}_{M,a}, g_{M,a})$ satisfies Assumptions G1 and G2, but not Assumption G3. In the case of $(\mathcal{M}_{M,a}, g_{M,a})$, the intersection of the hypersurfaces $\tilde{\Sigma}$, Σ , defined in Assumption G1, with the $1 + 1$ dimensional slice $\{\theta = \pi/2, \phi = 0\} \subset \mathcal{M}_{M,a}$ are schematically as depicted above

dence $\mathcal{D}^+(\Sigma), \mathcal{D}^+(\mathcal{S})$ of Σ, \mathcal{S} . Furthermore, the images of Σ, \mathcal{S} under the flow of T covers the whole of $\mathcal{M} \setminus \mathcal{H}^-$.

As in Section 2.1.1 of [30], we will extend the polar radius coordinate function $r : \mathcal{I}_{as} \rightarrow (0, +\infty)$ as a non-negative, smooth and T -invariant function on the whole of $\mathcal{M} \setminus \mathcal{H}^-$, such that $r > 0$ on $\mathcal{M} \setminus \mathcal{H}$ and $r|_{\mathcal{H}^+ \setminus \mathcal{H}^-} = 0, dr|_{\mathcal{H}^+ \setminus \mathcal{H}^-} \neq 0$. We will also define the function $t : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{R}$ by the relations

$$t|_{\Sigma} = 0 \quad \text{and} \quad T(t) = 1, \tag{2.2}$$

as well as the function $\bar{t} : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{R}$ by

$$\bar{t}|_{\mathcal{S}} = 0 \quad \text{and} \quad T(\bar{t}) = 1. \tag{2.3}$$

Note that $t = \bar{t}$ on $\{r \leq R_1\}$.

We will introduce the reference Riemannian metric

$$g_{ref} \doteq dt^2 + g_{\Sigma} \tag{2.4}$$

on $\mathcal{M} \setminus \mathcal{H}^- \simeq \mathbb{R} \times \Sigma$. We will denote the natural extension of g_{ref} on $\oplus_{l_1, l_2 \in \mathbb{N}} (\otimes^{l_1} T(\mathcal{M} \setminus \mathcal{H}^-) \otimes^{l_2} T^*(\mathcal{M} \setminus \mathcal{H}^-))$ also as g_{ref} .

2.1.2. Assumption G2 (Killing horizon with positive surface gravity). In the case $\mathcal{H}^+ \neq \emptyset$, we will assume that the Killing field T , when restricted to $\mathcal{H}^+ \setminus \mathcal{H}^-$, is parallel to the null generators of $\mathcal{H}^+ \setminus \mathcal{H}^-$. Furthermore, we will assume that there exists a T -invariant strictly timelike vector field N on $J^+(\Sigma)$, which, when restricted on $J^+(\Sigma) \cap \mathcal{H}$, satisfies

$$K^N(\psi) \geq c J_{\mu}^N(\psi) N^{\mu} \tag{2.5}$$

for some $c > 0$ and any $\psi \in C^1(\mathcal{M})$ (see Sect. 3 for the notation on vector field currents). We will extend N on the whole of $\mathcal{M} \setminus \mathcal{H}^-$ by the condition $[T, N] = 0$.

We will call the vector field N the *red shift* vector field. The reason for this name is that a vector field of that form was shown to exist for a general class of Killing horizons with positive surface gravity by Dafermos and Rodnianski in [13]. However, here we will just assume the existence of such a vector field without specifying the geometric origin of it.

Note that we can modify the vector field N away from the horizon \mathcal{H} , so that in the asymptotically flat region $\{r \gg 1\}$ (i.e. \mathcal{I}_{as}) it coincides with T , and still retain the bound (2.5) on $J^+(\Sigma) \cap \mathcal{H}$.³ We will hence assume without loss of generality that N has been chosen so that $N \equiv T$ in the region $\{r \gg 1\}$.

Due to the smoothness of N , there exists an $r_0 > 0$, such that (2.5) also holds (possibly with a smaller constant c on the right hand side) in a neighborhood of $\mathcal{H}^+ \setminus (\mathcal{H}^+ \cap \mathcal{H}^-)$ in \mathcal{M} of the form $\{r \leq r_0\}$. For $r \gg 1$, since $N \equiv T$ there, we have $K^T(\psi) \equiv 0$. Hence, due to the T -invariance of N and the compactness of the sets of the form $\{r \leq R\} \cap \Sigma$, there exists a (possibly large) constant $C > 0$ such that

$$|K^N(\psi)| \leq C \cdot J_{\mu}^N(\psi) N^{\mu} \tag{2.6}$$

everywhere on \mathcal{M} for any $\psi \in C^{\infty}(\mathcal{M})$.

³ The convexity of the cone of the future timelike vectors over each point of \mathcal{M} is used in this argument.

Without loss of generality, we will also assume that r_0 is sufficiently small so that

$$dr \neq 0 \tag{2.7}$$

on $\{r \leq 3r_0\}$. This is possible, since $dr|_{\mathcal{H}^+ \setminus \mathcal{H}^+ \cap \mathcal{H}^-} \neq 0$ and $\Sigma \cap \mathcal{H}^+$ is compact.

In the case $\mathcal{H}^+ = \emptyset$, we will fix N to be an arbitrary T -invariant timelike vector field on $\mathcal{M} \setminus \mathcal{H}^-$, such that $N \equiv T$ for $r \gg 1$, and we will set $r_0 = \frac{1}{4} \inf_{\Sigma} r$ (which is possible since $r > 0$ on Σ when $\mathcal{H}^+ = \emptyset$), so that $\{r \leq 3r_0\} = \emptyset$. In this case, (2.5), (2.6) and (2.7) are trivially satisfied.

2.1.3. Assumption G3 (Non-empty ergoregion avoiding the future event horizon). We will assume that the ergoregion of (\mathcal{M}, g) is non-empty, i.e.

$$\mathcal{E} \doteq \overline{\{g(T, T) > 0\}} \neq \emptyset, \tag{2.8}$$

and furthermore

$$\mathcal{E} \cap \mathcal{H}^+ = \emptyset. \tag{2.9}$$

Notice that the condition (2.9) is trivially satisfied when $\mathcal{H}^+ = \emptyset$. Note also that the subextremal Kerr exterior family with $a \neq 0$ has a non-empty ergoregion, but does not satisfy (2.9).

In the case when $\mathcal{H}^+ \neq \emptyset$, we will also assume that every connected component of $\mathcal{M} \setminus \mathcal{E}$ that intersects \mathcal{H}^+ also intersects the asymptotically flat region \mathcal{I}_{as} of (\mathcal{M}, g) .⁴

Remark. The assumption that every component of $\mathcal{M} \setminus \mathcal{E}$ intersecting \mathcal{H}^+ also intersects \mathcal{I}_{as} is not necessary for the results of this paper, which can also be established without this condition. The reason for adopting this assumption is that it leads to considerable simplifications in the proof of the Carleman-type estimates in Sect. 6.

We will assume that T is strictly timelike on the complement of $\mathcal{H} \cup \mathcal{E}$, i.e.:

$$g(T, T) < 0 \text{ on } \mathcal{M} \setminus (\mathcal{E} \cup \mathcal{H}). \tag{2.10}$$

Furthermore, we will assume that the boundary $\partial \mathcal{E}$ of \mathcal{E} is a smooth hypersurface of \mathcal{M} .

The complement $\mathcal{M} \setminus \mathcal{E}$ of \mathcal{E} might consist of more than one components. In view of our assumption that every component of $\mathcal{M} \setminus \mathcal{E}$ intersecting \mathcal{H}^+ also intersects \mathcal{I}_{as} , the connected components of $\mathcal{M} \setminus \mathcal{E}$ fall into two disjoint categories: The ones that intersect the asymptotically flat region \mathcal{I}_{as} and the future event horizon \mathcal{H}^+ , and the ones that intersect neither \mathcal{I}_{as} nor \mathcal{H}^+ . Let us call the union of the components of $\mathcal{M} \setminus \mathcal{E}$ falling into the last category the *enclosed* region of \mathcal{M} , and denote it by \mathcal{M}_{enc} .⁵ We will also introduce the notion of the *extended* ergoregion of \mathcal{M} defined by

$$\mathcal{E}_{ext} \doteq \mathcal{E} \cup \mathcal{M}_{enc}. \tag{2.11}$$

Note that, since $\mathcal{E}_{ext} \cap \mathcal{H}^+ = \emptyset$, we have $r > 0$ on \mathcal{E}_{ext} . Thus, in view of the T -invariance of \mathcal{E}_{ext} , we can assume without loss of generality that r_0 has been fixed sufficiently small so that $\{r \leq r_0\} \cap \mathcal{E}_{ext} = \emptyset$. Note also that $\partial \mathcal{E}_{ext} \subseteq \partial \mathcal{E}$.

⁴ Note that \mathcal{I}_{as} and \mathcal{H}^+ might have several components.

⁵ The reason for calling \mathcal{M}_{enc} enclosed is because $\partial \mathcal{M}_{enc} \subset \partial \mathcal{E}$, i.e. \mathcal{M}_{enc} is enclosed by the ergoregion.

2.1.4. Assumption A1 (Unique continuation around $p \in \partial \mathcal{E}_{ext}$). We will assume that there exists a point p on the boundary $\partial \mathcal{E}_{ext}$ of \mathcal{E}_{ext} and an open neighborhood \mathcal{U} of p in \mathcal{M} , such that for any $\psi \in H^1_{loc}(\mathcal{M} \setminus \mathcal{H}^-)$ solving the wave equation (1.2) and satisfying $\psi \equiv 0$ on $\mathcal{M} \setminus \mathcal{E}_{ext}$, we also have $\psi \equiv 0$ on \mathcal{U} . Since T is a Killing field of \mathcal{M} , the same result also holds on any T -translate of \mathcal{U} , and, for this reason, we will assume without loss of generality that \mathcal{U} is T -invariant. Furthermore, since $\partial \mathcal{E}_{ext} \subset \partial \mathcal{E}$, we will assume without loss of generality that \mathcal{U} is small enough so that $\mathcal{U} \cap \mathcal{E}_{ext} \subset \mathcal{E}$.

Remark. Assumption A1 is satisfied in the case when \mathcal{M} is axisymmetric with axisymmetric Killing field Φ , such that $[\Phi, T] = 0$ and the span of $\{\Phi, T\}$ is timelike, or in the case when there exists a point $p \in \partial \mathcal{E}$ such that g is real analytic on an open neighborhood of p in \mathcal{M} . See Sect. 2.3.

2.2. *The main results.* The main result of this paper is the following:

Theorem 2.1. *Let (\mathcal{M}^{d+1}, g) , $d \geq 2$, be a globally hyperbolic Lorentzian manifold satisfying Assumptions G1, G2, G3 and A1, and let the vector field T, N and the spacelike hypersurface Σ be as described in Assumptions G1-G2. Then, there exists a smooth function $\varphi : J^+(\Sigma) \rightarrow \mathbb{C}$ solving the wave equation (1.2) on $J^+(\Sigma)$ with compactly supported initial data on Σ , such that*

$$\limsup_{\tau \rightarrow +\infty} \int_{\Sigma_\tau} J_\mu^N(\varphi) n^\mu = +\infty. \tag{2.12}$$

The proof of Theorem 2.1 will be presented in Sect. 4.

Remark. The proof of Theorem 2.1 immediately generalises to the case when the boundary of the spacetime (\mathcal{M}, g) has a smooth, timelike and T -invariant component $\partial_{tim} \mathcal{M}$, such that $\Sigma \cap \partial_{tim} \mathcal{M}$ is compact and $\partial_{tim} \mathcal{M} \cap \mathcal{H} = \emptyset$, assuming that Dirichlet or Neumann boundary conditions are imposed for equation (1.2) on $\partial_{tim} \mathcal{M}$. In this case, we have to assume that the double $(\tilde{\mathcal{M}}, \tilde{g})$ of (\mathcal{M}, g) across $\partial_{tim} \mathcal{M}$ is a globally hyperbolic spacetime satisfying Assumptions G1, G2, G3 and A1 (see Sect. 6.9 for the relevant constructions).

Let us also note that we can readily replace the qualitative instability statement (2.12) with the following quantitative statement: For any $C > 0$, there exists a solution φ to equation (1.2) as in the statement of Theorem 2.1, such that

$$\limsup_{\tau \rightarrow +\infty} \left((\log(2 + \tau))^{-C} \int_{\Sigma_\tau} J_\mu^N(\varphi) n^\mu \right) = +\infty. \tag{2.13}$$

See the remark at the beginning of Sect. 4. However, we do not expect the logarithmic rate of growth in (2.13) to be sharp.

As a straightforward application of Theorem 2.1, we will obtain the following instability estimate for solutions to the acoustical wave equation on the hydronamic vortex $(\mathbb{R} \times \mathcal{V}_{hyd,\delta}, g_{hyd})$, where $\mathcal{V}_{hyd,\delta} \subset \mathbb{R}^3$ is the set $\{\bar{r} \geq \delta\}$ (in the cylindrical (\bar{r}, ϑ, z) coordinate system on \mathbb{R}^3) and g_{hyd} is given by the expression (1.9):

Corollary 2.1. *For any $\delta < 1$, there exist smooth and z -invariant solutions φ_D, φ_N to (1.2) on $(\mathbb{R} \times \mathcal{V}_{hyd,\delta}, g_{hyd})$, satisfying the boundary conditions*

$$\varphi_D|_{\{\bar{r}=\delta\}} = 0 \tag{2.14}$$

and

$$\partial_{\bar{r}} \varphi_N|_{\{\bar{r}=\delta\}} = 0 \tag{2.15}$$

and having smooth initial data at time $t = 0$ which are compactly supported when restricted on $\{z = 0\}$, such that (in the $(t, \bar{r}, \vartheta, z)$ coordinate chart on $\mathbb{R} \times \mathcal{V}_{\text{hyd},\delta}$):

$$\limsup_{\tau \rightarrow +\infty} \int_{\{t=\tau\} \cap \{z=0\} \cap \{\bar{r} \geq \delta\}} (|\partial_t \varphi_D|^2 + |\nabla_{\mathbb{R}^3} \varphi_D|^2) \bar{r} d\bar{r} d\vartheta = +\infty \tag{2.16}$$

and

$$\limsup_{\tau \rightarrow +\infty} \int_{\{t=\tau\} \cap \{z=0\} \cap \{\bar{r} \geq \delta\}} (|\partial_t \varphi_N|^2 + |\nabla_{\mathbb{R}^3} \varphi_N|^2) \bar{r} d\bar{r} d\vartheta = +\infty. \tag{2.17}$$

For the proof of Corollary 2.1, see Sect. 8.

2.3. *Discussion on Assumption A1.* There exists a class of natural geometric conditions, such that spacetimes (\mathcal{M}, g) satisfying these conditions (in addition to Assumptions G1–G3) automatically satisfy Assumption A1. Examples of such conditions are the following:

- Assumption A1 is always satisfied on spacetimes (\mathcal{M}, g) having an axisymmetric Killing field Φ , such that $[\Phi, T] = 0$ and the span of Φ, T on $\partial \mathcal{E}_{\text{ext}}$ contains a timelike direction. This is a consequence of Lemma 2.1 at the end of this section (choosing \mathcal{U} to be a suitable small neighborhood of a point $p \in \partial \mathcal{E}_{\text{ext}} \setminus \{\Phi = 0\}$ and $\mathcal{S} = \partial \mathcal{E}_{\text{ext}} \cap \mathcal{U}$ in the statement of Lemma 2.1).
- Assumption A1 is always satisfied on spacetimes (\mathcal{M}, g) on which there exists a point $p \in \partial \mathcal{E}_{\text{ext}}$ and an open neighborhood \mathcal{U} of p such that (\mathcal{U}, g) is a real analytic Lorentzian manifold and $\partial \mathcal{E}_{\text{ext}} \cap \mathcal{U}$ is a real analytic hypersurface. This is a consequence of Holmgren’s uniqueness theorem (see [24]).

On the other hand, we believe that Assumption A1 does not hold on all spacetimes satisfying Assumptions G1–G3. In particular, by adjusting the arguments of [1], we were able to construct a 3+1-dimensional spacetime (\mathcal{M}, g) , satisfying Assumptions G1–G3, as well as a suitable T -invariant smooth potential $V : \mathcal{M} \rightarrow \mathbb{C}$, so that Assumption A1 for equation

$$\square_g \varphi - V \varphi = 0 \tag{2.18}$$

in place of (1.2) is not satisfied.⁶ Note that such a construction is non-trivial, in view of the requirement that $T(V) = 0$; for instance, Assumption A1 always holds for equation (2.18) on stationary spacetimes without an ergoregion (see [37]). We will not pursue this issue any further in this paper.

Although we believe that Assumption A1 can be removed from the statement of Theorem 2.1, we were not able to do so.

The following lemma can be used to establish that Assumption A1 always holds in the presence of a second Killing field Φ on \mathcal{M} such that the span of T, Φ is timelike:

⁶ In view of the requirement $T(V) = 0$, this construction boils down to the adaptation of the construction of [1] on the quotient \mathcal{M}/T of \mathcal{M} by the orbits of T , with \square_g replaced by $\square_{g/T}$, where $\square_{g/T}$ is obtained from \square_g by dropping the terms involving T derivatives. Note that $\square_{g/T}$ is hyperbolic in $\text{int}(\mathcal{E})/T$ and elliptic on $(\mathcal{M} \setminus \mathcal{E})/T$.

Lemma 2.1. *Let \mathcal{U} be an open subset of a smooth spacetime (\mathcal{M}, g) with two Killing fields T, Φ such that $[T, \Phi] = 0$, $\Phi \neq 0$ on \mathcal{U} and the span of T, Φ contains a timelike direction everywhere on \mathcal{U} . Let also $\mathcal{S} \subset \mathcal{U}$ be a T, Φ -invariant smooth hypersurface, separating \mathcal{U} into two connected components $\mathcal{U}_1, \mathcal{U}_2$, so that \mathcal{U}_2 lies in the domain of dependence of \mathcal{U}_1 . Then, any $\psi \in H^2_{loc}(\mathcal{U})$ solving (1.2) on (\mathcal{U}, g) such that $\psi \equiv 0$ on \mathcal{U}_1 must vanish everywhere on \mathcal{U} .*

Proof. Since $[T, \Phi] = 0$ and $\Phi \neq 0$ on \mathcal{U} , we can assume without loss of generality (by shrinking \mathcal{U} if necessary) that \mathcal{U} is covered by a coordinate chart $(t, \varphi, x^2, \dots, x^d)$ such that:

1. $T(\varphi), T(x^2), \dots, T(x^d) = 0$,
2. $\Phi(t), \Phi(x^2), \dots, \Phi(x^d) = 0$,
3. $T(t) = \Phi(\varphi) = 1$,
4. $\mathcal{S} = \{x^1 = 0\}$.

In view of the fact that the span of Φ, T contains a timelike direction everywhere on \mathcal{U} , the wave operator (1.2) in the $(t, \varphi, x^2, \dots, x^d)$ coordinate system takes the form (using the shorthand notation $x = (x^2, \dots, x^d)$):

$$\begin{aligned} \square_g \psi &= \Delta_x \psi + \sum_{j=2}^d (a_{tj}^{(2)}(x) \partial_{x^j} \partial_t \psi + a_{\varphi j}^{(2)}(x) \partial_{x^j} \partial_\varphi \psi + a_{tj}^{(1)}(x) \partial_t \psi + a_{\varphi j}^{(1)}(x) \partial_\varphi \psi) \\ &\quad + a_{tt}(x) \partial_t^2 \psi + a_{t\varphi}(x) \partial_t \partial_\varphi \psi + a_{\varphi\varphi}(x) \partial_\varphi^2 \psi, \end{aligned} \tag{2.19}$$

where the operator Δ_x in the right hand side of (2.19) is a t, φ -invariant second order elliptic operator in the x^2, \dots, x^d variables. Since the coefficients of (2.19) are independent of t, φ and Δ_x is elliptic, the proof of the Lemma follows readily by the unique continuation result of Tataru [37]. \square

2.4. Examples of spacetimes satisfying Assumptions G1–G3 and A1. In this section, we will examine some explicit examples of spacetimes satisfying all of the Assumptions G1–G3 and A1.

An example with $\mathcal{H}^+ = \emptyset$. Our first example will be a simple spacetime with no event horizon. Let $\mathcal{M} = \mathbb{R}^{3+1}$, and let us fix two smooth functions $\chi_{\bar{r}} : [0, +\infty) \rightarrow [0, 1]$ and $\chi_\vartheta : [0, \pi] \rightarrow [0, 1]$, satisfying $\chi_{\bar{r}} \equiv 0$ on $[0, 3] \cup [6, +\infty)$, $\chi_{\bar{r}} \equiv 1$ on $[4, 5]$, $\chi_\vartheta \equiv 0$ on $[0, \frac{\pi}{6}] \cup [\frac{5\pi}{6}, \pi]$ and $\chi_\vartheta \equiv 1$ on $[\frac{\pi}{4}, \frac{3\pi}{4}]$. We will also assume that $\chi_{\bar{r}}, \chi_\vartheta$ have been chosen so that the set of zeros of the function

$$f(\bar{r}, \vartheta) \doteq 1 - (\chi_{\bar{r}}(\bar{r}) \chi_\vartheta(\vartheta))^2 \tag{2.20}$$

is a smooth curve without self-intersections in the open rectangle $(3, 6) \times (\frac{\pi}{6}, \frac{5\pi}{6})$, and the region $\{f \leq 0\} \subset (3, 6) \times (\frac{\pi}{6}, \frac{5\pi}{6})$ is simply connected.

We will consider the following metric on \mathcal{M} in the usual time-polar coordinate chart $(t, \bar{r}, \vartheta, \varphi)$ on \mathbb{R}^{3+1} :

$$g = -(1 - (\chi_{\bar{r}}(\bar{r}) \chi_\vartheta(\vartheta))^2) dt^2 - 1000 \chi_{\bar{r}}(\bar{r}) \chi_\vartheta(\vartheta) dt d\varphi + d\bar{r}^2 + \bar{r}^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2). \tag{2.21}$$

Note that g is everywhere non-degenerate, and has Lorentzian signature. Furthermore, (\mathcal{M}, g) is a globally hyperbolic spacetime, with Cauchy hypersurface $\{t = 0\}$, satisfying the following properties:

1. The vector field $T = \partial_t$ is a Killing field of (\mathcal{M}, g) . Furthermore, (\mathcal{M}, g) is asymptotically flat and satisfies Assumption G1. Notice that (\mathcal{M}, g) has no event horizon, since every point in \mathcal{M} can be connected with the asymptotically flat region $\mathcal{I}_{as} = \{\bar{r} \geq R_0 \gg 1\}$ through both a future directed and a past directed timelike curve, by following the flow of the timelike vector fields $\partial_{\bar{r}} + C(\partial_t + \frac{1}{10}\chi_{\bar{r}}(\bar{r})\chi_{\vartheta}(\vartheta)\partial_{\varphi})$ and $\partial_{\bar{r}} - C(\partial_t + \frac{1}{10}\chi_{\bar{r}}(\bar{r})\chi_{\vartheta}(\vartheta)\partial_{\varphi})$, respectively (for some fixed $C \gg 1$). The function $r : \mathcal{M} \rightarrow [0, +\infty)$, introduced in Assumption G1, can be chosen to be equal to $(1 + \bar{r}^2)^{1/2}$.
2. The spacetime (\mathcal{M}, g) has no event horizon, and, thus, it trivially satisfies Assumption G2.
3. The ergoregion $\mathcal{E} = \{g(T, T) > 0\}$ of (\mathcal{M}, g) is non-empty, and satisfies

$$\{4 \leq \bar{r} \leq 5\} \cap \{\frac{\pi}{4} \leq \vartheta \leq \frac{3\pi}{4}\} \subset \mathcal{E} \subset \{3 \leq \bar{r} \leq 6\} \cap \{\frac{\pi}{6} \leq \vartheta \leq \frac{5\pi}{6}\}. \quad (2.22)$$

Since $\mathcal{H}^+ = \emptyset$, we have $\mathcal{E} \cap \mathcal{H}^+ = \emptyset$. Furthermore, since we assumed that the region $\{f \leq 0\} \subset (3, 6) \times (\frac{\pi}{6}, \frac{5\pi}{6})$ for the function (2.20) is simply connected, we can readily infer that $\mathcal{M} \setminus \mathcal{E}$ is connected and, thus, $\mathcal{E}_{ext} = \mathcal{E}$. Furthermore, the boundary $\partial\mathcal{E}$ of \mathcal{E} is a smooth hypersurface of \mathcal{M} and (2.10) is satisfied, in view of our assumption on the set of zeros of the function (2.20). Therefore, (\mathcal{M}, g) satisfies Assumption G3.

4. The spacetime (\mathcal{M}, g) possesses an additional Killing field, i.e. $\Phi = \partial_{\varphi}$. The span of T, Φ contains the everywhere timelike vector field $T + \frac{1}{10}\chi_{\bar{r}}(\bar{r})\chi_{\vartheta}(\vartheta)\Phi$ and, thus, Lemma 2.1 implies that (\mathcal{M}, g) satisfies Assumption A1. In particular, any point $p \in \partial\mathcal{E}_{ext} \subseteq \partial\mathcal{E}$ and any open neighborhood \mathcal{U} of p in \mathcal{M} satisfy the unique continuation property of Assumption A1.

Therefore, (\mathcal{M}, g) satisfies all of the Assumptions G1–G3 and A1.

Remark. The hydrodynamic vortex $(\mathbb{R} \times \mathcal{V}_{hyd,\delta}, g_{hyd})$ of Corollary 2.1 is not a globally hyperbolic spacetime, since its boundary $\partial(\mathbb{R} \times \mathcal{V}_{hyd,\delta}) = \{\bar{r} = \delta\}$ is a timelike hypersurface. However, as we will show in the proof of Corollary 2.1, the double of $(\mathbb{R} \times \mathcal{V}_{hyd,\delta}, g_{hyd})$ across $\partial(\mathbb{R} \times \mathcal{V}_{hyd,\delta})$ is a globally hyperbolic spacetime without an event horizon, satisfying Assumptions G1–G3 and A1. In addition, the double of $(\mathbb{R} \times \mathcal{V}_{hyd,\delta}, g_{hyd})$ is an example of a spacetime having two asymptotically flat ends, with $(\mathbb{R} \times \mathcal{V}_{hyd,\delta}, g_{hyd}) \setminus \mathcal{E}$ having two connected components.

An example with $\mathcal{H}^+ \neq \emptyset$. We will now proceed to construct a slightly more complicated example of a spacetime satisfying Assumptions G1–G3 and A1, possessing in addition a non-empty event horizon. Note that, as we mentioned in Sect. 2.1, the subextremal Kerr exterior family $(\mathcal{M}_{M,a}, g_{M,a})$ does not satisfy G3, since the future event horizon \mathcal{H}^+ and the ergoregion \mathcal{E} of $(\mathcal{M}_{M,a}, g_{M,a})$ have a non-empty intersection.

For any $M > 0$, let \mathcal{M}_M be diffeomorphic to $\mathbb{R} \times (2M, +\infty) \times \mathbb{S}^2$. Let $\chi_{\bar{r}}, \chi_{\vartheta}$ be as before, assuming, in addition, that they have been chosen so that the set of zeros of the function

$$f_M(\bar{r}, \vartheta) \doteq \left(1 - \frac{2M}{\bar{r}} - (\chi_{\bar{r}}(M^{-1}\bar{r})\chi_{\vartheta}(\vartheta))^2\right) \quad (2.23)$$

is a smooth curve without self-intersections in the open rectangle $(3M, 6M) \times (\frac{\pi}{6}, \frac{5\pi}{6})$ and the region $\{f_M \leq 0\} \subset (3, 6) \times (\frac{\pi}{6}, \frac{5\pi}{6})$ is simply connected.

Let us consider the following metric in the $(t, \bar{r}, \vartheta, \varphi)$ coordinate chart on \mathcal{M}_M :

$$g_M = -\left(1 - \frac{2M}{\bar{r}} - (\chi_{\bar{r}}(M^{-1}\bar{r})\chi_{\vartheta}(\vartheta))^2\right)dt^2 - 1000M\chi_{\bar{r}}(M^{-1}\bar{r})\chi_{\vartheta}(\vartheta)dtd\varphi + \left(1 - \frac{2M}{\bar{r}}\right)^{-1}d\bar{r}^2 + \bar{r}^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2). \tag{2.24}$$

The metric g_M is everywhere non-degenerate, and has Lorentzian signature.

The spacetime (\mathcal{M}_M, g_M) is isometric to the Schwarzschild exterior spacetime $(\mathcal{M}_{M,Sch}, g_{M,Sch})$ outside the region $\{3M \leq \bar{r} \leq 6M\} \cap \{\frac{\pi}{6} \leq \vartheta \leq \frac{5\pi}{6}\}$ and, thus, it can be extended into a larger globally hyperbolic spacetime $(\widetilde{\mathcal{M}}_M, \widetilde{g}_M)$.⁷ This extension can be chosen to be the Schwarzschild maximal extension across $\bar{r} = 2M$ (see e.g. Section 2 [16]).⁸ Let us denote

$$\overline{\mathcal{M}}_M = i(\mathcal{M}_M) \cup \partial\mathcal{M}_M, \tag{2.25}$$

where $i : \mathcal{M}_M \rightarrow \widetilde{\mathcal{M}}_M$ is the natural inclusion of (\mathcal{M}_M, g_M) into its extension and $\partial\mathcal{M}_M$ is the boundary of $i(\mathcal{M}_M)$ inside $\widetilde{\mathcal{M}}_M$. Note that, in view of the properties of the maximally extended Schwarzschild spacetime, $(\overline{\mathcal{M}}_M, g_M)$ is a smooth Lorentzian manifold with piecewise smooth boundary $\partial\mathcal{M}_M$, consisting of two intersecting smooth null hypersurfaces. The functions $\bar{r}, \vartheta, \varphi$ can be smoothly extended on $\partial\mathcal{M}_M$, with $\bar{r}|_{\partial\mathcal{M}_M} = 2M$.

The spacetime $(\overline{\mathcal{M}}_M, g_M)$ is globally hyperbolic, with $\Sigma = \overline{\{t = 0\}}$ being a smooth Cauchy hypersurface, and satisfies the following properties:

1. The vector field $T = \partial_t$ on \mathcal{M}_M extends smoothly on $\partial\mathcal{M}_M$ and is a Killing vector field of $(\overline{\mathcal{M}}_M, g_M)$. Furthermore, the spacetime $(\overline{\mathcal{M}}_M, g_M)$ is asymptotically flat and satisfies Assumption G1. Note that the event horizon \mathcal{H} of $(\overline{\mathcal{M}}_M, g_M)$ coincides with $\partial\mathcal{M}_M$, since all the points in \mathcal{M}_M can be joined with the asymptotically flat region $\mathcal{I}_{as} = \{\bar{r} \geq R_0 \gg 1\}$ through both a future directed and a past directed timelike curve, by following the flow of the timelike vector fields $\partial_{\bar{r}} + C(\partial_t + \frac{1}{10M}\chi_{\bar{r}}(M^{-1}\bar{r})\chi_{\vartheta}(\vartheta)\partial_{\varphi})$ and $\partial_{\bar{r}} - C(\partial_t + \frac{1}{10M}\chi_{\bar{r}}(M^{-1}\bar{r})\chi_{\vartheta}(\vartheta)\partial_{\varphi})$, respectively (for some fixed $C \gg 1$). The function $r : \overline{\mathcal{M}}_M \rightarrow [0, +\infty)$, introduced in Assumption G1, can be chosen to be equal to $\bar{r} - 2M$.
2. There exists a T -invariant neighborhood \mathcal{V} of $\mathcal{H} = \partial\mathcal{M}_M$ in $\overline{\mathcal{M}}_M$, so that (\mathcal{V}, g_M) is isometric to a neighborhood of the event horizon $\mathcal{H}_{M,Sch}$ of Schwarzschild exterior spacetime. In particular, T is parallel to the null generators of $\mathcal{H}^+ \setminus \mathcal{H}^-$ and there exists a T -invariant timelike vector field N on $\overline{\mathcal{M}}_M$ as in Assumption G2, satisfying (2.5) (see [13, 16]). In particular, $(\overline{\mathcal{M}}_M, g_M)$ satisfies Assumption G2.
3. The ergoregion $\mathcal{E} = \{g(T, T) > 0\}$ of $(\overline{\mathcal{M}}_M, g_M)$ is non-empty, and satisfies

$$\{4M \leq \bar{r} \leq 5M\} \cap \left\{\frac{\pi}{4} \leq \vartheta \leq \frac{3\pi}{4}\right\} \subset \mathcal{E} \subset \{3M \leq \bar{r} \leq 6M\} \cap \left\{\frac{\pi}{6} \leq \vartheta \leq \frac{5\pi}{6}\right\}. \tag{2.26}$$

Thus, $\mathcal{E} \cap \mathcal{H}^+ = \emptyset$, since $\bar{r} > 2M$ on \mathcal{E} . Furthermore, since the function (2.23) was assumed to have the property that the set $\{f_M \leq 0\}$ is simply connected, we

⁷ Of course, the coordinate chart $(t, \bar{r}, \vartheta, \varphi)$ will not be regular up to the boundary of \mathcal{M}_M in this extension.

⁸ Note that the \bar{r} coordinate function on \mathcal{M}_M corresponds to the usual r coordinate function on Schwarzschild exterior $\mathcal{M}_{M,Sch}$.

can readily infer that $\overline{\mathcal{M}}_M \setminus \mathcal{E}$ is connected. Thus, $\mathcal{E}_{ext} = \mathcal{E}$ and \mathcal{H}^+ lies in the same connected component of $\overline{\mathcal{M}}_M \setminus \mathcal{E}$ as the asymptotically flat region \mathcal{I}_{as} . The boundary $\partial \mathcal{E}$ of \mathcal{E} is a smooth hypersurface of $\overline{\mathcal{M}}_M$ and (2.10) is satisfied, in view of our assumption on the set of zeros of the function (2.23). Therefore, $(\overline{\mathcal{M}}_M, g_M)$ satisfies Assumption G3.

4. In view of the fact that (\mathcal{M}_M, g_M) possesses an additional Killing field, namely $\Phi = \partial_\varphi$, and the span of T, Φ contains the vector field $T + \frac{1}{10M} \chi_{\bar{r}}(M^{-1}\bar{r})\chi_\vartheta(\vartheta)\Phi$ which is everywhere timelike on \mathcal{M}_M , Lemma 2.1 implies that $(\overline{\mathcal{M}}_M, g_M)$ satisfies Assumption A1. In particular, any point $p \in \partial \mathcal{E}_{ext} = \partial \mathcal{E}$ and any open neighborhood \mathcal{U} of p satisfy the unique continuation property of Assumption A1.

Thus, $(\overline{\mathcal{M}}_M, g_M)$ satisfies Assumptions G1–G3 and A1, and, in addition, $(\overline{\mathcal{M}}_M, g_M)$ has a non-empty future event horizon.

2.5. *A remark on the Carleman-type estimates in the proof of Theorem 2.1.* As we discussed in the introduction, a crucial step in the proof of Theorem 2.1 consists of showing that, under the assumption that

$$\limsup_{\tau \rightarrow +\infty} \int_{\Sigma_\tau} J_\mu^N(\varphi) n^\mu < +\infty \tag{2.27}$$

holds for every smooth function $\varphi : J^+(\Sigma) \rightarrow \mathbb{C}$ solving the wave equation (1.2) on $J^+(\Sigma)$ with compactly supported initial data on Σ , we also have that φ decays on $\mathcal{M} \setminus \mathcal{E}$; see Sect. 4 for more details. This fact is inferred using some suitable Carleman-type estimates on $(\mathcal{M} \setminus \mathcal{E}, g)$ for φ which are particularly useful when φ has localised frequency support in time (see Proposition 6.1 in Sect. 6; for the technical details related to the frequency decomposition of φ , see Sect. 5).

The aforementioned estimates could have been established using the techniques of our previous [30].⁹ However, we chose, instead, to provide an alternative proof, based entirely on the use of first order multipliers for equation (1.2). As a consequence, we obtain an alternative proof for the estimates of Section 7.1 of [30], as well as for the Carleman-type estimates established in [34] for the inhomogeneous Helmholtz equation

$$\Delta_{\bar{g}} u + \omega^2 u - V u = G, \tag{2.28}$$

$0 < Im(\omega) \ll 1, Re(\omega) \neq 0$, on an asymptotically conic Riemannian manifold (Σ, \bar{g}) , where the potential $V : \Sigma \rightarrow \mathbb{R}$ satisfies some suitable decay conditions on the asymptotically conic end of Σ . An alternative approach based on the positive commutator method and leading to similar Carleman-type inequalities on asymptotically flat backgrounds with a uniformly timelike Killing field has also been implemented in the recent [29]. For a more detailed statement of these results, see Sect. 6.

3. Notational Conventions and Hardy Inequalities

In this section, we will introduce some conventions on denoting constants and parameters that will appear throughout this paper. We will adopt similar conventions as in [30].

⁹ Although the techniques of [30] can be modified to yield the required Carleman-type estimates on the spacetimes under consideration, they can not be used as a black box in the current setting, since the geometry of the spacetime in a neighborhood of \mathcal{E} and the allowed range in the space dimension d (which includes the case $d = 2$) differ from the setting of [30].

3.1. *Constants and dependence on parameters.* We will adopt the following convention for denoting constants appearing in inequalities: We will use capital letters (e.g. C) to denote “large” constants, typically appearing on the right hand side of inequalities. (Such constants can be “freely” replaced by larger ones without rendering the inequality invalid.) Lower case letters (e.g. c) will be used to denote “small” constants (which can similarly freely be replaced by smaller ones). The same characters will be frequently used to denote different constants, even in adjacent lines.

We will assume that all non-explicit constants will depend on the specific geometric aspects of (\mathcal{M}, g) and we will not keep track of this dependence, except for some very specific cases. However, since we will introduce a plethora of parameters throughout this paper, we will always keep track of the dependence of all constants on each of these parameters. Once a parameter is fixed (which will be clearly stated in the text), the dependence of constants on it will be dropped.

3.2. *Inequality symbols.* We will use the notation $f_1 \lesssim f_2$ for two real functions f_1, f_2 as usual to imply that there exists some $C > 0$, such that $f_1 \leq C f_2$. This constant C might depend on free parameters, and these parameters will be stated clearly in each case. If nothing is stated regarding the dependence of this constant on parameters, it should be assumed that it only depends on the geometry of the spacetime (\mathcal{M}, g) under consideration.

We will denote $f_1 \sim f_2$ when we can bound $f_1 \lesssim f_2$ and $f_2 \lesssim f_1$. The notation $f_1 \ll f_2$ will be equivalent to the statement that $\frac{|f_1|}{|f_2|}$ can be bounded by some sufficiently small positive constant, the magnitude and the dependence of which on variable parameters will be clear in each case from the context. For any function $f : \mathcal{M} \rightarrow [0, +\infty)$, $\{f \gg 1\}$ will denote the subset $\{f \geq C\}$ of \mathcal{M} for some constant $C \gg 1$.

For functions $f_1, f_2 : [x_0, +\infty) \rightarrow \mathbb{R}$, the notation $f_1 = o(f_2)$ will mean that $\frac{|f_1|}{|f_2|}$ can be bounded by some continuous function $h : [x_0, +\infty) \rightarrow (0, +\infty)$ such that $h(x) \rightarrow 0$ as $x \rightarrow +\infty$. This bound h might depend on free parameters, and this fact will be clear in each case from the context.

3.3. *Some special subsets of \mathcal{M} .* The future event horizon of \mathcal{M} will be denoted by \mathcal{H}^+ , and the past event horizon by \mathcal{H}^- , i.e.

$$\begin{aligned} \mathcal{H}^+ &= \overline{J^+(\mathcal{I}_{as})} \cap \partial J^-(\mathcal{I}_{as}), \\ \mathcal{H}^- &= \overline{J^-(\mathcal{I}_{as})} \cap \partial J^+(\mathcal{I}_{as}). \end{aligned}$$

For any $\tau_1 \leq \tau_2$, we will denote

$$\mathcal{R}(\tau_1, \tau_2) \doteq \{\tau_1 \leq t \leq \tau_2\} \subset \mathcal{M} \setminus \mathcal{H}^- \tag{3.1}$$

and

$$\Sigma_\tau \doteq \{t = \tau\}, \tag{3.2}$$

where the function $t : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{R}$ is defined in Assumption G1.

The ergoregion of \mathcal{M} , defined by (2.8), will be denoted by \mathcal{E} . The boundary of \mathcal{E} (which is smooth, according to Assumption G3) will be denoted by $\partial \mathcal{E}$. We will fix a smooth T -invariant spacelike vector field $n_{\partial \mathcal{E}}$ in a small T -invariant neighborhood of

$\partial\mathcal{E}$, such that $n_{\partial\mathcal{E}}|_{\partial\mathcal{E}}$ is the unit normal of $\partial\mathcal{E}$. We will denote with \mathcal{E}_{ext} the extended ergoregion of (\mathcal{M}, g) , defined by (2.11). Notice that $\mathcal{E} \subseteq \mathcal{E}_{ext}$, but $\partial\mathcal{E}_{ext} \subseteq \partial\mathcal{E}$.

For any $\delta > 0$, we will denote

$$\mathcal{E}_\delta = \{x \in \mathcal{M} \setminus \mathcal{H}^- \mid \text{dist}_{g_{ref}}(x, \mathcal{E}_{ext}) \leq \delta\}. \tag{3.3}$$

Note that $\mathcal{E}_{ext} \subset \mathcal{E}_\delta$ for any $\delta > 0$, and $\bigcap_{\delta>0} \mathcal{E}_\delta = \mathcal{E}_{ext}$.

3.4. Notations on metrics, connections and integration. For any pseudo-Riemannian manifold $(\mathcal{N}, h_{\mathcal{N}})$ appearing in this paper, we will denote with $dh_{\mathcal{N}}$ the natural volume form associated with $h_{\mathcal{N}}$. Recall that in any local coordinate chart (x^1, x^2, \dots, x^k) on \mathcal{N} , $dh_{\mathcal{N}}$ is expressed as

$$dh_{\mathcal{N}} = \sqrt{|\det(h_{\mathcal{N}})|} dx^1 \dots dx^k.$$

We will also denote with $\nabla_{h_{\mathcal{N}}}$ the natural connection associated to $h_{\mathcal{N}}$. When $(\mathcal{N}, h_{\mathcal{N}}) = (\mathcal{M}, g)$, we will denote $\nabla_{h_{\mathcal{N}}}$ simply as ∇ . If $h_{\mathcal{N}}$ is Riemannian, $|\cdot|_{h_{\mathcal{N}}}$ will denote the associated norm on the tensor bundle of \mathcal{N} .

For any integer $l \geq 0$, we will denote with $(\nabla^{h_{\mathcal{N}}})^l$ or $\nabla_{h_{\mathcal{N}}}^l$ the higher order operator

$$\underbrace{\nabla_{h_{\mathcal{N}}} \dots \nabla_{h_{\mathcal{N}}}}_{l \text{ times}}. \tag{3.4}$$

Note that the product (3.4) is not symmetrised. We will also adopt the convention that we will always use Latin characters to denote such powers of covariant derivative operators. On the other hand, Greek characters will be used for the indices of a tensor in an abstract index notation.

For any smooth and spacelike hypersurface $\mathcal{S} \subset \mathcal{M}$, $g_{\mathcal{S}}$ will denote the induced (Riemannian) metric on \mathcal{S} , and $n_{\mathcal{S}}$ the future directed unit normal to \mathcal{S} .

Some examples of pseudo-Riemannian manifolds that will appear throughout this paper are (\mathcal{M}, g) , (\mathcal{M}, g_{ref}) and $(\Sigma_\tau, g_{\Sigma_\tau})$, where g_{ref} is the reference Riemannian metric (2.4). We will raise and lower indices of tensors on \mathcal{M} only with the use of g .

In some cases, we will omit the volume form dg or dg_{Σ_τ} when integrating over domains in \mathcal{M} or the hypersurfaces Σ_τ , respectively.

In the case of a smooth null hypersurface $\mathcal{H} \subset \mathcal{M}$, the volume form with which integration will be considered will as usual depend on the choice of a future directed null generator $n_{\mathcal{H}}$ for \mathcal{H} . For any such choice of $n_{\mathcal{H}}$, selecting an arbitrary vector field X on $T_{\mathcal{H}}\mathcal{M}$ such that $g(X, n_{\mathcal{H}}) = -1$ enables the construction of a non-degenerate top dimensional form on \mathcal{H} : $dvol_{\mathcal{H}} \doteq i_X dg$, which depends on the on the precise choice of $n_{\mathcal{H}}$, but not on the choice for X . In that case, $dvol_{\mathcal{H}}$ (or $dvol_{n_{\mathcal{H}}}$) will be the volume form on \mathcal{H} associated with $n_{\mathcal{H}}$.

3.5. Coordinate charts on $\mathcal{M} \setminus \mathcal{H}^-$. Using the function t as a projection, we can identify $\mathcal{M} \setminus \mathcal{H}^-$ with $\mathbb{R} \times \Sigma$. Under this identification, any local coordinate chart (x^1, \dots, x^d) on a subset \mathcal{V} of Σ can be extended to a coordinate chart (t, x^1, \dots, x^d) on $\mathbb{R} \times \mathcal{V} \subset \mathbb{R} \times \Sigma$, and in this chart, we have $\partial_t = T$. We will usually work in such coordinate charts throughout this paper.

In view of the flat asymptotics of (\mathcal{M}, g) and the fact that Σ intersects \mathcal{H}^+ transversally, the coarea formula yields that in the region $J^+(\Sigma)$, the volume forms dg and

$dt \wedge dg_\Sigma$ are equivalent, i.e. there exists a $C > 0$, such that for any integrable $\varphi : \mathcal{M} \rightarrow [0, +\infty)$ and any $0 \leq \tau_1 \leq \tau_2$ (identifying $\mathcal{M} \setminus \mathcal{H}^-$ with $\mathbb{R} \times \Sigma$):

$$C^{-1} \int_{\mathcal{R}(\tau_1, \tau_2)} \varphi dg \leq \int_{\tau_1}^{\tau_2} \left(\int_{\Sigma} \varphi(t, x) dg_\Sigma \right) dt \leq C \int_{\mathcal{R}(\tau_1, \tau_2)} \varphi dg. \tag{3.5}$$

Similarly, for any $\delta > 0$, there exists a $C_\delta > 0$ so that for any integrable $\varphi : \mathcal{M} \rightarrow [0, +\infty)$ and any $\tau_1 \leq \tau_2$ (not necessarily non-negative):

$$C_\delta^{-1} \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \geq \delta\}} \varphi dg \leq \int_{\tau_1}^{\tau_2} \left(\int_{\Sigma \cap \{r \geq \delta\}} \varphi(t, x) dg_\Sigma \right) dt \leq C_\delta \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \geq \delta\}} \varphi dg. \tag{3.6}$$

3.6. Notations for derivatives on \mathbb{S}^{d-1} . In this paper, we will frequently work in polar coordinates in the asymptotically flat region of (\mathcal{M}, g) or (Σ, g_Σ) . For this reason, we will adopt the same shorthand σ -notation for the angular variables in such a polar coordinate, as we did in [30,31]. See Section 3.6 of [30] for a detailed statement of this convention.

As an example of this convention, on subset \mathcal{U} of a spacetime \mathcal{M} covered by a polar coordinate chart $(u_1, u_2, \sigma) : \mathcal{U} \rightarrow \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{S}^{d-1}$, for any function $h : \mathcal{U} \rightarrow \mathbb{C}$ and any symmetric $(l, 0)$ -tensor b on \mathbb{S}^{d-1} , the following schematic notation for the contraction of the tensor $(\nabla_{g_{\mathbb{S}^{d-1}}})^l h(u_1, u_2, \cdot)$ with b will be frequently used:

$$b \cdot \partial_\sigma^l h(u_1, u_2, \cdot) \doteq b^{l_1 \dots l_l} (\nabla_{g_{\mathbb{S}^{d-1}}}^l)_{l_1 \dots l_l} h(u_1, u_2, \cdot), \tag{3.7}$$

where $g_{\mathbb{S}^{d-1}}$ is the standard metric on the unit sphere \mathbb{S}^{d-1} . Furthermore, we will also denote in this case

$$|\partial_\sigma^l h(u_1, u_2, \cdot)| \doteq |\nabla_{g_{\mathbb{S}^{d-1}}}^l h(u_1, u_2, \cdot)|_{g_{\mathbb{S}^{d-1}}}. \tag{3.8}$$

Notice, also, the following commutation relation holds:

$$[\mathcal{L}_{\partial_{u_i}}, \nabla^{\mathbb{S}^{d-1}}] = 0, \tag{3.9}$$

where ∂_{u_i} is the coordinate vector field associated to the coordinate function $u_i, i = 1, 2$. Therefore, we will frequently denote for any function $h : \mathcal{U} \rightarrow \mathbb{C}$:

$$\mathcal{L}_{\partial_{u_i}} \nabla^{\mathbb{S}^{d-1}} h \doteq \partial_{u_i} \partial_\sigma h, \tag{3.10}$$

and, in this notation, we will be allowed to commute ∂_{u_i} with ∂_σ , as if ∂_σ was a regular coordinate vector field. See Section 3.6 of [30] for more details.

3.7. *The $O_k(\cdot)$ notation.* For any integer $k \geq 0$ and any $b \in \mathbb{R}$, the notation $h = O_k(r^b)$ for some smooth function $h : \mathcal{M} \rightarrow \mathbb{C}$ will be used to denote that, in the (t, r, σ) polar coordinate chart on each connected component of the region $\{r \gg 1\}$ of \mathcal{M} (see Assumption 1):

$$\sum_{j=0}^k \sum_{j_1+j_2+j_3=j} r^{j_2+j_3} |\partial_\sigma^{j_1} \partial_t^{j_2} \partial_r^{j_3} h| \leq C \cdot r^b \tag{3.11}$$

for some constant $C > 0$ depending on k and h . The same notation (omitting the ∂_t derivatives) will also be used for functions on regions of manifolds cover by an (r, σ) polar coordinate chart.

Similarly, the notation $h = O_k^{\mathbb{S}^{d-1}}(r^b)$ will be used to denote a smooth tensor field h on \mathcal{M} such that, in the (t, r, σ) polar coordinate chart on each connected component of the region $\{r \gg 1\}$ of \mathcal{M} , h is tangential to the $\{r = \text{const}\}$ coordinate spheres (i. e. h contracted with ∂_r, ∂_t or $dr, d\sigma$, depending on its type, yields zero), and satisfies $|h|_{g_{\mathbb{S}^{d-1}}} = O_k(r^b)$. The type of the tensor h will always be clear from the context.

3.8. *Vector field multipliers and currents.* In this paper, we will frequently use the language of Lagrangean currents and vector field multipliers for equation (1.2): On any Lorentzian manifold (\mathcal{M}, g) , associated to the wave operator $\square_g = \frac{1}{\sqrt{-\det(g)}} \partial_\mu (\sqrt{-\det(g)} g^{\mu\nu} \partial_\nu)$ is a symmetric $(0, 2)$ -tensor called the *energy momentum tensor* Q . For any smooth function $\psi : \mathcal{M} \rightarrow \mathbb{C}$, the energy momentum tensor takes the form

$$Q_{\mu\nu}(\psi) = \frac{1}{2} (\partial_\mu \psi \cdot \partial_\nu \bar{\psi} + \partial_\mu \bar{\psi} \cdot \partial_\nu \psi) - \frac{1}{2} (\partial^\lambda \psi \cdot \partial_\lambda \bar{\psi}) g_{\mu\nu}. \tag{3.12}$$

For any continuous and piecewise C^1 vector field X on \mathcal{M} , the following associated currents can be defined almost everywhere:

$$J_\mu^X(\psi) = Q_{\mu\nu}(\psi) X^\nu, \tag{3.13}$$

$$K^X(\psi) = Q_{\mu\nu}(\psi) \nabla^\mu X^\nu. \tag{3.14}$$

The following divergence identity then holds almost everywhere on \mathcal{M} :

$$\nabla^\mu J_\mu^X(\psi) = K^X(\psi) + \text{Re} \left\{ (\square_g \psi) \cdot X \bar{\psi} \right\}. \tag{3.15}$$

3.9. *Hardy-type inequalities.* Frequently throughout this paper, we will need to control the weighted L^2 norm of some function u by some weighted L^2 norm of its derivative ∇u . This will always be accomplished with the use of some variant of the following Hardy-type inequality on \mathbb{R}^d (which is true for $d \geq 1$, although we will only need it for $d \geq 2$):

Lemma 3.1. *For any $a > 0$, there exists some $C_a > 0$ such that for any smooth and compactly supported function $u : \mathbb{R}^d \rightarrow \mathbb{C}$ and any $0 < R_1 < R_2$ we can bound*

$$\int_{\mathbb{R}^d \cap \{R_1 \leq r \leq R_2\}} r^{-d+a} |u|^2 dx + \int_{\{r=R_1\}} R_1^{-(d-1)+a} |u|^2 dg_{\{r=R_1\}}$$

$$\leq C_a \int_{\mathbb{R}^d \cap \{R_1 \leq r \leq R_2\}} r^{-(d-2)+a} |\partial_r u|^2 dx + \int_{\{r=R_2\}} R_2^{-(d-1)+a} |u|^2 dg_{\{r=R_2\}} \quad (3.16)$$

and

$$\begin{aligned} & \int_{\mathbb{R}^d \cap \{R_1 \leq r \leq R_2\}} r^{-d} |u|^2 dx + \int_{\{r=R_1\}} R_1^{-(d-1)} \log(R_1) |u|^2 dg_{\{r=R_1\}} \\ & \leq C \int_{\mathbb{R}^d \cap \{R_1 \leq r \leq R_2\}} r^{-(d-2)} (\log(r))^2 |\partial_r u|^2 dx \\ & \quad + \int_{\{r=R_2\}} R_2^{-(d-1)} \log(R_2) |u|^2 dg_{\{r=R_2\}}. \end{aligned} \quad (3.17)$$

In the above, r is the polar distance on \mathbb{R}^d , dx is the usual volume form on \mathbb{R}^d and $dg_{\{r=R\}}$ is the volume form of the induced metric on the sphere $\{r = R\} \subset \mathbb{R}^d$.

The proof of Lemma 3.1 is straightforward (see also Section 3.9 of [30]).

4. Proof of Theorem 2.1

The proof of Theorem 2.1 will proceed by contradiction: We will assume that all smooth solutions φ to (1.2) on $\mathcal{D}(\Sigma)$ with compactly supported initial data on Σ satisfy

$$\mathcal{E}[\varphi] \doteq \sup_{\tau \geq 0} \int_{\Sigma_\tau} J_\mu^N(\varphi) n^\mu < +\infty, \quad (4.1)$$

and we will reach a contradiction after choosing φ appropriately. To this end, we will need to establish a decay without a rate result outside the extended ergoregion \mathcal{E}_{ext} for solutions φ to (1.2), given the bound (4.1); see Proposition 4.1 in Sect. 4.2. This result is highly non-trivial and actually lies at the heart of the proof of Theorem 2.1, with Sects. 5–6 being devoted to the development of the necessary technical machinery for the proof of Proposition 4.1. In fact, the proof of Proposition 4.1 will be postponed until Sect. 7.

Remark. Instead of assuming (4.1), our proof of Theorem 2.1 also applies under the weaker assumption:

$$\sup_{\tau \geq 0} \left((\log(2 + \tau))^{-C} \int_{\Sigma_\tau} J_\mu^N(\varphi) n^\mu \right) < +\infty \quad (4.2)$$

for an arbitrary $C > 0$. Furthermore, as a consequence of the discussion in Sect. 6.9 (see also the remark below Proposition 4.1), the proof of Theorem 2.1 also applies without any significant change in the case when (\mathcal{M}, g) has a T -invariant timelike boundary component $\partial_{tim} \mathcal{M}$, with $\partial_{tim} \mathcal{M} \cap \Sigma$ compact and $\partial_{tim} \mathcal{M} \cap \mathcal{H} = \emptyset$, and φ is assumed to satisfy either Dirichlet or Neumann boundary conditions on $\partial_{tim} \mathcal{M}$ (see Sect. 6.9 for more details on the assumptions on the geometry of (\mathcal{M}, g) in this case).

In Sects. 4.1–4.3, we will establish some auxiliary results concerning the behaviour of solutions φ to (1.2), that will be used in the Sect. 4.4 to complete the proof of Theorem 2.1.

4.1. *Construction of initial data on Σ with negative higher order energy.* In this section, we will establish the following result:

Lemma 4.1. *There exists a smooth initial data set $(\varphi^{(0)}, \varphi^{(1)}) : \Sigma \rightarrow \mathbb{C}^2$ supported in $\Sigma \cap \mathcal{U}$ (where $\mathcal{U} \subset \mathcal{M}$ is the set described in Assumption A1) so that the function $\varphi : \mathcal{D}(\Sigma) \rightarrow \mathbb{C}$, defined by solving*

$$\begin{cases} \square_g \varphi = 0 \\ (\varphi|_{\Sigma}, T\varphi|_{\Sigma}) = (\varphi^{(0)}, \varphi^{(1)}), \end{cases} \quad \text{on } \mathcal{D}(\Sigma), \tag{4.3}$$

satisfies

$$\int_{\Sigma} J_{\mu}^T(T\varphi)n^{\mu} = -1. \tag{4.4}$$

Remark. Notice that the initial value problem (4.3) is well posed, since the vector field T , although not everywhere timelike, is everywhere transversal to Σ . For the construction of initial data sets $(\varphi^{(0)}, \varphi^{(1)})$ satisfying

$$\int_{\Sigma} J_{\mu}^T(\varphi)n^{\mu} = -1$$

instead of (4.4), see [19,20].

Proof. Since \mathcal{U} is an open subset of \mathcal{M} intersecting \mathcal{E} (according to Assumption A1), in view of the definition (2.8) of \mathcal{E} we infer that there exists a point $q \in \mathcal{U} \cap \Sigma$ and a contractible open neighborhood \mathcal{V} of q in \mathcal{M} such that T is strictly spacelike on \mathcal{V} . Therefore, provided \mathcal{V} is sufficiently small, there exists a vector field L on \mathcal{V} satisfying

$$g(L, L) = 0, \quad g(L, T) > 0 \quad \text{and} \quad \nabla L = 0. \tag{4.5}$$

The condition $\nabla L = 0$ on \mathcal{V} implies that there exists a function $w : \mathcal{V} \rightarrow \mathbb{R}$ such that

$$\nabla w = L. \tag{4.6}$$

Let us fix a smooth cut-off function $\chi : \mathcal{M} \rightarrow [0, 1]$ supported in \mathcal{V} such that $\chi(q) = 1$. Then, for any $l \gg 1$, the function

$$\tilde{\varphi}_l \doteq \chi e^{ilw} \tag{4.7}$$

on \mathcal{M} is supported in \mathcal{V} and satisfies [in view of (4.5) and (4.6)]

$$\square_g \tilde{\varphi}_l = \chi l^2 \partial_{\mu} w \partial^{\mu} w e^{ilw} + O(l) = O(l). \tag{4.8}$$

Furthermore, we compute:

$$\begin{aligned} & \int_{\Sigma} J^T(T\tilde{\varphi}_l)n^{\mu} \\ &= \int_{\Sigma} (n(T\tilde{\varphi}_l) \cdot T^2\tilde{\varphi}_l - \frac{1}{2}g(n, T)\partial_{\mu}T\tilde{\varphi}_l\partial^{\mu}T\tilde{\varphi}_l) dg_{\Sigma} \\ &= \int_{\Sigma \cap \mathcal{V}} \left((\chi l^2(nw)(Tw)e^{ilw} + O(l))(\chi l^2(Tw)^2e^{ilw} + O(l)) \right) \end{aligned}$$

$$\begin{aligned}
 & -\frac{1}{2}g(n, T)(\chi l^2(\partial_\mu w)(Tw)e^{ilw} + O(l))(\chi l^2(\partial^\mu w)(Tw)e^{ilw} + O(l)) \Big) dg_\Sigma \\
 & = \int_{\Sigma \cap \mathcal{V}} \left(\chi^2 l^4 (g(n, L)(g(T, L))^3 - \frac{1}{2}g(n, T)g(L, L)(g(T, L))^2) + O(l^3) \right) dg_\Sigma,
 \end{aligned} \tag{4.9}$$

which, in view of (4.5) (and the fact that $g(n, L) < 0$), yields:

$$\int_\Sigma J^T(T\tilde{\varphi}_l)n^\mu = -c_0 l^4 + O(l^3) \tag{4.10}$$

for some $c_0 > 0$.

Let us set

$$(\varphi^{(0)}, \varphi^{(1)}) \doteq (\tilde{\varphi}_l|_\Sigma, T\tilde{\varphi}_l|_\Sigma). \tag{4.11}$$

Note that $(\varphi^{(0)}, \varphi^{(1)})$ is supported in $\mathcal{V} \cap \Sigma \subset \mathcal{U} \cap \Sigma$. Then, the function

$$\check{\varphi} \doteq \varphi - \tilde{\varphi}_l, \tag{4.12}$$

where φ is defined by (4.3), satisfies (in view of (4.3), (4.8) and (4.11)):

$$\begin{cases} \square_g \check{\varphi} = O(l) & \text{on } \mathcal{D}(\Sigma), \\ (\check{\varphi}|_\Sigma, T\check{\varphi}|_\Sigma) = (0, 0). \end{cases} \tag{4.13}$$

In view of the fact that $\check{\varphi}|_\Sigma = 0$ and $\nabla \check{\varphi}|_\Sigma = 0$ (implying also that $\nabla_{g_\Sigma}^2 \check{\varphi}|_\Sigma = 0$ and $\nabla_{g_\Sigma} T\check{\varphi}|_\Sigma = 0$), the expression of the wave operator in a coordinate chart of the form (t, x) on \mathcal{V} readily yields

$$(\square_g \check{\varphi})|_\Sigma = (g^{00}T^2\check{\varphi})|_\Sigma = (g^{0\mu}\partial_\mu(T\check{\varphi}))|_\Sigma = \left(\frac{1}{g(n, T)}n(T\check{\varphi})\right)|_\Sigma. \tag{4.14}$$

Thus, from (4.13), (4.14) and the fact that $\square_g \check{\varphi}$ is supported in \mathcal{V} , we can readily bound

$$\int_\Sigma J_\mu^N(T\check{\varphi})n^\mu = O(l^2). \tag{4.15}$$

From (4.12), a Cauchy–Schwarz inequality implies:

$$\left| \int_\Sigma J_\mu^T(T\varphi)n^\mu - \int_\Sigma J_\mu^T(T\tilde{\varphi}_l)n^\mu \right| \leq C \int_\Sigma J_\mu^N(T\check{\varphi})n^\mu, \tag{4.16}$$

and thus, in view also of (4.10) and (4.15):

$$\int_\Sigma J_\mu^T(T\varphi)n^\mu = -c_0 l^4 + O(l^3) < 0, \tag{4.17}$$

provided $l \gg 1$. Multiplying $(\varphi^{(0)}, \varphi^{(1)})$ with a suitable non-zero constant, we can therefore achieve (4.4), and therefore the proof of the Lemma is complete. \square

4.2. *Decay outside the extended ergoregion.* The following proposition, establishing decay without a rate outside the ergoregion for solutions to Eq. (1.2), lies at the heart of the proof of Theorem 2.1:

Proposition 4.1. *Let $\varphi : \mathcal{D}(\Sigma) \rightarrow \mathbb{C}$ be a smooth function satisfying (1.2) with compactly supported initial data on Σ , and let us set $\psi = T\varphi$. Assume that the energy bound (4.1) holds for $\varphi, \psi, T\psi$ and $T^2\psi$,¹⁰ i.e.:*

$$\mathcal{E}[\varphi] + \mathcal{E}[\psi] + \mathcal{E}[T\psi] + \mathcal{E}[T^2\psi] < +\infty. \tag{4.18}$$

Then for any $0 < \epsilon < 1$, any $\delta_1 > 0$, any $R, \tau_* \gg 1$ and any $\bar{\tau}_0 \gg 1$, there exists a $\tau_{\bar{t}} \geq \bar{\tau}_0 + \tau_*$ depending on $\epsilon, \delta_1, R, \tau_*, \bar{\tau}_0, \mathcal{E}_{\log}[\varphi], \mathcal{E}_{\log}[\psi], \mathcal{E}_{\log}[T\psi]$ and $\mathcal{E}[T^2\psi]$ such that

$$\sum_{j=0}^1 \int_{(\mathcal{R}(\tau_{\bar{t}} - \tau_*, \tau_{\bar{t}} + \tau_*) \setminus \mathcal{E}_{\delta_1}) \cap \{r \leq R\}} (J_{\mu}^N(T^j\psi)n^{\mu} + |T^j\psi|^2) < \epsilon \tag{4.19}$$

(see (5.17) for the definition of the quantity $\mathcal{E}_{\log}[\cdot]$)

For the proof of Proposition 4.1, see Sect. 7.

Remark. The proof of Proposition 4.1 also applies when $\mathcal{E} = \emptyset$. Furthermore, in view of the discussion in Sect. 6.9, the proof of Proposition 4.1 in Sect. 7 also applies in the case when (\mathcal{M}, g) has a T -invariant timelike boundary component $\partial_{\text{tim}}\mathcal{M}$, with $\partial_{\text{tim}}\mathcal{M} \cap \Sigma$ compact and $\partial_{\text{tim}}\mathcal{M} \cap \mathcal{H} = \emptyset$, and φ is assumed to satisfy either Dirichlet or Neumann boundary conditions on $\partial_{\text{tim}}\mathcal{M}$.¹¹ As a consequence, the proof of Theorem 2.1 will also apply in this case as well (all the other steps in the proof of Theorem 2.1 immediately generalise in this case without any change).

4.3. *Limiting behaviour for solutions of (1.2).* We will need the following lemma on the behaviour of ψ asymptotically as $t \rightarrow +\infty$, following essentially from Proposition 4.1:

Lemma 4.2. *Let $\varphi, \psi : \mathcal{D}(\Sigma) \rightarrow \mathbb{C}$ be as in the statement of Proposition 4.1, and let us define, for any $\tau \geq 0$, the function $\psi_{\tau} : \mathcal{M} \setminus \mathcal{H}^{-} \rightarrow \mathbb{C}$ as follows:*

$$\psi_{\tau}(t, x) \doteq \begin{cases} \psi(t + \tau, x), & t \geq -\tau, \\ 0, & t < -\tau. \end{cases} \tag{4.20}$$

Then, there exists an increasing sequence $\{\tau_n\}_{n \in \mathbb{N}}$ of non-negative numbers and a function $\tilde{\psi} : \mathcal{M} \setminus \mathcal{H}^{-} \rightarrow \mathbb{C}$ with $\tilde{\psi}, T\tilde{\psi} \in H_{\text{loc}}^1(\mathcal{M} \setminus \mathcal{H}^{-})$, such that $\tilde{\psi}$ solves (1.2) on $\mathcal{M} \setminus \mathcal{H}^{-}$, satisfying in addition

$$\int_{-\tau_*}^{+\tau_*} \int_{\Sigma_{\tau}} (J_{\mu}^N(\tilde{\psi}) + J_{\mu}^N(T\tilde{\psi})n^{\mu}) d\tau < +\infty \text{ for any } \tau_* > 0, \tag{4.21}$$

$$\tilde{\psi} \equiv 0 \text{ on } \mathcal{M} \setminus (\mathcal{E}_{\text{ext}} \cup \mathcal{H}^{-}) \tag{4.22}$$

¹⁰ Note that, since T is a Killing field of (\mathcal{M}, g) , the functions $\psi, T\psi$ and $T^2\psi$ also solve (1.2) with compactly supported initial data on Σ .

¹¹ In this case, we have to assume that the double $(\tilde{\mathcal{M}}, \tilde{g})$ of (\mathcal{M}, g) across $\partial_{\text{tim}}\mathcal{M}$ is a globally hyperbolic spacetime satisfying Assumptions G1–G3 (note that Assumption A1 is not necessary for the proof of Proposition 4.1).

and $(\psi_{\tau_n}, T\psi_{\tau_n}) \rightarrow (\tilde{\psi}, T\tilde{\psi})$ weakly in $H^1_{loc}(\mathcal{M}\setminus\mathcal{H}^-) \times H^1_{loc}(\mathcal{M}\setminus\mathcal{H}^-)$ and strongly in $H^1_{loc}(\mathcal{M}\setminus(\mathcal{E}_{ext} \cup \mathcal{H}^-)) \times H^1_{loc}(\mathcal{M}\setminus(\mathcal{E}_{ext} \cup \mathcal{H}^-))$ and in $L^2_{loc}(\mathcal{M}\setminus\mathcal{H}^-) \times L^2_{loc}(\mathcal{M}\setminus\mathcal{H}^-)$ in the following sense:

- For any compactly supported test functions $\{\zeta_j\}_{j=0,1} \in L^2(\mathcal{M}\setminus\mathcal{H}^-)$ and compactly supported vector fields $\{X_j\}_{j=0,1}$ on $\mathcal{M}\setminus\mathcal{H}^-$ such that $|X_j|_{g_{ref}} \in L^2(\mathcal{M}\setminus\mathcal{H}^-)$:

$$\lim_{n \rightarrow +\infty} \sum_{j=0}^1 \int_{\mathcal{M}\setminus\mathcal{H}^-} \operatorname{Re} \left\{ g_{ref}(\nabla(T^j\psi_{\tau_n} - T^j\tilde{\psi}), X_j) + (T^j\psi_{\tau_n} - T^j\tilde{\psi})\zeta_j \right\} dg = 0. \tag{4.23}$$

- For any compact subset $\mathcal{K} \subset \mathcal{M}\setminus\mathcal{H}^-$ and any $\delta > 0$:

$$\lim_{n \rightarrow +\infty} \left(\sum_{j=0}^1 \int_{\mathcal{K}} |T^j\psi_{\tau_n} - T^j\tilde{\psi}|^2 dg + \sum_{j=0}^1 \int_{\mathcal{K}\setminus\mathcal{E}_\delta} |\nabla(T^j\psi_{\tau_n}) - \nabla(T^j\tilde{\psi})|_{g_{ref}}^2 dg \right) = 0. \tag{4.24}$$

Proof. Let us fix four sequences of positive numbers $\{\varepsilon_n\}_{n \in \mathbb{N}}, \{\delta_n\}_{n \in \mathbb{N}}, \{R_n\}_{n \in \mathbb{N}}$ and $\{\tau_n^*\}_{n \in \mathbb{N}}$ such that $\varepsilon_n, \delta_n \rightarrow 0$ and $R_n, \tau_n^* \rightarrow +\infty$ as $n \rightarrow +\infty$. We then define the sequence $\{\tau_n\}_{n \in \mathbb{N}}$ inductively: Setting $\tau_0 = 0$, τ_n is defined for any $n \geq 1$ as the value $\tau_{\bar{n}} > 0$ from Proposition 4.1 for ε_n in place of ε , δ_n in place of δ , R_n in place of R , τ_n^* in place of τ_* and τ_{n-1} in place of $\bar{\tau}_0$ (notice that the last condition guarantees that τ_n is an increasing sequence). Then, Proposition 4.1 applied for the pair $(\psi, T\psi)$ implies that the pair $(\psi_{\tau_n}, T\psi_{\tau_n})$ (which is merely a τ_n -translate of $(\psi, T\psi)$ in the region $\{t \geq -\tau_n\}$) satisfies the following estimate for any $n \in \mathbb{N}$:

$$\sum_{j=0}^1 \int_{-\tau_n^*}^{+\tau_n^*} \left(\int_{(\Sigma_t \setminus \mathcal{E}_{\delta_n}) \cap \{r \leq R_n\}} (J_\mu^N(T^j\psi_{\tau_n})n^\mu + |T^j\psi_{\tau_n}|^2) dt \right) < \varepsilon_n. \tag{4.25}$$

In view of the bounds (4.18) and 4.25, as well as the Poincare-type inequality

$$\begin{aligned} \int_{\mathcal{R}(\bar{\tau}_1, \bar{\tau}_2) \cap \{r \leq R\}} |\psi_{\tau_n}|^2 &\leq CR^2 \int_{\mathcal{R}(\bar{\tau}_1, \bar{\tau}_2) \cap \{r \leq 2R\}} J_\mu^N(\psi_{\tau_n})n^\mu \\ &\quad + C \int_{\mathcal{R}(\bar{\tau}_1, \bar{\tau}_2) \cap \{R \leq r \leq 2R\}} |\psi_{\tau_n}|^2 \end{aligned} \tag{4.26}$$

holding for any $\bar{\tau}_1 \leq \bar{\tau}_2$, we infer that, for any compact subset \mathcal{K} of $\mathcal{M}\setminus\mathcal{H}^-$, setting

$$n_0(\mathcal{K}) = \min \left\{ n \in \mathbb{N} : \mathcal{K} \text{ is contained in the set } \left\{ \max\{-\tau_n, -\tau_n^*\} < t < \tau_n^* \right\} \cap \{r \leq R_n\} \right\}, \tag{4.27}$$

there exists a $C = C_{\mathcal{K}}$ such that:

$$\sup_{n \geq n_0(\mathcal{K})} \left(\sum_{j=0}^1 \int_{\mathcal{K}} (J_\mu^N(T^j\psi_{\tau_n})N^\mu + |T^j\psi_{\tau_n}|^2) \right) \leq C_{\mathcal{K}}(\mathcal{E}[\psi] + \mathcal{E}[T\psi]) + \sup_{n \geq n_0(\mathcal{K})} \varepsilon_n < +\infty. \tag{4.28}$$

For any compact $\mathcal{K} \subset \mathcal{M}\setminus\mathcal{H}^-$, Rellich–Kondrachov’s theorem yields that the embedding $H^1(\mathcal{K}) \times H^1(\mathcal{K}) \hookrightarrow L^2(\mathcal{K}) \times L^2(\mathcal{K})$ is compact. Thus, (4.28) implies that for

any compact $\mathcal{K} \subset \mathcal{M} \setminus \mathcal{H}^-$ and any infinite subset $A \subseteq \mathbb{N}$, there exists an infinite subset $B_{\mathcal{K}, A} \subseteq A$ such that the subsequence $\{(\psi_{\tau_n}, T\psi_{\tau_n})\}_{n \in B_{\mathcal{K}, A}}$ of $\{(\psi_{\tau_n}, T\psi_{\tau_n})\}_{n \in \mathbb{N}}$ converges weakly in the $H^1(\mathcal{K}) \times H^1(\mathcal{K})$ norm and strongly in the $L^2(\mathcal{K}) \times L^2(\mathcal{K})$ to some limit pair $(\tilde{\psi}_{\mathcal{K}}, \tilde{\psi}_{\mathcal{K}})$ in $H^1(\mathcal{K}) \times H^1(\mathcal{K})$. Note that in this case, we necessarily have $\tilde{\psi}_{\mathcal{K}} = T\tilde{\psi}_{\mathcal{K}}$ in the sense of distributions.

Let $\{\mathcal{K}_m\}_{m \in \mathbb{N}}$ be a sequence of compact subsets of $\mathcal{M} \setminus \mathcal{H}^-$ such that $\mathcal{K}_m \subset \mathcal{K}_{m+1}$ and $\cup_{m \in \mathbb{N}} \mathcal{K}_m = \mathcal{M} \setminus \mathcal{H}^-$. Then, setting $A_{-1} = B_{\mathcal{K}_0, \mathbb{N}}$, $A_m = B_{\mathcal{K}_m, A_{m-1}}$ for $m \in \mathbb{N}$, and defining recursively

$$A = \cup_{m \in \mathbb{N}} \left\{ \min \left(A_m \setminus \{n : n < m\} \right) \right\}, \tag{4.29}$$

we infer that there exists a pair $(\tilde{\psi}, T\tilde{\psi}) \in H^1_{loc}(\mathcal{M} \setminus \mathcal{H}^-) \times H^1_{loc}(\mathcal{M} \setminus \mathcal{H}^-)$ such that the subsequence $\{\psi_{\tau_n}, T\psi_{\tau_n}\}_{n \in A}$ satisfies (4.23) and, for any compact $\mathcal{K} \subset \mathcal{M} \setminus \mathcal{H}^-$ (after permanently renumbering the indices of $\{\psi_{\tau_n}\}_{n \in A}$ through a map $\mathbb{N} \rightarrow A$):

$$\lim_{n \rightarrow +\infty} \sum_{j=0}^1 \int_{\mathcal{K}} |T^j \psi_{\tau_n} - T^j \tilde{\psi}|^2 dg = 0. \tag{4.30}$$

Since the functions ψ_{τ_n} solve (1.2) on $\{t > -\tau_n\}$, $\tilde{\psi}$ also solves (1.2) on $\mathcal{M} \setminus \mathcal{H}^-$ in the sense of distributions, in view of (4.23). Furthermore, in view of (4.18), we can bound for any $\tau_* > 0$

$$\sup_{n \in \mathbb{N}} \left\{ \sum_{j=0}^1 \int_{\max\{-\tau_*, -\tau_n\}}^{\tau_*} \left(\int_{\Sigma_\tau} J_\mu^N(T^j \psi) n^\mu \right) d\tau \right\} < +\infty \tag{4.31}$$

and, thus, (4.21) holds. The identity (4.22) follows by letting $n \rightarrow +\infty$ in (4.25). Finally, (4.24) follows from (4.22), (4.25) and (4.30). \square

4.4. Finishing the proof. Let us assume, for the sake of contradiction, that any smooth solution φ to (1.2) on $\mathcal{D}(\Sigma)$ with compactly supported initial data on Σ satisfies (4.1).

Let $\varphi : \mathcal{D}(\Sigma) \rightarrow \mathbb{C}$ be as in the statement of Lemma 4.1, and let us set

$$\psi = T\varphi. \tag{4.32}$$

In view of Lemma 4.1, $(\psi, T\psi)|_\Sigma$ is smooth and compactly supported in $\mathcal{U} \cap \Sigma$, and moreover

$$\int_\Sigma J_\mu^T(\psi) n^\mu = -1. \tag{4.33}$$

Let $\{\tau_n\}_{n \in \mathbb{N}}$ be the sequence defined by Lemma 4.2, and let $\psi_{\tau_n}, \tilde{\psi} : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{C}$ be the functions defined by Lemma 4.2.

We will make use of the following identity, appearing also in [20], holding for any acausal, inextendible and piecewise smooth hypersurface $\mathcal{S} \subset \mathcal{M} \setminus \mathcal{H}^-$ such that T is everywhere transversal to \mathcal{S} and any smooth function $\varphi_1 : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{C}$ such that $supp(\varphi_1) \cap \mathcal{S}$ is compact and $supp(\varphi_1) \cap \mathcal{S} \cap \mathcal{H}^+ = \emptyset$:

$$\begin{aligned} \int_{\mathcal{S}} J_{\mu}^T(\varphi_1) n_{\mathcal{S}}^{\mu} &= \int_{\mathcal{S}} \operatorname{Re}\{n_{\mathcal{S}}\varphi_1 \cdot T\bar{\varphi}_1 - \varphi_1 \cdot n_{\mathcal{S}}(T\bar{\varphi}_1)\} dg_{\mathcal{S}} \\ &\quad - \int_{\mathcal{S}} \operatorname{Re}\{\varphi_1 \square_g \bar{\varphi}_1\} g(n_{\mathcal{S}}, T) dg_{\mathcal{S}}, \end{aligned} \quad (4.34)$$

where $n_{\mathcal{S}}$ is the future directed unit normal to \mathcal{S} .

Proof of (4.34). One way to obtain (4.34) is the following: Since $\operatorname{supp}(\varphi_1) \cap \mathcal{S}$ is compact and $\operatorname{supp}(\varphi_1) \cap \mathcal{S} \cap \mathcal{H}^+ = \emptyset$, we can assume without loss of generality (by changing φ_1 away from \mathcal{S} if necessary) that φ_1 has compact support in $\mathcal{M} \setminus (\mathcal{H}^+ \cup \mathcal{H}^-)$. Then, integrating the identity

$$-2\operatorname{Re}\{T\varphi_1 \square_g \bar{\varphi}_1\} = -\operatorname{Re}\{T\varphi_1 \square_g \bar{\varphi}_1 - \varphi_1 \square_g (T\bar{\varphi}_1) + T(\varphi_1 \square_g \bar{\varphi}_1)\} \quad (4.35)$$

over $J^-(\mathcal{S})$, we readily obtain:

$$\begin{aligned} -2 \int_{J^-(\mathcal{S})} \operatorname{Re}\{T\varphi_1 \square_g \bar{\varphi}_1\} dg &= - \int_{J^-(\mathcal{S})} \operatorname{Re}\{T\varphi_1 \square_g \bar{\varphi}_1 - \varphi_1 \square_g (T\bar{\varphi}_1)\} dg \\ &\quad + \int_{\mathcal{S}} \operatorname{Re}\{\varphi_1 \square_g \bar{\varphi}_1\} g(n_{\mathcal{S}}, T) dg_{\mathcal{S}}. \end{aligned} \quad (4.36)$$

Using the identities

$$-2 \int_{J^-(\mathcal{S})} \operatorname{Re}\{T\varphi_1 \square_g \bar{\varphi}_1\} dg = \int_{\mathcal{S}} J_{\mu}^T(\varphi) n_{\mathcal{S}}^{\mu} \quad (4.37)$$

and

$$\begin{aligned} & - \int_{J^-(\mathcal{S})} \operatorname{Re}\{T\varphi_1 \square_g \bar{\varphi}_1 - \varphi_1 \square_g (T\bar{\varphi}_1)\} dg \\ &= \int_{\mathcal{S}} \operatorname{Re}\{n_{\mathcal{S}}\varphi_1 \cdot T\bar{\varphi}_1 - \varphi_1 \cdot n_{\mathcal{S}}(T\bar{\varphi}_1)\} dg_{\mathcal{S}} \end{aligned}$$

(holding because of the assumption that φ_1 has compact support in $\mathcal{M} \setminus (\mathcal{H}^+ \cup \mathcal{H}^-)$), we finally obtain (4.34).

We will also introduce the following (indefinite) inner product on the hypersurfaces Σ_{τ} : For any two functions $\varphi_1, \varphi_2 : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{C}$ such that for any $\tau_* > 0$:

$$\sup_{\tau \in [-\tau_*, \tau_*]} \sum_{j=1}^2 \int_{\Sigma_{\tau}} (J_{\mu}^N(\varphi_j) + J_{\mu}^N(T\varphi_j)) n^{\mu} < +\infty$$

and at least one of them has compact support in space (i.e. for any $\tau_* > 0$, its support in $[-\tau_* \leq t \leq \tau_*]$ is compact), we will define for any $\tau \in \mathbb{R}$:

$$\langle \varphi_1, \varphi_2 \rangle_{T, \tau} = \frac{1}{2} \int_{\Sigma_{\tau}} \operatorname{Re}\left\{ (n_{\mathcal{S}}\varphi_1 \cdot T\bar{\varphi}_2 + n_{\mathcal{S}}\varphi_2 \cdot T\bar{\varphi}_1) - (\varphi_1 \cdot n_{\mathcal{S}}(T\bar{\varphi}_2) + \varphi_2 \cdot n_{\mathcal{S}}(T\bar{\varphi}_1)) \right\}. \quad (4.38)$$

Note that, if both φ_1 and φ_2 solve equation (1.2) and at least one of them is supported away from \mathcal{H}^+ , then for any $\tau_1 \leq \tau_2$ the following identity holds:

$$\langle \varphi_1, \varphi_2 \rangle_{T, \tau_1} = \langle \varphi_1, \varphi_2 \rangle_{T, \tau_2}. \quad (4.39)$$

The equality (4.39) readily follows after integrating the identity

$$\frac{1}{2} \operatorname{Re} \left\{ (\square_g \varphi_1 T \bar{\varphi}_2 + \square_g \varphi_2 T \bar{\varphi}_1) - (\varphi_1 \square_g (T \bar{\varphi}_2) + \varphi_2 \square_g (T \bar{\varphi}_1)) \right\} = 0 \quad (4.40)$$

over $\mathcal{R}(\tau_1, \tau_2)$.

Remark. Note that, in the case when φ_1 and φ_2 solve equation (1.2) and at least one of them is supported away from \mathcal{H}^+ , the expression (4.38) is the inner product of φ_1, φ_2 associated to the $\int_{\Sigma_\tau} J_\mu^T(\cdot) n^\mu$ “norm”, in view of (4.34). Thus, (4.39) is a consequence of the conservation of the T -energy flux.

For any $\tau \geq 0$, the T -energy identity for ψ in the region $\mathcal{R}(0, \tau)$ combined with (4.33) yields:

$$\int_{\Sigma_\tau} J_\mu^T(\psi) n^\mu + \int_{\mathcal{H}^+ \cap \mathcal{R}(0, \tau)} J_\mu^T(\psi) n_{\mathcal{H}^+}^\mu = -1. \quad (4.41)$$

Since T is causal on $\mathcal{M} \setminus \mathcal{E}$, we can bound for any $\tau \geq 0$ and any $\delta > 0$:

$$\int_{\Sigma_\tau \setminus \mathcal{E}_\delta} J_\mu^T(\psi) n^\mu + \int_{\mathcal{H}^+ \cap \mathcal{R}(0, \tau)} J_\mu^T(\psi) n_{\mathcal{H}^+}^\mu \geq 0. \quad (4.42)$$

Therefore, (4.41) and (4.42) imply that for any $\tau \geq 0, \delta > 0$:

$$\int_{\Sigma_\tau \cap \mathcal{E}_\delta} J_\mu^T(\psi) n^\mu \leq -1. \quad (4.43)$$

Since the functions ψ_{τ_n} satisfy (4.20), from (4.43) we obtain for any $\delta > 0$, any $\tau > -\tau_n$, and any $n \in \mathbb{N}$:

$$\int_{\Sigma_\tau \cap \mathcal{E}_\delta} J_\mu^T(\psi_{\tau_n}) n^\mu \leq -1. \quad (4.44)$$

Let $\chi : \mathcal{M} \setminus \mathcal{H}^- \rightarrow [0, 1]$ be a smooth function of compact support such that $\chi \equiv 1$ on $\mathcal{R}(-1, 2) \cap \mathcal{E}_{\delta_0}$ for some $0 < \delta_0 < 1$ and $\operatorname{supp}(\chi) \cap \mathcal{H}^+ = \emptyset$. Applying the identity (4.34) for the function $\chi \psi_{\tau_n}$, and using the fact that ψ_{τ_n} solves (1.2), we obtain for any $n \in \mathbb{N}$ and any $0 < \tau_0 \leq 1$:

$$\begin{aligned} & \int_0^{\tau_0} \left(\int_{\Sigma_s} J_\mu^T(\chi \psi_{\tau_n}) n^\mu \right) ds \\ &= \int_0^{\tau_0} \left(\int_{\Sigma_s} \operatorname{Re} \{ n(\chi \psi_{\tau_n}) T(\chi \bar{\psi}_{\tau_n}) - (\chi \psi_{\tau_n}) n_S(T(\chi \bar{\psi}_{\tau_n})) \} dg_\Sigma \right) ds \\ & \quad - \int_0^{\tau_0} \left(\int_{\Sigma_s} \operatorname{Re} \{ \chi \psi_{\tau_n} (2\nabla^\mu \chi \nabla_\mu \bar{\psi}_{\tau_n} + (\square_g \chi) \bar{\psi}_{\tau_n}) \} g(n, T) dg_\Sigma \right) ds. \end{aligned} \quad (4.45)$$

In view of (4.44) and the fact that $\chi \equiv 1$ on $\mathcal{R}(-1, 2) \cap \mathcal{E}_{\delta_0}$, (4.45) yields:

$$\begin{aligned} & \int_0^{\tau_0} \left(\int_{\Sigma_s \cap \mathcal{E}_{\delta_0}} \operatorname{Re} \{ n \psi_{\tau_n} T \bar{\psi}_{\tau_n} - \psi_{\tau_n} n_S(T \bar{\psi}_{\tau_n}) \} dg_\Sigma \right) ds \\ & \leq -\tau_0 + C \sum_{j=0}^1 \int_{\operatorname{supp}(\chi) \setminus \mathcal{E}_{\delta_0}} (|\nabla T^j \psi_{\tau_n}|_{g_{\operatorname{ref}}}^2 + |T^j \psi_{\tau_n}|^2) dg. \end{aligned} \quad (4.46)$$

Let us examine the properties of (4.46) as $n \rightarrow +\infty$.

1. In view of (4.24) and the fact that $supp(\chi)$ is compact, the right hand side of (4.46) converges to $-\tau_0$ as $n \rightarrow +\infty$.
2. For any compact subset $\mathcal{K} \subset \mathcal{M} \setminus \mathcal{H}^+$ and any pair of sequences $(\varphi_n^{(1)}, \varphi_n^{(2)})_{n \in \mathbb{N}} \in L^2(\mathcal{K}) \times L^2(\mathcal{K})$ such that $\sup_n \|\varphi_n^{(1)}\|_{L^2(\mathcal{K})} < +\infty$, $\varphi_n^{(1)} \rightarrow \varphi^{(1)}$ weakly in $L^2(\mathcal{K})$ and $\varphi_n^{(2)} \rightarrow \varphi^{(2)}$ strongly in $L^2(\mathcal{K})$, one readily obtains that

$$\lim_{n \rightarrow +\infty} \int_{\mathcal{K}} \varphi_n^{(1)} \varphi_n^{(2)} dg = \int_{\mathcal{K}} \varphi^{(1)} \varphi^{(2)} dg. \tag{4.47}$$

Therefore, (4.18), (4.23) and (4.24) imply that:

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \int_0^{\tau_0} \left(\int_{\Sigma_s \cap \mathcal{E}_{\delta_0}} Re\{n\psi_{\tau_n} T\bar{\psi}_{\tau_n} - \psi_{\tau_n} n_{\mathcal{S}}(T\bar{\psi}_{\tau_n})\} dg_{\Sigma} \right) ds \\ &= \int_0^{\tau_0} \left(\int_{\Sigma_s \cap \mathcal{E}_{\delta_0}} Re\{n\tilde{\psi} T\bar{\tilde{\psi}} - \tilde{\psi} n_{\mathcal{S}}(T\bar{\tilde{\psi}})\} dg_{\Sigma} \right) ds. \end{aligned} \tag{4.48}$$

Thus, taking the limit $n \rightarrow +\infty$ in (4.46), we obtain for any $0 < \tau_0 \leq 1$:

$$\int_0^{\tau_0} \left(\int_{\Sigma_s \cap \mathcal{E}_{\delta_0}} Re\{n\tilde{\psi} T\bar{\tilde{\psi}} - \tilde{\psi} n_{\mathcal{S}}(T\bar{\tilde{\psi}})\} dg_{\Sigma} \right) ds \leq -\tau_0. \tag{4.49}$$

According to Lemma 4.2, $\tilde{\psi}$ belongs to $H^1_{loc}(\mathcal{M} \setminus \mathcal{H}^-)$ and vanishes outside \mathcal{E}_{ext} , and, thus, Assumption A1 implies that

$$\tilde{\psi} \equiv 0 \text{ on } \mathcal{U}. \tag{4.50}$$

Since $(\psi, T\psi)|_{\Sigma}$ is compactly supported in $\mathcal{U} \cap \Sigma$ and \mathcal{U} is open, in view of the finite speed of propagation property of Eq. (1.2), there exists some $0 < \tau_0 \leq 1$ (depending on the support of ψ on $\Sigma \cap \mathcal{U}$), such that for all $0 \leq \bar{\tau} \leq \tau_0$:

$$(\psi, T\psi) = (0, 0) \text{ on } \Sigma_{\bar{\tau}} \setminus \mathcal{U}. \tag{4.51}$$

In view of the fact that \mathcal{U} is translation invariant, (4.38), (4.50) and (4.51) imply that for any $\tau \in \mathbb{R}$:

$$\int_0^{\tau_0} \left\langle \psi, \mathcal{F}_{\bar{\tau}}^* \tilde{\psi} \right\rangle_{T, \bar{\tau}} d\bar{\tau} = 0 \tag{4.52}$$

[the expression (4.52) is well defined, in view of (4.21)], where

$$\mathcal{F}_{\bar{\tau}}^* \tilde{\psi}(t, x) \doteq \tilde{\psi}(t + \tau, x). \tag{4.53}$$

In view of Assumption G3, we have $\mathcal{E}_{ext} \cap \mathcal{H}^+ = \emptyset$. Thus, since $\tilde{\psi}$ vanishes outside \mathcal{E}_{ext} , we have $\tilde{\psi} \equiv 0$ on \mathcal{H}^+ . This fact, combined with (4.52) and the identity (4.39) [applied to a sequence of smooth approximations of $\tilde{\psi}$ in the norm defined by (4.21)] yields for any $s, \tau \in \mathbb{R}$:

$$\int_s^{s+\tau_0} \left\langle \psi, \mathcal{F}_{\bar{\tau}}^* \tilde{\psi} \right\rangle_{T, \bar{\tau}} d\bar{\tau} = 0. \tag{4.54}$$

In view of the definitions (4.20) and (4.53), the identity (4.54) for $s = \tau_n$ and $\tau = -s$ yields:

$$\int_0^{\tau_0} \left\langle \psi_{\tau_n}, \tilde{\psi} \right\rangle_{T, \bar{\tau}} d\bar{\tau} = 0. \tag{4.55}$$

Thus, since $\tilde{\psi}$ is supported in \mathcal{E}_{ext} and $\mathcal{R}(0, 1) \cap \mathcal{E}_{ext}$ is compact, (4.23) implies, after letting $n \rightarrow +\infty$ in (4.55):

$$\int_0^{\tau_0} \left\langle \tilde{\psi}, \tilde{\psi} \right\rangle_{T, \bar{\tau}} d\bar{\tau} = 0 \tag{4.56}$$

or, in view of (4.38):

$$\int_0^{\tau_0} \left(\int_{\Sigma_s} \operatorname{Re} \{ n \tilde{\psi} T \bar{\tilde{\psi}} - \tilde{\psi} n_{\mathcal{S}}(T \bar{\tilde{\psi}}) \} dg_{\Sigma_s} \right) ds = 0. \tag{4.57}$$

The contradiction now follows after comparing (4.57) with (4.49) (using also the fact that $\tilde{\psi}$ is supported in \mathcal{E}_{ext}). Thus, the proof of Theorem 2.1 is complete. \square

5. Frequency Decomposition

As we remarked in Sect. 4, Sects. 5–6 will be devoted to the development of the technical machinery required for the proof of Proposition 4.1. In particular, in this section, we will assume that we are given a smooth function $\psi : \mathcal{M} \rightarrow \mathbb{C}$ solving the wave equation (1.2) on $\mathcal{D}(\Sigma)$ (i.e. the domain of dependence of Σ) with compactly supported initial data on Σ , such that

$$\mathcal{E}[\psi] \doteq \sup_{\tau \geq 0} \int_{\Sigma_\tau} J^N(\psi) n^\mu < +\infty. \tag{5.1}$$

We will also introduce the frequency parameters $\omega_+ > 1$ and $0 < \omega_0 < 1$, and we will decompose the function ψ into components with localised frequency support (associated to the t variable). We will always identify $\mathcal{M} \setminus \mathcal{H}^-$ with $\mathbb{R} \times \Sigma$ under the flow of T as explained in Sect. 3.5. The constructions in this section will be similar to the associated constructions in Section 4 of [30].

5.1. Weighted energy estimates for ψ . Before proceeding to cut off ψ in the frequency space, we will first derive a few bounds for some suitable weighted energies of ψ .

In view of the finite speed of propagation for solutions to (1.2) and the fact that $(\psi, T\psi)|_{\Sigma_0}$ is compactly supported, we infer that $(\psi, T\psi)|_{\Sigma_\tau}$ is also compactly supported for any $\tau \geq 0$. The following lemma is a straightforward application of the finite speed of propagation property of Eq. (1.2):

Lemma 5.1. *For any $a > 0$, any $R \gg 1$ (so that T is timelike in $\{r \geq R\}$), any $\tau_1 \geq 0$ and any $\tau \in \mathbb{R}$:*

$$\begin{aligned} & \int_{\Sigma_\tau \cap \mathcal{D}(\Sigma_{\tau_1} \cap \{r \geq R\})} (\log(r))^a J_\mu^T(\psi) n^\mu \\ & \leq C_a (\log(2 + |\tau - \tau_1|))^{a+1} \int_{\Sigma_{\tau_1} \cap \{r \geq R\}} (\log(r))^a J_\mu^T(\psi) n^\mu \end{aligned} \tag{5.2}$$

and

$$\int_{\Sigma_\tau \cap \mathcal{D}(\Sigma_{\tau_1} \cap \{r \geq R\})} r^a J_\mu^T(\psi) n^\mu \leq C_a (1 + |\tau - \tau_1|)^a \int_{\Sigma_{\tau_1} \cap \{r \geq R\}} r^a J_\mu^T(\psi) n^\mu, \tag{5.3}$$

where $\mathcal{D}(\Sigma_{\tau_1} \cap \{r \geq R\}) \subset \{r \geq R\}$ is the domain of dependence of $\Sigma_\tau \cap \{r \geq R\}$ and $C_a > 0$ depends only on a and the geometry of (\mathcal{M}, g) .

Proof. Let us define for any $k \geq 1$ the sets

$$\mathcal{A}_k = \{2^k \leq r \leq 2^{k+1}\} \subset \mathcal{M}, \tag{5.4}$$

and let us set

$$\mathcal{A}_0 = \{r \leq 1\}. \tag{5.5}$$

Then, in view of the asymptotics (2.1) of g in each connected component of the asymptotically flat region \mathcal{I}_{as} , there exists a constant $C > 0$ depending on the geometry of (\mathcal{M}, g) such that for any $\tau_1 \geq 0$, $\tau \in \mathbb{R}$ and any $k \in \mathbb{N}$:

$$\Sigma_{\tau_1} \cap \left(J^+(\mathcal{A}_k \cap \Sigma_{\tau}) \cup J^-(\mathcal{A}_k \cap \Sigma_{\tau}) \right) \subset \bigcup_{n=\max\{0, k - \log_2(|\tau - \tau_1| + 1) - C\}}^{k + \log_2(|\tau - \tau_1| + 1) + C} \mathcal{A}_n \cap \Sigma_{\tau_1}. \tag{5.6}$$

Applying for any $k \in \mathbb{N}$ the conservation of the T -energy flux in the spacetime region $J^-(\mathcal{A}_k \cap \Sigma_{\tau}) \cap \mathcal{D}^+(\Sigma_{\tau_1} \cap \{r \geq R\})$, in the case $\tau \geq \tau_1$, or the region $J^+(\mathcal{A}_k \cap \Sigma_{\tau}) \cap \mathcal{D}^-(\Sigma_{\tau_1} \cap \{r \geq R\})$,¹² in the case $\tau \leq \tau_1$, we readily obtain in view of (5.6)(using also the fact that T is timelike for $r \geq R$):

$$\int_{\mathcal{A}_k \cap \Sigma_{\tau} \cap \mathcal{D}(\Sigma_{\tau_1} \cap \{r \geq R\})} J_{\mu}^T(\psi) n^{\mu} \leq \sum_{n=\max\{0, k - \log_2(|\tau - \tau_1| + 1) - C\}}^{k + \log_2(|\tau - \tau_1| + 1) + C} \int_{\mathcal{A}_n \cap \Sigma_{\tau_1} \cap \{r \geq R\}} J_{\mu}^T(\psi) n^{\mu}. \tag{5.7}$$

Multiplying (5.7) with k^a and summing over $k \in \mathbb{N}$, we obtain:

$$\begin{aligned} & \sum_{k=0}^{\infty} k^a \int_{\mathcal{A}_k \cap \Sigma_{\tau} \cap \mathcal{D}(\Sigma_{\tau_1} \cap \{r \geq R\})} J_{\mu}^T(\psi) n^{\mu} \\ & \leq \sum_{k=0}^{\infty} \left(k^a \sum_{n=\max\{0, k - \log_2(|\tau - \tau_1| + 1) - C\}}^{k + \log_2(|\tau - \tau_1| + 1) + C} \int_{\mathcal{A}_n \cap \Sigma_{\tau_1} \cap \{r \geq R\}} J_{\mu}^T(\psi) n^{\mu} \right) \\ & \leq C \sum_{k=0}^{\infty} \left(\left(\sum_{j=k}^{k + \log_2(|\tau - \tau_1| + 1) + C} j^a \right) \int_{\mathcal{A}_k \cap \Sigma_{\tau_1} \cap \{r \geq R\}} J_{\mu}^T(\psi) n^{\mu} \right) \\ & \leq C_a \sum_{k=0}^{\infty} \left((k + \log_2(|\tau - \tau_1| + 1) + C)^{a+1} \int_{\mathcal{A}_k \cap \Sigma_{\tau_1} \cap \{r \geq R\}} J_{\mu}^T(\psi) n^{\mu} \right). \end{aligned} \tag{5.8}$$

Inequality (5.2) follows readily from (5.8). Inequality (5.3) follows in the same way, after multiplying (5.7) with 2^{ka} and summing over $k \in \mathbb{N}$. \square

In view of (5.1) and the conservation of the T -energy flux in the region $\{t_- \geq 0\} \cap \{t \leq 0\}$, we can bound:

$$\sup_{\tau \in \mathbb{R}} \int_{\Sigma_{\tau} \cap \{t_- \geq 0\}} J_{\mu}^N(\psi) n^{\mu} \leq \mathcal{E}[\psi] \tag{5.9}$$

¹² Here, $\mathcal{D}^+(B)$ is the future domain of dependence of the set $B \subset \mathcal{M}$, while $\mathcal{D}^-(B)$ is the past domain of dependence.

(note that $\Sigma_\tau \cap \{t_- \geq 0\} = \Sigma_\tau$ when $\tau \geq 0$). Furthermore, in view of (5.2) for $\tau_1 = 0$ and the Hardy inequality (3.17), we can estimate:

$$\sup_{\tau \leq 0} \left((\log(2 + |\tau|))^{-3} \int_{\Sigma_\tau \cap \{t_- > 0\}} (1 + r)^{-2} |\psi|^2 \right) \leq C \int_{\Sigma_0} (\log(2 + r))^3 J_\mu^N(\psi) n^\mu. \tag{5.10}$$

5.2. *Frequency cut-off.* Let us fix a constant $R_1 \gg 1$ large in terms of the geometry of (\mathcal{M}, g) , as well as a smooth cut-off function $\chi_1 : [0, +\infty) \rightarrow [0, 1]$ satisfying $\chi_1(r) = 0$ for $r \leq R_1$ and $\chi_1(r) = 1$ for $r \geq R_1 + 1$. As in Section 4 of [30], we will define the following distorted time function on $\mathcal{M} \setminus \mathcal{H}^-$:

$$t_- = t + \frac{1}{2} \chi_1(r)(r - R_1). \tag{5.11}$$

Note that $\{t = 0\} \subset J^+(\{t_- = 0\})$.

We will also fix another smooth cut-off function $\chi_2 : \mathbb{R} \rightarrow [0, 1]$, satisfying $\chi_2 \equiv 0$ on $(-\infty, 0]$ and $\chi_2 \equiv 1$ on $[1, +\infty)$, and we will define the function $\psi_c : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{C}$ as

$$\psi_c \doteq \begin{cases} \chi_2(t_-) \cdot \psi, & t_- \geq 0, \\ 0, & t_- \leq 0. \end{cases} \tag{5.12}$$

Since ψ solves (1.2), ψ_c solves

$$\square_g \psi_c = F, \tag{5.13}$$

where

$$F = 2\partial^\mu \chi_2(t_-) \cdot \partial_\mu \psi + \square_g \chi_2(t_-) \cdot \psi \tag{5.14}$$

is supported in $\{0 \leq t_- \leq 1\}$.

Noting that $r \gtrsim |\tau|$ on $\{t = \tau\} \cap \{0 \leq t_- \leq 1\}$ for $\tau \leq 0$, combining (5.9) and (5.10) (in each asymptotically flat end of Σ_τ) with the Hardy-type inequality (obtained after averaging (3.17) over R_2 , using also a Poincare-type inequality in the near region $\{r \lesssim 1\}$):

$$\begin{aligned} \int_{\Sigma_\tau \setminus \mathcal{D}(\Sigma_0 \cap \{r \geq R\})} (1 + r)^{-2} |\psi_c|^2 &\leq C \int_{\Sigma_\tau \setminus \mathcal{D}(\Sigma_0 \cap \{r \geq 2R\})} (\log(2 + r))^2 J_\mu^N(\psi_c) n^\mu \\ &\quad + C \int_{\Sigma_\tau \cap \mathcal{D}(\Sigma_0 \cap \{R \leq r \leq 2R\})} (1 + r)^{-2} \log(r) |\psi_c|^2 \\ &\leq C (\log(2 + |\tau|))^2 \int_{\Sigma_\tau} J_\mu^N(\psi_c) n^\mu \\ &\quad + C \log(2 + |\tau|) \int_{\Sigma_\tau \cap \mathcal{D}(\Sigma_0 \cap \{r \geq R\})} (1 + r)^{-2} |\psi_c|^2, \end{aligned} \tag{5.15}$$

we obtain in view of (5.12):

$$\sup_{\tau \geq 0} \int_{\Sigma_\tau} J^N(\psi_c) n^\mu + \sup_{\tau \leq 0} \left(|\tau|^{-2} (\log(2 + |\tau|))^{-4} \int_{\Sigma_\tau} J_\mu^N(\psi_c) n^\mu \right)$$

$$+ \sup_{\tau \in \mathbb{R}} \left((\log(2 + |\tau|))^{-4} \int_{\Sigma_\tau} (1+r)^{-2} |\psi_c|^2 \right) \leq C \mathcal{E}_{\log}[\Psi], \tag{5.16}$$

where

$$\mathcal{E}_{\log}[\Psi] \doteq \mathcal{E}[\Psi] + \int_{\Sigma_0} (\log(2+r))^3 J_\mu^N(\Psi) n^\mu. \tag{5.17}$$

Remark. In dimensions $d \geq 3$, inequality (5.16), as well as most of the estimates of this section, holds without the logarithmic loss (since (5.10) holds without a logarithmic loss in this case).

We will now proceed to perform a cut-off procedure on ψ_c in the frequency domain. Let $0 < \omega_0 < 1$ be a (small) positive constant, and $\omega_+ \gg \omega_0$ a (large) positive constant, and let us set

$$n = \lceil \log_2 \frac{\omega_+}{\omega_0} \rceil \tag{5.18}$$

and, for any integer $1 \leq k \leq n$:

$$\omega_k = 2^k \omega_0. \tag{5.19}$$

Fixing a third smooth cut-off function $\chi_3 : \mathbb{R} \rightarrow [0, 1]$ satisfying $\chi_3 \equiv 1$ on $[-1, 1]$ and $\chi_3 \equiv 0$ on $(-\infty, -2] \cup [2, +\infty)$, we will define the following Schwartz functions on \mathbb{R} :

$$\begin{aligned} \zeta_0(t) &= \int_{-\infty}^{+\infty} e^{i\omega t} \chi_3(\omega_0^{-1}\omega) d\omega, \\ \zeta_k(t) &= \int_{-\infty}^{+\infty} e^{i\omega t} (\chi_3(\omega_k^{-1}\omega) - \chi_3(\omega_{k-1}^{-1}\omega)) d\omega, \text{ for } 1 \leq k \leq n \\ \zeta_{\leq \omega_+}(t) &= \sum_{k=0}^n \zeta_k(t). \end{aligned} \tag{5.20}$$

Notice that the Fourier transform of ζ_k is supported in $\{\omega_{k-1} \leq |\omega| \leq 2\omega_k\}$ (setting $\omega_{-1} = 0$), while the frequency support of $\zeta_{\leq \omega_+}$ is contained in $\{|\omega| \leq 4\omega_+\}$. Furthermore, the following Schwartz bounds hold for any integers $m, m' \in \mathbb{N}$ and $0 \leq k \leq n$:

$$\sup_{t \in \mathbb{R}} |\omega_k^{-1-m'} (1 + |\omega_k t|^m) (\frac{d}{dt})^{m'} \zeta_k(t)| \leq C_m \tag{5.21}$$

and

$$\sup_{t \in \mathbb{R}} |\omega_+^{-1-m'} (1 + |\omega_+ t|^m) (\frac{d}{dt})^{m'} \zeta_{\leq \omega_+}(t)| \leq C_m. \tag{5.22}$$

Using $\zeta_k, \zeta_{\leq \omega_+}$, we will define, for $0 \leq k \leq n$, the ‘‘frequency decomposed’’ components $\psi_k, \psi_{\leq \omega_+}, \psi_{\geq \omega_+} : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{C}$ of ψ through the following relations (identifying $\mathcal{M} \setminus \mathcal{H}^-$ with $\mathbb{R} \times \Sigma$ through the flow of T):

$$\psi_k(t, \cdot) = \int_{-\infty}^{+\infty} \zeta_k(t-s) \psi_c(s, \cdot) ds, \tag{5.23}$$

$$\psi_{\leq \omega_+}(t, \cdot) = \int_{-\infty}^{+\infty} \zeta_{\leq \omega_+}(t-s) \psi_c(s, \cdot) ds \tag{5.24}$$

and

$$\psi_{\geq\omega_+}(t, \cdot) = \psi_c(t, \cdot) - \psi_{\leq\omega_+}(t, \cdot). \tag{5.25}$$

Note that the integrals (5.23) and (5.24) do not necessarily converge pointwise for all $(t, x) \in \mathbb{R} \times \Sigma$, since the bound (5.1) does not suffice to exclude the pointwise exponential growth of ψ in the t variable. Instead, in view of (5.16), (5.21) and (5.22), the restrictions $(\psi_k, T\psi_k)|_{\Sigma_\tau}$, $(\psi_{\leq\omega_+}, T\psi_{\leq\omega_+})|_{\Sigma_\tau}$ and $(\psi_{\geq\omega_+}, T\psi_{\geq\omega_+})|_{\Sigma_\tau}$ are only defined as finite energy functions on Σ_τ for any $\tau \in \mathbb{R}$, satisfying the following bound for any $a > 0$ (derived from (5.16), (5.21), (5.22) and Young’s inequality):

$$\begin{aligned} & \sup_{\tau \geq 0} \left((1 + \omega_k^{-2-a})^{-1} \int_{\Sigma} (|N\psi_k(\tau, x)|^2 + |\nabla_{g_\Sigma} \psi_k(\tau, x)|_{g_\Sigma}^2) dg_\Sigma \right) \\ & + \sup_{\tau \leq 0} \left((1 + \omega_k^{-2-a})^{-1} |\tau|^{-2} (\log(2 + |\tau|))^{-4} \int_{\Sigma} (|N\psi_k(\tau, x)|^2 + |\nabla_{g_\Sigma} \psi_k(\tau, x)|_{g_\Sigma}^2) dg_\Sigma \right) \\ & + \sup_{\tau \in \mathbb{R}} \left((1 + \omega_k^{-a})^{-1} (\log(2 + |\tau|))^{-4} \int_{\Sigma} (1+r)^{-2} |\psi_k(\tau, x)|^2 dg_\Sigma \right) \leq C_a \mathcal{E}_{\log}[\psi], \end{aligned} \tag{5.26}$$

$$\begin{aligned} & \sup_{\tau \geq 0} \int_{\Sigma} (|N\psi_{\leq\omega_+}(\tau, x)|^2 + |\nabla_{g_\Sigma} \psi_{\leq\omega_+}(\tau, x)|_{g_\Sigma}^2) dg_\Sigma \\ & + \sup_{\tau \leq 0} \left(|\tau|^{-2} (\log(2 + |\tau|))^{-4} \int_{\Sigma} (|N\psi_{\leq\omega_+}(\tau, x)|^2 + |\nabla_{g_\Sigma} \psi_{\leq\omega_+}(\tau, x)|_{g_\Sigma}^2) dg_\Sigma \right) \\ & + \sup_{\tau \in \mathbb{R}} \left((\log(2 + |\tau|))^{-4} \int_{\Sigma} (1+r)^{-2} |\psi_{\leq\omega_+}(\tau, x)|^2 dg_\Sigma \right) \leq C_a \mathcal{E}_{\log}[\psi] \end{aligned} \tag{5.27}$$

and

$$\begin{aligned} & \sup_{\tau \geq 0} \int_{\Sigma} (|N\psi_{\geq\omega_+}(\tau, x)|^2 + |\nabla_{g_\Sigma} \psi_{\geq\omega_+}(\tau, x)|_{g_\Sigma}^2) dg_\Sigma \\ & + \sup_{\tau \leq 0} \left(|\tau|^{-2} (\log(2 + |\tau|))^{-4} \int_{\Sigma} (|N\psi_{\geq\omega_+}(\tau, x)|^2 + |\nabla_{g_\Sigma} \psi_{\geq\omega_+}(\tau, x)|_{g_\Sigma}^2) dg_\Sigma \right) \\ & + \sup_{\tau \in \mathbb{R}} \left((\log(2 + |\tau|))^{-4} \int_{\Sigma} (1+r)^{-2} |\psi_{\geq\omega_+}(\tau, x)|^2 dg_\Sigma \right) \leq C_a \mathcal{E}_{\log}[\psi]. \end{aligned} \tag{5.28}$$

Defining, similarly F_k , $F_{\leq\omega_+}$ and $F_{\geq\omega_+}$ in terms of F as in (5.23)–(5.25) (replacing ψ_c with F), in view of (5.13) we obtain the following relations (for any $0 \leq k \leq n$):

$$\square_g \psi_k = F_k, \tag{5.29}$$

$$\square_g \psi_{\leq\omega_+} = F_{\leq\omega_+} \tag{5.30}$$

and

$$\square_g \psi_{\geq\omega_+} = F_{\geq\omega_+}. \tag{5.31}$$

5.3. Bounds for the frequency-decomposed components. In this section, we will establish some useful estimates for the energy of ψ_k , $\psi_{\leq\omega_+}$, $\psi_{\geq\omega_+}$, as well as for the “error” terms F_k , $F_{\leq\omega_+}$, $F_{\geq\omega_+}$, in terms of $\mathcal{E}_{\log}[\psi]$.

We start with an estimate for weighted spacetime norms of the terms F_k , $F_{\leq\omega_+}$, $F_{\geq\omega_+}$.

Lemma 5.2. *We can bound for any $0 \leq k \leq n$, any $q, q' \in \mathbb{N}$ and any $0 \leq \tau_1 \leq \tau_2$:*

$$\int_{\mathcal{R}(\tau_1, \tau_2)} r^q |F_k|^2 \leq C_{qq'} (1 + \omega_k^{-q-2}) \cdot (1 + \omega_k \tau_1)^{-q'} \mathcal{E}_{\log}[\psi]. \quad (5.32)$$

The same inequality also holds for $F_{\leq \omega_+}$, $F_{\geq \omega_+}$ in place of F_k (with ω_+ in place of ω_k).

Proof. In view of (5.14) and the fact that

$$F_k(t, \cdot) = \int_{-\infty}^{+\infty} \zeta_k(t-s) F(s, \cdot) dt, \quad (5.33)$$

we can estimate (denoting with x the space variable in the splitting $\mathcal{M} \setminus \mathcal{H}^- = \mathbb{R} \times \Sigma$)

$$\begin{aligned} \int_{\mathcal{R}(\tau_1, \tau_2)} r^q |F_k|^2 &\leq C \int_{\Sigma} \int_{\tau_1}^{\tau_2} r^q \left| \int_{-\infty}^{\infty} \zeta_k(t-s) F(s, x) ds \right|^2 dt dg_{\Sigma} \\ &\leq C \left(\int_{\Sigma} r^q \int_{\tau_1}^{\tau_2} \left| \int_{-\frac{1}{2}\chi_1(r)(r-R_1)}^{1-\frac{1}{2}\chi_1(r)(r-R_1)} \zeta_k(t-s) F(s, x) ds \right|^2 dt dg_{\Sigma} \right). \end{aligned} \quad (5.34)$$

From the Schwartz bound (5.21) for $m = q' + q + 5$ and $m' = 0$ (using also (5.14)), we can estimate:

$$\begin{aligned} &\int_{\Sigma} r^q \int_{\tau_1}^{\tau_2} \left| \int_{-\frac{1}{2}\chi_1(r)(r-R_1)}^{1-\frac{1}{2}\chi_1(r)(r-R_1)} \zeta_k(t-s) F(s, x) ds \right|^2 dt dg_{\Sigma} \\ &\leq C_{qq'} \int_{\Sigma} r^q \int_{\tau_1}^{\tau_2} \left| \int_{-\frac{1}{2}\chi_1(r)(r-R_1)}^{1-\frac{1}{2}\chi_1(r)(r-R_1)} \frac{\omega_k}{(1 + \omega_k |t-s|)^{q'+q+5}} F(s, x) ds \right|^2 dt dg_{\Sigma} \\ &\leq C_{qq'} \int_{\Sigma} r^q \int_{\tau_1}^{\tau_2} \left(\int_{-\infty}^{+\infty} \frac{\omega_k}{(1 + \omega_k |\lambda|)^{q'+q+5}} d\lambda \right) \\ &\quad \left(\int_{-\frac{1}{2}\chi_1(r)(r-R_1)}^{1-\frac{1}{2}\chi_1(r)(r-R_1)} \frac{\omega_k}{(1 + \omega_k |t-s|)^{q'+q+5}} |F(s, x)|^2 ds \right) dt dg_{\Sigma} \\ &\leq C_{qq'} \int_{\Sigma} r^q \int_{\tau_1}^{\tau_2} \left(\int_{-\frac{1}{2}\chi_1(r)(r-R_1)}^{1-\frac{1}{2}\chi_1(r)(r-R_1)} \frac{\omega_k}{(1 + \omega_k |t-s|)^{q'+q+5}} |F(s, x)|^2 ds \right) dt dg_{\Sigma} \\ &\leq C_{qq'} \int_{\Sigma} r^q \left(\int_{\tau_1}^{\tau_2} \frac{\omega_k}{(1 + \omega_k |\tau + \frac{1}{2}\chi_1(r)(r-R_1)|)^{q'+q+5}} d\tau \right) \\ &\quad \left(\int_{-\frac{1}{2}\chi_1(r)(r-R_1)}^{1-\frac{1}{2}\chi_1(r)(r-R_1)} |F(s, x)|^2 ds \right) dg_{\Sigma} \\ &\leq C_{qq'} \frac{1 + \omega_k^{-q}}{(1 + \omega_k \tau_1)^{q'}} \int_{\Sigma} (1 + \omega_k r)^{-4} \left(\int_{-\frac{1}{2}\chi_1(r)(r-R_1)}^{1-\frac{1}{2}\chi_1(r)(r-R_1)} |F(t, x)|^2 dt \right) dg_{\Sigma} \\ &\leq C_{qq'} \frac{1 + \omega_k^{-q}}{(1 + \omega_k \tau_1)^{q'}} \int_{\Sigma} (1 + \omega_k r)^{-4} \left(\int_{-\frac{1}{2}\chi_1(r)(r-R_1)}^{1-\frac{1}{2}\chi_1(r)(r-R_1)} (J_{\mu}^N(\psi) n^{\mu} + |\psi|^2) dt \right) dg_{\Sigma} \end{aligned}$$

$$\leq C_{qq'} \frac{1 + \omega_k^{-q-2}}{(1 + \omega_k \tau_1)^{q'}} \int_{-\infty}^0 (1 + \tau)^{-2} \left(\int_{\Sigma_\tau \cap \{0 \leq t_- \leq 1\}} (J_\mu^N(\psi)n^\mu + (1+r)^{-2}|\psi|^2) \right) d\tau \tag{5.35}$$

(for the last inequality, we used the fact that $t \sim r$ on $\{0 \leq t_- \leq 1\}$). Therefore, from (5.34), (5.35), (5.9) and (5.10), we readily obtain (5.32).

The estimate for $F_{\leq \omega_+}$ and $F_{\geq \omega_+}$ follows in exactly the same way. \square

We will also need the following qualitative decay statement near spacelike infinity for the functions ψ_k , $\psi_{\leq \omega_+}$ and $\psi_{\geq \omega_+}$:

Lemma 5.3. *For any $q \in \mathbb{N}$, any $\tau \geq 0$ and any $0 \leq k \leq n$:*

$$\limsup_{R \rightarrow +\infty} \left(R^q \int_{\Sigma_\tau \cap \{R \leq r \leq R+1\}} (J_\mu^N(\psi_k)n^\mu + |\psi_k|^2) \right) = 0. \tag{5.36}$$

The relation (5.36) also holds for $\psi_{\leq \omega_+}$, $\psi_{\geq \omega_+}$ in place of ψ_k .

Proof. The proof of Lemma 5.3 is a straightforward consequence of the compact support of $(\psi, T\psi)|_\Sigma$ and the Schwartz bounds (5.21), (5.22).

Let $R_0(\psi)$ be sufficiently large, so that $(\psi, T\psi)|_\Sigma$ is supported in $\{r \leq R_0(\psi) - 1\}$. Then, in view of the finite speed of propagation property of equation (1.2), there exists a $C > 0$ (depending only on the geometry of (\mathcal{M}, g)), so that the function ψ is supported in $\{r \leq R_0(\psi) + C|t|\} \subset \mathcal{M}$. Thus,

$$\psi \equiv 0 \text{ on } \{|t| \geq C^{-1}(r - R_0(\psi))\}. \tag{5.37}$$

Then, in view of (5.12), (5.23), (5.21) and (5.37), we can bound for any $\tau \geq 0$, $R > R_0(\psi) + C\tau$ and $0 \leq k \leq n$:

$$\begin{aligned} & \int_{\Sigma_\tau \cap \{R \leq r \leq R+1\}} (J_\mu^N(\psi_k)n^\mu + |\psi_k|^2) \\ & \leq C \int_{\Sigma \cap \{R \leq r \leq R+1\}} \sum_{j=0}^1 \left| \int_{-\infty}^{+\infty} \zeta_k(\tau - s) \nabla^j \psi_c(s, x) ds \right|_{g_{ref}}^2 dg_\Sigma \\ & \leq C \left(\int_{-\infty}^{+\infty} |\zeta_k(s)| ds \right) \sum_{j=0}^1 \int_{\Sigma \cap \{R \leq r \leq R+1\}} \int_{-\infty}^{+\infty} |\zeta_k(\tau - s)| |\nabla^j \psi_c(s, x)|_{g_{ref}}^2 ds dg_\Sigma \\ & \leq C_q \sum_{j=0}^1 \int_{\Sigma \cap \{R \leq r \leq R+1\}} \int_{-\infty}^{+\infty} \frac{\omega_k}{1 + (\omega_k |\tau - s|)^{q+4}} |\nabla^j \psi_c(s, x)|_{g_{ref}}^2 ds dg_\Sigma \\ & \leq C_q \sum_{j=0}^1 \int_{\Sigma \cap \{R \leq r \leq R+1\}} \int_{C^{-1}(R - R_0(\psi))}^{+\infty} \frac{\omega_k}{1 + (\omega_k |\tau - s|)^{q+4}} |\nabla^j \psi(s, x)|_{g_{ref}}^2 ds dg_\Sigma. \end{aligned} \tag{5.38}$$

In view of the bounds (5.9) and (5.10), inequality (5.38) yields:

$$\int_{\Sigma_\tau \cap \{R \leq r \leq R+1\}} (J_\mu^N(\psi_k)n^\mu + |\psi_k|^2)$$

$$\begin{aligned} &\leq C_q R^2 \mathcal{E}_{\log[\Psi]} \int_{\Sigma \cap \{R \leq r \leq R+1\}} \int_{C^{-1}(R-R_0(\Psi))}^{+\infty} \frac{\omega_k}{1 + (\omega_k |\tau - s|)^{q+4}} (\log(2 + |s|))^3 ds dg_{\Sigma} \\ &\leq C_q \frac{\omega_k^{-1} R^2}{1 + (\omega_k (C^{-1}(R - R_0(\Psi)) - \tau))^{q+2}} \mathcal{E}_{\log[\Psi]}. \end{aligned} \tag{5.39}$$

Thus, (5.36) readily follows from (5.39).

The relation (5.36) for $\Psi_{\leq \omega_+}$, $\Psi_{\geq \omega_+}$ in place of Ψ_k follows in exactly the same way, using (5.22) in place of (5.21). \square

We will now proceed to obtain local in time estimates of the form $\int_{-\infty}^{\infty} |\partial_t \Psi_k|^2 dt \sim \omega_k^2 \int_{-\infty}^{\infty} |\Psi_k|^2 dt$. Let us define the following Schwartz functions on \mathbb{R} , similar to (5.20):

$$\begin{aligned} \xi_0(t) &= \int_{-\infty}^{+\infty} e^{i\omega t} \chi_3\left(\frac{1}{2}\omega_0^{-1}\omega\right) d\omega, \\ \xi_k(t) &= \int_{-\infty}^{+\infty} e^{i\omega t} \left(\chi_3\left(\frac{1}{2}\omega_k^{-1}\omega\right) - \chi_3\left(2\omega_{k-1}^{-1}\omega\right)\right) d\omega, \text{ for } 1 \leq k \leq n. \end{aligned} \tag{5.40}$$

Notice that, for any $0 \leq k \leq n$ (setting $\omega_{-1} = 0$), $\chi_3\left(\frac{1}{2}\omega_k^{-1}\omega\right) - \chi_3\left(2\omega_{k-1}^{-1}\omega\right) = 1$ for all $\omega \in \mathbb{R}$ such that $\chi_3\left(\omega_k^{-1}\omega\right) - \chi_3\left(\omega_{k-1}^{-1}\omega\right) \neq 0$, and thus:

$$\hat{\xi}_k = \hat{\xi}_k \cdot \hat{\xi}_k, \tag{5.41}$$

where $\hat{\cdot}$ denotes the Fourier transform operator on \mathbb{R} . Moreover, the following Schwartz bound holds for any integers $m, m' \in \mathbb{N}$ and $0 \leq k \leq n$:

$$\sup_{t \in \mathbb{R}} \left| \omega_k^{-1-m'} (1 + |\omega_k t|^m) \left(\frac{d}{dt}\right)^{m'} \xi_k(t) \right| \leq C_{m,m'}. \tag{5.42}$$

The relation (5.41), as well as the definition (5.20), implies for any $0 \leq k \leq n$ the following self reproducing formula for Ψ_k :

$$\Psi_k(t, \cdot) = \int_{-\infty}^{\infty} \xi_k(t - s) \cdot \Psi_k(s, \cdot) ds, \tag{5.43}$$

where, again, the integral in the right hand side of (5.43) converges with respect to the $\int_{\Sigma_t} J_{\mu}^N(\cdot) n^{\mu}$ norm (in view of (5.26), (5.42) and Young’s inequality).

For any $1 \leq k \leq n$, we will also introduce the anti-derivatives of ξ_k , defined as

$$\tilde{\xi}_k(t) = \int_{-\infty}^{+\infty} \frac{1}{i\omega} e^{i\omega t} \left(\chi_3\left(\frac{1}{2}\omega_k^{-1}\omega\right) - \chi_3\left(2\omega_{k-1}^{-1}\omega\right)\right) d\omega, \tag{5.44}$$

thus satisfying for any $m \in \mathbb{N}$ the Schwartz bound

$$\sup_{t \in \mathbb{R}} \left| (1 + |\omega_k t|^m) \tilde{\xi}_k(t) \right| \leq C_m, \tag{5.45}$$

as well as the frequency-domain identity:

$$\hat{\xi}_k = \hat{\tilde{\xi}}_k \cdot i\omega \hat{\xi}_k. \tag{5.46}$$

In view of (5.46), as well as the definition (5.20), we obtain for any $1 \leq k \leq n$:

$$\Psi_k(t, \cdot) = \int_{-\infty}^{\infty} \tilde{\xi}_k(t-s) \cdot T\Psi_k(s, \cdot) ds, \tag{5.47}$$

where the integral in the right hand side of (5.47) converges with respect to the $\int_{\Sigma_t} J_{\mu}^N(\cdot) n^{\mu}$ norm.

We can now establish the following lemma:

Lemma 5.4. *For any $1 \leq k \leq n$, any $0 \leq \tau_1 \leq \tau_2$, any T -invariant L_{loc}^{∞} function $\chi : \mathcal{M} \setminus \mathcal{H}^{-} \rightarrow [0, +\infty)$, any $R \geq 0$ and any $0 < a < 1$, we can bound*

$$\begin{aligned} & c\omega_k^2 \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq R\}} \chi |\Psi_k|^2 - C_a \omega_k^2 (1 + \omega_k^{-5-a}) \sup_{\{r \leq R\}} \chi \cdot \mathcal{E}_{log}[\Psi] \\ & \leq \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq R\}} \chi |T\Psi_k|^2 \leq C\omega_k^2 \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq R\}} \chi |\Psi_k|^2 \\ & \quad + C_a \omega_k^2 (1 + \omega_k^{-1-a}) (\log(2 + \tau_2))^4 R^2 \sup_{\{r \leq R\}} \chi \cdot \mathcal{E}_{log}[\Psi], \end{aligned} \tag{5.48}$$

and similarly for $k = 0$:

$$\begin{aligned} & \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq R\}} \chi |T\Psi_0|^2 \leq C\omega_0^2 \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq R\}} \chi |\Psi_0|^2 \\ & \quad + C_a \omega_0^2 (1 + \omega_0^{-1-a}) (\log(2 + \tau_2))^4 R^2 \sup_{\{r \leq R\}} \chi \cdot \mathcal{E}_{log}[\Psi]. \end{aligned} \tag{5.49}$$

Remark. Notice that the constant multiplying the error term in the right hand side of (5.48) depends on R and τ_2 , while this is not the case in the left hand side.

Proof. For any $0 \leq k \leq n$, from (5.43) and (5.42) (for $m = 5, m' = 1$) we can estimate for any $\tau \geq 0$:

$$\begin{aligned} & \int_{\Sigma \cap \{r \leq R\}} \chi(x) |T\Psi_k(\tau, x)|^2 dg_{\Sigma} \\ & = \int_{\Sigma \cap \{r \leq R\}} \chi(x) \left| \int_{-\infty}^{+\infty} \xi'_k(\tau-s) \Psi_k(s, x) ds \right|^2 dg_{\Sigma} \\ & \leq C \int_{\Sigma \cap \{r \leq R\}} \chi(x) \left| \int_{-\infty}^{+\infty} \frac{\omega_k^2}{1 + |\omega_k(\tau-s)|^5} \Psi_k(s, x) ds \right|^2 dg_{\Sigma} \\ & \leq C\omega_k^2 \int_{\Sigma \cap \{r \leq R\}} \chi(x) \left(\int_{-\infty}^{+\infty} \frac{\omega_k}{1 + |\omega_k(\tau-s)|^5} ds \right) \\ & \quad \left(\int_{-\infty}^{+\infty} \frac{\omega_k}{1 + |\omega_k(\tau-s)|^5} |\Psi_k(s, x)|^2 ds \right) dg_{\Sigma} \\ & \leq C\omega_k^2 \int_{\Sigma \cap \{r \leq R\}} \int_{-\infty}^{+\infty} \frac{\omega_k}{1 + |\omega_k(\tau-s)|^5} \chi(x) |\Psi_k(s, x)|^2 ds dg_{\Sigma}. \end{aligned} \tag{5.50}$$

Thus, integrating (5.50) over $\{\tau_1 \leq \tau \leq \tau_2\}$ we obtain:

$$\int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq R\}} \chi |T\Psi_k|^2$$

$$\begin{aligned}
 &\leq C\omega_k^2 \int_{\tau_1}^{\tau_2} \int_{\Sigma \cap \{r \leq R\}} \int_{-\infty}^{+\infty} \frac{\omega_k}{1 + |\omega_k(\tau - s)|^5} \chi(x) |\psi_k(s, x)|^2 ds dg_\Sigma d\tau \\
 &\leq C\omega_k^2 \left(\int_{\tau_1}^{\tau_2} \int_{\Sigma \cap \{r \leq R\}} \int_{\tau_1}^{\tau_2} \frac{\omega_k}{1 + |\omega_k(\tau - s)|^5} \chi(x) |\psi_k(s, x)|^2 ds dg_\Sigma d\tau \right. \\
 &\quad \left. + \int_{\tau_1}^{\tau_2} \int_{\Sigma \cap \{r \leq R\}} \int_{\mathbb{R} \setminus [\tau_1, \tau_2]} \frac{\omega_k}{1 + |\omega_k(\tau - s)|^5} \chi(x) |\psi_k(s, x)|^2 ds dg_\Sigma d\tau \right) \\
 &\leq C\omega_k^2 \left(\left(\int_{-\infty}^{+\infty} \frac{\omega_k}{1 + |\omega_k \lambda|^5} d\lambda \right) \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq R\}} \chi |\psi_k|^2 \right. \\
 &\quad \left. + \int_{\Sigma \cap \{r \leq R\}} \int_{-\infty}^{\tau_1} \left(\int_{\tau_1}^{\tau_2} \frac{\omega_k}{1 + |\omega_k(\tau - s)|^5} d\tau \right) \chi(x) |\psi_k(s, x)|^2 ds dg_\Sigma \right. \\
 &\quad \left. + \int_{\Sigma \cap \{r \leq R\}} \int_{\tau_2}^{+\infty} \left(\int_{\tau_1}^{\tau_2} \frac{\omega_k}{1 + |\omega_k(\tau - s)|^5} d\tau \right) \chi(x) |\psi_k(s, x)|^2 ds dg_\Sigma \right) \\
 &\leq C\omega_k^2 \left(\int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq R\}} \chi |\psi_k|^2 \right. \\
 &\quad \left. + \sup_{r \leq R} \chi \cdot \int_{\Sigma \cap \{r \leq R\}} \int_{\mathbb{R} \setminus [\tau_1, \tau_2]} \frac{1}{1 + (\omega_k \min\{|\tau_1 - s|, |\tau_2 - s|\})^4} |\psi_k(s, x)|^2 ds dg_\Sigma \right). \tag{5.51}
 \end{aligned}$$

From (5.26) we can readily estimate for any $0 < a < 1$:

$$\begin{aligned}
 &\int_{\Sigma \cap \{r \leq R\}} \int_{\mathbb{R} \setminus [\tau_1, \tau_2]} \frac{1}{1 + (\omega_k \min\{|\tau_1 - s|, |\tau_2 - s|\})^4} |\psi_k(s, x)|^2 ds dg_\Sigma \\
 &\leq C_a R^2 \left(\int_{\mathbb{R} \setminus [\tau_1, \tau_2]} \frac{1}{1 + (\omega_k \min\{|\tau_1 - s|, |\tau_2 - s|\})^4} (\log((2 + |s|)^4) ds) (1 + \omega_k^{-a}) \mathcal{E}_{\log}[\psi] \right. \\
 &\quad \left. \leq C_a R^2 (1 + \omega_k^{-1-2a}) (\log(2 + \tau_2))^4 \mathcal{E}_{\log}[\psi]. \right) \tag{5.52}
 \end{aligned}$$

Thus, from (5.51) and (5.52) we readily infer the right “half” of inequality (5.48), as well as inequality (5.49).

In order to establish the left “half” of inequality (5.48), we will work similarly, using formula (5.47) in place of (5.43). In particular, from (5.47) and (5.45) (for $m = 5$) we obtain for any $\tau \geq 0$ and any $1 \leq k \leq n$:

$$\begin{aligned}
 &\int_{\Sigma \cap \{r \leq R\}} \chi(x) |\psi_k(\tau, x)|^2 dg_\Sigma \\
 &= \int_{\Sigma \cap \{r \leq R\}} \chi(x) \left| \int_{-\infty}^{+\infty} \tilde{\xi}_k(\tau - s) T\psi_k(s, x) ds \right|^2 dg_\Sigma \\
 &\leq C \int_{\Sigma \cap \{r \leq R\}} \chi(x) \left| \int_{-\infty}^{+\infty} \frac{1}{1 + |\omega_k(\tau - s)|^5} T\psi_k(s, x) ds \right|^2 dg_\Sigma \\
 &\leq C\omega_k^{-2} \int_{\Sigma \cap \{r \leq R\}} \chi(x) \left(\int_{-\infty}^{+\infty} \frac{\omega_k}{1 + |\omega_k(\tau - s)|^5} ds \right) \\
 &\quad \left(\int_{-\infty}^{+\infty} \frac{\omega_k}{1 + |\omega_k(\tau - s)|^5} |T\psi_k(s, x)|^2 ds \right) dg_\Sigma \\
 &\leq C\omega_k^{-2} \int_{\Sigma \cap \{r \leq R\}} \int_{-\infty}^{+\infty} \frac{\omega_k}{1 + |\omega_k(\tau - s)|^5} \chi(x) |T\psi_k(s, x)|^2 ds dg_\Sigma. \tag{5.53}
 \end{aligned}$$

Integrating (5.53) over $\{\tau_1 \leq \tau \leq \tau_2\}$, we obtain:

$$\begin{aligned}
 & \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq R\}} \chi |\psi_k|^2 \\
 & \leq C \omega_k^{-2} \int_{\tau_1}^{\tau_2} \int_{\Sigma \cap \{r \leq R\}} \int_{-\infty}^{+\infty} \frac{\omega_k}{1 + |\omega_k(\tau - s)|^5} \chi(x) |T\psi_k(s, x)|^2 ds dg_\Sigma d\tau \\
 & \leq C \omega_k^{-2} \left(\int_{\tau_1}^{\tau_2} \int_{\Sigma \cap \{r \leq R\}} \int_{\tau_1}^{\tau_2} \frac{\omega_k}{1 + |\omega_k(\tau - s)|^5} \chi(x) |T\psi_k(s, x)|^2 ds dg_\Sigma d\tau \right. \\
 & \quad \left. + \int_{\tau_1}^{\tau_2} \int_{\Sigma \cap \{r \leq R\}} \int_{\mathbb{R} \setminus [\tau_1, \tau_2]} \frac{\omega_k}{1 + |\omega_k(\tau - s)|^5} \chi(x) |T\psi_k(s, x)|^2 ds dg_\Sigma d\tau \right) \\
 & \leq C \omega_k^{-2} \left(\left(\int_{-\infty}^{+\infty} \frac{\omega_k}{1 + |\omega_k \lambda|^5} d\lambda \right) \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq R\}} \chi |T\psi_k|^2 \right. \\
 & \quad \left. + \int_{\Sigma \cap \{r \leq R\}} \int_{-\infty}^{\tau_1} \left(\int_{\tau_1}^{\tau_2} \frac{\omega_k}{1 + |\omega_k(\tau - s)|^5} d\tau \right) \chi(x) |T\psi_k(s, x)|^2 ds dg_\Sigma \right. \\
 & \quad \left. + \int_{\Sigma \cap \{r \leq R\}} \int_{\tau_2}^{+\infty} \left(\int_{\tau_1}^{\tau_2} \frac{\omega_k}{1 + |\omega_k(\tau - s)|^5} d\tau \right) \chi(x) |T\psi_k(s, x)|^2 ds dg_\Sigma \right) \\
 & \leq C \omega_k^{-2} \left(\int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq R\}} \chi |\psi_k|^2 + \right. \\
 & \quad \left. + \sup_{r \leq R} \chi \cdot \int_{\Sigma \cap \{r \leq R\}} \int_{\mathbb{R} \setminus [\tau_1, \tau_2]} \frac{1}{1 + (\omega_k \min\{|\tau_1 - s|, |\tau_2 - s|\})^4} |T\psi_k(s, x)|^2 ds dg_\Sigma \right). \tag{5.54}
 \end{aligned}$$

From (5.26) we can estimate:

$$\begin{aligned}
 & \int_{\Sigma \cap \{r \leq R\}} \int_{\mathbb{R} \setminus [\tau_1, \tau_2]} \frac{1}{1 + (\omega_k \min\{|\tau_1 - s|, |\tau_2 - s|\})^4} |T\psi_k(s, x)|^2 ds dg_\Sigma \\
 & \leq C_a \left(\int_{\mathbb{R} \setminus [\tau_1, \tau_2]} \frac{(1 + \max\{0, -s\})^2 (\log(2 + \max\{0, -s\}))^4}{1 + (\omega_k \min\{|\tau_1 - s|, |\tau_2 - s|\})^4} ds \right) (1 + \omega_k^{-2-a}) \mathcal{E}_{\log}[\psi] \\
 & \leq C_a (1 + \omega_k^{-5-2a}) \mathcal{E}_{\log}[\psi]. \tag{5.55}
 \end{aligned}$$

Thus, the left “half” of inequality (5.48) follows from (5.54) and (5.55). \square

We will also need the following estimate in the case when ψ is of the form $T\varphi$, where φ is a smooth solution to the wave equation on $\mathcal{D}(\Sigma)$:

Lemma 5.5. *Let ψ be of the form*

$$\psi = T\varphi, \tag{5.56}$$

where $\varphi : \mathcal{D}(\Sigma) \rightarrow \mathbb{C}$ is a smooth function solving (1.2) with compactly supported initial data on Σ , such that $\mathcal{E}[\varphi] < +\infty$. Then, for any $0 \leq \tau_1 \leq \tau_2$, any $0 < a < 1$ and any $R \geq 0$ we can bound:

$$\begin{aligned}
 & \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq R\}} (J_\mu^N(\psi_0)N^\mu + |\psi_0|^2) \\
 & \leq C \omega_0^2 \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq R\}} (J_\mu^N(\varphi)N^\mu + |\varphi|^2) \\
 & \quad + C_a (\omega_0^2 (1 + \omega_0^{-1-a}) R^2 (\log(2 + |\tau_2|))^4 + (1 + \omega_0 \tau_1)^{-1} R^2) \mathcal{E}_{\log}[\varphi]. \tag{5.57}
 \end{aligned}$$

Proof. The bounds (5.9), (5.2) for φ in place of ψ (combined with the Hardy-type inequalities (3.17) and (5.15)) imply that

$$\sup_{\tau \in \mathbb{R}} \int_{\Sigma_\tau \cap \{t_- \geq 0\}} J^N(\varphi)n^\mu + \sup_{\tau \in \mathbb{R}} \left((\log(2 + |\tau|))^4 \int_{\Sigma_\tau \cap \{t_- \geq 0\}} (1 + r)^{-2} |\varphi|^2 \right) \leq C \mathcal{E}_{\log}[\varphi]. \tag{5.58}$$

From (5.56), (5.12) and (5.23) we calculate:

$$\begin{aligned} & \int_{\Sigma \cap \{r \leq R\}} \int_{\tau_1}^{\tau_2} |\psi_0(s, x)|^2 d\tau dg_\Sigma \\ &= \int_{\Sigma \cap \{r \leq R\}} \int_{\tau_1}^{\tau_2} \left| \int_{-\infty}^{+\infty} \zeta_0(t - s) \psi_c(s, x) ds \right|^2 dt dg_\Sigma \\ &= \int_{\Sigma \cap \{r \leq R\}} \int_{\tau_1}^{\tau_2} \left| \int_{-\infty}^{+\infty} \zeta_0(t - s) \chi_2(s + \frac{1}{2} \chi_1(r)(r - R_1)) \partial_t \varphi(s, x) ds \right|^2 dt dg_\Sigma \\ &= \int_{\Sigma \cap \{r \leq R\}} \int_{\tau_1}^{\tau_2} \left| - \int_{-\infty}^{+\infty} \frac{d}{ds} (\zeta_0(t - s) \chi_2(s + \frac{1}{2} \chi_1(r)(r - R_1))) \cdot \varphi(s, x) ds \right|^2 dt dg_\Sigma, \end{aligned} \tag{5.59}$$

noting that the integrating by parts in the last step of (5.59) is possible in view of the Schwartz bound (5.21) on ζ_0 and (5.58).

In view of (5.21), the relation (5.59) yields:

$$\begin{aligned} & \int_{\Sigma \cap \{r \leq R\}} \int_{\tau_1}^{\tau_2} |\psi_0(s, x)|^2 d\tau dg_\Sigma \\ & \leq C \int_{\Sigma \cap \{r \leq R\}} \int_{\tau_1}^{\tau_2} \left(\left| \int_{-\infty}^{+\infty} \frac{\omega_0^2}{1 + |\omega_0(t - s)|^3} \chi_2(s + \frac{1}{2} \chi_1(r)(r - R_1)) \cdot \varphi(s, x) ds \right|^2 \right. \\ & \quad \left. + \left| \int_{-\infty}^{+\infty} \frac{\omega_0}{1 + |\omega_0(t - s)|^3} \chi_2'(s + \frac{1}{2} \chi_1(r)(r - R_1)) \cdot \varphi(s, x) ds \right|^2 \right) dt dg_\Sigma \\ & \leq C \left(\int_{-\infty}^{+\infty} \frac{\omega_0}{1 + |\omega_0(t - s)|^3} ds \right) \\ & \quad \left\{ \omega_0^2 \int_{\Sigma \cap \{r \leq R\}} \int_{\tau_1}^{\tau_2} \int_{-\frac{1}{2} \chi_1(r)(r - R_1)}^{+\infty} \frac{\omega_0}{1 + |\omega_0(t - s)|^3} |\varphi(s, x)|^2 ds dt dg_\Sigma \right. \\ & \quad \left. + \int_{\Sigma \cap \{r \leq R\}} \int_{\tau_1}^{\tau_2} \int_{-\frac{1}{2} \chi_1(r)(r - R_1)}^{1 - \frac{1}{2} \chi_1(r)(r - R_1)} \frac{\omega_0}{1 + |\omega_0(t - s)|^3} |\varphi(s, x)|^2 ds dt dg_\Sigma \right\} \\ & \leq C \left\{ \omega_0^2 \int_{\Sigma \cap \{r \leq R\}} \int_{[\tau_1, \tau_2] \cap [-\frac{1}{2} \chi_1(r)(r - R_1), +\infty)} \left(\int_{\tau_1}^{\tau_2} \frac{\omega_0}{1 + |\omega_0(t - s)|^3} dt \right) |\varphi(s, x)|^2 ds dg_\Sigma \right. \\ & \quad \left. + \omega_0^2 \int_{\Sigma \cap \{r \leq R\}} \int_{[-\frac{1}{2} \chi_1(r)(r - R_1), +\infty) \setminus [\tau_1, \tau_2]} \left(\int_{\tau_1}^{\tau_2} \frac{\omega_0}{1 + |\omega_0(t - s)|^3} dt \right) |\varphi(s, x)|^2 ds dg_\Sigma \right. \\ & \quad \left. + \int_{\Sigma \cap \{r \leq R\}} \int_{-\frac{1}{2} \chi_1(r)(r - R_1)}^{1 - \frac{1}{2} \chi_1(r)(r - R_1)} \left(\int_{\tau_1}^{\tau_2} \frac{\omega_0}{1 + |\omega_0(t - s)|^3} dt \right) |\varphi(s, x)|^2 ds dg_\Sigma \right\} \end{aligned}$$

$$\begin{aligned}
 &\leq C \left\{ \omega_0^2 \int_{\Sigma \cap \{r \leq R\}} \int_{\tau_1}^{\tau_2} |\varphi(s, x)|^2 ds dg_\Sigma \right. \\
 &\quad + \omega_0^2 \int_{\Sigma \cap \{r \leq R\}} \int_{[-\frac{1}{2}\chi_1(r)(r-R_1), +\infty) \setminus [\tau_1, \tau_2]} \\
 &\quad \frac{1}{1 + \min\{|\omega_0(\tau_1 - s)|^2, |\omega_0(\tau_2 - s)|^2\}} |\varphi(s, x)|^2 ds dg_\Sigma \\
 &\quad \left. + \int_{\Sigma \cap \{r \leq R\}} \int_{-\frac{1}{2}\chi_1(r)(r-R_1)}^{1-\frac{1}{2}\chi_1(r)(r-R_1)} \frac{1}{1 + |\omega_0(\tau_1 - s)|^2} |\varphi(s, x)|^2 ds dg_\Sigma \right\}. \tag{5.60}
 \end{aligned}$$

From (5.58), we can estimate for any $0 < a < 1$:

$$\begin{aligned}
 &\omega_0^2 \int_{\Sigma \cap \{r \leq R\}} \int_{[-\frac{1}{2}\chi_1(r)(r-R_1), +\infty) \setminus [\tau_1, \tau_2]} \\
 &\quad \frac{1}{1 + \min\{|\omega_0(\tau_1 - s)|^2, |\omega_0(\tau_2 - s)|^2\}} |\varphi(s, x)|^2 ds dg_\Sigma \\
 &\leq C_a \omega_0^2 (1 + \omega_0^{-1-a}) R^2 (\log(2 + |\tau_2|))^4 \mathcal{E}_{\log}[\varphi] \tag{5.61}
 \end{aligned}$$

and

$$\begin{aligned}
 &\int_{\Sigma \cap \{r \leq R\}} \int_{-\frac{1}{2}\chi_1(r)(r-R_1)}^{1-\frac{1}{2}\chi_1(r)(r-R_1)} \frac{1}{1 + |\omega_0(\tau_1 - s)|^2} |\varphi(s, x)|^2 ds dg_\Sigma \\
 &\leq C_a (1 + \omega_0^{-1-a}) (1 + \omega_0 \tau_1)^{-1} R^2 \mathcal{E}_{\log}[\varphi]. \tag{5.62}
 \end{aligned}$$

Thus, from (5.60) we obtain for any $0 < a < 1$:

$$\begin{aligned}
 &\int_{\Sigma \cap \{r \leq R\}} \int_{\tau_1}^{\tau_2} |\psi_0(s, x)|^2 d\tau dg_\Sigma \\
 &\leq C \omega_0^2 \int_{\Sigma \cap \{r \leq R\}} \int_{\tau_1}^{\tau_2} |\varphi(s, x)|^2 ds dg_\Sigma \\
 &\quad + C_a (\omega_0^2 (\log(2 + |\tau_2|))^4 + (1 + \omega_0 \tau_1)^{-1}) (1 + \omega_0^{-1-a}) R^2 \mathcal{E}_{\log}[\varphi]. \tag{5.63}
 \end{aligned}$$

Repeating the same procedure with $T\psi_0$ and $\nabla_{g_\Sigma}\psi_0$ in place of ψ_0 , we similarly obtain:

$$\begin{aligned}
 &\int_{\Sigma \cap \{r \leq R\}} \int_{\tau_1}^{\tau_2} |T\psi_0(s, x)|^2 d\tau dg_\Sigma \\
 &\leq C \omega_0^2 \int_{\Sigma \cap \{r \leq R\}} \int_{\tau_1}^{\tau_2} |T\varphi(s, x)|^2 ds dg_\Sigma \\
 &\quad + C_a (\omega_0^2 (\log(2 + |\tau_2|))^4 + (1 + \omega_0 \tau_1)^{-1}) (1 + \omega_0^{-1-a}) R^2 \mathcal{E}_{\log}[\varphi] \tag{5.64}
 \end{aligned}$$

and

$$\begin{aligned} & \int_{\Sigma \cap \{r \leq R\}} \int_{\tau_1}^{\tau_2} |\nabla_{g_\Sigma} \psi_0(s, x)|_{g_\Sigma}^2 d\tau dg_\Sigma \\ & \leq C \omega_0^2 \int_{\Sigma \cap \{r \leq R\}} \int_{\tau_1}^{\tau_2} |\nabla_{g_\Sigma} \varphi(s, x)|_{g_\Sigma}^2 ds dg_\Sigma \\ & \quad + C_a (\omega_0^2 (\log(2 + |\tau_2|))^4 + (1 + \omega_0 \tau_1)^{-1} (1 + \omega_0^{-1-a}) R^2 \mathcal{E}_{\log}[\varphi]). \end{aligned} \tag{5.65}$$

Inequality (5.57) readily follows after adding (5.63), (5.64) and (5.65). \square

We will finally establish the following bound for the energy of the high frequency part $\psi_{\geq \omega_+}$ of ψ :

Lemma 5.6. *For any $\tau \geq 0$ and any $m \in \mathbb{N}$ such that*

$$\sum_{j=0}^m \mathcal{E}[T^j \psi] < +\infty, \tag{5.66}$$

there exists a constant $C_m > 0$ depending only on m such that:

$$\int_{\{t=\tau\}} J_\mu^N(\psi_{\geq \omega_+}) n^\mu \leq \frac{C_m}{\omega_+^{2m}} \left(\sum_{j=0}^m \mathcal{E}[T^j \psi] + \mathcal{E}_{\log}[\psi] \right). \tag{5.67}$$

Proof. We can assume without loss of generality that $m \geq 1$, since the $m = 0$ case is a direct consequence of (5.28). Let us introduce the function $\check{\xi}_m : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{C}$ by the formula

$$\check{\xi}_m(t) = \int_{-\infty}^{+\infty} (i\omega_+^{-1}\omega)^{-m} e^{i\omega t} (1 - \chi(\omega_+^{-1}\omega)) d\omega. \tag{5.68}$$

Note that, when $m = 1$, the right hand side of (5.68) diverges when $t = 0$. In view of the bound

$$\left| \int_1^{+\infty} \frac{1}{y^m} e^{i\lambda y} dy \right| \leq \begin{cases} C(|\log(\lambda)| + 1), & m = 1 \\ C, & m > 1, \end{cases} \tag{5.69}$$

as well as the relation

$$\begin{aligned} t \check{\xi}_m(t) &= i \int_{-\infty}^{+\infty} e^{i\omega t} \frac{d}{d\omega} ((i\omega)^{-m} (1 - \chi(\omega_+^{-1}\omega))) d\omega \\ &= -m \omega_+^{-1} \check{\xi}_{m+1}(t) - i \omega_+^{-1} \int_{-\infty}^{+\infty} e^{i\omega t} (i\omega_+^{-1}\omega)^{-m} \chi'(\omega_+^{-1}\omega) d\omega, \end{aligned} \tag{5.70}$$

from (5.68) we infer that for any integer $q \in \mathbb{N}$ and any $t \neq 0$:

$$|\check{\xi}_m(t)| \leq C_{qm} \omega_+ \frac{|\log(|\omega_+ t|)| + 1}{|\omega_+ t|^q + 1}. \tag{5.71}$$

Defining the tempered distribution

$$\zeta_{\geq \omega_+} \doteq \delta_D - \zeta_{\leq \omega_+}, \tag{5.72}$$

where δ_D is Dirac's delta function and $\zeta_{\leq \omega_+}$ is defined by (5.20), the Fourier transforms of $\check{\xi}_m$ and $\zeta_{\geq \omega_+}$ satisfy the relation:

$$\hat{\zeta}_{\geq \omega_+} = \omega_+^{-m} \widehat{\check{\xi}}_m \cdot (i\omega)^m \hat{\zeta}_{\geq \omega_+}, \tag{5.73}$$

yielding the following relation for $\psi_{\geq\omega_+}$ in physical space:

$$\psi_{\geq\omega_+}(t, \cdot) = \omega_+^{-m} \int_{-\infty}^{+\infty} \check{\xi}_m(t-s) T^m \psi_{\geq\omega_+}(s, \cdot) ds, \tag{5.74}$$

where, again, the integral in the right hand side of (5.74) converges in the $\int_{\Sigma_t} J_\mu^N(\cdot) n^\mu$ norm.

From (5.74) and (5.71) we can estimate for any $\tau \in \mathbb{R}$:

$$\begin{aligned} & \int_{\Sigma} (|T\psi_{\geq\omega_+}(\tau, x)|^2 + |\nabla_{g_\Sigma} \psi_{\geq\omega_+}(\tau, x)|_{g_\Sigma}^2) dg_\Sigma \\ &= \omega_+^{-2m} \int_{\Sigma} \left(\left| \int_{-\infty}^{+\infty} \check{\xi}_m(\tau-s) T^{m+1} \psi_{\geq\omega_+}(s, x) ds \right|^2 \right. \\ & \quad \left. + \left| \int_{-\infty}^{+\infty} \check{\xi}_m(\tau-s) \nabla_{g_\Sigma} T^m \psi_{\geq\omega_+}(s, x) ds \right|_{g_\Sigma}^2 \right) dg_\Sigma \\ &\leq C_m \omega_+^{-2m} \left\{ \int_{\Sigma} \left(\int_{-\infty}^{+\infty} \omega_+ \frac{|\log(|\omega_+(\tau-s)|)|+1}{|\omega_+(\tau-s)|^4+1} T^{m+1} \psi_{\geq\omega_+}(s, x) ds \right)^2 \right. \\ & \quad \left. + \left| \int_{-\infty}^{+\infty} \omega_+ \frac{|\log(|\omega_+(\tau-s)|)|+1}{|\omega_+(\tau-s)|^4+1} \nabla_{g_\Sigma} T^m \psi_{\geq\omega_+}(s, x) ds \right|_{g_\Sigma}^2 \right\} dg_\Sigma \\ &\leq C_m \omega_+^{-2m} \left(\int_{-\infty}^{+\infty} \omega_+ \frac{|\log(|\omega_+(\tau-s)|)|+1}{|\omega_+(\tau-s)|^4+1} ds \right) \\ & \quad \times \left(\int_{\Sigma} \int_{-\infty}^{+\infty} \omega_+ \frac{|\log(|\omega_+(\tau-s)|)|+1}{|\omega_+(\tau-s)|^4+1} (|T^{m+1} \psi_{\geq\omega_+}(s, x)|^2 \right. \\ & \quad \left. + |\nabla_{g_\Sigma} T^m \psi_{\geq\omega_+}(s, x)|_{g_\Sigma}^2) ds dg_\Sigma \right) \\ &\leq C_m \omega_+^{-2m} \left(\int_{\Sigma} \int_{-\infty}^{+\infty} \omega_+ \frac{|\log(|\omega_+(\tau-s)|)|+1}{|\omega_+(\tau-s)|^4+1} (|T^{m+1} \psi_{\geq\omega_+}(s, x)|^2 \right. \\ & \quad \left. + |\nabla_{g_\Sigma} T^m \psi_{\geq\omega_+}(s, x)|_{g_\Sigma}^2) ds dg_\Sigma \right). \end{aligned} \tag{5.75}$$

In view of (5.25) and the Schwartz bounds (5.22), we readily obtain that for any $\tau \in \mathbb{R}$:

$$\begin{aligned} & \int_{\Sigma} (|T^{m+1} \psi_{\geq\omega_+}(\tau, x)|^2 + |\nabla_{g_\Sigma} T^m \psi_{\geq\omega_+}(\tau, x)|_{g_\Sigma}^2) dg_\Sigma \\ &\leq C \int_{-\infty}^{+\infty} \frac{\omega_+}{(1+\omega_+|\tau-s|)^4} \left(\int_{\Sigma} (|T^{m+1} \psi_c(s, x)|^2 + |\nabla_{g_\Sigma} T^m \psi_c(s, x)|_{g_\Sigma}^2) dg_\Sigma \right) ds. \end{aligned} \tag{5.76}$$

In view of the definition (5.12) of ψ_c , (5.76) yields

$$\begin{aligned} & \int_{\Sigma} (|T^{m+1} \psi_{\geq\omega_+}(\tau, x)|^2 + |\nabla_{g_\Sigma} T^m \psi_{\geq\omega_+}(\tau, x)|_{g_\Sigma}^2) dg_\Sigma \\ &\leq C \sum_{j=0}^m \int_{-\infty}^{+\infty} \frac{\omega_+}{(1+\omega_+|\tau-s|)^4} \left\{ \int_{\Sigma \cap \{t \geq 0\}} (|T^{j+1} \psi(s, x)|^2 \right. \\ & \quad \left. + |\nabla_{g_\Sigma} T^j \psi(s, x)|_{g_\Sigma}^2) dg_\Sigma + \int_{\Sigma \cap \{0 \leq t \leq 1\}} |\psi(s, x)|^2 dg_\Sigma \right\} ds. \end{aligned} \tag{5.77}$$

Thus, (5.75), (5.77), (5.10), and (5.66) (combined with the conservation of the T -energy flux in the region $\{t_- \geq 0\} \cap \{t \leq 0\}$) imply:

$$\begin{aligned} & \int_{\Sigma} (|T\psi_{\geq\omega_+}(\tau, x)|^2 + |\nabla_{g_{\Sigma}}\psi_{\geq\omega_+}(\tau, x)|^2_{g_{\Sigma}}) dg_{\Sigma} \\ & \leq C_m \omega_+^{-2m} \left(\int_{-\infty}^{+\infty} \omega_+ \frac{|\log(|\omega_+(\tau-s)|)| + 1}{|\omega_+(\tau-s)|^4 + 1} (1 + |s|^2) (\log(2 + |s|))^4 ds \right) \\ & \quad \left(\sum_{j=0}^m \mathcal{E}[T^j \psi] + \mathcal{E}_{\log}[\psi] \right), \end{aligned} \tag{5.78}$$

from which (5.67) readily follows. \square

6. A Carleman-Type Estimate Outside the Extended Ergoregion

In this section, we will establish the following estimate for solutions φ to the inhomogeneous wave equation

$$\square_g \varphi = G \tag{6.1}$$

on (\mathcal{M}, g) :

Proposition 6.1. *For any $s, R \gg 1$ sufficiently large in terms of the geometry of (\mathcal{M}, g) and any $0 < \epsilon_0 < 1$, there exists a smooth T -invariant function $f : \mathcal{M} \setminus \mathcal{H}^- \rightarrow (0, +\infty)$ satisfying*

$$f = \begin{cases} e^{2s w_R} + e^{2s \tilde{w}_R}, & r \leq R, \\ C_s \left(\frac{r}{R} - \frac{9}{10} \log\left(\frac{r}{R}\right) \right) & r \geq R, \end{cases} \tag{6.2}$$

where the functions $w_R, \tilde{w}_R : \{r \leq R\} \rightarrow \mathbb{R}$ satisfy

1. $w_R \equiv \tilde{w}_R$ on $\{r \leq \frac{1}{4}R_0\} \cup \mathcal{E}_{ext} \cup \{r \geq \frac{1}{2}R_0\}$,
2. $\sup_{\{r \leq R\}} w_R - \inf_{\{r \leq R\}} w_R + \sup_{\{r \leq R\}} \tilde{w}_R - \inf_{\{r \leq R\}} \tilde{w}_R \leq C \epsilon_0^{-1} R^{3\epsilon_0}$ for some absolute constant $C > 0$,
3. $\inf_{\{\frac{1}{4}R_0 \leq r \leq R\} \setminus \mathcal{E}_{2\delta}} w_R \geq \max_{\mathcal{E}_{\delta}} w_R + c_{\delta} R^{-3\epsilon_0}$ and $\inf_{\{\frac{1}{4}R_0 \leq r \leq R\} \setminus \mathcal{E}_{2\delta}} \tilde{w}_R \geq \max_{\mathcal{E}_{\delta}} \tilde{w}_R + c_{\delta} R^{-3\epsilon_0}$ for any $0 < \delta \ll 1$,
4. $\sum_{j=1}^4 (|\nabla^j w_R|_{g_{ref}} + |\nabla^j \tilde{w}_R|_{g_{ref}}) \leq C$,

so that the following statement holds: For any $0 < \delta, \epsilon_0 \ll 1$, any $s, R \gg 1$ satisfying $\epsilon_0 s R^{-9\epsilon_0} \gg 1$, any $0 \leq \tau_1 \leq \tau_2$ and any smooth function $\varphi : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{C}$ solving (6.1) with compact support on the hypersurfaces $\{t = \tau\}$ for any $\tau_1 \leq \tau \leq \tau_2$, we can estimate:

$$\begin{aligned} & \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq R_0\} \setminus \mathcal{E}_{\delta}} (f + \inf_{\{r \geq \frac{1}{4}R_0\} \setminus \mathcal{E}} f) \left\{ s R^{-3\epsilon_0} |\nabla_{g_{\Sigma}} \varphi|^2_{g_{\Sigma}} - C_{\delta} s R^{-3\epsilon_0} |T\varphi|^2 + s^3 R^{-9\epsilon_0} |\varphi|^2 \right\} dg \\ & + \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{R_0 \leq r \leq \frac{1}{2}R\}} f \\ & \left\{ s R^{-3\epsilon_0} r^{-\frac{5}{2}} (|\partial_r \varphi|^2 + r^{-2} |\partial_{\sigma} \varphi|^2) + s R^{-3\epsilon_0} r^{-2} |T\varphi|^2 + \epsilon_0 s^3 R^{-9\epsilon_0} r^{-4} |\varphi|^2 \right\} dg \\ & + \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{2}R \leq r \leq R\}} f \left\{ r^{-\frac{5}{2}} (|\partial_r \varphi|^2 + r^{-2} |\partial_{\sigma} \varphi|^2) + R \partial_r w_R (c R^{-2} s |T\varphi|^2 - C R^{-4} s^3 |\varphi|^2) \right\} dg \end{aligned}$$

$$\begin{aligned}
 & + \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \geq R\}} f(R) \left\{ r^{-\frac{5}{2}} (|\partial_r \varphi|^2 + r^{-2} |\partial_\sigma \varphi|^2) + r^{-2} |T\varphi|^2 - CR^{-1} r^{-3} |\varphi|^2 \right\} dg \\
 & \leq C_\delta \int_{\mathcal{R}(\tau_1, \tau_2) \cap \mathcal{E}_\delta} f \left\{ s^2 R^{-6\epsilon_0} |\nabla \varphi|_{g_{ref}}^2 + s^4 R^{-12\epsilon_0} |\varphi|^2 \right\} dg \\
 & + C \left| \int_{\mathcal{R}(\tau_1, \tau_2)} G(\nabla^\mu f \nabla_\mu \bar{\varphi} + O\left(\sum_{j=1}^2 (1+r)^{j-2} |\nabla^j f|_{g_{ref}} \bar{\varphi}\right)) dg \right| \\
 & + C \sum_{j=1}^2 \int_{\Sigma_{\tau_j}} \left(|\nabla f|_{g_{ref}} |\nabla \varphi|_{g_{ref}}^2 + \left(\sum_{j=1}^3 (1+r)^{j-3} |\nabla^j f|_{g_{ref}}\right) |\varphi|^2 \right) dg_{\Sigma}. \tag{6.3}
 \end{aligned}$$

The proof of Proposition 6.1 will be given in Sect. 6.7. It will be based on the construction of a suitable multiplier for the inhomogeneous wave equation (6.1), which will be presented in Sects. 6.2–6.3, as well as an intricate integration-by-parts procedure, that will be performed in Sect. 6.4.

Remark. In fact, Proposition 6.1 also holds in the case when $\mathcal{E} = \emptyset$. We should also remark that Proposition 6.1 applies in the case when (\mathcal{M}, g) has a T -invariant timelike boundary component $\partial_{tim}\mathcal{M}$, with $\partial_{tim}\mathcal{M} \cap \Sigma$ compact and $\partial_{tim}\mathcal{M} \cap \mathcal{H} = \emptyset$, and φ is assumed to satisfy either Dirichlet or Neumann boundary conditions on $\partial_{tim}\mathcal{M}$ (see Sect. 6.9 for more details).

Furthermore, the proof of Proposition 6.1 applies without any change after replacing equation (6.1) with

$$\square_g \varphi - V\varphi = G, \tag{6.4}$$

for any smooth and T -invariant function $V : \mathcal{M} \rightarrow \mathbb{R}$ satisfying either $\partial_r V < 0$ and $V \rightarrow 0$ as $r \rightarrow +\infty$ in the asymptotically flat region of (\mathcal{M}, g) , or $\sup_{\mathcal{M}} \left((1+r)^{2+\eta} |V| \right) < +\infty$ for some $\eta > 0$. In the later case, the constants in the analogue of (6.3) can be chosen to depending only on η and $\sup_{\mathcal{M}} \left((1+r)^{2+\eta} |V| \right)$. In view of the fact that the proof of Proposition 6.1 also applies for equation (6.4), the degeneracy of (6.3) in the frequency regime $\{|\omega| \ll 1\}$ can not be fully removed without some additional argument depending on the fact that $V = 0$ in equation (6.1), or yielding an estimate depending qualitatively on V (see also the discussion in [34]). However, this degeneracy does not cause any problem in the proof of Theorem 2.1.

Finally, let us remark that the estimate (6.3) can be readily used to show that any smooth solution φ to equation (1.2) on \mathcal{M} of the form $\varphi = e^{-i\omega t} \varphi_\omega$ with $\omega \in \mathbb{R} \setminus \{0\}$, $T(\varphi_\omega) = 0$ and

$$\lim_{\rho \rightarrow +\infty} \int_{\{r=\rho\} \cap \{t=0\}} (|\varphi_\omega|^2 + |\nabla \varphi_\omega|^2) = 0 \tag{6.5}$$

vanishes identically on $\mathcal{M} \setminus \mathcal{E}_{ext}$.

As a corollary of Proposition 6.1, given $\omega_+ > 1$ and $0 < \omega_0 < 1$, we will establish the following estimate for the frequency localised components ψ_k of any solution ψ to the wave equation (1.2) on (\mathcal{M}, g) satisfying the bound (5.1) (see the relevant constructions in Sect. 5):

Corollary 6.1. *For any smooth solution ψ to (1.2) satisfying (5.1), any integer $1 \leq k \leq n$, any $0 < \delta_1, \delta_2, \epsilon_0 < 1$, any $R_1 \geq 0$, any $0 \leq \tau_1 \leq \tau_2$ we can bound:*

$$\begin{aligned} & \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq R_1\} \setminus \mathcal{E}_{2\delta_1}} \left(J_\mu^N(\psi_k) n^\mu + |\psi_k|^2 \right) \\ & \leq \delta_2 \int_{\mathcal{R}(\tau_1, \tau_2) \cap \mathcal{E}_{\delta_1}} \left(J_\mu^N(\psi_k) n^\mu + |\psi_k|^2 \right) \\ & \quad + C_{\epsilon_0 \delta_1 R_1} (1 + \omega_k^{-10}) (\log(2 + \tau_2))^4 \cdot e^{C_{\epsilon_0 \delta_1} \cdot \max\{|\omega_k|, |\omega_k|^{-\epsilon_0}, -\log \delta_2\}} \mathcal{E}_{\log}[\psi], \end{aligned} \tag{6.6}$$

where $C_{\epsilon_0 \delta_1 R_1}$ depends only on $\epsilon_0, \delta_1, R_1$ and the geometry of (\mathcal{M}, g) , while $C_{\epsilon_0 \delta_1}$ depends only on ϵ_0, δ_1 and the geometry of (\mathcal{M}, g) .

The proof of Corollary 6.1 will be presented in Sect. 6.8.

Finally, let us sketch an additional application of Proposition 6.1 in the Riemannian setting. Let (Σ^d, \bar{g}) , $d \geq 3$, be an asymptotically conic Riemannian manifold, with the asymptotics described in [34], and let us consider the unique solution $u \in L^2(\Sigma)$ of the inhomogeneous Helmholtz equation

$$\Delta_{\bar{g}} u + \omega^2 u - V u = G \tag{6.7}$$

on (Σ, \bar{g}) for a suitably decaying source term $G : \Sigma \rightarrow \mathbb{C}$, with $0 < \text{Im}(\omega) \ll 1$, $\text{Re}(\omega) \neq 0$ and a potential $V : \Sigma \rightarrow \mathbb{R}$ satisfying either $\partial_r V < 0$ in the asymptotically conic region of (Σ, \bar{g}) and $V \rightarrow 0$ as $r \rightarrow +\infty$ (where r is the radial coordinate function in the asymptotically conic region of Σ , extended to a positive function everywhere on Σ), or

$$\sup_{\Sigma} \left((r^{-2-\eta} + |\omega| r^{-1-\eta}) |V| \right) < +\infty. \tag{6.8}$$

Then, applying Proposition 6.1 on the product spacetime $(\mathbb{R} \times \Sigma, g = -dt^2 + \bar{g})$ for the function $\varphi = e^{-i\omega t} u$ solving (in view of (6.7))

$$\square_g \varphi - V \varphi = e^{-i\omega t} G, \tag{6.9}$$

and using the charge estimate

$$\text{Im}(\omega^2) \int_{\Sigma} |u|^2 d\bar{g} = \int_{\Sigma} \text{Im}(G\bar{u}) d\bar{g} \tag{6.10}$$

(combined with elliptic estimates for (6.7), as is done, for instance, in [34]) one readily obtains the (quantitative in V) global Carleman-type estimates of [34], albeit with a worse dependence on ω as $\text{Re}(\omega) \rightarrow 0$. Thus, the proof of Proposition 6.1 yields a proof of the Carleman-type estimates used in [30, 34] based entirely on the method of first order multipliers.

Remark. A multiplier-based proof of a similar set of Carleman-type estimates for equation (6.7) restricted, however, to the high frequency regime $\omega \gg 1$ was obtained previously in [18].

6.1. *Parameters and cut-off functions in the proof of Proposition 6.1.* Let $R_0 \gg 1$ be large in terms of the geometry of (\mathcal{M}, g) , such that $\{r \geq \frac{1}{8}R_0\} \subset \mathcal{I}_{as}(R_0)$ will be considered fixed and, thus, we will not use any special notation to denote the dependence of constants on R_0). In addition to the parameters $\delta, \varepsilon_0, s, R$ appearing in the statement of Proposition 6.1, we will introduce the parameters $R \gg R_0$ and $0 < \delta_0, \delta_1, \delta_2 \ll 1$. We will assume without loss of generality that $0 < \varepsilon_0 \ll 1$. These additional parameters will be fixed in the proof of Proposition 6.1.

In the region $\{r \geq \frac{1}{2}R_0\}$, the vector field ∂_r will simply denote the associated coordinate vector field in the (t, r, σ) coordinate chart in each connected component of this region.

Fixing a smooth function $\chi_4 : \mathbb{R} \rightarrow [0, 1]$ satisfying $\chi_4(x) = 0$ for $x \leq \frac{3}{4}$ and $\chi_4(x) = 1$ for $x \geq 1$, we will define the following smooth cut-off functions:

$$\chi_{\geq R_0}(r) \doteq \chi_4\left(\frac{r}{R_0}\right), \tag{6.11}$$

$$\chi_{\leq R}(r) \doteq \chi_4\left(\frac{R}{r}\right). \tag{6.12}$$

Remark. Note that $\chi_{\leq R} \equiv 1$ for $r \leq R$ and $\chi_{\leq R} \equiv 0$ for $r \geq \frac{4}{3}R$, while $\chi_{\geq R_0} \equiv 1$ for $r \geq R_0$ and $\chi_{\geq R_0} \equiv 0$ for $r \leq \frac{3}{4}R_0$.

6.2. *Construction of the auxiliary functions w_R, \tilde{w}_R .* In this section, we will construct the pair of functions $w_R, \tilde{w}_R : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{R}$ appearing in the statement of Proposition 6.1, depending on the parameters $\delta_0, \delta_1, \varepsilon_0, s, R$. These functions will be used extensively in the next sections.

First, we will establish the following lemma:

Lemma 6.1. *There exists a smooth and T -invariant function $\bar{w} : \mathcal{M} \setminus \mathcal{H} \rightarrow \mathbb{R}$ satisfying the following properties:*

1. *The restriction $\bar{w}|_\Sigma$ of \bar{w} on Σ is a Morse function on $\overline{\Sigma \setminus \mathcal{E}_{ext}}$, with no critical points on $\partial \mathcal{E}_{ext}$. Furthermore, none of the (at most finite) critical points $\{x_j\}_{j=1}^k$ of $\bar{w}|_\Sigma$ on $\Sigma \setminus \mathcal{E}_{ext}$ is a point of local maximum of $\bar{w}|_\Sigma$.*
2. *In the region $\{0 < r \leq \frac{1}{8}r_0\}$,¹³ \bar{w} is a function of r and satisfies*

$$\nabla^\mu r \nabla_\mu \bar{w} > 0. \tag{6.13}$$

3. *In the region $\{r \geq \frac{1}{2}R_0\}$, \bar{w} is a function of r , and satisfies (6.13).*

4. *On $(\overline{\mathcal{M} \setminus \mathcal{E}_{ext}} \cup \mathcal{H}) \setminus (\mathbb{R} \times \cup_{j=1}^k \{x_j\})$ we have*

$$\nabla^\mu \bar{w} \nabla_\mu \bar{w} > 0. \tag{6.14}$$

5. *For any $0 < \delta \ll 1$, we have*

$$\inf_{\{r \geq \frac{1}{4}r_0\} \setminus \mathcal{E}_{2\delta}} \bar{w} > \max_{\mathcal{E}_\delta} \bar{w}. \tag{6.15}$$

¹³ Recall that $\{r \leq \frac{1}{8}r_0\} \cap \Sigma$ is a neighborhood of $\mathcal{H}^+ \cap \Sigma$ in Σ .

Proof. The proof of Lemma 6.1 will be based on ideas from [30,34].

Let $R_0 \gg 1$ be a fixed constant large in terms of the geometry of (\mathcal{M}, g) . For any $0 \leq \gamma < 1$, let $\bar{w}_\gamma : (\Sigma \cap \{\frac{1}{6}r_0 \leq r \leq \frac{1}{4}R_0\}) \setminus \mathcal{E}_{ext} \rightarrow \mathbb{R}$ be the (unique) smooth solution of the elliptic boundary value problem:

$$\begin{cases} \Delta_{g_\Sigma} \bar{w}_\gamma = \gamma & \text{on } (\Sigma \cap \{\frac{1}{6}r_0 < r < \frac{1}{4}R_0\}) \setminus \mathcal{E}_{ext}, \\ \bar{w}_\gamma|_{r=\frac{1}{4}R_0} = 2, \\ \bar{w}_\gamma|_{r=\frac{1}{6}r_0} = 1, \\ \bar{w}_\gamma|_{\partial \mathcal{E}_{ext}} = 1. \end{cases} \tag{6.16}$$

Let us extend \bar{w}_γ on the whole of $\{\frac{1}{6}r_0 < r < \frac{1}{4}R_0\} \setminus \mathcal{E}_{ext} \subset \mathcal{M} \setminus \mathcal{H}^-$ by the requirement that $T\bar{w}|_\gamma = 0$.

Since g_Σ is smooth, \bar{w}_γ depends smoothly on γ (see [23]). In view of the fact that every connected component of $\mathcal{M} \setminus \mathcal{E}$ intersecting \mathcal{H}^+ also intersects \mathcal{I}_{as} (see Assumption G3), when $\gamma = 0$, the maximum principle and Hopf’s lemma (see [23]) imply that for any $\delta > 0$

$$\inf_{r=\frac{1}{4}R_0} (\nabla^\mu r \nabla_\mu \bar{w}_0), \quad \inf_{r=\frac{1}{6}r_0} (\nabla^\mu r \nabla_\mu \bar{w}_0), \quad \inf_{\partial \mathcal{E}_{ext}} (n_{\partial \mathcal{E}}(\bar{w}_0)), \quad (\inf_{\{r \geq \frac{1}{4}r_0\} \setminus \mathcal{E}_\delta} \bar{w}_0 - \max_{\partial \mathcal{E}_{ext}} \bar{w}_0) > 0 \tag{6.17}$$

(see Sect. 3.3 for the definition of $n_{\partial \mathcal{E}}$). Therefore, there exists a $\gamma_0 \in (0, 1)$ and a $c_0 > 0$, such that:

$$\inf_{r=\frac{1}{4}R_0} (\nabla^\mu r \nabla_\mu \bar{w}_{\gamma_0}), \quad \inf_{r=\frac{1}{6}r_0} (\nabla^\mu r \nabla_\mu \bar{w}_{\gamma_0}), \quad \inf_{\mathcal{E}_{c_0} \setminus \mathcal{E}_{ext}} (n_{\partial \mathcal{E}}(\bar{w}_{\gamma_0})) \geq c_0 > 0 \tag{6.18}$$

and, for all $0 < \delta < 1$ (and some fixed $c_1 > 0$):

$$\inf_{\{r \geq \frac{1}{4}r_0\} \setminus \mathcal{E}_\delta} \bar{w}_{\gamma_0} - \max_{\partial \mathcal{E}_{ext}} \bar{w}_{\gamma_0} > c_1 \delta > 0. \tag{6.19}$$

In view of 6.18 and (6.19), we can extend \bar{w}_{γ_0} as a T -invariant function on the whole of $\mathcal{M} \setminus \mathcal{H}^-$ in such a way, so that

$$\inf_{\{\frac{1}{16}r_0 \leq r \leq \frac{1}{6}r_0\} \cup \{\frac{1}{4}R_0 \leq r \leq R_0\}} (\nabla^\mu r \nabla_\mu \bar{w}_{\gamma_0}) \geq \frac{1}{10}c_0 > 0, \tag{6.20}$$

$$\inf_{\{\delta \leq r \leq \frac{1}{6}r_0\}} (\nabla^\mu r \nabla_\mu \bar{w}_{\gamma_0}) \geq c(\delta) > 0, \tag{6.21}$$

$$\inf_{\{r \geq \frac{1}{4}r_0\} \setminus \mathcal{E}_{2\delta}} \bar{w}_{\gamma_0} > \max_{\mathcal{E}_\delta} \bar{w}_{\gamma_0} + \frac{1}{2}c_1 \delta \tag{6.22}$$

for all $0 < \delta \ll 1$ (where $c(\delta)$ is a positive function of $\delta > 0$) and, in addition, \bar{w}_{γ_0} is a function of r in the region $\{r \leq \frac{1}{8}r_0\} \cup \{r \geq \frac{1}{2}R_0\}$. Notice also that, since $\bar{w}_{\gamma_0}|_{\partial \mathcal{E}_{ext}} = 1$ and $n_{\partial \mathcal{E}}(\bar{w}_{\gamma_0})|_{\partial \mathcal{E}_{ext}} \geq c_0$, we have

$$(\nabla^\mu \bar{w}_{\gamma_0} \nabla_\mu \bar{w}_{\gamma_0})|_{\partial \mathcal{E}_{ext}} \geq \frac{1}{2}c_0^2. \tag{6.23}$$

With \bar{w}_{γ_0} constructed as above, we can thus readily choose \bar{w} so that it satisfies the following conditions:

1. $\bar{w}|_\Sigma$ and $\bar{w}_{\gamma_0}|_\Sigma$ satisfy

$$|\bar{w}|_\Sigma - \bar{w}_{\gamma_0}|_\Sigma|_{C^2(\Sigma)} < \frac{1}{100} \min\{\gamma_0, c_0\}. \tag{6.24}$$

- 2. $\bar{w}|_\Sigma$ is a Morse function on an open neighborhood of $(\Sigma \cap \{\frac{1}{6}r_0 \leq r \leq \frac{1}{4}R_0\}) \setminus \mathcal{E}_{ext}$.
- 3. $\bar{w}|_\Sigma = \bar{w}_{\gamma_0}|_\Sigma$ in the region $\mathcal{E} \cup \{r \leq \frac{1}{6}r_0\} \cup \{r \geq \frac{1}{4}R_0\}$.
- 4. $T\bar{w} = 0$.

Remark. Note that the compatibility of conditions 1 and 2 follows from the density of the set of Morse functions on $(\Sigma \cap \{\frac{1}{6}r_0 \leq r \leq \frac{1}{4}R_0\}) \setminus \mathcal{E}_{ext}$ in $C^2(\overline{(\Sigma \cap \{\frac{1}{6}r_0 \leq r \leq \frac{1}{4}R_0\}) \setminus \mathcal{E}_{ext}})$.

In view of (6.16), (6.20), (6.21) and (6.24), \bar{w} satisfies

$$\Delta_{g_\Sigma} \bar{w}|_\Sigma > 0 \tag{6.25}$$

on $\overline{\{\frac{1}{6}r_0 < r < \frac{1}{4}R_0\} \setminus \mathcal{E}_{ext}}$ and

$$\inf_{\{\frac{1}{16}r_0 \leq r \leq \frac{1}{8}r_0\} \cup \{r \geq \frac{1}{4}R_0\}} (\nabla^\mu r \nabla_\mu \bar{w}) \geq \frac{1}{10} c_0 > 0, \tag{6.26}$$

$$\inf_{\{\delta \leq r \leq \frac{1}{8}r_0\}} (\nabla^\mu r \nabla_\mu \bar{w}) \geq c(\delta) > 0. \tag{6.27}$$

Therefore, none of the critical points of $\bar{w}|_\Sigma$ on $\overline{\Sigma \setminus \mathcal{E}_{ext}}$ is a point of local maximum. Furthermore, Conditions 2 and 3 imply that $\bar{w}|_\Sigma$ is a Morse function on $\overline{\Sigma \setminus \mathcal{E}_{ext}}$. Since $T(\bar{w}) = 0$ and T is strictly timelike on $\mathcal{M} \setminus (\mathcal{E}_{ext} \cup \mathcal{H})$, in view of (6.23) and Condition 1 we have

$$\nabla^\mu \bar{w} \nabla_\mu \bar{w} > 0 \text{ on } (\overline{\mathcal{M} \setminus \mathcal{E}_{ext} \cup \mathcal{H}}) \setminus (\mathbb{R} \times \cup_{j=1}^k \{x_j\}), \tag{6.28}$$

where $\{x_j\}_{j=1}^k$ are the (at most finite) critical points of $\bar{w}|_\Sigma$ on $\overline{\Sigma \setminus \mathcal{E}_{ext}}$, none of which lies on $\partial \mathcal{E}_{ext}$ (in view of (6.23) and Condition 1). Finally, in view of (6.18), (6.22) and (6.24), inequality (6.15) holds for all $0 < \delta \ll 1$. \square

Lemma 6.2. *For any $0 < \delta_0 \ll 1$ small in terms of the geometry of (\mathcal{M}, g) , there exists a pair of smooth and T -invariant functions $w, \tilde{w} : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{R}$, as well as a finite number of points $\{x_j\}_{j=1}^k, \{\tilde{x}_j\}_{j=1}^k \in \Sigma \cap \{r \leq \frac{1}{2}R_0 + 4\delta_0\} \setminus \mathcal{E}_{8\delta_0}$, such that the following statements hold:*

1. *Defining for any $\rho > 0$ the subsets*

$$\mathcal{B}_{crit}(\rho) = \mathbb{R} \times (\cup_{j=1}^k B_{g_\Sigma}(x_j, \rho)), \tag{6.29}$$

$$\tilde{\mathcal{B}}_{crit}(\rho) = \mathbb{R} \times (\cup_{j=1}^k B_{g_\Sigma}(\tilde{x}_j, \rho)) \tag{6.30}$$

of $\mathcal{M} \setminus \mathcal{H}^-$, where $B_{g_\Sigma}(x_j, \rho) \subset (\Sigma, g_\Sigma)$ is the closed Riemannian ball of radius ρ centered at x_j , we have:

$$\mathcal{B}_{crit}(\delta_0) \subset \tilde{\mathcal{B}}_{crit}(4\delta_0), \tag{6.31}$$

$$\tilde{\mathcal{B}}_{crit}(\delta_0) \subset \mathcal{B}_{crit}(4\delta_0), \tag{6.32}$$

$$\mathcal{B}_{crit}(\delta_0) \cap \tilde{\mathcal{B}}_{crit}(\delta_0) = \emptyset. \tag{6.33}$$

2. The functions w, \tilde{w} coincide outside $\mathcal{B}_{crit}(4\delta_0)$:

$$w \equiv \tilde{w} \text{ on } \mathcal{M} \setminus (\mathcal{B}_{crit}(4\delta_0) \cup \mathcal{H}^-). \tag{6.34}$$

3. The functions w, \tilde{w} satisfy the following non-degeneracy conditions for some absolute constant $c_0 > 0$ (independent of δ_0):

$$\inf_{\{r \geq \frac{1}{8}r_0\} \setminus (\mathcal{E}_{ext} \cup \mathcal{B}_{crit}(\delta_0))} \nabla^\mu \nabla^\nu w \nabla_\mu w \nabla_\nu w \geq c_0 > 0, \tag{6.35}$$

$$\inf_{\{r \geq \frac{1}{8}r_0\} \setminus (\mathcal{E}_{ext} \cup \tilde{\mathcal{B}}_{crit}(\delta_0))} \nabla^\mu \nabla^\nu \tilde{w} \nabla_\mu \tilde{w} \nabla_\nu \tilde{w} \geq c_0 > 0. \tag{6.36}$$

4. For any T -invariant vector fields X, \tilde{X} on $\{r \geq \frac{1}{8}r_0\} \setminus (\mathcal{E}_{ext} \cup \mathcal{B}_{crit}(\delta_0))$ such that $X(w) = 0$ and $\tilde{X}(\tilde{w}) = 0$, the following one sided bounds hold:

$$\nabla_\mu \nabla_\nu w X^\mu X^\nu > -\delta_0 \frac{\nabla^\mu \nabla^\nu w \nabla_\mu w \nabla_\nu w}{g_{ref}(dw, dw)} g_{ref}(X, X), \tag{6.37}$$

$$\nabla_\mu \nabla_\nu \tilde{w} \tilde{X}^\mu \tilde{X}^\nu > -\delta_0 \frac{\nabla^\mu \nabla^\nu \tilde{w} \nabla_\mu \tilde{w} \nabla_\nu \tilde{w}}{g_{ref}(d\tilde{w}, d\tilde{w})} g_{ref}(\tilde{X}, \tilde{X}), \tag{6.38}$$

where g_{ref} is the reference Riemannian metric (2.4).

5. The functions w, \tilde{w} satisfy

$$\max_{\mathcal{B}_{crit}(\delta_0)} w < \min_{\mathcal{B}_{crit}(\delta_0)} \tilde{w}, \tag{6.39}$$

$$\max_{\tilde{\mathcal{B}}_{crit}(\delta_0)} \tilde{w} < \min_{\tilde{\mathcal{B}}_{crit}(\delta_0)} w. \tag{6.40}$$

6. For any $0 < \delta \ll 1$:

$$\inf_{\{r \geq \frac{1}{4}r_0\} \setminus \mathcal{E}_{2\delta}} w > \max_{\mathcal{E}_\delta} w, \tag{6.41}$$

$$\inf_{\{r \geq \frac{1}{4}r_0\} \setminus \mathcal{E}_{2\delta}} \tilde{w} > \max_{\mathcal{E}_\delta} \tilde{w} \tag{6.42}$$

and

$$\inf_{\{r \geq \frac{1}{2}r_0\} \setminus \mathcal{E}_{ext}} w > \max_{\{r \leq \frac{1}{4}r_0\}} w, \tag{6.43}$$

$$\inf_{\{r \geq \frac{1}{2}r_0\} \setminus \mathcal{E}_{ext}} \tilde{w} > \max_{\{r \leq \frac{1}{4}r_0\}} \tilde{w} \tag{6.44}$$

Proof. Let $\bar{w} : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{R}$ be as in the statement of Lemma 6.1, and let $\{x_j\}_{j=1}^k$ be the (at most finite) critical points of $\bar{w}|_\Sigma$ in $\overline{\Sigma \setminus \mathcal{E}_{ext}}$. According to Lemma 6.1, none of these points lies on $\partial \mathcal{E}_{ext}$ or on $\{r \leq \frac{1}{8}r_0\} \cup \{r \geq \frac{1}{2}R_0\}$ and, thus, provided $\delta_0 \ll 1$, we have

$$\{x_j\}_{j=1}^k \in \Sigma \cap \{\frac{1}{8}r_0 \leq r \leq \frac{1}{2}R_0\} \setminus \mathcal{E}_{16\delta_0}. \tag{6.45}$$

Let $l > 0$ be large in terms of δ_0 , and let us define the T -invariant function $w : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{R}$ as

$$w = e^{l\bar{w}}. \tag{6.46}$$

Then, the one sided bounds (6.35) and (6.37) readily follow from the properties of \bar{w} (see Lemma 6.1), as well as the identities

$$\nabla_\mu w = l(\nabla_\mu \bar{w})e^{l\bar{w}} \tag{6.47}$$

and

$$\nabla_\mu \nabla_\nu w = (l^2(\nabla_\mu \bar{w})(\nabla_\nu \bar{w}) + l\nabla_\mu \nabla_\nu \bar{w})e^{l\bar{w}}, \tag{6.48}$$

provided l is sufficiently large in terms of δ_0 . Inequality (6.41) follows readily from (6.15).

Since the points $\{x_j\}_{j=1}^k$ satisfy (6.45) and none of them is a point of local maximum for $\bar{w}|_\Sigma$ (see Lemma 6.1), for any $0 < \delta_0 \ll 1$, there exists a diffeomorphism $\mathcal{X} : \Sigma \rightarrow \Sigma$ such that $\mathcal{X} = Id$ on $\Sigma \setminus \cup_{j=1}^k B_{g_\Sigma}(x_j, 4\delta_0)$ and for all $1 \leq j \leq k$:

$$2\delta_0 < \text{dist}_{g_\Sigma}(x_j, \mathcal{X}(x_j)) < 4\delta_0, \tag{6.49}$$

$$\mathcal{X}(B_{g_\Sigma}(x_j, \delta_0)) = B_{g_\Sigma}(\mathcal{X}(x_j), \delta_0), \tag{6.50}$$

$$\mathcal{X}(B_{g_\Sigma}(\mathcal{X}(x_j), \delta_0)) = B_{g_\Sigma}(x_j, \delta_0), \tag{6.51}$$

and

$$\max_{B_{g_\Sigma}(x_j, \delta_0)} \bar{w} < \min_{B_{g_\Sigma}(\mathcal{X}(x_j), \delta_0)} \bar{w}. \tag{6.52}$$

Setting $\tilde{x}_j \doteq \mathcal{X}(x_j)$ for $j = 1, \dots, k$, provided δ_0 is sufficiently small in terms of the geometry of (\mathcal{M}, g) , we have:

$$\{\tilde{x}_j\}_{j=1}^k \in \Sigma \cap \{r \leq \frac{1}{2}R_0 + 4\delta_0\} \setminus \mathcal{E}_{8\delta_0}. \tag{6.53}$$

Extending \mathcal{X} on the whole of $\mathcal{M} \setminus \mathcal{H}^-$ by the requirement that it commutes with the flow of T , i.e.:

$$\mathcal{L}_T \circ \mathcal{X} = \mathcal{X} \circ \mathcal{L}_T, \tag{6.54}$$

and defining the function $\tilde{w} : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{R}$ as

$$\tilde{w} \doteq w \circ \mathcal{X}, \tag{6.55}$$

we infer that, in view of (6.47), (6.48) and the properties of \mathcal{X} , the relations (6.34), (6.36) and (6.38) hold, provided l is sufficiently large in terms of δ and the precise choice of \mathcal{X} . Furthermore, in view of (6.52), inequalities (6.39) and (6.40) hold. Finally, inequalities (6.41) and (6.42) follow trivially from (6.15) and the fact that $\mathcal{X} = Id$ on \mathcal{E}_δ for $\delta \leq 4\delta_0$, while (6.43) and (6.44) follow from (6.13). \square

Lemma 6.3. *For any $R_0 \gg 1, 0 < \delta_0 \ll 1, 0 < \delta_1 \ll 1, s \gg 1, 0 < \epsilon_0 \ll 1$ and $R \gg \max\{R_0, \epsilon_0^{-1}\}$, there exists a pair of smooth and T -invariant function $w_R, \tilde{w}_R : \{r \leq R\} \subset \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{R}$ satisfying the following properties:*

1. *In the region $\{r \leq R_0\}$:*

$$w_R = R^{-3\epsilon_0} w \tag{6.56}$$

and

$$\tilde{w}_R = R^{-3\epsilon_0} \tilde{w}, \tag{6.57}$$

where w, \tilde{w} are the functions from Lemma 6.2.

2. In the region $\{R_0 \leq r \leq R^{\epsilon_0}\}$, w_R is a function of r and $\tilde{w}_R = w_R$. The following bounds are also satisfied for some constants depending only on R_0 and δ_0 (and the precise choice of w):

$$0 < cR^{-3\epsilon_0} \leq \partial_r w_R \leq C, \tag{6.58}$$

$$\partial_r^2 w_R + r^{-1} \partial_r w_R \geq cR^{-3\epsilon_0} + |r^{-\frac{1}{2}} \partial_r^2 w_R| + |r^{-\frac{3}{2}} \partial_r w_R|, \tag{6.59}$$

$$|\partial_r^2 w_R|, |\partial_r^3 w_R|, |\partial_r^4 w_R| \leq C. \tag{6.60}$$

3. In the region $\{R^{\epsilon_0} \leq r \leq \frac{1}{2}R\}$:

$$w_R = \tilde{w}_R = C_1 \epsilon_0^{-1} \left(\frac{r}{R}\right)^{\epsilon_0} + C_2 \tag{6.61}$$

for some constants C_1, C_2 depending only on R_0, δ_0 (and the precise choice of w).

4. In the region $\{\frac{1}{2}R \leq r \leq R\}$:

$$w_R = \tilde{w}_R = v_s \left(\frac{r}{R}\right) + C_3 \tag{6.62}$$

for some constant C_3 depending on R_0, δ_0, δ_1 (and the precise choice of w), where the function $v_s : [\frac{1}{2}, 1] \rightarrow \mathbb{R}$ depends on $s, \epsilon_0, \delta_0, \delta_1$ and satisfies (for some constants $c_{\delta_0}, C_{\delta_0} > 0$ depending on δ_0, R_0 and the precise choice of w):

$$\frac{dv_s}{dx} \geq c_{\delta_0} s^{-1}, \tag{6.63}$$

$$\left| \frac{d^2 v_s}{dx^2} \right| \leq C_{\delta_0} (\delta_1 s + \delta_1^{-1}) \frac{dv_s}{dx} \tag{6.64}$$

$$\left| \frac{d^2 v_s}{dx^2} \right|, \left| \frac{d^3 v_s}{dx^3} \right|, \left| \frac{d^4 v_s}{dx^4} \right| \leq C_{\delta_0} \delta_1^{-1} \tag{6.65}$$

and, for $x \in [\frac{3}{4}, 1]$:

$$v_s(x) = \frac{1}{2s} \log \left(x - \frac{9}{10} \log(x) \right). \tag{6.66}$$

Remark. Notice that we can bound on $\{R^{\epsilon_0} \leq r \leq \frac{1}{2}R\}$

$$\partial_r^2 w_R + r^{-1} \partial_r w_R > c_0 \epsilon_0 R^{-\epsilon_0} \cdot r^{-2+\epsilon_0} + |r^{-\frac{1}{2}} \partial_r^2 w_R| + |r^{-\frac{3}{2}} \partial_r w_R|, \tag{6.67}$$

$$\partial_r w_R > c_0 R^{-\epsilon_0} r^{-1+\epsilon_0}, \tag{6.68}$$

$$\sum_{j=1}^4 |r^j \partial_r^j w_R| < C_0 R^{-\epsilon_0} r^{\epsilon_0}. \tag{6.69}$$

Proof. The construction of w_R (and, similarly, \tilde{w}_R) can be readily performed in view of the following observations:

- In view of Condition 1 and the properties of the function w , for $r = R_0$ we have:

$$\partial_r w_R(R_0) \sim_{R_0} R^{-3\epsilon_0}, \tag{6.70}$$

$$\partial_r^2 w_R(R_0), r^{-1} \partial_r w_R(R_0) \sim_{R_0} R^{-3\epsilon_0}, \tag{6.71}$$

while Condition 3 requires that, for $r = R^{\varepsilon_0}$:

$$\partial_r w_R(R^{\varepsilon_0}) = C_1 R^{-2\varepsilon_0 + \varepsilon_0^2} \gg \partial_r w_R(R_0), \tag{6.72}$$

$$\partial_r^2 w_R(R^{\varepsilon_0}) = C_1(\varepsilon_0 - 1)R^{-3\varepsilon_0 + \varepsilon_0^2} \tag{6.73}$$

$$\partial_r^2 w_R(R^{\varepsilon_0}) + r^{-1} \partial_r w_R(R^{\varepsilon_0}) \geq c\varepsilon_0 R^{-3\varepsilon_0 + \varepsilon_0^2} + \left| r^{-\frac{1}{2}} \partial_r^2 w_R(R^{\varepsilon_0}) \right| + \left| r^{-\frac{3}{2}} \partial_r w_R(R^{\varepsilon_0}) \right|. \tag{6.74}$$

Therefore, we can readily construct the function $\partial_r w_R$ (as a function of r) on the interval $\{R_0 \leq r \leq R^{\varepsilon_0}\}$ (and then integrate in order to obtain w_R and the constant C_2 in (6.61)), so that (6.58)–(6.60) are satisfied. In particular, $\partial_r w_R$ can be constructed as an increasing function of r (i.e. with $\partial_r^2 w_R > 0$) up to $r = R^{\varepsilon_0} - 1$, while for $r \in [R^{\varepsilon_0} - 1, R^{\varepsilon_0}]$, $\partial_r w_R$ is constructed a smooth function of r extending (6.61) from $\{r \geq R^{\varepsilon_0}\}$ under the requirement that it satisfies the one sided bound

$$\partial_r^2 w_R(r) \geq -(1 + \varepsilon_0^2) |\partial_r^2 w_R(R^{\varepsilon_0})|. \tag{6.75}$$

- Let $\tilde{v} : [\frac{3}{5}, 1] \rightarrow \mathbb{R}$ be a smooth and strictly increasing function such that $\tilde{v}(x) = -(x - 1)^2 - 10$ for $x \in [\frac{3}{5}, \frac{7}{10}]$ and $\tilde{v}(x) = \log(x - \frac{9}{10} \log(x))$ for $x \in [\frac{3}{4}, 1]$. Then, provided $s \gg 1$ and $\delta_1 < 1$, it can be readily inferred that there exists a C^5 and piecewise C^6 function $\tilde{v}_s : [\frac{1}{2}, 1] \rightarrow \mathbb{R}$, which is smooth on $[\frac{1}{2}, 1] \setminus \{\frac{3}{5}\}$, satisfying $\frac{d\tilde{v}_s}{dx} \geq \frac{1}{10s} C_1$, $\sum_{j=1}^4 \left| \frac{d^j \tilde{v}_s}{dx^j} \right| \leq 10C_1 \delta_1^{-1}$ and

$$\tilde{v}_s(x) = \begin{cases} C_1 \varepsilon_0^{-1} x^{\varepsilon_0} + C_2 - C_3 & \text{for } x \in [\frac{1}{2}, \frac{11}{20}] \\ -\delta_1(x - \frac{3}{5})^6 - \frac{1}{2s}((x - 1)^2 + 10) & \text{for } x \in [\frac{23}{40}, \frac{3}{5}] \\ \frac{1}{2s} \tilde{v}(x), & \text{for } x \in [\frac{3}{5}, 1] \end{cases}$$

for a suitable constant $C_3 > 0$ depending on C_1, C_2 . The function v_s is then constructed by mollifying \tilde{v}_s around $x = \frac{3}{5}$. \square

6.3. *The seed functions f, \tilde{f}, h and \tilde{h} .* In this section, we will construct (using the auxiliary functions from the previous section) the seed functions for the multipliers that will be used in the proof of Proposition 6.1.

We will assume without loss of generality that $0 < \delta_0, \delta_1 \ll 1, s \gg 1, 0 < \varepsilon_0 \ll 1$ and $R \gg \max\{R_0, \varepsilon_0^{-1}\}$. Let $w_R, \tilde{w}_R : \{r \leq R\} \rightarrow \mathbb{R}$ be the functions from Lemma 6.3 (associated to the parameters $s, R, \varepsilon_0, \delta_0, \delta_1$). We define the smooth and T -invariant functions $f, \tilde{f} : \mathcal{M} \setminus \mathcal{H}^- \rightarrow (0, +\infty)$ as follows:

$$f = \begin{cases} e^{2s w_R}, & \text{on } \{r \leq R\} \\ C_4^{2s} \cdot \left(\frac{r}{R} - \frac{9}{10} \log\left(\frac{r}{R}\right) \right), & \text{on } \{r \geq R\}, \end{cases} \tag{6.76}$$

and

$$\tilde{f} = \begin{cases} e^{2s \tilde{w}_R}, & \text{on } \{r \leq R\} \\ C_4^{2s} \cdot \left(\frac{r}{R} - \frac{9}{10} \log\left(\frac{r}{R}\right) \right), & \text{on } \{r \geq R\}, \end{cases} \tag{6.77}$$

where $C_4 > 0$ is chosen so that f and \tilde{f} are smooth at $r = R$ [which is possible in view of (6.66)].

Let $h : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{R}$ be a smooth and T -invariant function satisfying the following conditions (provided $\delta_1, \delta_2 \ll 1$):

1. In the region $\{r \leq \frac{4}{3}R\}$:

$$h = -\chi_{\leq \frac{1}{2}R_0} s \delta_1 \frac{\nabla_\mu \nabla_\nu w_R \nabla^\mu w_R \nabla^\nu w_R}{g_{ref}(dw_R, dw_R)} e^{2sw_R} + \chi_{\geq R_0} (r^{-1} - r^{-\frac{3}{2}}) \partial_r f, \quad (6.78)$$

where

$$\chi_{\leq \frac{1}{2}R_0} \doteq \chi_4 \left(\frac{R_0}{2r} \right). \quad (6.79)$$

2. In the region $\{\frac{4}{3}R \leq r \leq \delta_2^{-1}R\}$, h is a function of r , satisfying

$$cf(R)r^{-2} < h \leq \min\{(r^{-1} - r^{-\frac{3}{2}})\partial_r f, (1 - r^{-\frac{1}{2}})\partial_r^2 f\} \quad (6.80)$$

and

$$-\square_g h \leq CR^{-4} f(R) \quad (6.81)$$

for some absolute constants $C, c > 0$.

3. In the region $\{r \geq \delta_2^{-1}R\}$:

$$h = \frac{1}{2} \partial_r^2 f. \quad (6.82)$$

Notice that h can indeed be defined as above on the interval $\{(1 + 2\delta_2)R \leq r \leq \delta_2^{-1}R\}$ (provided δ_2 is smaller than an absolute constant), in view of the fact that

$$\min\{\partial_r^2 f, r^{-1} \partial_r f\} \gtrsim f(R)r^{-2}, \quad (6.83)$$

$$\square_g h = (1 + O(r^{-1})) \partial_r^2 h + ((d - 1)r^{-1} + O(r^{-2})) \partial_r h \quad (6.84)$$

and

$$\sum_{j=1}^4 r^{j-4} |\partial_r^j f| \leq CR^{-4} f(R) \quad (6.85)$$

on that interval, while $(r^{-1} - r^{-\frac{3}{2}})\partial_r f < \partial_r^2 f$ for $R \leq r < \frac{4}{3}R$ (provided $R \gg 1$) and $r^{-1}\partial_r f > \partial_r^2 f$ for $r \geq \delta_2^{-1}R$ (provided $\delta_2 \ll 1$).

We also define $\tilde{h} : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{R}$ in the same way as h , but with \tilde{w}_R and \tilde{f} in place of w_R and f , respectively.

6.4. The integration-by-parts scheme. In this section, we will establish a general identity obtained from Eq. (6.1) and a suitable first order multiplier, after successively integrating by parts over $\mathcal{R}(\tau_1, \tau_2)$. This identity will lie at the core of the proof of Proposition 6.1.

Let f, h be as in Sect. 6.3. We introduce the following multiplier for equation (6.1)

$$2\nabla^\mu f \cdot \nabla_\mu \varphi + \square_g f \cdot \varphi. \quad (6.86)$$

Multiplying (5.29) with the complex conjugate of (6.86) and integrating by parts over $\mathcal{R}(\tau_1, \tau_2)$, we obtain:

$$\begin{aligned} & \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ 2\nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} - \frac{1}{2} \square_g^2 f |\varphi|^2 \right\} dg \\ &= - \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ G(2\nabla^\mu f \nabla_\mu \bar{\varphi} + (\square_g f) \bar{\varphi}) \right\} dg \end{aligned}$$

$$\begin{aligned}
 & - \sum_{j=1}^2 (-1)^j \int_{\Sigma_{\tau_j}} \operatorname{Re} \left\{ (2\nabla^\mu f \nabla_\mu \bar{\varphi} \nabla_\nu \varphi + (\square_g f) \varphi \nabla_\nu \bar{\varphi} - \nabla_\nu f \nabla^\mu \varphi \nabla_\mu \bar{\varphi} \right. \\
 & \left. - \frac{1}{2} (\nabla_\nu (\square_g f)) |\varphi|^2 \right\} n_{\Sigma_{\tau_j}}^\nu \} dg_{\Sigma_{\tau_j}} \\
 & - \int_{\mathcal{H}^+ \cap \mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ (2\nabla^\mu f \nabla_\mu \bar{\varphi} \nabla_\nu \varphi + (\square_g f) \varphi \nabla_\nu \bar{\varphi} - \nabla_\nu f \nabla^\mu \varphi \nabla_\mu \bar{\varphi} \right. \\
 & \left. - \frac{1}{2} (\nabla_\nu (\square_g f)) |\varphi|^2 \right\} n_{\mathcal{H}^+}^\nu \} d\operatorname{vol}_{\mathcal{H}^+}.
 \end{aligned} \tag{6.87}$$

Let us split the left hand side of (6.87) as

$$\begin{aligned}
 & \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ 2\nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} - \frac{1}{2} \square_g^2 f |\varphi|^2 \right\} dg \\
 & = + \int_{\mathcal{R}(\tau_1, \tau_2)} \chi_{\leq R} \operatorname{Re} \left\{ 2\nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} - \frac{1}{2} \square_g^2 f |\varphi|^2 \right\} dg \\
 & \quad + \int_{\mathcal{R}(\tau_1, \tau_2)} (1 - \chi_{\leq R}) \operatorname{Re} \left\{ 2\nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} - \frac{1}{2} \square_g^2 f |\varphi|^2 \right\} dg.
 \end{aligned} \tag{6.88}$$

Using the identity

$$\begin{aligned}
 \nabla_\mu \varphi \nabla_\nu \bar{\varphi} & = f^{-1} \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) \\
 & \quad - \frac{1}{2} f^{-1} (\nabla_\mu f \varphi \nabla_\nu \bar{\varphi} + \nabla_\nu f \bar{\varphi} \nabla_\mu \varphi) - \frac{1}{4} f^{-2} \nabla_\mu f \nabla_\nu f |\varphi|^2
 \end{aligned} \tag{6.89}$$

and integrating by parts in the $\varphi \nabla \bar{\varphi}$ terms, we have:

$$\begin{aligned}
 & \int_{\mathcal{R}(\tau_1, \tau_2)} \chi_{\leq R} \operatorname{Re} \left\{ 2\nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} - \frac{1}{2} \square_g^2 f |\varphi|^2 \right\} dg \\
 & = \int_{\mathcal{R}(\tau_1, \tau_2)} \chi_{\leq R} \operatorname{Re} \left\{ 2f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) + (\nabla_\nu (f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu f) \right. \\
 & \quad \left. - \frac{1}{2} f^{-2} \nabla^\mu \nabla^\nu f \nabla_\mu f \nabla_\nu f - \frac{1}{2} \square_g^2 f) |\varphi|^2 \right\} dg \\
 & \quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\nu \chi_{\leq R} \cdot f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu f |\varphi|^2 dg \\
 & \quad + \sum_{j=1}^2 (-1)^j \int_{\Sigma_{\tau_j}} \chi_{\leq R} \cdot f^{-1} \nabla_\mu \nabla_\nu f \nabla^\mu f |\varphi|^2 n_{\Sigma_{\tau_j}}^\nu \} dg_{\Sigma_{\tau_j}} \\
 & \quad + \int_{\mathcal{H}^+ \cap \mathcal{R}(\tau_1, \tau_2)} \chi_{\leq R} \cdot f^{-1} \nabla_\mu \nabla_\nu f \nabla^\mu f |\varphi|^2 n_{\mathcal{H}^+}^\nu \} d\operatorname{vol}_{\mathcal{H}^+}.
 \end{aligned} \tag{6.90}$$

Thus, in view of (6.90), the identity (6.88) yields:

$$\begin{aligned}
 & \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ 2 \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} - \frac{1}{2} \square_g^2 f |\varphi|^2 \right\} dg \\
 &= \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ 2 \chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} \right\} dg \\
 &+ \int_{\mathcal{R}(\tau_1, \tau_2)} \left(\chi_{\leq R} \nabla_\nu (f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu f) - \frac{1}{2} \chi_{\leq R} f^{-2} \nabla^\mu \nabla^\nu f \nabla_\mu f \nabla_\nu f - \frac{1}{2} \square_g^2 f \right) |\varphi|^2 dg \\
 &+ \int_{\mathcal{R}(\tau_1, \tau_2)} (\nabla_\nu \chi_{\leq R} \cdot f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu f) |\varphi|^2 dg \\
 &+ \sum_{j=1}^2 (-1)^j \int_{\Sigma_{\tau_j}} \chi_{\leq R} f^{-1} \nabla_\mu \nabla_\nu f \nabla^\mu f |\varphi|^2 n_{\Sigma_{\tau_j}}^\nu dg_{\Sigma_{\tau_j}} \\
 &+ \int_{\mathcal{H}^+ \cap \mathcal{R}(\tau_1, \tau_2)} \chi_{\leq R} \cdot f^{-1} \nabla_\mu \nabla_\nu f \nabla^\mu f |\varphi|^2 n_{\mathcal{H}^+}^\nu d\operatorname{vol}_{\mathcal{H}^+}. \tag{6.91}
 \end{aligned}$$

Adding to (6.91) the identity

$$\begin{aligned}
 0 &= \int_{\mathcal{R}(\tau_1, \tau_2)} \left(-2 \chi_{\geq R_0} \chi_{\leq R} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 + 2 \chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 \right. \\
 &\quad \left. + \chi_{\geq R_0} \chi_{\leq R} r^{-1} f^{-1} (\partial_r f)^2 \partial_r (|\varphi|^2) + \frac{1}{2} \chi_{\geq R_0} \chi_{\leq R} r^{-1} f^{-2} (\partial_r f)^3 |\varphi|^2 \right) dg \tag{6.92}
 \end{aligned}$$

(recall that ∂_r is the coordinate vector field in the (t, r, σ) coordinate chart in each connected component of the region $\{r \geq \frac{1}{4} R_0\}$) and integrating by parts in the $\partial_r (|\varphi|^2)$ term, we obtain:

$$\begin{aligned}
 & \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ 2 \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} - \frac{1}{2} \square_g^2 f |\varphi|^2 \right\} dg \\
 &= \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ 2 \chi_{\leq R} \left(f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 \right) \right. \\
 &\quad \left. + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + 2 \chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 + \mathcal{A}_f^{(R)} |\varphi|^2 \right\} dg \\
 &+ \sum_{j=1}^2 (-1)^j \int_{\Sigma_{\tau_j}} \chi_{\leq R} f^{-1} \nabla_\mu \nabla_\nu f \nabla^\mu f |\varphi|^2 n_{\Sigma_{\tau_j}}^\nu dg_{\Sigma_{\tau_j}} \\
 &+ \int_{\mathcal{H}^+ \cap \mathcal{R}(\tau_1, \tau_2)} \chi_{\leq R} \cdot f^{-1} \nabla_\mu \nabla_\nu f \nabla^\mu f |\varphi|^2 n_{\mathcal{H}^+}^\nu d\operatorname{vol}_{\mathcal{H}^+}, \tag{6.93}
 \end{aligned}$$

where

$$\begin{aligned}
 \mathcal{A}_f^{(R)} &\doteq \chi_{\leq R} \nabla_\nu (f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu f) - \frac{1}{2} \chi_{\leq R} f^{-2} \nabla^\mu \nabla^\nu f \nabla_\mu f \nabla_\nu f - \frac{1}{2} \square_g^2 f + \\
 &\quad - \chi_{\geq R_0} \chi_{\leq R} \partial_r (r^{-1} f^{-1} (\partial_r f)^2) - \chi_{\geq R_0} \chi_{\leq R} r^{-1} f^{-1} (\partial_r f)^2 \operatorname{div}(\partial_r) \\
 &\quad + \frac{1}{2} \chi_{\geq R_0} \chi_{\leq R} r^{-1} f^{-2} (\partial_r f)^3 - \\
 &\quad + \nabla_\nu \chi_{\leq R} \cdot f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu f - \partial_r (\chi_{\geq R_0} \chi_{\leq R}) r^{-1} f^{-1} (\partial_r f)^2. \tag{6.94}
 \end{aligned}$$

Thus, (6.87) and (6.93) yield:

$$\begin{aligned} & \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ 2\chi_{\leq R} \left(f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) \right) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 \right\} \\ & \quad + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 + \mathcal{A}_f^{(R)} |\varphi|^2 \Big\} dg \\ & = - \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ G(2\nabla^\mu f \nabla_\mu \bar{\varphi} + (\square_g f) \bar{\varphi}) \right\} dg - \mathcal{B}_f^{(R)}[\varphi; \tau_1, \tau_2], \end{aligned} \tag{6.95}$$

where

$$\begin{aligned} \mathcal{B}_f^{(R)}[\varphi; \tau_1, \tau_2] &= \sum_{j=1}^2 (-1)^j \int_{\Sigma_{\tau_j}} \operatorname{Re} \left\{ \left(2\nabla^\mu f \nabla_\mu \bar{\varphi} \nabla_\nu \varphi + (\square_g f) \varphi \nabla_\nu \bar{\varphi} - \nabla_\nu f \nabla^\mu \varphi \nabla_\mu \bar{\varphi} \right. \right. \\ & \quad \left. \left. + (\chi_{\leq R} f^{-1} \nabla_\mu \nabla_\nu f \nabla^\mu f - \frac{1}{2} (\nabla_\nu (\square_g f))) |\varphi|^2 \right) n_{\Sigma_{\tau_j}}^\nu \right\} dg_{\Sigma_{\tau_j}} \\ & \quad + \int_{\mathcal{H}^+ \cap \mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ \left(2\nabla^\mu f \nabla_\mu \bar{\varphi} \nabla_\nu \varphi + (\square_g f) \varphi \nabla_\nu \bar{\varphi} - \nabla_\nu f \nabla^\mu \varphi \nabla_\mu \bar{\varphi} \right. \right. \\ & \quad \left. \left. + (\chi_{\leq R} f^{-1} \nabla_\mu \nabla_\nu f \nabla^\mu f - \frac{1}{2} (\nabla_\nu (\square_g f))) |\varphi|^2 \right) n_{\mathcal{H}^+}^\nu \right\} d\operatorname{vol}_{\mathcal{H}^+}. \end{aligned} \tag{6.96}$$

Finally, for $h : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{R}$ as in Sect. 6.3, adding to (6.95) the Lagrangean identity

$$\begin{aligned} & \int_{\mathcal{R}(\tau_1, \tau_2)} \left(-2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + (\square_g h) |\varphi|^2 \right) dg = \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ G \cdot 2h \bar{\varphi} \right\} dg \\ & \quad + \sum_{j=1}^2 (-1)^j \int_{\Sigma_{\tau_j}} \operatorname{Re} \left\{ (2h \nabla_\nu \varphi \bar{\varphi} - \nabla_\nu h |\varphi|^2) n_{\Sigma_{\tau_j}}^\nu \right\} dg_{\Sigma_{\tau_j}} \\ & \quad + \int_{\mathcal{H}^+ \cap \mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ (2h \nabla_\nu \varphi \bar{\varphi} - \nabla_\nu h |\varphi|^2) n_{\mathcal{H}^+}^\nu \right\} d\operatorname{vol}_{\mathcal{H}^+}, \end{aligned} \tag{6.97}$$

we obtain:

$$\begin{aligned} & \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ 2\chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) \right. \\ & \quad - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 \\ & \quad + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 \\ & \quad \left. - 2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \mathcal{A}_{f,h}^{(R)} |\varphi|^2 \right\} dg \\ & = - \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ G(2\nabla^\mu f \nabla_\mu \bar{\varphi} + (\square_g f - 2h) \bar{\varphi}) \right\} dg - \mathcal{B}_{f,h}^{(R)}[\varphi; \tau_1, \tau_2], \end{aligned} \tag{6.98}$$

where

$$\begin{aligned} \mathcal{A}_{f,h}^{(R)} &\doteq \square_g h + \chi_{\leq R} \nabla_\nu (f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu f) - \frac{1}{2} \chi_{\leq R} f^{-2} \nabla^\mu \nabla^\nu f \nabla_\mu f \nabla_\nu f - \frac{1}{2} \square_g^2 f \\ & \quad - \chi_{\geq R_0} \chi_{\leq R} \partial_r (r^{-1} f^{-1} (\partial_r f)^2) - \chi_{\geq R_0} \chi_{\leq R} r^{-1} f^{-1} (\partial_r f)^2 \operatorname{div}(\partial_r) \end{aligned}$$

$$\begin{aligned}
 &+ \frac{1}{2} \chi_{\geq R_0} \chi_{\leq R} r^{-1} f^{-2} (\partial_r f)^3 \\
 &+ \nabla_\nu \chi_{\leq R} \cdot f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu f - \partial_r (\chi_{\geq R_0} \chi_{\leq R}) r^{-1} f^{-1} (\partial_r f)^2
 \end{aligned} \tag{6.99}$$

and

$$\begin{aligned}
 &\mathcal{B}_{f,h}^{(R)}[\varphi; \tau_1, \tau_2] \\
 &= \sum_{j=1}^2 (-1)^j \int_{\Sigma_{\tau_j}} \operatorname{Re} \left\{ \left(2 \nabla^\mu f \nabla_\mu \bar{\varphi} \nabla_\nu \varphi + (\square_g f - 2h) \varphi \nabla_\nu \bar{\varphi} - \nabla_\nu f \nabla^\mu \varphi \nabla_\mu \bar{\varphi} \right. \right. \\
 &\quad \left. \left. + (\chi_{\leq R} f^{-1} \nabla_\mu \nabla_\nu f \nabla^\mu f + \nabla_\nu h - \frac{1}{2} (\nabla_\nu (\square_g f))) |\varphi|^2 \right) n_{\Sigma_{\tau_j}}^\nu \right\} dg_{\Sigma_{\tau_j}} \\
 &\quad + \int_{\mathcal{H}^+ \cap \mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ \left(2 \nabla^\mu f \nabla_\mu \bar{\varphi} \nabla_\nu \varphi + (\square_g f - 2h) \varphi \nabla_\nu \bar{\varphi} - \nabla_\nu f \nabla^\mu \varphi \nabla_\mu \bar{\varphi} \right. \right. \\
 &\quad \left. \left. + (\chi_{\leq R} f^{-1} \nabla_\mu \nabla_\nu f \nabla^\mu f + \nabla_\nu h - \frac{1}{2} (\nabla_\nu (\square_g f))) |\varphi|^2 \right) n_{\mathcal{H}^+}^\nu \right\} d\operatorname{vol}_{\mathcal{H}^+}.
 \end{aligned} \tag{6.100}$$

In the next sections, we will establish a number of estimates for the left hand side of (6.98) that will lead to the proof of Proposition 6.1.

6.5. *Estimates for the zeroth order term.* In this section, we will establish some bounds for the coefficient $\mathcal{A}_{f,h}^{(R)}$ of the zeroth order term appearing in the left hand side of (6.98).

In view of the choice of the functions f, h in Sect. 6.3, we can readily calculate that on $\{r \leq R\}$ (where $\chi_R \equiv 1$), the quantity $\mathcal{A}_{f,h}^{(R)}$ in (6.99) has the form

$$\mathcal{A}_{f,h}^{(R)} = \left\{ \mathcal{A}_{w_R, 3} s^3 + \mathcal{A}_{w_R, 2} s^2 + \mathcal{A}_{w_R, 1} s \right\} e^{2s w_R}, \tag{6.101}$$

where

$$\begin{aligned}
 \mathcal{A}_{w_R, 3} &= \left(4 - \chi_{\leq \frac{1}{2} R_0} \delta_1 \frac{\nabla^\alpha w_R \nabla_\alpha w_R}{g_{ref}(dw_R, dw_R)} \right) \nabla^\mu \nabla^\nu w_R \nabla_\mu w_R \nabla_\nu w_R \\
 &\quad + 4 \chi_{\geq R_0} r^{-1} (1 + O(r^{-\frac{1}{2}})) (\partial_r w_R)^3,
 \end{aligned} \tag{6.102}$$

$$\begin{aligned}
 \mathcal{A}_{w_R, 2} &= 4 \nabla_\nu \nabla^\mu \nabla^\nu w_R \nabla_\mu w_R - 4 \nabla_\nu \nabla^\nu \nabla^\mu w_R \nabla_\mu w_R \\
 &\quad - 4 \nabla^\mu (\square_g w_R) \nabla_\mu w_R - 2 (\square_g w_R)^2 \\
 &\quad + 4 \chi_{\geq R_0} r^{-1} (1 + O(r^{-\frac{1}{2}})) \partial_r^2 w_R \partial_r w_R + 4 \chi_{\geq R_0} r^{-2} (1 + O(r^{-\frac{1}{2}})) (\partial_r w_R)^2 \\
 &\quad + O(\delta_1) \chi_{\leq \frac{1}{2} R_0} \sum_{j=0}^1 |\nabla^{2+j} w_R|_{g_{ref}} |\nabla^{2-j} w_R|_{g_{ref}} \\
 &\quad + O(|\nabla \chi_{\leq R_0}|_{g_{ref}} + |\nabla \chi_{\leq \frac{1}{2} R_0}|_{g_{ref}}) (|\nabla^2 w_R|_{g_{ref}}^2 + |\nabla w_R|_{g_{ref}}^2),
 \end{aligned} \tag{6.103}$$

$$\begin{aligned}
 \mathcal{A}_{w_R, 1} &= -\square_g^2 w_R + 2 \chi_{\geq R_0} r^{-1} (1 + O(r^{-\frac{1}{2}})) \partial_r^3 w \\
 &\quad + 2(d-3) \chi_{\geq R_0} r^{-2} (1 + O(r^{-\frac{1}{2}})) \partial_r^2 w_R \\
 &\quad - 2(d-3) \chi_{\geq R_0} r^{-3} (1 + O(r^{-\frac{1}{2}})) \partial_r w_R
 \end{aligned}$$

$$\begin{aligned}
 &+ O(\delta_1)\chi_{\leq \frac{1}{2}R_0} \sum_{j_1+j_2+j_3=1}^2 \frac{|\nabla^{2+j_1} w_R|_{g_{ref}} |\nabla^{1+j_2} w_R|_{g_{ref}} |\nabla^{1+j_3} w_R|_{g_{ref}}}{|\nabla w_R|_{g_{ref}}^2} \\
 &+ 2(\square_g \chi_{\leq R_0})r^{-1} \partial_r w_R - \delta_1(\square_g \chi_{\leq \frac{1}{2}R_0}) \nabla^\mu \nabla^\nu w_R \nabla_\mu w_R \nabla_\nu w_R \frac{\nabla^\alpha w_R \nabla_\alpha w_R}{g_{ref}(dw_R, dw_R)} \\
 &+ O\left(\sum_{j=1}^2 (|\nabla^j \chi_{\leq R_0}|_{g_{ref}} + |\nabla^j \chi_{\leq \frac{1}{2}R_0}|_{g_{ref}})\right) (|\nabla^2 w_R|_{g_{ref}} + |\nabla w_R|_{g_{ref}}) \tag{6.104}
 \end{aligned}$$

(with the constants implicit in the $O(\cdot)$ notation depending only on the geometry of (\mathcal{M}, g)).

Remark. Notice the cancellation of the $O(s^4)$ terms that were expected to appear in (6.101).

6.5.1. *Bound on $\{r \leq R_0\}$.* In view of (6.101)–(6.104), the properties of the function w_R (see Lemma 6.3) and the form (2.1) of the metric g in the region $r \gg 1$ imply that in the region $\{r \leq R_0\}$:

$$\mathcal{A}_{f,h}^{(R)} = \left\{ \mathcal{A}_{w,3;R_0} R^{-9\epsilon_0} s^3 + O_{\delta_0}(1) R^{-6\epsilon_0} s^2 + O_{\delta_0}(1) R^{-3\epsilon_0} s \right\} e^{2s w_R}, \tag{6.105}$$

where

$$\begin{aligned}
 \mathcal{A}_{w,3;R_0} &= \left(4 - \chi_{\leq \frac{1}{2}R_0} \delta_1 \frac{\nabla^\alpha w \nabla_\alpha w}{g_{ref}(dw, dw)} \right) \nabla^\mu \nabla^\nu w \nabla_\mu w \nabla_\nu w \\
 &+ 4\chi_{\geq R_0} r^{-1} (1 + O(r^{-\frac{1}{2}})) (\partial_r w)^3, \tag{6.106}
 \end{aligned}$$

w is the function from Lemma 6.2 and the constants implicit in the $O_{\delta_0}(1)$ notation depend only on R_0, δ_0 .

6.5.2. *Bounds on $\{R_0 \leq r \leq R\}$.* In the region $\{R_0 \leq r \leq R\}$, the expressions (6.102), (6.103) and (6.104) simplify as follows, in view of (2.1) and the fact that w_R is a function of r for $r \geq R_0$:

$$\mathcal{A}_{w_R,3} = 4(1 + O(r^{-\frac{1}{2}})) \partial_r^2 w_R (\partial_r w_R)^2 + 4r^{-1} (1 + O(r^{-\frac{1}{2}})) (\partial_r w_R)^3, \tag{6.107}$$

$$\begin{aligned}
 \mathcal{A}_{w_R,2} &= -4(1 + O(r^{-\frac{1}{2}})) \partial_r w_R \partial_r^3 w_R - 2(1 + O(r^{-\frac{1}{2}})) (\partial_r^2 w_R)^2 \\
 &+ (4 - 8(d - 1))r^{-1} (1 + O(r^{-\frac{1}{2}})) \partial_r^2 w_R \partial_r w_R \\
 &+ (4(d - 2) - 2(d - 1)^2)r^{-2} (1 + O(r^{-\frac{1}{2}})) (\partial_r w_R)^2, \tag{6.108}
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{A}_{w_R,1} &= -(1 + O(r^{-\frac{1}{2}})) \partial_r^4 w_R - 2(d - 1)r^{-1} (1 + O(r^{-\frac{1}{2}})) \partial_r^3 w_R \\
 &- (d - 3)^2 r^{-1} (1 + O(r^{-\frac{1}{2}})) \partial_r^2 w_R + (d - 3)^2 r^{-3} (1 + O(r^{-\frac{1}{2}})) \partial_r w_R. \tag{6.109}
 \end{aligned}$$

Therefore, the properties of the function w_R (see Lemma 6.3) imply the following relations for $\mathcal{A}_{f,h}^{(R)}$ on $\{R_0 \leq r \leq R\}$ (provided R_0 is sufficiently large in terms of the geometry of (\mathcal{M}, g)):

1. In the region $\{R_0 \leq r \leq R^{\varepsilon_0}\}$, (6.58)–(6.60) yield:

$$\mathcal{A}_{f,h}^{(R)} \geq \left\{ c_{\delta_0} R^{-9\varepsilon_0} s^3 - C_{\delta_0} s^2 - C_{\delta_0} s \right\} e^{2s w_R} \quad (6.110)$$

for some constants $c_{\delta_0}, C_{\delta_0} > 0$ depending on δ_0, R_0 .

2. In the region $\{R^{\varepsilon_0} \leq r \leq \frac{1}{2}R\}$, (6.67)–(6.69) yield:

$$\mathcal{A}_{f,h}^{(R)} \geq \left\{ c_{\delta_0} \varepsilon_0 R^{-3\varepsilon_0} \cdot r^{-4+3\varepsilon_0} s^3 - C_{\delta_0} R^{-2\varepsilon_0} r^{-4+2\varepsilon_0} s^2 - C_{\delta_0} R^{-\varepsilon_0} r^{-4+\varepsilon_0} s \right\} e^{2s w_R}. \quad (6.111)$$

3. In the region $\{\frac{1}{2}R \leq r \leq R\}$, (6.62)–(6.65) yield:

$$\mathcal{A}_{f,h}^{(R)} \geq -C_{\delta_0} R^{-4} \left\{ v'_s \left(\frac{r}{R} \right) s^3 + s^2 + s \right\} e^{2s w_R}, \quad (6.112)$$

where $C_{\delta_0} > 0$ depends only on δ_0, R_0 .

6.5.3. *Bound on $\{r \geq R\}$.* In the region $\{R \leq r \leq \delta_2^{-1}R\}$, (6.76), (6.78) and (6.81) yield:

$$\mathcal{A}_{f,h}^{(R)}(r) \geq -CR^{-4}f(R) \quad (6.113)$$

for some absolute constant $C > 0$, while for $r \geq \delta_2^{-1}R$ we have (provided $\delta_2 \ll 1$):

$$\begin{aligned} \mathcal{A}_{f,h}^{(R)} &= -\frac{(d-1)}{2}r^{-1}(1+O(r^{-\frac{1}{2}}))\partial_r^3 f - \frac{(d-1)(d-3)}{2}r^{-2}(1+O(r^{-\frac{1}{2}}))\partial_r^2 f \\ &\quad + \frac{(d-1)(d-3)}{2}r^{-3}(1+O(r^{-\frac{1}{2}}))\partial_r f \\ &= \frac{1}{2}(d-1)r^{-4}f(R)\left((d-3)\left(\frac{r+O(r^{\frac{1}{2}})}{R}\right) + \frac{9}{5} + O(r^{-\frac{1}{2}})\right) \\ &\geq f(R)\left(\frac{1}{2}(d-1)(d-3)R^{-1}r^{-3}(1+O(r^{-\frac{1}{2}})) + cr^{-4}\right) \end{aligned} \quad (6.114)$$

for some absolute constant $c > 0$.

6.6. *Estimates for the first order terms.* In this Section, we will establish various bounds for the quantity

$$\begin{aligned} &Re \left\{ 2\chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 \right. \\ &\quad \left. + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 - 2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} \right\} \end{aligned} \quad (6.115)$$

appearing in the integral in left hand side of (6.98). Thus, combined with the bounds of Sect. 6.5 for the zeroth order term $\mathcal{A}_{f,h}^{(R)}|\varphi|^2$ in left hand side of (6.98), the results of this section will provide all the necessary estimates leading to the proof of Proposition 6.1.

Let us denote with g^{-1} the natural extension of the metric (2.1) on the cotangent bundle $T^*\mathcal{M}$ of \mathcal{M} . Since we have identified $\mathcal{M} \setminus \mathcal{H}^-$ with $\mathbb{R} \times \Sigma$ under the flow of T , g^{-1} splits naturally in any local coordinate chart (t, x^1, \dots, x^d) on $\mathbb{R} \times \Sigma$ as

$$g^{-1} = g^{00}T \otimes T + \frac{1}{2}g^{0i}(T \otimes \partial_{x^i} + \partial_{x^i} \otimes T) + (g^{-1})_{\Sigma}^{ij} \partial_{x^i} \otimes \partial_{x^j}, \quad (6.116)$$

where $(g^{-1})_\Sigma$ is a symmetric $(2, 0)$ -tensor on Σ . In view of Assumption G3, the expression (6.116) and the fact that g^{-1} is non-degenerate and has Lorentzian signature imply that $(g^{-1})_\Sigma$ has Riemannian signature on $\Sigma \setminus (\mathcal{E} \cup \mathcal{H}^+)$ and Lorentzian signature on $\Sigma \cap \text{int}(\mathcal{E})$, while $(g^{-1})_\Sigma$ degenerates on $\Sigma \cap (\partial\mathcal{E} \cup \mathcal{H}^+)$. Using the tensor $(g^{-1})_\Sigma$, we can conveniently bound for any $\varphi \in C^1(\mathcal{M} \setminus \mathcal{H}^-)$ (for some constant $C > 0$ depending only on the geometry of (\mathcal{M}, g)):

$$\nabla^\mu \varphi \nabla_\mu \bar{\varphi} \geq (g^{-1})_\Sigma^{ij} \partial_i \varphi \partial_j \bar{\varphi} - C |\nabla_{g_\Sigma} \varphi|_{g_\Sigma} |T \varphi| - C |T \varphi|^2, \tag{6.117}$$

where the indices i, j in the abstract index notation $(g^{-1})_\Sigma^{ij} \partial_i \varphi \partial_j \bar{\varphi}$ run over the variables $\{x^i\}_{i=1}^d$ in any local coordinate chart on $(\mathcal{M} \setminus \mathcal{H}^-)$ of the form (t, x^1, \dots, x^d) .

6.6.1. *Bound on $\mathcal{E}_{\text{ext}} \cup \{r \leq \frac{1}{4}r_0\}$.* For any $0 \leq \tau_1 \leq \tau_2$, we can readily bound from above on $\mathcal{R}(\tau_1, \tau_2) \cap (\mathcal{E}_{\text{ext}} \cup \{r \leq \frac{1}{4}r_0\})$ in view of (6.76) and (6.78):

$$\begin{aligned} & \int_{\mathcal{R}(\tau_1, \tau_2) \cap (\mathcal{E}_{\text{ext}} \cup \{r \leq \frac{1}{4}r_0\})} \\ & \text{Re} \left\{ 2\chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 \right. \\ & \quad + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 \\ & \quad \left. - 2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \mathcal{A}_{f,h}^{(R)} |\varphi|^2 \right\} dg \\ & \leq C_{\delta_0} \int_{\mathcal{R}(\tau_1, \tau_2) \cap (\mathcal{E}_{\text{ext}} \cup \{r \leq \frac{1}{4}r_0\})} e^{2s w_R} \left((R^{-6\epsilon_0} s^2 + 1) |\nabla \varphi|_{g_{\text{ref}}}^2 + (R^{-12\epsilon_0} s^4 + 1) |\varphi|^2 \right) dg. \end{aligned} \tag{6.118}$$

6.6.2. *Bound on $\{\frac{1}{4}r_0 \leq r \leq \frac{1}{2}R_0\} \setminus \mathcal{E}_{\text{ext}}$.* In the region $\{\frac{1}{4}r_0 \leq r \leq \frac{1}{2}R_0\}$, in view of (6.37) and (6.76), we can estimate (using a Cauchy–Schwarz inequality):

$$\begin{aligned} & \text{Re} \left\{ 2f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) \right\} \\ & = e^{2s w_R} \left\{ 8s^2 |\nabla^\mu w_R \nabla_\mu \varphi|^2 + 16s^3 \nabla^\mu w_R \nabla_\mu w_R \text{Re} \{ \varphi \cdot \nabla^\nu w_R \nabla_\nu \bar{\varphi} \} \right. \\ & \quad + 8s^4 (\nabla^\mu w_R \nabla_\mu w_R)^2 |\varphi|^2 + 4s \nabla^\mu \nabla^\nu w_R \nabla_\mu \varphi \nabla_\nu \bar{\varphi} \\ & \quad \left. + 8s \nabla^\mu \nabla^\nu w_R \nabla_\mu w_R \text{Re} \{ \varphi \cdot \nabla_\nu \bar{\varphi} \} + 4s^3 \nabla^\mu \nabla^\nu w_R \nabla_\mu w_R \nabla_\nu w_R |\varphi|^2 \right\} \\ & \geq e^{2s w_R} \left\{ 8s^2 |\nabla^\mu w_R \nabla_\mu \varphi + s^2 \nabla^\mu w_R \nabla_\mu w_R \cdot \varphi|^2 \right. \\ & \quad - C s \delta_0 \frac{\nabla^\mu \nabla^\nu w_R \nabla_\mu w_R \nabla_\nu w_R}{g_{\text{ref}}(dw_R, dw_R)} |\nabla \varphi|_{g_{\text{ref}}}^2 \\ & \quad \left. + 3s^3 \nabla^\mu \nabla^\nu w_R \nabla_\mu w_R \nabla_\nu w_R |\varphi|^2 \right\} \end{aligned} \tag{6.119}$$

for some absolute constant $C > 0$. Thus, (6.119), (6.78) and (6.117) imply that we can bound from below on the region $\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{4}r_0 \leq r \leq \frac{1}{2}R_0\} \setminus \mathcal{E}_{\text{ext}}$:

$$\begin{aligned}
 & \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{4}r_0 \leq r \leq \frac{1}{2}R_0\} \setminus \mathcal{E}_{ext}} \operatorname{Re} \left\{ 2f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - 2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} \right\} dg \\
 & \geq \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{4}r_0 \leq r \leq \frac{1}{2}R_0\} \setminus \mathcal{E}_{ext}} e^{2s w_R} \left\{ \frac{\nabla^\mu \nabla^\nu w_R \nabla_\mu w_R \nabla_\nu w_R}{g_{ref}(dw_R, dw_R)} \left(cs \delta_1 (g^{-1})_{\Sigma}^{ij} \partial_i \varphi \partial_j \bar{\varphi} + cs^3 g_{ref}(dw_R, dw_R) |\varphi|^2 \right) \right. \\
 & \quad \left. - Cs \left(\sum_{j=1}^2 |\nabla^j w_R|_{g_{ref}} \right) |\nabla_{g_\Sigma} \varphi|_{g_\Sigma} |T\varphi| - Cs \left(\sum_{j=1}^2 |\nabla^j w_R|_{g_{ref}} \right) |T\varphi|^2 \right\} dg. \tag{6.120}
 \end{aligned}$$

for some absolute constants $c, C > 0$.

6.6.3. *Bound on $\{\frac{1}{2}R_0 \leq r \leq R_0\}$.* In the region $\{r \geq \frac{1}{2}R_0\}$, the functions f, h depend only on r . In particular, we compute in the (t, r, σ) coordinate system in each connected component of the region $\{r \geq \frac{1}{2}R_0\}$ for any function $\psi \in C^1(\mathcal{M})$:

$$\begin{aligned}
 \nabla^\mu \nabla^\nu f \nabla_\mu \psi \nabla_\nu \bar{\psi} &= ((1 + O(r^{-1})) \partial_r^2 f + O(r^{-2}) \partial_r f) |\partial_r \psi|^2 \\
 & \quad + r^{-3} (1 + O(r^{-1})) \partial_r f |\partial_\sigma \psi|^2 + O(r^{-2}) \partial_r f |T\psi|^2. \tag{6.121}
 \end{aligned}$$

Therefore, (6.76) and (6.78) yield the following lower bound:

$$\begin{aligned}
 & \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{2}R_0 \leq r \leq R_0\}} \operatorname{Re} \left\{ 2\chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 \right. \\
 & \quad \left. + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 - 2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} \right\} dg \\
 & \geq c \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{2}R_0 \leq r \leq R_0\}} e^{2s w_R} \left\{ (s^2 (\partial_r w_R)^2 + O(1)s \sum_{j=1}^2 |\nabla^j w_R|_{g_{ref}}) e^{-2s w_R} |\partial_r (e^{s w_R} \varphi)|^2 \right. \\
 & \quad + s (\partial_r w_R) (\chi_{\geq R_0} (r^{-\frac{3}{2}} + O(r^{-2})) + O(\delta_1) |\nabla w_R|_{g_{ref}}^{-1} |\nabla^2 w_R|_{g_{ref}}) |\partial_r \varphi|^2 \\
 & \quad + s (\partial_r w_R) (r^{-3} + O(r^{-4}) + \chi_{\geq R_0} (r^{-\frac{7}{2}} - r^{-3})) \\
 & \quad + O(\delta_1) |\nabla w_R|_{g_{ref}}^{-1} |\nabla^2 w_R|_{g_{ref}} |\partial_\sigma \varphi|^2 \\
 & \quad \left. + s (\partial_r w_R) (\chi_{\geq R_0} (r^{-1} + O(r^{-\frac{3}{2}})) + O(\delta_1) |\nabla w_R|_{g_{ref}}^{-1} |\nabla^2 w_R|_{g_{ref}}) |T\varphi|^2 \right\} dg \tag{6.122}
 \end{aligned}$$

for some $c > 0$ depending on the geometry of (\mathcal{M}, g) .

6.6.4. *Bound on $\{R_0 \leq r \leq R\}$.* In the region $\{R_0 \leq r \leq R\}$, in view of (6.76), (6.78) and (6.121), we can readily estimate from below:

$$\begin{aligned}
 & \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{R_0 \leq r \leq R\}} \\
 & \operatorname{Re} \left\{ 2\chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 \right. \\
 & \quad \left. + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 - 2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} \right\} dg \\
 & \geq c \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{R_0 \leq r \leq R\}} \\
 & \quad e^{2s w_R} \left\{ (s^2 (1 + O(r^{-1})) (\partial_r w_R)^2 + s (\partial_r^2 w_R + O(r^{-2}) \partial_r w_R)) e^{-2s w_R} |\partial_r (e^{s w_R} \varphi)|^2 \right. \\
 & \quad \left. + s (\partial_r w_R) (r^{-\frac{3}{2}} + O(r^{-2})) |\partial_r \varphi|^2 + s (\partial_r w_R) (r^{-\frac{7}{2}} + O(r^{-4})) |\partial_\sigma \varphi|^2 \right. \\
 & \quad \left. + c s (\partial_r w_R) (r^{-1} + O(r^{-\frac{3}{2}})) |T \varphi|^2 \right\} dg \tag{6.123}
 \end{aligned}$$

for some $c > 0$ depending on the geometry of (\mathcal{M}, g) .

6.6.5. *Bound on $\{r \geq R\}$.* In the region $\{r \geq R\}$, we can estimate in view of in view of (6.121):

$$\begin{aligned}
 & \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \geq R\}} \\
 & \operatorname{Re} \left\{ 2\chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 \right. \\
 & \quad \left. + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 - 2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} \right\} dg \\
 & \geq \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \geq R\}} \\
 & \quad \left\{ \chi_{\leq R} (2(1 + O(r^{-1})) \partial_r^2 f - 2(r^{-1} - r^{-\frac{3}{2}} + O(r^{-2})) \partial_r f) f^{-1} |\partial_r (f^{\frac{1}{2}} \varphi)|^2 \right. \\
 & \quad \left. + (2(1 - \chi_{\leq R}) ((1 + O(r^{-1})) \partial_r^2 f + O(r^{-2}) \partial_r f) - 2(1 + O(r^{-1})) h) |\partial_r \varphi|^2 \right. \\
 & \quad \left. + (2r^{-3} (1 + O(r^{-1})) \partial_r f - 2(r^{-2} + O(r^{-3})) h) |\partial_\sigma \varphi|^2 \right. \\
 & \quad \left. + (2h + O(r^{-2}) \partial_r f) |T \varphi|^2 \right\} dg. \tag{6.124}
 \end{aligned}$$

6.7. *Proof of Proposition 6.1.*

1. In view of (6.105), (6.106), (6.120) and Lemma 6.2, we can bound:

$$\begin{aligned}
 & \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{4} r_0 \leq r \leq \frac{1}{2} R_0\} \setminus \mathcal{E}_{\text{ext}}} \\
 & \operatorname{Re} \left\{ 2\chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 \right. \\
 & \quad \left. + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 - 2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \mathcal{A}_{f,h}^{(R)} |\varphi|^2 \right\} dg \\
 & \geq \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{4} r_0 \leq r \leq \frac{1}{2} R_0\} \setminus (\mathcal{E}_{\text{ext}} \cup \mathcal{B}_{\text{crit}}(\delta_0))} \\
 & \quad e^{2s w_R} \left\{ c_{\delta_0 \delta_1} s R^{-3\epsilon_0} (g^{-1})_{\Sigma}^{ij} \partial_i \varphi \partial_j \bar{\varphi} - C s R^{-3\epsilon_0} |\nabla_{g_\Sigma} \varphi|_{g_\Sigma} |T \varphi| \right.
 \end{aligned}$$

$$\begin{aligned}
& -CsR^{-3\epsilon_0}|T\varphi|^2 + (c_{\delta_0}s^3R^{-9\epsilon_0} - C_{\delta_0}s^2R^{-6\epsilon_0} - C_{\delta_0}sR^{-3\epsilon_0})|\varphi|^2 \Big\} dg \\
& -C_{\delta_0} \int_{\mathcal{R}(\tau_1, \tau_2) \cap \mathcal{B}_{crit}(\delta_0)} e^{2sw_R} \left\{ sR^{-3\epsilon_0} |\nabla\varphi|_{g_{ref}}^2 + s^3R^{-9\epsilon_0} |\varphi|^2 \right\} dg. \tag{6.125}
\end{aligned}$$

Repeating the same procedure for \tilde{f} in place of f , \tilde{h} in place of h and \tilde{w}_R in place of w_R (see Lemma 6.3), from (the analogues of) (6.105), (6.106), (6.120) for w_R in place of \tilde{w}_R we obtain:

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{4}r_0 \leq r \leq \frac{1}{2}R_0\} \setminus \mathcal{E}_{ext}} \\
& Re \left\{ 2\chi_{\leq R} \tilde{f}^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (\tilde{f}^{\frac{1}{2}} \varphi) \nabla_\nu (\tilde{f}^{\frac{1}{2}} \bar{\varphi}) - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} \tilde{f}^{-1} (\partial_r \tilde{f}) |\partial_r (\tilde{f}^{\frac{1}{2}} \varphi)|^2 \right. \\
& \quad + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu \tilde{f} \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r \tilde{f}) |\partial_r \varphi|^2 \\
& \quad \left. - 2\tilde{h} \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \mathcal{A}_{\tilde{f}, \tilde{h}}^{(R)} |\varphi|^2 \right\} dg \\
& \geq \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{4}r_0 \leq r \leq \frac{1}{2}R_0\} \setminus (\mathcal{E}_{ext} \cup \tilde{\mathcal{B}}_{crit}(\delta_0))} \\
& \quad e^{2s\tilde{w}_R} \left\{ c_{\delta_0\delta_1} sR^{-3\epsilon_0} (g^{-1})_{\Sigma}^{ij} \partial_i \varphi \partial_j \bar{\varphi} - CsR^{-3\epsilon_0} |\nabla_{g_\Sigma} \varphi|_{g_\Sigma} |T\varphi| \right. \\
& \quad \left. - CsR^{-3\epsilon_0} |T\varphi|^2 + (c_{\delta_0}s^3R^{-9\epsilon_0} - C_{\delta_0}s^2R^{-6\epsilon_0} - C_{\delta_0}sR^{-3\epsilon_0}) |\varphi|^2 \right\} dg \\
& \quad - C_{\delta_0} \int_{\mathcal{R}(\tau_1, \tau_2) \cap \tilde{\mathcal{B}}_{crit}(\delta_0)} e^{2s\tilde{w}_R} \left\{ sR^{-3\epsilon_0} |\nabla\varphi|_{g_{ref}}^2 + s^3R^{-9\epsilon_0} |\varphi|^2 \right\} dg. \tag{6.126}
\end{aligned}$$

Adding (6.125) and (6.126) and using (6.39) and (6.40), we obtain provided s is large in terms of δ_0 :

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{4}r_0 \leq r \leq \frac{1}{2}R_0\} \setminus \mathcal{E}_{ext}} \\
& Re \left\{ 2\chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 \right. \\
& \quad + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 \\
& \quad \left. - 2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \mathcal{A}_{f, h}^{(R)} |\varphi|^2 \right\} dg \\
& \geq \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{4}r_0 \leq r \leq \frac{1}{2}R_0\} \setminus \mathcal{E}_{ext}} \\
& Re \left\{ 2\chi_{\leq R} \tilde{f}^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (\tilde{f}^{\frac{1}{2}} \varphi) \nabla_\nu (\tilde{f}^{\frac{1}{2}} \bar{\varphi}) - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} \tilde{f}^{-1} (\partial_r \tilde{f}) |\partial_r (\tilde{f}^{\frac{1}{2}} \varphi)|^2 \right. \\
& \quad + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu \tilde{f} \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r \tilde{f}) |\partial_r \varphi|^2 \\
& \quad \left. - 2\tilde{h} \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \mathcal{A}_{\tilde{f}, \tilde{h}}^{(R)} |\varphi|^2 \right\} dg \\
& \geq \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{4}r_0 \leq r \leq \frac{1}{2}R_0\} \setminus \mathcal{E}_{ext}} (e^{2sw_R} + e^{2s\tilde{w}_R}) \\
& \quad \left\{ c_{\delta_0\delta_1} sR^{-3\epsilon_0} (g^{-1})_{\Sigma}^{ij} \partial_i \varphi \partial_j \bar{\varphi} - CsR^{-3\epsilon_0} |\nabla_{g_\Sigma} \varphi|_{g_\Sigma} |T\varphi| \right. \\
& \quad \left. - CsR^{-3\epsilon_0} |T\varphi|^2 + (c_{\delta_0}s^3R^{-9\epsilon_0} - C_{\delta_0}s^2R^{-6\epsilon_0} - C_{\delta_0}sR^{-3\epsilon_0}) |\varphi|^2 \right\} dg. \tag{6.127}
\end{aligned}$$

2. In view of (6.105), (6.106), (6.122) and Lemma 6.2, we can estimate

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{2}R_0 \leq r \leq R_0\}} \\
& \operatorname{Re} \left\{ 2\chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 \right. \\
& \quad + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 \\
& \quad \left. - 2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \mathcal{A}_{f,h}^{(R)} |\varphi|^2 \right\} dg \\
& \geq \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{2}R_0 \leq r \leq R_0\}} \\
& \quad e^{2s_{wR}} \left\{ (c_{\delta_0} s^2 R^{-6\epsilon_0} - C_{\delta_0} s R^{-3\epsilon_0}) e^{-2s_{wR}} |\partial_r (e^{s_{wR}} \varphi)|^2 \right. \\
& \quad + s R^{-3\epsilon_0} (c_{\delta_0} \chi_{\geq R_0} (r^{-\frac{3}{2}} + O(r^{-2})) - C_{\delta_0} \delta_1) |\partial_r \varphi|^2 \\
& \quad + s R^{-3\epsilon_0} (c_{\delta_0} (r^{-3} + O(r^{-4})) + \chi_{\geq R_0} (r^{-\frac{7}{2}} - r^{-3})) - C_{\delta_0} \delta_1) |\partial_\sigma \varphi|^2 \\
& \quad \left. - C_{\delta_0} s R^{-3\epsilon_0} |T\varphi|^2 + (c_{\delta_0} s^3 R^{-9\epsilon_0} - C_{\delta_0} s^2 R^{-6\epsilon_0} - C_{\delta_0} s R^{-3\epsilon_0}) |\varphi|^2 \right\} dg. \tag{6.128}
\end{aligned}$$

Provided $sR^{-3\epsilon_0}$ is sufficiently large in terms of δ_0 , we can estimate

$$\begin{aligned}
& (c_{\delta_0} s^2 R^{-6\epsilon_0} - C_{\delta_0} s R^{-3\epsilon_0}) e^{-2s_{wR}} |\partial_r (e^{s_{wR}} \varphi)|^2 - C_{\delta_0} \delta_1 s R^{-3\epsilon_0} |\partial_r \varphi|^2 \\
& \geq c_{\delta_0} \delta_1 s R^{-3\epsilon_0} |\partial_r \varphi|^2 - C_{\delta_0} \delta_1 s^3 R^{-9\epsilon_0} |\varphi|^2 \tag{6.129}
\end{aligned}$$

and thus (6.128) yields:

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{2}R_0 \leq r \leq R_0\}} \\
& \operatorname{Re} \left\{ 2\chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 \right. \\
& \quad + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 \\
& \quad \left. - 2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \mathcal{A}_{f,h}^{(R)} |\varphi|^2 \right\} dg \\
& \geq \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{2}R_0 \leq r \leq R_0\}} \\
& \quad e^{2s_{wR}} \left\{ c_{\delta_0} \delta_1 s R^{-3\epsilon_0} |\partial_r \varphi|^2 + s R^{-3\epsilon_0} (c_{\delta_0} r^{-\frac{7}{2}} - C_{\delta_0} \delta_1) |\partial_\sigma \varphi|^2 \right. \\
& \quad - C_{\delta_0} s R^{-3\epsilon_0} |T\varphi|^2 + (c_{\delta_0} - C_{\delta_0} \delta_1) s^3 R^{-9\epsilon_0} \\
& \quad \left. - C_{\delta_0} s^2 R^{-6\epsilon_0} - C_{\delta_0} s R^{-3\epsilon_0}) |\varphi|^2 \right\} dg. \tag{6.130}
\end{aligned}$$

3. In view of (6.110), (6.123) and Lemma 6.3, we can bound:

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{R_0 \leq r \leq R^{\epsilon_0}\}} \\
& \operatorname{Re} \left\{ 2\chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 \right. \\
& \quad + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 \\
& \quad \left. - 2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \mathcal{A}_{f,h}^{(R)} |\varphi|^2 \right\} dg
\end{aligned}$$

$$\begin{aligned}
 &\geq \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{R_0 \leq r \leq R^{\varepsilon_0}\}} \\
 &\quad e^{2sw_R} \left\{ (c_{\delta_0} s^2 R^{-6\varepsilon_0} - C_{\delta_0} s) e^{-2sw_R} |\partial_r (e^{sw_R} \varphi)|^2 \right. \\
 &\quad + c_{\delta_0} s R^{-3\varepsilon_0} r^{-\frac{3}{2}} |\partial_r \varphi|^2 + c_{\delta_0} s R^{-3\varepsilon_0} r^{-\frac{7}{2}} |\partial_\sigma \varphi|^2 \\
 &\quad \left. + c_{\delta_0} s R^{-3\varepsilon_0} r^{-1} |T\varphi|^2 + (c_{\delta_0} R^{-9\varepsilon_0} s^3 - C_{\delta_0} s^2 - C_{\delta_0} s) |\varphi|^2 \right\} dg. \tag{6.131}
 \end{aligned}$$

4. In view of (6.111), (6.123) and (6.67)–(6.69), we can bound:

$$\begin{aligned}
 &\int_{\mathcal{R}(\tau_1, \tau_2) \cap \{R^{\varepsilon_0} \leq r \leq \frac{1}{2}R\}} \\
 &\quad Re \left\{ 2\chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 \right. \\
 &\quad + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 \\
 &\quad \left. - 2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \mathcal{A}_{f,h}^{(R)} |\varphi|^2 \right\} dg \\
 &\geq \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{R^{\varepsilon_0} \leq r \leq \frac{1}{2}R\}} \\
 &\quad e^{2sw_R} \left\{ r^{-2} (c_{\delta_0} r^{2\varepsilon_0} s^2 R^{-2\varepsilon_0} - C_{\delta_0} s (r^{\varepsilon_0} R^{-\varepsilon_0} + r^{2\varepsilon_0} R^{-2\varepsilon_0})) e^{-2sw_R} |\partial_r (e^{sw_R} \varphi)|^2 \right. \\
 &\quad + c_{\delta_0} r^{-\frac{5}{2} + \varepsilon_0} s R^{-\varepsilon_0} |\partial_r \varphi|^2 + c_{\delta_0} r^{-\frac{9}{2} + \varepsilon_0} s R^{-\varepsilon_0} |\partial_\sigma \varphi|^2 + c_{\delta_0} r^{-2 + \varepsilon_0} s R^{-\varepsilon_0} |T\varphi|^2 \\
 &\quad \left. + r^{-4} (c_{\delta_0} \varepsilon_0 r^{3\varepsilon_0} s^3 R^{-3\varepsilon_0} - C_{\delta_0} r^{2\varepsilon_0} s^2 R^{-2\varepsilon_0} - C_{\delta_0} r^{\varepsilon_0} s R^{-\varepsilon_0}) |\varphi|^2 \right\} dg. \tag{6.132}
 \end{aligned}$$

5. In view of (6.112), (6.123) and (6.63)–(6.65), we can bound:

$$\begin{aligned}
 &\int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{2}R \leq r \leq R\}} \\
 &\quad Re \left\{ 2\chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 \right. \\
 &\quad + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 \\
 &\quad \left. - 2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \mathcal{A}_{f,h}^{(R)} |\varphi|^2 \right\} dg \\
 &\geq \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{2}R \leq r \leq R\}} \\
 &\quad e^{2sw_R} \left\{ R^{-2} (c_{\delta_0} - C_{\delta_0} (\delta_1 + s^{-1} \delta_1^{-1})) e^{-2sw_R} |\partial_r (e^{sw_R} \varphi)|^2 \right. \\
 &\quad + c_{\delta_0} R^{-\frac{5}{2}} |\partial_r \varphi|^2 + c_{\delta_0} R^{-\frac{9}{2}} |\partial_\sigma \varphi|^2 \\
 &\quad \left. + c_{\delta_0} R^{-2} v'_s \left(\frac{r}{R}\right) s |T\varphi|^2 - C_{\delta_0} R^{-4} \left(v'_s \left(\frac{r}{R}\right) s^3 + s^2 + s\right) \right\} dg. \tag{6.133}
 \end{aligned}$$

6. In view of (6.113), (6.124), (6.76), (6.78) and (6.80), we can estimate:

$$\begin{aligned}
 &\int_{\mathcal{R}(\tau_1, \tau_2) \cap \{R \leq r \leq \delta_2^{-1}R\}} \\
 &\quad Re \left\{ 2\chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 \right. \\
 &\quad + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 \\
 &\quad \left. - 2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \mathcal{A}_{f,h}^{(R)} |\varphi|^2 \right\} dg
 \end{aligned}$$

$$\begin{aligned}
 &\geq \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{R \leq r \leq \delta_2^{-1} R\}} \left\{ 2\chi_{\leq R} \frac{R^2}{r^2} \left(\frac{9}{5} - \frac{r}{R} \right) R^{-2} f(R) \cdot f^{-1} |\partial_r (f^{\frac{1}{2}} \varphi)|^2 \right. \\
 &\quad + \left((1 - \chi_{\leq R}) f(R) r^{-\frac{5}{2}} - 2\chi_{\leq R} \left(\frac{R^2}{r^2} \left(\frac{r}{R} - \frac{9}{10} \right) (R^{-2} + O(R^{-\frac{5}{2}})) f(R) \right) \right) |\partial_r \varphi|^2 \\
 &\quad \left. + c f(R) r^{-\frac{9}{2}} |\partial_\sigma \varphi|^2 + c f(R) r^{-2} |T \varphi|^2 - C R^{-4} f(R) |\varphi|^2 \right\} dg. \tag{6.134}
 \end{aligned}$$

Remark. Notice that the positivity of the coefficient of $|\partial_r (f^{\frac{1}{2}} \varphi)|^2$ in the right hand side of (6.134) follows from the fact that, in view of (6.76), provided R is sufficiently large in terms of the geometry of (\mathcal{M}, g) , we can bound for $R \leq r \leq \frac{4}{3} R$ (i.e. on $\text{supp}(\chi_{\leq R}) \cap \{r \geq R\}$):

$$\partial_r^2 f + r^{-\frac{3}{2}} \partial_r f > (r^{-1} + O(r^{-2})) \partial_r f + O(r^{-1}) \partial_r^2 f + f(R) \left(\frac{9}{5} - \frac{r}{R} \right) R^{-2}. \tag{6.135}$$

Applying the product rule and a Cauchy–Schwarz inequality on the first term of the right hand side of (6.134), we obtain:

$$\begin{aligned}
 &2\chi_{\leq R} \frac{R^2}{r^2} \left(\frac{9}{5} - \frac{r}{R} \right) R^{-2} f(R) \cdot f^{-1} |\partial_r (f^{\frac{1}{2}} \varphi)|^2 \\
 &\geq 2\chi_{\leq R} \frac{R^2}{r^2} \left(\frac{9}{5} - \frac{r}{R} - \frac{1}{100} \right) R^{-2} f(R) |\partial_r \varphi|^2 - C \chi_{\leq R} R^{-4} f(R) |\varphi|^2 \tag{6.136}
 \end{aligned}$$

for some absolute constant $C > 0$. Thus, (6.134) yields (provided $R \gg 1$):

$$\begin{aligned}
 &\int_{\mathcal{R}(\tau_1, \tau_2) \cap \{R \leq r \leq \delta_2^{-1} R\}} \\
 &\quad \text{Re} \left\{ 2\chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - 2\chi_{\geq R} \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 \right. \\
 &\quad \left. + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 \right. \\
 &\quad \left. - 2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \mathcal{A}_{f,h}^{(R)} |\varphi|^2 \right\} dg \\
 &\geq \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{R \leq r \leq \delta_2^{-1} R\}} f(R) \\
 &\quad \left\{ c r^{-\frac{5}{2}} |\partial_r \varphi|^2 + c r^{-\frac{9}{2}} |\partial_\sigma \varphi|^2 + c r^{-2} |T \varphi|^2 - C R^{-4} |\varphi|^2 \right\} dg. \tag{6.137}
 \end{aligned}$$

7. In view of (6.114), (6.124), (6.76) and (6.82), we can bound:

$$\begin{aligned}
 &\int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \geq \delta_2^{-1} R\}} \\
 &\quad \text{Re} \left\{ 2\chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 \right. \\
 &\quad \left. + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 \right. \\
 &\quad \left. - 2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \mathcal{A}_{f,h}^{(R)} |\varphi|^2 \right\} dg \\
 &\geq \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \geq \delta_2^{-1} R\}} f(R) \\
 &\quad \left\{ c r^{-2} |\partial_r \varphi|^2 + c R^{-1} r^{-3} |\partial_\sigma \varphi|^2 + c r^{-2} |T \varphi|^2 - C R^{-1} r^{-3} |\varphi|^2 \right\} dg. \tag{6.138}
 \end{aligned}$$

In view of (6.118), (6.127), (6.130), (6.131), (6.132), (6.137), (6.138), as well as the fact that $\tilde{f} \equiv f$ and $\tilde{h} \equiv h$ on $\mathcal{M} \setminus (\mathcal{B}_{crit}(4\delta_0) \cup \mathcal{H}^-)$, we obtain from (6.98) (provided $\delta_0, \delta_2 > 0$ are sufficiently small in terms of R_0 and the geometry of (\mathcal{M}, g) , $\delta_1 > 0$ is sufficiently small in terms of δ_0 and R_0 , s is sufficiently large in terms of δ_0, δ_1 and $\varepsilon_0 s R^{-9\varepsilon_0}$ is sufficiently large in terms of δ_0, δ_1):

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{4}r_0 \leq r \leq \frac{1}{2}R_0\} \setminus \mathcal{E}_{ext}} (f + \tilde{f}) \\
& \left\{ c_{\delta_0 \delta_1} s R^{-3\varepsilon_0} (g^{-1})_{\Sigma}^{ij} \partial_i \varphi \partial_j \bar{\varphi} - C s R^{-3\varepsilon_0} |\nabla_{g_{\Sigma}} \varphi|_{g_{\Sigma}} |T\varphi| - C s R^{-3\varepsilon_0} |T\varphi|^2 \right. \\
& \quad \left. + c_{\delta_0} s^3 R^{-9\varepsilon_0} |\varphi|^2 \right\} dg \\
& + c_{\delta_0} \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \geq \frac{1}{2}R_0\}} (f + \tilde{f}) \\
& \left\{ \chi_{\{\frac{1}{2}R_0 \leq r \leq R_0\}} \delta_1 s R^{-3\varepsilon_0} + \chi_{\{R_0 \leq r \leq R^{\varepsilon_0}\}} s R^{-3\varepsilon_0} r^{-\frac{3}{2}} + \chi_{\{R^{\varepsilon_0} \leq r \leq \frac{1}{2}R\}} s R^{-\varepsilon_0} r^{-\frac{5}{2} + \varepsilon_0} \right. \\
& \quad \left. + \chi_{\{\frac{1}{2}R \leq r \leq \delta_2^{-1}R\}} r^{-\frac{5}{2}} + \chi_{\{r \geq \delta_2^{-1}R\}} r^{-2} \right\} (|\partial_r \varphi|^2 + r^{-2} |\partial_{\sigma} \varphi|^2) dg \\
& + \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{2}R_0 \leq r \leq \frac{1}{2}R\}} (f + \tilde{f}) \\
& \left\{ \chi_{\{\frac{1}{2}R_0 \leq r \leq R_0\}} \left(-C_{\delta_0} s R^{-3\varepsilon_0} |T\varphi|^2 + c_{\delta_0} s^3 R^{-9\varepsilon_0} |\varphi|^2 \right) \right. \\
& \quad + c_{\delta_0} \chi_{\{R_0 \leq r \leq R^{\varepsilon_0}\}} \left(s R^{-3\varepsilon_0} r^{-1} |T\varphi|^2 + s^3 R^{-9\varepsilon_0} |\varphi|^2 \right) \\
& \quad \left. + \chi_{\{R^{\varepsilon_0} \leq r \leq \frac{1}{2}R\}} \left(s R^{-\varepsilon_0} r^{-2 + \varepsilon_0} |T\varphi|^2 + c_{\delta_0} \varepsilon_0 s^3 R^{-3\varepsilon_0} r^{-4 + 3\varepsilon_0} |\varphi|^2 \right) \right\} dg \\
& + \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{2}R \leq r \leq R\}} (f + \tilde{f}) v'_s \left(\frac{r}{R} \right) \left(c_{\delta_0} R^{-2} s |T\varphi|^2 - C_{\delta_0} R^{-4} s^3 |\varphi|^2 \right) dg \\
& + \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \geq R\}} (f(R) + \tilde{f}(R)) \left(c r^{-2} |T\varphi|^2 - C R^{-1} r^{-3} |\varphi|^2 \right) dg \\
& \leq C_{\delta_0} \int_{\mathcal{R}(\tau_1, \tau_2) \cap (\mathcal{E}_{ext} \cup \{r \leq \frac{1}{4}r_0\})} (f + \tilde{f}) \left\{ s^2 R^{-6\varepsilon_0} |\nabla \varphi|_{g_{ref}}^2 + s^4 R^{-12\varepsilon_0} |\varphi|^2 \right\} dg \\
& - \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ G(2(\nabla^{\mu} f + \nabla^{\mu} \tilde{f}) \nabla_{\mu} \bar{\varphi} + (\square_g f + \square_g \tilde{f} - 2h - 2\tilde{h}) \bar{\varphi}) \right\} dg \\
& - \mathcal{B}_{f, h}^{(R)}[\varphi; \tau_1, \tau_2] - \mathcal{B}_{\tilde{f}, \tilde{h}}^{(R)}[\varphi; \tau_1, \tau_2],
\end{aligned} \tag{6.139}$$

where, for any set $A \subset \mathcal{M}$, we denote with χ_A the characteristic function of A .

In view of the fact that $(g^{-1})_{\Sigma}$ is positive definite on $\mathcal{M} \setminus \mathcal{E} \cup \mathcal{H}$, we can estimate on $\mathcal{M} \setminus \mathcal{E}_{\delta} \cup \{r \geq \frac{1}{4}r_0\}$ for any $\delta > 0$:

$$c_{\delta_0 \delta_1} s R^{-3\varepsilon_0} (g^{-1})_{\Sigma}^{ij} \partial_i \varphi \partial_j \bar{\varphi} - C s R^{-3\varepsilon_0} |\nabla_{g_{\Sigma}} \varphi|_{g_{\Sigma}} |T\varphi|$$

$$\geq c_{\delta_0\delta_1} s R^{-3\epsilon_0} |\nabla_{g_\Sigma} \varphi|_{g_\Sigma} - C_\delta s R^{-3\epsilon_0} |T\varphi|^2. \tag{6.140}$$

Furthermore, if $\chi_{r_0} : \mathcal{M} \setminus \mathcal{H}^- \rightarrow [0, 1]$ is a smooth T -invariant function supported in $\{r \leq r_0\}$ such that $\chi_{r_0} \equiv 1$ on $\{r \leq \frac{1}{2}r_0\}$, then, after integrating by parts in the identity

$$\begin{aligned} & \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ N(\chi_{r_0} \bar{\varphi}) \square_g (\chi_{r_0} \varphi) \right\} dg \\ &= \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ N(\chi_{r_0} \bar{\varphi}) (\chi_{r_0} G + 2\nabla^\mu \chi_{r_0} \nabla_\mu \varphi + \square_g \chi_{r_0} \varphi) \right\} dg, \end{aligned} \tag{6.141}$$

using also the bounds (2.5) and (2.6) from Assumption G2 (as well as a Poincare-type inequality), we readily obtain the red-shift-type estimate

$$\begin{aligned} & c \int_{\mathcal{H}^+ \cap \mathcal{R}(\tau_1, \tau_2)} (J_\mu^N(\varphi) n_{\mathcal{H}^+}^\mu + |\varphi|^2) \operatorname{vol}_{\mathcal{H}^+} + c \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq \frac{1}{2}r_0\}} (|\nabla \varphi|_{g_{ref}}^2 + |\varphi|^2) dg \\ & \leq C \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{2}r_0 \leq r \leq r_0\}} (|\nabla \varphi|_{g_{ref}}^2 + |\varphi|^2) dg \\ & \quad + C \int_{\Sigma_{\tau_1} \cap \{r \leq r_0\}} |\nabla \varphi|_{g_{ref}}^2 - \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ \chi_{r_0} N(\chi_{r_0} \bar{\varphi}) \cdot G \right\} dg. \end{aligned} \tag{6.142}$$

Therefore, in view of (6.140), (6.142) and the fact that

$$\frac{\sup_{\{r \leq \frac{1}{4}r_0\}} (e^{2s w_R} + e^{2s \tilde{w}_R})}{\inf_{\{r \geq \frac{1}{2}r_0\}} (e^{2s w_R} + e^{2s \tilde{w}_R})} > e^{c s R^{-3\epsilon_0}} \tag{6.143}$$

(following from (6.43), (6.44), (6.76) and (6.77)), (6.139) yields (provided $s R^{-3\epsilon_0} \gg 1$):

$$\begin{aligned} & \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq \frac{1}{2}R_0\} \setminus \mathcal{E}_\delta} (f + \tilde{f} + \sup_{\{r \leq \frac{1}{4}r_0\}} f) \\ & \quad \left\{ c_{\delta_0\delta_1} s R^{-3\epsilon_0} |\nabla_{g_\Sigma} \varphi|_{g_\Sigma}^2 - C_\delta s R^{-3\epsilon_0} |T\varphi|^2 + c_{\delta_0} s^3 R^{-9\epsilon_0} |\varphi|^2 \right\} dg \\ & \quad + c_{\delta_0} \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \geq \frac{1}{2}R_0\}} (f + \tilde{f}) \\ & \quad \left\{ \chi_{\{\frac{1}{2}R_0 \leq r \leq R_0\}} \delta_1 s R^{-3\epsilon_0} + \chi_{\{R_0 \leq r \leq R^{\epsilon_0}\}} s R^{-3\epsilon_0} r^{-\frac{3}{2}} + \chi_{\{R^{\epsilon_0} \leq r \leq \frac{1}{2}R\}} s R^{-\epsilon_0} r^{-\frac{5}{2} + \epsilon_0} \right. \\ & \quad \left. + \chi_{\{\frac{1}{2}R \leq r \leq \delta_2^{-1}R\}} r^{-\frac{5}{2}} + \chi_{\{r \geq \delta_2^{-1}R\}} r^{-2} \right\} (|\partial_r \varphi|^2 + r^{-2} |\partial_\sigma \varphi|^2) dg \end{aligned}$$

$$\begin{aligned}
& + \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{2}R_0 \leq r \leq \frac{1}{2}R\}} (f + \tilde{f}) \\
& \left\{ \chi_{\{\frac{1}{2}R_0 \leq r \leq R_0\}} \left(-C_{\delta_0} s R^{-3\epsilon_0} |T\varphi|^2 + c_{\delta_0} s^3 R^{-9\epsilon_0} |\varphi|^2 \right) \right. \\
& \quad + c_{\delta_0} \chi_{\{R_0 \leq r \leq R^{\epsilon_0}\}} \left(s R^{-3\epsilon_0} r^{-1} |T\varphi|^2 + s^3 R^{-9\epsilon_0} |\varphi|^2 \right) \\
& \quad \left. + \chi_{\{R^{\epsilon_0} \leq r \leq \frac{1}{2}R\}} \left(s R^{-\epsilon_0} r^{-2+\epsilon_0} |T\varphi|^2 + c_{\delta_0} \epsilon_0 s^3 R^{-3\epsilon_0} r^{-4+3\epsilon_0} |\varphi|^2 \right) \right\} dg \\
& + \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{2}R \leq r \leq R\}} (f + \tilde{f}) v'_s \left(\frac{r}{R} \right) \left(c_{\delta_0} R^{-2} s |T\varphi|^2 - C_{\delta_0} R^{-4} s^3 |\varphi|^2 \right) dg \\
& + \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \geq R\}} (f(R) + \tilde{f}(R)) \left(c r^{-2} |T\varphi|^2 - C R^{-1} r^{-3} |\varphi|^2 \right) dg \\
& \leq C_{\delta_0} \delta \int_{\mathcal{R}(\tau_1, \tau_2) \cap \mathcal{E}_\delta^c} (f + \tilde{f}) \left\{ s^2 R^{-6\epsilon_0} |\nabla \varphi|_{g_{ref}}^2 + s^4 R^{-12\epsilon_0} |\varphi|^2 \right\} dg \\
& \quad - \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ G \left(2(\nabla^\mu f + \nabla^\mu \tilde{f}) \nabla_\mu \bar{\varphi} + (\square_g f + \square_g \tilde{f} - 2h - 2\tilde{h}) \bar{\varphi} \right) \right\} dg \\
& + C \bar{\mathcal{B}}_{f, h; \tilde{f}, \tilde{h}}^{(R)}[\varphi; \tau_1, \tau_2], -\mathcal{B}_{f, h}^{(R)}[\varphi; \tau_1, \tau_2] - \mathcal{B}_{\tilde{f}, \tilde{h}}^{(R)}[\varphi; \tau_1, \tau_2] \\
& + C \sup_{\{r \leq \frac{1}{4}r_0\}} f \int_{\Sigma_{\tau_1} \cap \{r \leq r_0\}} |\nabla \varphi|_{g_{ref}}^2, \tag{6.144}
\end{aligned}$$

where

$$\begin{aligned}
& \bar{\mathcal{B}}_{f, h; \tilde{f}, \tilde{h}}^{(R)}[\varphi; \tau_1, \tau_2] \\
& \doteq \sum_{j=1}^2 \left| \int_{\Sigma_{\tau_j}} \operatorname{Re} \left\{ \left(2\nabla^\mu f \nabla_\mu \bar{\varphi} \nabla_\nu \varphi + (\square_g f - 2h) \varphi \nabla_\nu \bar{\varphi} - \nabla_\nu f \nabla^\mu \varphi \nabla_\mu \bar{\varphi} \right. \right. \right. \\
& \quad \left. \left. + (\chi_{\leq R} f^{-1} \nabla_\mu \nabla_\nu f \nabla^\mu f + \nabla_\nu h - \frac{1}{2} (\nabla_\nu (\square_g f))) |\varphi|^2 \right) n_{\Sigma_{\tau_j}}^\nu \right\} dg_{\Sigma_{\tau_j}} \left| \right. \\
& \quad + \sum_{j=1}^2 \left| \int_{\Sigma_{\tau_j}} \operatorname{Re} \left\{ \left(2\nabla^\mu \tilde{f} \nabla_\mu \bar{\varphi} \nabla_\nu \varphi + (\square_g \tilde{f} - 2\tilde{h}) \varphi \nabla_\nu \bar{\varphi} - \nabla_\nu \tilde{f} \nabla^\mu \varphi \nabla_\mu \bar{\varphi} \right. \right. \right. \\
& \quad \left. \left. + (\chi_{\leq R} \tilde{f}^{-1} \nabla_\mu \nabla_\nu \tilde{f} \nabla^\mu \tilde{f} + \nabla_\nu \tilde{h} - \frac{1}{2} (\nabla_\nu (\square_g \tilde{f}))) |\varphi|^2 \right) n_{\Sigma_{\tau_j}}^\nu \right\} dg_{\Sigma_{\tau_j}} \left| \right. \\
& \quad + \sup_{\{r \leq \frac{1}{4}r_0\}} f \int_{\Sigma_{\tau_1} \cap \{r \leq r_0\}} |\nabla \varphi|_{g_{ref}}^2 \\
& \tag{6.145}
\end{aligned}$$

(note that the boundary terms on \mathcal{H}^+ in the right hand side of (6.139) were absorbed by the term in the left hand side of (6.142)). Inequality (6.3) now readily follows from (6.144) in view of (6.76), (6.77). \square

6.8. *Proof of Corollary 6.1.* For any $1 \leq k \leq n$ and $0 < \delta_1, \delta_2, \varepsilon_0 \ll 1$, let us choose the parameters R, s to be sufficiently large in terms of $\delta_1, \delta_2, \varepsilon_0$ and the geometry of (\mathcal{M}, g) , satisfying in addition:

$$R \geq C_{\delta_1 \varepsilon_0} \max \{1, \omega_k^{-\frac{1}{1-9\varepsilon_0}}, (-\log \delta_2)^{\frac{1}{1-9\varepsilon_0}}\}, \tag{6.146}$$

$$C_{\delta_1 \varepsilon_0}^{\frac{1}{3}} \max \{(1 + \omega_k)R^{9\varepsilon_0}, -\log \delta_2\} \leq s \leq C_{\delta_1 \varepsilon_0}^{-\frac{1}{3}} R \omega_k, \tag{6.147}$$

for some constant $C_{\delta_1 \varepsilon_0} > 1$ large in terms of δ_1, ε_0 and the geometry of (\mathcal{M}, g) (notice that the bound (6.146) guarantees that an s satisfying (6.147) exists).

By approximating the functions $\psi_k, 1 \leq k \leq n$, by smooth solutions to (1.2) with compact support in space and using Lemma 5.3 on the decay of ψ_k as $r \rightarrow +\infty$, we infer that Proposition 6.1 also applies for the functions ψ_k . Therefore, using the values of s, R chosen above, we obtain for any $0 \leq \tau_1 \leq \tau_2$ and any $1 \leq k \leq n$:

$$\begin{aligned} & \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq R_0\} \setminus \mathcal{E}_{\delta_1}} \left(f + \inf_{\{r \geq \frac{1}{4}r_0\} \setminus \mathcal{E}} f \right) \\ & \left\{ s R^{-3\varepsilon_0} |\nabla_{g_\Sigma} \psi_k|_{g_\Sigma}^2 - C_{\delta_1} s R^{-3\varepsilon_0} |T \psi_k|^2 + s^3 R^{-9\varepsilon_0} |\psi_k|^2 \right\} dg \\ & + \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{R_0 \leq r \leq \frac{1}{2}R\}} f \\ & \left\{ s R^{-3\varepsilon_0} r^{-\frac{5}{2}} (|\partial_r \psi_k|^2 + r^{-2} |\partial_\sigma \psi_k|^2) + s R^{-3\varepsilon_0} r^{-2} |T \psi_k|^2 + \varepsilon_0 s^3 R^{-9\varepsilon_0} r^{-4} |\psi_k|^2 \right\} dg \\ & + \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{2}R \leq r \leq R\}} f \\ & \left\{ r^{-\frac{5}{2}} (|\partial_r \psi_k|^2 + r^{-2} |\partial_\sigma \psi_k|^2) + R \partial_r w_R (c R^{-2} s |T \psi_k|^2 - C R^{-4} s^3 |\psi_k|^2) \right\} dg \\ & + \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \geq R\}} f(R) \\ & \left\{ r^{-\frac{5}{2}} (|\partial_r \psi_k|^2 + r^{-2} |\partial_\sigma \psi_k|^2) + r^{-2} |T \psi_k|^2 - C R^{-1} r^{-3} |\psi_k|^2 \right\} dg \\ & \leq C_{\delta_1} \int_{\mathcal{R}(\tau_1, \tau_2) \cap \mathcal{E}_{\delta_1}} f \left\{ s^2 R^{-6\varepsilon_0} |\nabla \psi_k|_{g_{ref}}^2 + s^4 R^{-12\varepsilon_0} |\psi_k|^2 \right\} dg \\ & + C \left| \int_{\mathcal{R}(\tau_1, \tau_2)} F_k (\nabla^\mu f \nabla_\mu \bar{\psi}_k + O(\sum_{j=1}^2 (1+r)^{j-2} |\nabla^j f|_{g_{ref}}) \bar{\psi}_k) dg \right| \\ & + C \sum_{j=1}^2 \int_{\Sigma_{\tau_j}} \left(|\nabla f|_{g_{ref}} |\nabla \psi_k|_{g_{ref}}^2 + (\sum_{j=1}^3 (1+r)^{j-3} |\nabla^j f|_{g_{ref}}) |\psi_k|^2 \right) dg_\Sigma. \end{aligned} \tag{6.148}$$

In view Lemma 5.4, the bound (6.148) implies:

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \leq R_0\} \setminus \mathcal{E}_{\delta_1}} (f + \inf_{\{r \geq \frac{1}{4}r_0\} \setminus \mathcal{E}} f) \\
& \left\{ sR^{-3\epsilon_0} |\nabla_{g_\Sigma} \Psi_k|_{g_\Sigma}^2 + sR^{-3\epsilon_0} |T\Psi_k|^2 + (s^3 R^{-9\epsilon_0} - C_{\delta_1} \omega_k^2 s R^{-3\epsilon_0}) |\Psi_k|^2 \right\} dg \\
& + \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{R_0 \leq r \leq \frac{1}{2}R\}} f \\
& \left\{ sR^{-3\epsilon_0} r^{-\frac{5}{2}} (|\partial_r \Psi_k|^2 + r^{-2} |\partial_\sigma \Psi_k|^2) + sR^{-3\epsilon_0} r^{-2} |T\Psi_k|^2 + \epsilon_0 s^3 R^{-9\epsilon_0} r^{-4} |\Psi_k|^2 \right\} dg \\
& + \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{\frac{1}{2}R \leq r \leq R\}} f \\
& \left\{ r^{-\frac{5}{2}} (|\partial_r \Psi_k|^2 + r^{-2} |\partial_\sigma \Psi_k|^2) \right. \\
& \quad \left. + R^{-1} s \partial_r w_R |T\Psi_k|^2 + R^{-1} s \partial_r w_R (c\omega_k^2 - CR^{-2}s^2) |\Psi_k|^2 \right\} dg \\
& + \int_{\mathcal{R}(\tau_1, \tau_2) \cap \{r \geq R\}} f(R) \\
& \left\{ r^{-\frac{5}{2}} (|\partial_r \Psi_k|^2 + r^{-2} |\partial_\sigma \Psi_k|^2) + cr^{-2} |T\Psi_k|^2 + (c\omega_k^2 - CR^{-2}) r^{-2} |\Psi_k|^2 \right\} dg \\
& \leq C_{\delta_1} \int_{\mathcal{R}(\tau_1, \tau_2) \cap \mathcal{E}_{\delta_1}} f \left\{ s^2 R^{-6\epsilon_0} |\nabla \Psi_k|_{g_{ref}}^2 + s^4 R^{-12\epsilon_0} |\Psi_k|^2 \right\} dg \\
& + C \left| \int_{\mathcal{R}(\tau_1, \tau_2)} F_k(\nabla^\mu f \nabla_\mu \bar{\Psi}_k + O(\sum_{j=1}^2 (1+r)^{j-2} |\nabla^j f|_{g_{ref}}) \bar{\Psi}_k) dg \right| \\
& + C \sum_{j=1}^2 \int_{\Sigma^{\tau_j}} (|\nabla f|_{g_{ref}} |\nabla \Psi_k|_{g_{ref}}^2 + (\sum_{j=1}^3 (1+r)^{j-3} |\nabla^j f|_{g_{ref}}) |\Psi_k|^2) dg_\Sigma \\
& + C \int_{\mathcal{R}(\tau_1, \tau_2) \cap \mathcal{H}^+} (|\nabla f|_{g_{ref}} J_\mu^N(\Psi_k) n_{\mathcal{H}^+}^\mu + (\sum_{j=1}^3 (1+r)^{j-3} |\nabla^j f|_{g_{ref}}) |\Psi_k|^2) dvol_{\mathcal{H}^+}. \\
& + C \omega_k^2 (1 + \omega_k^{-2}) (\log(2 + \tau_2))^4 R_0^2 \sup_{\{r \leq R_0\}} f \cdot \mathcal{E}_{\log}[\Psi] \\
& + C \omega_k^2 (1 + \omega_k^{-6}) \sup_{\{r \leq R\}} f \cdot \mathcal{E}_{\log}[\Psi]. \tag{6.149}
\end{aligned}$$

In view of the bound (6.147) for the parameters R, s , as well as the properties of the function (6.2), inequality (6.149) yields [using also (5.26) and Lemma 5.2, combined with a Cauchy–Schwarz inequality, to estimate the second and third terms in the right hand side of (6.149)]:

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2) \setminus \mathcal{E}_{2\delta_1}} ((1+r)^{-\frac{5}{2}} |\nabla \Psi_k|_{g_{ref}}^2 + (\omega_k^2 r^{-2} + r^{-4}) |\Psi_k|^2) dg \\
& \leq \sum_{j=1}^4 (|\nabla^j w_R|_{g_{ref}} + |\nabla^j \tilde{w}_R|_{g_{ref}})
\end{aligned}$$

$$\left\{ C_{\delta_1} \frac{\sup_{\mathcal{E}_{\delta_1}} (e^{2s w_R} + e^{2s \tilde{w}_R})}{\inf_{\{r \geq \frac{1}{4} r_0\} \setminus \mathcal{E}_{2\delta_1}} (e^{2s w_R} + e^{2s \tilde{w}_R})} (s R^{-3\epsilon_0}) \int_{\mathcal{R}(\tau_1, \tau_2) \setminus \mathcal{E}_{\delta_1}} (|\nabla \Psi_k|_{g_{ref}}^2 + |\Psi_k|^2) dg + C_{\delta_1} (1 + \omega_k^{-10}) (\log(2 + \tau_2))^4 \frac{\sup_{\{r \leq R\}} (e^{2s w_R} + e^{2s \tilde{w}_R})}{\inf_{\{r \leq R\}} (e^{2s w_R} + e^{2s \tilde{w}_R})} \mathcal{E}_{\log[\Psi]} \right\}. \tag{6.150}$$

In view of the properties of the function 6.2, we can estimate

$$\sup_{\{r \leq R\}} w_R - \inf_{\{r \leq R\}} w_R + \sup_{\{r \leq R\}} \tilde{w}_R - \inf_{\{r \leq R\}} \tilde{w}_R \leq C \epsilon_0^{-1} R^{3\epsilon_0}, \tag{6.151}$$

$$\inf_{\{\frac{1}{4} r_0 \leq R\} \setminus \mathcal{E}_{2\delta_1}} w_R \geq \max_{\mathcal{E}_{\delta_1}} w_R + c_{\delta_1} R^{-3\epsilon_0}, \tag{6.152}$$

$$\inf_{\{\frac{1}{4} r_0 \leq r \leq R\} \setminus \mathcal{E}_{2\delta_1}} \tilde{w}_R \geq \max_{\mathcal{E}_{\delta_1}} \tilde{w}_R + c_{\delta_1} R^{-3\epsilon_0} \tag{6.153}$$

and

$$\sum_{j=1}^4 (|\nabla^j w_R|_{g_{ref}} + |\nabla^j \tilde{w}_R|_{g_{ref}}) \leq C. \tag{6.154}$$

Therefore, inequality (6.6) readily follows from 6.150, provided $C_{\delta_1 \epsilon_0}$ in (6.147) is sufficiently large in terms of δ_1, ϵ_0 . \square

6.9. *Proof of Proposition 6.1 in the case of Dirchlet or Neumann boundary conditions.* In this section, we will briefly sketch how the proof of Proposition 6.1 can be applied to the case when the boundary $\partial \mathcal{M}$ of (\mathcal{M}, g) is allowed to have a non-trivial time-like component $\partial_{tim} \mathcal{M}$ and equation (6.1) is supplemented with Dirichlet or Neumann boundary conditions for φ on $\partial_{tim} \mathcal{M}$.

We will first describe the class of Lorentzian manifolds with such a boundary component on which Proposition 6.1 will apply. Let $(\mathcal{M}^{d+1}, g), d \geq 2$, be a smooth Lorentzian manifold with piecewise smooth boundary $\partial \mathcal{M}$ splitting as

$$\partial \mathcal{M} = \partial_{hor} \mathcal{M} \cup \partial_{tim} \mathcal{M}, \tag{6.155}$$

where $\partial_{hor} \mathcal{M}$ has the structure of a piecewise smooth null hypersurface and $\partial_{tim} \mathcal{M}$ is a smooth timelike hypersurface, with $\partial_{hor} \mathcal{M} \cap \partial_{tim} \mathcal{M} = \emptyset$. For the discussion of this section, we will assume that $\partial_{tim} \mathcal{M} \neq \emptyset$, but $\partial_{hor} \mathcal{M}$ will be allowed to be empty. Let $(\tilde{\mathcal{M}}, \tilde{g})$ be the double of (\mathcal{M}, g) across $\partial_{tim} \mathcal{M}$, which is defined as the disjoint union of two copies of (\mathcal{M}, g) glued along $\partial_{tim} \mathcal{M}$ (for the relevant definitions, see e.g. [28]). Let $i_1, i_2 : \mathcal{M} \rightarrow \tilde{\mathcal{M}}$ be the two natural isometric embeddings of (\mathcal{M}, g) into $(\tilde{\mathcal{M}}, \tilde{g})$. Note that $\tilde{\mathcal{M}} = i_1(\mathcal{M}) \cup i_2(\mathcal{M})$ and $i_1(\partial_{tim} \mathcal{M}) = i_2(\partial_{tim} \mathcal{M})$. Furthermore, $\tilde{\mathcal{M}}$ is a smooth manifold, and the metric \tilde{g} is continuous and piecewise smooth on $\tilde{\mathcal{M}}$ and smooth on $\tilde{\mathcal{M}} \setminus i_1(\partial_{tim} \mathcal{M})$.¹⁴ We will always identify \mathcal{M} with $i_1(\mathcal{M}) \subset \tilde{\mathcal{M}}$.

We will assume that $(\tilde{\mathcal{M}}, \tilde{g})$ is a globally hyperbolic Lorentzian manifold (with the regularity of \tilde{g} as described before), satisfying Assumptions G1, G2 and G3 of Sect. 2 (for the discussion of this Section, we can also allow the case $\mathcal{E} = \emptyset$). Additionally, we will assume that the stationary Killing field T of $\tilde{\mathcal{M}}$ (defined by Assumption G1)

¹⁴ The metric \tilde{g} is continuous across $i_1(\partial_{tim} \mathcal{M})$, but fails to be C^1 at all the points of $i_1(\partial_{tim} \mathcal{M})$ on which the second fundamental form of $i_1(\partial_{tim} \mathcal{M})$ is non-zero.

is tangent to $i_1(\partial_{tim}\mathcal{M})$. Let also $\Sigma_{\widetilde{\mathcal{M}}}, \mathcal{S}_{\widetilde{\mathcal{M}}}, \mathcal{H}_{\widetilde{\mathcal{M}}}^\pm \subset \widetilde{\mathcal{M}}, t_{\widetilde{\mathcal{M}}} : \widetilde{\mathcal{M}} \setminus \mathcal{H}_{\widetilde{\mathcal{M}}}^- \rightarrow \mathbb{R}$ and $r_{\widetilde{\mathcal{M}}} : \widetilde{\mathcal{M}} \setminus \mathcal{H}_{\widetilde{\mathcal{M}}}^- \rightarrow [0, +\infty)$ be as defined under Assumption G1. We will assume without loss of generality that $\Sigma_{\widetilde{\mathcal{M}}}, \mathcal{S}_{\widetilde{\mathcal{M}}}$ intersect $i_1(\partial_{tim}\mathcal{M})$ transversally, and that $\Sigma_{\widetilde{\mathcal{M}}} \cap i_1(\partial_{tim}\mathcal{M}), \mathcal{S}_{\widetilde{\mathcal{M}}} \cap i_1(\partial_{tim}\mathcal{M})$ are compact. Note that the restriction of $\mathcal{H}_{\widetilde{\mathcal{M}}}$ on \mathcal{M} coincides with $\partial_{hor}\mathcal{M}$.

Remark. We will use the notation $\Sigma, \mathcal{S}, \mathcal{H}^\pm, t$ and r for the restriction of the hypersurfaces $\Sigma_{\widetilde{\mathcal{M}}}, \mathcal{S}_{\widetilde{\mathcal{M}}}, \mathcal{H}_{\widetilde{\mathcal{M}}}^\pm$ and the functions $t_{\widetilde{\mathcal{M}}}, r_{\widetilde{\mathcal{M}}}$ on $\mathcal{M} \simeq i_1(\mathcal{M})$.

For any $F \in C^\infty(\mathcal{M})$ and any $(\varphi_0, \varphi_1) \in C^\infty(\Sigma) \times C^\infty(\Sigma)$, the initial-boundary value problem

$$\begin{cases} \square_g \varphi = G & \text{on } \{t \geq 0\} \\ (\varphi, T\varphi) = (\varphi_0, \varphi_1) & \text{on } \{t = 0\} \\ \varphi = 0 & \text{on } \partial_{tim}\mathcal{M} \end{cases} \tag{6.156}$$

is well posed on $\{t \geq 0\} \subset \mathcal{M}$. This follows from the assumption that $(\widetilde{\mathcal{M}}, \widetilde{g})$ is globally hyperbolic. The Dirichlet boundary condition $\varphi|_{\partial_{tim}\mathcal{M}} = 0$ in (6.156) can also be replaced by the Neumann boundary condition

$$n_{\partial_{tim}\mathcal{M}}(\varphi)|_{\partial_{tim}\mathcal{M}} = 0, \tag{6.157}$$

where $n_{\partial_{tim}\mathcal{M}}$ is the unit normal vector field on $\partial_{tim}\mathcal{M}$, pointing towards the interior of \mathcal{M} .

On a spacetime (\mathcal{M}, g) as above, we will extend Proposition 6.1 as follows:

Proposition 6.2. *Let (\mathcal{M}, g) be a Lorentzian manifold with boundary as above. For any $s, R \gg 1$ sufficiently large in terms of the geometry of (\mathcal{M}, g) and any $0 < \epsilon_0 < 1$, there exists a smooth T -invariant function $f : \mathcal{M} \setminus \mathcal{H}^- \rightarrow (0, +\infty)$ as in Proposition 6.1, so that (provided $\epsilon_0 s R^{-9\epsilon_0} \gg 1$), for any $0 < \delta \ll 1$, any $0 \leq \tau_1 \leq \tau_2$ and any smooth function $\varphi : \mathcal{M} \setminus \mathcal{H}^- \rightarrow \mathbb{C}$ with compact support on the hypersurfaces $\{t = \text{const}\}$ solving (6.1) and satisfying on $\partial_{tim}\mathcal{M}$ either the Dirichlet condition $\varphi = 0$ or the Neumann condition $n_{\partial_{tim}\mathcal{M}}(\varphi) = 0$, the estimate (6.3) holds.*

Proof. The proof of Proposition 6.2 follows in almost exactly the same way as the proof of Proposition 6.1, the only difference being the following: When using the multiplier (6.86) for equation (6.1) as in Sect. 6.4 and after performing the same integration-by-parts procedure, one obtains instead of (6.98) the following relation:

$$\begin{aligned} & \int_{\mathcal{R}(\tau_1, \tau_2)} \text{Re} \left\{ 2\chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) \right. \\ & \quad - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 \\ & \quad + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 \\ & \quad \left. - 2h \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \mathcal{A}_{f,h}^{(R)} |\varphi|^2 \right\} dg \\ & = - \int_{\mathcal{R}(\tau_1, \tau_2)} \text{Re} \left\{ G(2\nabla^\mu f \nabla_\mu \bar{\varphi} + (\square_g f - 2h)\bar{\varphi}) \right\} dg \\ & \quad - \mathcal{B}_{f,h}^{(R)}[\varphi; \tau_1, \tau_2] - \mathcal{B}_{f,h}^{(b)}[\varphi; \tau_1, \tau_2], \end{aligned} \tag{6.158}$$

where

$$\begin{aligned} \mathcal{B}_{f,h}^{(b)}[\varphi; \tau_1 \tau_2] &= \int_{\partial_{tim}\mathcal{M} \cap \mathcal{R}(\tau_1, \tau_2)} \\ &Re \left\{ \left(2\nabla^\mu f \nabla_\mu \bar{\varphi} \nabla_\nu \varphi + (\square_g f - 2h) \varphi \nabla_\nu \bar{\varphi} - \nabla_\nu f \nabla^\mu \varphi \nabla_\mu \bar{\varphi} \right. \right. \\ &\quad \left. \left. + (f^{-1} \nabla_\mu \nabla_\nu f \nabla^\mu f + \nabla_\nu h - \frac{1}{2} (\nabla_\nu (\square_g f))) |\varphi|^2 \right) n_{\partial_{tim}\mathcal{M}}^\nu \right\} dg_{\partial_{tim}\mathcal{M}}, \end{aligned} \quad (6.159)$$

$g_{\partial_{tim}\mathcal{M}}$ being the induced (Lorentzian) metric on $\partial_{tim}\mathcal{M}$. Notice that (6.158) differs from (6.98) only by the term (6.159) in the right hand side.

Let us assume, without loss of generality, that the function \bar{w} of Lemma 6.1 has been chosen so that it additionally satisfies $n_{\partial_{tim}\mathcal{M}}(\bar{w}) > 0$, with \bar{w} being constant on $\partial_{tim}\mathcal{M}$ (it can be readily checked that Lemma 6.1 can be established under this additional assumption). In the case when φ satisfies the Dirichlet boundary condition $\varphi|_{\partial_{tim}\mathcal{M}} = 0$, it is straightforward to check that this choice of \bar{w} implies (in view of the choice of the functions f, h in Sect. 6.3) that the term (6.159) is non-negative, and in particular

$$\begin{aligned} &\int_{\partial_{tim}\mathcal{M} \cap \mathcal{R}(\tau_1, \tau_2)} \\ &Re \left\{ \left(2\nabla^\mu f \nabla_\mu \bar{\varphi} \nabla_\nu \varphi + (\square_g f - 2h) \varphi \nabla_\nu \bar{\varphi} - \nabla_\nu f \nabla^\mu \varphi \nabla_\mu \bar{\varphi} \right. \right. \\ &\quad \left. \left. + (f^{-1} \nabla_\mu \nabla_\nu f \nabla^\mu f + \nabla_\nu h - \frac{1}{2} (\nabla_\nu (\square_g f))) |\varphi|^2 \right) n_{\partial_{tim}\mathcal{M}}^\nu \right\} dg_{\partial_{tim}\mathcal{M}} \\ &\geq c \int_{\partial_{tim}\mathcal{M} \cap \mathcal{R}(\tau_1, \tau_2)} n_{\partial_{tim}\mathcal{M}}(f) |n_{\partial_{tim}\mathcal{M}}(\varphi)|^2 dg_{\partial_{tim}\mathcal{M}} \geq 0 \end{aligned} \quad (6.160)$$

for some $c > 0$. Thus, the term (6.159) can be dropped from the right hand side of (6.158) (thus yielding (6.98)) and one can proceed as before to establish (6.3).

In the case when φ satisfies the Neumann boundary condition $n_{\partial_{tim}\mathcal{M}}(\varphi)|_{\partial_{tim}\mathcal{M}} = 0$, (6.159) is not necessarily non-negative, since the term

$$n_{\partial_{tim}\mathcal{M}}(f) \nabla^\mu \varphi \nabla_\mu \bar{\varphi}$$

in (6.159) does not necessarily have a sign (as is the case when $\varphi|_{\partial_{tim}\mathcal{M}} = 0$). In order to absorb this term, we proceed as follows: Let $\mathcal{U} \subset \mathcal{M}$ be a (small) T -invariant tubular neighborhood of $\partial_{tim}\mathcal{M}$ (so that $\mathcal{U} \cap (\mathcal{H} \cup \mathcal{B}_{crit}(8\delta_0)) = \emptyset$), split as $\mathcal{U} \simeq [0, 1] \times \partial_{tim}\mathcal{M}$, where the projection onto the factor $[0, 1]$ is given by a smooth function $\bar{r} : \mathcal{U} \rightarrow [0, 1]$ such that $\nabla^\mu \bar{r}|_{\partial_{tim}\mathcal{M}} = n_{\partial_{tim}\mathcal{M}}^\mu$, and the projection onto $\partial_{tim}\mathcal{M}$ is given by a smooth map $\bar{\sigma} : \mathcal{U} \rightarrow \partial_{tim}\mathcal{M}$. We will extend $n_{\partial_{tim}\mathcal{M}}$ on the whole of \mathcal{U} by the relation

$$n_{\partial_{tim}\mathcal{M}}^\mu = \nabla^\mu \bar{r}. \quad (6.161)$$

Let $\chi_c : [0, 1] \rightarrow [0, 1]$ be a smooth function satisfying $\chi_c \equiv 1$ on $[0, \frac{1}{4}]$ and $\chi_c \equiv 0$ on $[\frac{1}{2}, 1]$, and let us define the function $\check{f} : \mathcal{M} \rightarrow \mathbb{R}$ by the relation

$$\check{f}(\bar{r}, \bar{\sigma}) \doteq \chi_c(\bar{r}) \cdot (n_{\partial_{tim}\mathcal{M}}(f))|_{\partial_{tim}\mathcal{M}}(\bar{\sigma}) \text{ on } \mathcal{U} \simeq [0, 1] \times \partial_{tim}\mathcal{M}$$

(where $(n_{\partial_{tim}\mathcal{M}}(f))|_{\partial_{tim}\mathcal{M}}$ is the value of $n_{\partial_{tim}\mathcal{M}}(f)$ on $\{\bar{r} = 0\}$) and

$$\check{f} \equiv 0 \text{ on } \mathcal{M} \setminus \mathcal{U}.$$

Adding to (6.158) the identity

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ 2\nabla^\mu (\check{f} n_{\partial_{\text{im}}\mathcal{M}}^\nu) \nabla_\mu \varphi \nabla_\nu \bar{\varphi} - \nabla_\nu (\check{f} n_{\partial_{\text{im}}\mathcal{M}}^\nu) \nabla^\mu \varphi \nabla_\mu \bar{\varphi} \right\} dg \\
&= - \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ G \cdot 2\check{f} n_{\partial_{\text{im}}\mathcal{M}}(\bar{\varphi}) \right\} dg \\
&\quad - \int_{\partial_{\text{im}}\mathcal{M} \cap \mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ (2\check{f} n_{\partial_{\text{im}}\mathcal{M}}(\bar{\varphi}) \nabla_\nu \varphi - \check{f} (n_{\partial_{\text{im}}\mathcal{M}})_\nu \nabla^\mu \varphi \nabla_\mu \bar{\varphi}) n_{\partial_{\text{im}}\mathcal{M}}^\nu \right\} dg_{\partial_{\text{im}}\mathcal{M}}
\end{aligned} \tag{6.162}$$

and using the Neumann condition $n_{\partial_{\text{im}}\mathcal{M}}(\varphi)|_{\partial_{\text{im}}\mathcal{M}} = 0$, we thus infer:

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ 2\chi_{\leq R} f^{-1} \nabla^\mu \nabla^\nu f \nabla_\mu (f^{\frac{1}{2}} \varphi) \nabla_\nu (f^{\frac{1}{2}} \bar{\varphi}) + 2\nabla^\mu (\check{f} n_{\partial_{\text{im}}\mathcal{M}}^\nu) \nabla_\mu \varphi \nabla_\nu \bar{\varphi} \right. \\
&\quad - 2\chi_{\leq R} \chi_{\geq R_0} r^{-1} f^{-1} (\partial_r f) |\partial_r (f^{\frac{1}{2}} \varphi)|^2 \\
&\quad + 2(1 - \chi_{\leq R}) \nabla^\mu \nabla^\nu f \nabla_\mu \varphi \nabla_\nu \bar{\varphi} + 2\chi_{\geq R_0} \chi_{\leq R} r^{-1} (\partial_r f) |\partial_r \varphi|^2 \\
&\quad \left. - (2h + \nabla_\nu (\check{f} n_{\partial_{\text{im}}\mathcal{M}}^\nu)) \nabla^\mu \varphi \nabla_\mu \bar{\varphi} + \mathcal{A}_{f,h}^{(R)} |\varphi|^2 \right\} dg \\
&= - \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ G (2\nabla^\mu f \nabla_\mu \bar{\varphi} + (\square_g f - 2h) \bar{\varphi}) \right\} dg \\
&\quad - \mathcal{B}_{f,h}^{(R)}[\varphi; \tau_1, \tau_2] - \check{\mathcal{B}}_{f,h}^{(b)}[\varphi; \tau_1, \tau_2],
\end{aligned} \tag{6.163}$$

where

$$\begin{aligned}
& \check{\mathcal{B}}_{f,h}^{(b)}[\varphi; \tau_1, \tau_2] \\
&= \int_{\partial_{\text{im}}\mathcal{M} \cap \mathcal{R}(\tau_1, \tau_2)} (f^{-1} \nabla_\mu \nabla_\nu f \nabla^\mu f + \nabla_\nu h - \frac{1}{2} (\nabla_\nu (\square_g f))) |\varphi|^2 n_{\partial_{\text{im}}\mathcal{M}}^\nu dg_{\partial_{\text{im}}\mathcal{M}}.
\end{aligned} \tag{6.164}$$

Notice that, if $sR^{-3\epsilon_0} \gg 1$, the term (6.164) is non-negative (in view of the properties of the functions f, h , see Sect. 6.3), and thus it can be dropped from the right hand side of (6.163). Furthermore, if $l \gg 1$ in Lemma 6.1 and $sR^{-3\epsilon_0} \gg 1$, the terms $2\nabla^\mu (\check{f} n_{\partial_{\text{im}}\mathcal{M}}^\nu) \nabla_\mu \varphi \nabla_\nu \bar{\varphi}$ and $\nabla_\nu (\check{f} n_{\partial_{\text{im}}\mathcal{M}}^\nu) \nabla^\mu \varphi \nabla_\mu \bar{\varphi}$ in the left hand side of (6.163) (restricted to the complement of \mathcal{E}_{ext}) can be absorbed into the right hand side of 6.120. Thus, following exactly the same steps as we did in order to obtain (6.3) from (6.98) in the case $\partial_{\text{im}}\mathcal{M} = \emptyset$, we can also obtain (6.3) from (6.163) in the case when $\partial_{\text{im}}\mathcal{M} \neq \emptyset$ and $n_{\partial_{\text{im}}\mathcal{M}}(\varphi)|_{\partial_{\text{im}}\mathcal{M}} = 0$. \square

7. Proof of Proposition 4.1

Let us introduce the parameters $0 < \omega_0 \ll 1$, $\omega_+ \gg 1$ and $\tau_1 \geq \bar{\tau}_0 + \varepsilon^{-2} \tau_*$ depending on $\varepsilon, \delta_1, R, \tau_*, \bar{\tau}_0, \mathcal{E}_{\log}[\varphi], \mathcal{E}_{\log}[\psi], \mathcal{E}_{\log}[T\psi]$ and $\mathcal{E}[T^2\psi]$ in the statement of Proposition 4.1 (we will fix ω_0, ω_+ and τ_1 later), and, for $n = \lceil \log_2(\frac{\omega_+}{\omega_0}) \rceil$, let us decompose

ψ and $T\psi$ into their frequency localised components $\{\psi_k\}_{k=0}^n$, $\psi_{\geq\omega_+}$ and $\{(T\psi)_k\}_{k=0}^n$, $(T\psi)_{\geq\omega_+}$, respectively, as in Sect. 5.2 (notice that (5.1) is satisfied in view of (4.18)).

In view of Lemma 5.5, (4.1) (and (4.18)), as well as a Hardy-type inequality (of the form (5.15)), we obtain for any $\tau \geq 2\tau_*$ and any $0 < a \ll 1$:

$$\begin{aligned} & \int_{\mathcal{R}(\tau-\tau_*, \tau+\tau_*) \cap \{r \leq R\}} (J_\mu^N(\psi_0)n^\mu + |\psi_0|^2) \\ & \leq C\omega_0^2 \int_{\mathcal{R}(\tau-\tau_*, \tau+\tau_*) \cap \{r \leq R\}} (J_\mu^N(\varphi)n^\mu + |\varphi|^2) \\ & \quad + C_a \left(\omega_0^2 (\log(2+\tau))^4 + (1+\omega_0\tau)^{-1} \right) (1+\omega_0^{-1-a}) R^2 \mathcal{E}_{\log}[\varphi] \\ & \leq C_a \left\{ \omega_0^2 R^2 \tau_* (\log(2+\tau))^4 + \omega_0^2 (\log(2+\tau))^4 (1+\omega_0^{-1-a}) R^2 \right. \\ & \quad \left. + (1+\omega_0\tau)^{-1} (1+\omega_0^{-1-a}) R^2 \right\} \mathcal{E}_{\log}[\varphi]. \end{aligned} \tag{7.1}$$

From Lemmas 5.4 and 5.6, we obtain for any $\tau > 0$ and any $\delta > 0$:

$$\begin{aligned} & \sum_{j=0}^{\lfloor \frac{\delta-1}{2} \rfloor} \int_{\mathcal{R}(\tau+2j\tau_*, \tau+2(j+1)\tau_*) \cap \{r \leq R\}} (J_\mu^N(\psi_{\geq\omega_+})n^\mu + |\psi_{\geq\omega_+}|^2) \\ & \leq \int_{\mathcal{R}(\tau, \tau+\delta^{-1}\tau_*) \cap \{r \leq R\}} (J_\mu^N(\psi_{\geq\omega_+})n^\mu + |\psi_{\geq\omega_+}|^2) \\ & \leq C\delta^{-1} \tau_* \omega_+^{-2} (\mathcal{E}_{\log}[\psi] + \mathcal{E}[T\psi]) + C\mathcal{E}_{\log}[\psi]. \end{aligned} \tag{7.2}$$

We will assume that ω_+ is sufficiently large in terms of ε , τ_* , $\mathcal{E}_{\log}[\psi]$, $\mathcal{E}[T\psi]$ so that

$$\tau_* \omega_+^{-2} (\mathcal{E}_{\log}[\psi] + \mathcal{E}[T\psi]) \ll \varepsilon. \tag{7.3}$$

Let us also use the ansatz

$$\omega_0 = \frac{\bar{\omega}_0}{(\log(2+\tau_1))^\delta}, \tag{7.4}$$

and let us assume that $\bar{\omega}_0$ is sufficiently small in terms of ε , R , τ_* , $\mathcal{E}_{\log}[\varphi]$, and τ_1 is sufficiently large in terms of ε , R , $\mathcal{E}_{\log}[\varphi]$, $\bar{\omega}_0$, so that for any $\tau_1 \leq \tau \leq 100\tau_1$ (having fixed an $a \in (0, 1)$):

$$\begin{aligned} & (\omega_0^2 R^2 \tau_* (\log(2+\tau))^4 + \omega_0^2 (\log(2+\tau))^4 (1+\omega_0^{-1-a}) \\ & \quad R^2 + (1+\omega_0\tau)^{-1} (1+\omega_0^{-1-a}) R^2) \mathcal{E}_{\log}[\varphi] \ll \varepsilon \end{aligned} \tag{7.5}$$

(later, we will also need to assume that τ_1 is also sufficiently large in terms of ω_+). Then, (7.1), (7.2), (7.5) and (7.3) imply that for any $\tau \geq 0$:

$$\begin{aligned} & \sum_{l=0}^{\lfloor \frac{\delta-1}{2} \rfloor} \int_{\mathcal{R}(\tau+2l\tau_*, \tau+2(l+1)\tau_*) \cap \{r \leq R\}} (J_\mu^N(\psi_0)n^\mu + J_\mu^N(\psi_{\geq\omega_+})n^\mu + |\psi_0|^2 + |\psi_{\geq\omega_+}|^2) \\ & \leq \frac{1}{20} \varepsilon \delta^{-1} + C\mathcal{E}_{\log}[\psi]. \end{aligned} \tag{7.6}$$

Repeating the same procedure for $T\psi$ in place of ψ and adding the result to (7.6), we obtain for any $\delta > 0$ (provided $\bar{\omega}_0$ is fixed sufficiently small in terms of $\varepsilon, R, \tau_*, \mathcal{E}_{\log}[\varphi], \mathcal{E}_{\log}[\psi]$, τ_1 is fixed sufficiently large in terms of $\varepsilon, R, \mathcal{E}_{\log}[\varphi], \mathcal{E}_{\log}[\psi]\bar{\omega}_0$ and ω_+ is fixed sufficiently large in terms of $\varepsilon, \tau_*, \mathcal{E}_{\log}[\psi], \mathcal{E}_{\log}[T\psi], \mathcal{E}[T^2\psi]$):

$$\begin{aligned} & \sum_{l=0}^{\lfloor \frac{\delta-1}{2} \rfloor} \sum_{j=0}^1 \int_{\mathcal{R}(\tau+2l\tau_*, \tau+2(l+1)\tau_*) \cap \{r \leq R\}} \\ & \quad \left(J_\mu^N((T^j\psi)_0)n^\mu + J_\mu^N((T^j\psi)_{\geq \omega_+})n^\mu + |(T^j\psi)_0|^2 + |(T^j\psi)_{\geq \omega_+}|^2 \right) \\ & \leq \frac{1}{10} \varepsilon \delta^{-1} + C \sum_{j=0}^1 \mathcal{E}_{\log}[T^j\psi]. \end{aligned} \tag{7.7}$$

In view of Corollary 6.1, for any $1 \leq k \leq n$, any $0 < \delta_1, \delta_2 < 1$, any $0 < \varepsilon_0 < 1$ and any $\bar{\tau} \geq \tau_1$, we can bound:

$$\begin{aligned} & \int_{(\mathcal{R}(\tau_1, \bar{\tau}) \setminus \mathcal{E}_{\delta_1}) \cap \{r \leq R\}} \left(J_\mu^N(\psi_k)n^\mu + |\psi_k|^2 \right) \\ & \leq C_R \delta_2 \int_{\mathcal{R}(\tau_1, \bar{\tau}) \cap \mathcal{E}_{\delta_1/2}} \left(J_\mu^N(\psi_k)n^\mu + |\psi_k|^2 \right) \\ & \quad + C_{\varepsilon_0 \delta_1 R} \omega_0^{-10} (\log(2 + \bar{\tau}))^4 e^{C_{\varepsilon_0 \delta_1 \omega_+} \max\{\omega_0^{-\varepsilon_0}, -\log \delta_2\}} \mathcal{E}_{\log}[\psi]. \end{aligned} \tag{7.8}$$

Let us set

$$\delta_2 = \omega_0^3 \omega_+^{-1} \bar{\delta}_2, \tag{7.9}$$

where $\bar{\delta}_2$ is sufficiently small in terms of $\varepsilon, \varepsilon_0, R, \tau_*, \mathcal{E}_{\log}[\psi]$. Assuming also that $\bar{\omega}_0$ in (7.4) has been fixed sufficiently small in terms of $\varepsilon, \varepsilon_0, R, \tau_*, \mathcal{E}_{\log}[\psi]$, from (7.8), (5.26), (4.18) and the Poincare inequality

$$\int_{\mathcal{R}(\tau_1, \bar{\tau}) \cap \mathcal{E}_{\delta_1}} |\psi_k|^2 \leq C \int_{\mathcal{R}(\tau_1, \bar{\tau}) \cap \{r \leq R\}} J_\mu^N(\psi_k)n^\mu + C \int_{(\mathcal{R}(\tau_1, \bar{\tau}) \setminus \mathcal{E}_{\delta_1}) \cap \{r \leq R\}} |\psi_k|^2, \tag{7.10}$$

we obtain after summing over all $k \in \{1, \dots, n\}$ provided $\bar{\delta}_2$ is sufficiently small in terms of $\varepsilon, \varepsilon_0, R, \tau_*, \mathcal{E}_{\log}[\psi]$ (recall that $n \sim \log(\omega_0^{-1} \omega_+)$):

$$\begin{aligned} & \sum_{k=1}^n \int_{(\mathcal{R}(\tau_1, \bar{\tau}) \setminus \mathcal{E}_{\delta_1}) \cap \{r \leq R\}} \left(J_\mu^N(\psi_k)n^\mu + |\psi_k|^2 \right) \\ & \leq C_{\varepsilon_0 R} (\bar{\tau} - \tau_1) \delta_2 \omega_0^{-2} \log(\omega_0^{-1} \omega_+) \mathcal{E}_{\log}[\psi] \\ & \quad + C_{\varepsilon_0 \delta_1 R} \omega_0^{-10} (\log(2 + \bar{\tau}))^4 e^{C_{\varepsilon_0 \delta_1 \omega_+} \max\{\omega_0^{-\varepsilon_0}, -\log \delta_2\}} \mathcal{E}_{\log}[\psi] \\ & \leq \frac{1}{40\tau_*} \varepsilon \bar{\tau} + C_1 (\log(2 + \bar{\tau}))^{14} e^{C_1 (\log(2 + \tau_1))^{8\varepsilon_0}}, \end{aligned} \tag{7.11}$$

where C_1 depends on $\varepsilon, \varepsilon_0, \delta_1, R, \tau_*, \mathcal{E}_{\log}[\psi], \omega_+$.

Repeating the same procedure for $T\psi$ in place of ψ , we obtain the following analogue of (7.11):

$$\begin{aligned} & \sum_{k=1}^n \int_{(\mathcal{R}(\tau_1, \bar{\tau}) \setminus \mathcal{E}_{\delta_1}) \cap \{r \leq R\}} \left(J_{\mu}^N ((T\psi)_k) n^{\mu} + |(T\psi)_k|^2 \right) \\ & \leq \frac{1}{40\tau_*} \varepsilon \bar{\tau} + C_2 (\log(2 + \bar{\tau}))^{14} e^{C_2 (\log(2 + \tau_1))^{8\varepsilon_0}}, \end{aligned} \tag{7.12}$$

where C_2 depends on $\varepsilon, \varepsilon_0, \delta_1, R, \tau_*, \mathcal{E}_{\log}[T\psi], \omega_+$.

From (7.11) and (7.12) we obtain for any $\delta > 0$ (setting $\bar{\tau} = \tau_1 + \delta^{-1}\tau_*$)

$$\begin{aligned} & \sum_{l=0}^{\lfloor \frac{\delta^{-1}-1}{2} \rfloor} \left\{ \sum_{k=1}^n \sum_{j=0}^1 \int_{\mathcal{R}(\tau_1+2l\tau_*, \tau_1+2(l+1)\tau_*) \setminus \mathcal{E}_{\delta_1}) \cap \{r \leq R\}} \left(J_{\mu}^N ((T^j\psi)_k) n^{\mu} + |(T^j\psi)_k|^2 \right) \right\} \\ & \leq \frac{1}{20\tau_*} \varepsilon (\tau_1 + \delta^{-1}\tau_*) + C_3 (\log(\tau_1 + \delta^{-1}\tau_*))^{14} e^{C_3 (\log(2 + \tau_1))^{8\varepsilon_0}}, \end{aligned} \tag{7.13}$$

where $C_3 = C_1 + C_2$. Adding (7.7) (for $\tau = \tau_1$) and (7.13), we therefore obtain for any $\delta > 0$:

$$\begin{aligned} & \sum_{l=0}^{\lfloor \frac{\delta^{-1}-1}{2} \rfloor} \left\{ \sum_{j=0}^1 \int_{\mathcal{R}(\tau_1+2l\tau_*, \tau_1+2(l+1)\tau_*) \setminus \mathcal{E}_{\delta_1}) \cap \{r \leq R\}} \left(J_{\mu}^N (T^j\psi) n^{\mu} + |T^j\psi|^2 \right) \right\} \\ & \leq \frac{1}{10} \varepsilon \delta^{-1} + C \sum_{j=0}^1 \mathcal{E}_{\log}[T^j\psi] + \frac{1}{20\tau_*} \varepsilon (\tau_1 + \delta^{-1}\tau_*) \\ & \quad + C_3 (\log(\tau_1 + \delta^{-1}\tau_*))^{14} e^{C_3 (\log(2 + \tau_1))^{8\varepsilon_0}}. \end{aligned} \tag{7.14}$$

Applying the pigeonhole principle on (7.14) (assuming that $\delta \ll 1$), we infer that there exists some $l_0 \in \{0, \dots, \lfloor \frac{\delta^{-1}-1}{2} \rfloor\}$ such that

$$\begin{aligned} & \sum_{j=0}^1 \int_{\mathcal{R}(\tau_1+2l_0\tau_*, \tau_1+2(l_0+1)\tau_*) \setminus \mathcal{E}_{\delta_1}) \cap \{r \leq R\}} \left(J_{\mu}^N (T^j\psi) n^{\mu} + |T^j\psi|^2 \right) \\ & \leq \lfloor \frac{\delta^{-1}-1}{2} \rfloor^{-1} \left(\frac{1}{5} \varepsilon \delta^{-1} + \frac{1}{20} \varepsilon \tau_*^{-1} \tau_1 + C \sum_{j=0}^1 \mathcal{E}_{\log}[T^j\psi] \right) \\ & \quad + C_3 (\log(\tau_1 + \delta^{-1}\tau_*))^{14} e^{C_3 (\log(2 + \tau_1))^{8\varepsilon_0}} \\ & \leq \frac{\varepsilon}{2} (1 + \tau_*^{-1} \delta \tau_1) + C \delta \sum_{j=0}^1 \mathcal{E}_{\log}[T^j\psi] + C_3 \delta (\log(\tau_1 + \delta^{-1}\tau_*))^{14} e^{C_3 (\log(2 + \tau_1))^{8\varepsilon_0}}. \end{aligned} \tag{7.15}$$

Thus, provided

$$\delta = \frac{\bar{\delta}}{\tau_1}, \tag{7.16}$$

where $\bar{\delta}$ is small in terms of ε , τ_* , $\mathcal{E}_{log}[\psi]$, $\mathcal{E}_{log}[T\psi]$ and the precise choice of the constants C_1 , C_2 , and that τ_1 is chosen sufficiently large in terms of ε , ε_0 , τ_* and the precise choice of C_1 , C_2 (assuming also that ε_0 has been fixed so that $0 < \varepsilon_0 < \frac{1}{8}$), from (7.15) we infer:

$$\sum_{j=0}^1 \int_{\mathcal{R}(\tau_1+2l_0\tau_*, \tau_1+2(l_0+1)\tau_*) \setminus \mathcal{S}_{\bar{\delta}_1} \cap \{r \leq R\}} \left(J_{\mu}^N(T^j\psi)n^{\mu} + |T^j\psi|^2 \right) < \varepsilon. \tag{7.17}$$

Setting $\tau_{\natural} = \tau_1 + (2l_0 + 1)\tau_*$ (and thus $\tau_1 + 2l_0\tau_* = \tau_{\natural} - \tau_*$ and $\tau_1 + 2(l_0 + 1)\tau_* = \tau_{\natural} + \tau_*$), (7.17) yields (4.19). \square

8. Proof of Corollary 2.1

The proof of Corollary 2.1 follows immediately from Theorem 2.1 applied to the quotient of $(\mathbb{R} \times \mathcal{V}_{hyd,\delta}, g_{hyd})$ by the translations in the z -direction, i.e. the $2 + 1$ dimensional spacetime $(\mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta}, \bar{g}_{hyd})$, where $\bar{\mathcal{V}}_{hyd,\delta} = \mathbb{R}^2 \setminus \{\bar{r} \leq \delta\}$ (in the polar (\bar{r}, ϑ) coordinate system) and

$$\bar{g}_{hyd} = -\left(1 - \frac{C^2}{\bar{r}^2}\right)dt^2 + d\bar{r}^2 - 2Cdt d\vartheta + \bar{r}^2 d\vartheta^2 \tag{8.1}$$

(see also the remark below Theorem 2.1, as well as Sect. 6.9, regarding the Dirichlet or Neumann boundary conditions on $\{\bar{r} = \delta\}$).

In particular, in the language of Sect. 6.9, $(\mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta}, \bar{g}_{hyd})$ is a smooth Lorentzian manifold with smooth timelike boundary

$$\partial_{tim}(\mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta}) = \{\bar{r} = \delta\}. \tag{8.2}$$

The double $(\mathbb{R} \times \widetilde{\bar{\mathcal{V}}}_{hyd,\delta}, \widetilde{\bar{g}}_{hyd})$ of $(\mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta}, \bar{g}_{hyd})$ across the boundary $\partial_{tim}(\mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta})$ is diffeomorphic to $\mathbb{R} \times \mathbb{R} \times \mathbb{S}^1$, with the metric $\widetilde{\bar{g}}_{hyd}$ in the (t, \bar{r}, ϑ) coordinate chart of $\mathbb{R} \times \mathbb{R} \times \mathbb{S}^1$ having the form:

$$\widetilde{\bar{g}}_{hyd} = -\left(1 - \frac{C^2}{(|\bar{r} - \delta| + \delta)^2}\right)dt^2 + d\bar{r}^2 - 2Cdt d\vartheta + (|\bar{r} - \delta| + \delta)^2 d\vartheta^2 \tag{8.3}$$

Notice that $(\mathbb{R} \times \widetilde{\bar{\mathcal{V}}}_{hyd,\delta}, \widetilde{\bar{g}}_{hyd})$ is a globally hyperbolic spacetime without boundary, with Cauchy hypersurface $\{t = 0\}$. Let $i_1, i_2 : (\mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta}, \bar{g}_{hyd}) \rightarrow (\mathbb{R} \times \widetilde{\bar{\mathcal{V}}}_{hyd,\delta}, \widetilde{\bar{g}}_{hyd})$ be the two natural inclusions (see Sect. 6.9). Then, in the coordinate charts (t, \bar{r}, ϑ) on $\mathbb{R} \times [\delta, +\infty) \times \mathbb{S}^1 \simeq \mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta}$ and $\mathbb{R} \times \mathbb{R} \times \mathbb{S}^1 \simeq \mathbb{R} \times \widetilde{\bar{\mathcal{V}}}_{hyd,\delta}$, we have $i_1((t, \bar{r}, \vartheta)) = (t, \bar{r}, \vartheta)$ and $i_2((t, \bar{r}, \vartheta)) = (t, \delta - \bar{r}, \vartheta)$.

Note that $\widetilde{\bar{g}}_{hyd}$ is smooth everywhere except on $i_1(\partial_{tim}(\mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta})) = \{\bar{r} = \delta\}$. Notice also that $(\mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta}, \bar{g}_{hyd})$ has no event horizon \mathcal{H} (and thus, trivially, $\mathcal{H} \cap i_1(\partial_{tim}(\mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta})) = \emptyset$), and $i_1(\partial_{tim}(\mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta})) \cap \{t = 0\}$ is compact. Thus, in view of the remark below Theorem 2.1 on spacetimes with timelike boundary, it only remains to verify that $(\mathbb{R} \times \widetilde{\bar{\mathcal{V}}}_{hyd,\delta}, \widetilde{\bar{g}}_{hyd})$ satisfies Assumptions G1–G3 and A1, and that $i_1(\partial_{tim}(\mathbb{R} \times \bar{\mathcal{V}}_{hyd,\delta}))$ is invariant with respect to the stationary Killing field of $(\mathbb{R} \times \widetilde{\bar{\mathcal{V}}}_{hyd,\delta}, \widetilde{\bar{g}}_{hyd})$.

1. The vector field ∂_t (in the (t, \bar{r}, ϑ) coordinate system for $(\mathbb{R} \times \widetilde{\bar{V}}_{hyd,\delta}, \widetilde{g}_{hyd})$ is Killing, and the metric (8.1) is asymptotically flat (with the asymptotically flat region $\widetilde{\mathcal{I}}_{as} = \{\bar{r} \geq R_0 \gg 1\}$ consisting of two connected components) and satisfies Assumption G1. Furthermore, $\partial_{tim}(\mathbb{R} \times \widetilde{V}_{hyd,\delta})$ is ∂_t -invariant.
2. The spacetime $(\mathbb{R} \times \widetilde{V}_{hyd,\delta}, \widetilde{g}_{hyd})$ has no event horizon \mathcal{H} , and thus Assumption G2 is trivially satisfied.
3. The spacetime $(\mathbb{R} \times \widetilde{V}_{hyd,\delta}, \widetilde{g}_{hyd})$ has a non empty ergoregion $\widetilde{\mathcal{E}} = \{2\delta - C < \bar{r} \leq C\}$. The boundary $\partial\widetilde{\mathcal{E}} = \{\bar{r} = 2\delta - C\} \cup \{\bar{r} = C\}$ of $\widetilde{\mathcal{E}}$ is a smooth hypersurface of $\mathbb{R} \times \widetilde{V}_{hyd,\delta}$, and $\mathbb{R} \times \widetilde{V}_{hyd,\delta} \setminus \widetilde{\mathcal{E}}$ consists of two connected components, each containing one asymptotically flat end of $\mathbb{R} \times \widetilde{V}_{hyd,\delta}$ (and, thus, $\widetilde{\mathcal{E}}_{ext} = \widetilde{\mathcal{E}}$). In particular, Assumption G3 is satisfied.
4. Assumption A1 is readily satisfied in view of the fact that $(\mathbb{R} \times \widetilde{V}_{hyd,\delta}, \widetilde{g}_{hyd})$ is also axisymmetric, with axisymmetric Killing field ∂_ϑ such that $[\partial_\vartheta, \partial_t] = 0$ and the span of $\partial_\vartheta, \partial_t$ contains a timelike direction (see the discussion in Sect. 2.3).

Thus, the proof of Corollary 2.1 is complete.

9. Aside: Discussion on Friedman’s Heuristic Argument

In this Section, we will briefly sketch the heuristic arguments developed by Friedman in [20], and we will discuss their connections with the methods used in this paper.

9.1. *Friedman’s argument.* As we already explained in the introduction, on any globally hyperbolic, stationary and asymptotically flat spacetime (\mathcal{M}, g) with a non-empty ergoregion \mathcal{E} and no future event horizon \mathcal{H}^+ , Friedman constructed, in [20], a class of smooth solutions ψ to the wave equation (1.2) satisfying

$$\int_{\Sigma} J_{\mu}^T(\psi)n^{\mu} = -1, \tag{9.1}$$

where Σ is a Cauchy hypersurface of (\mathcal{M}, g) , T is the stationary Killing field of (\mathcal{M}, g) and n is the future directed unit normal to Σ . In view of the conservation of the T -energy flux for solutions to (1.2) on (\mathcal{M}, g) and the fact that $J_{\mu}^T(\psi)n^{\mu} \geq 0$ on $\mathcal{M} \setminus \mathcal{E}$, from (9.1) Friedman inferred that for any $\tau \geq 0$:

$$\int_{\Sigma_{\tau} \cap \mathcal{E}} J_{\mu}^T(\psi)n^{\mu} \leq -1, \tag{9.2}$$

where Σ_{τ} is defined as in Sect. 3 (i.e. the image of Σ under the flow of T for time τ).

Proceeding to study the consequences of the bound (9.2) on the (in)stability properties of equation (1.2), Friedman first noted the following dichotomy for the energy flux through the future null infinity \mathcal{I}^+ of any solution ψ to (1.2), satisfying (9.1)¹⁵: Either

$$\int_{\mathcal{I}^+} J_{\mu}^T(\psi)n_{\mathcal{I}^+}^{\mu} = +\infty, \tag{9.3}$$

¹⁵ See [31] for the definition of the Friedlander radiation field and the energy flux of φ through \mathcal{I}^+ on general asymptotically flat spacetimes.

in which case (in view of (9.1) and the conservation of the J^T -flux) there exists a sequence of hyperboloidal hypersurfaces \mathcal{S}_{τ_n} terminating at \mathcal{I}^+ such that

$$\limsup_{n \rightarrow +\infty} \int_{\mathcal{S}_{\tau_n}} J_\mu^T(\psi) n_{\mathcal{S}_{\tau_n}}^\mu = +\infty, \tag{9.4}$$

or

$$\int_{\mathcal{I}^+} J_\mu^T(\psi) n_{\mathcal{I}^+}^\mu < +\infty. \tag{9.5}$$

In case the first scenario (9.4) holds, one immediately obtains an energy instability statement for equation (1.2). In case the second scenario (9.5), Friedman argued (see [20]) that ψ “settles down” to a “non-radiative state” $\tilde{\psi}$, which is to be interpreted as a solution to (1.2) such that

$$\int_{\mathcal{I}^+} J_\mu^T(\tilde{\psi}) n_{\mathcal{I}^+}^\mu = 0. \tag{9.6}$$

Furthermore, in view of (9.2), Friedman argued that $\tilde{\psi}$ should also satisfy for all $\tau \geq 0$:

$$\int_{\Sigma_\tau \cap \mathcal{E}} J_\mu^T(\tilde{\psi}) n^\mu \leq -1. \tag{9.7}$$

Assuming that (\mathcal{M}, g) is globally real analytic and that the metric g has a proper asymptotic expansion in powers of r^{-1} in a neighborhood of \mathcal{I}^+ , Friedman inferred from (9.6) (using an adaptation of Holmgren’s uniqueness theorem for analytic linear partial differential equations, see [24]) that

$$\tilde{\psi} \equiv 0 \tag{9.8}$$

on (\mathcal{M}, g) . Thus, (9.7) and (9.8) yield a contradiction, implying that the scenario (9.5) should not occur on such spacetimes.

9.2. Comparison with the proof of Theorem 2.1. In general terms, the proof of Theorem 2.1 (see Sect. 4) follows the roadmap of the heuristic arguments of Friedman. In particular, our proof proceeds by contradiction, assuming the energy bound (4.1) on the $\{t = \tau\}$ hypersurfaces, which is a slightly stronger assumption than the energy bound (9.5) on \mathcal{I}^+ in the second scenario considered by Friedman.

In Lemma 4.2, we show that, under the assumption (4.1), a function ψ solving (1.2) with compactly supported initial data indeed “settles down” to a function $\tilde{\psi}$ (in a well defined way), such that $\tilde{\psi}$ vanishes identically outside the extended ergoregion \mathcal{E}_{ext} . This result makes use (through Proposition 4.1) of the Carleman-type estimates of Sect. 6, as well as the bound (4.1). Here, assuming merely the bound (9.5) on \mathcal{I}^+ would not be enough. Note that, in the argument of [20], no justification is provided (even at the heuristic level) of why a function ψ solving (1.2) and satisfying (9.5) is expected to “settle down” to a non-radiating solution $\tilde{\psi}$ of (1.2).

The fact that $\tilde{\psi}$ vanishes outside \mathcal{E}_{ext} follows from the estimates of Sect. 6, without any need to impose a real analyticity assumption on (\mathcal{M}, g) or a complete asymptotic expansion for g on \mathcal{I}^+ . In general, however, it can not be inferred that $\tilde{\psi}$ vanishes also on \mathcal{E} .¹⁶ Thus, a contradiction can not be reached following the argument of Friedman

¹⁶ We can in fact construct spacetimes (\mathcal{M}^{d+1}, g) , $d \geq 3$, with a smooth solution $\tilde{\psi}$ to an equation of the form $\square_g \tilde{\psi} + V \tilde{\psi} = 0$, such that $T(V) = 0$, $\tilde{\psi} \equiv 0$ on $\mathcal{M} \setminus \mathcal{E}$ and $\tilde{\psi}$ not identically 0 in \mathcal{E} .

in this setting. Instead, after restricting ourselves to spacetimes (\mathcal{M}, g) satisfying the unique continuation assumption A1, which guarantees that $\tilde{\psi}$ vanishes on $(\mathcal{M} \setminus \mathcal{E}_{ext}) \cup \mathcal{U}$, we reach the desired contradiction by exploiting our freedom to choose the initial data for ψ appropriately: We choose $(\psi, T\psi)|_{\Sigma}$ to be supported in $\Sigma \cap \mathcal{U}$, so that the support of $(\psi, T\psi)|_{\Sigma}$ will be disjoint from $\mathcal{M} \setminus \mathcal{U}$, where the support of all the time translates of $(\psi, T\psi)|_{\Sigma}$ is contained. Therefore, ψ and all the time translates of $\tilde{\psi}$ are orthogonal with respect to the (indefinite) T -inner product (4.38). This fact leads to relation (4.57), from which a contradiction follows readily in view of (4.49).

Acknowledgements. I would like to thank my advisor Mihalis Dafermos for suggesting this problem to me and providing me with valuable assistance while this paper was being written. I would also like to thank Igor Rodnianski for many insightful suggestions.

References

1. Alinhac, S., Baouendi, M.: A non uniqueness result for operators of principal type. *Math. Z.* **220**(1), 561–568 (1995)
2. Andersson, L., Blue, P.: Hidden symmetries and decay for the wave equation on the Kerr spacetime. *Ann. Math.* **182**(3), 787–853 (2015)
3. Blue, P., Soffer, A.: Semilinear wave equations on the Schwarzschild manifold I: local decay estimates. *Adv. Differ. Equ.* **8**(3), 595–614 (2003)
4. Blue, P., Sterbenz, J.: Uniform decay of local energy and the semi-linear wave equation on Schwarzschild space. *Commun. Math. Phys.* **268**(2), 481–504 (2006)
5. Butterworth, E., Ipser, J.: On the structure and stability of rapidly rotating fluid bodies in general relativity. I-The numerical method for computing structure and its application to uniformly rotating homogeneous bodies. *Astrophys. J.* **204**, 200–223 (1976)
6. Cardoso, V., Pani, P., Cadoni, M., Cavaglia, M.: Ergoregion instability of ultracompact astrophysical objects. *Phys. Rev. D* **77**(12), 124044 (2008)
7. Chandrasekhar, S.: Solutions of two problems in the theory of gravitational radiation. *Phys. Rev. Lett.* **24**(11), 611–615 (1970)
8. Christodoulou, D., Klainerman, S.: *The Global Nonlinear Stability of the Minkowski Space*, volume 1 of Princeton Mathematical Series. Princeton University Press, Princeton (1993)
9. Comins, N., Schutz, B.: On the ergoregion instability. In: *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, vol. 364, pp. 211–226. The Royal Society (1978)
10. Dafermos, M., Holzegel, G., Rodnianski, I.: The linear stability of the Schwarzschild solution to gravitational perturbations (2016). arXiv preprint [arXiv:1601.06467](https://arxiv.org/abs/1601.06467)
11. Dafermos, M., Rodnianski, I.: A proof of Price's law for the collapse of a self-gravitating scalar field. *Invent. Math.* **162**(2), 381–457 (2005)
12. Dafermos, M., Rodnianski, I.: A note on energy currents and decay for the wave equation on a Schwarzschild background (2007). arXiv preprint [arXiv:0710.0171](https://arxiv.org/abs/0710.0171)
13. Dafermos, M., Rodnianski, I.: The red-shift effect and radiation decay on black hole spacetimes. *Commun. Pure Appl. Math.* **62**(7), 859–919 (2009)
14. Dafermos, M., Rodnianski, I.: Decay for solutions of the wave equation on Kerr exterior spacetimes I-II: the cases $l \leq M$ or axisymmetry (2010). arXiv preprint [arXiv:1010.5132](https://arxiv.org/abs/1010.5132)
15. Dafermos, M., Rodnianski, I.: A proof of the uniform boundedness of solutions to the wave equation on slowly rotating Kerr backgrounds. *Invent. Math.* **185**(3), 467–559 (2011)
16. Dafermos, M., Rodnianski, I.: Lectures on black holes and linear waves. In: *Evolution Equations*, Clay Mathematics Proceedings, vol. 17, pp. 97–205 (2013)
17. Dafermos, M., Rodnianski, I., Shlapentokh-Rothman, Y.: Decay for solutions of the wave equation on Kerr exterior spacetimes III: the full subextremal case $l < M$. *Ann. Math.* **183**(3), 787–913 (2016)
18. Datchev, K.: Quantitative limiting absorption principle in the semiclassical limit. *Geom. Funct. Anal.* **24**(3), 740–747 (2014)
19. Eskin, G.: Superradiance initiated inside the ergoregion. *Rev. Math. Phys.* **28**(10), 1650025 (2016)
20. Friedman, J.L.: Ergosphere instability. *Commun. Math. Phys.* **63**(3), 243–255 (1978)
21. Friedman, J.L., Lockitch, K.: Gravitational-wave driven instability of rotating relativistic stars. *Prog. Theor. Phys. Suppl.* **136**, 121–134 (1999)
22. Friedman, J.L., Schutz, B.: Secular instability of rotating Newtonian stars. *Astrophys. J.* **222**, 281–296 (1978)

23. Gilbarg, D., Trudinger, N.S.: *Elliptic Partial Differential Equations of Second Order*, vol. 224. Springer, Berlin (2001)
24. Holmgren, E.: Über Systeme von linearen partiellen Differentialgleichungen. *Öfversigt Af Kongl. Vetenskaps-Academien Förhandlingar* **58**, 91–103 (1901)
25. Kay, B.S., Wald, R.M.: Linear stability of Schwarzschild under perturbations which are non-vanishing on the bifurcation 2-sphere. *Class. Quantum Gravity* **4**(4), 893 (1987)
26. Kokkotas, K., Ruoff, J., Andersson, N.: w-mode instability of ultracompact relativistic stars. *Phys. Rev. D* **70**(4), 043003 (2004)
27. Kokkotas, K., Schutz, B.: Normal modes of a model radiating system. *Gen. Relativ. Gravit.* **18**(9), 913–921 (1986)
28. Lee, John M.: *Introduction to Smooth Manifolds*. Springer, New York (2012)
29. Metcalfe, J., Sterbenz, J., Tataru, D.: Local energy decay for scalar fields on time dependent non-trapping backgrounds (2017). arXiv preprint [arXiv:1703.08064](https://arxiv.org/abs/1703.08064)
30. Moschidis, G.: Logarithmic local energy decay for scalar waves on a general class of asymptotically flat spacetimes. *Ann. PDE* **2**, 5 (2016). doi:[10.1007/s40818-016-0010-8](https://doi.org/10.1007/s40818-016-0010-8)
31. Moschidis, G.: The r^p -weighted energy method of Dafermos and Rodnianski in general asymptotically flat spacetimes and applications. *Ann. PDE* **2**, 6 (2016). doi:[10.1007/s40818-016-0011-7](https://doi.org/10.1007/s40818-016-0011-7)
32. Moschidis, G.: Superradiant instabilities for short-range non-negative potentials on kerr spacetimes and applications (2016). arXiv preprint [arXiv:1608.02041](https://arxiv.org/abs/1608.02041)
33. Oliveira, L., Cardoso, V., Crispino, L.: Ergoregion instability: the hydrodynamic vortex. *Phys. Rev. D* **89**(12), 124008 (2014)
34. Rodnianski, I., Tao, T.: Effective limiting absorption principles, and applications. *Commun. Math. Phys.* **333**, 1–95 (2015)
35. Shlapentokh-Rothman, Y.: Exponentially growing finite energy solutions for the Klein–Gordon equation on sub-extremal Kerr spacetimes. *Commun. Math. Phys.* **329**(3), 859–891 (2013)
36. Shlapentokh-Rothman, Y.: Quantitative mode stability for the wave equation on the Kerr spacetime. *Annales Henri Poincaré* **16**(1), 289–345 (2015)
37. Tataru, D.: Unique continuation for solutions to PDE’s; between Hörmander’s theorem and Holmgren’s theorem. *Commun. Partial Differ. Equ.* **20**(5-6), 855–884 (1995)
38. Tataru, D., Tohaneanu, M.: A local energy estimate on Kerr black hole backgrounds. *Int. Math. Res. Not.* **2011**(2), 248–292 (2011)
39. Yoshida, S., Eriguchi, Y.: Ergoregion instability revisited—a new and general method for numerical analysis of stability. *Mon. Not. R. Astron. Soc.* **282**(2), 580–586 (1996)

Communicated by P. T. Chrusciel