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Spinning Black Hole Binary Dynamics

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Abstract

This thesis reviews and expands on the methods used for computing classical gravitational observables in spinning black hole binary dynamics, using modern amplitude-based techniques. Working within the framework of Heavy Mass Effective Field Theory, and including the minimally coupled spinning three-point and four-point amplitudes, we provide all the building blocks needed for the computation of classical observables, up to third post-Minkowskian order. A novel calculation of the eikonal is presented, at all orders in spin, up to third post-Minkowskian order and first order in the angle between the angular momentum and spin, in the probe limit. This calculation is a stepping stone towards observable computations that are exact in spin at higher orders in PM.

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1 Introduction

The history of the two-body problem in gravitational dynamics, goes back a long way and throughout its course, the study of this rather simple setting keeps driving forward our understanding of the nature of gravity and its interactions. In the context of Newtonian gravity, the motion of two, interacting, point masses was famously solved in closed form [1], yielding the well-known Keplerian orbits. Furthermore, proper understanding of two body dynamics lead to the prediction of the existence of the planet Neptune [2, 3]. In the times of Einstein and his remarkable General Theory of Relativity, the study of Schwarzschild geodesics was able to explain the anomalous perihelion advance of the planet mercury [4]. In modern times, with the detection of gravitational waves from binary black hole mergers [5, 6], there has been a renewed effort to develop efficient and high-precision methods for the accurate modeling of binary systems. The accurate modeling of binary black hole mergers, will help us further our understanding of gravitational waves, which will in turn provide us with the opportunity to explore the range of validity of the General Theory of Relativity. Moreover, having the tools to precisely model gravitational wave signals presents a great avenue for probing the structure of astrophysical objects.

Traditionally, the post-Newtonian (PN) expansion [7–16] has been the primary tool used to compute gravitational observables in the weak-field, slow-velocity regime. While powerful, PN techniques become increasingly cumbersome at high orders and are limited in their ability to describe ultra-relativistic or strong-field scenarios. An alternative approach, known as the post-Minkowskian (PM) expansion [17–19], avoids the low-velocity assumption and instead performs a perturbative expansion in Newtons constant G_n. The PM formalism is well-suited for studying high-energy scattering of compact objects and offers a systematic way to explore conservative dynamics without the need to solve Einstein's equations directly.

In recent years, a new perspective has emerged from the field of high-energy physics: the use of modern methods for particle scattering, that was developed for the computation of observables in particle physics. The first step towards a quantum field theory approach to general relativity, is to consider it as an effective field theory, which was pioneered in [20]. This approach is well suited for capturing the classical effects which take place at large distances. The next step is to consider a weak field approximation, which allows for the calculation of observables, in a perturbative manner, organized in powers of G_N. This emergent route, consistent with the PM formalism, has received plenty of attention in the recent years, and several works [21–38] have contributed to the computation of observables in both conservative and radiative dynamics. At the time of writing, the state of the art (spinless) calculations are of fifth post-Minkowskian order, presented in [39].

An interesting direction is the study of spinning black hole binaries, which can be realized through the inclusion of spin effects to the observables. This task is crucial, if we want to have a realistic picture of the interaction of black hole binary systems in our universe. In simple cases, such as the scattering of a Schwarzschild probe, off a heavy Kerr black hole, in the equatorial plane, (aligned spin with angular momentum) the bending angle can be calculated, to relatively high PM orders, using the general theory of relativity directly [40]. In that reference, the result was expanded to accommodate a spinning probe as well, up to and including second order in the probe spin and any order in the PM expansion.

In order to incorporate results beyond the probe limit, and most importantly beyond the aligned spin configuration, quantum field theory techniques are needed. To that end, research [41–48] within the worldline quantum field theory (WQFT) formalism [25], was able to provide results for both radiative and conservative observables up to fourth PM order, however typically, truncated at quadratic order in spin. The state of the art calculation so far, is the calculation of conservative observables at fourth PM order, and linear in spin, presented in [44]. On the other hand amplitude-based methods, were able to provide results [49–56], for both radiative and conservative observables, up to high order in spin, but typically lower order in PM. In [51] an all order in spin calculation was performed at two PM as well, in conservative dynamics. Moreover, calculations of waveforms are provided in [57–60], up to quadratic order in spin.

Turning our attention in conservative dynamics, this thesis, aims to bridge the gap between the high order in PM, low in spin calculations of WQFT and low order in PM, high in spin computations of amplitude based methods, by providing an all orders in spin result of the scattering of a Schwarzschild probe, off a heavy Kerr black hole at third order in PM and leading order in the angle between the spin of the Kerr black hole and the angular momentum of the probe. This computation is a stepping stone towards exact-in-spin calculations in higher PM order.

In this project, we work within the amplitude based methods and in particular, we adopt the Heavy Mass Effective Field Theory (HEFT) formalism developed in [23, 24]. The key observation is that in a scattering process between two heavy compact objects, the momentum exchange q is small compared to their masses. This separation of scales is reminiscent of Heavy Quark Effective Theory (HQET) in QCD [61–64] and motivates a similar effective field theory treatment of gravity. The resulting theory, exhibits a double copy structure as was presented in [24], and the tree amplitudes that enter the computation through unitarity cuts are not only manifestly gauge invariant, but also their expressions are very compact. Since we are interested in probe limit calculations, spin will enter through the minimally coupled, graviton matter 3-point amplitude, at tree level, with one graviton, and two massive legs presented in [50].

This thesis is organized as follows. In chapter 2, we introduce the effective field theory approach to general relativity and review the use of unitarity in computing classical observables. In chapter 3, we present the HEFT formalism, discuss its relation to HQET, and describe how to compute gravitational amplitudes using a heavy mass expansion. Chapter 4 extends this framework to include spin, providing explicit expressions for the 3-point and 4-point spinning tree amplitudes, with two massive legs. In chapter 5, we review the loop integration techniques used in our computations, namely, IBP reduction, differential equations and tensor integral decomposition. Chapters 6 and 7 contain the main results: after a warm up calculation of the scattering at 2PM, we present the calculation of spinning spinless scattering at 3PM and at leading order in the angle between spin and angular momentum. Finally, we conclude with a discussion of possible approaches towards an exact result and open directions.

2 Classical Gravity Observables From Scattering Amplitudes

In this chapter, we will lay the groundwork for the developments presented in the following chapters. We begin with a brief overview of general relativity viewed as an effective field theory. Following this introduction, the discussion becomes more technical. We then review the basics of the unitarity method, which allows us to extract the pieces of scattering amplitudes relevant to our analysis. Finally, we describe how these amplitudes relate to observables in classical gravity.

2.1 General Relativity as an Effective Field Theory

As with any other quantum field theory, the starting point to describe gravitational interactions from the quantum mechanical point of view is a Lagrangian. Treating general relativity (GR) as an effective field theory [20], the appropriate Lagrangian, for pure GR (excluding the cosmological term Λ) can be written as:

$$\mathcal{L}_{\text{grav}} = \sqrt{-g}\left(\frac{2R}{16\pi G_N} + c_1 R^2 + c_2 R^{\mu\nu} R_{\mu\nu} + \dots\right), \tag{2.1}$$

where G_N is Newtons constant, and g is the metric of the tensor field $g^{\mu\nu}$, and the Riemann tensor $R^{\mu}_{\nu\alpha\beta}$, is defined as usual:

$$R^{\mu}_{\nu\alpha\beta} = \partial_\alpha \Gamma^{\mu}_{\nu\beta} - \partial_\beta \Gamma^{\mu}_{\nu\alpha} + \Gamma^{\mu}_{\sigma\alpha}\Gamma^{\sigma}_{\nu\beta} - \Gamma^{\mu}_{\sigma\beta}\Gamma^{\sigma}_{\alpha\beta}. \tag{2.2}$$

The infinite higher curvature terms in eq. 2.1 ensure that any divergences that the loop diagrams may generate can be associated and thus absorbed by one of the (infinite) terms in the Lagrangian. Moreover they organize interactions according to energy scales: in the Christoffel symbol, $\Gamma^{\mu}_{\alpha\beta}$, derivatives appear only in first order and in curvature, derivatives are of second order. In momentum space, derivatives turn into factors of momentum, and thus higher curvature terms contribute higher powers of momentum. At the energies that characterize the classical regime, these higher terms will contribute negligibly and for that matter, we may truncate the Lagrangian to the Einstein term:

$$\mathcal{L}_{\text{EH}} = \sqrt{-g}\left(\frac{2R}{16\pi G_N}\right). \tag{2.3}$$

Of course, this is the Hilbert-Einstein action.

We may couple real massive scalar fields of mass m to gravity by considering the Lagrangian:

$$\mathcal{L}_{\text{matter}} = \sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right). \tag{2.4}$$

In order to be able to do perturbation theory, we may define a weak field expansion of the metric $g_{\mu\nu}$, around the Minkowski metric $\eta^{\mu\nu}$ as :

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \sqrt{32G_N}h_{\mu\nu}(x). \tag{2.5}$$

The scattering of two black holes, (evidently), falls within the $2 \to 2$ scattering processes. The relevant amplitudes are therefore the four-point amplitudes. Given the weak field expansion of the metric as defined in eq. 2.5, higher-loop corrections to the classical components of these amplitudes appear as higher order in the perturbative expansion in G_N. That is, our perturbative expansion is consistent with the post-Minkowskian expansion.

2.2 Implications of Unitarity

So far we have a Lagrangian, that describes (effectively) matter coupled to a gravitational field. One may take this Lagrangian, derive the Feynman Rules, and start calculating amplitudes and observables. However, when it comes to calculating scattering amplitudes, there are more efficient techniques, and we are about to describe one. That is, the method of unitarity [65]. Colloquially, the method of unitarity is able to relate loop amplitudes to products of tree amplitudes. Starting by the requirement that the S-matrix is *unitary*, we will be able to relate the discontinuity of a loop amplitude to products of lower loops and tree level amplitudes. This implies that with the method of unitarity one is able to obtain all the non-analytic pieces of an amplitude [66]. When considering the dynamics of a black hole binary, in the long distance regime, the parts of the amplitudes that contribute to our observables are the non-analytic parts [66]. Therefore, unitarity is very well suited for our purposes.

2.2.1 The Unitarity of The S-Matrix

The scattering of two black holes can be described, from the quantum mechanical point of view, as the scattering of two scalar fields coupled to gravity. These single particle states $|p_i\rangle$, are eigensates of the momentum operator $\mathbb{P}\,|p_i\rangle = p_i\,|p_i\rangle$, and thus we are interested in a scattering from two initial states $|p_1\rangle$ and $|p_2\rangle$, to two final states $|p_3\rangle$ and $|p_4\rangle$. We may write down the scattering amplitude:

$$\langle p_3, p_4|\,\mathbb{T}\,|p_1, p_2\rangle = \mathcal{M}\left(p_1, p_2 \to p_3, p_4\right)\hat{\delta}^{(4)}\left(p_1 + p_2 - p_3 - p_4\right), \tag{2.6}$$

where we have defined the n-fold delta function $\hat{\delta}^{(n)}(x)$ as:

$$\hat{\delta}^{(n)}(x) = (2\pi)^n \delta^{(n)}(x). \tag{2.7}$$

The matrix $\mathbb{T}$ is related to the $\mathbb{S}$-*Matrix*, through the usual decomposition: $\mathbb{S} = \mathbb{1} + i\mathbb{T}$.

The $\mathbb{S}$-Matrix is unitary:

$$\begin{aligned} \mathbb{S}^\dagger\mathbb{S} = 1 \Rightarrow \left(\mathbb{1} - i\mathbb{T}^\dagger\right)\left(\mathbb{1} + i\mathbb{T}\right) = 1 \\ \therefore \mathbb{T}^\dagger\mathbb{T} = -i\left(\mathbb{T} - \mathbb{T}^\dagger\right). \end{aligned} \tag{2.8}$$

Sandwiching both sides of the last line of eq. 2.8 with $\langle p_{34}|$ and $|p_{12}\rangle$ (where we wrote $|p_{ij}\rangle = |p_i p_j\rangle$ for brevity), and using the completeness relation:

$$\mathbb{1} = \sum\!\!\!\!\!\!\int |p_{nn'}\rangle\,\langle p_{nn'}|\,, \tag{2.9}$$

where the sum is over all the intermediate states $|p_{nn'}\rangle$ and their helicities, and the integral is over the intermediate-state phase space [67], we obtain [68]:

$$\begin{aligned} &\mathcal{M}\left(p_{12} \to p_{34}\right) - \mathcal{M}^*\left(p_{34} \to p_{12}\right) = \\ &\qquad i \sum\!\!\!\!\!\!\int \hat{\delta}^{(n)}\left(p_{12} - p_{nn'}\right)\mathcal{M}\left(p_{12} \to p_{nn'}\right)\mathcal{M}^*\left(p_{34} \to p_{nn'}\right). \end{aligned} \tag{2.10}$$

Considering the special case where $|p_{12}\rangle = |p_{34}\rangle = |v\rangle$ (see Chapter 3), we may write schematically[1]:

$$\text{disc}\mathcal{M} = \mathcal{M}^\dagger\mathcal{M}, \tag{2.11}$$

[1]We will return to the state sums in the next sub-section

where disc$\mathcal{M} = 2\text{Im}\mathcal{M}$. Importantly, equations 2.10 and 2.11, hold order by order in perturbation theory. For a four point amplitude $\mathcal{M}_4$ we have the following perturbative expansion for the coupling g:

$$\mathcal{M}_4 = g^2\mathcal{M}_4^{(0)} + g^4\mathcal{M}_4^{(1)} + g^6\mathcal{M}_4^{(2)} + \ldots, \tag{2.12}$$

which when inserted to eq. 2.11 yields:

$$\begin{aligned} \text{disc}\mathcal{M}_4^{(0)} &= 0 \\ \text{disc}\mathcal{M}_4^{(1)} &= \mathcal{M}_4^{(0)\,\dagger}\mathcal{M}_4^{(0)} \\ \text{disc}\mathcal{M}_4^{(2)} &= \mathcal{M}_4^{(0)\,\dagger}\mathcal{M}_4^{(1)} + \mathcal{M}_4^{(1)\,\dagger}\mathcal{M}_4^{(0)} + \mathcal{M}_4^{(0)\,\dagger}\mathcal{M}_4^{(0)}, \end{aligned} \tag{2.13}$$

where the superscript (n), denotes the number of loops, with $n = 0$, being the tree level. It is apparent, that the requirement that the $\mathbb{S}$ matrix is unitary, has allowed us to relate the discontinuity of an L loop amplitude to tree level amplitudes and amplitudes of lower loops. In the sub-section that follows we will review how to use this in order to construct the pieces of the integrand that are relevant to our purposes.

2.2.2 Polarization Sums

Before we discuss the sums over all states and helicities (which we will call polarization sums since we are only interested in graviton cuts) let us examine where do the discontinuities in an amplitude come from. Following [68], we will evaluate the imaginary part of a Feynman propagator:

$$\text{Im}\frac{1}{p^2 - m^2 + i\epsilon} = \frac{1}{2i}\left(\frac{1}{p^2 - m^2 + i\epsilon} - \frac{1}{p^2 - m^2 - i\epsilon}\right) = \frac{-\epsilon}{(p^2 - m^2)^2 + \epsilon^2}. \tag{2.14}$$

In the limit $\epsilon \to 0$, $\text{Im}\frac{1}{p^2 - m^2 + i\epsilon}$ vanishes unless the particle goes on shell $p^2 \to m^2$. This fact, implies that

$$\text{Im}\frac{1}{p^2 - m^2 + i\epsilon} = -\pi\delta\left(p^2 - m^2\right). \tag{2.15}$$

The conclusion we draw from this analysis, is that discontinuities in loop amplitudes, stem from particles going on shell.

Let us now consider an one loop amplitude $\mathcal{A}_{\text{ex}}$ (see fig. 2.1) and a discontinuity (cut) in the channel that the red dashed lines denote. Based on our discussion so far, the

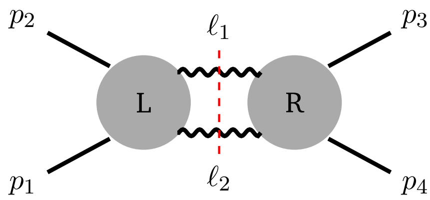


Figure 2.1.: Example of a one-loop amplitude illustrating a unitarity cut (shown as red dashed lines) through propagators with loop momenta ℓ_1 and ℓ_2. The cut separates the diagram into two tree-level sub-amplitudes, $\mathcal{A}_L$ and $\mathcal{A}_R$, corresponding to the left and right parts of the diagram.

discontinuity in $\mathcal{A}_{\text{ex}}$ will be written as:

$$\text{disk}\mathcal{A}_{\text{ex}}\Big|_{\ell_{12}} = \sum_{\text{pols.}} \int \frac{d^D\ell_1}{\pi^{D/2}} \delta^+(\ell_1^2)\delta^+(\ell_2^2)\mathcal{A}_{\text{L}}^{h_1,h_2}\left(\ell_1,\ell_2,p_1,p_2\right)\mathcal{A}_{\text{R}}^{-h_1,-h_2}\left(-\ell_1,-\ell_2,p_1,p_2\right), \tag{2.16}$$

where $\mathcal{A}_{L,R}$ are the four point trees with two massive legs and two gravitons, on the left and the right of the cut, and $\ell_2 = q - \ell_1$, $q = \ell_1 + \ell_2$. Moreover, the delta functions $\delta^+(\ell_i^2)$ denote that the momentum ℓ_i is on shell (cut), with $\ell_i^0 > 0$, and we have replaced the $\displaystyle\not\int$ symbol with explicit integration and summation. The discussion regarding loop integration techniques will be postponed to Chapter 5. The topic of the current discussion will be the summation. Firstly, we are summing over all polarizations $h_1 = \pm 1$ and $h_2 = \pm 1$ of the graviton legs. Moreover, the polarization sum has to be performed separately for legs 1 and 2.

Let us consider, the polarization sum for leg 1 (ℓ_1). We may do the following decomposition:

$$\mathcal{A}_L = \mathcal{C}_{\text{L}}^{\mu\nu}\varepsilon_\mu^{(+)}\varepsilon_\nu^{(+)}, \quad \mathcal{A}_R = \mathcal{C}_{\text{R}}^{\alpha\beta}\varepsilon_\alpha^{(-)}\varepsilon_\beta^{(-)}, \tag{2.17}$$

where we used the shorthand notation $\varepsilon^{(\pm)}$ for $\varepsilon(\pm\ell_1)$. The quantities $\mathcal{C}_L$ and $\mathcal{C}_R$ are composed by the polarization vector of the second leg and the external momenta, and they obey the ward identity:

$$\begin{aligned} \mathcal{C}_L^{\mu\nu}\ell_{1\mu} &= \mathcal{C}_L^{\mu\nu}\ell_{1\nu} = 0 \\ \mathcal{C}_R^{\alpha\beta}\ell_{1\alpha} &= \mathcal{C}_R^{\alpha\beta}\ell_{1\beta} = 0. \end{aligned} \tag{2.18}$$

Followig [69], the polarization sum reeds :

$$\begin{aligned}&\sum_{\text{pols.}} \mathcal{A}_{\mathrm{L}}^{h_1,h_2}\left(\ell_1,\ell_2,p_1,p_2\right)\mathcal{A}_{\mathrm{R}}^{-h_1,-h_2}\left(-\ell_1,-\ell_2,p_1,p_2\right)=\\ &\mathcal{C}_L^{\mu\nu}\mathcal{C}_R^{\alpha\beta}\sum_{\text{pols.}}\varepsilon_\mu^{(+)}\varepsilon_\nu^{(+)}\varepsilon_\alpha^{(-)}\varepsilon_\beta^{(-)}=\mathcal{C}_L^{\mu\nu}\mathcal{C}_R^{\alpha\beta}\mathcal{P}_{\alpha\beta\mu\nu}\left(\ell_1,q\right).\end{aligned} \tag{2.19}$$

The physical state projector in gravity $\mathcal{P}_{\mu\nu\alpha\beta}$ is related to the gauge theory physical state projector $\mathcal{P}^{\mu\nu}$, as:

$$\begin{aligned}\mathcal{P}^{\mu\nu\alpha\beta}\left(\ell,q\right)&=\sum_{\text{pols.}}\varepsilon_{(-)}^{\mu}\varepsilon_{(-)}^{\nu}\varepsilon_{(+)}^{\alpha}\varepsilon_{(+)}^{\beta}=\frac{1}{2}\left(\mathcal{P}^{\mu\alpha}\mathcal{P}^{\nu\beta}+\mathcal{P}^{\nu\alpha}\mathcal{P}^{\mu\beta}-\frac{1}{D-2}\mathcal{P}^{\mu\nu}\mathcal{P}^{\alpha\beta}\right),\\ \mathcal{P}^{\mu\nu}\left(\ell,q\right)&=\sum_{\text{pols.}}\varepsilon_{(-)}^{\mu}\varepsilon_{(+)}^{\nu}=\eta^{\mu\nu}-\frac{q^\mu\ell^\nu+q^\nu\ell^\mu}{q\cdot\ell},\end{aligned} \tag{2.20}$$

for some reference null momentum q^μ.

The pole $q\cdot\ell$ may seem worrisome. However, gauge invariance of $\mathcal{C}_L$ and $\mathcal{C}_R$, ensures us that the term $\frac{q^\mu\ell^\nu+q^\nu\ell^\mu}{q\cdot\ell}$, drops out, when contracted with $\mathcal{C}_L$ and $\mathcal{C}_R$. The fact that the spuriously singular term drops out of the computations enables us to (effectively) write:

$$\sum_{\text{pols.}}\varepsilon_{(-)}^{\mu}\varepsilon_{(-)}^{\nu}\varepsilon_{(+)}^{\alpha}\varepsilon_{(+)}^{\beta}\doteq\frac{1}{2}\left(\eta^{\mu\alpha}\eta^{\nu\beta}+\eta^{\nu\alpha}\eta^{\mu\beta}-\frac{2}{D-2}\eta^{\mu\nu}\eta^{\alpha\beta}\right), \tag{2.21}$$

where the dotted equality $\doteq$ means that this relation holds when both sides of eq. 2.21, are contracted with $\mathcal{C}_L^{\mu\nu}\mathcal{C}_R^{\alpha\beta}$.

2.3 Observables

Now that we have an effective description of gravity, as well as a handy method of calculating the parts of the amplitudes that are relevant for long range interactions, we may discuss how to obtain observables from our amplitudes. In this work, we employ the Heavy Mass Effective Filed Theory (HEFT) formalism, [23, 24, 70]. Through the classical part of the scattering amplitude $\mathcal{M}_{2\to2}^{\text{HEFT}}$, one may obtain the eikonal phase χ as :

$$\chi=\int\frac{d^Dq}{(2\pi)^{D-2}}\frac{e^{ibq}\delta\left(q\cdot v_1\right)\delta\left(q\cdot v_2\right)}{4m_1m_2}\mathcal{M}_{2\to2}^{\text{HEFT}}, \tag{2.22}$$

which can be understood as the generator of scattering observables [71, 72]. The exchanged momentum is q, and $p_1=mv_1$, $p_2=mv_2$ are the momenta of the two particles. When the scattering takes place in a plane (i.e aligned spin configuration) the bending angle may be obtained as:

$$\theta^{3PM} = -\frac{\partial}{\partial J}\chi, \tag{2.23}$$

where J is the total angular momentum of the system.

3 Heavy Mass Effective Field Theory

In this chapter we present the formalism we use to compute our main observable: the bending angle of two black holes. In a typical black hole scattering process, since we are interested in long-range (classical) interactions, there exists a hierarchy of scales due to the fact that the momentum q exchanged by the two heavy (point-like) objects is small relative to their mass[70]. Such a hierarchy is reminiscent of the Heavy Quark Effective Theory (HQET) [61–64], which describes the interactions of heavy, and light quarks, in a limit where the energy scale of the interactions is comparable to the QCD scale Λ_{QCD}, a scale that is small compared to the mass of the heavy quarks. It is therefore natural for classical $2 \to 2$ gravitational scattering to work within an effective field theory (EFT) framework, and in particular in a Heavy Mass Effective Field Theory (HEFT) [8, 24, 70] that is similar to HQET[1].

Long range gravitational effects, stem from the parts of the amplitudes where the exchanged gravitons are almost on shell [66]. Such parts are non analytic [20, 73], and can by obtained by the method of generalized unitarity [65], making this method particularly suitable for the task of calculating classical observables. Another advantage of generalized unitarity method is the fact that allows for the computation of loop amplitudes with the use of only tree level amplitudes (trees). In the context of HEFT, it was shown in [23] that the needed trees are those with two massive scalar legs and an arbitrary number of massless graviton legs, to leading order in an inverse mass expansion.

Traditionally, the HEFT trees for gravity can be computed from the corresponding QCD trees, using the color kinematics duality and the double copy presented in [74] and taking a heavy mass limit. However, a novel gauge invariant double copy was proposed in [24], that was able to generate the HEFT trees for gravity directly from the HEFT trees for QCD (these are the HQET trees), without the need for a heavy mass expansion. This was made possible by considering the algebraic structure of the

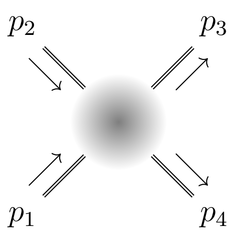
[1]Based on this terminology, HQET is a HEFT for quarks and gluons.

numerators consistent with the color kinematics duality. Despite the intellectual appeal of the concepts presented in that reference, providing a comprehensive review is out of the scope of this work. Be that as it may, we will list the results for the 3-point, 4-point and 5-point trees, since they are crucial for constructing the 1-loop and 2-loop integrands.

We have organized this chapter as follows: After discussing the kinematics of $2 \to 2$ scattering, we will present a brief review of HQET. We will then provide the results for the 3-point, 4-point, and 5-point trees, and discuss the heavy mass expansion, for gravity amplitudes and how one can obtain the loop amplitudes, relevant to classical physics.

3.1 Kinematics of The Scattering

In what follows, the discussions will be in the context of binary black hole scattering and more specifically, we study the scattering of a light Schwarzschild and a heavy Kerr black hole. Therefore in this section we review our conventions and the kinematics of such processes.



$$\begin{aligned} p_1^\mu &\equiv \bar{p}_1^\mu + \tfrac{q^\mu}{2} = (E_1, \vec{p} + \vec{q}/2)\,, \\ p_2^\mu &\equiv \bar{p}_2^\mu - \tfrac{q^\mu}{2} = (E_2, -\vec{p} - \vec{q}/2)\,, \\ p_3^\mu &\equiv \bar{p}_2^\mu + \tfrac{q^\mu}{2} = (E_3, -\vec{p} + \vec{q}/2)\,, \\ p_4^\mu &\equiv \bar{p}_1^\mu - \tfrac{q^\mu}{2} = (E_4, \vec{p} - \vec{q}/2)\,. \end{aligned} \tag{3.1}$$

The momentum exchange q is spacelike, and it follows that $\bar{p}_i \cdot q = 0$. The Mandelstam variables are :

$$s \equiv (p_1 + p_2)^2 = (E_1 + E_2)^2 \equiv E^2, \qquad q^2 \equiv (p_1 + p_4)^2 = -\vec{q}^{\,2}\,. \tag{3.2}$$

We introduce the usual barred masses as:

$$\bar{p}_1^2 = \bar{p}_4^2 \equiv \bar{m}_1^2 = m_1^2 - \frac{q^2}{4}\,, \qquad \bar{p}_2^2 = \bar{p}_3^2 \equiv \bar{m}_2^2 = m_2^2 - \frac{q^2}{4}\,, \tag{3.3}$$

and the relativistic velocities: $p_i \equiv m_i v_i$, $v_i^2 = 1$, with their scalar product being the Lorentz factor:

$$y_1 \equiv v_1 v_2. \tag{3.4}$$

The spin of the Kerr black hole in the heavy mass limit[2] is described by the tensor $S^{\mu\nu}$ and the Pauli Lubanski vector a^μ:

$$\begin{aligned} S^{\mu\nu} &= -\epsilon^{\mu\nu\rho\sigma}\bar{p}_\rho a_\sigma, \\ \bar{p}_i &= \bar{m}_i\bar{v}_i. \end{aligned} \tag{3.5}$$

We constrain a_μ to be perpendicular to the velocities of the two black holes, we have $a \cdot v_1 = a \cdot v_2 = 0$, and considering the impact parameter of the scattering b^μ, we can parametrize the process, using the following Lorentz Invariant scalars:

$$\begin{aligned} &y_1 = v_1 \cdot v_2, \quad y_2 = a \cdot a, \\ &y_3 = b \cdot b, \qquad y_4 = b \cdot a, \\ &q \cdot q, \qquad\qquad q \cdot a. \end{aligned} \tag{3.6}$$

At this point we can calculate the ever occurring contraction of the Levi-Civita tensor with the spin , impact parameter and the velocities $\epsilon(a, b, v_1, v_2) = \epsilon^{\mu\nu\rho\sigma}a_\mu b_\nu(v_1)_\rho(v_2)_\sigma$. Following [58], we will consider the rest frame of v_2 (the spinning black hole), such that[3]:

$$\begin{aligned} v_1 &= (y_1, \sqrt{y_1^2 - 1}, 0, 0), \qquad v_2 = (1, 0, 0, 0), \\ a &= \sqrt{-y_2}(0, 0, \cos\psi, \sin\psi), \quad b = \sqrt{-y_3}(0, 0, 0, 1), \end{aligned} \tag{3.7}$$

where $y_2 = -|a|^2$ and $y_3 = -|b|^2$, with $|a|, |b| > 0$, being the norms of a^μ and b^μ in turn, and ψ is the angle of the spin vector a^μ in the plane normal to the plane defined by v_1 and v_2 (see fig. 3.1. It is easy to verify that $a \cdot v_1 = 0$ and $a \cdot v_2 = 0$, and so the desired result is

$$\begin{aligned} &\epsilon(a, b, v_1, v_2) = \cos\psi\sqrt{-y_2}\sqrt{-y_3}\sqrt{y_1^2 - 1}, \\ &y_2^2, b^2 < 0. \end{aligned} \tag{3.8}$$

The quantity $\cos\psi$, can be related to y_4 through the fundamental identity $\cos\psi = \sqrt{1 - \sin^2\psi}$, and fact that $y_4 = \sqrt{-y_2}\sqrt{-y_3}\sin\psi$.

The angle ψ, as well as the vectors $\vec{a}$, $\vec{b}$, and $\vec{v}_1$, composed by the spatial components of the four vectors a^μ, b^μ, v_1^μ, can be seen in figure 3.1. Since $b \cdot v_1 = 0$, $\vec{b} \perp \vec{v}_1$, and thus we can align the 3 axes, with the vectors $\vec{b}$, $\vec{v}_1$, and $\vec{v}_1 \times \vec{b} \propto \vec{J}$, where $\vec{J}$ is the angular momentum of the probe. The fact that $a \cdot v_1 = a \cdot v_2 = 0$, constrains a in the plane perpendicular to v_1 and v_2 (the $J\,b$ plane in fig. 3.1).

[2]see the discussion in the next chapter, for the heavy mass expansion of $S^{\mu\nu}$.

[3]Note that formally, a, v_1 and v_2 should be barred. However, in the classical limit the barred and unbarred variables coincide.

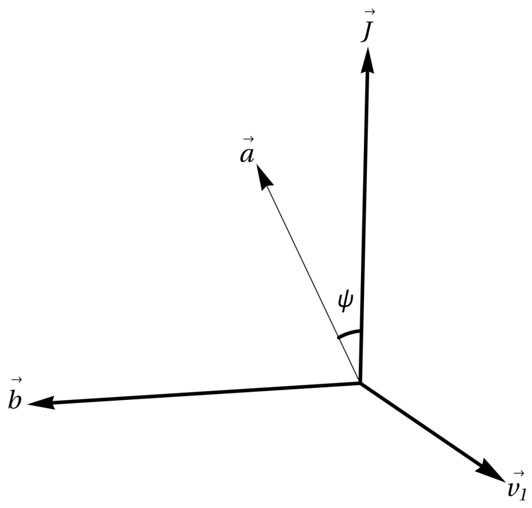


Figure 3.1.: The angle ψ between the spin vector $\vec{a}$ and the plane normal to the plane defined by $\vec{v_1}$ and $\vec{v_2}$. In the reference frame of the spinning black hole, $\vec{v_2}$, ψ is the angle between the $\vec{a}$ and the angular momentum of the probe $\vec{J}$.

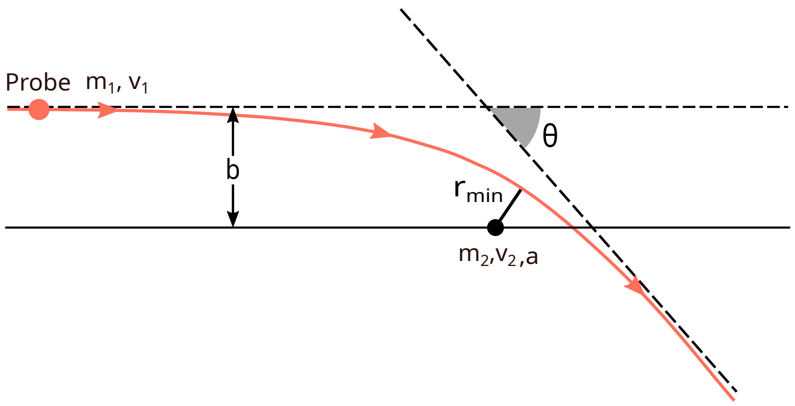


Figure 3.2.: The bending angle θ, in planar scattering.

In the case where the spin a, of the heavy particle, and the angular momentum of the probe, v_1, are aligned the scattering can be characterized by the bending angle, which can be seen in fig. 3.2.

3.2 Heavy Quark Effective Theory

When considering heavy quark scattering in the HQET formalism, the kinematic setup is very similar to the one we just reviewed: We consider an incoming quark with mass m and momentum $p^\mu = mv^\mu$. After the scattering, the momentum changes by

$p^\mu = mv^\mu + k^\mu$, with $k << \Lambda_{QCD}$. The momentum p^μ is considered as on-shell and thus yields the following constraint: $u \cdot k = -k^2/(2m) \approx 0 + \mathcal{O}\left(1/m^2\right)$ in the large mass limit.

The goal is now to derive an effective Lagrangian, that captures the heavy quark interactions at the heavy mass limit. To do so we will follow the pedagogical treatment presented in [68].

3.2.1 The HQET Lagrangian

Our starting point is to consider a quark ψ, of mass m coupled to a gluon A_μ^a which is described by the Lagrangian:

$$\mathcal{L}_{QG} = \bar{\psi}\left(i\not{D} - m\right)\psi, \tag{3.9}$$

with the covariant derivative defined as $D = \partial_\mu - igA_\mu^a$, and we use the slashed notation $\not{y} \equiv \gamma^\mu y_\mu$. An effective description, is a *low energy* description of heavy mass quarks. This suggests the decomposition of the ψ field to high energy and low energy degrees of freedom:

$$\psi \to Q_{\text{LE}} + Q_{\text{HE}}. \tag{3.10}$$

On the other hand, in the limit of large m, quantum fluctuations of ψ around its classical value are suppressed and thus, to leading order one can assume that ψ satisfies the equations of motion. In momentum space, the Dirac equation [75] reads:

$$\left(\not{u} - 1\right)\psi = 0 + \mathcal{O}\left(1/m\right). \tag{3.11}$$

We may then proceed to integrate out[4], the high energy degrees of freedom Q_{HE} as follows:

$$\psi = \psi + \frac{\not{\psi}}{2} - \frac{\not{\psi}}{2} = \frac{1+\not{\psi}}{2}\psi - \frac{1-\not{\psi}}{2}\psi = \frac{1+\not{\psi}}{2}\psi + \mathcal{O}\left(1/m\right). \tag{3.12}$$

Replacing $\psi \to e^{-imv\cdot x}\frac{1+\not{v}}{2}Q$ in eq. 3.9 yields the heavy mass expansion:

$$\mathcal{L}_{m\to\infty} = i\bar{Q}v \cdot D\frac{1+\not{v}}{2}Q + \mathcal{O}\left(1/m\right). \tag{3.13}$$

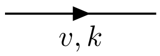
[4]A general way to obtain an effective Lagrangian, is to work in the path integral formulation and *integrate out* the undesired fields, achieving thus an effective description of the theory [68]. Colloquially, when a field is in the full Lagrangian but not in the effective one, we say that it has been integrated out.

The complete HQET Lagrangian[5] reads:

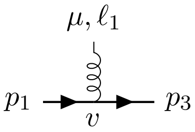
$$\mathcal{L}_{\text{HQET}} = -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 + \bar{Q} v \cdot D \frac{1+\not{v}}{2} Q + \mathcal{O}\left(1/m\right), \tag{3.14}$$

with $F^a_{\mu\nu}$ being the gluon field strength $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$, and f^{abc} being the structure constants of $SU(3)$.

The new Feynman rules for the quark propagator and the quark-gluon vertex, can be read directly from the Lagrangian:



$$\frac{i}{v \cdot k + i\varepsilon} \frac{1+\not{v}}{2},$$

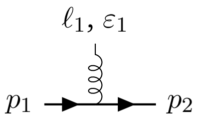


$$i g T^a v_\mu \frac{1+\not{v}}{2}, \tag{3.15}$$

where we have decomposed $A_\mu = A^a_\mu T^a$, in terms of the generators T^a of SU(3)(or SU(N) in general)[6]. The Feynman rules for the gluon-gluon vertices as well as the gluon propagator are the standard ones (see for example [68, 76, 77]). Finally we remark that these Feynman rules are universal, to leading order, since they are independent of the mass and the spin of the quark.

Having the Feynman rules at hand, we can start computing the gluon-matter (G-M) HEFT trees. For example the 3-point tree can be computed as:

$$A_3^{\text{G-M}}(123) = \tag{3.16}$$



$$= T^a \bar{Q}\left(\frac{1+\not{\varepsilon}}{2}\right) Q(\varepsilon_1 \cdot v) = T^a \bar{Q} Q(\varepsilon_1 \cdot v) = T^a m \varepsilon_1 \cdot v\ ,$$

where we use the shorthand ε_1 for $\varepsilon(\ell_1)$ for the polarization vector. It is evident that this amplitude, can be (trivially) separated to a color factor T^a containing only information regarding the gauge group of the theory, and a kinematic factor containing only information regarding the kinematics of the scattering. This is no coincidence. In

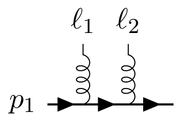
[5] we exclude light quarks since they are irrelevant for our purpose
[6] For a detailed treatment of non abelian gauge theory see [68, 76, 77]

a general gauge theory, one can decompose a tree amplitude with 2 quarks and $n-2$ gluons as [67] :

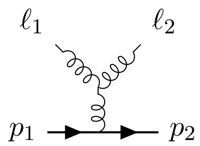
$$\mathcal{A}^{\text{tree}} = g^{n-2} \sum_{\sigma \in S_{n-2}} (T^{a_{\sigma(3)}} \ldots T^{a_{\sigma(n)}})_{i2}^{\bar{j}1} A^{\text{tree}} \left(1_{\bar{q}}^{\lambda_1}, 2_q^{\lambda_2}, \sigma\left(3^{\lambda_3}\right), \ldots, \sigma\left(n^{\lambda_n}\right)\right), \tag{3.17}$$

where the helicity of each particle has been kept explicit and denoted by $\lambda_i = \pm$. $(T^a)_i^{\bar{j}}$ are the generators of SU(N), and the sum is over the set of all permutations of n objects, S. The (color-striped) amplitude A^{tree} is referred to as partial amplitude [67, 78, 79], and it can be further decomposed to a linear combination of *color ordered* amplitudes [79]. These are amplitudes that receive contributions from diagrams with a specific ordering of gluons. Since our primary focus is on gravity amplitudes, we are interested in the color-stripped amplitudes instead of the full ones. For example, the color-stripped 4-point is given by:

$A_4^{\text{G-M}}(1234) =$



$+$



$$= 2m\Big(-\frac{\varepsilon_1 \cdot \ell_2 v \cdot \varepsilon_2}{s_{12}} - \frac{\varepsilon_1 \cdot \varepsilon_2 v \cdot \ell_1}{s_{12}} + \frac{\varepsilon_2 \cdot \ell_1 v \cdot \varepsilon_1}{s_{12}} + \frac{v \cdot \varepsilon_1 v \cdot \varepsilon_2}{2v \cdot \ell_1}\Big) . \tag{3.18}$$

The next step is to present the corresponding graviton-matter (GR-M) HEFT amplitudes from the gluon-matter ones described above.

3.2.2 Graviton-Matter HEFT Amplitudes

The graviton-matter HEFT amplitudes are tree amplitudes with 2 massive scalar and n massless graviton legs, that one obtains from a HEFT, approach to gravity, similar the one we described for fermions and gluons above. We refer interested reader to [80, 81] as well as [70] for a Lagrangian treatment.

In [24] it was shown how to construct manifestly gauge invariant BCJ numerators for the gluon-matter HEFT amplitudes. When cast in this form, one may use a simple prescription to obtain the corresponding graviton-matter HEFT trees. A prescription as such is called a double copy and it is a consequence of a duality between the color factors, and the kinematic factors of specific diagrams (with cubic or higher point vertices) in an amplitude. This duality is called the color-kinematics duality [82, 83]. Even though it is out of the scope of this work, we refer the interested reader in [84] for a review.

As we will see in the next section, the graviton-matter trees we need for loop calculations at 2PM and 3PM are the 3-point, 4-point, and 5-point trees, A_3^{GR-M}, A_4^{GR-M}, A_5^{GR-M}, which we will summarize below:

$$
\begin{aligned}
A_3^{\text{G-M}}(1,v) &= m\varepsilon_2\cdot v \xrightarrow{\text{D.C}} A_3^{\text{GR-M}}(1,v) = m^2(\epsilon_1\cdot v)^2,\\
A_4^{\text{G-M}}(12,v) &= \frac{m}{s_{12}}\left(\frac{v\cdot F_1\cdot F_2\cdot v}{v\cdot\ell_2}\right) \xrightarrow{\text{D.C}} A_4^{\text{GR-M}}(12,v) = \frac{m^2}{s_{12}}\left(\frac{v\cdot F_1\cdot F_2\cdot v}{v\cdot\ell_2}\right)^2,\\
A_5^{\text{G-M}}(123,v) &= \frac{\mathcal{N}_5([[1,2],3],v)}{s_{123}s_{12}} + \frac{\mathcal{N}_5([1,[2,3]],v)}{s_{123}s_{23}} \xrightarrow{\text{D.C}}\\
A_5^{\text{GR-M}}(123,v) &= \frac{\left[\mathcal{N}_5([[1,2],3],v)\right]^2}{s_{123}s_{12}} + \frac{\left[\mathcal{N}_5([[1,3],2],v)\right]^2}{s_{123}s_{13}} + \frac{\left[\mathcal{N}_5([[2,3],1],v)\right]^2}{s_{123}s_{23}}\,,
\end{aligned}
\tag{3.19}
$$

with $F_i^{\mu\nu} = \ell_i^\mu\varepsilon_i^\nu - \ell_i^\nu\varepsilon_i^\mu$, and $s_{ij\ldots} = (\ell_i+\ell_j+\ldots)^2$. Moreover, we use the notation $A\,(1,2,...)$, which is shorthand for $A(\ell_1,\ell_2\ldots)$.

The (BCJ) numerator $\mathcal{N}_n(([\ldots[[2,3],4],\ldots,n-1],v)$ (with $n=5$ in the last line of eq.3.19), can be obtained in terms of the *pre-numerator* $\mathcal{N}\,(1,2,3,\ldots,n-1,v)$, by acting on it with the operator [85–87]:

$$
\mathbb{L}(i_1,i_2,\ldots,i_r) \equiv \left[\mathbb{I}-\mathbb{P}_{(i_2i_1)}\right]\left[\mathbb{I}-\mathbb{P}_{(i_3i_2i_1)}\right]\cdots\left[\mathbb{I}-\mathbb{P}_{(i_r\ldots i_2i_1)}\right], \tag{3.20}
$$

as

$$
\mathcal{N}_n([\ldots[[2,3],4],\ldots,n-1],v) \equiv \mathbb{L}(2,3,4,\ldots,n-1)\circ\mathcal{N}_n(234\ldots n-1,v). \tag{3.21}
$$

The operator $\mathbb{I}$ is the identity element and the $\mathbb{P}_{(j_1,j_2,\ldots,j_m)}$ operator denotes the permutation $j_1\to j_2$, $j_2\to j_3$, …, $j_m\to j_1$:

$$
\begin{aligned}
\mathbb{P}_{(12)}\circ\mathcal{N}_5(123,v) &= \mathcal{N}_5(213,v),\\
\mathbb{P}_{(123)}\circ\mathcal{N}_5(123,v) &= \mathcal{N}_5(231,v).
\end{aligned}
\tag{3.22}
$$

There is not a unique choice for the pre-numerator $\mathcal{N}_n$. For instance in [23] and [24] the following pre-numerator was proposed:

$$\mathcal{N}_5(1,2,3,v) = 4m\frac{v\cdot F_1\cdot F_2\cdot V_2\cdot F_3\cdot v}{v\cdot \ell_2 v\cdot \ell_3}\,, \tag{3.23}$$

while in [88]:

$$\begin{aligned}\mathcal{N}_5\left(123,v_1\right) = m\frac{v_1\cdot F_1\cdot F_2\cdot F_3\cdot v_1}{v_1\cdot \ell_1} - m\frac{v_1\cdot F_1\cdot F_2\cdot V_{12}\cdot F_3\cdot v_1}{v_1\cdot \ell_1 v_1\cdot \ell_{12}} \\ - m\frac{v_1\cdot F_1\cdot F_3\cdot V_1\cdot F_2\cdot v_1}{v_1\cdot \ell_1 v_1\cdot \ell_{13}},\end{aligned} \tag{3.24}$$

with $V_\tau^{\mu\nu} = v_1^\mu \sum_{j\in\tau} \ell_j^\nu = v_1^\mu \ell_\tau^\nu$ and $\ell_{ij}^\mu = \ell_i^\mu + \ell_j^\mu$. For our computations, we use the latter.

3.3 Heavy Mass Expansion For Gravity Amplitudes

Having the graviton-matter trees at hand, we will now describe how to build integrands, using these trees as building blocks. This can be achieved by considering a heavy mass expansion of a full tree level gravity amplitude in terms of the HEFT trees. A loop amplitude may then be constructed with the terms of these expanded trees, with the use of unitarity cuts.

As we will see, not all the integrands that can be created contribute to classical physics. To decide whether an amplitude is classical or not, one rescales $q \to \hbar q$ and $\ell \to \hbar \ell$, and counts the overall $\hbar$ scaling. Following [70], we require a classical amplitude to scale as the Newtonian potential in momentum space. Using the non-relativistic normalization of external states:

$$\langle p_1|p_2\rangle = (2\pi)^3\,\delta^3\left(\vec{p_2}-\vec{p_1}\right), \tag{3.25}$$

the Newtonian potential[7] in momentum space reads:

$$V = -\frac{4\pi G_n m_1 m_2}{\hbar^3 q^2}. \tag{3.26}$$

[7] The Newtonian potential can be obtained as the leading order non relativistic correction to the graviton exchange diagram at tree-level.

We note that Newton's constant G_N contributes $\hbar^{-1}$, while q, contributes $\hbar$. Thus, the overall scaling is $\hbar^3$. This scaling should hold at all loops. Therefore, a classical L-Loop integral (not integrand) should scale as $\hbar^{L-2}$.

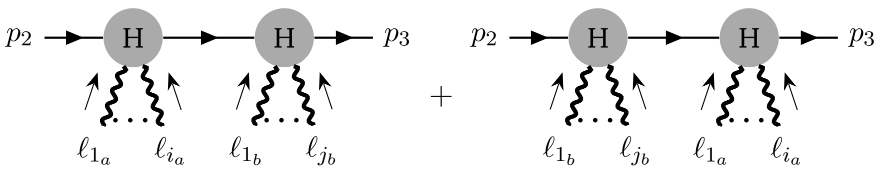
Keeping this discussion in mind, we will proceed to show the heavy mass expansion (HEFT expansion) of gravity trees. This expansion, was presented in [23] which we will follow.

3.3.1 The HEFT expansion

The starting point, is to explore how to join two HEFT amplitudes together, to a larger tree. Since gravity amplitudes are not color ordered, one has to consider all the possible orderings of the gravitons. Moreover, we use the scalar propagator which is proportional to:

$$\frac{1}{(p+\mathrm{L})^2 - m^2 + i\epsilon}. \tag{3.27}$$

Schematically, the new amplitude will be :



(3.28)

Since we are looking for a mass expansion, we need to expand the propagator in eq. 3.27, with $p = p_2$, and L being the sum of the graviton momenta. We will first rewrite eq. 3.27, using the barred variables introduced in section 3.1, $p = \bar{p} \pm \frac{q}{2}$ where $\bar{p} \cdot q = 0$ and $\bar{p}^2 = m^2 - \frac{q^2}{4}$:

$$\frac{1}{\left(\bar{p} \pm \frac{q}{2} + \mathrm{L}\right)^2 + i\varepsilon - m^2} = \frac{1}{2\bar{p} \cdot \mathrm{L} \pm q \cdot \mathrm{L} + \mathrm{L}^2 + i\varepsilon}. \tag{3.29}$$

Now we notice that $\mathrm{L}^2 \sim \ell^2 << 1$. Moreover, since $q << \bar{p}$ and $q \cdot \ell << \bar{p} \cdot \ell$, we can expand eq. 3.29 for small $\mathrm{L}^2 \pm q \cdot \mathrm{L}$ to get:

$$\frac{1}{2\bar{p} \cdot \mathrm{L} \pm q \cdot \mathrm{L} + \mathrm{L}^2 + i\varepsilon} = \frac{1}{2\bar{p}\mathrm{L} + i\epsilon}\left(1 - \frac{\pm q \cdot \mathrm{L} + \mathrm{L}^2}{2\bar{p} \cdot \mathrm{L}} + \ldots\right) \approx \frac{1}{2\bar{p}\mathrm{L} + i\varepsilon}. \tag{3.30}$$

Considering now the sum of the two diagrams in eq. 3.28, we have:

$$\begin{aligned}&\left(\frac{1}{2\bar{p}\cdot \mathrm{L}_a\,+\,i\,\varepsilon}+\frac{1}{-2\bar{p}\cdot \mathrm{L}_a\,+\,i\,\varepsilon}\right)A_{i+2}(1_a\ldots i_a,\bar{v}_2)\,A_{j+2}(1_b\ldots j_b,\bar{v}_2)\\&=-(2\pi i)\,\delta\big(2\bar{p}\cdot\mathrm{L}_a\big)\,A_{i+2}(1_a\ldots i_a,\bar{v}_2)\,A_{j+2}(1_b\ldots j_b,\bar{v}_2)\,,\end{aligned}\tag{3.31}$$

with $\mathrm{L}_a=\ell_{a_1}+\ldots\ell_{a_r}$ and $\mathrm{L}_b=\ell_{a_1}+\ldots\ell_{a_r}=-\mathrm{L}_a+q$. This suggest the decomposition of a full gravity amplitude with 2 massive legs and $n-2$ gravitons, to a sum over all possible partitions of $n-2$ gravitons, $\mathsf{P}(n-2,h)$ into h non empty sets, joined with the procedure described above :

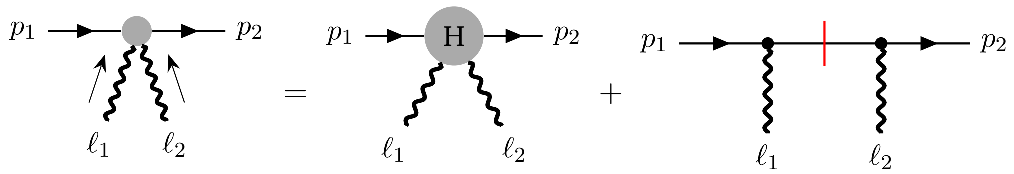
$$\begin{aligned}&\sum_{h=1}^{n-2}\sum_{\mathbf{P}\in\mathsf{P}(n-2,h)}\left(\prod_{j=1}^{h-1}(-2\pi i)\delta\left(2\bar{m}_2\bar{v}_2\cdot\ell_{\mathbf{P}_j}\right)\right)A_{i_1+2}(\mathbf{P}_1,\bar{v}_2)\ldots A_{i_h+2}(\mathbf{P}_h,\bar{v}_2)\\&+\ldots\end{aligned}\tag{3.32}$$

It is evident that for $h=1$ we get a term without any delta functions. This term is the HEFT amplitude. Moreover, this expansion is indeed an inverse mass expansion: the HEFT trees A_{i+2} scale as m^2 and thus, the expansion goes as $m^{n-1}+\ldots+m^2+\mathcal{O}(m)$.

Before we discuss loops, let us give as examples, the decomposition of the 4-point and 5-point trees with two massive and 2(3) gravitons:

Example: HEFT expansion of the 4-point and the 5-point trees

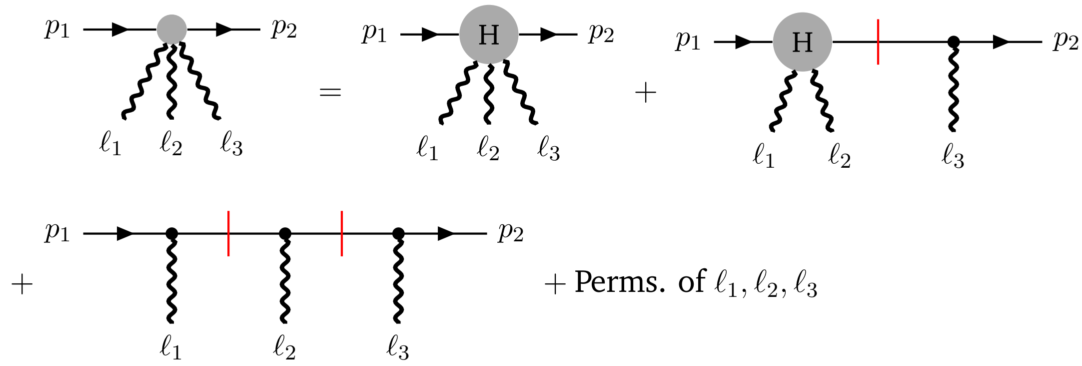
For the 4-point, we have the following schematic decomposition:



$$\begin{aligned}&=-(i\pi)\bar{m}_2^3\delta(\bar{v}_1\cdot\ell_1)(\bar{v}_1\cdot\varepsilon_1)^2(\bar{v}_1\cdot\varepsilon_2)^2\\&+\frac{\bar{m}_2^2}{\ell_{12}^2}\Big(\frac{\bar{v}_1\cdot f_1\cdot f_2\cdot\bar{v}_1}{\bar{v}_1\cdot\ell_2}\Big)^2+\cdots\end{aligned}\tag{3.33}$$

The gray blob is the full gravity amplitude, while the gray blobs with an H are the HEFT trees. The red line denotes the delta function $\delta(\bar{v}_2\cdot\ell_1)$.

The 5-point tree has a similar decomposition:



$$
\begin{aligned}
&= (-i\pi)^2\, \bar{m}_2^4 \delta(\bar{v}_1\cdot\ell_1)\delta(\bar{v}_1\cdot\ell_2)(\bar{v}_1\cdot\varepsilon_1)^2(\bar{v}_2\cdot\varepsilon_2)^2(\bar{v}_1\cdot\varepsilon_3)^2 \\
&- \frac{i\pi\, \bar{m}_2^3}{\ell_{12}^2}\delta(\bar{v}_1\cdot\ell_{12})\left(\frac{\bar{v}_1\cdot F_1\cdot F_2\cdot\bar{v}_1}{\bar{v}_1\cdot\ell_2}\right)^2(\bar{v}_1\cdot\varepsilon_3)^2 \\
&- \frac{i\pi\, \bar{m}_2^3}{\ell_{23}^2}\delta(\bar{v}_1\cdot\ell_{23})\left(\frac{\bar{v}_1\cdot F_2\cdot F_3\cdot\bar{v}_1}{\bar{v}_1\cdot\ell_3}\right)^2(\bar{v}_1\cdot\varepsilon_1)^2 \\
&- \frac{i\pi\, \bar{m}_2^3}{\ell_{13}^2}\delta(\bar{v}_1\cdot\ell_{13})\left(\frac{\bar{v}_1\cdot F_1\cdot F_3\cdot\bar{v}_1}{\bar{v}_1\cdot\ell_3}\right)^2(\bar{v}_1\cdot\varepsilon_2)^2 + A_5(123,\bar{v}_1) + \cdots
\end{aligned}
\tag{3.34}
$$

In the next section we will use this decomposition to build the loop integrals.

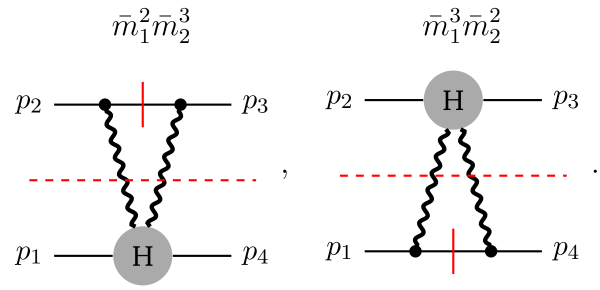
3.4 Loops

As we have already discussed, we are interested in the classical contributions to our observables and thus, the classical diagrams. These are the ones who scale as $\hbar^{L-2}$, at L-Loops. Since the HEFT trees in gravity A_n are scaleless in $\hbar$, we can determine the classical contributions by considering the number of graviton cuts, and the number of delta functions $\delta(v\cdot\ell)$ that can give the correct $\hbar$ scaling.

At 1-loop, we want our integral to go as $\hbar^{-1}$. Thus, the only possible way, is a two graviton cut and one delta function[8]. Looking at our expansions, we see that

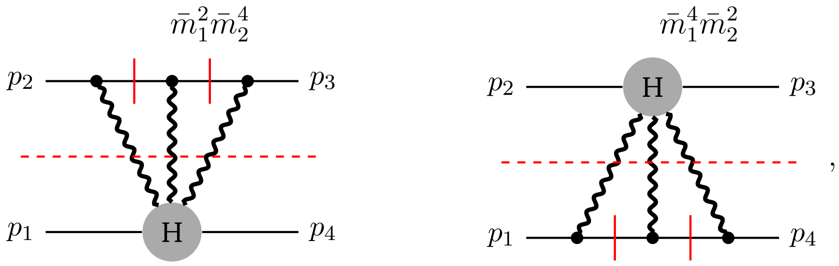
[8]Don't forget to take into account the integration measure in this analysis

the classical diagrams at 1-loop are of order $\mathcal{O}(m^5)$ in the mass expansion. The corresponding diagrams are:

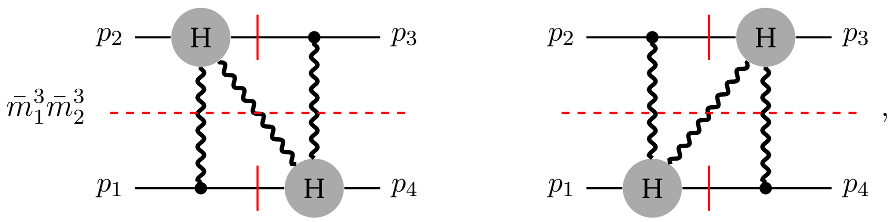


(3.35)

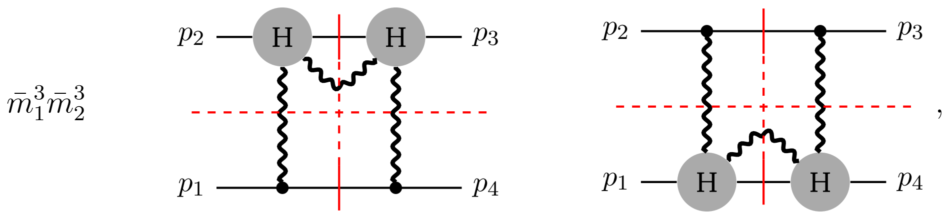
At 2-loops, the integrals should scale as $\hbar^0$, and thus, the classical diagrams are the ones with three graviton cuts and two delta functions (or of order $\mathcal{O}(m^6)$):



(3.36)

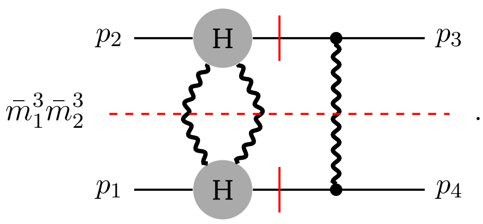


(3.37)



(3.38)

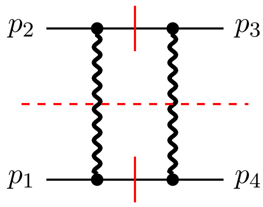
and



(3.39)

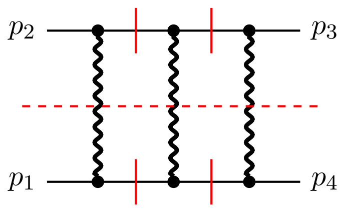
Technically, the diagram with a four graviton cut (and an additional four graviton vertex) and two delta functions, is also of the correct order in the mass expansion ($\mathcal{O}\left(m^6\right)$) and thus classical. However it was shown in [23], that it doesn't contribute anything new to the classical observables.

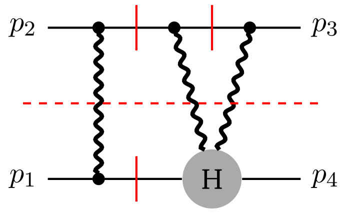
The diagrams with the classical scaling, are not the leading order terms in the mass expansion. Indeed looking at the expansions, in eqs. 3.33 and 3.34 we see that in 1-loop the leading diagram is:

$\bar{m}_1^3\bar{m}_2^3$ 

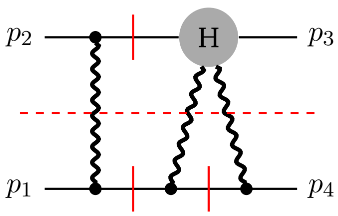
. (3.40)

At 2-loops the leading diagrams are:

$\bar{m}_1^4\bar{m}_2^4$


$\bar{m}_1^3\bar{m}_2^4$


,

$\bar{m}_1^4\bar{m}_2^3$ 

. (3.41)

These diagrams are two-massive-particle reducible, and even though their scaling is *hyper-classical*, expanding $\bar{m}$ in the heavy mass limit, within these contributions produces classical feed-down terms. This happens because the barred variables $\bar{m}^2 = m^2 - q^2/2$ mix quantum and classical terms. Even though it may seem as a problem, it is not. As we will see in the next section, two-massive-particle reducible diagrams trivially factorize in the impact parameter space, and thus their contributions can be absorbed by the classical diagrams of equations 3.35, 3.36,3.37,3.38 and 3.39.

3.5 The HEFT Phase

In the first chapter, it was discussed that observables in HEFT, may be obtained by Fourier transforming an L-loop amplitude $\mathcal{M}(q)$, to the impact parameter space (IPS). This gives an exponentiated expression for the $\mathbb{S}$-matrix:

$$\mathbb{S} = 1 + \tilde{\mathbb{S}} = e^{i\chi}. \tag{3.42}$$

Expanding this to a certain loop order, L, one can calculate the corresponding contribution to the bending angle, $\theta^{(L)}$ as:

$$\theta^{(L)} = -\frac{\partial \mathrm{Re}\chi}{\partial J}. \tag{3.43}$$

In this section we will formalize the computation of observables in the ocntext of HEFT. We remind the reader that we are still following [23].

3.5.1 Two-Massive-Particle Reducible Diagrams And The HEFT Phase

In HEFT, the IPS Fourier transform is defined in terms of the barred variables $\bar{m}_i$ and $\bar{p}_i$ as:

$$\tilde{\mathcal{M}}(b) \equiv \int \frac{d^D q}{(2\pi)^{D-2}} \delta\left(2\bar{p}_1 \cdot q\right) \delta\left(2\bar{p}_2 \cdot q\right) e^{ibq} \mathcal{M}(q). \tag{3.44}$$

The resulting exponentiated expression for the S-Matrix reads:

$$\begin{gathered} \mathbb{S} = 1 + \tilde{\mathbb{M}} = e^{i\chi_{\text{HEFT}}}, \\ \chi_{\text{HEFT}} = \bar{\chi}^{(0)} + \bar{\chi}^{(1)} + \bar{\chi}^{(2)} + \dots \end{gathered} \tag{3.45}$$

The bars indicate that the resulting HEFT eikonal $\bar{\chi}^{(i)}$ is expressed in the barred variables.

Now that we have refined the computation of the eikonal, it is easy to see that a two-massive-particle reducible diagram is a convolution integral in momentum space and thus factorize in IPS as follows [23]:

$$\int \frac{d^D \ell}{(2\pi)^D} (-2\pi i)^2 \delta(\bar{p}_1 \cdot \ell) \delta(\bar{p}_2 \cdot \ell) \mathcal{M}_{\mathrm{L}}(\ell) \mathcal{M}_{\mathrm{R}}(q - \ell) \xrightarrow{\text{IPS}} -\widetilde{\mathcal{M}}_{\mathrm{L}}(b) \widetilde{\mathcal{M}}_{\mathrm{R}}(b). \tag{3.46}$$

Most importantly, this factorization, is the product of two-massive-particle *irreducible*(2MPI) diagrams. These are the classical (and quantum but we are not interested in those) diagrams of equations 3.35, 3.36,3.37,3.38 and 3.39, that we saw before. It is thus evident that the two-massive-particle reducible diagrams do not contribute any new information to classical physics. More concretely, in [23], it was shown that if one considers the contribution of all the 2MPI and two-massive-particle reducible diagrams, up to 2-loops, the S-Matrix takes the form:

$$\mathbb{S} = e^{i\mathcal{M}^{(0)} + i\mathcal{M}^{(1)}_{(2\text{MPI})} + i\mathcal{M}^{(2)}_{(2\text{MPI})} + \text{Quantum Terms}}. \tag{3.47}$$

The eikonal in HEFT, may now be defined the exponent in eq. 3.47. That is, the sum of all the 2MPI diagrams.

Having a clear picture of what amplitudes contribute to our observables, we may now perform one last step. Since we are working in the heavy mass limit, where $m \gg q$, we may drop the bar from our variables, since $\bar{m}^2 = m^2 + q^2 \approx m^2$. Thus in the heavy mass limit, $\bar{p}_i, \bar{m}_i, \bar{y}_i \to p_i, m_i, y_i$. Even though the barred variables may appear again in the next chapter, where we will extend the current formalism to incorporate spinning trees, in chapters 6 and 7 we will only use the unbarred variables.

4 Spinning Heavy Mass Effective Theory

To make our study of black hole binary dynamics more realistic, we must incorporate scattering amplitudes that describe objects with classical spin. In the previous chapter we saw that, to study the scattering of two black holes, it is natural to work in an effective theory framework similar to that of heavy quarks. In particular, we discussed that one can find manifestly gauge invariant BCJ numerators for the gluon-matter HEFT amplitudes. These trees, when expressed in terms of the BCJ numerators, exhibit a simple double copy structure that allowed us to obtain the graviton-matter amplitudes in the heavy mass limit. The advantage of this framework is the fact that the expressions for the graviton-matter trees are compact. Moreover, as we saw in section 3.5, the number of diagrams one has to compute is minimal, since only the 2MPI diagrams contribute to the eikonal phase. These properties make HEFT very handy for loop computations, and for that reason we would like to extend it in order to facilitate the calculation of spinning observables as well. That is, we want to be able to write down Spinning-HEFT (S-HEFT) trees.

The 3-point spinning amplitude, with two massive legs and one graviton, in the heavy mass limit (the 3-point spinning-HEFT tree) was given in [89–92]. Higher point trees in spinning-HEFT were approached, using a bootstrap, by general considerations on symmetry and locality requirements, in [49, 50]. In particular, in [49, 50], the 4-point spinning-HEFT amplitude was given, where the contact contributions where determined by matching with physical data [93, 94] at $\mathcal{O}\left(a^4\right)$ and at $\mathcal{O}\left(a^5\right)$ and then extrapolating an all order in spin contribution. A discussion in the agreement of this result with known results in the bibliography was given in [49] and furthered in [52].

In what follows, we will review the spinning-HEFT $3-$point and the $4-$point trees following mostly [49, 50].

4.1 The Pauli Lubanski Pseudo Vector

The first step in order to write down spinning amplitudes, is to decide how to parametrize the (classical) spin of our particles. In the context of the formalism that we use, authors typically parametrize the classical spin vector of a particle in its (asymptotical) rest frame (see for example [49–52, 58]), with the use of the Pauli-Lubanski vector, which may be obtained by inverting the relation:

$$S^{\mu\nu}(p) = -\frac{1}{m_i}\epsilon^{\mu\nu\rho\sigma}p_\rho s_\sigma\left(p_i\right), \quad S^{\mu\nu}p_\nu = 0. \tag{4.1}$$

From a technical perspective, eq. 4.1 is all we need in order to talk about spinning amplitudes. However, before we do that, we will briefly present a discussion that we found interesting on what operator in quantum mechanics is related to the classical spin of a particle in eq. 4.1. We found this discussion in [22].

The authors of [22], propose that the classical spin vector of eq. 4.1, is the expectation value of the Pauli-Lubanski operator $\mathbb{W}$:

$$\langle s^\mu\rangle = \frac{1}{m}\langle\mathbb{W}^\mu\rangle, \tag{4.2}$$

with

$$\mathbb{W}_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathbb{P}^\nu\mathbb{J}^{\rho\sigma}. \tag{4.3}$$

The operator $\mathbb{P}^\nu$ is the translation generator and $\mathbb{J}^{\rho\sigma}$ is the Lorentz generator, and they obey the commutation relations:

$$\begin{aligned}[\mathbb{J}^{\mu\nu},\mathbb{P}^\rho] &= i(\eta^{\mu\rho}\mathbb{P}^\nu - \eta^{\nu\rho}\mathbb{P}^\mu),\\ [\mathbb{J}^{\mu\nu},\mathbb{J}^{\rho\sigma}] &= i(\eta^{\nu\rho}\mathbb{J}^{\mu\sigma} - \eta^{\mu\rho}\mathbb{J}^{\nu\sigma} - \eta^{\nu\sigma}\mathbb{J}^{\mu\rho} + \eta^{\mu\rho}\mathbb{J}^{\mu\sigma}),\end{aligned} \tag{4.4}$$

from which one can show that:

$$[\mathbb{P}^\mu,\mathbb{W}^\nu] = 0. \tag{4.5}$$

Moreover, contracting eq. 4.3 with $\mathbb{P}^\mu$ and using the fact that $\epsilon^{\alpha\beta\mu\nu}\mathbb{P}_\alpha\mathbb{P}_\beta = 0$, it becomes apparent that :

$$\mathbb{P}^\mu\mathbb{W}_\nu = 0. \tag{4.6}$$

This relation, along with the well-known property of commutators $[A, BC] = A[B, C] + [A, C]B$ can be used, to show that:

$$
\begin{aligned}
&0 = [\mathbb{J}_{\mu\nu}, \mathbb{W}^\rho \mathbb{P}_\rho] = \mathbb{W}^\rho[\mathbb{J}_{\mu\nu}, \mathbb{P}_\rho] + [\mathbb{J}_{\mu\nu}, \mathbb{W}^\rho]\mathbb{P}_\rho = [\mathbb{J}_{\mu\nu}, \mathbb{W}^\rho]\mathbb{P}_\rho - i\mathbb{W}^\rho\left(\eta_{\mu\rho}\mathbb{P}_\nu - \eta_{\nu\rho}\mathbb{P}_\mu\right) \\
&\quad \therefore [\mathbb{J}^{\mu\nu}, \mathbb{W}^\rho] = i(\eta^{\mu\rho}\mathbb{W}^\nu - \eta^{\nu\rho}\mathbb{W}^\mu).
\end{aligned}
\tag{4.7}
$$

Combining eqs. 4.5,4.7 with 4.4, it follows that:

$$
[\mathbb{W}^\mu, \mathbb{W}^\nu] = -i\epsilon^{\mu\nu\rho\sigma}\mathbb{W}_\rho\mathbb{P}_\sigma. \tag{4.8}
$$

As presented in [22], the last relation expressed in a single particle's rest frame (where $\mathbb{W}^0 = 0$), becomes:

$$
[\mathbb{W}^i, \mathbb{W}^j] = -i\epsilon^{ijk}\mathbb{W}^k, \tag{4.9}
$$

which makes it evident that the components $\mathbb{W}^\mu$ are the generators of the little group.

4.1.1 Spin in the Heavy Mass Limit

In section 3.1, we introduced the classical spin tensor as $S^{\mu\nu} = -\epsilon^{\mu\nu\rho\sigma}\bar{p}_\rho a_\sigma$. Now that we have a concrete picture of what our spin vector is, we may proceed and derive this relation by considering the spin tensor in equation 4.1 in the heavy mass limit. The main idea is that a Lorentz boost $\Lambda^\mu_\nu \bar{p}_i^\nu = \bar{p}_i + q^\mu/2$, in the heavy mass limit $\bar{p} \gg q$, is infinitesimal as explained in [22]. However the derivation we are going to show now can be found in [58].

Our starting point is the spin tensor for a massive particle with incoming momentum p_i, outgoing momentum p'_i:

$$
\begin{aligned}
S_i^{\mu\nu}(p_i) &= -\frac{1}{m_i}\epsilon^{\mu\nu\rho\sigma}p_{i\rho}s_{i\sigma}(p_i)\,, \\
S_i^{\mu\nu}(p'_i) &= -\frac{1}{m_i}\epsilon^{\mu\nu\rho\sigma}p'_{i\rho}s_{i\sigma}(p'_i)\,.
\end{aligned}
\tag{4.10}
$$

Before we continue, we remind the reader that the momentum of an incoming particle is $p_{i=1,2} = \bar{p}_i \pm \frac{q^\mu}{2}$ and the momentum of an outgoing particle is $p_{i=3,4} = \bar{p}_i \pm \frac{q^\mu}{2}$.

We want to express the spin vector in the barred variables $s_i^\mu(\bar{p} + q/2)$ and to do so we notice that this object is a Lorentz boost of $s_i^\mu(\bar{p})$. Let us consider an infinitesimal

Lorentz transformation $\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu$, with $\Lambda^\mu_\nu \bar{p}^\nu_i = \bar{p}_i + q^\mu/2$. It follows that $\omega^\mu_\nu \bar{p}^\nu = q^\mu/2$, and one may choose the following form for ω [22]:

$$\omega^\mu_\nu = \frac{1}{2\bar{m}^2}\left(\bar{p}^\mu q_\nu - q^\mu \bar{p}_\nu\right). \tag{4.11}$$

We now perform a boost on the spin vector $s^\mu_i\left(\bar{p}_i\right)$:

$$\begin{aligned} s^\mu_i\left(\bar{p} \pm q/2\right) &= \left(\delta^\mu_\nu \pm \omega^\mu_\nu\right) s^\nu_i\left(\bar{p}\right) \\ &= \left(\delta^\mu_\nu \mp \frac{1}{2\bar{m}_i^2}\left(\bar{p}^\mu_i q_\nu - q^\mu \bar{p}_{i\nu}\right)\right) s^\nu_i\left(\bar{p}\right) \\ &= s^\mu_i\left(\bar{p}_i\right) \mp \frac{\bar{p}^\mu_i}{2\bar{m}_i^2} q \cdot s_i\left(\bar{p}\right) + \mathcal{O}\left(\bar{m}^{-3}\right), \end{aligned} \tag{4.12}$$

where we have used the fact that $\bar{p}_i \cdot s_i\left(\bar{p}\right) = 0$. Going back to eq. 4.10, we may now insert the heavy mass expansion of the spin we just derived, to get:

$$\begin{aligned} S^{\mu\nu}_i(p_i) &= -\frac{1}{m_i}\epsilon^{\mu\nu\rho\sigma} p_{i\rho} s_{i\sigma}\left(p_i\right) = \\ &- \frac{1}{\bar{m}_i}\epsilon^{\mu\nu\rho\sigma}\left(\left(\bar{p}_{i\rho} + \frac{q^\rho}{2}\right)\left(s_{i\sigma}\left(\bar{p}\right) - \frac{\bar{p}_{i\sigma}}{2\bar{m}_i^2} q \cdot s_i\left(\bar{p}\right) + \mathcal{O}\left(\bar{m}^{-3}\right)\right)\right) \\ &= -\frac{1}{\bar{m}_i}\epsilon^{\mu\nu\rho\sigma}\bar{p}_{i\rho}\bar{s}_{i\sigma} - \frac{1}{2\bar{m}_i}\epsilon^{\mu\nu\rho\sigma} q_\rho \bar{s}_{i\sigma} + \mathcal{O}\left(\bar{m}^{-2}\right), \end{aligned} \tag{4.13}$$

where we have defined $\bar{s}_i = s_i\left(\bar{p}_i\right)$, and we work exactly in the same fashion for $S^{\mu\nu}_i(p'_i)$.

We now define the classical spin parameter a^μ as

$$a^\mu_i = \frac{\bar{s}^\mu_i}{\bar{m}_i}, \tag{4.14}$$

and the spin tensors can be re-written as:

$$\begin{aligned} S^{\mu\nu}_i(p_i) &= -\epsilon^{\mu\nu\rho\sigma}\left(\bar{p}_{i\rho} + \frac{q_\rho}{2}\right) a_{i\sigma} + \mathcal{O}\left(\bar{m}^{-2}\right), \\ S^{\mu\nu}_i(p'_i) &= -\epsilon^{\mu\nu\rho\sigma}\left(\bar{p}_{i\rho} - \frac{q_\rho}{2}\right) a_{i\sigma} + \mathcal{O}\left(\bar{m}^{-2}\right). \end{aligned} \tag{4.15}$$

The final step is to drop the q, since in the heavy mass limit $\bar{p} \gg q$, and we obtain the expression of eq. 3.5:

$$S^{\mu\nu}_i = -\epsilon^{\mu\nu\rho\sigma}\bar{p}_{i\rho} a_{i\sigma} + \mathcal{O}\left(\bar{m}^{-2}\right). \tag{4.16}$$

Now that we have a well-defined spin vector in the heavy mass limit, we can use it to parametrize the spin for the tree amplitudes in spinning-HEFT. In the next section we will present the 3-point and 4-point tree amplitudes in spinning-HEFT. Since the goal of this project is to explore the 3PM calculations involving spin, and due to the fact that the 5-point tree in spinning-HEFT is known only up to $\mathcal{O}(a^2)$, we will restrict ourselves to spinning-spinless binary dynamics at 3PM in the probe limit. For this purpose we only need the spinning-HEFT 3-point. However, in order to have a complete presentation, we will also present the 4-point.

4.2 Tree Level Spinning-HEFT Amplitudes

4.2.1 Spinning-HEFT 3-Point

We will start by presenting the minimally coupled 3-point amplitude $\mathcal{A}_a(1,v)$[1], in gauge theory, for arbitrary spin a, expressed in terms of the color kinematic numerator $\mathcal{N}_a(1,v)$, as shown in [89–92]:

$$\mathcal{N}_a(1,v) = m(v\cdot\varepsilon_1)\exp\left(-i\frac{\ell_1\cdot S\cdot\varepsilon_1}{mv\cdot\varepsilon_1}\right), \tag{4.17}$$

where S is defined at eq. 4.16. We may now define the 3-point tree in the heavy mass limit through equation 4.17, as [50]:

$$\lim_{m\to\infty}\mathcal{A}_a(1,\bar{2},\bar{3}) = \mathcal{N}_a(1,v), \tag{4.18}$$

where the barred numebers $\bar{1},\bar{2}$ represent massive particles. This amplitude, presents a double copy structure as shown in [95], that we may exploit to obtain the corresponding gravity 3-point $\mathcal{M}_a(1,v)$:

$$\mathcal{M}_a(1,v) = \mathcal{N}_0(1,v)\mathcal{N}_a(1,v), \tag{4.19}$$

where $\mathcal{N}_0(1,v)$ is the spinless gluon-matter HEFT tree given in eq. 3.19.

Before we proceed any further, let us note that the use of minimally coupled amplitudes is due to the fact that these kind of amplitudes where able to reproduce the known results of scattering from a Kerr black hole. More precisely, in [96], the impulse that

[1]we are using again the shorthand $\mathcal{A}_a(1,2,\dots,)$ for $\mathcal{A}_a(\ell_1,\ell_2,\dots,)$.

a probe particle experiences due to a Kerr black hole was derived, using minimally coupled 3-point amplitudes in the large mass limit. Interestingly enough, that result was derived from the impulse that a probe feels due to the scattering from a charge distribution[2] through the double copy.

We now turn our attention, back to the expression for $\mathcal{N}_a$ in eq. 4.17 and in particular, to the singularity that appears in the denominator of the exponent. Following [50], we will prove that this pole is spurious, by removing it.

We begin by writing the exponential in its power series representation, separating the even from the odd powers:

$$\begin{aligned}\exp\left(-i\frac{\ell_1\cdot S\cdot\varepsilon_1}{mv\cdot\varepsilon_1}\right) &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(i\frac{\ell_1\cdot S\cdot\varepsilon_1}{mv\cdot\varepsilon_1}\right)^n \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{\ell_1\cdot S\cdot\varepsilon_1}{mv\cdot\varepsilon_1}\right)^{2n} \\ &- i\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{\ell_1\cdot S\cdot\varepsilon_1}{mv\cdot\varepsilon_1}\right)^{2n+1}.\end{aligned} \tag{4.20}$$

The products $(\ell_1\cdot S\cdot\varepsilon_1)^2 n$ can be simplified using the identity :

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta} = -\delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta}. \tag{4.21}$$

As a matter of fact, one has to work only to simplify the product $(\ell_1\cdot S\cdot\varepsilon_1)^2$. Inserting the expression for the spin tensor of eq. 3.5, and the identity of eq. 4.21, yields:

$$(\ell_1\cdot S\cdot\varepsilon_1)^2 = -m^2(v\cdot\varepsilon_1)^2(p_1\cdot a)^2. \tag{4.22}$$

Note the on-shell conditions: $\ell_1^2 = 0$, $\ell_1\cdot\varepsilon_1 = 0$, $\varepsilon_1^2 = 0$, and $v\cdot\ell_1 = 0$. Higher even powers can be (trivially) computed as:

$$(\ell_1\cdot S\cdot\varepsilon_1)^{2n} = \left((\ell_1\cdot S\cdot\varepsilon_1)^2\right)^n = (-1)^n m^{2n}(v\cdot\varepsilon_1)^{2n}(p_1\cdot a)^{2n}. \tag{4.23}$$

The odd powers can be reorganized by pulling one power out of the sum:

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{\ell_1\cdot S\cdot\varepsilon_1}{mv\cdot\varepsilon_1}\right)^{2n+1} = \left(\frac{\ell_1\cdot S\cdot\varepsilon_1}{mv\cdot\varepsilon_1}\right)\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{\ell_1\cdot S\cdot\varepsilon_1}{mv\cdot\varepsilon_1}\right)^{2n}. \tag{4.24}$$

[2]The corresponding electromagnetic field, derived from the (minimally coupled) 3-points was termed by the authors of [96] as $\sqrt{\text{Kerr}}$.

Putting it all together, the exponential can be rewritten as:

$$\begin{aligned}\exp\left(-i\frac{\ell_1\cdot S\cdot\varepsilon_1}{mv\cdot\varepsilon_1}\right) &= \sum_{n=0}^{\infty}\frac{(\ell_1\cdot a)^{2n}}{(2n)!} - i\left(\frac{\ell_1\cdot S\cdot\varepsilon_1}{mv\cdot\varepsilon_1}\right)\sum_{n=0}^{\infty}\frac{(\ell_1\cdot a)^{2n+1}}{(2n+1)!} \\ &= \cosh(\ell_1\cdot a) - i\left(\frac{\ell_1\cdot S\cdot\varepsilon_1}{mv\cdot\varepsilon_1}\right)G(\ell_1\cdot a),\end{aligned} \tag{4.25}$$

where G is the entire function:

$$G(x) \equiv \frac{\sinh x}{x}. \tag{4.26}$$

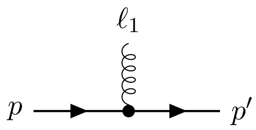
Note that as we will see in chapter 6, it is useful to introduce the integral representation of $G(x)$ as:

$$G(x) = \int_0^1 d\sigma \cosh\left(x\sigma\right). \tag{4.27}$$

This concludes the removal of the (spurious) singularity from the exponent at eq. 4.17.

We can use eq. 4.25, to write the 3-point, in gauge theory, and in gravity in the compact forms:

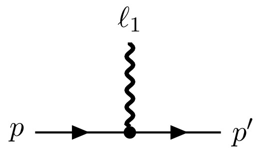
$$\mathcal{A}_a(1,v) =$$



$$= \bar{w}\cdot\varepsilon_1, \tag{4.28}$$

and,

$$\mathcal{M}_a(1,v) =$$



$$= \bar{m}(v\cdot\varepsilon_1)(\bar{w}\cdot\varepsilon_1). \tag{4.29}$$

with $p = \bar{p} + q/2$, $p' = \bar{p} - q/2$, and the vector w^μ is defined as:

$$w^\mu = m\cosh(\ell_1\cdot a)v^\mu - iG(\ell_1\cdot a)\left(\ell_1\cdot S\right)^\mu. \tag{4.30}$$

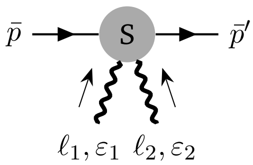
The bar in $\bar{w}$ signifies that w is written in terms of the barred variables.

Note that in the limit $a\to 0$, one recovers the spinless tree $\mathcal{A}_0(1,v)$, since $S\to 0$ in that limit. Moreover, gauge invariance is manifest, due to the on shell condition $\ell_1\cdot v = 0$.

4.2.2 Spinning-HEFT 4-Point

For completeness, we will now present the 4-point tree amplitude, in the heavy mass limit. A covariant form of this amplitude was presented in [49, 50] where the interested reader can find detailed discussions regarding its derivation. In this sub-section we will provide the result for the amplitude as well as an overview of its derivation.

To have a clear picture of the involved kinematics, we show a diagram of the 4-point amplitude in spinning-HEFT below:



$$. \tag{4.31}$$

It was shown in [49] that this amplitude receives three contributions, and schematically it looks as follows:

$$\mathcal{M}_4(1,2,\bar{p}',\bar{p}) = -\frac{\mathcal{N}_a(1,2,\bar{p}',\bar{p})\,\mathcal{N}_0(1,2,\bar{p},\bar{p}')}{2(\ell_1\cdot\ell_2)} + \frac{\mathcal{N}_{\rm r}(1,2,\bar{p}',\bar{p})}{4(\bar{p}\cdot\ell_1)(\bar{p}\cdot\ell_2)} + \mathcal{N}_{\rm c}(1,2,\bar{p}',\bar{p}). \tag{4.32}$$

The first term

$$\frac{\mathcal{N}_a(1,2,\bar{p}',\bar{p})\,\mathcal{N}_0(1,2,\bar{p},\bar{p}')}{2(\ell_1\cdot\ell_2)}, \tag{4.33}$$

as its form suggest, it exhibits a double copy structure. The spinless numerator appears also in the 4-point gluon-matter HEFT tree (see equation 3.19). The classical spin numerator, in gauge theory, has the following form [49]:

$$\begin{aligned}\mathcal{N}_a(1,2,\bar{p}',\bar{p}) = -\frac{w_1\cdot F_1\cdot F_2\cdot w_2}{(\ell_1\cdot\bar{p})} + \frac{\ell_1\cdot\bar{p}-\ell_2\cdot\bar{p}}{2(p_1\cdot\bar{p})}\times \\ \Big(iG_2\left(x_1,x_2\right)\left(a\cdot F_1\cdot F_2\cdot S\cdot\ell_2\right) \\ + iG_2(x_1,x_2)(a\cdot F_2\cdot F_1\cdot S\cdot\ell_1) \\ + iG_1\left(x_{12}\right)\mathrm{tr}\left(F_1\cdot S\cdot F_2\right) \\ + G_1(x_1)G_1(x_2)\times \\ \Big((a\cdot F_1\cdot\bar{p})(a\cdot F_2\cdot\ell_1)-(a\cdot F_1\cdot p_2)(a\cdot F_2\cdot\bar{p}) \\ -\frac{\ell_2\cdot\bar{p}-\ell_1\cdot\bar{p}}{2}(a\cdot F_1\cdot F_2\cdot a)\Big)\Big),\end{aligned} \tag{4.34}$$

where $x_{i=1,2} = a \cdot \ell_i$ and $x_{ij} = x_i + x_j$. The function $G_2(x_1, x_2)$, is (a generalization of $G(x)$) an entire function as well, defined as[3]:

$$G_2(x_1; x_2) = \frac{1}{x_2} \left(G_1(x_{12}) - \cosh(x_2) G_1(x_1) \right) . \tag{4.35}$$

As noted in [51], for computations it is useful to introduce its integral representation as well:

$$G_2(x_1; x_2) = \int_0^1 d\sigma_1 d\sigma_2 \left(\sigma_1 \sinh(\sigma_1 x_1 + \sigma_1 \sigma_2 x_2) - \sinh(\sigma_2 x_2) \cosh(\sigma_1 x_1) \right) . \tag{4.36}$$

This numerator was obtained by first separating the analytic part from the singular part. The non analytic part, containing a massive pole in $v \cdot \ell_1$, was fixed by the requirement to factorize correctly on that pole. On the other hand, the analytic part was fixed by considering an ansatz containing the G_1 and G_2 functions and the requirement that the amplitude needs to have to correct factorization on the massless pole $\ell_{12} \to 0$ [50].

The term constructed from the double copy, is able to give the correct contributions up to quadratic order in spin, beyond which, spin flip effects arise. Hence, the second term in eq. 4.32, which accounts exactly for these effects. This term, was bootstrapped by the requirement factorize correctly on the massive poles [49], and reads:

$$\begin{aligned} \mathcal{N}_{\rm r}(1,2,\bar{p}',\bar{p}) = & \frac{\Big((\partial_{x_1} - \partial_{x_2}) G_1(x_1) G_1(x_2) \Big)}{4(\bar{p}\cdot\ell_1)(\bar{p}\cdot\ell_2)} \times \\ & \Big(\bar{p}\cdot\ell_2 (\bar{p}^2 (a\cdot F_1\cdot F_2\cdot a)(a\cdot F_2\cdot F_1\cdot\bar{p}) \\ & + a^2 (\bar{p}\cdot F_1\cdot F_2\cdot\bar{p})(a\cdot F_1\cdot F_2\cdot\bar{p})) - (1\leftrightarrow 2) \Big) \\ & + \left(\frac{i(\partial_{x_1} - \partial_{x_2}) G_2(x_1, x_2)}{4(\bar{p}\cdot\ell_1)(\bar{p}\cdot\ell_2)} \right) \times \\ & \Big((\bar{p}\cdot\ell_2)(a\cdot F_2\cdot F_1\cdot\bar{p})((a\cdot F_2\cdot\bar{p})(a\cdot\tilde{F}_1\cdot\bar{p}) \\ & - (a\cdot F_1\cdot\bar{p})(a\cdot\tilde{F}_2\cdot\bar{p})) + (1\leftrightarrow 2) \Big) . \end{aligned} \tag{4.37}$$

[3]For a review of the properties of the G functions, see [49].

The final part of the amplitude, $\mathcal{N}_c$, contains no poles and thus cannot be deduced by factorization requirements. In [50], a form for $\mathcal{N}_c$ was proposed, by matching with physical data [93, 94] at $\mathcal{O}(a^4)$ and at $\mathcal{O}(a^5)$. The proposed form reeds:

$$\begin{aligned}\mathcal{N}_c(1,2,\bar{p}',\bar{p}) = &\Big(\frac{(\partial_{x_1}-\partial_{x_2})^2}{2!}G_1(x_1)G_1(x_2)\Big)\times \\ &\Big((a\cdot F_1\cdot\bar{p})(a\cdot F_2\cdot\bar{p})(a\cdot F_1\cdot F_2\cdot a) \\ &-\frac{a^2}{2}\big((a\cdot F_1\cdot F_2\cdot p)(a\cdot F_2\cdot F_1\cdot\bar{p}) \\ &+(a\cdot F_1\cdot F_2\cdot a)(\bar{p}\cdot F_1\cdot F_2\cdot\bar{p})\big)\Big) \\ &+\Big(\frac{i(\partial_{x_1}-\partial_{x_2})^2}{2!}G_2(x_1,x_2)\Big)\times \\ &\Big(-\frac{1}{2}\big((a\cdot F_1\cdot F_2\cdot a)(a\cdot F_2\cdot\bar{p})(a\cdot\tilde{F}_1\cdot\bar{p})-(1\leftrightarrow 2)\big)\Big).\end{aligned} \tag{4.38}$$

The proposal for the spinning-HEFT 4-point, presented in eq. 4.32, with the numerators in eqs. 4.34, 4.37 and 4.38, was used (along with the spinning-HEFT 3-point) for the computation of the scattering angle of two Kerr black holes at 2PM, in [51, 52]. In [52] the authors present a discussion on the comparison between the result in eq. 4.32, (which we will denote as $\mathcal{M}_{\text{SP}}$), the result from higher-spin theory [97] (M_{HS}) and the 4-point $M_{\text{TS-FZ}}$, containing the contributions obtained by imposing the far-zone asymptotic behaviors to the amplitude extracted from the solution of the Teukolsky equation [98].

In the comparison of $\mathcal{M}_{\text{SP}}$ with the 4-point obtained from the higher-spin theory M_{HS}, the authors in [52] found that the two amplitudes agree, up to $\mathcal{O}(a^{20})$, in the limit where the spheroidicity parameter z, present in M_{HS}, goes to 0. This parameter is defined as $z = 2\sqrt{-a\cdot a}(\bar{p}\cdot\ell_1)/m$, and enters M_{HS} through the two entire functions that this amplitude contains. Furthermore, it is speculated that z contributes to near-zone physics effects.

On the other hand, when comparing $\mathcal{M}_{\text{SP}}$ with $M_{\text{TS-FZ}}$, the authors found that at order $\mathcal{O}(a^5)$, additional contact terms are needed, in order to make the two amplitudes consistent. These extra contributions depend on z described above, and found to vanish in the limit $z\to 0$. This was verified up to order $\mathcal{O}(a^8)$.

Contrary to the work of [52], where the authors calculate the eikonal order by order in spin, the work presented in [51] proposes a novel method for computing the resulting integrals[4] using the amplitude in eq. 4.32, without the need to expand the integrand in spin. In the scattering of a Kerr test black hole in a Schwarzschild background, their results for the eikonal show the expected singular behavior.

We will close this chapter, by noting that, apart from the probe limit at 3PM, the 3-point and 4-point spinning-HEFT trees we presented are enough for calculations up to 3PM. Moreover, the rest of the formalism that was reviewed in the previous chapter is valid for spinning amplitudes as well. That is, one can expand spinning amplitudes in the full theory, in terms of the spinning-HEFT trees using the HEFT expansion that we reviewed in section 3.3. On the diagrammatical level, the amplitudes relevant to the scattering of two Kerr black holes, are given by eqs. 3.35, 3.36, and 3.38 with the the HEFT amplitudes replaced with the corresponding spinning-HEFT ones.

[4]This method is relevant to this work and it will be reviewed in chapter 6.

5 Computing Feynman Integrals

Feynman integrals play a crucial role in any amplitude computation, and therefore deriving efficient techniques for their computation is a stepping stone to higher-loop and precision calculations. In this chapter we will review the most powerful tools in multi-loop computations: Integration By Parts (IBP) reduction [99, 100], the method of differential equations[101] and tensor reduction [102].

5.1 Integration By Parts Reduction

Let us consider an L loop Feynman Integral (FI), that depends on N external vectors p_i^μ, $i = 1, 2 \dots N$:

$$\begin{aligned} I\left(\{p_i\}\right) &= \int d^D\ell_1 \dots d^D\ell_L \frac{1}{\mathcal{N}_1^{n_1} \dots \mathcal{N}_Q^{n_Q}} = \int d^D\ell_1 \dots d^D\ell_L \mathcal{F}(\ell_i), \\ \mathcal{F}(\ell_i) &= \frac{1}{\mathcal{N}_1^{n_1} \dots \mathcal{N}_Q^{n_Q}}, \end{aligned} \tag{5.1}$$

with $n_i \in \mathbb{Z}$, $i = 1, \dots Q$, and $Q = L(L+1)/2 + LN$ is the number of scalar products s_{ij} that depend on the loop momenta, $\mathcal{N}_a$ are linear polynomials of s_{ij} and $D = 4 - 2\varepsilon$. Additionally we assume that any s_{ij} can be written in terms of $\mathcal{N}_a$ and any non zero linear combination of $\mathcal{N}_a$ is a function of the loop momenta. We say that $\mathcal{N}_a$ are linearly independent and form a complete basis [103].

Instead of viewing eq. 5.1 as a function of the external momenta, we can view it as a function of the powers of the denominators $\mathcal{N}_a$: $I = I(n_1, \dots, n_Q)$. That is, if we consider a Q dimensional lattice space, $\mathbb{Z}^Q$ then we can associate each point $(n_1, \dots, n_Q)$ on the lattice with one integral I. As we will see below we can then associate different integrals in different points of the lattice with each other, using IBP relations between them. This will allow us to maximally reduce a generic FI of the form of eq. 5.1 to a finite set of irreducible integrals, called Master Integrals (MIs). We will then see that the MIs form a complete basis that spans all integrals in $\mathbb{Z}^Q$.

5.1.1 Properties of Feynman Integrals in Dimensional Regularization

The starting point of deriving IBP relations between FIs, is their properties in dimensional regularization. Thus, let us first review some of these properties. Firstly, an integral of the form of eq. 5.1 is a Lorentz scalar, and as such it is invariant under Lorentz transformations. As a Lorentz scalar it can only depend on the external scales $y_{ij} = p_i \cdot p_j$, which are also Lorentz scalars, and we will make this dependence explicit by writing $I = I\left(\{y_{ij}\}\right)$.

Let us consider now a rescaling of the external vectors $p_i \to \lambda p_i$, $\lambda \in \mathbb{R} \setminus \{0\}$. Then we have $y_{ij} \to \lambda^2 y_{ij}$, and the integral will scale as:

$$\begin{aligned} &I\left(\lambda^2 \{y_{ij}\}\right) \to \lambda^{LD-2\nu} I\left(\{y_{ij}\}\right), \\ &\nu = \sum_{i=1}^{Q} n_i, \end{aligned} \tag{5.2}$$

where $LD - 2\nu = 4L - 2\nu - 2\varepsilon L \neq 0 \forall L, \nu$, is the mass dimension of the integral. The fact that an integral that does not depend on any external scale (scaleless integral) needs to be consistent with eq. 5.2, means that *scaleless integrals vanish in dimensional regularization*. Moreover, all integrals without denominators, $I = I(n_1, \ldots n_Q)$, $n_1, \ldots n_Q < 0$, vanish as well, since they can be written as a linear combination of scaleless integrals [104].

The next property we will derive is based on the fact that in dimensional regularization integrals are invariant under linear changes of variables [104]. Following [105], we consider an infinitesimal change of variables of the form $\ell_i^\mu \to \ell_i^\mu + \varepsilon v^\mu$, with v^μ being an arbitrary vector. Of course, $d^D\ell_i \to d^D\ell_i$ and the integrand in 5.1 transforms as:

$$\begin{aligned} &\mathcal{F}(\ell_i) \to \mathcal{F}(\ell_i \to \ell_i + \varepsilon v) \approx \mathcal{F}(\ell_i) + \varepsilon(\partial_{\ell_i} \cdot v)\mathcal{F}(\ell_i) + \mathcal{O}\left(\varepsilon^2\right), \\ &(\partial_{\ell_i} \cdot v) = \frac{\partial}{\partial \ell_i^\mu} v^\mu. \end{aligned} \tag{5.3}$$

The integral becomes:

$$I = I + \varepsilon \int d^D\ell_1 \ldots d^D\ell_L (\partial_{\ell_i} \cdot v)\mathcal{F}(\ell_i) + \mathcal{O}\left(\varepsilon^2\right), \tag{5.4}$$

and translation invariance of I implies that

$$\int d^D\ell_1 \dots d^D\ell_L (\partial_{\ell_i} \cdot v)\mathcal{F}(\ell_i) = 0 \forall v^\mu. \tag{5.5}$$

We can *integrate by parts* the LHS of equation 5.5, and express scalar products in terms of the propagators, to derive recursive relations between integrals of the same family. These relations can be solved in terms of the master integrals.

Note that apart from IBP relations, there is another class of identities called Lorentz Invariance (LI) identities [106]:

$$p_i^\mu p_j^\nu \left(\sum_k p_{k[n} \frac{\partial}{\partial p_k^{\mu]}} \right) I(n_1, \dots, n_Q). \tag{5.6}$$

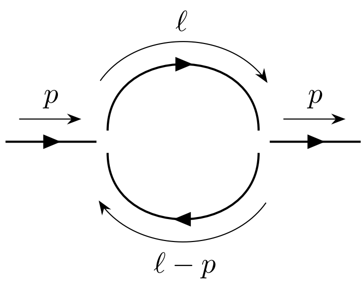
These are based on the fact that FIs, are Lorentz scalars. However, it was shown in [105] that these identities can be expressed as linear combinations of IBP identities.

Before we delve into the properties of bases of MIs let us see an example of IBP reduction.

Example: One loop bubble

Let us consider the one loop bubble scalar diagram, with one massive and one massless propagator:

$F =$



$$= \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{(\ell^2 - m^2 + i\epsilon)\,((p-\ell)^2 + i\epsilon)}. \tag{5.7}$$

We define the integral family:

$$I(n_1, n_2) = \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{(\ell^2 - m^2 + i\epsilon)^{n_1}\,((p-\ell)^2 + i\epsilon)^{n_2}}, \tag{5.8}$$

and we have that $F = I(1,1)$. We can obtain useful relations, by using eq.5.5, with $v^\mu = \ell^\mu, p^\mu$:

$$\begin{aligned}
&\int \frac{d^D\ell}{(2\pi)^D}\frac{\partial}{\partial \ell^\mu}\frac{\ell^\mu}{(\ell^2-m^2+i\epsilon)^{n_1}((p-\ell)^2+i\epsilon)^{n_2}} \\
&=\int \frac{d^D\ell}{(2\pi)^D}\left(\frac{d-2n_1-n_2}{(\ell^2-m^2+i\epsilon)^{n_1}((p-\ell)^2+i\epsilon)^{n_2}}-\frac{2n_1m^2}{(\ell^2-m^2+i\epsilon)^{n_1+1}((p-\ell)^2+i\epsilon)^{n_2}}\right. \\
&\left.-\frac{n_2(m^2-q^2)}{(\ell^2-m^2+i\epsilon)^{n_1}((p-\ell)^2+i\epsilon)^{n_2+1}}-\frac{n_2}{(\ell^2-m^2+i\epsilon)^{n_1-1}((p-\ell)^2+i\epsilon)^{n_2+1}}\right) \\
&=(d-2n_1-n_2)I(n_1,n_2)-2n_1m^2I(n_1+1,n_2)-n_2\left(m^2-p^2\right)I(n_1,n_2+1) \\
&-n_2I(n_1-1,n_2+1)=0,
\end{aligned} \tag{5.9}$$

similarly we have:

$$\begin{aligned}
&\int \frac{d^D\ell}{(2\pi)^D}\frac{\partial}{\partial \ell^\mu}\frac{p^\mu}{(\ell^2-m^2+i\epsilon)^{n_1}((p-\ell)^2+i\epsilon)^{n_2}} \\
&=(n_2-n_1)I(n_1,n_2)+n_1I(n_1+1,n_2-1)-n_1(m^2+p^2)I(n_1+1,n_2) \\
&-n_2I(n_1-1,n_2+1)-n_2(m^2-p^2)I(n_1,n_2+1)=0,
\end{aligned} \tag{5.10}$$

and we have used $\ell^2 = \ell^2 - m^2 + m^2$ as well as $(\ell - p^2) = \ell^2 - 2\ell \cdot p + p^2$ and $\delta^\mu_\mu = D$.

Before we try to find the master integrals, it is useful to define the notion of a sector. We follow [105], and define the sector $(\theta_1, \ldots, \theta_Q)$, $\theta_i = 0, 1$, as the collection of points $(n_1, \ldots n_Q) \in \mathbb{N}^Q$ that obey the following condition:

$$\text{sign}\left(n_i - \frac{1}{2}\right) = 2\theta_i - 1. \tag{5.11}$$

Of course $(n_1 = \theta_1, \ldots, n_Q = \theta_Q) \in (\theta_1, \ldots, \theta_Q)$ and this point is called the corner point of the sector.

Coming back to our example, in eq. 5.7, we see that the family 5.8, splits in four sectors, namely: $(0,0), (0,1), (1,0), (1,1)$. The sector $(0,0)$, contains the integrals $I(n_1, n_2)$,with n_1, $n_2 \leq 0$, and such integrals are zero, since they only have numerators. The sector $(0,1)$ contains the integrals $I(n_1, n_2)$, with $n_1 \leq 0$, $n_2 > 0$. However such integrals are also zero since they are scaleless. Such sectors are called 0 sectors. The non zero sectors are the sector $(1,0)$, that contains all the integrals $I(n_1, n_2)$, with $n_1 > 0$, $n_2 \leq 0$, and finally the sector $(1,1)$ contains the integrals $I(n_1, n_2)$, with $n_1 > 0$, $n_2 > 0$.

Let us analyze first, the sector $(1,0)$. We can use the relation in eq. 5.9, for $n_2 = 0$ to get the equation:

$$(D-2n_1)I(n_1,0) - 2m^2 n_1 I(1+n1,0) = 0, \tag{5.12}$$

with solution:

$$I(n_1,0) = -\frac{m^2\left(-\frac{1}{m^2}\right)^{n_1}\left(1-\frac{D}{2}\right)_{n_1-1}}{(1)_{n_1-1}} I(1,0), \tag{5.13}$$

where $(X)_n = \Gamma(X+n)/\Gamma(x)$, is the Pochhammer Symbol. We now find the rest of the integrals of the sector, recursively with equation 5.10. For example we see that for $n_2 = 0$, and $n_1 \to n_1 - 1$ we can get $I(n_1,-1)$ in terms of $I(n_1,0)$:

$$J(n_1,-1) = \left(m^2+p^2\right) J(n_1,0) + J(n_1-1,0), \tag{5.14}$$

similarly for $n_2 = -1$, we can get $I(n_1,-2)$:

$$\begin{aligned} &J(n_1,-2) \\ &= \frac{(p^2-m^2)\,J(n_1-1,0) + (n_1-1)\,(m^2+p^2)\,J(n_1,-1) - J(n_1-2,0) + n_1 J(n_1-1,-1)}{n_1-1}, \end{aligned} \tag{5.15}$$

and we can continue for more n_2. It is now evident, that all integrals in this sector can be expressed in terms of $I(1,0)$, that is, $I(1,0)$ is a master integral.

The reduction of sector $(1,1)$ is a bit more involved. We can use eq. 5.9 with $n_1 = 1$, eqs. 5.10 with $n_1 = 1$ and 5.10, again, with $n_1 = 1$ and $n_2 = n_2 - 1$ to get the equation:

$$\begin{aligned} &(D-2n_2-1)\left(m^2+p^2\right) I(1,n_2) + (-D+n_2+1)I(1,n_2-1) \\ &+ n_2\left(m^2-p^2\right)^2 I(1,n_2+1) = 0, \end{aligned} \tag{5.16}$$

from which we can generate all the integrals $I(1,n_2)$, in terms of $I(1,1)$ and $I(1,0)$, for example, for $n_2 = 1$ we have:

$$\frac{(D-3)I(1,1)\,(m^2+p^2) + (2-D)I(1,0) + I(1,2)\,(m^2-p^2)^2}{m} = 0. \tag{5.17}$$

For the integrals, $I(n_1,1)$, we subtract eq. 5.9 from eq. 5.10 for $n_2 = 1$ to get:

$$(-D+n_1+2)J(n_1,1) + n_1\left(\left(m^2-p^2\right) J(n_1+1,1) + J(n_1+1,0)\right) = 0, \tag{5.18}$$

and we see that again all the integrals $I(n_1, 1)$ can be generated in terms of $I(1,1)$ and $I(1,0)$.

Even in this simple example, deriving the IBP relations and the master integrals, was a rather involved procedure. Of course in integral families with more denominators, such procedure gets more complicated. Luckily, one can get the IBP relations in an algorithmic way, (see for example Laporta's algorithm [107]) and there are a lot of computer algebra systems, designed for this job. In this work we use the Mathematica program, LiteRed [103, 108].

We conclude this section, by mentioning that the number of master integrals is always finite, and the proof can be found in [109, 110]. The fact that the number of master integrals is never infinite ensure us that we can always use IBP reduction for scalar integrals.

5.2 The Method of Differential Equations

We have seen in the previous section how we can reduce any integral in a given family to a set of MIs. Evidently, the next step is to develop the techniques one can use to solve these integrals. For one loop computations, there are many efficient ways one can solve an integral: one can use Feynman or Schwinger parameterization [111, 112], or cast the integral in the Mellin-Barnes representation [113]. However, for higher loops such techniques fail to deliver the required efficacy. There is one method that works for an arbitrary number of loops and has been able to deliver impressive results: the method of differential equations [101, 114–117].

Once the system of differential equations has been derived for a set of MIs, the solution of that system is not always simple. Nevertheless, as the reader will notice in the following subsections, dimensional regularization, will provide an excellent framework for solving the differential equations perturbatively in the parameter ε.

5.2.1 Deriving the Differential Equations

We have already seen that as a Lorentz scalar, an integral $I(\{y_{ij}\})$ of the form of eq. 5.1, depends on the scalar products of the external vectors and scales y_{ij}. We can attempt to derive a set of differential equations (DE) for I by acting with the differential operators $\partial_{y_{ij}} \equiv \frac{\partial}{\partial y_{ij}}$, on the RHS of eq. 5.1. In the case where y_{ij} is some external scale, like a propagator mass, m^2, it is easy to see that:

$$\partial_{m^2} I(n_1, \ldots, n_Q) \propto I(n'_1 \ldots n'_Q). \tag{5.19}$$

If $I(n_1, \ldots, n_Q)$ is a master integral, then using IBP relations, we can express $I(n'_1 \ldots n'_Q)$ in terms of master integrals as well. Repeating this procedure for the rest of the MIs of the set, returns a system of DEs for the MIs in terms of m^2.

We can extend this to the scalar products of external vectors as well. Using the chain rule we can write [103]:

$$\begin{aligned}
\frac{\partial}{\partial (p_l \cdot p_k)} J(n_1 \ldots n_Q) &= \sum \left[\mathbf{G}^{-1}\right]_{ik} p_i \cdot \partial_{p_l} J(n_1 \ldots n_Q) \\
&= \sum \left[\mathbf{G}^{-1}\right]_{il} p_i \cdot \partial_{p_k} J(n_1 \ldots n_Q), \\
\frac{\partial}{\partial (p_l^2)} J(n_1, \ldots n_Q) &= \frac{1}{2} \sum \left[\mathbf{G}^{-1}\right]_{il} p_i \cdot \partial_{p_l} J(n_1, \ldots n_Q),
\end{aligned} \tag{5.20}$$

where $\mathbf{G}_{ij} = p_i \cdot p_j$ is the Gramm matrix. Evaluating the right hand side, and applying the IBP relations will yield the desired differential equations for the master integrals.

Example: Deriving the DEs for the One Loop Bubble MIs

Let us illustrate how we can derive differential equations by an example. Returning to the example in section 5.1, we will derive the DEs for the MIs we found, namely $I(1,1)$ and $I(1,0)$. The external scales are $y_i = (m^2, p^2)$, and we define $\vec{I} = (I(1,0), I(1,1))^T$, then we get the following differential equations:

$$\begin{aligned}
\partial_{y_1} \vec{I} &= \begin{pmatrix} I(2,0) \\ I(2,1) \end{pmatrix} = \begin{pmatrix} \frac{(D-2)}{2y_1} & 0 \\ \frac{D-2}{2y_1(y_1-y_2)} & \frac{D-3}{y_1-y_2} \end{pmatrix} \vec{I}, \\
\partial_{y_2} \vec{I} &= \begin{pmatrix} \frac{(y_1+y_2)I(1,2)}{2y_2} - \frac{I(1,1)}{2y_2} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{(D-2)(y_1+y_2)}{2(y_1-y_2)^2 y_2} & -\frac{(D-2)y_1^2+2(D-4)y_1y_2+(D-2)y_2^2}{2y_2(y_1-y_2)^2} \\ 0 & 0 \end{pmatrix} \vec{I}.
\end{aligned} \tag{5.21}$$

5.2.2 Boundary Conditions

In order to derive a unique solution of a differential equation (or a system of differential equations), boundary conditions should be provided. In the case where the functions that satisfy the differential equations are Feynman integrals, the boundary conditions are the values of the integrals with their integrands evaluated in a specific limit of the external scales. For a given limit, a Feynman integral collects contributions from several physical regions: Since the components of the loop momenta ℓ_i are being integrated in the interval $(-\infty, +\infty)$, the relative scaling between external parameters will change on different parts of the integration intervals [118]. Let us illustrate this by an example: Let us consider the massive one loop bubble integral, with two equal masses $m_1 = m_2 = m$, in the limit $|m^2/p^2| \equiv \lambda \ll 1$:

$$\int \frac{d^D \ell_1}{\pi^{D/2}} \frac{1}{(\ell^2 - m^2)\left((p-\ell)^2 - m^2\right)}. \tag{5.22}$$

One might try to naively perform an expansion of the integrand, by writing

$$\frac{1}{(\ell^2 - m^2)\left((\ell-p)^2 - m^2\right)} = \frac{1}{\ell^2 (\ell - p)^2}\left(1 + \frac{\lambda^2}{\ell^2} + \dots\right)\left(1 + \frac{p^2\lambda^2}{(\ell+p)^2} + \dots\right). \tag{5.23}$$

However, since the components of ℓ are integrated over the integrval $(-\infty, +\infty)$, there will exist a region such that $|\ell| \sim m$, thus making the above expansion invalid in that region.

It is now apparent, that in order to evaluate a Feynman Integral in a given kinematic limit, one has to know the physical regions that contribute to this limit. In what follows we will review a systematic way of obtaining the asymptotic expansion of master integrals, that was first proposed in [119, 120].

We begin by considering an L loop integral of the form of eq. 5.1 in the Feynman parametrization:

$$I(n_1, \dots n_Q) = c \int_0^1 dx_1 \dots d_Q \delta\left(1 - x_1 - \dots - x_Q\right) x_1^{n_1 - 1} x_Q^{n_Q - 1} U^\alpha F^\beta, \tag{5.24}$$

along with a small parameter λ. The constant c and the exponents α, β depend on the number of loops L, the dimension D and the integers n_i. The Symanzik polynomials [121], U and F are homogeneous polynomials in the Feynman parameters x_i, of the form $\lambda^{r_0} x_1^{r_1} \dots x_Q^{r_Q}$ with $r_0 = 0$ for the terms of U. Moreover, we are interested in the limit $\lambda \to 0$.

The different regions that contribute to the limit $\lambda \to 0$ are characterized by how the loop momenta ℓ_i scale, and thus the key conjecture in [119, 120], is that if we perform a rescaling in the integration variables x_i, by some power of the expansion parameter λ, and expand the integrand, we will obtain the leading terms of the expansion in the given region. The problem is then to find all the rescalings $x_i \to \lambda^{\nu_i} x_i$ and the corresponding leading terms in the limit $\lambda \to 0$. As we will see below, this problem can be translated to a geometric problem which can be solved algorithmically.

We begin by associating a vector $\vec{r} : \lambda^{r_0} x_1^{r_1} \dots x_Q^{r_Q} \to (r_0, r_1, \dots, r_Q)$, with each term of U and F, and we may define the sets $\mathbb{F}$ and $\mathbb{U}$ that contain all the points that correspond to the terms in F and U respectively. The F polynomial is uniform in x_i and thus, the components of every $\vec{r} \in \mathbb{F}$ must satisfy the equation:

$$r_1 + \dots + r_Q = L + 1, \tag{5.25}$$

which defines a Q-dimensional plane P, parallel to the r_0 direction. Of course, all the points in $\mathbb{F}$ live in P.

A rescaling $x_i \to \lambda^{\nu_i} x_i$ will transform the terms in F and U as follows:

$$\lambda^{r_0} x_1^{r_1} \dots x_Q^{r_Q} \to x_1^{r_1} \dots x_Q^{r_Q} \lambda^{r_0 + \nu_1 r_1 + \dots \nu_Q r_Q} \sim \lambda^{\vec{v} \cdot \vec{r}}, \vec{v} = (1, \nu_1, \dots, \nu_Q)\,, \tag{5.26}$$

and after expanding the integrand, the leading terms will have the same scaling in λ. This translates to the following constraint equation:

$$\vec{v}\,(\vec{r}_i - \vec{r}_j) = 0, \quad \forall \vec{r}_i, \vec{r}_j \in \mathbb{U}' \cup \mathbb{F}', \tag{5.27}$$

where we have defined $\mathbb{U}'$ and $\mathbb{F}'$ as the sets of points associated with the leading terms in the integrand, after the rescaling. The equation 5.27 defines a hyperplane, P' normal to $\vec{v}$, that all points $\vec{r} \in \mathbb{U}' \cup \mathbb{F}'$ live in.

If the dimensionality of P' is lower that Q, then the leading contribution is a scaleless integral. In Feynman parameters, integrals are scaleless if rescaling a subset of Feynman variables x_i, leaves the U and F polynomials invariant, up to a global factor:

$$U\left(\{x_j\}, \{a x_i\}\right) = a^u U\left(\{x_j\}\right), \quad F\left(\{x_j\}, \{a x_i\}\right) = a^f F\left(\{x_j\}\right). \tag{5.28}$$

This ensures that points corresponding to leading terms in a given region (leading points) will live in a Q-dimensional plane. Points corresponding to sub-leading terms (sub-leading points), in that region carry higher powers of λ (and thus larger r_0 coordinate), and therefore, will be located above the leading points. Therefore, the leading points in each region, will live in the outermost, bottommost[1] hyperplanes that can be constructed from the points of the set $\mathbb{U} \cup \mathbb{F}$. These hyperplanes are the bottom faces of the convex hull of the set $\mathbb{U} \cup \mathbb{F}$, and the components (excluding the component in the r_0 direction) of their normal vectors, correspond the relative scales of x_i and thus define the expansion regions.

To summarize, in order to find all the regions that contribute to a limit $\lambda \to 0$, one may perform the following steps:

- Starting from a Feynman parametrized integral of equation 5.24, construct the sets $\mathbb{F}$, $\mathbb{U}$ as well as the convex hull C, of $\mathbb{F} \cup \mathbb{U}$. One may achieve this by using Mathematica's build in function ConvexHullMesh, or more sophisticated codes like Qhul[122].

- Having the convex hull, determine the (inwards pointing) normal vectors $\vec{v}$ of each face of C. The ones who are normal to the bottom faces of C are those with a positive component v_0, in the r_0 direction (i.e the ones who point up). The normalization of $\vec{v}$ is such that $v_0 = 1$.

- The expanded integral will then have the form:

  $$I \xrightarrow{\lambda \to 0} \sum_R I^{LD}_{(R)}, \tag{5.29}$$

  Where R is the number of regions (up pointing vectors), and I_{LD} is leading terms of the integrand, after performing the rescaling $x_i \to \lambda^{\nu_i} x_i$, prescribed by the components $(\nu_1, \ldots \nu_Q)$, of the normal vectors $\vec{v}$, and expanding in $\lambda \to 0$.

[1]The sense of direction, (up or down), is with respect to the r_0 axis.

Example: Asymptotics of the one loop bubble integral

Let us return, once again, to the integral of equation 5.7. We will use the method described above to derive the contributing regions in the limit $m \ll 1$. The integral in Feynman parameters read:

$$\begin{aligned} &\Gamma\left(2+\frac{D}{2}\right)\int_0^{+\infty} dx_1 dx_2 \delta\left(1-x_1-x_2\right) U^{2-D} F^{D/2-2}, \\ &U = x_1 + x_2, F = m^2 x_1^2 + m^2 x_1 x_2 - p^2 x_1 x_2, \end{aligned} \tag{5.30}$$

and the corresponding sets $\mathbb{F}$ and $\mathbb{U}$ read:

$$\mathbb{F} = \{(1,2,0),(1,1,1),(0,1,1)\}\,, \quad \mathbb{U} = \{(0,1,0),(0,0,1)\}\,. \tag{5.31}$$

The convex hull C of $\mathbb{U} \cup \mathbb{F}$ (the blue shaded polyhedron), along with the points $p_f \in \mathbb{F}$(red) and $p_U \in \mathbb{U}$(blue) is shown in figure 5.1.

The normal vectors of each of the faces of C read:

$$\vec{v}_1 = (0,-1,-1), \quad \vec{v}_2 = (0,0,-1), \quad \vec{v}_3 = (1,-1,0), \quad \vec{v}_4 = (1,0,0), \quad \vec{v}_5 = (-1,1,1), \tag{5.32}$$

and we see that the two regions of expansion are characterized by $\vec{v}_3$ and $\vec{v}_4$, which can also be seen in eq. 5.1. In the literature, the region corresponding to $\vec{v}_4$ is called the hard region, and the region corresponding to $\vec{v}_3$ is called a collinear region.

Even though we presented this method in the context of defining boundary conditions for our differential equations, there more scenarios where one has to expand an integral in some limit of its external invariants. As we will see in the following chapters, when considering the dynamics of spinning bodies, one useful way to compute the integrals involved, is to expand the integrand in the region $|a| \to 0$, where $|a|$ is the norm of the spin vector. In this case since the spin dependence is in the numerator, there exist only one region that coincides with the naive Taylor expansion. The resulting spin expanded integrands, are easier to compute, since they can be reduced to spinless master integrals.

Asymptotic expansions of Feynman integrals, are also useful in the study of effective field theories where one can show that the dimensionally regularized Feynman diagrams of the effective field theory under study, are into one-to-one correspondence with the asymptotically expanded integrals of the full theory [118].

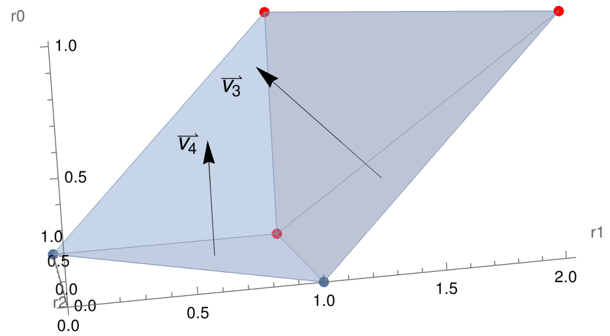


Figure 5.1.: The points $p_f \in \mathbb{F}$ in red and $p_U \in \mathbb{U}$ in blue, along with the convex hull of $\mathbb{U} \cup \mathbb{F}$, depicted as the blue shaded polyhedron.

5.2.3 Solving the Differential Equations

In the simplest case, one can solve the system of differential equations, directly. However, in most cases the systems of DEs one encounters are complicated and in order to solve them, one can try to simplify the system by performing linear transformations.

Let us consider a system of differential equations for some scale y_i:

$$\partial_{y_i} \vec{I} = \mathbf{M}(\vec{y}, D)\vec{I}, \tag{5.33}$$

where $\mathbf{M}(\vec{y}, D)$ is a $p \times p$ matrix of coefficients and $\vec{I}(\vec{y}) = (I_1(\vec{y}), \ldots I_p(\vec{y}))$ is a basis of MIs. One may perform a linear transformation by changing basis of master integrals from $\vec{I}$ to $\vec{J}$. Considering, both are bases of the same vector space, there exists an invertible matrix $\mathbf{T}(\vec{y}, D)$ such that $\vec{J} = \mathbf{T}\vec{I}$. The system of equations that $\vec{J}$ satisfies will be:

$$\begin{aligned} \partial_{y_i} \vec{J} &= \left(\mathbf{TMT}^{-1} + \frac{d\mathbf{T}(\vec{y}, D)}{y_i} \mathbf{T}^{-1} \right) \vec{J}, \\ \partial_{y_i} \vec{J} &= \tilde{\mathbf{M}}(\vec{y}, D)\vec{J}, \end{aligned} \tag{5.34}$$

and the goal of the transformation is for $\tilde{\mathbf{M}}$ to be in the *canonical form*[123]:

$$\tilde{\mathbf{M}}(\vec{y}, \varepsilon) = \varepsilon \mathbf{A}(\vec{y}), \quad \lim_{\varepsilon \to 0} \mathbf{A}(\vec{y}) = \mathbf{A}(\vec{y}), \tag{5.35}$$

that is, all the dependence on the dimensional regulator ε can be factored out of $\tilde{\mathbf{M}}$. The solution of the DEs will then be:

$$\vec{J}(y_i) = \mathbf{C} \exp\left(\int dy_i \mathbf{A}(\vec{y}) \right) \vec{J}_0(y_i). \tag{5.36}$$

Starting from a system of DEs, of the form of eq. 5.33, one can realize a linear transformation that brings the system into a canonical form with the following steps:

1. Diagonal transformation:

$$\mathbf{T}_1 = \text{diag}\left(c_i f_1(y_i), \ldots, f_p(y_i)\right), \tag{5.37}$$

where p is the size of the system of DEs. The goal of this transformation is to bring $\mathbf{M}(\vec{y})$ in the form $\mathbf{M}(\vec{y}) \to \mathbf{M}_1(\vec{y}) + \varepsilon \mathbf{M}_2(\vec{y})$, with $\mathbf{M}_1$ and $\mathbf{M}_2$ independent of ε. This can be implemented by choosing the functions $f_i(y_i)$ such that they kill any non ε dependence in the diagonal of $\mathbf{M}$. The constants c_i can then be chosen such that they cancel off any ε dependence in the denominators of the off diagonal elements of $\mathbf{M}$, leaving at most a linear factor of ε in the numerator[2].

2. Non Diagonal transformation $\mathbf{T}_2$. Here $\mathbf{T}_2$ does not depend on ε and satisfies the following equation:

$$\frac{d\mathbf{T}_2}{dy_i} + \mathbf{T}_2 \mathbf{M}_2 = 0. \tag{5.38}$$

It is evident that the transformation $\vec{J} = \mathbf{T}_2 \mathbf{T}_1 \vec{I}$, will bring eq.5.33 into canonical form.

Example: Canonical Form For The One Loop Bubble DEs.

Coming back to our one loop bubble example, we will illustrate how these transformations work, on the system of DEs for the mass dependence y_1. We see that for $D \to 4 - 2\varepsilon$, the system is in the form where $\mathbf{M} = \mathbf{M}_1 + \varepsilon \mathbf{M}_2$:

$$\partial_{y_1} \vec{I} = \begin{pmatrix} \frac{(1-\varepsilon)}{y_1} & 0 \\ \frac{1-\varepsilon}{y_1(y_1-y_2)} & \frac{1-2\varepsilon}{y_1-y_2} \end{pmatrix} \vec{I}, \tag{5.39}$$

with

$$\begin{aligned} \mathbf{M}_1 &= \begin{pmatrix} \frac{1}{y_1} & 0 \\ \frac{1}{y_1(y_1-y_2)} & \frac{1}{(y_1-y_2)} \end{pmatrix}, \text{ and} \\ \mathbf{M}_2 &= \begin{pmatrix} \frac{-1}{y_1} & 0 \\ \frac{-1}{y_1(y_1-y_2)} & \frac{-2}{(y_1-y_2)} \end{pmatrix}. \end{aligned} \tag{5.40}$$

[2]Note that when we perform this step, to cast a system of DEs in the ε-regular form (see the next discussion), it is not necessary (and often not possible), to choose c_i such that they cancel any ε dependence in the off-diagonal denominators and leave at least a linear term of ε. Typically, in such cases, just canceling the ε dependency from the denominator will be enough to bring the system into ε-regular form

We may now jump to step no. 2 immediately, performing a transformation:

$$\mathbf{T}_2 = \begin{pmatrix} f_1(y_1) & 0 \\ f_2(y_1) & f_3(y_1) \end{pmatrix}, \text{ with}$$

$$\frac{d\mathbf{T}_2}{dy_1} + \mathbf{T}_2\mathbf{M}_2 \tag{5.41}$$

$$= \begin{pmatrix} \frac{f_1(y_1)}{y_1} + \frac{df_1(y_1)}{dy_1} & 0 \\ \frac{f_2(y_1)}{y_1} + \frac{f_3(y_1)}{y_1(y_1-y_2)} + \frac{df_2(y_1)}{dy_1} & \frac{f_3(y_1)}{(y_1-y_2)} + \frac{df_3(y_1)}{dy_1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Solving the differential equations for f_1, f_2 and f_3, we get the following for $\mathbf{T}_2$:

$$\mathbf{T}_2 = \begin{pmatrix} \frac{C_1}{y_1} & 0 \\ \frac{-C_1}{y_1(y_1-y_2)} + \frac{c_1}{y_1} & \frac{c_1}{(y_2-y_1)} \end{pmatrix}, \tag{5.42}$$

and we have chosen all the constants of integration to be c_1. Now we perofrm the change of basis $\vec{J} = \mathbf{T}_2\vec{I}$, and the system becomes:

$$\partial_{y_1}\vec{J} = \varepsilon \begin{pmatrix} \frac{-1}{y_1} & 0 \\ \frac{y_1^2 - y_2(1+y_2)}{y_1(y_1-y_2)^2} & \frac{2}{(y_2-y_1)} \end{pmatrix} \vec{J}. \tag{5.43}$$

It is not known if every system of DEs, can be cast into the canonical form [104], and there are many cases where the algorithm described above may not work. In such cases, the strategy is to bring the system into the following form (this form is called ε-regular in some references e.g. [124].):

$$\mathbf{M}(\vec{y}) = \mathbf{M}_0(\vec{y}) + \mathbf{M}_1(\vec{y}, \varepsilon), \quad \lim_{\varepsilon \to 0} \mathbf{M}_1(\vec{y}, \varepsilon) = 0, \tag{5.44}$$

and solve the system perturbatively in ε, either directly or using the Frobenius method.

Typically performing step no. 1, is sufficient to bring the system into an ε-regular form. However a general algorithm can be found in [124]. In that same reference, one can find also the proof that every system of differential equations can be cast into the epsilon regular form, thus one can always rely on this method to solve the system of DEs.

We will close off this section by presenting a brief review of the systematics of perturbation expansions in differential equations. For what follows, we follow [125].

Assuming a system of differential equations of the form of eq. 5.33, where $\mathbf{M}$ is in the ε-regular form, we expand $\vec{I}$ and $\mathbf{M}$ for small ε:

$$\begin{aligned} \vec{I}\,(\vec{y},\varepsilon) &= \sum_{k=0}^{N} \vec{I}_{(k)}\varepsilon^{k}, \\ \mathbf{M}\,(\vec{y},\varepsilon) &= \sum_{k=0}^{N} \mathbf{M}_{(k)}\varepsilon^{k}. \end{aligned} \tag{5.45}$$

This discussion will be restricted to a finite expansion of $\mathbf{M}$. This means that the $\mathbf{M}_1$ in eq. 5.44 doesn't have ε-poles of the form $1/\varepsilon$. Typically this can be achieved by rescaling the master integrals with the appropriate overall ε scaling.

Plugging equations 5.45 into 5.33, and collecting terms order by order in ε, we obtain the differential equation for the k-th order term in ε:

$$\partial_{y_i}\vec{I}_{(k)} = \mathbf{M}_{(0)}\vec{I}_k + \sum_{j=0}^{k-1}\mathbf{M}_{(k-j)}\vec{I}_{(j)}, \tag{5.46}$$

which is a differential equation of the form

$$\partial_x\vec{f}(x) = \mathbf{A}(x)\vec{f}(x) + \vec{b}, \tag{5.47}$$

with $x \to y_i$, $f \to I$, $\mathbf{A} \to \mathbf{M}$ and $b \to \sum_{j=0}^{k-1}\mathbf{M}_{(k-j)}\vec{I}_{(j)}$. Assuming that the homogeneous part $\partial_x\vec{g}(x) = \mathbf{A}(x)\vec{g}(x)$, of eq. 5.47 is solvable, one can construct from the solutions $\vec{g}$ the matrix of solutions $\mathbf{G}$ such that[3]:

$$\partial\mathbf{G} = \mathbf{M}\mathbf{G}. \tag{5.48}$$

Next we consider the matrix equation:

$$\partial\mathbf{F} = \mathbf{M}\mathbf{F} + \mathbf{B}, \tag{5.49}$$

where $\mathbf{B} = \frac{1}{p}\left(\vec{b},\ldots,\vec{b}\right)$. One notes that if we set $\mathbf{F} = \mathbf{G}\mathbf{H}$, with $\mathbf{G}$ satisfying the homogeneous equation 5.48, then $\mathbf{H}$ satisfies :

$$\mathbf{G}\partial\mathbf{H} = \mathbf{B} \Rightarrow \mathbf{H} = \mathbf{C} + \left(\int \mathbf{G}^{-1}\mathbf{B}\right), \tag{5.50}$$

[3] See [125] for how to construct the matrix of solutions in case where one solved the homogeneous system using the Frobenius method.

with $\mathbf{C} = \mathrm{diag}\,(c_1, \ldots, c_p)$, being constants of integration. Then the solution to eq. 5.50 is $\mathbf{F} = \mathbf{G}\,(\mathbf{C} + \int \mathbf{G}^{-1}\mathbf{B})$ and thus, the general solution to equation 5.47 is given by:

$$\vec{f} = \sum_{j=1}^{p} \vec{F}_k, \tag{5.51}$$

with $\vec{F}_k$ being the k-th column of $\mathbf{F}$.

5.3 Tensor Reduction

So far, we have reviewed the most common techniques that deal with scalar integrals. However, these are not the only integrals one encounters during a typical computation. Another class of multi-loop integrals, is the one of tensor integrals and unfortunately the methods we have developed so far are not directly applicable. When dealing with such integrals, the main strategy is to reduce them in scalar integrals which can then be tamed by IBP reduction and differential equations. In this section, we will describe a modern take on Passarino-Veltman (PV) reduction [126], that was proposed in [127].

As the reader will see below, the generic methods of tensor reduction, are not the most efficient as the rank of the tensor integrals increases. Oftentimes, one may be able to find shortcuts depending on the problem at hand.

5.3.1 A Modern Take on Passarino-Veltman Reduction

Let us consider a general L-loop rank-R tensor integral :

$$I^{\mu_1 \cdots \mu_R} \equiv \int d^D\ell_1 \ldots d^D\ell_L \tau\left(\{\ell_i\}\right) f\left(\{\ell_i\}, p_1, \ldots, p_N\right), \tag{5.52}$$

where $p_{i=1,N}$ are the external momenta, f is a scalar function and τ is a tensor function of the loop momenta $\ell_{i=1,N}$. In a typical Passarino-Veltman reduction, one decomposes eq. 5.52, to a sum of all rank R tensors $T^{\mu_1,\cdots\mu_L}$ that can can be made from the external vectors p_i and the metric $\eta^{\mu\nu}$:

$$I^{\mu_1 \cdots \mu_R} = \sum_j I_j T_j^{\mu_1,\cdots\mu_R}. \tag{5.53}$$

The form factors I_j can be obtained if we consider the dual $\langle T_j \rangle$ of T_j, which satisfies $\langle T_i \rangle \cdot T_j = \langle T_i \rangle^{\mu_a \ldots \mu_L} (T_j)_{\mu_a \ldots \mu_L} = \delta_{ij}$. Then for the form factors we get:

$$I_j = I \cdot \langle T_j \rangle, \tag{5.54}$$

and thus equation 5.53 becomes:

$$I^{\mu_1 \ldots \mu_R} = \sum_j I \cdot \langle T_j \rangle T_j^{\mu_1, \ldots \mu_R} = I_{\alpha_1 \ldots \alpha_R} \sum_j \langle T_j^{\alpha_1 \ldots \alpha_R} \rangle T_j^{\mu_1, \ldots \mu_R}. \tag{5.55}$$

We will show now how we construct the dual basis $\langle \mathcal{B} \rangle = \{\langle T_a \rangle\}$, from a basis $\mathcal{B} = \{T_a\}$.

Starting from a basis of the external momenta $p_{i=N}$, we construct the Gram matrix, $\Pi_{ij} = p_i \cdot p_j$. Then the dual momenta $\langle p_i \rangle$ are:

$$\langle p_i \rangle = \sum_{j=1}^{N} \Pi_{ij}^{-1} p_j, \tag{5.56}$$

and thus, for a rank R tensor product of external momenta, $T^{\mu_1 \ldots \mu_R} = p_{i_1}^{\mu_1} \ldots p_{i_R}^{\mu_R}$ we write the dual $\langle T \rangle^{\mu_1 \ldots \mu_R}$ as:

$$\langle T \rangle^{\mu_1 \ldots \mu_R} = \langle p_{i_1}^{\mu_1} \ldots p_{i_R}^{\mu_R} \rangle = \langle p_{i_1}^{\mu_1} \rangle \ldots \langle p_{i_R}^{\mu_R} \rangle. \tag{5.57}$$

The keen reader may have noticed that one cannot include the metric tensor to $\mathcal{B}$, since contracting $\eta^{\mu\nu}$ with eq. 5.57 doesn't give 0. Therefore, the needed metric has to be transverse to the external momenta:

$$\eta_\perp^{\mu\nu} p_{i,\mu} = \eta_\perp^{\mu\nu} \langle p_{i,\mu} \rangle = 0. \tag{5.58}$$

It is evident that if we define a unit tensor $u^{\mu\nu}$, acting as the unity in the space of the external momenta then, $\eta_\perp$ can be defined as:

$$\begin{aligned} \eta_\perp^{\mu\nu} &= \eta^{\mu\nu} - u^{\mu\nu}, \\ u^{\mu\nu} &= \sum_{i=1}^{N} p_i^\mu \langle p_i^\nu \rangle, \\ \eta_\perp^{\mu\nu} (\eta_\perp)_{\mu\nu} &= D_\perp = D - N. \end{aligned} \tag{5.59}$$

For the dual of a rank R tensor product of external momenta with (transverse) metrics we have:

$$\begin{aligned}\langle T\rangle^{\mu_1\cdots\mu_R} &= \langle \eta_\perp^{\mu_1\mu_2}\ldots\eta_\perp^{\mu_{m-1}\mu_m} p_{i_{m+1}}^{\mu_{m+1}}\ldots p_{i_R}^{\mu_R}\rangle \\ &= \langle \eta_\perp^{\mu_1\mu_2}\ldots\eta_\perp^{\mu_{m-1}\mu_m}\rangle\langle p_{i_{m+1}}^{\mu_{m+1}}\ldots p_{i_R}^{\mu_R}\rangle.\end{aligned} \tag{5.60}$$

Unfortunately, the dual of a product of metrics is not equal to the product of dual metrics. To construct such duals one has to consider an ansatz based on the index symmetries and fix the parameters using the fact that $\langle T_i\rangle^{\mu_a\cdots\mu_L}\,(T_j)_{\mu_a\ldots\mu_L} = \delta_{ij}$. As an example let us consider $\langle\eta_\perp^{\mu_1\mu_2}\eta_\perp^{\mu_3\mu_4}\rangle$. We see that this object is symmetric under the change $\mu_1 \leftrightarrow \mu_2$, as well as $\mu_3 \leftrightarrow \mu_4$ and $(\mu_1, \mu_2) \leftrightarrow (\mu_3, \mu_4)$. Thus, we construct the ansatz:

$$\langle\eta_\perp^{\mu_1\mu_2}\eta_\perp^{\mu_3\mu_4}\rangle = \alpha\eta_\perp^{\mu_1\mu_2}\eta_\perp^{\mu_3\mu_4}, \beta\left(\eta_\perp^{\mu_1\mu_3}\eta_\perp^{\mu_2\mu_4} + \eta_\perp^{\mu_1\mu_4}\eta_\perp^{\mu_2\mu_3}\right), \tag{5.61}$$

with the conditions:

$$\begin{aligned}\langle\eta_\perp^{\mu_1\mu_2}\eta_\perp^{\mu_3\mu_4}\rangle(\eta_\perp)_{\mu_1\mu_2}(\eta_\perp)_{\mu_3\mu_4} &= 1,\\ \langle\eta_\perp^{\mu_1\mu_2}\eta_\perp^{\mu_3\mu_4}\rangle(\eta_\perp)_{\mu_1\mu_3}(\eta_\perp)_{\mu_2\mu_4} &= 0,\end{aligned} \tag{5.62}$$

and we find that:

$$\begin{aligned}\alpha &= \frac{D_\perp + 1}{D_\perp\left(D_\perp - 1\right)\left(D_\perp + 1\right)},\\ \beta &= -\frac{1}{D_\perp\left(D_\perp - 1\right)\left(D_\perp + 1\right)}.\end{aligned} \tag{5.63}$$

We can now try and evaluate eq. 5.55 using the above constructions. As we have already seen, one can write:

$$\sum_{T\in\mathcal{B}}\langle T^{\alpha_1\ldots\alpha_R}\rangle T^{\mu_1,\ldots\mu_R} = \sum_{T\in\mathcal{B}(p_i)}\langle T^{\alpha_1\ldots\alpha_R}\rangle T^{\mu_1,\ldots\mu_R} + \sum_{T\in\mathcal{B}(pi,\eta_\perp)}\langle T^{\alpha_1\ldots\alpha_R}\rangle T^{\mu_1,\ldots\mu_R}, \tag{5.64}$$

where $\mathcal{B}(p_i) \subset \mathcal{B}$ contains all the tensors, products of the external momenta and $\mathcal{B}(p_i, \eta_\perp) \subset \mathcal{B}$ contains all the tensors products of the external momenta with the metric. Using eq.5.57 we can write:

$$\sum_{T\in\mathcal{B}(p_i)}\langle T^{\alpha_1\ldots\alpha_R}\rangle T^{\mu_1,\ldots\mu_R}\sum_{i_1,\ldots i_R=1}^{N}\langle p_{i_1}^{\alpha_1}\rangle\ldots\langle p_{i_R}^{\alpha_R}\rangle p_{i_1}^{\mu_1}\ldots p_{i_R}^{\mu_R} = \prod_{n=1}^{R} u^{a_k\mu_k}. \tag{5.65}$$

Similarly using eq. 5.60, we can write

$$
\begin{aligned}
&\sum_{T\in\mathcal{B}(pi,\eta_\perp)} \langle T^{\alpha_1\ldots\alpha_R}\rangle T^{\mu_1,\ldots\mu_R} \\
&= \sum_{m=2}^{\lfloor R/2\rfloor} \sum_{i_1,\ldots i_R=1}^{N} \langle \eta_\perp^{\alpha_1\alpha_2}\ldots\eta_\perp^{\alpha_{m-1}\alpha_m}\rangle\langle p_{i_{m+1}}^{\alpha_{m+1}}\ldots p_{i_R}^{\alpha_R}\rangle \eta_\perp^{\mu_1\mu_2}\ldots\eta_\perp^{\mu_{m-1}\mu_m} p_{i_{m+1}}^{\mu_{m+1}}\ldots p_{i_R}^{\mu_R} \\
&= \prod_{i=1}^{n/2} \overline{u^{\alpha_{2i-1},\mu_{2i-1}} u}^{\alpha_{2i},\mu_{2i}} \prod_{k=n+1}^{R} u^{a_k\mu_k},
\end{aligned}
\tag{5.66}
$$

with the product of contractions[4] $\prod_{i=} \overline{u^{\alpha_i,\mu_i} u}^{\beta_i,\nu_i}$ being defined as:

$$
\prod_{i=} \overline{u^{\alpha_i,\mu_i} u}^{\beta_i,\nu_i} = \left(\prod_i \eta_\perp^{\alpha_i\beta_i}\right)\left\langle\prod_i \eta_\perp^{\mu_i\nu_i}\right\rangle. \tag{5.67}
$$

We can now sum together eqs. 5.66 and 5.67 by defining an ordering symbol $\mathcal{T}$ as :

$$
\mathcal{T}\left\{u^{\mu_1\nu_1}\ldots u^{\mu_n\nu_n}\right\} = u^{\mu_1\nu_1}\ldots u^{\mu_n\nu_n} + \text{all contractions}. \tag{5.68}
$$

The result for tensor reduction can thus be written in the compact form:

$$
I^{\mu_1\ldots\mu_R} = I_{\alpha_1\ldots\alpha_R}\mathcal{T}\left\{\prod_{i=1}^{R} u^{\alpha_i\mu_i}\right\}. \tag{5.69}
$$

5.3.2 Examples From the Two Loop Calculation

Let us illustrate the above method by providing some examples from the calculations we will present in the chapters 6 and 7.

In our case[5], we define a basis of 3 external vectors $p_i^\mu = (v_1^\mu, v_2^\mu, q^\mu)$ with their duals being:

$$
\langle p^\mu\rangle = \left(\frac{y_1 {v_2}^\mu - {v_1}^\mu}{y_1^2-1}, \frac{y_1 {v_1}^\mu - {v_2}^\mu}{y_1^2-1}, \frac{q^\mu}{-q^2}\right). \tag{5.70}
$$

[4] in the sense of wick contractions.

[5] We prompt the reader in sections 3.1, where we describe the kinematics and variables of our scattering problem and in chapter 7, for the origin of the tensor integrals.

The transverse metric and the unit element read:

$$\begin{aligned}\eta_{\perp}^{\mu\nu} &= \eta^{\mu\nu} - \frac{q^\mu q^\nu}{-q^2} - \frac{{v_2}^\mu\left(y_1 {v_1}^\nu - {v_2}^\nu\right)}{y_1^2-1} - \frac{{v_1}^\mu\left(y_1 {v_2}^\nu - {v_1}^\nu\right)}{y_1^2-1},\\ u^{\mu\nu} &= \frac{q^\mu q^\nu}{-q^2} + \frac{{v_2}^\mu\left(y_1 {v_1}^\nu - {v_2}^\nu\right)}{y_1^2-1} + \frac{{v_1}^\mu\left(y_1 {v_2}^\nu - {v_1}^\nu\right)}{y_1^2-1}.\end{aligned} \tag{5.71}$$

A typical 2-loop integral from the $\mathcal{O}\left(a\right)$ part looks like:

$$I^\mu = \int \frac{d^D\ell_1}{\pi^{D/2}}\frac{d^D\ell_3}{\pi^{D/2}}\frac{\delta(\ell_1\cdot v_2)\delta(\ell_3\cdot v_2)\left(\ell_1\cdot\ell_3\right)^2(\ell_3\cdot v_1)\ell_3^\mu}{\ell_1^2\ell_3^2\left(q-\ell_3\right)^2\left(-\ell_1-\ell_3+q\right)^2(\ell_1\cdot v_1)}, \tag{5.72}$$

with the free index ℓ_3^μ being contracted with a Levi-Civita tensor, $\epsilon(a,\mu,v_1,v_2)$, and we have stripped I^μ from its mass dependence and any quantity that doesn't depend on ℓ_1 and ℓ_3 for clarity. Before we present the reduction, let us comment, that we can treat internals with cut propagators (delta functions) by the method of *reverse unitarity*[128, 129] as:

$$\frac{2\pi i}{(-1)^n n!}\delta^{(n)}(x) = \frac{1}{(x-i\epsilon)^{n+1}} - \frac{1}{(x+i\epsilon)^{n+1}}, \tag{5.73}$$

where with $\delta^{(n)}(x)$, we denote the nth derivative, of $\delta(x)$. One may then use the standard techniques we have reviewed in this chapter[6].

At rank one we have:

$$\mathcal{T}\left(u^{\mu\nu}\right) = u^{\mu\nu} = \frac{q^\mu q^\nu}{-q^2} + \frac{{v_2}^\mu\left(y_1 {v_1}^\nu - {v_2}^\nu\right)}{y_1^2-1} + \frac{{v_1}^\mu\left(y_1 {v_2}^\nu - {v_1}^\nu\right)}{y_1^2-1}, \tag{5.74}$$

and thus:

$$\begin{aligned}I^\mu = I_\alpha\mathcal{T}\left(u^{\alpha\mu}\right) = \frac{q^\mu}{-q^2}\int\frac{d^D\ell_1}{\pi^{D/2}}\frac{\delta(\ell_1\cdot v_2)\delta(\ell_3\cdot v_2)\left(\ell_1\cdot\ell_3\right)^2(\ell_3\cdot v_1)(q\cdot\ell_3)}{\ell_1^2\ell_3^2\left(q-\ell_3\right)^2\left(-\ell_1-\ell_3+q\right)^2(\ell_1\cdot v_1)} + \\ \frac{\left(y_1 {v_2}^\nu - {v_1}^\nu\right)}{y_1^2-1}\int\frac{d^D\ell_1}{\pi^{D/2}}\frac{\delta(\ell_1\cdot v_2)\delta(\ell_3\cdot v_2)\left(\ell_1\cdot\ell_3\right)^2(\ell_3\cdot v_1)(v_1\cdot\ell_3)}{\ell_1^2\ell_3^2\left(q-\ell_3\right)^2\left(-\ell_1-\ell_3+q\right)^2(\ell_1\cdot v_1)},\end{aligned} \tag{5.75}$$

where we have used $\ell_3\cdot v_2 = 0$ due to the delta constraint $\delta\left(\ell_3\cdot v_2\right)$. Now both integrals in eq. 5.75, can be reduced with IBP identities using LiteRed.

[6] In chapter 3, when we reviewed the heavy mass expansion for gravity amplitudes, we saw that, in the heavy mass limit, the expansion contains delta functions (eq. 3.31. Moreover, when we discussed the method of unitarity in chapter 2, we saw that on-shell propagators, are replaced with delta functions as well. It is thus evident, that the relevant integrals for our computations, will contain delta functions in the numerators.

Aa another example we consider a rank 2 integral from the $\mathcal{O}\left(a^3\right)$ part:

$$I^{\mu_1\mu_2} = \int \frac{d^D\ell_1}{\pi^{D/2}}\frac{d^D\ell_3}{\pi^{D/2}}\frac{\delta\left(\ell_1\cdot v_2\right)\delta\left(\ell_3\cdot v_2\right)\left(q\cdot a\right)\left(q\cdot\ell_1\right)^2\ell_3{}^{\mu_1}\ell_3{}^{\mu_2}}{2\ell_1^2\ell_3^2\left(q-\ell_3\right)^2\left(-\ell_1-\ell_3+q\right)^2\left(\ell_1\cdot v_1\right)}, \tag{5.76}$$

with the two free indices μ_1 and μ_2 being contracted with the spin vector a^μ and the Levi-Civita tensor, $\epsilon\left(a,q,\mu,v_2\right)$. For rank 2 we have:

$$\begin{aligned}\mathcal{T}\left\{u^{\alpha_1\mu_1}u^{\alpha_2\mu_2}\right\} = u^{\alpha_1\mu_1}u^{\alpha_2\mu_2} + \overline{u^{\alpha_1\mu_1}u}^{\alpha_2\mu_2} = \\ \left(\frac{q^{\alpha_1}q^{\mu_1}}{-q^2}+\frac{v_2{}^{\alpha_1}\left(y_1v_1{}^{\mu_1}-v_2{}^{\mu_1}\right)}{y_1^2-1}+\frac{v_1{}^{\alpha_1}\left(y_1v_2{}^{\mu_1}-v_1{}^{\mu_1}\right)}{y_1^2-1}\right)\times \\ \left(\frac{q^{\alpha_2}q^{\mu_2}}{-q^2}+\frac{v_2{}^{\alpha_2}\left(y_1v_1{}^{\mu_2}-v_2{}^{\mu_2}\right)}{y_1^2-1}+\frac{v_1{}^{\alpha_2}\left(y_1v_2{}^{\mu_2}-v_1{}^{\mu_2}\right)}{y_1^2-1}\right) \\ +\frac{1}{D-3}\left(\eta^{\alpha_1\alpha_2}-\frac{q^{\alpha_1}q^{\alpha_2}}{-q^2}-\frac{v_2{}^{\alpha_1}\left(y_1v_1{}^{\alpha_2}-v_2{}^{\alpha_2}\right)}{y_1^2-1}\right. \\ \left.-\frac{v_1{}^{\alpha_1}\left(y_1v_2{}^{\alpha_2}-v_1{}^{\alpha_2}\right)}{y_1^2-1}\right)\times \\ \left(\eta^{\mu_1\mu_2}-\frac{q^{\mu_1}q^{\mu_2}}{-q^2}-\frac{v_2{}^{\mu_1}\left(y_1v_1{}^{\mu_2}-v_2{}^{\mu_2}\right)}{y_1^2-1}\right. \\ \left.-\frac{v_1{}^{\mu_1}\left(y_1v_2{}^{\mu_2}-v_1{}^{\mu_2}\right)}{y_1^2-1}\right),\end{aligned} \tag{5.77}$$

and

$$\begin{aligned}I^{\mu_1\mu_2} &= I_{\alpha_1\alpha_2}\mathcal{T}\left\{u^{\alpha_1\mu_1}u^{\alpha_2\mu_2}\right\} \\ &= C_1^{\mu_1\mu_2}\left(I\cdot q\cdot q\right)+C_2^{\mu_1\mu_2}\left(I\cdot v_1\cdot v_1\right)+C_3^{\mu_1\mu_2}\left(I\cdot v_1\cdot q\right).\end{aligned} \tag{5.78}$$

With the constants $C_i^{\mu\nu}$ defined as:

$$\begin{aligned}C_1^{\mu_1\mu_2} &= -\frac{\eta^{\mu_1\mu_2}}{(D-3)(-q^2)}+\frac{v_2^{\mu_1}\left(y_1v_1^{\mu_2}-v_2^{\mu_2}\right)}{(D-3)(-q^2)\left(y_1^2-1\right)}+\left(\frac{1}{D-3}+1\right)\frac{q^{\mu_1}q^{\mu_2}}{q^4}, \\ C_2^{\mu_1\mu_2} &= \frac{v_1^{\mu_1}\left(v_1^{\mu_2}-y_1v_2^{\mu_2}\right)}{\left(y_1^2-1\right)^2}+\frac{y_1v_2^{\mu_1}\left(y_1v_2^{\mu_2}-v_1^{\mu_2}\right)}{\left(y_1^2-1\right)^2}, \\ C_3^{\mu_1\mu_2} &= \frac{q^{\mu_1}\left(y_1v_2^{\mu_2}-v_1^{\mu_2}\right)}{\left(-q^2\right)\left(y_1^2-1\right)}+\frac{q^{\mu_2}\left(y_1v_2^{\mu_1}-v_1^{\mu_1}\right)}{\left(-q^2\right)\left(y_1^2-1\right)}.\end{aligned} \tag{5.79}$$

5.3.3 Different Problem Dependent Strategies

Even though the method of tensor reduction we have reviewed so far is a very powerful method that works for an arbitrary number of loops and tensor ranks, it is a "maximally agnostic" method, in the sense that it doesn't take into account the various problem specific structures and constraints. This is not a problem for low rank tensors. However, with increasing rank and for higher loops, the number of terms eq. 5.69 generates will start to become a bottleneck in the computation. As an alternative route, one can try and look into the problem at hand, and try to find shortcuts that simplify the computation.

For example, let us look at the integrals in eqs. 5.75 and 5.76. The free indices of the loop momenta are contracted with the spin vector a^μ and the Levi-Civita tensor $\epsilon\,(a, \mu, \nu, v_2)$ and as we have already seen in section 5.1, we can reduce integrals with scalar products of the form $a \cdot l$, by including such scalar products in our IBP basis. The only remaining structure is then $\epsilon\,(a, \mu, \nu, v_2)$, and we can decompose it using a basis of our external vectors a^μ, q^μ, v_1^μ, v_2^μ as follows: since $\epsilon\,(a, \mu, \nu, v_2)$ is anti-symmetric in $\mu \leftrightarrow \nu$ we begin by making the following ansatz:

$$\epsilon\,(a, \mu, \nu, v_2) = c_1 a^{[\mu} v_1^{\nu]} + c_2 a^{[\mu} q^{\nu]} + c_3 v_1^{[\mu} q^{\nu]} + c_4 v_1^{[\mu} v_2^{\nu]} + c_5 a^{[\mu} v_2^{\nu]} + c_6 q^{[\mu} v_2^{\nu]}. \tag{5.80}$$

We find the constants c_i by contracting both sides in eq. 5.80 with the external vectors, in order to derive a system of linear equations for the constants c_i. The solutions are:

$$\begin{aligned}
c_1 &= -\frac{(q \cdot a)\epsilon\,(a, q, v_1, v_2)}{(y_1^2 - 1)\left((q \cdot a)^2 - y_2(-q^2)\right))}, \\
c_2 &= 0, \\
c_3 &= \frac{y_2 \epsilon\,(a, q, v_1, v_2)}{(y_1^2 - 1)\left(y_2(-q^2) - (q \cdot a)^2\right)}, \\
c_4 &= 0, \\
c_5 &= \frac{y_1 (q \cdot a)\epsilon\,(a, q, v_1, v_2)}{(y_1^2 - 1)\left((q \cdot a)^2 - y_2(-q^2)\right)}, \\
c_6 &= \frac{y_1 y_2 \epsilon\,(a, q, v_1, v_2)}{(y_1^2 - 1)\left(y_2(-q^2) - (q \cdot a)^2\right)},
\end{aligned} \tag{5.81}$$

and thus, the decomposition of $\epsilon\,(a, \mu, \nu, v_2)$ is:

$$\begin{aligned}
\epsilon\,(a, \mu, \nu, v_2) = \epsilon\,(a, q, v_1, v_2) \times \\
\frac{({v_1}^\nu - y_1 {v_2}^\nu)\left(y_2 q^\mu - (q \cdot a) a^\mu\right) - ({v_1}^\mu - y_1 {v_2}^\mu)\left(y_2 q^\nu - (q \cdot a) a^\nu\right)}{(y_1^2 - 1)\left(y_2 q^2 + (q \cdot a)^2\right)}.
\end{aligned} \tag{5.82}$$

We can use this result to calculate the integral in eq. 5.76. Contracting the free indices with a^{μ} and $\epsilon\,(a,q,\nu,v_2)$ we get:

$$
\begin{aligned}
&I^{\mu_1\mu_2}a_{\mu_1}\epsilon\,(a,q,\mu_2,v_2) = \\
&\qquad \int \frac{d^D\ell_1}{\pi^{D/2}}\frac{d^D\ell_3}{\pi^{D/2}}\frac{\delta\,(l_1\cdot v_2)\,\delta\,(l_3\cdot v_2)\,(q\cdot a)\,(q\cdot l_1)^{\,2}(a\cdot l_3)\epsilon\,(a,q,l_3,v_2)}{2l_1^2l_3^2\,(q-l_3)^{\,2}\,(-l_1-l_3+q)^{\,2}(l_1\cdot v_1)}.
\end{aligned}
\tag{5.83}
$$

We then reduce $\epsilon\,(a,q,l_3,v_2)$ to:

$$
\epsilon(a,q,l_3,v_2) = \frac{(l_3\cdot v_1)\epsilon(a,q,v_1,v_2)}{1-y_1^2},
\tag{5.84}
$$

by contracting eq. 5.82 with q^{μ} and l_3^{ν}. The integral in eq. 5.83 reduces to:

$$
\int \frac{d^D\ell_1}{\pi^{D/2}}\frac{d^D\ell_3}{\pi^{D/2}}\frac{\delta\,(l_1\cdot v_2)\,\delta\,(l_3\cdot v_2)\,(q\cdot a)\,(q\cdot l_1)^{\,2}(a\cdot l_3)(l_3\cdot v_1)\epsilon(a,q,v_1,v_2)}{2l_1^2l_3^2\,(q-l_3)^{\,2}\,(-l_1-l_3+q)^{\,2}(l_1\cdot v_1)(1-y_1^2)},
\tag{5.85}
$$

which can be reduced further using IBP.

6 Spinning-Spinless Scattering At 2PM

We saw, in chapter 4, that in spinning amplitudes, spin enters through the hyperbolic trigonometric functions $\cosh a \cdot \ell$ and $\sinh a \cdot \ell / a \cdot \ell$. In the present and the following chapter, we will develop the most common techniques to deal with loop integrals involving spin, through these functions, with the ultimate goal of applying these techniques to the computation of gravity observables to two loops. We will see that the most straightforward method to deal with such computations is to Taylor expand the hyperbolic trigonometric functions, and solve the integrals order by order in spin. In this approach, the involved master integrals, are typically simpler and one can obtain exact-in-spin results by resuming the result.

A different approach is to work in the exponential representation of hyperbolic functions and rewrite the exponential by introducing the delta function:

$$e^{ia\cdot\ell} = \int_{-\infty}^{+\infty} dt e^{it} \delta\left(a \cdot \ell - t\right), \tag{6.1}$$

and treat the resulting delta function as a cut propagator. This method was presented in [51], and allows the use of the standard multi-loop computation techniques presented in the chapter 5 to obtain exact-in-spin results. However, the resulting function space for the master integrals is more complicated than that of the non spinning integrals. The virtue of this method becomes apparent when dealing with scenarios where the number of external scalar products that the master integrals can depend on increases. One can imagine, that in such cases re-summation might be a hard task to accomplish, and a more systematic method, such as rewriting the exponential as a delta function might be optimal.

As a warm up for the 3PM computation, we will present here, the calculation of the 2PM contribution of the bending angle for the spinning-spinless black hole scattering problem. We will restrict ourselves to the case where the spin is aligned with the angular momentum of the probe, as it is sufficient to showcase the methods and ideas we use to include spin into the calculations of black hole scattering.

6.1 Setting Up The Problem

Following the discussion in the chapters 2 and 3, we will calculate the eikonal χ^{2PM} at 2PM:

$$\chi^{2PM} = \int \frac{d^D q}{(2\pi)^{D-2}} \frac{e^{ibq}\delta\left(q\cdot v_1\right)\delta\left(q\cdot v_2\right)}{4m_1 m_2} \mathcal{M}^{2PM}_{2\to 2}\left(q, a, u_1, u_2\right), \tag{6.2}$$

from which we can derive the bending angle:

$$\begin{aligned} \theta^{\text{2PM}} &= -\frac{\partial}{\partial J}\chi^{\text{2PM}}, \\ J &= \frac{m_1 m_2 b\sqrt{y_1^2 - 1}}{\sqrt{s}}, \end{aligned} \tag{6.3}$$

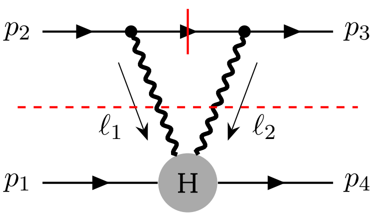
where J, is the conserved angular momentum of the system. Moreover, since we are working in the probe limit where $m_2 \gg m_1$, we may expand the Mandelstam s as:

$$\sqrt{s} = \sqrt{m_1^2 + m_2^2 + 2m_1 m_2 y_1} \approx m_2. \tag{6.4}$$

We remind the reader that the kinematics of the scattering were defined in section 3.1 of chapter 3.

The classical one loop amplitude is constructed using unitarity cuts as discussed in chapter 3, and is shown in the diagram below:

$$\mathcal{M}^{\text{2PM}}_{2\to 2}\left(q, a, v_1, v_2\right) =$$



$$= \frac{1}{2!}\frac{(32\pi G_N)^2}{(4\pi)^{D/2}}(-\pi)\int \frac{d^D\ell_1}{\pi^{D/2}}\delta(-m_2 v_2\cdot\ell_1)\times \sum_{h_1,h_2}\frac{\mathcal{M}_4^{h_1,h_2}(\ell_1,\ell_2,v_1)\mathcal{M}_a^{-h_1}(-\ell_1,v_2)\mathcal{M}_a^{-h_2}(-\ell_2,v_2)}{\ell_1^2\ell_2^2}. \tag{6.5}$$

The dotted red lines, represent the unitarity cuts, and the solid red lines, represent the propagators in the heavy mass expansion(i.e delta functions). The HEFT 4-point tree amplitude and the spinning-HEFT 3-points are given by eqs. 3.19 and 4.17. We remind the reader that we use eq. 2.21 in order to perform the polarization sums[1].

[1] Do not confuse the polarization vector $\varepsilon^{\mu}_{\pm}$ with the dimensional regulator ε

6.2 Spin Expansion

We now proceed to expand the integrand of eq. 6.5, in powers of spin. To perform the expansion, we decompose the spin vector a^μ as $a^\mu = |a|\hat{\alpha}^\mu$ where $|a| > 0$ and $\hat{\alpha}^\mu$ is a spacelike unit vector with $\hat{\alpha} \cdot \hat{\alpha} = -1$. We then evaluate the Taylor expansion of eq. 6.5, around $|a| = 0$. At even orders in spin the integrand contains only scalar products of the loop momentum with the spin parameter, $a \cdot \ell_1$, and at odd orders in spin, the integrand contains the loop momentum contracted with one Levi Civita tensor: $\epsilon(a, \ell_1, X, v_2) = \epsilon_{\alpha\beta\gamma\delta} a^\alpha \ell_1^\beta X^\gamma v_2^\delta$, $X^\mu \in \{q^\mu, v_1^\mu\}$. The integral structures we find, are summarized below:

$$
\begin{aligned}
I_1\left(1,1,1,\lambda_1,\lambda_2\right) &= \int \frac{d^D\ell_1}{\pi^{D/2}} \frac{\delta\left(\ell_1 \cdot v_2\right)\left(a \cdot \ell_1\right)^{\lambda_1}}{\ell_1^2 \left(q-\ell_1\right)^2 \left(\ell_1 \cdot v_1\right)^{\lambda_2}}, \\
I_2\left(X;1,1,1,\lambda_1,\lambda_2\right) &= \int \frac{d^D\ell_1}{\pi^{D/2}} \frac{\delta\left(\ell_1 \cdot v_2\right)\left(a \cdot \ell_1\right)^{\lambda_1} \epsilon(\ell_1, a, X, v_2)}{\ell_1^2 \left(q-\ell_1\right)^2 \left(\ell_1 \cdot v_1\right)^{\lambda_2}}.
\end{aligned} \tag{6.6}
$$

6.2.1 Master Integrals

We compute the first integral of eq. 6.6 using IBP identities, with LiteRed [103, 108] including the scalar products $a \cdot \ell_1$ and $a \cdot \ell_3$ to the IBP basis. The second integral is computed by decomposing the Levi-Civita tensor[2] $\epsilon\left(a, \mu, \nu, v_2\right)$, with μ and ν being free indices, to:

$$
\begin{aligned}
\epsilon\left(a,\mu,\nu,v_2\right) = \epsilon\left(a,q,v_1,v_2\right) \times& \\
&\frac{\left(v_1{}^\nu - y_1 v_2{}^\nu\right)\left(y_4' a^\mu - y_2 q^\mu\right) + \left(v_1{}^\mu - y_1 v_2{}^\mu\right)\left(y_2 q^\nu - y_4' a^\nu\right)}{\left(y_1^2 - 1\right)\left(y_2 y_3' - (y_4')^2\right)},
\end{aligned} \tag{6.7}
$$

where the variables y_i, $i = 1, 2$ defined in eq. 3.6. Moreover, we have defined $y_3' = q \cdot q$ and $y_4' = q \cdot a$. After contracting eq. 6.7 with ℓ_1 and v_1, the resulting scalar products can be decomposed by LiteRed.

At each order in spin, the loop integrals, reduce to the spinless triangle master integral, found, for example, in [23] :

$$
\mathcal{I} \equiv I_1\left(1,1,1,0,0\right) = \int \frac{d^D\ell_1}{\pi^{D/2}} \frac{\delta\left(\ell_1 \cdot v_2\right)}{\ell_1^2 \left(q-\ell_1\right)^2}, \tag{6.8}
$$

[2] See chapter 5 for the derivation and a discussion for different approaches in tensor integrals.

and the differential equations it satisfies are:

$$\partial_{y_3'}\mathcal{I} = \frac{(D-5)\mathcal{I}}{2y_3'}, \quad \partial_{y_{i\neq 3}}\mathcal{I} = 0, \tag{6.9}$$

with solution:

$$\mathcal{I}\left(y_3'\right) = c_1(y_3')^{\frac{D-5}{2}}. \tag{6.10}$$

We fix the constant of integration by matching with the known result in [23] and we can write our full result as:

$$\mathcal{I}\left(y_3'\right) = \frac{2^{4-d}\pi(-1)^{\frac{d-5}{2}}(y_3')^{\frac{d-5}{2}}\sec\left(\frac{\pi d}{2}\right)}{\Gamma\left(\frac{d}{2}-1\right)}. \tag{6.11}$$

After the loop integration, a typical impact parameter space Fourier integral can be computed as:

$$\begin{aligned}&\int \frac{d^D q}{(2\pi)^{D-2}} e^{ibq}\delta\left(q\cdot v_1\right)\delta\left(q\cdot v_2\right)(y_3')^{\alpha} q^{\mu_1}\dots q^{\mu_n}\\ &= (-i\partial_b)^{\mu_1}\dots(-i\partial_b)^{\mu_n}\hat{\mathcal{I}}\left(y_3\right),\end{aligned} \tag{6.12}$$

using the integral that can be found in [130, 131]:

$$\begin{aligned}\hat{\mathcal{I}}\left(y_3\right) &= \int \frac{d^D q}{(2\pi)^{D-2}} e^{ibq}\delta\left(q\cdot v_1\right)\delta\left(q\cdot v_2\right)(y_3)^{\alpha}\\ &= \frac{\pi^{-(D-2)/2}2^{D-2+2\alpha}e^{-\frac{1}{2}i\pi(4\alpha+d-2)}}{\sqrt{y_1^2-1}\,(y_3)^{(D-2)/2+\alpha}}\frac{\Gamma\left(\frac{D-2}{2}+\alpha\right)}{\Gamma\left(-\alpha\right)}.\end{aligned} \tag{6.13}$$

The four vectors q^μ (or the derivatives $(-i\partial_b)^\mu = -i\left(\frac{\partial}{\partial_b}\right)^\mu$) in eq.6.12, are contracted with either the spin vector a^μ or a Levi-Civita tensor $\epsilon\left(a,\mu,v_1,v_2\right)$ (or both). After all integrations the remaining Levi-Civita tensor, is $\epsilon\left(a,b,v_1,v_2\right)$, which was calculated in eq. 3.8.

below, we summarize the result for the eikonal up to $\mathcal{O}\left(a^8\right)$:

$$
\begin{aligned}
\chi^{(0)} &= i\pi G_N^2 m_1 m_2^2 \frac{3\left(5y_1^2-1\right)}{4\sqrt{y_1^2-1}\sqrt{y_3}}, \\
\chi^{(1)} &= -i\pi G_N^2 m_1 m_2^2 \frac{y_1\left(5y_1^2-3\right)\sqrt{y_2}}{\left(y_1-1\right)\left(y_1+1\right)y_3}, \\
\chi^{(2)} &= i\pi G_N^2 m_1 m_2^2 \frac{\left(95y_1^4-102y_1^2+15\right)y_2}{16\left(y_1^2-1\right)^{3/2} y_3^{3/2}}, \\
\chi^{(3)} &= -i\pi G_N^2 m_1 m_2^2 \frac{3y_1\left(9y_1^2-5\right)y_2^{3/2}}{4\left(y_1^2-1\right)y_3^2}, \\
\chi^{(4)} &= i\pi G_N^2 m_1 m_2^2 \frac{\left(239y_1^4-250y_1^2+35\right)y_2^2}{32\left(y_1^2-1\right)^{3/2} y_3^{5/2}}, \\
\chi^{(5)} &= -i\pi G_N^2 m_1 m_2^2 \frac{5y_1\left(13y_1^2-7\right)y_2^{5/2}}{8\left(y_1^2-1\right)y_3^3}, \\
\chi^{(6)} &= i\pi G_N^2 m_1 m_2^2 \frac{5\left(3\left(149y_1^4-154y_1^2+21\right)\right)y_2^3}{256\left(y_1^2-1\right)^{3/2} y_3^{7/2}}, \\
\chi^{(7)} &= -i\pi G_N^2 m_1 m_2^2 \frac{35y_1\left(17y_1^2-9\right)y_2^{7/2}}{64\left(y_1^2-1\right)y_3^4}, \\
\chi^{(8)} &= i\pi G_N^2 m_1 m_2^2 \frac{7\left(719y_1^4-738y_1^2+99\right)y_2^4}{512\left(y_1^2-1\right)^{3/2} y_3^{9/2}}.
\end{aligned}
\tag{6.14}
$$

Our result agrees with the computation in [52] which was done using HEFT in the spin expansion as well (when taking the aligned spin limit on their result). Moreover, we see that the resulting spin expanded bending angle that we get using eq. 6.3 (the reader can find this in the appendix A), agrees up to conventions, with [132], a calculation made using quantum higher spin theory.

6.2.2 Re-Summation

It is natural to re-sum the even and odd part of the eikonal separately, and add the two results together. Starting by the even order, it is easy to see that the polynomials[3]

[3]Note that we multiplied and divided χ^0 by $(y_1-1)(y_1+1)$.

$R_1^{\text{even}}(y_1) = 3(5y_1^2-1)(y_1-1)(y_1+1)$, $R_2^{\text{even}}(y_1) = 95y_1^4 - 102y_1^2 + 15, \ldots$ follow the linear relation:

$$\begin{aligned}
R_1^{\text{even}}(y_2) - R_0^{\text{even}}(y_2) &= 80y_1^4 - 84y_1^2 + 12, \\
R_2^{\text{even}}(y_2) - R_1^{\text{even}}(y_2) &= 144y_1^4 - 148y_1^2 + 20, \\
R_3^{\text{even}}(y_2) - R_2^{\text{even}}(y_2) &= 208y_1^4 - 212y_1^2 + 28, \\
&\vdots \\
R_{n+1}^{\text{even}}(y_2) - R_n^{\text{even}}(y_2) &= (64(n-1)+80)y_1^4 \\
&\quad - (64(n-1)+84)y_1^2 + 8(n-1) + 12,
\end{aligned} \tag{6.15}$$

with solution:

$$R_n^{\text{even}}(y_1) = 4n^2\left(8y_1^4 - 8y_1^2 + 1\right) - 4ny_1^2\left(4y_1^2 - 3\right) + 3\left(5y_1^4 - 6y_1^2 + 1\right). \tag{6.16}$$

The prefactors $1/4, 1/16, 1/32, 5/256, \ldots$, can be expressed through the sequences OEIS A098597 and OEIS A101926 [133] as :

$$Q_n^{\text{even}} = \frac{\text{A098597}}{\text{A101926}} = \frac{2^{-2n-3}}{2n+1}\binom{2(n+1)}{n+1}, \tag{6.17}$$

where:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \tag{6.18}$$

is the binomial coefficient. We can see that the even terms of the eikonal can be generated by the series:

$$\chi^{(2n)} = i\pi G_N^2 m_1 m_2^2 \frac{Q_n^{\text{even}} R_n^{\text{even}}\left(y_1\right) y_2^n}{\left(y_1^2-1\right)^{3/2} y_3^{n+1/2}}, n = 0, 1, 2 \ldots \tag{6.19}$$

Looking at the odd terms, one can identify that the polynomials $R_1^{\text{odd}} = (5y_1^2 - 3)$, $R_2^{\text{odd}} = (9y_1^2 - 5)$, $R_3^{\text{odd}} = (13y_1^2 - 7)$, $\ldots$, are generated by the sequence:

$$R_n^{\text{odd}} = (4n+5)y_1^2 - 2n - 3, n = 0, 1, \ldots, \tag{6.20}$$

and the prefactors $1, 3/4, 5/8, 35/64, \ldots$ can be generated by the following sequences:

$$Q_n^{\text{odd}} = \frac{\text{A001790}}{\text{A120777}} = 2^{-1-2n}\binom{2(n+1)}{n+1}, n = 0, 1, \ldots \tag{6.21}$$

Therefore, the odd part of the eikonal has the following form:

$$\chi^{(2n+1)} = -i\pi G_N^2 m_1 m_2^2 \frac{Q_n^{\text{odd}} R_n^{\text{odd}}(y_1) y_1 y_2^{n+\frac{1}{2}}}{(y_1^2-1)\, y_3^{n+1}}, n = 0, 1, \ldots, \tag{6.22}$$

and the re-summed eikonal at 2PM has the following form:

$$\begin{aligned} \chi^{\text{2PM}} &= \sum_{n=0}^{\infty} \chi^{2n} + \chi^{2n+1} = i\pi G_N^2 m_1 m_2^2 \times \\ &\left(\sum_{n=0}^{\infty} \frac{Q_n^{\text{even}} R_n^{\text{even}}(y_1)\, y_2^n}{(y_1^2-1)^{\,3/2} y_3^{n+1/2}} - \sum_{n=0}^{\infty} \frac{Q_n^{\text{odd}} R_n^{\text{odd}}(y_1) y_1 y_2^{n+\frac{1}{2}}}{(y_1^2-1)\, y_3^{n+1}} \right) \\ &= \pi G_N^2 m_1 m_2^2 \frac{\left(\sqrt{\beta}\left(4|a|y_1 u^{3/2}\beta - |b|\beta u^2\right) + (|a|y_1 - |b|\sqrt{u})^{\,4}\right)}{2|a|^2 u^{3/2}\beta^{3/2}}, \end{aligned} \tag{6.23}$$

where we have replaced the variables $y_2 = -|a|^2$, $a > 0$ and $y_3 = -|b|^2$, $|a|$, $|b| > 0$, and have defined:

$$u \equiv (-1 + y_1^2), \quad \beta \equiv (-|a|^2 + |b|^2). \tag{6.24}$$

The exact-in-spin bending angle will then be:

$$\begin{aligned} \theta^{\text{2PM}} &= -(G_N m_2)^2 \pi \times \\ &\frac{4|a|^5 u y_1^3 - 3|a|^4 b P_1 \sqrt{u} y_1^2 + 4|a|^3 b^2 P_3 y_1}{2|a|^2 \beta^{5/2} u^{5/2}} \\ &+ \frac{-2|a|^2|b|^3 P_2 u^{3/2} + |b|^5 u^{5/2} - \beta^{5/2} u^{5/2}}{2|a|^2 \beta^{5/2} u^{5/2}}, \end{aligned} \tag{6.25}$$

with

$$\begin{aligned} P_1 &= 5y_1^2 - 4, \\ P_2 &= 5y_1^2 - 2, \\ P_3 &= 5y_1^4 - 8y_1^2 + 3. \end{aligned} \tag{6.26}$$

We note that this result, agrees with the result presented in [40].

6.3 Exact-in-Spin Calculation

In this section we will show how to evaluate the exact-in-spin eikonal directly without expanding the integrand. As we have already discussed, we can rewrite the spin depended exponentials as delta functions, and treat them as cut propagators, using the standard techniques. This allows up to treat the impact parameter space Fourier transform simultaneously with the loop integration:

$$
\begin{aligned}
&\int \frac{d^D q}{(2\pi)^{D-2}} \frac{d^D \ell_1}{\pi^{D/2}} \frac{\delta\left(v_1 \cdot q\right) \delta\left(v_2 \cdot q\right) \delta\left(v_2 \cdot \ell_1\right)}{D_1^{\lambda_1} \ldots D_N^{\lambda_N}} e^{ib\cdot q} e^{a\cdot \ell_1} \\
&= \int \frac{d^D q}{(2\pi)^{D-2}} \frac{d^D \ell_1}{\pi^{D/2}} \frac{\delta\left(v_1 \cdot q\right) \delta\left(v_2 \cdot q\right) \delta\left(v_2 \cdot \ell_1\right)}{D_1^{\lambda_1} \ldots D_N^{\lambda_N}} e^{ib\cdot q + i\tilde{a}\cdot \ell_1} \\
&= \int_{-\infty}^{+\infty} dt e^{it} \int \frac{d^D q}{(2\pi)^{D-2}} \frac{d^D \ell_1}{\pi^{D/2}} \frac{\delta\left(b \cdot q + \tilde{a} \cdot \ell_1 - t\right) \delta\left(v_1 \cdot q\right) \delta\left(v_2 \cdot q\right) \delta\left(v_2 \cdot \ell_1\right)}{D_1^{\lambda_1} \ldots D_N^{\lambda_N}},
\end{aligned}
\tag{6.27}
$$

where D_n are propagators of the problem, and we have introduced the analytically continued spin variable $\tilde{a}^\mu = ia^\mu$.

Before we jump into the calculations, there is one subtlety that needs to be discussed. We saw in chapter 4, that the spin dependency does not enter only through the $\cosh a \cdot \ell$ function, but also from the $G(a \cdot \ell) = \frac{\sinh a\cdot\ell}{a\cdot\ell}$ function which, after writing $\sinh$ in terms of exponentials and introducing the delta function, will leave a spurious singularity behind. We can avoid this by using its integral representation:

$$
G(x_1)G(x_2) = \frac{\sinh(x_1)\sinh(x_2)}{x_1 x_2} = \int_0^1 d\sigma_1 d\sigma_2 \cosh x_1\sigma_1 \cosh x_2\sigma_2, \tag{6.28}
$$

and we can do this integration as the last step of the computation.

6.3.1 Organizing the Integrand in Sectors

After the introduction of the integral representation of the G functions, the original integrand splits into a sum of integrands, and each of them can be classified into distinct sectors $(\alpha) = (a_1, \ldots, a_n)$, based on the kinematic invariants it contains. Then, the eikonal can be reorganized as a sum over these sectors:

$$
\chi^{\text{2PM}} = \mathcal{G} \sum_a \int_0^1 \int \prod_i dK_i \frac{e^{\left(i\sum_j K_j \cdot \alpha_j\right)} \delta\left(u_1 \cdot q\right) \delta\left(u_2 \cdot q\right) \delta\left(u_2 \cdot \ell_1\right)}{D_1^{\lambda_1} \ldots D_N^{\lambda_N}}, \tag{6.29}
$$

where we kept the σ integrations explicit. $D_i = \left\{\ell_1^2,\ (q-\ell_1)^2,\ \ell_1 \cdot v_1\right\}$, are the propagators, $K_i = \{\ell, q, \sigma_1.\sigma_2\}$, $\mathcal{G} = \frac{1}{2!}\frac{(32\pi G_N)^2}{(4\pi)^{D/2}}(-\pi)\frac{m_1^2 m_2^3}{4m_1 m_2}$ and α_j represents the variables present in each sector (kinematic invariants, σ_1,σ_2). For this computation, it is natural to introduce the following variables:

$$y_1 = v_1 \cdot v_2, \quad y_2 = \frac{a \cdot a}{-b \cdot b}, \quad y_3 = b \cdot b, \tag{6.30}$$

and the sectors are defined by the variables:

$$\begin{aligned} \left(\hat{b}, \hat{a}\right) &\equiv \left(b + c_1\left(\sigma\right)\tilde{a}, c_2\left(\sigma\right)\tilde{a}\right), \\ \left(\vec{\hat{y}}\right) &\equiv \left(\hat{y}_1, \hat{y}_2, \hat{y}_3, \sigma\right), \end{aligned} \tag{6.31}$$

with $\tilde{a} = ia$ and the hatted variables $\hat{a}, \hat{b}, \vec{\hat{y}}$ being defined in a similar fashion to eq. 6.30.

We found the following 14 sectors:

$$\begin{aligned}
i\hat{b}_1 &= ib - i\tilde{a}, & i\hat{a}_1 &= 2i\tilde{a}, \\
i\hat{b}_2 &= ib + i\tilde{a}, & i\hat{a}_2 &= -2i\tilde{a}, \\
i\hat{b}_3 &= ib - i\sigma_2\tilde{a}, & i\hat{a}_3 &= i(\sigma_2 - \sigma_1)\tilde{a}, \\
i\hat{b}_4 &= ib + i\sigma_2\tilde{a}, & i\hat{a}_4 &= -i(\sigma_2 + \sigma_1)\tilde{a}, \\
i\hat{b}_5 &= ib - i\sigma_2\tilde{a}, & i\hat{a}_5 &= i(\sigma_2 + \sigma_1)\tilde{a}, \\
i\hat{b}_6 &= ib + i\sigma_2\tilde{a}, & i\hat{a}_6 &= -i(\sigma_2 - \sigma_1)\tilde{a},
\end{aligned}
\qquad
\begin{aligned}
i\hat{b}_7 &= i\hat{b}_2, & i\hat{a}_7 &= i(\sigma_1 - 1)\tilde{a}, \\
i\hat{b}_8 &= i\hat{b}_1, & i\hat{a}_8 &= i(\sigma_1 + 1)\tilde{a}, \\
i\hat{b}_9 &= i\hat{b}_1, & i\hat{a}_9 &= i(-\sigma_1 + 1)\tilde{a}, \\
i\hat{b}_{10} &= i\hat{b}_2, & i\hat{a}_{10} &= -i(\sigma_1 + 1)\tilde{a}, \\
i\hat{b}_{11} &= i\hat{b}_3, & i\hat{a}_{11} &= i(\sigma_1 - 1)\tilde{a}, \\
i\hat{b}_{12} &= i\hat{b}_3, & i\hat{a}_{12} &= i(\sigma_1 + 1)\tilde{a}, \\
i\hat{b}_{13} &= i\hat{b}_4, & i\hat{a}_{13} &= i(-\sigma_1 + 1)\tilde{a}, \\
i\hat{b}_{14} &= i\hat{b}_4, & i\hat{a}_{14} &= -i(\sigma_1 + 1)\tilde{a}.
\end{aligned} \tag{6.32}$$

The integrals that belong in each sector have the following form:

$$\begin{aligned}
I_1^{(\alpha)} &= \int \frac{d^D q}{(2\pi)^{D-2}} \frac{d^D \ell_1}{\pi^{D/2}} \frac{\delta\left(v_2 \cdot l_1\right)\delta\left(q \cdot v_1\right)\delta\left(q \cdot v_2\right)e^{i\hat{a}\cdot l_1 + i\hat{b}\cdot q}}{l_1^2\left(q - l_1\right)^2} \sum_i \frac{\mathcal{N}_i^{(\alpha)}\left(\vec{\hat{y}}, \ell_1, \ell_3, q\right)}{\left(\ell_1 \cdot v_1\right)^{\lambda_i}\left(q^2\right)^{\kappa_i}}, \\
I_2^{(\alpha)} &= \int \frac{d^D q}{(2\pi)^{D-2}} \frac{d^D \ell_1}{\pi^{D/2}} \frac{\delta\left(v_2 \cdot l_1\right)\delta\left(q \cdot v_1\right)\delta\left(q \cdot v_2\right)e^{i\hat{b}\cdot q}}{l_1^2\left(q - l_1\right)^2} \sum_i \frac{\mathcal{N}_i'^{(\alpha)}\left(\vec{\hat{y}}, \ell_1, \ell_3, q\right)}{\left(\ell_1 \cdot v_1\right)^{\lambda_i}\left(q^2\right)},
\end{aligned} \tag{6.33}$$

where the numerators $\mathcal{N}_i^{(\alpha)}, \mathcal{N}_i'^{(\alpha)}$, as well as the integers λ_i and κ_i, are summarized in the appendix B

In the next subsection, we will discuss the introduction of the delta function, as well as the reduction of the integrals to master integrals. The discussion is going to be general and thus applicable to all sectors α. For that reason, we will drop the hats in the variables $\vec{\hat{y}}$ and $\hat{a}$, $\hat{b}$.

6.3.2 Master Integrals

We begin by defining the integral families $\tilde{I}_1$ and $\tilde{I}_2$ in which the integrals I_1 and I_2 (note that we have dropped the sector dependence α) in eq. 6.33, belong to:

$$\begin{aligned}
&\tilde{I}_1(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7,\lambda_8) = \\
&\int \frac{d^D q}{(2\pi)^{D-2}} \frac{d^D \ell_1}{\pi^{D/2}} \frac{\delta^{\lambda_6}(a\cdot l_1 + b\cdot q - t)\,\delta^{\lambda_3}(v_2\cdot l_1)\,\delta^{\lambda_4}(q\cdot v_1)\,\delta^{\lambda_5}(q\cdot v_2)}{(l_1^2)^{\lambda_1}\left((q-l_1)^2\right)^{\lambda_2}(q^2)^{\lambda_7}(\ell_1\cdot v_1)^{\lambda_8}}, \\
&\tilde{I}_2(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7,\lambda_8) = \\
&\int \frac{d^D q}{(2\pi)^{D-2}} \frac{d^D \ell_1}{\pi^{D/2}} \frac{\delta^{\lambda_6}(b\cdot q - t)\,\delta^{\lambda_3}(v_2\cdot l_1)\,\delta^{\lambda_4}(q\cdot v_1)\,\delta^{\lambda_5}(q\cdot v_2)}{(l_1^2)^{\lambda_1}\left((q-l_1)^2\right)^{\lambda_2}(q^2)^{\lambda_7}(\ell_1\cdot v_1)^{\lambda_8}}.
\end{aligned} \tag{6.34}$$

The integral $\tilde{I}_2$ reduces to one master integral:

$$\tilde{J} = \tilde{I}_2(1,1,1,1,1,1,0,0,0,0), \tag{6.35}$$

and differential equations it satisfies are summarized below:

$$\begin{aligned}
&\partial_t \tilde{J}(t,y_1,y_3) = \frac{2(-4+d)\tilde{J}(t,y_1,y_3)}{t}, \quad \partial_{y_3}\tilde{J}(t,y_1,y_3) = \frac{(7-2d)\tilde{J}(t,y_1,y_3)}{2y_3}, \\
&\partial_{y_1}\tilde{J}(t,y_1,y_3) = \frac{\partial_{y_1}\tilde{J}(t,y_1,y_3)}{\frac{1}{y_1}-y_1},
\end{aligned} \tag{6.36}$$

The solution for $\tilde{J}$ is :

$$\tilde{J}(t,y_1,y_3) = \frac{i t^{2(D-4)}}{4\pi(D-4)\sqrt{y_1^2-1\sqrt{y_3}}}, \tag{6.37}$$

and was evaluated to match the q-integrated spinless triangle integral, which also appeared in section 6.2:

$$\begin{aligned}\int_{\infty}^{+\infty} dte^{it}\tilde{J}\left(t,y_1,y_3\right) &= \int \frac{d^Dq}{(2\pi)^{D-2}}\frac{d^D\ell_1}{\pi^{D/2}}\frac{e^{ib\cdot q}\delta\left(q\cdot v_1\right)\delta\left(q\cdot v_2\right)\delta\left(\ell_1\cdot v_2\right)}{\ell_1^2\left(q-\ell_1\right)^2}\\ &= \frac{2^{4-D}\pi\sec\left(\frac{\pi D}{2}\right)}{\Gamma\left(\frac{D}{2}-1\right)}\int\frac{d^Dq}{(2\pi)^{D-2}}e^{ib\cdot q}\delta\left(q\cdot v_1\right)\delta\left(q\cdot v_2\right)\left(-q^2\right)^{\frac{D-5}{2}}\\ &= -\frac{i2^{D-6}e^{-i\pi D}y_3^{\frac{7}{2}-D}\sec\left(\frac{\pi D}{2}\right)\Gamma\left(D-\frac{7}{2}\right)}{\pi\sqrt{y_1^2-1}\Gamma\left(\frac{5-D}{2}\right)\Gamma\left(\frac{D}{2}-1\right)}\\ &= -\frac{i}{4\pi\sqrt{y_3}\sqrt{y_1^2-1}}+\mathcal{O}\left(D-4\right).\end{aligned} \tag{6.38}$$

The integral $\tilde{I}_1$ of eq. 6.34 reduces to a total of 6 master integrals. However only 3 of them contribute to the computation and we summarize them below:

$$\begin{aligned}&\hat{J}_1=\tilde{I}_1(1,1,1,1,1,1,0,0),\quad \hat{J}_2=\tilde{I}_1(2,1,1,1,1,1,0,0),\quad \hat{J}_3=\tilde{I}_1(1,1,1,1,1,1,0,1),\\ &\vec{\hat{J}}=\left(\hat{J}_1,\hat{J}_2,\hat{J}_3\right).\end{aligned} \tag{6.39}$$

The system of differential equations for the t-dependence is independent from $\vec{y}$:

$$\partial_t\vec{\hat{J}}=\begin{pmatrix}\frac{2(D-4)}{t} & 0 & 0\\ 0 & \frac{2(D-5)}{t} & 0\\ 0 & 0 & \frac{2(D-5)}{t}\end{pmatrix}\begin{pmatrix}\hat{J}_1(t,\vec{y})\\ \hat{J}_2(t,\vec{y})\\ J_4(t,\vec{y})\end{pmatrix}, \tag{6.40}$$

with solution:

$$\begin{aligned}\vec{\hat{J}}&=\left(f_1(t)\hat{J}_1'(\vec{y}),f_2(t)\hat{J}_2'(\vec{y}),f_3(t)\hat{J}_3'(\vec{y})\right)^T,\\ \vec{f}(t)=\left(f_1(t),f_2(t),f_3(t)\right)^T&=\left(\frac{t^{2(d-4)}}{d-4},t^{2(d-5)},t^{2(d-5)},\right)^T.\end{aligned} \tag{6.41}$$

The fact that the t dependent part is independent from the rest, means that one can separate the t from $\vec{y}$ dependence in the master integrals and thus factor the t integration out of the loop integrations. To proceed further we will define a new set of master integrals:

$$\mathcal{J}_1=\frac{D-4}{t^{2(D-4)}}\hat{J}_1,\quad \mathcal{J}_2=\frac{1}{t^{2(D-8)}}\hat{J}_2,\quad \mathcal{J}_3=\frac{1}{t^{2(D-8)}}\hat{J}_3,\quad \vec{\mathcal{J}}=\left(\mathcal{J}_1,\mathcal{J}_2,\mathcal{J}_3\right), \tag{6.42}$$

which satisfy the following differential equations:

$$\partial_{y_i}\vec{\mathcal{J}} = \mathbf{A}(t)^{-1}\mathbf{B}_i(\vec{y},t)\mathbf{A}(t)\vec{\mathcal{J}}, \tag{6.43}$$

with,

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} \frac{2(D-4)}{t} & 0 & 0 \\ 0 & \frac{2(D-5)}{t} & 0 \\ 0 & 0 & \frac{2(D-5)}{t} \end{pmatrix}, \\
\mathbf{B}_1 &= \begin{pmatrix} \frac{1}{\frac{1}{y_1}-y_1} & 0 & 0 \\ 0 & \frac{1}{\frac{1}{y_1}-y_1} & 0 \\ 0 & 0 & \frac{1}{\frac{1}{y_1}-y_1} \end{pmatrix}, \\
\mathbf{B}_2 &= \begin{pmatrix} -\frac{(D-4)(3y_2-2)}{2(y_2-1)y_2} & \frac{(D-5)t^2}{2(D-4)(y_2-1)y_2y_3} & 0 \\ -\frac{(D-4)(2D-9)y_3}{2t^2} & 0 & 0 \\ \frac{(D-4)(2D-9)y_3}{2(D-5)t^2} & -\frac{1}{2y_2} & -\frac{D-4}{2y_2} \end{pmatrix}, \\
\mathbf{B}_3 &= \begin{pmatrix} \frac{7-2D}{2y_3} & 0 & 0 \\ 0 & \frac{9-2D}{2y_3} & 0 \\ 0 & 0 & \frac{9-2D}{2y_3} \end{pmatrix}.
\end{aligned} \tag{6.44}$$

The solutions for $\vec{\mathcal{J}}$, in D dimensions are:

$$\mathcal{J}_1 = \frac{h_1(y_2)\,y_3^{\frac{7}{2}-D}}{\sqrt{1-y_1^2}}, \quad \mathcal{J}_2 = \frac{h_2(y_2)\,y_3^{\frac{9}{2}-D}}{\sqrt{1-y_1^2}}, \quad \mathcal{J}_3 = \frac{h_3(y_2)\,y_3^{\frac{9}{2}-D}}{\sqrt{1-y_1^2}}. \tag{6.45}$$

The function $h_3(y_2)$ is the solution of the y_2 dependence of $\mathcal{J}_3$ and reads:

$$\begin{aligned}
h_3(y_2) = &-\frac{2c_2e^{-\frac{1}{2}i\pi D}y_2^{5-D}\cdot {}_3F_2\left(\frac{1}{2},\ 3-\frac{D}{2},\ \frac{7}{2}-\frac{D}{2};\ 6-D,\ 4-\frac{D}{2};\ y_2\right)}{D-6} \\
&-\frac{2c_3e^{\frac{i\pi D}{2}}\cdot {}_3F_2\left(\frac{D}{2}-2,\ \frac{D}{2}-\frac{3}{2},\ D-\frac{9}{2};\ \frac{D}{2}-1,\ D-4;\ y_2\right)}{D-4} \\
&+c_1y_2^{2-\frac{D}{2}},
\end{aligned} \tag{6.46}$$

where ${}_3F_2(a_1,a_1,a_3;b_1,b_2;z)$, is the generalized hyper-geometric function.

Similarly to h_3, the y_2 dependency of $\mathcal{J}_1$ and $\mathcal{J}_2$, h_1 and h_2 was determined as follows:

$$
\begin{aligned}
h_1(y_2) &= -\frac{c_2 y_2^{4-D} e^{-\frac{i\pi D}{2}}}{2D-9}\Gamma(6-D)\Big(g_1\left(\varrho_1 y_2^2 + 2\varrho_2 - 3y_2\right) + g_2\left(\varrho_4 y_2 + 2\varrho_2\right)\varrho_3\Big) \\
&\quad - c_3 e^{\frac{i\pi D}{2}}\Gamma(D-4)\left(g_3\left(y_2-1\right)\varrho_5 + 2g_4\right), \\
h_2(y_2) &= \frac{e^{-\frac{i\pi D}{2}}}{5-D}\Bigg(c_2 y_2^{5-D}\Gamma(6-D)\Big(g_2\varrho_3\left(\varrho_4 y_2 + \varrho_2\right) + g_1\left(y_2-1\right)\left(\varrho_1 y_2 - \varrho_2\right)\Big) \\
&\quad + c_3 e^{i\pi D}\Gamma(D-4)\left(g_3\left(y_2-1\right)y_2\varrho_5\varrho_6 + 2g_4\left(\varrho_6 y_2 - \varrho_2\right)\right)\Bigg).
\end{aligned} \tag{6.47}
$$

The functions g_1, g_2, g_3 and g_4 are the regularized hyper-geometric functions ${}_2\tilde{F}_1\left(a,b;c;z\right) = {}_2F_1\left(a,b;c;z\right)/\Gamma(c)$ and are defined as:

$$
\begin{aligned}
g_1 &= {}_2\tilde{F}_1\left(\frac{1}{2}, \frac{9-d}{2}; 7-d; y_2\right), \\
g_2 &= {}_2\tilde{F}_1\left(-\frac{1}{2}, \frac{9-d}{2}; 7-d; y_2\right), \\
g_3 &= {}_2\tilde{F}_1\left(d-\frac{7}{2}, \frac{d-1}{2}; d-3; y_2\right), \\
g_4 &= {}_2\tilde{F}_1\left(d-\frac{9}{2}, \frac{d-3}{2}; d-4; y_2\right),
\end{aligned} \tag{6.48}
$$

the constants $\varrho_{i=1,\ldots,6}$ carry the dependence on the dimension D and read:

$$
\begin{aligned}
&\varrho_1 = (D-8)(d-6), \quad &&\varrho_2 = (D-5), \quad &&\varrho_3 = (2D-13), \\
&\varrho_4 = (d-6), &&\varrho_5 = (D-3), &&\varrho_6 = (2D-9),
\end{aligned} \tag{6.49}
$$

and c_1, c_2, c_3 are the constants of integration that will be determined promptly.

6.3.3 Boundary Conditions

In order to determine the integration constants, one has to solve the boundary integrals. We can determine the boundary conditions by requiring the master integrals $\mathcal{I}_1, \mathcal{I}_2$ and $\mathcal{I}_3$ to reduce to the ones found in the spinless case, and we note that we evaluate the boundary values using the IBP relations for the $\tilde{I}_2$ family in eq. 6.34. Thus, we have the following boundary conditions:

$$\begin{aligned}
\mathcal{J}_1^{y_2\to 0} &= -\frac{i}{4\pi\sqrt{y_3}\sqrt{y_1^2-1}} + \mathcal{O}\left(D-4\right), \\
\mathcal{J}_2^{y_2\to 0} &= \int \frac{d^Dq}{(2\pi)^{D-2}}\frac{d^D\ell_1}{\pi^{D/2}}\frac{e^{ib\cdot q}\delta\left(q\cdot v_1\right)\delta\left(q\cdot v_2\right)\delta\left(\ell_1\cdot v_2\right)}{\left(\ell_1^2\right)^2\left(q-\ell_1\right)^2} = 0, \\
\mathcal{J}_3^{y_2\to 0} &= \int \frac{d^Dq}{(2\pi)^{D-2}}\frac{d^D\ell_1}{\pi^{D/2}}\frac{e^{ib\cdot q}\delta\left(q\cdot v_1\right)\delta\left(q\cdot v_2\right)\delta\left(\ell_1\cdot v_2\right)}{\ell_1^2\left(q-\ell_1\right)^2 q^2} \\
&= \frac{i\sqrt{y_3}}{2\pi\sqrt{y_1^2-1}} + \mathcal{O}\left(D-4\right).
\end{aligned} \tag{6.50}$$

There are two points that need to be clarified here. The first one is that in order to get the spinless integrals, we decomposed $a^\mu = |a|\hat{\alpha}$, with $|a|>0$, in the same way we did for the spin expansion in section 6.2. Then, after taking the limit $|a|\to 0$, the integrals of the $\tilde{I}_1$ family in eq. 6.34, reduce to the corresponding integrals of the $\tilde{I}_2$ family in eq. 6.34.

For the second point, let us clarify the following: the integrals we evaluate the boundary conditions for have the form:

$$\int dte^{it}f_i(t)\mathcal{J}_i(\vec{y}), \tag{6.51}$$

with $f(t)_{i=1,\ldots,3}$ and $\mathcal{J}_i(\vec{y})$ being found in eqs. 6.41, 6.42 and 6.45. Since the t dependence decouples from the rest, the t integral can be treated as an overall constant ρ:

$$\int dte^{it}f_i(t)\mathcal{J}_i(\vec{y}) = \rho\mathcal{J}_i(\vec{y}), \tag{6.52}$$

which can be absorbed by the constants c_1, c_2 and c_3, in eqs. 6.46, 6.46 and 6.47. Therefore, one can determine these constants by matching the $\mathcal{J}$s to the boundaries in eq.6.50 directly.

We are now in position to take the spinless limit $y_2\to 0$ in eq. 6.45. We do so by considering the Taylor expansions of $\mathcal{J}_i$ around $d\to 4$ and (evidently) $y_2\to 0$. To fix the constants we only need to consider the expansion of $\mathcal{J}_1$ and $\mathcal{J}_3$, since this limit is trivial for $\mathcal{J}_2$, and we have:

$$\begin{aligned}
\mathcal{J}_1(\vec{y})\Big|_{y_2\to 0; D\to 4} &= \frac{1}{D-4}\left(\frac{c_3}{\sqrt{1-y_1^2}\sqrt{y_3}} + O\left(y_2^1\right)\right) \\
&+ \left(\frac{-2c_3\log(y_3) + 8c_2 + (2-2\gamma+i\pi+\mathcal{H}_1)c_3}{2\sqrt{1-y_1^2}\sqrt{y_3}} + O\left(y_2^1\right)\right) \\
&+ O\left((D-4)\right), \\
\mathcal{J}_3(\vec{y})\Big|_{y_2\to 0; D\to 4} &= -\frac{2\left(c_3\sqrt{y_3}\right)}{(D-4)\sqrt{1-y_1^2}} \\
&+ \left(\frac{\sqrt{y_3}\left(2c_3\log(y_3) + c_1 - i\pi c_3 - c_3\mathcal{H}_2\right)}{\sqrt{1-y_1^2}} + O\left(y_2^1\right)\right) \\
&+ O\left((D-4)\right).
\end{aligned} \tag{6.53}$$

The constants $\mathcal{H}_1$ and $\mathcal{H}_2$ contain solely derivatives of the hyper-geometric functions and are irrelevant for this computation. We can make this limit finite by setting $c_3 = 0$. Then we have:

$$\begin{aligned}
\mathcal{J}_1(\vec{y})\Big|_{y_2\to 0; D\to 4} &= \left(\frac{4c_2}{\sqrt{1-y_1^2}\sqrt{y_3}} + O\left(y_2^1\right)\right) + O\left((D-4)^1\right), \\
\mathcal{J}_2(\vec{y})\Big|_{y_2\to 0; D\to 4} &= \left(\frac{c_1\sqrt{y_3}}{\sqrt{1-y_1^2}} + O\left(y_2^1\right)\right) + O\left((D-4)^1\right),
\end{aligned} \tag{6.54}$$

and by comparison to eq. 6.50 we see that $c_1 = -8c_2$ and $c_2 = 1/16\pi$. Thus the master integrals in $d = 4$ dimensions are:

$$\begin{aligned}
\mathcal{J}_1(\vec{y}) &= -\frac{iK(y_2)}{2\pi^2\sqrt{y_1^2-1}\sqrt{y_3}}, \\
\mathcal{J}_2(\vec{y}) &= -\frac{i\sqrt{y_3}\left((y_2-1)K(y_2) + E(y_2)\right)}{2\pi^2\sqrt{y_1^2-1}}, \\
\mathcal{J}_3(\vec{y}) &= \frac{i\sqrt{y_3}\left(\pi + 2E(y_2)\right)}{4\pi^2\sqrt{y_1^2-1}},
\end{aligned} \tag{6.55}$$

where $K(x)$ and $E(x)$ are the complete elliptic integrals. Note that these solutions where found also in [51]. In that work the authors did a computation similar to ours. Namely, they calculated the scattering of two Kerr black holes with spins a_1 and a_2, in a slight generalization of the aligned spin scenario: $a = a_2 = \xi a_1$, with $b \cdot a_i \neq 0$. The first two MIs we found match directly with theirs (eq. 23) in their paper. The third MI they present carries the $b \cdot a$ dependence, and it can be seen, that setting $b \cdot a \to 0$ in eq. 27 of the their paper, returns our result in eq. 6.55.

6.3.4 Putting the Result Together

The exact-in-spin, spinning-spinless eikonal, at 2PM, has the following form:

$$\chi^{(2\text{PM})} = \int d\sigma_1 d\sigma_2 \int dt e^{it} \sum_{\alpha=1,2} \left(\hat{I}_1^{(\alpha)} + \hat{I}_2^{(\alpha)}\right) + \sum_{\alpha=3}^{14} \hat{I}_1^{(\alpha)}, \tag{6.56}$$

with $I_{1,2}^{(\alpha)}$ (which can be found in the appendix B) being the integrals found in each sector in eq. 6.3.1, and we have kept the σ and t integrations explicit. After we reduce this to the master integrals found in the previous section, the t integrals take the form:

$$\int_{-\infty}^{+\infty} dt e^{it} t^{j-4\varepsilon}, j = 0, \ldots, 6. \tag{6.57}$$

We can do this integration[4] by expanding $t^j - 4\varepsilon \approx t^j - 4t^j \log|t|\varepsilon + \mathcal{O}(\varepsilon^2)$. The first integral is:

$$\int_{-\infty}^{+\infty} dt e^{it} t^j \equiv \mathcal{T}_j, \tag{6.58}$$

and can be computed by noticing that the total derivative $\partial_t = (e^{it}t^j)$, vanishes at the boundaries $-\infty, +\infty$. Thus we can integrate eq. 6.58 by parts to get:

$$i\mathcal{T}_j + j\mathcal{T}_{j-1} = 0, \quad \mathcal{T}_j = i^j (1)_j \mathcal{T}_0, \tag{6.59}$$

where $(1)_j$ is the Pochhammer symbol. Since the integral $\mathcal{T}_0 = \int_{-\infty}^{+\infty} dt e^{it}$ vanishes, all integrals $\mathcal{T}_j$ vanish as well, and the integral in eq. 6.57 becomes:

$$\int_{-\infty}^{+\infty} dt e^{it} t^{j-4\varepsilon} = -4\varepsilon \int_{-\infty}^{+\infty} dt e^{it} t^j \log|t| + \mathcal{O}\left(\varepsilon^2\right), \tag{6.60}$$

and restricting j to non negative integers we get :

$$\int_{-\infty}^{+\infty} dt e^{it} t^{j-4\varepsilon} = 4\pi (i)^j (j!)\varepsilon + \mathcal{O}\left(\varepsilon^2\right). \tag{6.61}$$

[4]For an alternative, exact-in-ε derivation of this result see C

It is interesting to note that the leading piece of eq. 6.61, is linear in ε. Moreover, the IBP reduction will generate terms that look like $t^{2j-4\varepsilon}/\varepsilon$, and thus, after the t integration the result is finite.

Let us showcase the calculation of the integrals of the first 2 sectors which can be written as:

$$\int dte^{it}\Bigg(\int \frac{d^Dq}{(2\pi)^{D-2}}\frac{d^D\ell_1}{\pi^{D/2}}\frac{\delta\left(v_2\cdot l_1\right)\delta\left(q\cdot v_1\right)\delta\left(q\cdot v_2\right)}{l_1^2\left(q-l_1\right)^2}e^{i\hat{a}\cdot l_1+i\hat{b}\cdot q}\tilde{\mathcal{P}}\left(q,\ell\right) + \int \frac{d^Dq}{(2\pi)^{D-2}}\frac{d^D\ell_1}{\pi^{D/2}}\frac{\delta\left(v_2\cdot l_1\right)\delta\left(q\cdot v_1\right)\delta\left(q\cdot v_2\right)}{l_1^2\left(q-l_1\right)^2}e^{i+i\hat{b}\cdot q}\tilde{\mathcal{P}}\left(q,\ell\right)\Bigg), \tag{6.62}$$

with $\tilde{\mathcal{P}}\left(q,\ell\right)$ being:

$$\tilde{\mathcal{P}} = \frac{q^2\left(-\frac{y_1^2}{8(D-2)}+\frac{1}{16(D-2)^2}+\frac{y_1^4}{16}\right)}{\left(l_1\cdot v_1\right)^2}+\frac{\left(\frac{1}{4}-\frac{1}{4(D-2)}\right)\left(l_1\cdot v_1\right)^2}{q^2} + \frac{1}{4(D-2)^2}-\frac{y_1^2}{4}. \tag{6.63}$$

The first integral in eq. 6.62, reduces to:

$$-\frac{1}{8(D-2)^2\left(y_1^2-1\right)\left(y_2-1\right)y_2}\times \left(t^{2(D-4)}\mathcal{J}_1\mathcal{C}_1+\frac{(-5+D)t^{2(D-4)}\left(\mathcal{J}_2\mathcal{C}_2+\mathcal{J}_3\mathcal{C}_3\right)}{(-4+D)(-9+2D)y_3}\right), \tag{6.64}$$

where the t-dependence, and the $D-4$ pole have been kept explicit. The master integrals $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$, are given in eq. 6.55, and the (finite for $D=4$) quantities $\mathcal{C}_i$, depend on D, y_1, y_2 and their value for $D=4$ is given below:

$$\begin{aligned}
\mathcal{C}_1 &= -4\left(y_1^2-1\right)^2-6y_2\left(y_1^2-1\right)+\left(4y_1^4-2y_1^2-2\right)y_2^2,\\
\mathcal{C}_2 &= \left(3-4y_1^2\right)y_2-4\left(y_1^2-1\right)^2,\\
\mathcal{C}_3 &= 2\left(y_1^2-1\right)^2\left(y_2-1\right).
\end{aligned} \tag{6.65}$$

The t integration, for $D=4-2\varepsilon$, yields

$$\int dte^{it}t^{-4\varepsilon} = 4\pi\varepsilon+O\left(\varepsilon^2\right), \tag{6.66}$$

and we see how it cancels the $-4+D=-2\varepsilon$ pole on the denominator of eq. 6.64. Now we can take the limit $\varepsilon \to 0$ to get:

$$\begin{aligned}
&\int dte^{it}\int \frac{d^Dq}{(2\pi)^{D-2}}\frac{d^D\ell_1}{\pi^{D/2}}\frac{\delta\left(v_2\cdot l_1\right)\delta\left(q\cdot v_1\right)\delta\left(q\cdot v_2\right)}{l_1^2\left(q-l_1\right)^2}e^{i\hat{a}\cdot l_1+i\hat{b}\cdot q}\mathcal{P}\left(q,\ell\right)\\
&=\frac{i\left(y_2-1\right)\left(\pi\left(y_1^2-1\right)^2-\left(4y_1^4-6y_1^2+1\right)y_2K\left(y_2\right)\right)}{32\pi\left(y_1-1\right)\left(y_1+1\right)\sqrt{y_1^2-1}\left(y_2-1\right)y_2\sqrt{y_3}}\\
&+\frac{i\left(2\left(y_2+1\right)y_1^4-4y_1^2-y_2+2\right)E\left(y_2\right)}{32\pi\left(y_1-1\right)\left(y_1+1\right)\sqrt{y_1^2-1}\left(y_2-1\right)y_2\sqrt{y_3}}.
\end{aligned} \tag{6.67}$$

It may seem worrisome that there is a factor of y_2 in the denominator. However, this is a spurious singularity, since the leading order of the limit $y_2 \to 0$ is

$$-\frac{3i\left(5y_1^2-1\right)}{128\sqrt{y_1^2-1}\sqrt{y_3}}+O\left(y_2^1\right). \tag{6.68}$$

The second integral in eq. 6.62, is simpler and reduces to

$$\begin{aligned}
&\int dte^{it}\int \frac{d^Dq}{(2\pi)^{D-2}}\frac{d^D\ell_1}{\pi^{D/2}}\frac{\delta\left(v_2\cdot l_1\right)\delta\left(q\cdot v_1\right)\delta\left(q\cdot v_2\right)}{l_1^2\left(q-l_1\right)^2}e^{i+i\hat{b}\cdot q}\mathcal{P}\left(q,\ell\right)\\
&=-\int dte^{it}\frac{(D-3)\tilde{J}\left(t,y_1,y_3\right)\left((2D-5)y_1^2\left((2D-3)y_1^2-6\right)+3\right)}{16(D-2)^2\left(y_1^2-1\right)}\\
&=-\frac{3i\left(5y_1^2-1\right)}{128\sqrt{y_1^2-1}\sqrt{y_3}},
\end{aligned} \tag{6.69}$$

with $\tilde{J}$ defined given in eq. 6.37, and the t-integral was the same as before. We see now that in the spinless limit, the two integrals coincide as they should.

If we were to recover the complete spinless limit, we would have to add the results in eqs. 6.68 and 6.69 together. Moreover, we would need an overall factor of two, since only the first two sectors contribute and at $|a|\to 0$ their integrals coincide. Lastly, we would need to include the overall factor $\mathcal{G}=-8\pi G_N^2 m_1 m_2^2$. The final result then reads:

$$\left(\chi'\right)^{(0)}=\pi G_N^2 m_1 m_2^2\frac{3i\left(5y_1^2-1\right)}{4\sqrt{y_1^2-1}\sqrt{y_3}}, \tag{6.70}$$

which agrees with the known result (e.g [23]).

The only thing that we haven't addressed so far is the σ integrations and unfortunately these cannot be done analytically. We may expand the σ integrands for small $|a|$ and do the integration (analytically) order by order. We present the first 3 orders of the eikonal :

$$
\begin{aligned}
(\chi')^{(0)} &= \pi G_N^2 m_1 m_2^2 \frac{3\left(5y_1^2-1\right)}{4|b|\sqrt{y_1^2-1}},\\
(\chi')^{(1)} &= -\pi G_N^2 m_1 m_2^2 \frac{|a| y_1\left(5y_1^2-3\right)}{|b|^2\left(y_1^2-1\right)},\\
(\chi')^{(2)} &= \pi G_N^2 m_1 m_2^2 \frac{|a|^2\left(95y_1^4-102y_1^2+15\right)}{16|b|^3\left(y_1^2-1\right)^{3/2}},
\end{aligned}
\tag{6.71}
$$

where we used the notation χ' to distinguish the eikonal as obtained, from the spin expansion in eq. 6.14. We see an overall agreement up to a factor of i in eq. 6.14, which will be eliminated, when one goes from the variables $y_2,\ y_3$ to $|a|,\ |b|$.

7 Spinning-Spinless Scattering At 3PM

In the present chapter, we will present the computation of the bending angle in the scattering of a Schwarzschild probe off a heavy Kerr black hole. We will relax the aligned spin condition by considering a similar problem, where the spin forms a slight angle with the angular momentum of the probe. The fact that this angle is small means that the dot product of the spin and the impact parameter will also be small, and this will allow us to truncate the spin expanded eikonal keeping only the leading term. This simplification provides the opportunity to get the exact-in-spin eikonal up to leading order in $b \cdot a$. In the final section, we will discuss possible strategies in order to obtain an all things considered, exact result.

In the previous chapter, we illustrated the two methods we have in our arsenal, in order to perform calculations that involve black holes with spin. As we saw, the method of spin expansion was simpler and required less steps compared to working with the spin resumed integrands. Moreover, we discovered that even in one loop, the introduction of the σ parameters, drove us away from an analytical result. The application of this method to the two loop problem, increased the complexity of the computation exponentially, and the time we would need to complete the calculation, would lie beyond the deadline of this work. For that reason, in this chapter we will work only with the spin expansion.

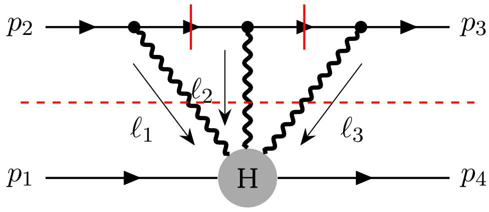
7.1 Setting Up The Problem

Let us begin by reminding the reader that we parametrize the final result, using the following external Lorentz invariant quantities:

$$\begin{aligned} y_1 &= v_1 \cdot v_2, \quad y_2 = a \cdot a, \\ y_3 &= b \cdot b, \quad\ \ y_4 = b \cdot a. \end{aligned} \tag{7.1}$$

The relevant amplitude for this computation, is shown below.

$$\mathcal{M}_{2\to2}^{\text{3PM}}\left(q,a,v_1,v_2\right) =$$



$$= \frac{1}{3!}\frac{(32\pi G_N)^3}{(4\pi)^D}\pi^2\int\frac{d^D\ell_1}{\pi^{D/2}}\frac{d^D\ell_3}{\pi^{D/2}}\delta(-m_2v_2\cdot\ell_1)\delta(-m_2v_2\cdot\ell_3)\times$$
$$\sum_{h_1,h_2,h_3}\frac{\mathcal{M}_5^{h_1,h_2,h_3}(\ell_1,\ell_2,\ell_3,v_1)\mathcal{M}_a^{-h_1}(-\ell_1,v_2)\mathcal{M}_a^{-h_2}(-\ell_2,v_2)\mathcal{M}_a^{-h_3}(-\ell_3,v_2)}{\ell_1^2\ell_2^2\ell_3^2}. \tag{7.2}$$

The involved tree level amplitudes are the HEFT 5-point and the spinning-HEFT 3-points and are given by equations 3.19 and 4.17. The polarization sum is performed with the use of equation 2.21 and the resulting integrand is expanded in spin by decomposing the spin vector as $a^\mu = |a|\hat{\alpha}$.

Similarly to the one loop case, we find only scalar products of the loop momenta with the spin parameter, $a\cdot\ell_1$ and $a\cdot\ell_3$, at even orders in spin. At odd orders, we find the loop momenta contracted with one Levi Civita tensor: $\epsilon(\ell_i,X,Y,Z) = \epsilon_{\alpha\beta\gamma\delta}\ell_i^\alpha X^\beta Y^\gamma Z^\delta$, $X^\mu, Y^\mu, Z^\mu \in \{q^\mu, v_1^\mu, v_2^\mu, \ell_{j\neq i}^\mu\}$. The relevant integral structures, are summarized below:

$$\begin{aligned}
&\int\frac{d^D\ell_1}{\pi^{D/2}}\frac{d^D\ell_3}{\pi^{D/2}}\frac{\delta\left(\ell_1\cdot v_2\right)\delta\left(\ell_3\cdot v_2\right)\left(a\cdot\ell_1\right)^{\lambda_1}\left(a\cdot\ell_3\right)^{\lambda_2}}{\ell_1^2\ell_3^2\left(q-\ell_1-\ell_3\right)^2\left(\ell_1\cdot v_1\right)^{\lambda_3}D_1^{\lambda_4}D_2^2},\\
&\int\frac{d^D\ell_1}{\pi^{D/2}}\frac{d^D\ell_3}{\pi^{D/2}}\frac{\delta\left(\ell_1\cdot v_2\right)\delta\left(\ell_3\cdot v_2\right)\left(a\cdot\ell_1\right)^{\lambda_1}\left(a\cdot\ell_3\right)^{\lambda_2}\epsilon(\ell_1,a,X,Y)}{\ell_1^2\ell_3^2\left(q-\ell_1-\ell_3\right)^2\left(\ell_1\cdot v_1\right)^{\lambda_3}D_1^{\lambda_4}D_2^2},\\
&\int\frac{d^D\ell_1}{\pi^{D/2}}\frac{d^D\ell_3}{\pi^{D/2}}\frac{\delta\left(\ell_1\cdot v_2\right)\delta\left(\ell_3\cdot v_2\right)\left(a\cdot\ell_1\right)^{\lambda_1}\left(a\cdot\ell_3\right)^{\lambda_2}\epsilon(\ell_3,a,X,Y)}{\ell_1^2\ell_3^2\left(q-\ell_1-\ell_3\right)^2\left(\ell_1\cdot v_1\right)^{\lambda_3}D_1^{\lambda_4}D_2^2},\\
&\int\frac{d^D\ell_1}{\pi^{D/2}}\frac{d^D\ell_3}{\pi^{D/2}}\frac{\delta\left(\ell_1\cdot v_2\right)\delta\left(\ell_3\cdot v_2\right)\left(a\cdot\ell_1\right)^{\lambda_1}\left(a\cdot\ell_3\right)^{\lambda_2}\epsilon(\ell_1,\ell_3,a,X)}{\ell_1^2\ell_3^2\left(q-\ell_1-\ell_3\right)^2\left(\ell_1\cdot v_1\right)^{\lambda_3}D_1^{\lambda_4}D_2^2},\\
&D_1 = \left(\ell_3\cdot v_1\right),\ \left(\ell_1\cdot v_1+\ell_3\cdot v_1\right),\\
&D_2 = \left(q-\ell_3\right)^2,\ \left(q-\ell_1\right)^2,\ \left(\ell_1+\ell_3\right)^2.
\end{aligned} \tag{7.3}$$

We tackle all the aforementioned integrals with the use of IBP reduction with LiteRed2, after we reduce the Levi-Civita tensor to equation 6.7.

At each order in spin, the loop integrals, reduce to the double triangle master integral:

$$\mathcal{K} \equiv \int \frac{d^D\ell_1}{\pi^{D/2}} \frac{d^D\ell_3}{\pi^{D/2}} \frac{\delta\left(\ell_1 \cdot v_2\right) \delta\left(\ell_3 \cdot v_2\right)}{\ell_1^2 \ell_3^2 \left(q - \ell_1 - \ell_3\right)^2}. \tag{7.4}$$

This integral appears in the spinless probe limit calculation at 3PM (see [23] for instance), and it can easily be solved with differential equations. Its only dependence is q^2 and the corresponding differential equation reads:

$$\partial_{q^2}\mathcal{K} = \frac{(-4+D)}{(-q^2)}\mathcal{K}, \quad \mathcal{K} = (-q^2)^{d-4} c_1. \tag{7.5}$$

The integration constant, can be matched with the value of $\mathcal{K}$ in [23] to:

$$\mathcal{K} = \frac{\pi^2 2^{7-D}(-q^2)^{D-4}}{4-D}. \tag{7.6}$$

The next step is the IPS Fourier transform of eq. 7.2, which is performed by replacing q^μ in scalar products of $q \cdot a$ and $\epsilon\left(a, q, v_1, v_2\right)$ to $q^\mu \to (\partial_b)^\mu = \frac{\partial}{\partial b^\mu}$, and using eq. 6.13, in parallel to the one loop computation.

The value of the remaining Levi-Civita tensor, $\epsilon\left(a, b, v_1, v_2\right)$, was calculated in eq. 3.8.

Since we are interested in a $b \cdot a$ expansion, we Taylor-expand the Levi Civita tensor in eq. 3.8, by expressing $\cos\psi$ in terms of y_2, y_3 and y_4 as[1]:

$$\cos\psi = \sqrt{1 - \sin^2\psi} = \sqrt{1 - \frac{y_4^2}{y_2 y_3}}. \tag{7.7}$$

This expansion is valid. Relaxing the aligned spin condition, makes ψ small and thus $\sin\psi = \frac{y_4}{\sqrt{-y_2}\sqrt{-y_3}} \ll 1$. The expanded $\epsilon(a, b, v_1, v_2)$, reads:

$$\begin{aligned}\epsilon(a, b, v_1, v_2) &= \cos\psi \sqrt{-y_2}\sqrt{-y_3}\sqrt{y_1^2 - 1} \\ &= \sqrt{-y_2}\sqrt{-y_3}\sqrt{y_1^2 - 1}\left(1 - \frac{y_4^2}{2\left(y_2 y_3\right)} + O\left(y_4^4\right)\right).\end{aligned} \tag{7.8}$$

[1] We remind the reader that as we discussed in section 3.1, ψ is the angle of the spin vector in the plane perpendicular to the plane defined by the velocities v_1 and v_2. Thus, the scalar product $a \cdot b$, can be written in terms of ψ as $\sqrt{-y_2}\sqrt{-y_3}\sin\psi$.

Below we summarize the spin expanded eikonal (with each term being exact in y_4) $\chi^{3\text{PM}} = \sum_i \hat{\chi}_i a^i \equiv \sum_i \chi^{(i)}$ up to $\mathcal{O}(a^4)$[2]:

$$\begin{aligned}
\chi^{(0)} &= -G_N^3 m_2^3 m_1 \frac{64y_1^6 - 120y_1^4 + 60y_1^2 - 5}{3\,(y_1^2-1)^{5/2} y_3}, \\
\chi^{(1)} &= G_N^3 m_2^3 m_1 \frac{4y_1\,(16y_1^4 - 20y_1^2 + 5)\,\epsilon\,(a,b,v_1,v_2)}{(y_1-1)^2\,(y_1+1)^2\sqrt{y_1^2-1}\,y_3^2}, \\
\chi^{(2)} &= G_N^3 m_2^3 m_1 \frac{(128y_1^6 - 216y_1^4 + 96y_1^2 - 7)\,(4y_4^2 - 3y_2y_3)}{3\,(y_1^2-1)^{5/2} y_3^3}, \\
\chi^{(3)} &= G_N^3 m_2^3 m_1 \frac{8y_1\,(80y_1^4 - 92y_1^2 + 21)\,(y_2y_3 - 2y_4^2)\,\epsilon\,(a,b,v_1,v_2)}{3\,(y_1^2-1)^{5/2} y_3^4}, \\
\chi^{(4)} &= -G_N^3 m_2^3 m_1 \frac{(64y_1^6 - 104y_1^4 + 44y_1^2 - 3)\,(16y_4^4 - 20y_2y_3y_4^2 + 5y_2^2y_3^2)}{(y_1^2-1)^{5/2} y_3^5}.
\end{aligned} \tag{7.9}$$

It is evident, that each term $\chi^{(n)}$ in the spin expansion, can be expanded in the variable y_4: $\chi^{(n)} = \sum_m \chi^{(n,m)}$, with $\chi^{(n,m)} \equiv \hat{\chi}_{n,m}(y_2)^{n/2} y_4^m$, and n, m count the order in y_2 and y_4 in turn. Since we are considering the angle ψ to be small, we can truncate the y_4 expansion keeping only the leading order in y_4:

$$\chi^{3\text{PM}} = \sum_n \sum_m \chi^{(n,m)} = \sum_n \hat{\chi}_{n,0}(y_2)^{n/2} + y_4^2 \sum_n \hat{\chi}_{n,2}(y_2)^{n/2} + \mathcal{O}\left(y_4^4\right). \tag{7.10}$$

The next step is to derive analytical expressions for the sums $\mathcal{A}_0 \equiv \sum_n \hat{\chi}_{n,0}(y_2)^{n/2}$ and $\mathcal{A}_2 = \sum_n \hat{\chi}_{n,2}(y_2)^{n/2}$, and it will be discussed in the forthcoming sections.

[2] An expansion up to $\mathcal{O}(a^8)$ can be found in appendix D

7.2 The Aligned Spin Contribution

We begin by presenting the first 8 terms $A_0^{(n)}$ of the leading contribution to $\chi^{3\text{PM}}$, $\mathcal{A}_0 = \sum_n A_0^{(n)}$ (we have dropped the constant $G_n^3 m_2^3 m_1$ for clarity.):

$$
\begin{aligned}
A_0^{(0)} &= \frac{-64y_1^6 + 120y_1^4 - 60y_1^2 + 5}{3\left(y_1^2 - 1\right)^{5/2} y_3}, \\
A_0^{(1)} &= \frac{4y_1\left(16y_1^4 - 20y_1^2 + 5\right)\sqrt{y_2}}{\left(y_1^2 - 1\right)^2 y_3^{3/2}}, \\
A_0^{(2)} &= \frac{\left(-128y_1^6 + 216y_1^4 - 96y_1^2 + 7\right) y_2}{\left(y_1^2 - 1\right)^{5/2} y_3^2}, \\
A_0^{(3)} &= \frac{8y_1\left(80y_1^4 - 92y_1^2 + 21\right) y_2^{3/2}}{3\left(y_1^2 - 1\right)^2 y_3^{5/2}}, \\
A_0^{(4)} &= \frac{5\left(-64y_1^6 + 104y_1^4 - 44y_1^2 + 3\right) y_2^2}{\left(y_1^2 - 1\right)^{5/2} y_3^3}, \\
A_0^{(5)} &= \frac{4y_1\left(112y_1^4 - 124y_1^2 + 27\right) y_2^{5/2}}{\left(y_1^2 - 1\right)^2 y_3^{7/2}}, \\
A_0^{(6)} &= \frac{7\left(-256y_1^6 + 408y_1^4 - 168y_1^2 + 11\right) y_2^3}{3\left(y_1^2 - 1\right)^{5/2} y_3^4}, \\
A_0^{(7)} &= \frac{16y_1\left(48y_1^4 - 52y_1^2 + 11\right) y_2^{7/2}}{\left(y_1^2 - 1\right)^2 y_3^{9/2}}, \\
A_0^{(8)} &= \frac{3\left(-320y_1^6 + 504y_1^4 - 204y_1^2 + 13\right) y_2^4}{\left(y_1^2 - 1\right)^{5/2} y_3^5}.
\end{aligned}
\tag{7.11}
$$

We note that $\mathcal{A}_0$ is the, aligned spin, contribution to the eikonal at 3PM, and the corresponding bending angle up to $\mathcal{O}\left(a^8\right)$ can be found in the appendix D. In [27], the computation up to second order in spin is presented, using the world-line formalism, and we can verify that the first two terms of our resulting bending angle match these results, after taking the probe limit and the aligned spins limit on eq. 17b and 17c of that paper.

We proceed now to re-sum eq. 7.11. We will employ the same tactic we used in the one loop computation, and we will work with the even and odd terms separately.

Starting from the even terms, we first multiply and divide by 3, the terms that do not have a 3 in the denominator, namely, $A_0^{(2)}$, $A_0^{(4)}$ and $A_0^{(8)}$, to force the sequence $\mathcal{Q}_n^{\text{even}} = 1, 3, 5, \ldots = 1 + 2n$ on the numerators. Next, the polynomials $\mathcal{R}_n^{\text{even}} =$

$\{(-64y_1^6+120y_1^4-60y_1^2+5), (-128y_1^6+216y_1^4-96y_1^2+7), 3(-64y_1^6+104y_1^4-44y_1^2+3), \ldots\}$ have a constant difference:

$$
\begin{aligned}
\mathcal{R}_2^{\text{even}} - \mathcal{R}_1^{\text{even}} &= -64y_1^6 + 96y_1^4 - 36y_1^2 + 2, \\
\mathcal{R}_3^{\text{even}} - \mathcal{R}_2^{\text{even}} &= -64y_1^6 + 96y_1^4 - 36y_1^2 + 2, \\
&\vdots \\
\mathcal{R}_n^{\text{even}} - \mathcal{R}_{n-1}^{\text{even}} &= -64y_1^6 + 96y_1^4 - 36y_1^2 + 2,
\end{aligned}
\tag{7.12}
$$

and solving the resulting difference equation yields :

$$
\mathcal{R}_n^{\text{even}} = (n+1)\left(-64y_1^6 + 96y_1^4 - 36y_1^2 + 2\right) + 3\left(8y_1^4 - 8y_1^2 + 1\right), \tag{7.13}
$$

and thus the even part of $\mathcal{A}_0$, can be generated by the sequence:

$$
\mathcal{A}_0^{(2n)} = \frac{\mathcal{Q}_n^{\text{even}} \mathcal{R}_n^{\text{even}} y_2^n}{3(y_1^2-1)^{5/2} y_3^{n+1}}, \tag{7.14}
$$

and it sums to:

$$
\begin{aligned}
\mathcal{A}_0^{\text{even}} &= \sum_{n=0}^{\infty} \frac{\mathcal{Q}_n^{\text{even}} \mathcal{R}_n^{\text{even}} y_2^n}{3(y_1^2-1)^{5/2} y_3^{n+1}} \\
&= \frac{3\left(8y_1^4 - 8y_1^2 + 1\right) y_2^2 + 6\left(32y_1^6 - 48y_1^4 + 18y_1^2 - 1\right) y_3 y_2}{3\left(y_1^2-1\right)^{5/2} \left(y_2 - y_3\right)^3} \\
&+ \frac{\left(64y_1^6 - 120y_1^4 + 60y_1^2 - 5\right) y_3^2}{3\left(y_1^2-1\right)^{5/2} \left(y_2 - y_3\right)^3}.
\end{aligned}
\tag{7.15}
$$

The procedure for the odd part is identical: we first multiply and divide the terms that do not have a factor of 3 in the denominator, and the emergent prefactros on the numerators follow the sequence $\{\mathcal{Q}^{\text{odd}}\} = \{4, 8, 12, 16, \ldots\} = \{4(n+1)\}$. The y_1 polynomials $\mathcal{R}_n^{\text{odd}}$ have constant differences as well:

$$
\begin{aligned}
\mathcal{R}_2^{\text{odd}} - \mathcal{R}_1^{\text{odd}} &= 6 - 32y1^2 + 32y1^4, \\
\mathcal{R}_3^{\text{odd}} - \mathcal{R}_2^{\text{odd}} &= 6 - 32y1^2 + 32y1^4, \\
&\vdots \\
\mathcal{R}_n^{\text{odd}} - \mathcal{R}_{n-1}^{\text{odd}} &= 6 - 32y1^2 + 32y1^4,
\end{aligned}
\tag{7.16}
$$

$$
\therefore \mathcal{R}_n^{\text{odd}} = n\left(32y_1^4 - 32y_1^2 + 6\right) + 48y_1^4 - 60y_1^2 + 15, \tag{7.17}
$$

and the odd part has the form:

$$\mathcal{A}_0^{(2n+1)} = \frac{\mathcal{Q}_n^{\text{odd}}\mathcal{R}_n^{\text{odd}} y_1 y_2^{1/2+n}}{3(y_1^2-1)^2 y_3^{n+3/2}}. \tag{7.18}$$

The sum $\mathcal{A}^{\text{odd}}$ evaluates to:

$$\begin{aligned}\mathcal{A}_0^{\text{odd}} &= \sum_{n=0}^{\infty} \frac{\mathcal{Q}_n^{\text{odd}}\mathcal{R}_n^{\text{odd}} y_1 y_2^{1/2+n}}{3(y_1^2-1)^2 y_3^{n+3/2}} = \\ &- \frac{4y_1\sqrt{y_2}\sqrt{y_3}\left(\left(16y_1^4-4y_1^2-3\right)y_2+3\left(16y_1^4-20y_1^2+5\right)y_3\right)}{3\left(y_1^2-1\right)^2\left(y_2-y_3\right)^3}.\end{aligned} \tag{7.19}$$

The full result will be:

$$\begin{aligned}\chi_0^{\text{3PM}} &= \mathcal{A}_0 = \mathcal{A}_0^{\text{even}} + \mathcal{A}_0^{\text{odd}} = \\ &- \frac{3|a|^4 P_1(y_1) + 4|a|^3|b| P_2(y_1) + 6|a|^2|b|^2 P_3(y_1)}{3\left(y_1^2-1\right)^{5/2}\left(a^2-b^2\right)^3} \\ &+ \frac{-12|a||b|^3 P_4(y_1) + |b|^4 P_5(y_1)}{3\left(y_1^2-1\right)^{5/2}\left(a^2-b^2\right)^3},\end{aligned} \tag{7.20}$$

where we replaced $y_2 = -|a|^2$ and $y_3 = -|b|^2$. Moreover, we have defined the following algebraic functions:

$$\begin{aligned}P_1(y_1) &= 8y_1^4 - 8y_1^2 + 1, \\ P_2(y_1) &= y_1\sqrt{y_1^2-1}\left(-16y_1^4+4y_1^2+3\right), \\ P_3(y_1) &= 32y_1^6 - 48y_1^4 + 18y_1^2 - 1, \\ P_4(y_1) &= y_1\sqrt{y_1^2-1}\left(16y_1^4-20y_1^2+5\right), \\ P_5(y_1) &= 64y_1^6 - 120y_1^4 + 60y_1^2 - 5.\end{aligned} \tag{7.21}$$

In the align spin case, the scattering even can be fully specified by the scattering angle, which we provide below: (note that we re-introduced the missing $G_n^3 m_2^2$ factor):

$$\begin{aligned}\theta_0^{\text{3PM}} &= -\frac{2(G_n m_2)^3\left(-2|a|^5 P_2(y_1) - 3|a|^4|b|\left(4P_3(y_1) - P_5(y_1)\right)\right)}{3\left(y_1^2-1\right)^3\left(|a|^2-|b|^2\right)^4} \\ &+ \frac{2(G_n m_2)^3\left(2|a|^3|b|^2\left(9P_4(y_1) - 5P_2(y_1)\right)\right)}{3\left(y_1^2-1\right)^3\left(|a|^2-|b|^2\right)^4} \\ &- \frac{2(G_n m_2)^3\left(2|a|^2|b|^3\left(-6P_3(y_1) - P_5(y_1)\right) + 18|a||b|^4 P_4(y_1) - |b|^5 P_5(y_1)\right)}{3\left(y_1^2-1\right)^3\left(|a|^2-|b|^2\right)^4},\end{aligned} \tag{7.22}$$

with the $P_i(y_1)$ functions defined in equation 7.21. The result for the aligned spin bending angle agrees with [40]. This is already a remarkable result. Even though it merely reproduces what was already known, it is a a spin-resumed result at 3PM.

7.3 The Leading Contribution

The first 7 terms $A_2^{(n)}$, of the leading contribution to $\chi^{3\text{PM}}$, $\mathcal{A}_2 = \sum_n A_2^{(n)}$, read:

$$
\begin{aligned}
A_2^{(0)} &= -\frac{2y_1\left(16y_1^4-20y_1^2+5\right)}{\left(y_1^2-1\right)^2\sqrt{y_2}y_3^{5/2}}, \\
A_2^{(1)} &= \frac{4\left(128y_1^6-216y_1^4+96y_1^2-7\right)}{3\left(y_1^2-1\right)^{5/2}y_3^3}, \\
A_2^{(2)} &= -\frac{20y_1\left(80y_1^4-92y_1^2+21\right)\sqrt{y_2}}{3\left(y_1^2-1\right)^2y_3^{7/2}}, \\
A_2^{(3)} &= \frac{20\left(64y_1^6-104y_1^4+44y_1^2-3\right)y_2}{\left(y_1^2-1\right)^{5/2}y_3^4}, \\
A_2^{(4)} &= -\frac{70y_1\left(112y_1^4-124y_1^2+27\right)y_2^{3/2}}{3\left(y_1^2-1\right)^2y_3^{9/2}}, \\
A_2^{(5)} &= \frac{56\left(256y_1^6-408y_1^4+168y_1^2-11\right)y_2^2}{3\left(y_1^2-1\right)^{5/2}y_3^5}, \\
A_2^{(6)} &= -\frac{168y_1\left(48y_1^4-52y_1^2+11\right)y_2^{5/2}}{\left(y_1^2-1\right)^2y_3^{11/2}}, \\
A_2^{(7)} &= \frac{40\left(320y_1^6-504y_1^4+204y_1^2-13\right)y_2^3}{\left(y_1^2-1\right)^{5/2}y_3^6},
\end{aligned}
\tag{7.23}
$$

and they present very similar patterns to the terms $A_0^{(n)}$ of eq. 7.11. Namely, we can identify the y_1 polynomials in the odd terms $\hat{\mathcal{R}}_n^{\text{odd}}$ as $\hat{\mathcal{R}}_n^{\text{odd}} = \mathcal{R}_{n+1}^{\text{even}}$. Moreover, looking at the numerators of the odd terms we can identify the sequence $\{\hat{\mathcal{Q}}_n^{\text{odd}}\}$ as $\{\hat{\mathcal{Q}}_n^{\text{odd}}\} = \{4, 20, 56, 120, \dots\} = 2/3(6+13n+9n^2+2n^3)$. Putting it all together we have:

$$
\mathcal{A}_2^{(2n+1)} = \frac{\hat{\mathcal{Q}}_n^{\text{odd}}\hat{\mathcal{R}}_n^{\text{odd}}\,y_2^n}{3(y_1^2-1)^{5/2}y_3^{n+3}}, \tag{7.24}
$$

and the sum evaluates to:

$$\begin{aligned}\mathcal{A}_2^{\text{odd}} &= \sum_{n=0}^{\infty} \mathcal{A}_2^{(2n+1)} = \\ &- \frac{4\left(\left(64y_1^6 - 72y_1^4 + 12y_1^2 + 1\right)y_2^2 + 10\left(32y_1^6 - 48y_1^4 + 18y_1^2 - 1\right)y_3y_2\right)}{3\left(y_1^2 - 1\right)^{5/2}\left(y_2 - y_3\right)^5} \\ &+ \frac{4\left(\left(128y_1^6 - 216y_1^4 + 96y_1^2 - 7\right)y_3^2\right)}{3\left(y_1^2 - 1\right)^{5/2}\left(y_2 - y_3\right)^5}.\end{aligned} \tag{7.25}$$

The case is similar for the even terms as well, as we can identify $\hat{\mathcal{R}}_n^{\text{even}} = \mathcal{R}_n^{\text{odd}}$, and the sequence $\{\hat{\mathcal{Q}}_n^{\text{even}}\} = \{2, 20, 70, 168, \ldots\} = 2/3\left((-1 - n + 4(1+n)^3\right)$. We thus have the following result:

$$\mathcal{A}_2^{(2n)} = \frac{\hat{\mathcal{Q}}_n^{\text{even}} \hat{\mathcal{R}}_n^{\text{even}} y_1 y_2^{n-1/2}}{3(y_1^2-1)^2 y_3^{n+5/2}}, \tag{7.26}$$

$$\begin{aligned}\mathcal{A}_2^{\text{even}} = \sum_{n=0}^{\infty} \mathcal{A}_2^{(2n)} &= \frac{2y_1\left(\left(16y_1^4 - 4y_1^2 - 3\right)y_2^3 + 5\left(80y_1^4 - 68y_1^2 + 9\right)y_3y_2^2\right)}{3\left(y_1^2 - 1\right)^2\sqrt{y_2}\left(y_2 - y_3\right)^5\sqrt{y_3}} \\ &+ \frac{2y_1\left(5\left(112y_1^4 - 124y_1^2 + 27\right)y_3^2y_2\right)}{3\left(y_1^2 - 1\right)^2\sqrt{y_2}\left(y_2 - y_3\right)^5\sqrt{y_3}} \\ &+ \frac{2y_1\left(3\left(16y_1^4 - 20y_1^2 + 5\right)y_3^3\right)}{3\left(y_1^2 - 1\right)^2\sqrt{y_2}\left(y_2 - y_3\right)^5\sqrt{y_3}}.\end{aligned} \tag{7.27}$$

The leading order contribution to the eikonal reads:

$$\begin{aligned}\mathcal{A}_2 &= \mathcal{A}^{\text{even}} + \mathcal{A}^{\text{odd}} = \\ &- \frac{2\left(|a|^6\tilde{P}_1(y_1) + |a|^5|b|\tilde{P}_2(y_1) + |a|^4|b|^2\tilde{P}_3(y_1) + |a|^3|b|^3\tilde{P}_4(y_1)\right)}{3|a||b|\left(y_1^2 - 1\right)^{5/2}\left(|a|^2 - |b|^2\right)^5} \\ &+ \frac{2\left(|a|^2|b|^4\tilde{P}_5(y_1) + |a||b|^5\tilde{P}_6(y_1) + b^6\tilde{P}_7(y_1)\right)}{3|a||b|\left(y_1^2 - 1\right)^{5/2}\left(|a|^2 - |b|^2\right)^5},\end{aligned} \tag{7.28}$$

and we replaced $y_2 = -|a|^2$ and $y_3 = -|b|^2$, $|a|, |b| > 0$, as usual, and we have defined the following algebraic functions:

$$\begin{aligned}
\tilde{P}_1(y_1) &= \sqrt{y_1^2 - 1}\left(16y_1^5 - 4y_1^3 - 3y_1\right), \\
\tilde{P}_2(y_1) &= -128y_1^6 + 144y_1^4 - 24y_1^2 - 2, \\
\tilde{P}_3(y_1) &= \sqrt{y_1^2 - 1}\left(400y_1^5 - 340y_1^3 + 45y_1\right), \\
\tilde{P}_4(y_1) &= -640y_1^6 + 960y_1^4 - 360y_1^2 + 20, \\
\tilde{P}_5(y_1) &= \sqrt{y_1^2 - 1}\left(560y_1^5 - 620y_1^3 + 135y_1\right), \\
\tilde{P}_6(y_1) &= -256y_1^6 + 432y_1^4 - 192y_1^2 + 14, \\
\tilde{P}_7(y_1) &= \sqrt{y_1^2 - 1}\left(48y_1^5 - 60y_1^3 + 15y_1\right).
\end{aligned} \tag{7.29}$$

Let us highlight that to our knowledge, this is an entirely new result and thus we have no literature, to compare it with.

7.4 Towards An Exact Eikonal

So far we have seen how working in the spin expansion, has been very fruitful. It enabled us to get the bending angle for the aligned spin configuration in equation 7.22, with ease, and by considering a second expansion in $b \cdot a$, we were able to obtain results beyond the aligned spin configuration. Of course, the next logical step would be to attempt to find a solution for the general problem where the angle between the spin vector a and the impact parameter b is arbitrary. That is, the scalar product $b \cdot a$, is not small anymore. This is a rather hard problem and providing a solution, is beyond the scope of this work. However, we will close this chapter by discussing possible directions, on a journey towards an exact Eikonal.

The obvious approach would be to keep working in the $b \cdot a$ expansion, obtaining higher order terms, until a pattern becomes apparent, and then try to re-sum the result. However, the complicated expressions we have gotten so far, in equations 7.20 and 7.28, seem to indicate that identifying a pattern would by a highly non trivial objective.

A different approach, might be to consider finding a systematic way to produce the polynomials in y_2, y_3, and y_4, that appear in equation 7.9. We can rewrite these polynomials, by multiplying and dividing each of them by a factor of y_4 raised to the highest power that it appears in each polynomial. We may then introduce the dimensionless variable $x \equiv \frac{y_2 y_3}{y_4^2}$. The first 7 polynomials (starting from the $\mathcal{O}\left(a^2\right)$ term) read:

$$\begin{aligned}
\mathcal{P}_0 &= 4 - 3x, \\
\mathcal{P}_1 &= 8x - 16, \\
\mathcal{P}_2 &= -5x^2 + 20x - 16, \\
\mathcal{P}_3 &= 12x^2 - 64x + 64, \\
\mathcal{P}_4 &= -7x^3 + 56x^2 - 112x + 64, \\
\mathcal{P}_5 &= 16x^3 - 160x^2 + 384x - 256, \\
\mathcal{P}_6 &= -9x^4 + 120x^3 - 432x^2 + 576x - 256.
\end{aligned} \tag{7.30}$$

One thing that is apparent, is the fact that the degree of the polynomial $\mathcal{P}_{2n}$, is the same as the polynomial $P_{(2n+1)}$. Of course, this is no coincidence, and it will be true for all the polynomials of the sequence. One can see this by the fact that for a term of the eikonal, of even spin order, $\chi^{(2n)}, n \geq 2$ the spin dependence comes through the variables y_2 and y_4, as a polynomial of the form :

$$\chi^{2n} \propto c_1 y_2^n + c_2 y_2^{n-1} y_4^2 + \cdots + c_n y_4^{2n}. \tag{7.31}$$

On the other hand, the next term, $\chi^{(2n+1)}$, will be of odd spin, and out of the $2n+1$ powers of spin it caries, the Levi-Civita tensor $\epsilon\left(a, b, v_1, v_2\right)$ will contribute 1 of them. Therefore, the rest $2n$ powers of spin has to be contributed by the variables y_2 and y_4 as a polynomial of the form of eq. 7.31. It is evident that the corresponding polynomial in x, of the terms $\chi^{(2n)}$ and $\chi^{(2n+1)}$, will be of the same degree.

The fact that $\mathcal{P}_{2n}$ and $\mathcal{P}_{2n+1}$ are of the same degree means that their quotient will be a number that does not depend on x. Indeed looking at the quotients of the terms of equal degree one finds the pattern:

$$\mathbb{Q}\left(\mathcal{P}_{2n+1}, \mathcal{P}_{2n}\right) = \{-\frac{8}{3} - \frac{12}{5}, -\frac{16}{7}, -\frac{20}{9}, \ldots\} = -\frac{4(n+1)}{2n+1}. \tag{7.32}$$

Moreover, it is expected that the quotients of polynomials whose order differs by one, to be a first degree polynomial. Looking at the quotients of $\mathcal{P}_{2n+2}$, $\mathcal{P}_{2n}$ we find the pattern

$$\begin{aligned}
\mathbb{Q}'\left(\mathcal{P}_{2n+2}, \mathcal{P}_{2n}\right) &= \left\{\frac{5x}{3} - \frac{40}{9}, \frac{7x}{5} - \frac{28}{5}, \frac{9x}{7} - \frac{48}{7}, \ldots\right\} \\
&= -\frac{(2n+3)(4n-3x+4)}{6n+3},
\end{aligned} \tag{7.33}$$

and we have checked that these properties hold up to the $\mathcal{P}_9$.

Despite the tentative character of this discussion, the fact that we have already been able to identify simple patterns in the polynomials of equation 7.30, signifies the potential of this approach.

8 Conclusions and Outlook

In this work, we reviewed and furthered the research on computing classical gravitational observables in spinning black hole binary dynamics using modern amplitude-based techniques. By employing the Heavy Mass Effective Field Theory formalism, including the spinning three point tree level amplitude, presented in [50], we calculated the all orders in spin eikonal for the scattering of a Schwarzschild probe, off a heavy Kerr black hole, at third order in PM and leading order in the angle between the spin of the Kerr black hole and the angular momentum of the probe. This calculation is a stepping stone towards, observable computations that are exact in spin, at higher orders in PM.

Through our calculations at second PM order, we illustrated the two main techniques at our disposal for the computation of the loop integrals, relevant for spinning observables. Namely, one can expand the integrand in spin, and calculate the eikonal order by order in spin up to some high order and then re-sum the result, obtaining thus, an exact result. As we saw, one may perform a second expansion by considering the angle between the angle between the angular momentum and spin, to be small. This way we were able to obtain exact-in-spin results beyond the aligned spin configuration. The advantage of this method is the fact that the resulting integrals one has to compute are typically simpler, and in some cases, well studied in the literature. On the other hand, since spin enters the calculation through the hyperbolic trigonometric functions, one can replace the resulting exponentials with delta functions. This way, they can be treated as cut propagators allowing for the use of the standard multi-loop techniques. This method provides exact-in-spin results directly, without the need for re-summation. However, we saw that in the problems of spinning-spinless dynamics, working in spin expansion, was more efficient.

The study of spinning-spinless dynamics presents a rather controllable environment. We believe that staying within this class of problems will be useful for furthering our study of the techniques we used for tackling the integrals that occur in spinning dynamics. For instance, we are definitely interested in exploring further our calculations within the framework of spin expansion, in order to incorporate exact-in-spin results beyond the aligned spin configuration, without the need for hierarchical expansions, as we discussed in the last section of chapter 7.

Another interesting direction, would be to explore the resulting function space of the integrals that occur when one works within the systematic framework of re-summed integrands at third PM order. In chapter 6 we explored a part of the resulting function space at second PM order, in the aligned spin configuration. The master integrals we encountered, were solved in terms of the complete elliptic integrals. The 2PM function space for the problem beyond the aligned spin configuration, was explored in [51]. The dependence on the direction of spin entered through incomplete elliptic integrals. The differential equations we encountered at 3PM, seemed particularly challenging at first glance. We are excited to return to this problem and investigate them further.

Moreover, in this work we constrained spin to be perpendicular to the velocity of the probe. Even though this simplification was sufficient to obtain results beyond the aligned spin configuration, it would be particularly interesting to relax this constraint and explore the problem of a totally generic spin alignment.

The framework we have reviewed in this thesis, as well as the tree level amplitudes provided in eqs. 4.17 and 4.32, are enough to tackle calculations of spinning-spinning scattering beyond the probe limit up to third PM order. It would be of interest to explore these calculations, both in the spin expansion framework, and within the systematic framework of resumed integrands. We are also interested in studying the extension of these techniques for the computation of waveforms. We leave these exciting questions for future research.

A

Spin Expanded Bending Angle at 2PM

In this Appendix we will summarize the spin expanded bending angle at 2PM $\theta^{2\mathrm{PM}} = \sum_i \theta^{(i)}$ up to $\mathcal{O}\left(a^8\right)$. This result agrees with the result presented in [132], up to conventions.

$$
\begin{aligned}
\theta^{(0)} &= \pi\left(G_N m_2\right)^2 \frac{3\left(5y_1^2-1\right)}{4|b|^2\left(y_1^2-1\right)}, \\
\theta^{(1)} &= -\pi\left(G_N m_2\right)^2 \frac{2|a| y_1\left(5y_1^2-3\right)}{|b|^3\left(y_1^2-1\right)^{3/2}}, \\
\theta^{(2)} &= \pi\left(G_N m_2\right)^2 \frac{3|a|^2\left(95y_1^4-102y_1^2+15\right)}{16|b|^4\left(y_1^2-1\right)^{2}}, \\
\theta^{(3)} &= -\pi\left(G_N m_2\right)^2 \frac{3|a|^3 y_1\left(9y_1^2-5\right)}{|b|^5\left(y_1^2-1\right)^{3/2}}, \\
\theta^{(4)} &= \pi\left(G_N m_2\right)^2 \frac{5|a|^4\left(239y_1^4-250y_1^2+35\right)}{32|b|^6\left(y_1^2-1\right)^{2}}, \\
\theta^{(5)} &= -\pi\left(G_N m_2\right)^2 \frac{15|a|^5 y_1\left(13y_1^2-7\right)}{4|b|^7\left(y_1^2-1\right)^{3/2}}, \\
\theta^{(6)} &= \pi\left(G_N m_2\right)^2 \frac{105|a|^6\left(149y_1^4-154y_1^2+21\right)}{256|b|^8\left(y_1^2-1\right)^{2}}, \\
\theta^{(7)} &= -\pi\left(G_N m_2\right)^2 \frac{35|a|^7 y_1\left(17y_1^2-9\right)}{8|b|^9\left(y_1^2-1\right)^{3/2}}, \\
\theta^{(8)} &= \pi\left(G_N m_2\right)^2 \frac{63|a|^8\left(719y_1^4-738y_1^2+99\right)}{512|b|^{10}\left(y_1^2-1\right)^{2}},
\end{aligned}
\tag{A.1}
$$

Integrals in Each Sector for the One Loop Calculation

B

In this Appendix we summarize the integrands and the integrals that appear, in each sector, in the computation of the 2PM bending angle, presented in section 6.3. In the first section we will sumarise the relevant integrands, and in the second section we will present their solutions.

B.1 Integrands

The involved integrals hae the form:

$$\begin{aligned} I_1^{(\alpha)} &= \int \frac{d^D q}{(2\pi)^{D-2}} \frac{d^D \ell_1}{\pi^{D/2}} \frac{\delta\left(v_2 \cdot l_1\right) \delta\left(q \cdot v_1\right) \delta\left(q \cdot v_2\right) e^{i\hat{a}\cdot l_1 + i\hat{b}\cdot q}}{l_1^2 \left(q - l_1\right)^2} \sum_i \frac{\mathcal{N}_i^{(\alpha)}\left(\hat{\vec{y}}, \ell_1, \ell_3, q\right)}{\left(\ell_1 \cdot v_1\right)^{\lambda_i} \left(q^2\right)^{\kappa_i}}, \\ I_2^{(\alpha)} &= \int \frac{d^D q}{(2\pi)^{D-2}} \frac{d^D \ell_1}{\pi^{D/2}} \frac{\delta\left(v_2 \cdot l_1\right) \delta\left(q \cdot v_1\right) \delta\left(q \cdot v_2\right) e^{i\hat{b}\cdot q}}{l_1^2 \left(q - l_1\right)^2} \sum_i \frac{\mathcal{N}_i^{\prime(\alpha)}\left(\hat{\vec{y}}, \ell_1, \ell_3, q\right)}{\left(\ell_1 \cdot v_1\right)^{\lambda_i} \left(q^2\right)}, \end{aligned} \tag{B.1}$$

and we will list the numerators $\mathcal{N}^{(\alpha)}$ and $\mathcal{N}^{\prime(\alpha)}$, as well as the powers λ_i, κ_i below:

$$\begin{aligned} &\alpha = 1, 2: \\ &\mathcal{N}_1 = \frac{1}{4(D-2)^2} - \frac{y_1^2}{4}, && \lambda_1 = 0 && \kappa_1 = 0 \\ &\mathcal{N}_2 = -\frac{y_1^2}{8(D-2)} + \frac{1}{16(D-2)^2} + \frac{y_1^4}{16}, && \lambda_1 = 2, && \kappa_1 = -1 \\ &\mathcal{N}_3 = \frac{1}{4} - \frac{1}{4(D-2)}, && \lambda_2 = -2, && \kappa_2 = 1 \\ &\mathcal{N}_1' = \mathcal{N}_1, && \lambda_1' = \lambda_1, && \kappa_1' = \kappa_1, \\ &\mathcal{N}_2' = \mathcal{N}_2, && \lambda_2' = \lambda_2, && \kappa_2' = \kappa_2, \\ &\mathcal{N}_3' = \mathcal{N}_3, && \lambda_3' = \lambda_3, && \kappa_3' = \kappa_3. \end{aligned} \tag{B.2}$$

$$
\begin{aligned}
&\alpha = 3,4,5,6: \\
&\mathcal{N}_1 = -\frac{(a\cdot\ell_1)(a\cdot q)}{8} + \frac{(a\cdot\ell_1)^{\,2}}{8} + \frac{(a\cdot q)^2}{8}, && \lambda_1 = -2, \quad \kappa_1 = 1, \\
&\mathcal{N}_2 = \frac{y_2 y_3}{8}, && \lambda_2 = -2, \quad \kappa_2 = 0, \\
&\mathcal{N}_3 = \left(\frac{y_1^2}{16} - \frac{y_1^4}{16}\right)(a\cdot\ell_1)^{\,2} + \left(\frac{y_1^4}{16} - \frac{y_1^2}{16}\right)(a\cdot\ell_1)(a\cdot q), && \lambda_3 = 2, \quad \kappa_3 = -1, \\
&\mathcal{N}_4 = \frac{1}{32} y_1^4 y_2 y_3 - \frac{1}{32} y_1^2 y_2 y_3 && \lambda_4 = 2, \quad \kappa_4 = -2, \\
&\mathcal{N}_5 = \left(\frac{y_2 y_3}{32} - \frac{5}{32} y_1^2 y_2 y_3\right), && \lambda_5 = 0, \quad \kappa_5 = -1, \\
&\mathcal{N}_6 = \left(\frac{y_1^2}{8} + \frac{1}{16}\right)(a\cdot\ell_1)(a\cdot q) + \left(-\frac{y_1^2}{8} - \frac{1}{16}\right)(a\cdot\ell_1)^{\,2} - \frac{1}{8} y_1^2 (a\cdot q)^2, && \lambda_6 = 0, \quad \kappa_6 = 0, \\
&\mathcal{N}' = 0.
\end{aligned}
\tag{B.3}
$$

$$
\begin{aligned}
&\alpha = 7,8,9,10: \\
&\mathcal{N}_1 = \left(\frac{i y_1^3}{8} - \frac{i y_1}{8(D-2)}\right)\epsilon\,(a,\ell_1,q,v_2)\,, && \lambda_1 = 1, \quad \kappa_1 = 0, \\
&\mathcal{N}_2 = i y_1 \epsilon\,(a,\ell_1,q,v_2)\,, && \lambda_2 = -1, \quad \kappa_2 = 1, \\
&\mathcal{N}_3 = \left(\frac{i y_1}{16(D-2)} - \frac{i y_1^3}{16}\right)\epsilon\,(a,\ell_1,v_1,v_2)\,, && \lambda_3 = 2, \quad \kappa_3 = -1, \\
&\mathcal{N}_4 = \frac{i y_1}{8}\epsilon\,(a,\ell_1,v_1,v_2)\,, && \lambda_4 = 0, \quad \kappa_4 = 0, \\
&\mathcal{N}' = 0.
\end{aligned}
\tag{B.4}
$$

$$
\begin{aligned}
&\alpha = 11,12,13,14: \\
&\mathcal{N}_1 = \left(\frac{i y_1^3}{8} - \frac{i y_1}{8(D-2)}\right)\epsilon\,(a,\ell_1,q,v_2)\,, && \lambda_1 = 1, \quad \kappa_1 = 0, \\
&\mathcal{N}_2 = -\frac{i y_1}{4}\epsilon\,(a,\ell_1,q,v_2)\,, && \lambda_2 = -1 \quad \kappa_2 = 1, \\
&\mathcal{N}_3 = \left(\frac{i y_1}{16(D-2)} - \frac{i y_1^3}{16}\right)\epsilon\,(a,q,v_1,v_2)\,, && \lambda_3 = 2, \quad \kappa_3 = -1, \\
&\mathcal{N}_4 = \frac{i y_1}{8}\epsilon\,(a,q,v_1,v_2)\,, && \lambda_4 = 0, \quad \kappa_4 = 0, \\
&\mathcal{N}_5 = -\left(\frac{i y_1}{16(D-2)} - \frac{i y_1^3}{16}\right)\epsilon\,(a,\ell_1,v_1,v_2)\,, && \lambda_5 = 2, \quad \kappa_5 = -1, \\
&\mathcal{N}_6 = -\frac{i y_1}{8}\epsilon\,(a,\ell_1,v_1,v_2)\,, && \lambda_6 = 0, \quad \kappa_6 = 0, \\
&\mathcal{N}' = 0.
\end{aligned}
\tag{B.5}
$$

B.2 Integrals

In this section we will summarize the IBP reduced integrals presented in the previous section. Following the previous definitions, we will quote here the IBP reduced values, (in $D = 4$ dimensions) of the integrals $I_1^{(\alpha)}$ and $I_2^{(\alpha)}$ defined in 6.33.

$$
\begin{aligned}
I_1^{(1,2)} &= \frac{i\left(y_2-1\right)\left(\pi\left(y_1^2-1\right)^2-\left(4y_1^4-6y_1^2+1\right)y_2K\left(y_2\right)\right)}{32\pi\left(y_1-1\right)\left(y_1+1\right)\sqrt{y_1^2-1}\left(y_2-1\right)y_2\sqrt{y_3}} \\
&+\frac{i\left(2\left(y_2+1\right)y_1^4-4y_1^2-y_2+2\right)E\left(y_2\right)}{32\pi\left(y_1-1\right)\left(y_1+1\right)\sqrt{y_1^2-1}\left(y_2-1\right)y_2\sqrt{y_3}}, \\
I_2^{1,2} &= \frac{3i\left(5y_1^2-1\right)}{128\sqrt{y_1^2-1}\sqrt{y_3}}, \\
I_1^{(3,\ldots,6)} &= \frac{i\left(\left(-11y_2^2+14y_2-7\right)y_1^2+7\left(y_2-1\right)^2\right)E\left(y_2\right)}{2\sqrt{y_1^2-1}\left(y_2-1\right)^2y_2\sqrt{y_3}} \\
&+\frac{i\left(3\pi\left(y_1^2-1\right)\left(y_2-1\right)-\left(\left(4y_2^2-3y_2+1\right)y_1^2-2y_2^2+3y_2-1\right)K\left(y_2\right)\right)}{2\sqrt{y_1^2-1}\left(y_2-1\right)y_2\sqrt{y_3}}, \\
I_1^{(7,\ldots,10)} &= \frac{iy_1\epsilon(a,b,v_1,v_2)}{16\sqrt{y_1^2-1}y_2y_3^{3/2}} \\
&-\frac{iy_1\left(4\left(y_2^2-y_2+1\right)y_1^2-3y_2^2+5y_2-4\right)E\left(y_2\right)\epsilon(a,b,v_1,v_2)}{16\pi\left(y_1-1\right)\left(y_1+1\right)\sqrt{y_1^2-1}\left(y_2-1\right)^2y_2y_3^{3/2}} \\
&-\frac{iy_1\left(2\left(y_2^2-y_2+1\right)y_1^2-y_2^2+2y_2-2\right)K\left(y_2\right)\epsilon\left(a,b,v_1,v_2\right)}{16\pi\left(y_1-1\right)\left(y_1+1\right)\sqrt{y_1^2-1}\left(y_2-1\right)y_2y_3^{3/2}}, \\
I_1^{(11,\ldots,14)} &= \frac{i\pi^2y_1\epsilon(a,b,v_1,v_2)}{16\sqrt{y_1^2-1}y_2y_3^{3/2}} \\
&+\frac{iy_1\left(y_1^2\left(y_2+1\right)-1\right)E\left(y_2\right)\epsilon\left(a,b,v_1,v_2\right)}{8\pi\left(y_1^2-1\right)^{3/2}\left(y_2-1\right)^2y_3^{3/2}} \\
&-\frac{i\pi y_1\left(2\left(y_2^2-y_2-1\right)y_1^2-2y_2^2+y_2+2\right)K\left(y_2\right)\epsilon\left(a,b,v_1,v_2\right)}{16\pi\sqrt{y_1^2-1}\left(y_2-1\right)y_2y_3^{3/2}}.
\end{aligned}
\tag{B.6}
$$

C

An All Order in ε Derivation of the t-Integrals

In this appendix we will provide an alternative calculation of the t integrals in 6.57, for all orders in ε.

We will do this integration first for even j, $j \to 2n$, $n = 0, 1, 2, 3$. We begin by rewriting $t^{2n-4\varepsilon} = (t^2)^{n-2\varepsilon} = (t^2)^{-(2\varepsilon-n)}$, and assume that $\varepsilon > 0$, such that $2\varepsilon - n > 0$. We now introduce a Schwinger parameter, s as:

$$\frac{1}{(t^2)^{2\varepsilon-n}} = \frac{1}{\Gamma(2\varepsilon-n)} \int_0^{+\infty} ds s^{2\varepsilon-n-1} e^{-st^2}, \tag{C.1}$$

and the integral of 6.57 becomes:

$$\int_{-\infty}^{+\infty} dt e^{it} \left(t^2\right)^{-(2\varepsilon-n)} = \frac{1}{\Gamma(2\varepsilon-n)} \int_0^{+\infty} ds s^{2\varepsilon-n-1} \int_{-\infty}^{+\infty} dt e^{it-st^2}. \tag{C.2}$$

The t integration can now be performed by completing the square:

$$\int_{-\infty}^{+\infty} dt e^{it-st^2} = \int_{-\infty}^{+\infty} dt e^{-\frac{1}{4s}-s\left(t-\frac{i}{2s}\right)^2} = \frac{e^{-\frac{1}{4s}}}{\sqrt{s}}, \tag{C.3}$$

and the s integration reads:

$$\int_0^{\infty} ds \frac{s^{2\varepsilon-n-1} e^{-\frac{1}{4s}}}{\sqrt{s}} = 2^{1-4\varepsilon+2n} \sqrt{\pi} \Gamma\left(\frac{1-4\varepsilon+2n}{2}\right). \tag{C.4}$$

Thus the full result is:

$$\begin{aligned} \int_{-\infty}^{+\infty} dt e^{it} \left(t^2\right)^{-(2\varepsilon-n)} &= \frac{2^{1-4\varepsilon+2n} \sqrt{\pi} \Gamma\left(\frac{1-4\varepsilon+2n}{2}\right)}{\Gamma(2\varepsilon-n)} \\ &= -2\sin(\pi n - 2\pi\varepsilon)\Gamma(2n-4\varepsilon+1). \end{aligned} \tag{C.5}$$

For the last step we used the identity:

$$\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z). \tag{C.6}$$

For odd j, we write $j \to 2n+1$, and $t^{2n+1-4\varepsilon} \to t\left(t^2\right)^{-(2\varepsilon-n)}$, and introduce a Schwinger parameter as before in C.1. We proceed with steps similar to the even case and the result reeds:

$$\begin{aligned}\int_{-\infty}^{+\infty} dt e^{it} t \left(t^2\right)^{-(2\varepsilon-n)} &= \frac{i 2^{2+2n-4\varepsilon}\sqrt{\pi}\Gamma\left(n-2\varepsilon+\frac{3}{2}\right)}{\Gamma(2\varepsilon-n)} \\ &= -2i\sin(\pi n - 2\pi\varepsilon)\Gamma(2n-4\varepsilon+2).\end{aligned} \tag{C.7}$$

By virtue of the identity theorem, the results render valid in the vicinity of ε, around 0, and thus we can expand our results C.5 and C.7 for $\varepsilon \to 0$, while restricting n to non negative integers $n = 0, 1, 2, \ldots$:

$$\begin{aligned}\int_{-\infty}^{+\infty} dt e^{it}\left(t^2\right)^{-(2\varepsilon-n)} &= 4\pi\varepsilon(-1)^n(2n)! + O\left(\varepsilon^2\right), \\ \int_{-\infty}^{+\infty} dt e^{it} t \left(t^2\right)^{-(2\varepsilon-n)} &= 4i\pi\varepsilon(-1)^n(2n+1)! + O\left(\varepsilon^2\right).\end{aligned} \tag{C.8}$$

We can encapsulate both the above expressions for both $j = 2n$ and $j = 2n+1$ with expression:

$$4\varepsilon\pi(i)^j j!, j = 0, 1, 2\ldots, \tag{C.9}$$

which is exactly the result in 6.61

D Spin Expanded Eikonal and Bending Angle at 3PM

Here we will summarize the spin expanded bending angle for the aligned spin case, as well as the spin expanded eikonal, up to $\mathcal{O}\left(a^8\right)$.

D.1 Bending Angle for Aligned Spin

$$
\begin{aligned}
\theta^{(0)} &= G_N^3 m_2^3 \frac{2\left(64y_1^6 - 120y_1^4 + 60y_1^2 - 5\right)}{3|b|^3\left(y_1^2-1\right)^3}, \\
\theta^{(1)} &= -G_N^3 m_2^3 \frac{12|a|y_1\left(16y_1^4 - 20y_1^2 + 5\right)}{|b|^4\left(y_1^2-1\right)^{5/2}}, \\
\theta^{(2)} &= G_N^3 m_2^3 \frac{4|a|^2\left(128y_1^6 - 216y_1^4 + 96y_1^2 - 7\right)}{|b|^5\left(y_1^2-1\right)^3}, \\
\theta^{(3)} &= -G_N^3 m_2^3 \frac{40|a|^3 y_1\left(80y_1^4 - 92y_1^2 + 21\right)}{3|b|^6\left(y_1^2-1\right)^{5/2}}, \\
\theta^{(4)} &= G_N^3 m_2^3 \frac{30|a|^4\left(64y_1^6 - 104y_1^4 + 44y_1^2 - 3\right)}{|b|^7\left(y_1^2-1\right)^3}, \\
\theta^{(5)} &= -G_N^3 m_2^3 \frac{28|a|^5 y_1\left(112y_1^4 - 124y_1^2 + 27\right)}{|b|^8\left(y_1^2-1\right)^{5/2}}, \\
\theta^{(6)} &= G_N^3 m_2^3 \frac{56|a|^6\left(8y_1^2\left(32y_1^4 - 51y_1^2 + 21\right) - 11\right)}{3|b|^9\left(y_1^2-1\right)^3}, \\
\theta^{(7)} &= -G_N^3 m_2^3 \frac{144|a|^7 y_1\left(48y_1^4 - 52y_1^2 + 11\right)}{|b|^{10}\left(y_1^2-1\right)^{5/2}}, \\
\theta^{(8)} &= G_N^3 m_2^3 \frac{30|a|^8\left(320y_1^6 - 504y_1^4 + 204y_1^2 - 13\right)}{|b|^{11}\left(y_1^2-1\right)^3}.
\end{aligned}
\tag{D.1}
$$

D.2 Eikonal

In order to present the eikonal, we define $\mathcal{G} = G_N^3 m_2^3 m_1$, and we have:

$$
\begin{aligned}
\chi^{(0)} &= -\mathcal{G}\frac{64y_1^6 - 120y_1^4 + 60y_1^2 - 5}{3\left(y_1^2-1\right)^{5/2} y_3},\\
\chi^{(1)} &= \mathcal{G}\frac{4y_1\left(16y_1^4 - 20y_1^2 + 5\right)\epsilon\left(a,b,v_1,v_2\right)}{\left(y_1-1\right)^2\left(y_1+1\right)^2\sqrt{y_1^2-1}\,y_3^2},\\
\chi^{(2)} &= \mathcal{G}\frac{\left(128y_1^6 - 216y_1^4 + 96y_1^2 - 7\right)\left(4y_4^2 - 3y_2y_3\right)}{3\left(y_1^2-1\right)^{5/2} y_3^3},\\
\chi^{(3)} &= \mathcal{G}\frac{8y_1\left(80y_1^4 - 92y_1^2 + 21\right)\left(y_2y_3 - 2y_4^2\right)\epsilon\left(a,b,v_1,v_2\right)}{3\left(y_1^2-1\right)^{5/2} y_3^4},\\
\chi^{(4)} &= -\mathcal{G}\frac{\left(64y_1^6 - 104y_1^4 + 44y_1^2 - 3\right)\left(16y_4^4 - 20y_2y_3y_4^2 + 5y_2^2y_3^2\right)}{\left(y_1^2-1\right)^{5/2} y_3^5},\\
\chi^{(5)} &= \mathcal{G}\frac{4y_1\left(112y_1^4 - 124y_1^2 + 27\right)\left(16y_4^4 - 16y_2y_3y_4^2 + 3y_2^2y_3^2\right)\epsilon\left(a,b,v_1,v_2\right)}{3\left(y_1^2-1\right)^{5/2} y_3^6},\\
\chi^{(6)} &= \mathcal{G}\frac{\left(256y_1^6 - 408y_1^4 + 168y_1^2 - 11\right)\left(64y_4^6 - 112y_2y_3y_4^4 + 56y_2^2y_3^2y_4^2 - 7y_2^3y_3^3\right)}{3\left(y_1^2-1\right)^{5/2} y_3^7},\\
\chi^{(7)} &= \mathcal{G}\frac{16y_1\left(48y_1^4 - 52y_1^2 + 11\right)\left(-16y_4^6 + 24y_2y_3y_4^4 - 10y_2^2y_3^2y_4^2 + y_2^3y_3^3\right)\epsilon\left(a,b,v_1,v_2\right)}{\left(y_1^2-1\right)^{5/2} y_3^8},\\
\chi^{(8)} &= -\mathcal{G}\frac{\left(320y_1^6 - 504y_1^4 + 204y_1^2 - 13\right)\left(256y_4^8 - 576y_2y_3y_4^6 + 432y_2^2y_3^2y_4^4 - 120y_2^3y_3^3y_4^2 + 9y_2^4y_3^4\right)}{3\left(y_1^2-1\right)^{5/2} y_3^9}.
\end{aligned}
\tag{D.2}
$$

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