UNIVERSITY OF COPENHAGEN
FACULTY OF SCIENCE

Master's Thesis

On Carrollian Expansion of Gravity

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Abstract

Carrollian geometry is a type of non-Lorentzian geometry that has gained considerable traction in recent years. It has important applications to flat space holography due to its relation to null hypersurfaces as well as to the physics of black holes and cosmology. One arrives at Carrollian geometry by taking a small speed of light expansion of Riemannian geometry and one arrives at Carrollian gravity in the same manner via the small speed of light limit of general relativity. The goal of this thesis is threefold: to complete the full derivation of NLO equations of motion of the Carroll expansion of GR (previously done to truncated order), to show that the Carrollian expansion of Schwarzschild black holes can be derived from general vacuum solutions to the evolution equation of the theory, and to expand the Kerr metric for both the electric and magnetic limit of the Carrollian expansion. In order to expand the Kerr metric, one needs to include odd powers of c in the Carrollian expansion which will be explored in this thesis as well.

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Contents

1. Introduction

Albert Einstein's theory of General Relativity (GR) published in 1915 is among the most successful models within modern theoretical physics. It describes how the curvature of spacetime induces what is often called the force of gravity, thus showing that it is not actually a force but rather a manifestation of the geometry of spacetime. Einstein's field equations in 3+1 dimensions are a set of 10 nonlinear partial differential equations that relate the matter distribution within a spacetime to its geometry. They are not exactly solvable in general, thus requiring various simplifications or assumptions in order to find analytic solutions. These include assumptions of symmetry, such as spherical symmetry for the Schwarzschild solution, and limits such as the infinite speed of light (c) limit, often referred to as the non-relativistic (NR) limit of GR.

In 1923 Élie J. Cartan formulated a geometric theory of Newtonian gravity, referred to as Newton-Cartan (NC) geometry [1, 2]. This framework makes manifest the connection between Newtonian gravity and GR and allows one to study the NR limit of GR. In particular one has the post-Newtonidan (PN) expansion for small speeds and weak gravity which has had great success within astrophysics and cosmology [3–5]. The NR limit $c \to \infty$ of gravity is made possible by the PN expansion where one expands in powers of $1/c$. In theory, one is able to take the opposite limit, $c \to 0$ and expand GR in powers of c. At first this might seem counter-intuitive, considering that the limit might imply that the velocity of the observer is infinite compared to that of light, which is impossible according to causality. However, one can instead interpret this limit as the ultra-local limit of GR that collapses the light cone to a single line such that all spatially separated points become causally disconnected. Jean-Marc Lévy-Leblond was the first to consider the possibility of taking this limit for the Poincaré group in 1965 [6] and coined the resulting structure Carrollian geometry. The Carroll limit of GR was first considered in 1979 by Marc Henneaux [7] and recently in [8] using a more modern approach of taking the Carroll limit of the Lorentz geometry.

Carrollian geometry and Newton-Cartan geometry are examples of non-Lorentzian geometries, which have seen a resurgence of interest in recent years. These theories allow for probing corners of GR that have useful applications of their own as well as hinting at directions for more general solutions, this is especially relevant for theories of quantum gravity. Holographic dualities are among the most promising theories of quantum gravity that relate gravitational theories to field theories. The most developed realization is that of the AdS/CFT correspondence [9] which establishes a link between strongly coupled conformal field theories (CFTs), that live on the boundary of Anti-de Sitter (AdS) space, and weakly coupled gravitational theories in asymptotically AdS spacetime (one refers to the gravitational theories as existing in the bulk of the theory). The properties of AdS spacetime endow the AdS/CFT correspondence with a lot of elegance and simplicity which makes it a very attractive model to study. However, there is great interest in establishing other types of holographic dualities that are better suited to describing real-world observations in certain areas. Lifshitz holography [10, 11] makes use of an extension of NC geometry, called torsional Newton-Cartan geometry that describes the boundary of the theory, to arrive at a non-relativistic holography that is well adapted to condensed matter problems.

Another reason for wanting to consider novel types of holographic dualities is the fact that AdS spacetimes are not compatible with observations of our own universe. This has spurred many researchers to try and construct de-Sitter or flat space holographic dualities. One of the main hints to Carrollian geometry being relevant for flat space holography is that on null hypersurfaces in Lorentzian theories, one naturally finds Carrollian geometry. Incidentally, at light-like infinity, that is to say on the conformal boundary of asymptotically flat spacetimes, one finds a null hypersurface. It would, therefore, not be surprising if Carrollian geometry appeared in flat space holographic dualities, which is indeed the case. The Carroll group has been shown to be relevant to flat space holography [12] and conformal Carrollian field theory in 3D is applied in celestial holograph where it describes gravity in 4D asymptotically flat spacetime [13–16].

Apart from potential applications in holography, Carrollian geometry and gravity has become an active field of research in the last few years [17–45], as well as having spawned the research subject of Carrollian field theory [12, 17, 37, 42, 46–52]. Carrollian gravity has been shown to have great potential for describing dynamics of various limits of gravity, both with regards to black hole physics [25, 26, 47] as well as cosmology [46, 47]. Null hypersurfaces are induced on the event horizon of a black hole and, in accordance with the discussion above, Carrollian geometry is thus a candidate for describing that structure. The connection to cosmology is clearly seen when considering recessional velocities far outside the Hubble sphere, there the velocity satisfies $v \gg c$ and thus corresponding with the Carroll limit $c \to 0$. Carroll symmetry has also found another application in string theory, where it has been shown that various theories exhibit Carrollian worldsheet structure [53–61].

For both Newton-Cartan geometry and Carrollian geometry, one encounters a split into a temporal and a spatial part, reminiscent of the 3+1 formalism of GR [62]. Before expanding both theories one accounts for this split by introducing pre-non-relativistic (PNR) variables for the NC theory and pre-ultra-local (PUL) for the Carrollian theory. These two approaches lead to similar structure, the leading order of the PNR variables gives NC geometry while the leading order of the PUL variables yields Carrollian geometry, where both expansions have further geometric fields appearing at higher orders. The PNR formalism is described in [63, 64] and the PUL in [8]. A duality that can be formulated between the leading orders of the two theories is explored in [17, 65].

Although sharing many similarities, the NC and Carroll expansions differ a lot when one begins to analyze the content of the expansion orders. The LO of NC geometry only serves to constrain the theory and a kinetic term containing extrinsic curvature first appears at next-to-next-to-leading order (NNLO). For Carrollian geometry, one has interesting dynamics appearing at LO where such a kinetic term already appears. Furthermore, The LO Carrollian geometry can be written in the form of constraint and evolution equations as is done in the 3+1 formalism, which allows one to take initial data that satisfies the LO equations and evolve it in time [8]. It turns out that, at LO one does not yet have mass or energy present but at the order of c^2 in the expansion one is able to consider massive solutions due to the curvature terms that appear at that order in the theory. In much of the literature on both expansions, one runs into the assumption of the expansions being analytic in even powers of the expansion parameter, i.e. expanding in powers of $1/c^2$ for NC geometry and c^2 for Carrollian geometry. We will, however, encounter an example where this assumption breaks down for the Carrollian geometry, specifically when considering the Carroll expansion of the Kerr metric, and thus we will spend some effort into developing an expansion for all powers of c. This has been done before for NC geometry in [66]. Thus, great care has to be taken when discussing higher orders of the theories. As an example, the curvature terms discussed above appear at

NLO in the c^2 expansion but get shifted to the NNLO order in the c expansion.

Having derived the LO and NLO of the Carroll expansion allows one to consider the so-called electric and magnetic sectors of the theory, the name coming from a comparison to Maxwell theory [17, 67] where only the electric field or the magnetic field survive in each respective limit. The leading-order of the Carroll expansion will be identified as the electric theory while the magnetic theory is that of the truncated next-to-leading-order. The truncation is imposed via a constraint on the LO, $K_{\mu\nu} = 0$, that makes the LO vanish and the NLO then effectively becomes the 'new' leading-order. This has been explored in detail in [46, 47]. These truncated NLO EOMs for Carrollian geometry have, until now, been the only presentation of the Carrollian NLO.

1.1. Overview

This thesis begins with a review of a few key topics from general relativity in Section 2. The concept of taking derivatives on a manifold is discussed in 2.1 and the Lagrangian formulation of gravitational theories, specifically via the Einstein-Hilbert action, are reviewed in 2.2. The section concludes with a review of the vielbein formalism 2.3. Section 3 presents a review of the main concepts of Carrollian physics. We begin by introducing Carroll transformations in 3.1 which will then be related to the Poincaré algebra. In 3.2 we derive the Carroll algebra by taking the Carroll limit of the Poincaré algebra. Then in 3.3 we will first gauge the Poincaré algebra and thereafter move onto gauging the Carroll algebra to arrive at both the data for Carrollian geometry as well as showing how one can, in a relatively simple manner, derive curvature terms using said procedure. This is referred to as a first-order approach for obtaining Carrollian geometry and is presented in [21]. For later illustrating the electric and magnetic sectors of Carrollian geometry we will in 3.4 consider a simple example of a Carrollian scalar field theory that can then be related to the Carroll expansion of GR.

In this thesis we will consider two ways in which to arrive at Carrollian geometry and the second approach is that of a ultra-local expansion in the metric formulation of GR around $c = 0$, this will be the main approach considered in this thesis. In order to prepare for the Carroll expansion we follow [8] by introducing PUL variables that perform a split into a spatial and temporal part of the theory which can then be Carroll expanded. We describe this process in Section 4 where in 4.1 we introduce transformations of the PUL variables, the PUL and Carroll connections in 4.2 which are then related to the Levi-Civita connection of GR in 4.3. We then show a few identities for the connections in 4.4 along with Appendix A.1, define the variables of the curvature in 4.5 and finally construct the PUL Einstein Hilbert action in 4.6.

Having defined the PUL expansion, we move on to deriving the Carrollian geometry in Section 5. In anticipation of problems we encounter later in the thesis related to the assumption that the expansion is analytic in c^2, we introduce two different approaches to the expansion in 5.1. One where we expand in even powers of c and one where we consider all powers of c, which we call the c^2 and c expansion respectively. Some details of these derivations have been relocated to Appendix B. Subsequently, we review the leading-order of the Carrollian geometry in 5.2, following [8] and then in 5.3 we derive the next-to-leading-order theory. This has not been done before in pre-existing literature and will be the main result of the thesis. Again, detailed derivations for these sections are found in Appendix C.

In addition to presenting the derivation of the LO and NLO equations of motion we also show their spatial and temporal projections for both the c and c^2 expansion in Section 6, the LO in 6.1 and NLO in 6.2. The Carroll expansion of GR spacetimes will be considered in 6.3. The expansion of the Schwarzschild solution has been studied before in [8, 47] and in this thesis the connection between those solutions and the evolution equation of the LO theory will be made explicit. Furthermore, an attempt will be made to derive the Carroll expansion of the Kerr metric where we will, as mentioned before, have to consider the inclusion of odd powers in c. There also turns out to be an issue with the electric limit of the Carrollian Kerr metric owing to super-leading terms appearing in the expansion. The relation of these solutions to the general Carroll expansion will be discussed and put into perspective. We conclude the thesis with Section 7 where the results of the thesis are discussed as well as further directions for research.

2. General relativity

Gravity arises due to the effects of the geometry of spacetime. This is well known and was famously formulated by Einstein in his theory of general relativity. The theory uses the Riemann curvature tensor as the geometric invariants along with an affine connection that is both torsion-free and metric compatible. See [68, 69] for detailed reviews of conventional general relativity. Later it was shown that one can relax both the assumption of metric compatibility and the zero torsion of the connection, see e.g. the Palatini formulation of gravity [70] and Einstein-Cartan theory [71], to find alternative theories of gravity. This section shows some aspects and formalism of gravitational theories that will be of use in this thesis.

2.1. Differentiation on a manifold

A general metric-affine theory is defined by three objects, the manifold $\mathcal{M}$, the metric tensor $g_{\mu\nu}$ and a metric affine connection which can be written as

$$\bar{\Gamma}^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\mu\nu} + K^{\rho}{}_{\mu\nu} + L^{\rho}{}_{\mu\nu}, \tag{2.1}$$

which defines a covariant derivative of an arbitrary tensor X

$$\bar{\nabla}_{\rho} X^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_k} = \partial_{\mu} X^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_k} + \bar{\Gamma}^{\mu_1}{}_{\rho\lambda} X^{\lambda \dots \mu_k}{}_{\nu_1 \dots \nu_k} + \dots - \bar{\Gamma}^{\lambda}{}_{\rho\nu_1} X^{\mu_1 \dots \mu_k}{}_{\lambda \dots \nu_k} - \dots \tag{2.2}$$

Here, $\Gamma^{\rho}{}_{\mu\nu}$ is the Levi-Civita connection, $K^{\rho}{}_{\mu\nu}$ is usually called the contortion tensor and $L^{\rho}{}_{\mu\nu}$ the disformation tensor. Their explicit expressions are

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} \left(\partial_{\mu} g_{\lambda\nu} + \partial_{\nu} g_{\mu\lambda} - \partial_{\lambda} g_{\mu\nu} \right), \tag{2.3a}$$

$$K^{\rho}{}_{\mu\nu} = \frac{1}{2} \left(T_{\mu}{}^{\rho}{}_{\nu} + T_{\nu}{}^{\rho}{}_{\mu} - T^{\rho}{}_{\mu\nu} \right), \tag{2.3b}$$

$$L^{\rho}{}_{\mu\nu} = \frac{1}{2} \left(Q^{\rho}{}_{\mu\nu} - Q_{\mu}{}^{\rho}{}_{\nu} - Q_{\nu}{}^{\rho}{}_{\mu} \right), \tag{2.3c}$$

where $T^{\rho}{}_{\mu\nu} = 2 \bar{\Gamma}^{\rho}{}_{[\mu\nu]}$ is the torsion tensor and $Q_{\rho\mu\nu} = \bar{\nabla}_{\rho} g_{\mu\nu}$ the non-metricity tensor. From this it is evident that imposing zero torsion $T^{\rho}{}_{\mu\nu} = 0$ and metric compatibility $\bar{\nabla}_{\rho} g_{\mu\nu} = 0$, as for Einstein's GR, we arrive at the Levi-Civita connection. We define the covariant derivative in that case via ∇ without the bar on top. The contortion tensor and non-metricity tensor are the geometric objects for theories of gravity along with the Riemann curvature tensor

$$R_{\mu\sigma\nu}{}^{\rho} = 2 \partial_{[\sigma} \bar{\Gamma}^{\rho}{}_{\mu]\nu} + 2 \bar{\Gamma}^{\rho}{}_{[\sigma|\lambda|} \bar{\Gamma}^{\lambda}{}_{\mu]\nu} \tag{2.4}$$

and in general one can consider their role for various theories of gravity. The formal study of these formulations of gravity is called the Geometric Trinity of Gravity, see [72] for a useful review, but that is outside the scope of this thesis. Here, we merely highlight the existence of torsion and non-metricity for later reference.

The motivation for defining a covariant derivative, as in (2.2), is often shown by taking a partial derivative of a tensor, $\partial_{\mu} X^{\alpha}{}_{\beta}$, and showing that such an object does not transform as a tensor under a general coordinate transformation. However, a more fundamental argument is found by considering that a tensor at a point P lives on the tangent plane to the manifold at P. Thus, when wanting to compare that tensor at two different points, one runs into the problem of what it actually means to compare the two since they live on two different tangent planes. We have to account for a change of basis of the tangent space between the two points and this requires us to specify a connection between the two, which we can further constrain via consistency conditions. The covariant derivative thus requires the introduction of a new structure, $\bar{\Gamma}^{\rho}{}_{\mu\nu}$. We can, however, define a different type of derivative that also transforms as a tensor, called the Lie derivative:

$$\pounds_{\xi} X^{\alpha}{}_{\beta} = \xi^{\sigma}\nabla_{\sigma}X^{\alpha}{}_{\beta} - X^{\sigma}{}_{\beta}\nabla_{\sigma}\xi^{\alpha} - \xi^{\sigma}T^{\alpha}{}_{\sigma\lambda}X^{\lambda}{}_{\beta} + X^{\alpha}{}_{\sigma}\nabla_{\beta}\xi^{\sigma} + \xi^{\sigma}T^{\lambda}{}_{\sigma\beta}X^{\alpha}{}_{\lambda}, \tag{2.5}$$

that does not require any extra structure. When the Lie derivative of a tensor field vanishes, $\pounds_u X^{\alpha\dots}{}_{\beta\dots} = 0$, it is said to be Lie transported along the integral curve of ξ^{α}.

2.2. Lagrangian formulation of theories of gravity

Under a coordinate transformation $x^{\alpha} \to x^{\alpha'}(x^{\alpha})$, the volume element for $d+1$ dimensions transforms as $\mathrm{d}^{d+1}x = J\,\mathrm{d}^{d+1}x'$, where $J = \det[\partial x^{\alpha}/\partial x^{\alpha'}]$. Writing the transformation of the metric as

$$g_{\alpha'\beta'} = \frac{\partial x^{\alpha}}{\partial x^{\alpha'}}\frac{\partial x^{\beta}}{\partial x^{\beta'}}g_{\alpha\beta}, \tag{2.6}$$

and taking the determinant on both sides, we have

$$g' = J^2 g \qquad \Rightarrow \qquad J = \sqrt{g'/g}, \tag{2.7}$$

where $g = \det[g_{\alpha\beta}]$. Now considering the transformation to be from a local Lorentz frame $x^{\alpha'}$ to an arbitrary coordinate system x^{α}, we have $g' = \det[\eta_{\alpha\beta}] = -1$, and the transformation of the volume element becomes

$$\mathrm{d}^{d+1}x' = J^{-1}\,\mathrm{d}^{d+1}x = \sqrt{-g}\,\mathrm{d}^{d+1}x. \tag{2.8}$$

Thus, for an arbitrary region $\mathcal{M}$ of a spacetime manifold that is bounded by a closed hypersurface $\partial\mathcal{M}$ we can write up an action functional

$$S = \int_{\mathcal{M}} \mathcal{L}(q, \nabla_{\alpha}q)\sqrt{-g}\,\mathrm{d}^{d+1}x, \tag{2.9}$$

of a Lagrangian density $\mathcal{L}(q, \partial_{\alpha}q)$ for a field $q(x^{\alpha})$. Requiring that the variation of the action vanishes, we arrive at the Euler-Lagrange equations of motion

$$\frac{\partial\mathcal{L}}{\partial q} - \nabla_{\mu}\frac{\partial\mathcal{L}}{\partial(\nabla_{\mu}q)} = 0. \tag{2.10}$$

2.2.1. Einstein-Hilbert action

Here, we will give a heuristic argument for why the Einstein-Hilbert action is the simplest possible scalar action for general relativity. For constructing a scalar action for general relativity we start by considering our dynamical variable, the metric $g_{\mu\nu}$. According to the Einstein equivalence principle, one can always find a local inertial frame where the metric reduces to the Minkowski metric $\eta_{\mu\nu} = (-1, 1, 1, \ldots)$ and the first derivatives of the metric are all zero. In other words, locally flat spacetime. Thus, we know that we need a scalar that depends, at least, on the second derivatives of the metric. The Riemann curvature tensor is the obvious candidate and it can be shown that the Ricci scalar is the only independent scalar one can construct from the Riemann tensor. See various textbooks on general relativity for more in-depth explanations.

It was David Hilbert who first argued this and presented the Einstein-Hilbert action:

$$S_{\mathrm{EH}} = \frac{c^4}{16\pi G} \int_M R\sqrt{-g}\, \mathrm{d}^{d+1}x, \tag{2.11}$$

where R is the Ricci scalar

$$R = g^{\mu\nu} R_{\mu\nu}, \tag{2.12a}$$

$$R_{\mu\nu} = R_{\mu\rho\nu}{}^{\rho} = 2\, \partial_{[\rho} \bar{\Gamma}^{\rho}_{\mu]\nu} + 2\, \bar{\Gamma}^{\rho}_{[\rho|\lambda|} \bar{\Gamma}^{\lambda}_{\mu]\nu}, \tag{2.12b}$$

and $R_{\mu\nu}$ the Ricci tensor, which is a contraction of the Riemann curvature tensor (2.4). Later in this thesis we will expand the Einstein-Hilbert action to arrive at Carrollian geometry.

2.3. Vielbein structure

When working with manifolds one has to choose coordinates for the tangent spaces, often they are chosen as $\hat{e}_{(\mu)}$, i.e. based on our preexisting coordinate charts. It can be convenient, however, to choose a set of vectors that comprise an orthonormal basis, $\hat{e}_{(a)}$, which is defined as such:

$$g\left(\hat{e}_{(a)}, \hat{e}_{(b)}\right) = \eta_{ab}, \tag{2.13}$$

where $g\left(\hat{e}_{(a)}, \hat{e}_{(b)}\right)$ is a general metric and η_{ab} the Minkowski metric. What this means is that locally at each point the spacetime metric is the Minkowski metric. We can thus express the old basis vectors using the new ones as

$$\hat{e}_{(\mu)} = e_{\mu}{}^{a}\, \hat{e}_{(a)}, \tag{2.14}$$

where $e_{\mu}{}^{a}$ is an invertible $n \times n$ matrix which we call the tetrad or vielbein. Their inverse is defined such that

$$e^{\mu}{}_{a}\, e_{\nu}{}^{a} = \delta^{\mu}_{\nu}, \qquad e_{\mu}{}^{a}\, e^{\mu}{}_{b} = \delta^{a}_{b}. \tag{2.15}$$

Using these, (2.13) becomes:

$$g_{\mu\nu}\, e^{\mu}{}_{a}\, e^{\nu}{}_{b} = \eta_{ab}, \tag{2.16a}$$

$$g_{\mu\nu} = e_{\mu}{}^{a}\, e_{\nu}{}^{b}\, \eta_{ab}, \tag{2.16b}$$

where the second line is just a rewriting of the first.

Similarly, we can set up an orthonormal basis of one forms

$$\hat{\theta}^{(\mu)} = e^{\mu}{}_{a}\, \hat{\theta}^{(a)}, \qquad \hat{\theta}^{(a)} = e_{\mu}{}^{a}\, \hat{\theta}^{(\mu)}, \tag{2.17}$$

where we have made them compatible with the basis vectors in the following way:

$$\hat{\theta}^{(a)}\hat{e}_{(b)} = \delta^a_b. \tag{2.18}$$

This results in allowing us to refer to a vector V (or a one-form) in whichever basis we find convenient, $V^\mu\,\hat{e}_{(\mu)}$ or $V^a\hat{e}_{(a)}$. Therefore, we can write tensors with multiple indices in either basis as:

$$V^a{}_b = e_\mu{}^a\,V^\mu{}_b = e^\nu{}_b\,V^a{}_\nu = e_\mu{}^a\,e^\nu{}_b\,V^\mu{}_\nu\,. \tag{2.19}$$

From (2.16) we can deduce that the Latin indices correspond to those of the flat Minkowski metric, η_{ab}. Thus, allowing us to raise and lower said indices via the flat metric. Moreover, the Greek indices are then raised and lowered with the curved metric, $g_{\mu\nu}$:

$$e^\mu{}_a = g^{\mu\nu}\eta_{ab}\,e_\nu{}^b. \tag{2.20}$$

Having introduced a new basis to work in, we have to consider its transformation properties. The orthonormal basis is not derived from any coordinate system and is therefore independent of the coordinates. The basis can therefore be changed at will and the transformation only has to preserve (2.13). The transformations that preserve the flat metric in Lorentzian signature are the Lorentz transformations:

$$\hat{e}_{(a)} \longrightarrow \hat{e}_{(a')} = \Lambda^a{}_{a'}(x)\hat{e}_{(a)}. \tag{2.21}$$

We call these local Lorentz transformations (LLTs) because the tangent space is defined in a single point p and the transformations are thus only valid locally around said point. The transformation of a tensor with mixed coordinate and orthonormal indices then becomes:

$$X^{a'\mu'}{}_{b'\nu'} = \Lambda^{a'}{}_a\frac{\partial x^{\mu'}}{\partial x^\mu}\Lambda^b{}_{b'}\frac{\partial x^\nu}{\partial x^{\nu'}}X^{a\mu}{}_{b\nu}. \tag{2.22}$$

Now, that we have such tensors with mixed indices, it is important to account for them appropriately in the covariant derivative. Just as we have the affine connection for the coordinate based (Greek) indices (2.2), we now have a different connection for the noncoordinate (Latin) indices called the spin- or Ehresmann connection, $\omega_\mu{}^a{}_b$. Thus, taking the covariant derivative of the mixed index tensor in (2.22) yields:

$$\mathcal{D}_\rho X^{a\mu}{}_{b\nu} = \partial_\rho X^{a\mu}{}_{b\nu} + \bar{\Gamma}^\mu_{\rho\sigma}X^{a\sigma}{}_{b\nu} - \bar{\Gamma}^\sigma_{\rho\nu}X^{a\mu}{}_{b\sigma} - \omega_\rho{}^a{}_c X^{c\mu}{}_{b\nu} + \omega_\rho{}^c{}_b X^{a\mu}{}_{c\nu}, \tag{2.23}$$

where both connections appear.

Using the fact that a tensor should be independent of the basis we choose, we can find a relationship between the spin connection and the affine connection. Considering a covariant derivative of a vector, writing it first in a coordinate basis

$$\begin{aligned}\mathcal{D}V &= \left(\mathcal{D}_\mu V^\nu\right)\hat{\theta}^{(\mu)}\otimes\hat{e}_{(\nu)}\\ &= \left(\partial_\mu V^\nu + \bar{\Gamma}^\nu_{\mu\rho}V^\rho\right)\hat{\theta}^{(\mu)}\otimes\hat{e}_{(\nu)},\end{aligned} \tag{2.24}$$

and then in a mixed basis and converting to coordinate basis:

$$\mathcal{D}V = \left(\mathcal{D}_\mu V^a\right)\hat{\theta}^{(\mu)}\otimes\hat{e}_{(a)}$$

$$
\begin{aligned}
&= \left(\partial_\mu V^a - \omega_\mu{}^a{}_b V^b\right) \hat{\theta}^{(\mu)} \otimes \hat{e}_{(a)} \\
&= \left(\partial_\mu (e_\nu{}^a V^\nu) - \omega_\mu{}^a{}_b e_\rho{}^b V^\rho\right) \hat{\theta}^{(\mu)} \otimes e^\sigma{}_a \hat{e}_{(\sigma)} \\
&= e^\sigma{}_a \left(e_\nu{}^a \partial_\mu V^\nu + V^\nu \partial_\mu e_\nu{}^a - \omega_\mu{}^a{}_b e_\rho{}^b V^\rho\right) \hat{\theta}^{(\mu)} \otimes \hat{e}_{(\sigma)} \\
&= \left(\partial_\mu V^\sigma + V^\nu e^\sigma{}_a \partial_\mu e_\nu{}^a - \omega_\mu{}^a{}_b e^\sigma{}_a e_\rho{}^b V^\rho\right) \hat{\theta}^{(\mu)} \otimes \hat{e}_{(\sigma)}.
\end{aligned}
\tag{2.25}
$$

We can now compare this with (2.24) which gives a relationship between the two connections:

$$
\bar{\Gamma}^\sigma_{\mu\rho} = e^\sigma{}_a \partial_\mu e_\rho{}^a - \omega_\mu{}^a{}_b e^\sigma{}_a e_\rho{}^b, \tag{2.26a}
$$

$$
\omega_\mu{}^a{}_b = e^\rho{}_b \partial_\mu e_\rho{}^a - \bar{\Gamma}^\sigma_{\mu\rho} e_\sigma{}^a e^\rho{}_b, \tag{2.26b}
$$

where the second equation is merely a rearrangement of the first. Taking the covariant derivative of the vielbein and inserting (2.26b)

$$
\begin{aligned}
\mathcal{D}_\mu e_\nu{}^a &= \partial_\mu e_\nu{}^a - \bar{\Gamma}^\alpha_{\mu\nu} e_\alpha{}^a - \omega_\mu{}^a{}_b e_\nu{}^b \\
&= \partial_\mu e_\nu{}^a - \bar{\Gamma}^\alpha_{\mu\nu} e_\alpha{}^a - \left(e^\rho{}_b \partial_\mu e_\rho{}^a - \bar{\Gamma}^\sigma_{\mu\rho} e_\sigma{}^a e^\rho{}_b\right) e_\nu{}^b \\
&= \partial_\mu e_\nu{}^a - \bar{\Gamma}^\alpha_{\mu\nu} e_\alpha{}^a - \delta^\rho_\nu \partial_\mu e_\rho{}^a + \bar{\Gamma}^\sigma_{\mu\rho} e_\sigma{}^a \delta^\rho_\nu = 0, \\
\Rightarrow \quad \mathcal{D}_\mu e_\nu{}^a &= 0,
\end{aligned}
\tag{2.27}
$$

and we have arrived at the so-called vielbein postulate, i.e. that the vielbein vanishes when taking the covariant derivative.

3. Carroll symmetry and geometry

A standard procedure for simplifying general relativity is to take the infinite speed of light limit, $c \rightarrow \infty$, which flattens the light cones, thus making information propagate instantaneously. One often refers to this as the Galilean limit and one can interpret it using a dimensionless parameter c/v_c with some characteristic velocity $v_c \rightarrow 0$, i.e. c is infinite compared to v_c. This characteristic velocity is that of an observer moving at non-relativistic speeds and thus corresponds to the realm of classical physics of the everyday world, Newtonian physics. Another limit one might take to simplify GR is the $c \rightarrow 0$ limit. This was first done by Lévy-Leblond where he took the $c \rightarrow 0$ limit of the Poincaré group and called the result the Carroll group. The Carroll group gets its name from the Red Queen's race from Lewis Carroll's *Through the Looking Glass* [73] where Alice and the Red Queen run as fast as they can while standing still in the same spot. Unlike the Galilean limit, one does not interpret the Carroll limit as having $v_c \rightarrow \infty$, since this would mean a characteristic velocity greater than the speed of light. Instead, one sets $c = \hat{c}\,\epsilon$ where the dimensionless ϵ is expanded around zero. Thus the Carroll limit collapses the light cones to a line, making all spatially separated points causally disconnected, i.e. ultra-local. Carrollian causal structure therefore implies that space is absolute and points cannot move even when boosted. It is from this behavior one can draw parallels to Carroll's Red Queen's race. This section will present an introduction to Carroll symmetry and transformations which will then be used to derive the Carroll algebra. We then gauge the Carroll algebra and further hint at a derivation of Carrollian geometry via first-order formalism. Lastly, we introduce the electric and magnetic sectors of Carroll theory via an example of a scalar field theory.

3.1. Carroll transformations

In general relativity the spacetime metric can be locally approximated by the Minkowski metric, thus exhibiting Lorentz symmetry and translational symmetry:

$$x^\mu \longrightarrow x^{\mu'} = \Lambda^\mu{}_\nu x^\nu + a^\mu, \tag{3.1}$$

where $\Lambda^\mu{}_\nu$ are the Lorentz transformation matrices and a^μ is a constant vector in spacetime. This transformation can be split into its temporal and spatial part using $x^\mu = (ct, x^i)$:

$$ct \longrightarrow ct' = \Lambda^0{}_\nu x^\nu + a^0 = c\left(\Lambda^0{}_0 t + \Lambda^0{}_i \frac{x^i}{c}\right) + a^0, \tag{3.2a}$$

$$x^i \longrightarrow x^{i'} = \Lambda^i{}_\nu x^\nu + a^i = c\left(\Lambda^i{}_0 t + \Lambda^i{}_j \frac{x^j}{c}\right) + a^i. \tag{3.2b}$$

This split results in $\Lambda^i{}_\nu$ only corresponding to Lorentz rotations. The symmetries presented here are captured by the Poincaré group, whose Lie algebra is defined by

$$[J_{AB}, P_C] = \eta_{AC}P_B - \eta_{BC}P_A = 2\,\eta_{C[A}P_{B]}, \tag{3.3a}$$

$$\begin{aligned}[J_{AB}, J_{CD}] &= \eta_{AC}J_{BD} + \eta_{BD}J_{AC} - \eta_{BC}J_{AD} - \eta_{AD}J_{BC} \\ &= 2\big(\eta_{C[A}J_{B]D} - \eta_{D[A}J_{B]C}\big) = 4\,\eta_{[C[A}J_{B]D]},\end{aligned} \tag{3.3b}$$

with all the other commutation relations being zero. The algebra has been generalized to $d+1$ dimension with $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, \ldots, 1)$, i.e. the uppercase indices are as such:

$A = 0, 1, \ldots, d$. The square brackets around indices represent anti-symmetrization in the respective indices with a normalization factor of $1/n!$ in front of it (n representing the number of indices being anti-symmetrized). Now, the generators of the algebra are defined as

$$P_A = \partial_A, \tag{3.4a}$$

$$J_{AB} = x_B \partial_A - x_A \partial_B. \tag{3.4b}$$

Now specifying further the transformations (3.2) to Lorentz boosts along the x direction we get

$$ct \longrightarrow ct' = \gamma\left(ct - \beta_i x^i\right), \tag{3.5a}$$

$$x^i \longrightarrow x^{i'} = \gamma\left(x^i - \beta^i ct\right), \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \tag{3.5b}$$

with the Lorentz boost parameter β^i taking the values $0 \leq |\beta| \leq 1$. If we wanted to take the Galilean limit we would now define a boost parameter $b^i = c\beta^i$ and take the limit $c \longrightarrow \infty$ which would result in $t' = t$, i.e. time being absolute, and we would arrive at the Galilei algebra. Instead we define the boost parameter as $b^i = \beta^i / c$ and get

$$t' = \gamma\left(t - b_i x^i\right) \quad \xrightarrow{c \to 0} \quad t' = t - b_i x^i, \tag{3.6a}$$

$$x^{i'} = \gamma\left(x^i - b^i c^2 t\right) \quad \xrightarrow{c \to 0} \quad x^{i'} = x^i, \tag{3.6b}$$

where the second term of the second equation vanishes due to the limit $c \longrightarrow 0$, thus space being absolute, and $\gamma \longrightarrow 1$ due to $\beta^i \longrightarrow 0$. We have now derived the Carroll boosts. Using these we further derive how partial derivatives transform under Carroll boosts:

$$\frac{\partial}{\partial t'} = \frac{\partial x}{\partial t'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial t'}\frac{\partial}{\partial t} = \frac{\partial}{\partial t} \tag{3.7a}$$

$$\frac{\partial}{\partial x^{i'}} = \frac{\partial x}{\partial x^{i'}}\frac{\partial}{\partial x} + \frac{\partial t}{\partial x^{i'}}\frac{\partial}{\partial t} = \frac{\partial}{\partial x^i} + b_i \frac{\partial}{\partial t} \tag{3.7b}$$

The Carroll boosts, along with spatial rotations and translations, define the Carroll group. For a more in-depth treatment see [17, 46].

3.2. Carroll algebra

The Carroll algebra can, however, also be derived via a contraction of the Poincaré algebra (3.3). We define

$$P_0 = \epsilon^{-1} H, \qquad J_{0a} = \epsilon^{-1} C_a, \tag{3.8}$$

where we have introduced a split into a temporal part, represented by the generators defined above where the parameter $\epsilon \sim c$ reintroduces factors of c, and the spatial parts P_a and J_{ab}. The lowercase indices, $a = 1, 2, \ldots, d$, are just the spatial part of the uppercase indices from before. Since we are using the mostly plus signature, we can raise and lower the lowercase indices via the Kronecker-δ symbol. Using the Poincaré algebra, changing to lowercase indices and exchanging η_{ab} with δ_{ab}, we get the following commutation relations:

$$
\begin{aligned}
[J_{ab}, P_c] &= 2\,\eta_{c[a}P_{b]} = 2\,\delta_{c[a}P_{b]}, && \text{(3.9a)}\\
[J_{ab}, P_0] &= 2\,\eta_{0[a}P_{b]} = 2\,\eta_{0[a}P_{b]} = 0, && \text{(3.9b)}\\
[J_{0a}, P_b] &= 2\,\eta_{b[0}P_{a]} = \underbrace{\eta_{b0}P_a}_{=0} - \delta_{ba}P_0, && \text{(3.9c)}\\
[J_{0a}, P_0] &= 2\,\eta_{0[0}P_{a]} = -P_a - \underbrace{\eta_{0a}P_0}_{=0}, && \text{(3.9d)}\\
[J_{ab}, J_{cd}] &= 4\,\eta_{[c[a}J_{b]d]} = 4\,\delta_{[c[a}J_{a]d]}, && \text{(3.9e)}\\
[J_{ab}, J_{0c}] &= 4\,\eta_{[0[a}J_{b]c]} = 2\,(\underbrace{\eta_{0[a}J_{b]c}}_{=0} - \delta_{c[a}J_{b]0}), && \text{(3.9f)}\\
[J_{0a}, J_{0b}] &= 4\,\eta_{[0[0}J_{a]b]} = -J_{ab}, && \text{(3.9g)}
\end{aligned}
$$

where the terms including η_{a0} (with a mixture of 0 and lowercase index) vanish, since the lowercase indices can never equal 0. We now rewrite the equations using the previously defined contractions (3.8):

$$
\begin{aligned}
[P_a, J_{0b}] &= \delta_{ab}P_0 &&\Rightarrow && \epsilon^{-1}[P_a, C_b] = \delta_{ab}\epsilon^{-1}H, && \text{(3.10a)}\\
[P_0, J_{0a}] &= P_a &&\Rightarrow && \epsilon^{-2}[H, C_a] = P_a, && \text{(3.10b)}\\
[J_{ab}, J_{c0}] &= 2\,\delta_{c[a}J_{0b]} &&\Rightarrow && \epsilon^{-1}[J_{ab}, C_c] = 2\epsilon^{-1}\delta_{c[a}C_{b]}, && \text{(3.10c)}\\
[J_{0a}, J_{0b}] &= -J_{ab} &&\Rightarrow && \epsilon^{-2}[C_a, C_b] = -J_{ab}, && \text{(3.10d)}
\end{aligned}
$$

and thus, taking the limit $\epsilon \to 0$, we arrive at the Carroll algebra:

$$
\begin{aligned}
[J_{ab}, P_c] &= 2\,\delta_{c[a}P_{b]}, && \text{(3.11a)}\\
[J_{ab}, J_{cd}] &= 4\,\delta_{[c[a}J_{b]d]}, && \text{(3.11b)}\\
[P_a, C_b] &= \delta_{ab}H, && \text{(3.11c)}\\
[J_{ab}, C_c] &= 2\,\delta_{c[a}C_{b]}, && \text{(3.11d)}\\
[H, C_a] &= 0, && \text{(3.11e)}\\
[C_a, C_b] &= 0. && \text{(3.11f)}
\end{aligned}
$$

3.3. Gauging algebras

Gauge symmetry describes a redundancy in the description of a system's configuration [74], allowing the freedom to choose a specific gauge that makes it easier to examine certain aspects of a theory and often simplifies calculations. To construct gauge theories one can start with a Lie algebra and extend its global symmetries to local symmetries. This process introduces gauge fields and connections, both of which have some freedom of choice called gauge freedom. In this section we will first present a method to obtain pseudo-Riemannian geometry by gauging the Poincaré algebra and thereafter the same procedure is used for obtaining the Carrollian geometry by gauging the Carroll algebra. For a review of gauging algebras see e.g. [75].

3.3.1. Gauging the Poincaré algebra

For Einstein's theory of general relativity we expect every point in a curved spacetime to locally exhibit symmetries under Lorentz transformations and translations, i.e. every point can locally be approximated by flat Minkowski spacetime. Thus, by taking the Poincaré algebra and gauging it (making the global properties local), one can arrive at GR. This can be done by various approaches but here we will use the that of [65].

A non-Abelian Lie algebra g has in general a gauge field defined as

$$A_\mu = A_\mu{}^a T_a, \quad a = 1, \ldots, \dim(g), \tag{3.12}$$

where T_a are the generators of the algebra. The field A_μ used in gauging the Poincaré algebra will thus consist of the generators of the algebra contracted with associated gauge field components to leave only one coordinate space index:

$$A_\mu = P_A e_\mu{}^A + \frac{1}{2} J_{AB}\, \omega_\mu{}^{AB}, \tag{3.13}$$

where $\omega_\mu{}^{AB} = -\omega_\mu{}^{BA}$ is imposed due to the antisymmetry of J_{AB}. The field will transform as

$$A_\mu(x) \longrightarrow A'_\mu(x) = U^\dagger(x) A_\mu(x) U(x) + U(x)^\dagger \partial_\mu U(x), \tag{3.14}$$

with the local infinitesimal transformation:

$$U(x) = e^{\Lambda(x)} = \mathbb{1} + \Lambda(x) + \mathcal{O}(\Lambda^2), \tag{3.15}$$

thus giving:

$$\begin{aligned} A'_\mu &= (\mathbb{1} - \Lambda) A_\mu (\mathbb{1} + \Lambda) + (\mathbb{1} - \Lambda)\partial_\mu(\mathbb{1} + \Lambda) \\ &= A_\mu - \Lambda A_\mu \Lambda - \Lambda A_\mu + A_\mu \Lambda + \partial_\mu \Lambda - \Lambda \partial_\mu \Lambda \\ &= A_\mu + \left[A_\mu, \Lambda\right] + \partial_\mu \Lambda + \mathcal{O}(\Lambda^2) \\ \Rightarrow \quad \delta A_\mu &= \partial_\mu \Lambda + \left[A_\mu, \Lambda\right]. \end{aligned} \tag{3.16}$$

Writing out the variation of (3.13) gives

$$\delta A_\mu = P_A \delta e_\mu{}^A + \frac{1}{2} J_{AB}\, \delta\omega_\mu{}^{AB}. \tag{3.17}$$

Now we can relate the two expressions. Defining $\Lambda(x)$ also in terms of the Poincaré generators but contracted with some other tensors

$$\Lambda(x) = P_A \zeta^A(x) + \frac{1}{2} J_{AB}\, \sigma^{AB}(x), \tag{3.18}$$

we insert it along with (3.13) into (3.16) and get:

$$\begin{aligned} \delta A_\mu &= P_A \partial_\mu \zeta^A + \frac{1}{2} J_{AB}\, \partial_\mu \sigma^{AB} + \left[P_A e_\mu{}^A + \frac{1}{2} J_{BC}\, \omega_\mu{}^{BC}, \quad P_D \zeta^D + \frac{1}{2} J_{EF}\, \sigma^{EF} \right] \\ &= P_A \partial_\mu \zeta^A + \frac{1}{2} J_{AB}\, \partial_\mu \sigma^{AB} + e_\mu{}^A \zeta^D \underbrace{[P_A, P_D]}_{=0} + \frac{1}{2} e_\mu{}^A \sigma^{EF} [P_A, J_{EF}] \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\omega_\mu{}^{BC}\zeta^D[J_{BC},P_D]+\frac{1}{4}\omega_\mu{}^{BC}\sigma^{EF}[J_{BC},J_{EF}]\\
&=P_A\partial_\mu\zeta^A+\frac{1}{2}J_{AB}\,\partial_\mu\sigma^{AB}+\frac{1}{2}e_\mu{}^A\sigma^{FE}2\,\eta_{A[E}P_{F]}\\
&\quad-\frac{1}{2}\omega_\mu{}^{CB}\zeta^D 2\,\eta_{D[B}P_{C]}+\frac{1}{4}\omega_\mu{}^{BC}\sigma^{EF}4\,\eta_{[E[B}J_{C]F]}\\
&=P_A\partial_\mu\zeta^A+\frac{1}{2}J_{AB}\,\partial_\mu\sigma^{AB}+e_\mu{}^A\sigma^F{}_AP_F-\omega_\mu{}^C{}_D\zeta^DP_C\\
&\quad+\frac{1}{2}\,\omega_\mu{}^{BC}\sigma_B{}^FJ_{CF}-\frac{1}{2}\,\omega_\mu{}^{BC}\sigma_B{}^EJ_{EC}\\
&=P_A\Big(\partial_\mu\zeta^A+e_\mu{}^B\sigma^A{}_B-\omega_\mu{}^A{}_B\zeta^B\Big)+\frac{1}{2}J_{AB}\Big(\partial_\mu\sigma^{AB}+\omega_\mu{}^{CA}\sigma_C{}^B-\omega_\mu{}^{CB}\sigma_C{}^A\Big).
\end{aligned}
\tag{3.19}
$$

Comparing this to (3.17) we find the following variations:

$$
\delta e_\mu{}^A=\partial_\mu\zeta^A+e_\mu{}^B\sigma^A{}_B-\omega_\mu{}^A{}_B\zeta^B \tag{3.20a}
$$

$$
\delta\omega_\mu{}^{AB}=\partial_\mu\sigma^{AB}-\omega_\mu{}^{AC}\sigma_C{}^B+\omega_\mu{}^{BC}\sigma_C{}^A, \tag{3.20b}
$$

which shows that $e_\mu{}^A$ and $\omega_\mu{}^{AB}$ are the vielbein and spin connection respectively.

The transformation $\Lambda(x)$ as defined in (3.18) is a local transformation that exhibits local spacetime translations that we would, however, like to replace with diffeomorphisms. In order to do that, we can denote a new set of local transformations by $\bar{\delta}$ where we replace the parameter relating to local spacetime translations ζ_A with a spacetime vector ξ^μ defined in the following way:

$$
\zeta^A=\xi^\mu e_\mu{}^A. \tag{3.21}
$$

Then we define

$$
\Sigma=\frac{1}{2}J_{AB}\Big(\sigma^{AB}-\xi^\mu\omega_\mu{}^{AB}\Big)=\frac{1}{2}J_{AB}\lambda^{AB}, \tag{3.22}
$$

which results in us being able to write:

$$
\begin{aligned}
\Lambda&=\xi^\mu A_\mu+\Sigma\\
&=\xi^\mu\left(P_Ae_\mu{}^A+\frac{1}{2}J_{AB}\,\omega_\mu{}^{AB}\right)+\Sigma\\
&=P_A\zeta^A+\cancel{\frac{1}{2}J_{AB}\,\xi^\mu\omega_\mu{}^{AB}}+\frac{1}{2}J_{AB}\sigma^{AB}-\cancel{\frac{1}{2}\xi^\mu J_{AB}\omega_\mu{}^{AB}},
\end{aligned}
\tag{3.23}
$$

where the last line recovers the original definition of $\Lambda(x)$ in (3.18). Inserting this new definition of $\Lambda(x)$ into (3.16), we have:

$$
\begin{aligned}
\delta A_\mu&=\big[A_\mu,\Lambda\big]+\partial_\mu\Lambda\\
&=\big[A_\mu,\,\xi^\nu A_\nu+\Sigma\big]+\partial_\mu(\xi^\nu A_\nu+\Sigma)\\
&=\xi^\nu\big[A_\mu,A_\nu\big]+\big[A_\mu,\Sigma\big]+A_\nu\partial_\mu\xi^\nu+\xi^\nu\partial_\mu A_\nu+\partial_\mu\Sigma.
\end{aligned}
\tag{3.24}
$$

The general field strength tensor is defined as

$$
F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+\big[A_\mu,A_\nu\big], \tag{3.25}
$$

and looking at the last line, we notice that we have two out of three terms of the field strength tensor contracted with ξ^ν. The transformation can thus be written as:

$$\begin{aligned}\delta A_\mu &= \left[A_\mu, \Sigma\right] + A_\nu \partial_\mu \xi^\nu + \partial_\mu \Sigma + \xi^\nu F_{\mu\nu} + \xi^\nu \partial_\nu A_\mu \\ &= \left[A_\mu, \Sigma\right] + \partial_\mu \Sigma + \xi^\nu F_{\mu\nu} + \pounds_\xi A_\mu, \end{aligned} \tag{3.26}$$

where in the last line, two of the terms have been gathered into a Lie derivative of a one-form. From this, the new local transformation is defined as

$$\bar{\delta} A_\mu = \delta A_\mu - \xi^\nu F_{\mu\nu} \tag{3.27a}$$

$$= \pounds_\xi A_\mu + \partial_\mu \Sigma + \left[A_\mu, \Sigma\right], \tag{3.27b}$$

where the Lie derivative of A_μ with respect to ξ^μ shows clearly that A_μ transforms under diffeomorphisms. In other words, we have gone from having a local infinitesimal transformation (3.16) using $\Lambda(x)$ that consists of translations and rotations, as per (3.18), to having a local infinitesimal transformation (3.27b) using Σ which only consists of Lorentz transformations, as can be seen in (3.22), as well as the diffeomorphism $\pounds_\xi A_\mu$. We have turned the translational symmetry into a coordinate transformation.

Repeating the steps done to arrive at (3.19) but this time for $\bar{\delta} A_\mu$ using (3.22) and (3.13), we have

$$\bar{\delta} A_\mu = P_A \bar{\delta} e_\mu{}^A + \frac{1}{2} J_{AB}\, \bar{\delta} \omega_\mu{}^{AB} \tag{3.28}$$

$$\begin{aligned} &= \pounds_\xi A_\mu + \partial_\mu \Sigma + \left[A_\mu, \Sigma\right] \\ &= \pounds_\xi \left(P_A e_\mu{}^A + \frac{1}{2} J_{AB}\, \omega_\mu{}^{AB}\right) + \partial_\mu \left(\frac{1}{2} J_{AB} \lambda^{AB}\right) + \left[P_A e_\mu{}^A + \frac{1}{2} J_{AB}\, \omega_\mu{}^{AB}, \quad \frac{1}{2} J_{CD} \lambda^{CD}\right] \\ &= P_A \pounds_\xi e_\mu{}^A + \frac{1}{2} J_{AB} \pounds_\xi \omega_\mu{}^{AB} + \frac{1}{2} J_{AB}\, \partial_\mu \lambda^{AB} + \frac{1}{2} \lambda^{CD} \left(e_\mu{}^A [P_A, J_{CD}] + \frac{1}{2} \omega_\mu{}^{AB} [J_{AB}, J_{CD}]\right) \\ &= P_A \pounds_\xi e_\mu{}^A + \frac{1}{2} J_{AB} \pounds_\xi \omega_\mu{}^{AB} + \frac{1}{2} J_{AB}\, \partial_\mu \lambda^{AB} + \lambda^A{}_B e_\mu{}^B P_A + \lambda_C{}^B \omega_\mu{}^{CA} J_{AB} \\ &= P_A \left(\pounds_\xi e_\mu{}^A + \lambda^A{}_B e_\mu{}^B\right) + \frac{1}{2} J_{AB} \left(\pounds_\xi \omega_\mu{}^{AB} + \partial_\mu \lambda^{AB} + 2\, \lambda_C{}^{[B} \omega_\mu{}^{|C|A]}\right), \end{aligned} \tag{3.29}$$

where the last term in the last step becomes antisymmetrised since J_{AB} is antisymmetric. This allows us to identify:

$$\bar{\delta} e_\mu{}^A = \pounds_\xi e_\mu{}^A + \lambda^A{}_B e_\mu{}^B, \tag{3.30a}$$

$$\bar{\delta} \omega_\mu{}^{AB} = \pounds_\xi \omega_\mu{}^{AB} + \partial_\mu \lambda^{AB} + 2\, \lambda_C{}^{[B}\, \omega_\mu{}^{|C|A]}, \tag{3.30b}$$

i.e. $e_\mu{}^A$ as the vielbein and $\omega_\mu{}^{AB}$ the spin connection, both transforming under $\bar{\delta}$ with the infinitesimal local Lorentz transformation Λ^{AB}. Taking the covariant derivative, as defined in (2.23), of the vielbein

$$\mathcal{D}_\mu e_\nu{}^A = \partial_\mu e_\nu{}^A - \Gamma^\rho_{\mu\nu} e_\rho{}^A - \omega_\mu{}^A{}_B e_\nu{}^B, \tag{3.31}$$

(here we ignore the overbar on the connection that was used in Section 2) we know that it should transform in the same way as the vielbein in (3.30a). Comparing that with doing the $\bar{\delta}$ transformation directly:

$$\bar{\delta}\left(\mathcal{D}_\mu e_\nu{}^A\right) = \partial_\mu \left(\bar{\delta} e_\nu{}^A\right) - \bar{\delta}\left(\Gamma^\rho_{\mu\nu} e_\rho{}^A\right) - \bar{\delta}\left(\omega_\mu{}^A{}_B e_\nu{}^B\right) \tag{3.32a}$$

$$= \pounds_\xi\left(\mathcal{D}_\mu e_\nu{}^A\right) + \lambda^A{}_B\left(\mathcal{D}_\mu e_\nu{}^B\right), \tag{3.32b}$$

we expand all terms in both lines and find a constraint for them to be equal.

Starting with one term at a time in the first line (3.32a):

$$\begin{aligned} i)\qquad \partial_\mu\left(\bar{\delta} e_\nu{}^A\right) &= \partial_\mu\left(\xi^\sigma\partial_\sigma e_\nu{}^A + e_\sigma{}^A\partial_\nu\,\xi^\sigma + \lambda^A{}_B\,e_\nu{}^B\right) \\ &= \partial_\mu\xi^\sigma\partial_\sigma e_\nu{}^A + \xi^\sigma\partial_\mu\partial_\sigma e_\nu{}^A + \partial_\mu e_\sigma{}^A\partial_\nu\,\xi^\sigma \\ &\quad + e_\sigma{}^A\partial_\mu\partial_\nu\,\xi^\sigma + e_\nu{}^B\partial_\mu\lambda^A{}_B + \lambda^A{}_B\partial_\mu e_\nu{}^B, \end{aligned} \tag{3.33a}$$

$$\begin{aligned} ii)\quad -\bar{\delta}\left(\Gamma^\rho_{\mu\nu}e_\rho{}^A\right) &= -e_\rho{}^A\bar{\delta}\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\mu\nu}\bar{\delta}e_\rho{}^A \\ &= -e_\rho{}^A\bar{\delta}\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\mu\nu}\left(\xi^\sigma\partial_\sigma e_\rho{}^A + e_\sigma{}^A\partial_\rho\,\xi^\sigma + \lambda^A{}_B\,e_\rho{}^B\right), \end{aligned} \tag{3.33b}$$

$$\begin{aligned} iii) - \bar{\delta}\left(\omega_\mu{}^A{}_B e_\nu{}^B\right) &= -\omega_\mu{}^A{}_B\bar{\delta}e_\nu{}^B - e_{\nu B}\bar{\delta}\omega_\mu{}^{AB} \\ &= -\omega_\mu{}^A{}_B\left(\xi^\sigma\partial_\sigma e_\nu{}^B + e_\sigma{}^B\partial_\nu\,\xi^\sigma + \lambda^B{}_C\,e_\nu{}^C\right) \\ &\quad - e_{\nu B}\left(\xi^\sigma\partial_\sigma\omega_\mu{}^{AB} + \omega_\sigma{}^{AB}\partial_\mu\xi^\sigma + \partial_\mu\lambda^{AB} + \lambda_C{}^B\omega_\mu{}^{CA} - \lambda_C{}^A\omega_\mu{}^{CB}\right), \end{aligned} \tag{3.33c}$$

and then expanding the two terms in the second line (3.32b):

$$\begin{aligned} iv)\ \pounds_\xi\left(\mathcal{D}_\mu e_\nu{}^A\right) &= \pounds_\xi\left(\partial_\mu e_\nu{}^A - \Gamma^\rho_{\mu\nu}e_\rho{}^A - \omega_\mu{}^A{}_B e_\nu{}^B\right) \\ &= \xi^\sigma\partial_\sigma\partial_\mu e_\nu{}^A + \partial_\sigma e_\nu{}^A\partial_\mu\xi^\sigma + \partial_\mu e_\sigma{}^A\partial_\nu\xi^\sigma - \xi^\sigma\partial_\sigma\left(\Gamma^\rho_{\mu\nu}e_\rho{}^A\right) - \Gamma^\rho_{\sigma\nu}e_\rho{}^A\partial_\mu\xi^\sigma \\ &\quad - \Gamma^\rho_{\mu\sigma}e_\rho{}^A\partial_\nu\xi^\sigma - \xi^\sigma\partial_\sigma\left(\omega_\mu{}^A{}_B e_\nu{}^B\right) - \omega_\sigma{}^A{}_B e_\nu{}^B\partial_\mu\xi^\sigma - \omega_\mu{}^A{}_B e_\sigma{}^B\partial_\nu\xi^\sigma, \end{aligned} \tag{3.33d}$$

$$v)\ \lambda^A{}_B\left(\mathcal{D}_\mu e_\nu{}^B\right) = \lambda^A{}_B\left(\partial_\mu e_\nu{}^B - \Gamma^\rho_{\mu\nu}e_\rho{}^B - \omega_\mu{}^B{}_C e_\nu{}^C\right). \tag{3.33e}$$

Setting the two lines equal and canceling common terms, we are left with

$$\begin{aligned} &e_\rho{}^A\left(\partial_\mu\partial_\nu\,\xi^\rho - \bar{\delta}\Gamma^\rho_{\mu\nu} - \Gamma^\sigma_{\mu\nu}\partial_\sigma\,\xi^\rho\right) - \cancel{\omega_\mu{}^A{}_B\lambda^B{}_C\,e_\nu{}^C} + e_\nu{}^C\left(-\cancel{\lambda^B{}_C\omega_\mu{}^A{}_B} + \cancel{\lambda^A{}_B\omega_\mu{}^B{}_C}\right) \\ &\qquad = e_\rho{}^A\left(-\xi^\sigma\partial_\sigma\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\sigma\nu}\partial_\mu\xi^\sigma - \Gamma^\rho_{\mu\sigma}\partial_\nu\xi^\sigma\right) - \cancel{\lambda^A{}_B\omega_\mu{}^B{}_C e_\nu{}^C}, \end{aligned} \tag{3.34}$$

which can be rearranged into the $\bar{\delta}$ transformation of the affine connection:

$$\begin{aligned} \bar{\delta}\Gamma^\rho_{\mu\nu} &= \partial_\mu\partial_\nu\,\xi^\rho - \Gamma^\sigma_{\mu\nu}\partial_\sigma\,\xi^\rho + \xi^\sigma\partial_\sigma\Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\sigma\nu}\partial_\mu\xi^\sigma + \Gamma^\rho_{\mu\sigma}\partial_\nu\xi^\sigma, & (3.35a) \\ &= \partial_\mu\partial_\nu\,\xi^\rho + \pounds_\xi\Gamma^\rho_{\mu\nu}. & (3.35b) \end{aligned}$$

It is evident that the affine connection transforms via a diffeomorphism under the $\bar{\delta}$ transformation.

Poincaré curvature

Since A_μ is expressed in terms of the Poincaré generators P_A and J_{AB}, we should also be able to express the curvature $F_{\mu\nu}$ in terms of those as well, contracted with some curvature tensors:

$$F_{\mu\nu} = P_A R_{\mu\nu}{}^{A}(P) + \frac{1}{2} J_{AB} R_{\mu\nu}{}^{AB}(J). \tag{3.36}$$

These curvature terms, $R_{\mu\nu}{}^{A}$ and $R_{\mu\nu}{}^{AB}$, are then specified by inserting (3.13) into the definition of the field strength tensor and using the Poincaré algebra as defined in (3.3):

$$\begin{aligned}
F_{\mu\nu} &= 2\,\partial_{[\mu}A_{\nu]} + \left[A_\mu, A_\nu\right] \\
&= 2\,\partial_{[\mu}\left(P_A e_{\nu]}{}^{A} + \frac{1}{2} J_{AB}\,\omega_{\nu]}{}^{AB}\right) + \left[P_A e_\mu{}^A + \frac{1}{2} J_{BC}\,\omega_\mu{}^{BC}, \quad P_D e_\nu{}^D + \frac{1}{2} J_{EF}\,\omega_\nu{}^{EF}\right] \\
&= 2\,P_A \partial_{[\mu} e_{\nu]}{}^{A} + J_{AB}\,\partial_{[\mu}\omega_{\nu]}{}^{AB} + e_\mu{}^A e_\nu{}^D \underbrace{[P_A, P_D]}_{=0} + \frac{1}{2} e_\mu{}^A \omega_\nu{}^{EF}[P_A, J_{EF}] \\
&\quad + \frac{1}{2}\omega_\mu{}^{BC} e_\nu{}^D [J_{BC}, P_D] + \frac{1}{4}\omega_\mu{}^{BC}\omega_\nu{}^{EF}[J_{BC}, J_{EF}] \\
&= 2\,P_A \partial_{[\mu} e_{\nu]}{}^{A} + J_{AB}\,\partial_{[\mu}\omega_{\nu]}{}^{AB} - e_\mu{}^A \omega_\nu{}^{EF}\eta_{A[E}P_{F]} \\
&\quad + \omega_\mu{}^{BC} e_\nu{}^D \eta_{D[B}P_{C]} + \omega_\mu{}^{BC}\omega_\nu{}^{EF}\eta_{[E[B}J_{C]F]} \\
&= 2\,P_A \partial_{[\mu} e_{\nu]}{}^{A} + J_{AB}\,\partial_{[\mu}\omega_{\nu]}{}^{AB} + e_{\mu B}\,\omega_\nu{}^{AB} P_A - \omega_\mu{}^{AB} e_{\nu B} P_A \\
&\quad + \frac{1}{2}\left(\omega_\nu{}^{CA}\omega_\mu{}^{B}{}_{C} J_{AB} - \omega_\mu{}^{CA}\omega_\nu{}^{B}{}_{C} J_{AB}\right) \\
&= 2\,P_A\left(\partial_{[\mu} e_{\nu]}{}^{A} - \omega_{[\mu}{}^{AB} e_{\nu]B}\right) + J_{AB}\left(\partial_{[\mu}\omega_{\nu]}{}^{AB} - \omega_{[\mu}{}^{CA}\omega_{\nu]}{}^{B}{}_{C}\right).
\end{aligned} \tag{3.37}$$

Comparing with (3.36) we identify:

$$R_{\mu\nu}{}^{A}(P) = 2\,\partial_{[\mu} e_{\nu]}{}^{A} - 2\,\omega_{[\mu}{}^{AB} e_{\nu]B}, \tag{3.38a}$$

$$R_{\mu\nu}{}^{AB}(J) = 2\,\partial_{[\mu}\omega_{\nu]}{}^{AB} - 2\,\omega_{[\mu}{}^{CA}\omega_{\nu]}{}^{B}{}_{C}. \tag{3.38b}$$

Returning to the vielbein postulate (2.27) and rearranging:

$$\Gamma^{\alpha}_{\mu\nu} e_\alpha{}^A = \partial_\mu e_\nu{}^A - \omega_\mu{}^A{}_B e_\nu{}^B, \tag{3.39}$$

as well as contracting it with the inverse vielbein $e^\rho{}_A$:

$$\begin{aligned}
\Gamma^{\alpha}_{\mu\nu} e_\alpha{}^A e^\rho{}_A &= e^\rho{}_A\left(\partial_\mu e_\nu{}^A - \omega_\mu{}^A{}_B e_\nu{}^B\right) \\
\Rightarrow \quad \Gamma^{\rho}_{\mu\nu} &= e^\rho{}_A \partial_\mu e_\nu{}^A - e^\rho{}_A \omega_\mu{}^A{}_B e_\nu{}^B,
\end{aligned} \tag{3.40}$$

we have the affine connection in terms of vielbeine and the spin connection. Now taking the antisymmetric part of the vielbein postulate

$$\partial_{[\mu} e_{\nu]}{}^A - \Gamma^{\rho}_{[\mu\nu]} e_\rho{}^A - \omega_{[\mu}{}^A{}_B e_{\nu]}{}^B = 0, \tag{3.41}$$

and comparing to $R_{\mu\nu}{}^A(P)$, (3.38a), we find the following relationship:

$$R_{\mu\nu}{}^{A}(P) = 2\,\partial_{[\mu} e_{\nu]}{}^{A} - 2\,\omega_{[\mu}{}^{AB} e_{\nu]B} = 2\,\Gamma^{\rho}_{[\mu\nu]} e_\rho{}^A. \tag{3.42}$$

Thus, $R_{\mu\nu}{}^A(P)$ has been identified as the torsion tensor.

The Riemann curvature tensor, as defined in (2.4), has the following relation to the commutator of the covariant derivative:

$$R_{\mu\nu\sigma}{}^{\rho}V_{\rho} = \left[\nabla_{\mu}, \nabla_{\nu}\right]V^{\rho} + 2\,\Gamma^{\rho}_{[\mu\nu]}\nabla_{\rho}V_{\sigma}. \tag{3.43}$$

First, showing that

$$\begin{aligned}\partial_{\mu}\left(e^{\rho}{}_{A}e_{\sigma}{}^{A}\right) &= e_{\sigma}{}^{A}\partial_{\mu}e^{\rho}{}_{A} + e^{\rho}{}_{A}\partial_{\mu}e_{\sigma}{}^{A}\\ &= \partial_{\mu}\delta^{\rho}_{\sigma} = 0\\ \Rightarrow\quad e_{\sigma}{}^{A}\partial_{\mu}e^{\rho}{}_{A} &= -e^{\rho}{}_{A}\partial_{\mu}e_{\sigma}{}^{A},\end{aligned} \tag{3.44}$$

which will be used in the following calculations. We then insert the relation found from the vielbein postulate (3.40) and find:

$$\begin{aligned}R_{\mu\nu\sigma}{}^{\rho} &= -2\,\partial_{[\mu}\left(e^{\rho}{}_{A}\partial_{\nu]}e_{\sigma}{}^{A} - e^{\rho}{}_{A}\omega_{\nu]}{}^{A}{}_{B}e_{\sigma}{}^{B}\right)\\ &\quad -2\left(e^{\rho}{}_{A}\partial_{[\mu}e_{\lambda}{}^{A} - e^{\rho}{}_{A}\omega_{[\mu}{}^{A}{}_{B}e_{\lambda}{}^{B}\right)\left(e^{\lambda}{}_{C}\partial_{\nu]}e_{\sigma}{}^{C} - e^{\lambda}{}_{C}\omega_{\nu]}{}^{C}{}_{D}e_{\sigma}{}^{D}\right)\\ &= -2\Big[\partial_{[\mu}e^{\rho}{}_{A}\partial_{\nu]}e_{\sigma}{}^{A} - e_{\sigma}{}^{B}\partial_{[\mu}e^{\rho}{}_{A}\omega_{\nu]}{}^{A}{}_{B} - e_{\sigma}{}^{B}e^{\rho}{}_{A}\partial_{[\mu}\omega_{\nu]}{}^{A}{}_{B} - e^{\rho}{}_{A}\partial_{[\mu}e_{|\sigma|}{}^{B}\omega_{\nu]}{}^{A}{}_{B}\\ &\quad + e^{\rho}{}_{A}\partial_{[\mu}e_{|\lambda|}{}^{A}e^{\lambda}{}_{C}\partial_{\nu]}e_{\sigma}{}^{C} - e^{\rho}{}_{A}\partial_{[\mu}e_{|\lambda|}{}^{A}e^{\lambda}{}_{C}\omega_{\nu]}{}^{C}{}_{D}e_{\sigma}{}^{D}\\ &\quad - e^{\rho}{}_{A}\omega_{[\mu}{}^{A}{}_{B}e_{|\lambda|}{}^{B}e^{\lambda}{}_{C}\partial_{\nu]}e_{\sigma}{}^{C} + e^{\rho}{}_{A}\omega_{[\mu}{}^{A}{}_{B}e_{|\lambda|}{}^{B}e^{\lambda}{}_{C}\omega_{\nu]}{}^{C}{}_{D}e_{\sigma}{}^{D}\Big]\\ &= -2\Big[\partial_{[\mu}e^{\rho}{}_{A}\partial_{\nu]}e_{\sigma}{}^{A} - e_{\sigma}{}^{B}\partial_{[\mu}e^{\rho}{}_{A}\omega_{\nu]}{}^{A}{}_{B} - e_{\sigma}{}^{B}e^{\rho}{}_{A}\partial_{[\mu}\omega_{\nu]}{}^{A}{}_{B} - e^{\rho}{}_{A}\partial_{[\mu}e_{|\sigma|}{}^{B}\omega_{\nu]}{}^{A}{}_{B}\\ &\quad - \underbrace{e_{\lambda}{}^{A}e^{\lambda}{}_{C}}_{=\delta^{A}_{C}}\,\partial_{[\mu}e^{\rho}{}_{A}\partial_{\nu]}e_{\sigma}{}^{C} + \underbrace{e^{\lambda}{}_{C}e_{\lambda}{}^{A}}_{=\delta^{A}_{C}}\,\partial_{[\mu}e^{\rho}{}_{A}\omega_{\nu]}{}^{C}{}_{D}e_{\sigma}{}^{D}\\ &\quad - \underbrace{e_{\lambda}{}^{B}e^{\lambda}{}_{C}}_{\delta^{B}_{C}}\,e^{\rho}{}_{A}\omega_{[\mu}{}^{A}{}_{B}\partial_{\nu]}e_{\sigma}{}^{C} + \underbrace{e_{\lambda}{}^{B}e^{\lambda}{}_{C}}_{\delta^{B}_{C}}\,e^{\rho}{}_{A}e_{\sigma}{}^{D}\omega_{[\mu}{}^{A}{}_{B}\omega_{\nu]}{}^{C}{}_{D}\Big]\\ &= -e^{\rho}{}_{B}e_{\sigma A}\left(2\,\partial_{[\mu}\omega_{\nu]}{}^{AB} - 2\,\omega_{[\mu}{}^{CA}\omega_{\nu]}{}^{B}{}_{C}\right),\end{aligned} \tag{3.45}$$

where we have used the identity (3.44). Now in comparison with (3.38b) this gives:

$$R_{\mu\nu\sigma}{}^{\rho} = -e^{\rho}{}_{B}e_{\sigma A}R_{\mu\nu}{}^{AB}(J). \tag{3.46}$$

Thus we have shown that the curvature tensor $R^{\rho}{}_{\sigma\mu\nu}$ from (3.36) is the Riemann curvature two-form and we have therefore constructed the building blocks of GR via gauging the Poincaré algebra.

3.3.2. Gauging the Carroll algebra

The process of gauging the Carroll algebra is conceptually the same as for gauging the Poincaré algebra, albeit with some modifications. The goal is to derive from the Carroll algebra the data of the theory and then further Carrollian geometry. Here we will follow a similar route as in [18]. We start by defining the gauge field A_{μ} in terms of the generators of the Carroll algebra contracted with the associated gauge field components in the following way:

$$A_\mu = H\tau_\mu + P_a e_\mu{}^a + C_a \omega_\mu{}^a + \frac{1}{2} J_{ab}\omega_\mu{}^{ab}, \tag{3.47}$$

with the transformation matrix defined as:

$$\Lambda = \xi^\mu A_\mu + \Sigma, \qquad \Sigma = C_a\lambda^a + \frac{1}{2}J_{ab}\lambda^{ab}. \tag{3.48}$$

We have here introduced a split of the Σ from before into a spatial and temporal part. Likewise, in (3.47), we have split up the vielbein from the previous section, $e_\mu{}^A$, into a temporal part τ_μ and a spatial part $e_\mu{}^a$

$$e_\mu{}^A = (\tau_\mu, e_\mu{}^a). \tag{3.49}$$

Then, we introduce another frame $e^\mu{}_A = (v^\mu, e^\mu{}_a)$ consisting of the inverses of the two vielbeine with the following properties:

$$v^\mu\tau_\mu = -1, \quad v^\mu e_\mu{}^a = 0, \quad \tau_\mu e^\mu{}_a = 0, \quad e_\mu{}^a e^\mu{}_b = \delta^a_b. \tag{3.50}$$

The contraction $e_\mu{}^a e^\nu{}_a$ can be found via:

$$\begin{aligned} e^\mu{}_A e_\nu{}^A &= -v^\mu\tau_\nu + e^\mu{}_a e_\nu{}^a \\ \Rightarrow \quad e^\mu{}_a e_\nu{}^a &= \delta^\mu_\nu + v^\mu\tau_\nu. \end{aligned} \tag{3.51}$$

The transformation of the field will again be as in (3.16)

$$\delta A_\mu = \partial_\mu\Lambda + [A_\mu, \Lambda], \tag{3.52}$$

which is then modified to

$$\bar\delta A_\mu = \pounds_\xi A_\mu + \partial_\mu\Sigma + [A_\mu, \Sigma], \tag{3.53}$$

identical to (3.27b). Using the given ingredients we now write the previous equation explicitly:

$$\begin{aligned} \bar\delta A_\mu &= H\bar\delta\tau_\mu + P_a\bar\delta e_\mu{}^a + C_a\bar\delta\omega_\mu{}^a + \frac{1}{2}J_{ab}\bar\delta\omega_\mu{}^{ab} \\ &= H\pounds_\xi\tau_\mu + P_a\pounds_\xi e_\mu{}^a + C_a\pounds_\xi\omega_\mu{}^a + \frac{1}{2}J_{ab}\pounds_\xi\omega_\mu{}^{ab} + C_a\partial_\mu\lambda^a + \frac{1}{2}J_{ab}\partial_\mu\lambda^{ab} \\ &\quad + \Big[\underbrace{H\tau_\mu}_{=0} + P_a e_\mu{}^a + C_a\omega_\mu{}^a + \frac{1}{2}J_{ab}\omega_\mu{}^{ab}, \quad C_c\lambda^c + \frac{1}{2}J_{cd}\lambda^{cd}\Big] \\ &= H\pounds_\xi\tau_\mu + P_a\pounds_\xi e_\mu{}^a + C_a\left(\pounds_\xi\omega_\mu{}^a + \partial_\mu\lambda^a\right) + \frac{1}{2}J_{ab}\left(\pounds_\xi\omega_\mu{}^{ab} + \partial_\mu\lambda^{ab}\right) \\ &\quad + e_\mu{}^a\lambda^c\underbrace{[P_a, C_c]}_{=\delta_{ac}H} + \frac{1}{2}e_\mu{}^a\lambda^{cd}\underbrace{[P_a, J_{cd}]}_{=-2\delta_{a[c}P_{d]}} + \frac{1}{2}\omega_\mu{}^a\lambda^{cd}\underbrace{[C_a, J_{cd}]}_{=-2\delta_{a[c}C_{d]}} \\ &\quad + \frac{1}{2}\omega_\mu{}^{ab}\lambda^c\underbrace{[J_{ab}, C_c]}_{=2\delta_{c[a}C_{b]}} + \frac{1}{4}\omega_\mu{}^{ab}\lambda^{cd}\underbrace{[J_{ab}, J_{cd}]}_{=4\delta_{[c[a}J_{b]d]}} \end{aligned} \tag{3.54}$$

$$
\begin{aligned}
&= H\Big(\pounds_\xi \tau_\mu + e_\mu{}^a \lambda_a\Big) + P_a \pounds_\xi e_\mu{}^a + C_a\Big(\pounds_\xi \omega_\mu{}^a + \partial_\mu \lambda^a\Big) + \frac{1}{2} J_{ab}\Big(\pounds_\xi \omega_\mu{}^{ab} + \partial_\mu \lambda^{ab}\Big) \\
&\quad + e_\mu{}^a \lambda^d{}_a P_d + \omega_\mu{}^a \lambda^d{}_a C_d + \omega_\mu{}^{ab} \lambda_a C_b - \omega_\mu{}^{ab} \lambda^c{}_a J_{bc} \\
&= H\Big(\pounds_\xi \tau_\mu + e_\mu{}^a \lambda_a\Big) + P_a\Big(\pounds_\xi e_\mu{}^a + e_\mu{}^b \lambda^a{}_b\Big) + \frac{1}{2} J_{ab}\Big(\pounds_\xi \omega_\mu{}^{ab} + \partial_\mu \lambda^{ab} - 2\,\omega_\mu{}^{c[a} \lambda^{b]}{}_c\Big) \\
&\quad + C_a\Big(\pounds_\xi \omega_\mu{}^a + \partial_\mu \lambda^a + \omega_\mu{}^b \lambda^a{}_b + \omega_\mu{}^{ba} \lambda_b\Big),
\end{aligned}
\tag{3.55}
$$

and see that we get the following transformation rules:

$$\bar{\delta}\tau_\mu = \pounds_\xi \tau_\mu + e_\mu{}^a \lambda_a, \tag{3.56a}$$

$$\bar{\delta} e_\mu{}^a = \pounds_\xi e_\mu{}^a + e_\mu{}^b \lambda^a{}_b, \tag{3.56b}$$

$$\bar{\delta}\omega_\mu{}^a = \pounds_\xi \omega_\mu{}^a + \partial_\mu \lambda^a + \omega_\mu{}^b \lambda^a{}_b + \omega_\mu{}^{ba} \lambda_b, \tag{3.56c}$$

$$\bar{\delta}\omega_\mu{}^{ab} = \pounds_\xi \omega_\mu{}^{ab} + \partial_\mu \lambda^{ab} - 2\,\omega_\mu{}^{c[a} \lambda^{b]}{}_c. \tag{3.56d}$$

Using the inverse relations in (3.50), we then find that the inverses transform as:

$$\bar{\delta} v^\mu = \pounds_\xi v^\mu, \tag{3.56e}$$

$$\bar{\delta} e^\mu{}_a = \pounds_\xi e^\mu{}_a + v^\mu \lambda_a + \lambda_a{}^b e^\mu{}_b. \tag{3.56f}$$

We can repeat the same procedure as in the Poincaré case where one transforms the covariant derivatives in the same way as the tensors themselves and then compares to transforming it directly. But note that since the transformations of $e_\mu{}^a$ and $\omega_\mu{}^{ab}$ are identical to the expressions found when gauging the Poincaré algebra, see (3.30), we will therefore find the same transformation of the affine connection as in (3.35a):

$$\bar{\delta}\Gamma^\rho_{\mu\nu} = \partial_\mu \partial_\nu\, \xi^\rho + \pounds_\xi \Gamma^\rho_{\mu\nu}. \tag{3.57}$$

Again, since $e_\mu{}^a$ transforms in the same way as before, the covariant derivative of $e_\mu{}^a$ will be the same as before (3.31)

$$\mathcal{D}_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a - \Gamma^\rho_{\mu\nu} e_\rho{}^a - \omega_\mu{}^a{}_b e_\nu{}^b. \tag{3.58}$$

Writing

$$\mathcal{D}_\mu \tau_\nu = \partial_\mu \tau_\nu - \Gamma^\rho_{\mu\nu} \tau_\rho - \omega_{\mu a} e_\nu{}^a, \tag{3.59}$$

we can show that $\mathcal{D}_\mu \tau_\nu$ transforms just like τ_μ:

$$
\begin{aligned}
\bar{\delta}\big(\mathcal{D}_\mu \tau_\nu\big) &= \partial_\mu \bar{\delta}\tau_\nu - \tau_\rho \bar{\delta}\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\mu\nu} \bar{\delta}\tau_\rho - e_\nu{}^a \bar{\delta}\omega_{\mu a} - \omega_{\mu a} \bar{\delta} e_\nu{}^a \\
&= \partial_\mu(\pounds_\xi \tau_\nu + e_\nu{}^a \lambda_a) - \tau_\rho\Big(\partial_\mu \partial_\nu\, \xi^\rho + \pounds_\xi \Gamma^\rho_{\mu\nu}\Big) - \Gamma^\rho_{\mu\nu}\Big(\pounds_\xi \tau_\rho + e_\rho{}^a \lambda_a\Big) \\
&\quad - e_\nu{}^a\Big(\pounds_\xi \omega_{\mu a} + \partial_\mu \lambda_a - \cancel{\omega_{\mu b} \lambda^b{}_a} + \omega_\mu{}^b{}_a \lambda_b\Big) - \omega_{\mu a}\Big(\pounds_\xi e_\nu{}^a + \cancel{e_\nu{}^b \lambda^a{}_b}\Big) \\
&= \partial_\mu(\pounds_\xi \tau_\nu) + \cancel{e_\nu{}^a \partial_\mu \lambda_a} - \tau_\rho \partial_\mu \partial_\nu\, \xi^\rho + \pounds_\xi\Big(-\Gamma^\rho_{\mu\nu} \tau_\rho - \omega_{\mu a} e_\nu{}^a\Big) \\
&\quad - \cancel{e_\nu{}^a \partial_\mu \lambda_a} + \lambda_a\Big(\partial_\mu e_\nu{}^a - \Gamma^\rho_{\mu\nu} e_\rho{}^a - \omega_\mu{}^a{}_b e_\nu{}^b\Big) \\
&= \pounds_\xi\big(\mathcal{D}_\mu \tau_\nu\big) + \lambda_a\big(\mathcal{D}_\mu e_\nu{}^a\big),
\end{aligned}
$$

which can also be done for v^ν and $e^\nu{}_a$:

$$\mathcal{D}_\mu v^\nu = \partial_\mu v^\nu + \Gamma^\nu_{\mu\rho} v^\rho, \tag{3.60a}$$

$$\mathcal{D}_\mu e^\nu{}_a = \partial_\mu e^\nu{}_a + \Gamma^\nu_{\mu\rho} e^\rho{}_a - v^\nu \omega_{\mu a} - \omega_{\mu a}{}^b e^\nu{}_b. \tag{3.60b}$$

At this point we have derived the transformation rules and covariant derivatives of the vielbeine for the Carroll algebra. This is sufficient for the remainder of the thesis and we instead rely on an expansion around $c = 0$ to derive Carrollian geometry in Section 4. For completeness, however, we will show how one can derive Carrollian geometry using a gauging procedure.

Carroll curvature

Considering now the curvature $F_{\mu\nu}$ and expressing it as:

$$F_{\mu\nu} = HR_{\mu\nu}(H) + P_a R_{\mu\nu}{}^a(P) + C_a R_{\mu\nu}{}^a(C) + \frac{1}{2} J_{ab} R_{\mu\nu}{}^{ab}(J), \tag{3.61}$$

and comparing with

$$\begin{aligned}
F_{\mu\nu} &= 2\,\partial_{[\mu} A_{\nu]} + \left[A_\mu, A_\nu\right] \\
&= 2\left(H\partial_{[\mu}\tau_{\nu]} + P_a \partial_{[\mu} e_{\nu]}{}^a + C_a \partial_{[\mu}\omega_{\nu]}{}^a + \frac{1}{2} J_{ab}\partial_{[\mu}\omega_{\nu]}{}^{ab}\right) \\
&\quad + \Big[\underbrace{H\tau_\mu}_{=0} + P_a e_\mu{}^a + C_a \omega_\mu{}^a + \frac{1}{2} J_{ab}\omega_\mu{}^{ab}, \quad \underbrace{H\tau_\nu}_{=0} + P_c e_\nu{}^c + C_c \omega_\nu{}^c + \frac{1}{2} J_{cd}\omega_\nu{}^{cd}\Big] \\
&= 2\left(H\partial_{[\mu}\tau_{\nu]} + P_a \partial_{[\mu} e_{\nu]}{}^a + C_a \partial_{[\mu}\omega_{\nu]}{}^a + \frac{1}{2} J_{ab}\partial_{[\mu}\omega_{\nu]}{}^{ab}\right) \\
&\quad + e_\mu{}^a \omega_\nu{}^c \delta_{ac} H - e_\mu{}^a \omega_\nu{}^{cd}\delta_{ac} P_d - \omega_\mu{}^a e_\nu{}^c \delta_{ac} H - \omega_\mu{}^a \omega_\nu{}^{cd}\delta_{ac} C_d \\
&\quad + \omega_\mu{}^{ab} e_\nu{}^c \delta_{ca} P_b + \omega_\mu{}^{ab}\omega_\nu{}^c \delta_{ca} C_b - \omega_\mu{}^{ab}\omega_\nu{}^c{}_a J_{bc} \\
&= H\left(2\,\partial_{[\mu}\tau_{\nu]} + 2\, e_{[\mu}{}^a \omega_{\nu]a}\right) + P_a\left(2\,\partial_{[\mu} e_{\nu]}{}^a + 2\, e_{[\mu}{}^b \omega_{\nu]}{}^a{}_b\right) \\
&\quad + C_a\left(2\,\partial_{[\mu}\omega_{\nu]}{}^a + 2\,\omega_{[\mu}{}^{ba}\omega_{\nu]b}\right) + \frac{1}{2} J_{ab}\left(2\,\partial_{[\mu}\omega_{\nu]}{}^{ab} + 2\,\omega_{[\mu}{}^{ac}\omega_{\nu]}{}^b{}_c\right),
\end{aligned} \tag{3.62}$$

we have the following:

$$R_{\mu\nu}(H) = 2\,\partial_{[\mu}\tau_{\nu]} + 2\, e_{[\mu}{}^a \omega_{\nu]a}, \tag{3.63a}$$

$$R_{\mu\nu}{}^a(P) = 2\,\partial_{[\mu} e_{\nu]}{}^a + 2\, e_{[\mu}{}^b \omega_{\nu]}{}^a{}_b, \tag{3.63b}$$

$$R_{\mu\nu}{}^a(C) = 2\,\partial_{[\mu}\omega_{\nu]}{}^a + 2\,\omega_{[\mu}{}^{ba}\omega_{\nu]b}, \tag{3.63c}$$

$$R_{\mu\nu}{}^{ab}(J) = 2\,\partial_{[\mu}\omega_{\nu]}{}^{ab} + 2\,\omega_{[\mu}{}^{ac}\omega_{\nu]}{}^b{}_c. \tag{3.63d}$$

The derivatives defined in (3.58) and (3.59) can now be set to zero to derive vielbeine postulates:

$$\Gamma^\rho_{\mu\nu} e_\rho{}^a = \partial_\mu e_\nu{}^a - e_\nu{}^b \omega_\mu{}^a{}_b, \tag{3.64a}$$

$$\Gamma^\rho_{\mu\nu}\tau_\rho = \partial_\mu \tau_\nu - e_\nu{}^a \omega_{\mu a}, \tag{3.64b}$$

and contracting the first of these with ${e^\sigma}_a$:

$$
\begin{aligned}
{e^\sigma}_a \Gamma^\rho_{\mu\nu} {e_\rho}^a &= {e^\sigma}_a \partial_\mu {e_\nu}^a - {e^\sigma}_a {e_\nu}^b {\omega_\mu}^a{}_b \\
&= \Gamma^\rho_{\mu\nu}\left(\delta^\sigma_\rho + v^\sigma \tau_\rho\right) \\
&= \Gamma^\sigma_{\mu\nu} + v^\sigma \Gamma^\rho_{\mu\nu}\tau_\rho \\
&= \Gamma^\sigma_{\mu\nu} + v^\sigma\left(\partial_\mu \tau_\nu - {e_\nu}^a \omega_{\mu a}\right) \\
\Rightarrow \quad \Gamma^\sigma_{\mu\nu} &= {e^\sigma}_a \partial_\mu {e_\nu}^a - {e^\sigma}_a {e_\nu}^b {\omega_\mu}^a{}_b - v^\sigma \partial_\mu \tau_\nu + v^\sigma {e_\nu}^a \omega_{\mu a},
\end{aligned}
\tag{3.65}
$$

gives us the affine connection written in terms of the vielbeine. Here we have inserted (3.64b) into the penultimate line. Antisymmetrising (3.65):

$$
\Gamma^\sigma_{[\mu\nu]} = -v^\sigma\left(\partial_{[\mu}\tau_{\nu]} - \omega_{[\mu a}{e_{\nu]}}^a\right) + {e^\sigma}_a\left(\partial_{[\mu}{e_{\nu]}}^a - {\omega_{[\mu}}^a{}_b {e_{\nu]}}^b\right),
\tag{3.66}
$$

we see that the two terms in brackets can be identified with $R_{\mu\nu}(H)$ and ${R_{\mu\nu}}^a(P)$:

$$
2\,\Gamma^\rho_{[\mu\nu]} = -v^\rho R_{\mu\nu}(H) + {e^\rho}_a {R_{\mu\nu}}^a(P),
\tag{3.67}
$$

where we have then found a relationship between two of the curvature terms and the affine connection.

Now taking the Riemann tensor (2.4) and inserting (3.65):

$$
\begin{aligned}
{R_{\mu\nu\sigma}}^\rho &= -2\,\partial_{[\mu}\Gamma^\rho_{\nu]\sigma} - 2\,\Gamma^\rho_{[\mu|\lambda|}\Gamma^\lambda_{\nu]\sigma} \\
&= -2\,\partial_{[\mu}\left({e^\rho}_a(\partial_{\nu]}{e_\sigma}^a - {e_\sigma}^b {\omega_{\nu]}}^a{}_b) - v^\rho(\partial_{\nu]}\tau_\sigma - {e_\sigma}^a \omega_{\nu]a})\right) \\
&\quad - 2\left({e^\rho}_a(\partial_{[\mu}{e_{|\lambda|}}^a - {e_\lambda}^b {\omega_{[\mu}}^a{}_b) - v^\rho(\partial_{[\mu}\tau_{|\lambda|} - {e_\lambda}^a \omega_{[\mu a})\right) \\
&\qquad \times\left({e^\lambda}_c(\partial_{\nu]}{e_\sigma}^c - {e_{|\sigma|}}^d {\omega_{\nu]}}^c{}_d) - v^\lambda(\partial_{\nu]}\tau_\sigma - {e_{|\sigma|}}^c \omega_{\nu]c})\right) \\
&= -2\Big[\partial_{[\mu}{e^\rho}_a \partial_{\nu]}{e_\sigma}^a + \underbrace{{e^\rho}_a \partial_{[\mu}\partial_{\nu]}{e_\sigma}^a}_{=0} - {e_\sigma}^b \partial_{[\mu}{e^\rho}_a {\omega_{\nu]}}^a{}_b \\
&\qquad - {e^\rho}_a \partial_{[\mu}{e_\sigma}^b {\omega_{\nu]}}^a{}_b - {e^\rho}_a {e_\sigma}^b \partial_{[\mu}{\omega_{\nu]}}^a{}_b - \partial_{[\mu}v^\rho \partial_{\nu]}\tau_\sigma \\
&\qquad - \underbrace{v^\rho \partial_{[\mu}\partial_{\nu]}\tau_\sigma}_{=0} + {e_\sigma}^a \partial_{[\mu}v^\rho \omega_{\nu]a} + v^\rho \partial_{[\mu}{e_\sigma}^a \omega_{\nu]a} + v^\rho {e_\sigma}^a \partial_{[\mu}\omega_{\nu]a}\Big] \\
&\quad - 2\Big[{e^\rho}_a {e^\lambda}_b \partial_{[\mu}{e_{|\lambda|}}^a \partial_{\nu]}{e_\sigma}^b - {e^\rho}_a {e^\lambda}_c {e_\sigma}^b \partial_{[\mu}{e_{|\lambda|}}^a {\omega_{\nu]}}^c{}_b - {e^\rho}_a {\omega_{[\mu}}^a{}_b \partial_{\nu]}{e_\sigma}^b \\
&\qquad + {e^\rho}_a {e_\sigma}^c {\omega_{[\mu}}^a{}_b {\omega_{\nu]}}^b{}_c - {e^\rho}_a v^\lambda \partial_{[\mu}{e_{|\lambda|}}^a \partial_{\nu]}\tau_\sigma + {e^\rho}_a v^\lambda {e_\sigma}^b \partial_{[\mu}{e_{|\lambda|}}^a \omega_{\nu]b} \\
&\qquad - v^\rho {e^\lambda}_a \partial_{[\mu}\tau_{|\lambda|}\partial_{\nu]}{e_\sigma}^a + v^\rho {e^\lambda}_a {e_\sigma}^b \partial_{[\mu}\tau_{|\lambda|}{\omega_{\nu]}}^a{}_b + v^\rho \omega_{[\mu a}\partial_{\nu]}{e_\sigma}^a \\
&\qquad - v^\rho {e_\sigma}^b \omega_{[\mu a}{\omega_{\nu]}}^a{}_b + v^\rho v^\lambda \partial_{[\mu}\tau_{|\lambda|}\partial_{\nu]}\tau_\sigma - v^\rho v^\lambda {e_\sigma}^c \partial_{[\mu}\tau_{|\lambda|}\omega_{\nu]c}\Big] \\
&= -2\,v^\rho e_{\sigma a}\left(\partial_{[\mu}{\omega_{\nu]}}^a + {\omega_{[\mu}}^{ba}\omega_{\nu]b}\right) - 2\,{e^\rho}_b e_{\sigma a}\left(\partial_{[\mu}{\omega_{\nu]}}^{ab} + {\omega_{[\mu}}^{ac}{\omega_{\nu]}}^b{}_c\right) \\
&= -v^\rho e_{\sigma a}{R_{\mu\nu}}^a(C) - {e^\rho}_b e_{\sigma a}{R_{\mu\nu}}^{ab}(J),
\end{aligned}
\tag{3.68}
$$

we find a type of Carrollian curvature. This is a relatively easy way of obtaining Carrollian geometry and can be used in formulating a dynamical theory, see [21], by finding the Ricci tensor and scalar and inserting into the Eintein-Hilbert action (2.11). We will not be using this formulation of Carrollian geometry further in this thesis.

3.4. Carrollian field theory

Having derived the Carroll algebra and geometry along with the Carroll transformations we are now in a position to explore a Carrollian field theory. For the purpose of this thesis we explore here the simple example of a real scalar field in order to illustrate the so called electric and magnetic sectors of the Carroll limit. The names for the sectors come from considering limits of Maxwell theory, see the seminal work [67] as well as [76], first for $c \longrightarrow \infty$ and later for $c \longrightarrow 0$ as in [46, 47]. Here, we follow the presentation in [46].

3.4.1. Scalar field

We start with a Lagrangian for a relativistic scalar field ϕ

$$\mathcal{L} = \frac{1}{2c^2}(\partial_t\phi)^2 - \frac{1}{2}(\partial_i\phi)^2 - V(\phi), \qquad V(\phi) = \frac{1}{2}\frac{m^2c^2}{\hbar^2}\phi^2, \tag{3.69}$$

where we have assumed the potential to be quadratic, moreover, the scalar field transforms under Lorentz boosts as

$$\delta\phi = ct\beta^i\partial_i\phi + \frac{1}{c}\beta_i x^i\partial_t\phi. \tag{3.70}$$

Expanding the field around $c = 0$ where we assume that the expansion is analytic in c^2, we get

$$\phi = c^\Delta\Big(\phi_0 + c^2\phi_1 + c^4\phi_2 + O\big(c^6\big)\Big), \tag{3.71a}$$

$$\begin{aligned}\delta\phi &= c^{\Delta-1}\Big(\beta_i x^i\partial_t\phi_0 + c^2\Big(\beta_i x^i\partial_t\phi_1 + t\beta^i\partial_i\phi_0\Big) + O(c^4)\Big) \\ &= c^{\Delta-1}\Big(\delta\phi_0 + c^2\delta\phi_1 + O(c^4)\Big).\end{aligned} \tag{3.71b}$$

The factor of c^Δ is there to show that one moves all factors of c preceding the leading order term of the expansion outside the brackets. Now, by expanding (3.70) in the same way as above, we can generalize the Carroll boosts of the scalar field as:

$$\delta\phi_0 = \beta_i x^i\partial_t\phi_0, \quad \delta\phi_n = \beta_i x^i\partial_t\phi_n + t\beta^i\partial_i\phi_{n-1}, \qquad n > 0. \tag{3.72}$$

The assumption of analyticity in c^2 will later on in the thesis be shown to be inadequate in certain cases, see Section 5.1, but for this example it is perfectly fine.

We write the Lagrangian for the free theory as

$$\begin{aligned}\mathcal{L} &= c^{2\Delta}\frac{1}{c^2}\left[\frac{1}{2}(\partial_t\phi)^2 - \frac{c^2}{2}(\partial_i\phi)^2 - \frac{1}{2}\frac{m^2c^4}{\hbar^2}\phi^2\right] \\ &= c^{2\Delta-2}\left[\frac{1}{2}\dot\phi_0^2 + c^2\Big(\dot\phi_0\dot\phi_1 - \frac{1}{2}(\partial_i\phi_0)^2\Big) + O\big(c^4\big)\right]\end{aligned} \tag{3.73}$$

$$= c^{2\Delta-2}\Big[\mathcal{L}_0 + c^2\mathcal{L}_1 + \mathcal{O}\Big(c^4\Big)\Big], \tag{3.74}$$

where we have:

$$\mathcal{L}_0 = \frac{1}{2}\dot{\phi}_0^2, \qquad \mathcal{L}_1 = \dot{\phi}_0\dot{\phi}_1 - \frac{1}{2}\big(\partial_i\phi_0\big)^2. \tag{3.75}$$

We now want to check whether the Lagrangians are Carroll invariant. The rotational and translational invariance is almost trivial since there are, of course, no free indices in the Lagrangian. For checking the boost invariance we use the Carroll boosts (3.6) and the transformation rules for the derivatives (3.7) defined before along with the fact that the scalar field transforms as $\phi'(x') = \phi(x)$:

$$\mathcal{L}_0 \to \mathcal{L}_0' = \frac{1}{2}\big(\partial_{t'}\phi_0'\big)^2 = \frac{1}{2}\big(\partial_t\phi_0\big)^2 = \mathcal{L}_0 \tag{3.76a}$$

$$\begin{aligned}\mathcal{L}_1 \to \mathcal{L}_1' &= \partial_{t'}\phi_0'\partial_{t'}\phi_1' - \frac{1}{2}\big(\partial_{i'}\phi_0'\big)^2\\ &= \partial_t\phi_0\partial_t\phi_1 - \frac{1}{2}\big((\partial_i + b_i\partial_t)\phi_0\big)^2\\ &= \partial_t\phi_0\partial_t\phi_1 - \frac{1}{2}\big(\partial_i\phi_0\big)^2 - b^i\partial_i\phi_0\partial_t\phi_0 - \frac{1}{2}b^2\big(\partial_t\phi_0\big)^2\\ &= \mathcal{L}_1 - \dot{\phi}_0 b^i\partial_i\phi_0 - \frac{1}{2}b^2\dot{\phi}_0^2.\end{aligned} \tag{3.76b}$$

From this it is clear that the leading-order Lagrangian $\mathcal{L}_0$ is Carroll invariant while $\mathcal{L}_1$ is not, due to the extra terms that can not be written on the form of a total derivative. However, when setting $\dot{\phi}_0 = 0$ via a Lagrange multiplier, essentially setting $\mathcal{L}_0 = 0$, $\mathcal{L}_1$ becomes Carroll invariant as well. What this effectively does is to kill the LO Lagrangian while truncating the NLO Lagrangian so that it becomes the new leading order term. Moreover, if one instead truncates (3.72) in the same manner, then $\mathcal{L}_1$ is invariant under the Carroll boosts. Drawing parallels from [67], we call the LO Lagrangian the electric theory and the truncated NLO Lagrangian the magnetic theory. Later in this thesis, see Section 5, we will derive the LO and NLO EOMs for the Carroll expansion and define the electric and magnetic sector in an analogous manner to the discussion here.

4. PUL expansion

Carrollian geometry can be derived from Lorentzian geometry by expanding around the point $c = 0$, i.e. the small speed of light expansion as presented in [8]. As explained before, in this limit the light cones close up and the theory becomes ultra-local. Anticipating this behaviour, we introduce a pre-ultra-local (PUL) parameterisation of the Lorentzian metric and vielbein that splits the metric and vielbein into a spatial and temporal part. We can then consider the transformations of the PUL vielbeine and define a suitable connection, which yields a so-called Carroll connection upon taking the Carroll limit. We then relate it to the Levi-Civita connection, show some identities of the PUL and Carroll connections and define a Ricci tensor and scalar using the Carroll connection that we then further use to define the PUL Einstein-Hilbert action for the theory.

4.1. Transformations of PUL vielbeine

The PUL parameterisation of the vielbein and its inverse looks like:

$$E_\mu{}^A = \left(cT_\mu,\, E_\mu{}^a\right), \qquad E^\mu{}_A = \left(-\frac{1}{c}V^\mu,\, E^\mu{}_a\right), \tag{4.1}$$

where T_μ and V^μ are a timelike one-form and vector, and $E_\mu{}^a$ and $E^\mu{}_a$ are the spatial parts with the index $a = 1, 2, \ldots, d$, while the indices for the full vielbeine are $A = 0, 1, \ldots, d$. This is the split of the vielbein into temporal and spatial vielbene. The vielbein and its inverse transform under local Lorentz transformations $\Lambda^A{}_B$ as:

$$\delta E_\mu{}^A = \Lambda^A{}_B E_\mu{}^B, \quad \delta E^\mu{}_A = -\Lambda^B{}_A E^\mu{}_B, \tag{4.2}$$

with $\Lambda^{AB} = -\Lambda^{BA}$. Moreover, they are related to the metric in the following way:

$$g_{\mu\nu} = \eta_{AB} E_\mu{}^A E_\nu{}^B, \qquad g^{\mu\nu} = \eta^{AB} E^\mu{}_A E^\nu{}_B, \tag{4.3}$$

which upon insertion of (4.1) gives the PUL parameterisation of the metric:

$$g_{\mu\nu} = -c^2 T_\mu T_\nu + \delta_{ab} E_\mu{}^a E_\nu{}^b = -c^2 T_\mu T_\nu + \Pi_{\mu\nu}, \tag{4.4a}$$

$$g^{\mu\nu} = -\frac{1}{c^2} V^\mu V^\nu + \delta^{ab} E^\mu{}_a E^\nu{}_b = -\frac{1}{c^2} V^\mu V^\nu + \Pi^{\mu\nu}, \tag{4.4b}$$

where $\Pi_{\mu\nu}$ and $\Pi^{\mu\nu}$ are symmetric tensors that represent the spatial part of the metric. The variables just introduced then have the following properties:

$$T_\mu V^\mu = -1, \quad T_\mu E^\mu{}_a = 0, \quad E_\mu{}^a V^\mu = 0, \quad E_\mu{}^a E^\mu{}_b = \delta^a_b, \tag{4.5a}$$

$$T_\mu \Pi^{\mu\nu} = 0, \quad \Pi_{\mu\nu} V^\mu = 0, \quad E_\mu{}^A E^\nu{}_A = \delta^\nu_\mu = -T_\mu V^\nu + \Pi_{\mu\rho}\Pi^{\rho\nu}. \tag{4.5b}$$

Inserting (4.1) into (4.2):

$$\begin{aligned} \delta E_\mu{}^A &= \left(c\delta T_\mu,\, \delta E_\mu{}^a\right) \\ &= \left(\Lambda^0{}_b E_\mu{}^b,\, c\Lambda^a{}_0 T_\mu + \Lambda^a{}_b E_\mu{}^b\right), \end{aligned} \tag{4.6}$$

$$
\begin{aligned}
\delta E^{\mu}{}_{A} &= \left(-\frac{1}{c}\delta V^{\mu},\, \delta E^{\mu}{}_{a}\right) \\
&= \left(-\Lambda^{b}{}_{0}E^{\mu}{}_{b},\, \frac{1}{c}\Lambda^{0}{}_{a}V^{\mu} - \Lambda^{b}{}_{a}E^{\mu}{}_{b}\right),
\end{aligned}
\tag{4.7}
$$

and identifying that the transformation has also been split into its spatial part $\Lambda^{a}{}_{b}$ and temporal part $c\Lambda_a = \Lambda^{0}{}_{a}$ (which are both finite in the $c \to 0$ limit), we find that the PUL variables transform as:

$$
\delta T_\mu = \Lambda_a E_\mu{}^a, \qquad \delta V^\mu = c^2 \Lambda^a E^\mu{}_a, \tag{4.8a}
$$

$$
\delta E_\mu{}^a = \Lambda^a{}_b E_\mu{}^b + c^2\Lambda^a T_\mu, \qquad \delta E^\mu{}_a = -\Lambda^b{}_a E^\mu{}_b + \Lambda_a V^\mu. \tag{4.8b}
$$

From these we further find:

$$
\begin{aligned}
\delta\Pi_{\mu\nu} &= \delta_{ab}\left(\delta E_\mu{}^a E_\nu{}^b + E_\mu{}^a \delta E_\nu{}^b\right) \\
&= \delta_{ab}\left(E_\nu{}^b\left(\Lambda^a{}_c E_\mu{}^c + c^2\Lambda^a T_\mu\right) + E_\mu{}^a\left(\Lambda^b{}_c E_\nu{}^c + c^2\Lambda^b T_\nu\right)\right) \\
&= 2\,c^2\Lambda_a T_{(\mu}E_{\nu)}{}^a + \cancel{E_\nu{}^a\Lambda_{ac}E_\mu{}^c} + \cancel{E_\mu{}^a\Lambda_{ac}E_\nu{}^c},
\end{aligned}
\tag{4.8c}
$$

$$
\delta\Pi^{\mu\nu} = 2\,\Lambda^a V^{(\mu}E^{\nu)}{}_a, \tag{4.8d}
$$

giving us the complete list of PUL vielbeine transformations. In a similar manner as we defined the anti-symmetrization of indices, here the parentheses around indices represent symmetrization with the same normalization factor $1/n!$. Having split the vielbein (and its inverse) into the spatial and temporal PUL vielbeine, we are now in a position to start considering the Carroll expansion around $c = 0$.

Throughout the literature one mostly assumes that the PUL variables are analytic in c^2 and can thus be expanded purely in even powers of c. Later in this thesis we show that this is not always the case and therefore, we have to consider the full expansion by including odd powers of c. This has not been done before for the Carroll expansion of GR but a similar treatment for non-relativistic GR can be found in [66]. For now, we define the general expansion as such:

$$
X = x_{(0)} + \sum_{j=1}^{\infty} c^{Nj} x_{(j)}, \tag{4.9}
$$

where we can either choose $N = 1$ for an expansion in c or $N = 2$ for c^2, as will be done in Section 5.1. For the PUL variables we then have:

$$
V^\mu = v^\mu_{(0)} + \sum_{j=1}^{\infty} c^{Nj} v^\mu_{(j)}, \qquad T_\mu = \tau^{(0)}_\mu + \sum_{j=1}^{\infty} c^{Nj}\tau^{(j)}_\mu, \tag{4.10a}
$$

$$
E^\mu{}_a = e_{(0)}{}^\mu{}_a + \sum_{j=1}^{\infty} c^{Nj} e_{(j)}{}^\mu{}_a, \qquad E_\mu{}^a = e^{(0)}{}_\mu{}^a + \sum_{j=1}^{\infty} c^{Nj} e^{(j)}{}_\mu{}^a, \tag{4.10b}
$$

$$
\Pi^{\mu\nu} = h^{\mu\nu}_{(0)} + \sum_{j=1}^{\infty} c^{Nj} h^{\mu\nu}_{(j)}, \qquad \Pi_{\mu\nu} = h^{(0)}_{\mu\nu} + \sum_{j=1}^{\infty} c^{Nj} h^{(j)}_{\mu\nu}, \tag{4.10c}
$$

and the transformation parameters are defined as:

$$\Lambda^a = \lambda^a_{(0)} + \sum_{j=1}^{\infty} c^{Nj}\lambda^a_{(j)}, \tag{4.11a}$$

$$\Lambda^a{}_b = \lambda_{(0)}{}^a{}_b + \sum_{j=1}^{\infty} c^{Nj}\lambda_{(j)}{}^a{}_b. \tag{4.11b}$$

Now, we insert these expansions into (4.8), giving the leading order transformations:

$$\delta\tau^{(0)}_\mu = \lambda_{(0)a} e^{(0)}{}_\mu{}^a, \qquad \delta v^\mu_{(0)} = 0, \tag{4.12a}$$

$$\delta e_{(0)}{}_\mu{}^a = \lambda_{(0)}{}^a{}_b e^{(0)}{}_\mu{}^b, \qquad \delta e_{(0)}{}^\mu{}_a = -\lambda_{(0)}{}^b{}_a e^{(0)\mu}{}_b + \lambda_{(0)a} v^\mu_{(0)}, \tag{4.12b}$$

$$\delta h^{(0)}_{\mu\nu} = 0, \qquad \delta h^{\mu\nu}_{(0)} = 2\lambda^a_{(0)} v^{(\mu}_{(0)} e_{(0)}{}^{\nu)}{}_a, \tag{4.12c}$$

which are found in the following manner:

$$\begin{aligned} \delta T_\mu &= \lambda_{(0)a} e^{(0)}{}_\mu{}^a + c^N\left(\lambda_{(0)a} e^{(1)}{}_\mu{}^a + \lambda_{(1)a} e^{(0)}{}_\mu{}^a\right) + \mathcal{O}\left(c^{2N}\right) \\ &= \delta\tau^{(0)}_\mu + \sum_{j=1}^{\infty} c^{Nj}\delta\tau^{(j)}_\mu, \end{aligned} \tag{4.13}$$

by matching terms with no powers of c. Comparing these LO vielbeine transformations to the transformations acquired through gauging the Carroll algebra, (3.56), they are evidently the same with regards to Lorentz rotations and boosts, i.e. ignoring the diffeomorphisms.

4.2. The PUL and Carroll connection

The transformations (4.12) show clearly that $v^\mu_{(0)}$ and $h^{(0)}_{\mu\nu}$, the timelike vector and spatial metric, vanish while their inverses, $\tau^{(0)}_\mu$ and $h^{\mu\nu}_{(0)}$, do not. This means that the latter pair are not invariant under Carroll boosts and in turn, the vanishing of covariant derivatives can therefore not be ensured in all reference frames. In light of this, we choose an affine connection $\tilde{\Gamma}$ such that $v^\mu_{(0)}$ and $h^{(0)}_{\mu\nu}$ vanish with respect to the covariant derivative

$$\tilde{\nabla}_\rho v^\mu_{(0)} = 0, \qquad \tilde{\nabla}_\rho h^{(0)}_{\mu\nu} = 0. \tag{4.14}$$

This is the Carrollian version of the metric compatibility, $\nabla_\rho g_{\mu\nu} = 0$, mentioned in Section 2.1. It is, however, impossible to enforce vanishing torsion on the theory, since intrinsic torsion is always part of the general structure [77], even though the intrinsic torsion can in some cases be zero.

Starting with the non-expanded PUL variables V^ν and $\Pi_{\mu\nu}$, we define a connection $\tilde{C}^\rho_{\mu\nu}$, called the PUL connection, associated with a covariant derivative $\overset{(\tilde{C})}{\nabla}_\mu$ such that

$$\overset{(\tilde{C})}{\nabla}_\mu V^\nu = 0, \qquad \overset{(\tilde{C})}{\nabla}_\rho \Pi_{\mu\nu} = 0. \tag{4.15}$$

As presented in [8] the connection can be written as

$$\tilde{C}^{\rho}_{\mu\nu} = -V^{\rho}\partial_{(\mu}T_{\nu)} - V^{\rho}T_{(\mu}\pounds_V T_{\nu)} + \frac{1}{2}\Pi^{\rho\lambda}\left[\partial_\mu\Pi_{\nu\lambda} + \partial_\nu\Pi_{\lambda\mu} - \partial_\lambda\Pi_{\mu\nu}\right] - \Pi^{\rho\lambda}T_\nu\mathcal{K}_{\mu\lambda}, \quad (4.16)$$

where $\mathcal{K}_{\mu\nu} = -\frac{1}{2}\pounds_V\Pi_{\mu\nu}$ is the extrinsic curvature which is purely spatial since

$$\begin{aligned} V^\mu\mathcal{K}_{\mu\nu} &= -\frac{1}{2}V^\mu\pounds_V\Pi_{\mu\nu} \\ &= -\frac{1}{2}\Big(V^\mu V^\sigma\partial_\sigma\Pi_{\mu\nu} + V^\mu\Pi_{\sigma\nu}\partial_\mu V^\sigma + \underbrace{V^\mu\Pi_{\mu\sigma}}_{=0}\partial_\nu V^\sigma\Big) \\ &= -\frac{1}{2}\Big(\partial_\sigma\left(V^\mu V^\sigma\Pi_{\mu\nu}\right) - V^\mu\Pi_{\mu\nu}\partial_\sigma V^\sigma\Big) = 0, \end{aligned} \quad (4.17)$$

and symmetric since $\Pi_{\mu\nu}$ is symmetric. Moreover, since the extrinsic curvature is purely spatial we are allowed to define

$$\mathcal{K}^{\mu\nu} = \Pi^{\mu\rho}\Pi^{\nu\sigma}\mathcal{K}_{\rho\sigma}, \quad (4.18)$$

i.e. the indices on the extrinsic curvature can be raised with the spatial metric $\Pi^{\mu\nu}$. This is only defined for notational convenience and should not to be interpreted as anything more than that.

We define the leading order of the connection as

$$\begin{aligned} \tilde{\Gamma}^{\rho}_{\mu\nu} = \tilde{C}^{\rho}_{\mu\nu}\Big|_{c=0} &= -v^{\rho}_{(0)}\partial_{(\mu}\tau^{(0)}_{\nu)} - v^{\rho}_{(0)}\tau^{(0)}_{(\mu}\pounds_{v_{(0)}}\tau^{(0)}_{\nu)} \\ &\quad + \frac{1}{2}h^{\rho\lambda}_{(0)}\left[\partial_\mu h^{(0)}_{\nu\lambda} + \partial_\nu h^{(0)}_{\lambda\mu} - \partial_\lambda h^{(0)}_{\mu\nu}\right] - h^{\rho\lambda}_{(0)}\tau^{(0)}_\nu K_{\mu\lambda}, \end{aligned} \quad (4.19)$$

where $K_{\mu\nu} = -\frac{1}{2}\pounds_{v_{(0)}}h^{(0)}_{\mu\nu}$. The connection $\tilde{\Gamma}^{\rho}_{\mu\nu}$, called the Carroll connection, is then associated with the covariant derivatives (4.15).

4.3. The Levi-Civita connection

Having found the PUL connection, we can now relate it to the Levi-Civita connection of GR. We begin by Carroll expanding the Levi-Civita connection via (4.4):

$$\begin{aligned} \Gamma^{\rho}_{\mu\nu} &= \frac{1}{2}g^{\rho\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) \\ &= \frac{1}{2}\left(-\frac{1}{c^2}V^\rho V^\sigma + \Pi^{\rho\sigma}\right)\Big(-c^2\Big(\partial_\mu(T_\sigma T_\nu) + \partial_\nu\left(T_\mu T_\sigma\right) - \partial_\sigma\left(T_\mu T_\nu\right)\Big) \\ &\qquad + \partial_\mu\Pi_{\sigma\nu} + \partial_\nu\Pi_{\mu\sigma} - \partial_\sigma\Pi_{\mu\nu}\Big) \\ &= \frac{1}{2}\Big(\frac{1}{c^2}V^\rho\Big(\underbrace{\Pi_{\sigma\nu}\partial_\mu V^\sigma + \Pi_{\mu\sigma}\partial_\nu V^\sigma + V^\sigma\partial_\sigma\Pi_{\mu\nu}}_{=\pounds_V\Pi_{\mu\nu}}\Big) + V^\rho\underbrace{V^\sigma T_\sigma}_{-1}\,2\,\partial_{(\mu}T_{\nu)} \\ &\qquad - V^\rho\Big(\underbrace{T_\nu T_\sigma\partial_\mu V^\sigma + V^\sigma T_\nu\partial_\sigma T_\mu}_{=T_\nu\pounds_V T_\mu} + \underbrace{T_\mu T_\sigma\partial_\nu V^\sigma + V^\sigma T_\mu\partial_\sigma T_\nu}_{=T_\mu\pounds_V T_\nu}\Big) \end{aligned}$$

$$
\begin{aligned}
&+ \Pi^{\rho\sigma}\left(2\,\partial_{(\mu}\Pi_{\nu)\sigma} - \partial_\sigma \Pi_{\mu\nu}\right) \\
&- 2\,c^2 \Pi^{\rho\sigma}\Big(\underbrace{T_\sigma}_{=0}\left(\partial_{(\mu}T_{\nu)}\right) - T_\mu \partial_{[\sigma}T_{\nu]} - T_\nu \partial_{[\sigma}T_{\mu]}\Big)\Big) \\
= c^{-2}\frac{1}{2}V^\rho \pounds_V \Pi_{\mu\nu} - V^\rho \partial_{(\mu}T_{\nu)} - V^\rho T_{(\mu}\pounds_V T_{\nu)} + \Pi^{\rho\sigma}\left(\partial_{(\mu}\Pi_{\nu)\sigma} - \frac{1}{2}\partial_\sigma \Pi_{\mu\nu}\right) \\
+ \frac{c^2}{2}\Pi^{\rho\sigma}\left(T_\mu T_{\sigma\nu} + T_\nu T_{\sigma\mu}\right), \qquad (4.20)
\end{aligned}
$$

where we have defined

$$
T_{\mu\nu} = 2\,\partial_{[\mu}T_{\nu]}. \qquad (4.21)
$$

We further see that the Levi-Civita connection can be split up into three terms of different orders in c:

$$
\Gamma^\rho_{\mu\nu} = c^{-2}\overset{(-2)}{C}{}^\rho_{\mu\nu} + \overset{(0)}{C}{}^\rho_{\mu\nu} + c^2\overset{(2)}{C}{}^\rho_{\mu\nu}, \qquad (4.22)
$$

where

$$
\overset{(-2)}{C}{}^\rho_{\mu\nu} = \frac{1}{2}V^\rho \pounds_V \Pi_{\mu\nu} = -V^\rho \mathcal{K}_{\mu\nu}, \qquad (4.23a)
$$

$$
\overset{(0)}{C}{}^\rho_{\mu\nu} = -V^\rho\left(\partial_{(\mu}T_{\nu)} + T_{(\mu}\pounds_V T_{\nu)}\right) + \Pi^{\rho\sigma}\left(\partial_{(\mu}\Pi_{\nu)\sigma} - \frac{1}{2}\partial_\sigma \Pi_{\mu\nu}\right), \qquad (4.23b)
$$

$$
\overset{(2)}{C}{}^\rho_{\mu\nu} = \frac{1}{2}\Pi^{\rho\sigma}\left(T_\mu T_{\sigma\nu} + T_\nu T_{\sigma\mu}\right). \qquad (4.23c)
$$

The numbers in parentheses on top of the tensors indicate the power of c preceding the term. On comparison with (4.16) we see that $\overset{(0)}{C}{}^\rho_{\mu\nu}$ can be further decomposed as:

$$
\overset{(0)}{C}{}^\rho_{\mu\nu} = \tilde{C}^\rho_{\mu\nu} + S^\rho_{\mu\nu}, \qquad S^\rho_{\mu\nu} = -\frac{1}{2}\Pi^{\rho\lambda}T_\nu \pounds_V \Pi_{\mu\lambda} = \Pi^{\rho\lambda}T_\nu \mathcal{K}_{\mu\lambda}. \qquad (4.24)
$$

One can in fact think of $S^\rho_{\mu\nu}$ as a 'shift' tensor that shifts $\overset{(0)}{C}{}^\rho_{\mu\nu}$ to produce the PUL connection. This then yields the following expression, relating the PUL and Levi-Civita connections:

$$
\Gamma^\rho_{\mu\nu} = c^{-2}\overset{(-2)}{C}{}^\rho_{\mu\nu} + \tilde{C}^\rho_{\mu\nu} + S^\rho_{\mu\nu} + c^2\overset{(2)}{C}{}^\rho_{\mu\nu}. \qquad (4.25)
$$

4.4. Connection identities

Before proceeding further, we will show a few useful identities for the PUL connection we have just defined. We write the antisymmetrised version of the connection, or $\frac{1}{2}$ of the torsion, as such:

$$
\begin{aligned}
\tilde{C}^\rho_{[\mu\nu]} &= \frac{1}{2}\Pi^{\rho\lambda}\left(\cancel{\partial_{[\mu}\Pi_{\nu]\lambda}} + \cancel{\partial_{[\nu}\Pi_{|\lambda|\mu]}} - 2\,T_{[\nu}\mathcal{K}_{\mu]\lambda}\right) \\
&= \Pi^{\rho\lambda}T_{[\mu}\mathcal{K}_{\nu]\lambda} = -S^\rho_{[\mu\nu]}. \qquad (4.26)
\end{aligned}
$$

From this it is evident that the combination $\tilde{C}^\rho_{\mu\nu} + S^\rho_{\mu\nu}$ will cancel out the antisymmetric parts of the tensors and thus yield a symmetric tensor, as can also clearly be seen in

(4.23b), thus confirming how one can relate the torsionful connection $\tilde{C}^{\rho}_{\mu\nu}$ to the torsion-free Levi-Civita connection. The trace of the connection can be shown to be:

$$\begin{aligned}
\tilde{C}^{\rho}_{\rho\nu} &= -V^{\rho}\left(\partial_{(\rho}T_{\nu)} + T_{(\rho}\pounds_V T_{\nu)}\right) + \frac{1}{2}\Pi^{\rho\lambda}\left[\partial_{(\rho}\Pi_{\nu)\lambda} - \partial_{\lambda}\Pi_{\rho\nu}\right] - \Pi^{\rho\lambda}T_{\nu}\mathcal{K}_{\rho\lambda} \\
&= \frac{1}{2}\left(-\cancel{V^{\rho}\partial_{\rho}T_{\nu}} - V^{\rho}\partial_{\nu}T_{\rho} + \cancel{V^{\sigma}\partial_{\sigma}T_{\nu}} + T_{\sigma}\partial_{\nu}V^{\sigma}\right) - V^{\rho}T_{\nu}\left(V^{\sigma}\partial_{\sigma}T_{\rho} + T_{\sigma}\partial_{\rho}V^{\sigma}\right) \\
&\quad + \frac{1}{2}\Pi^{\rho\lambda}\left[\cancel{\partial_{\rho}\Pi_{\nu\lambda}} + \partial_{\nu}\Pi_{\rho\lambda} - \cancel{\partial_{\lambda}\Pi_{\rho\nu}}\right] - T_{\nu}\mathcal{K} \\
&= -V^{\rho}\partial_{\nu}T_{\rho} + \frac{1}{2}\Pi^{\rho\lambda}\partial_{\nu}\Pi_{\rho\lambda} - T_{\nu}\mathcal{K} - T_{\nu}\underbrace{V^{\rho}V^{\sigma}\left(\partial_{\sigma}T_{\rho} - \partial_{\rho}T_{\sigma}\right)}_{=0} \\
&= \frac{1}{E}\partial_{\nu}E - T_{\nu}\mathcal{K},
\end{aligned} \tag{4.27}$$

where $\mathcal{K} = \Pi^{\mu\nu}\mathcal{K}_{\mu\nu}$ is the trace of the extrinsic curvature and we have defined $E = \det\left[\left(T_{\mu}, E_{\mu}{}^{a}\right)\right]$, called the vielbein determinant, such that:

$$\begin{aligned}
\frac{1}{E}\partial_{\alpha}E &= \det\left[\left(V^{\mu}, E^{\mu}{}_{a}\right)\right]\partial_{\alpha}\det[(T_{\nu}, E_{\nu}{}^{a})] \\
&= \det\left[\left(V^{\mu}, E^{\mu}{}_{a}\right)\right]\det[(T_{\nu}, E_{\nu}{}^{a})] \\
&\quad \times \mathrm{Tr}\left[\left(V^{\mu}, E^{\mu}{}_{a}\right)\partial_{\alpha}(T_{\nu}, E_{\nu}{}^{a})\right] \\
&= -V^{\mu}\partial_{\alpha}T_{\mu} + E^{\mu}{}_{a}\partial_{\alpha}E_{\mu}{}^{a} \\
&= -V^{\mu}\partial_{\alpha}T_{\mu} + \frac{1}{2}\Pi^{\mu\nu}\partial_{\alpha}\Pi_{\mu\nu}.
\end{aligned} \tag{4.28}$$

In the last line we have used the following:

$$\begin{aligned}
\Pi^{\mu\nu}\partial_{\alpha}\Pi_{\mu\nu} &= E^{\mu}{}_{a}E^{\nu a}\partial_{\alpha}\left(E_{\mu}{}^{b}E_{\nu b}\right) \\
&= E^{\mu}{}_{a}E^{\nu a}\left(E_{\mu}{}^{b}\partial_{\alpha}E_{\nu b} + E_{\nu b}\partial_{\alpha}E_{\mu}{}^{b}\right) \\
&= 2\,E^{\nu}{}_{b}\partial_{\alpha}E_{\nu}{}^{b}.
\end{aligned} \tag{4.29}$$

For the covariant derivative associated with the PUL connection, we have:

$$\overset{(\tilde{C})}{\nabla}_{\sigma}X^{\rho}_{\mu\nu} = \partial_{\sigma}X^{\rho}_{\mu\nu} + \tilde{C}^{\rho}_{\sigma\lambda}X^{\lambda}_{\mu\nu} - \tilde{C}^{\lambda}_{\sigma\mu}X^{\rho}_{\lambda\nu} - \tilde{C}^{\lambda}_{\sigma\nu}X^{\rho}_{\mu\lambda}, \tag{4.30}$$

from which we can derive a few useful identities (see Appendix A.1):

$$\overset{(\tilde{C})}{\nabla}_{\sigma}T_{\mu} = \frac{1}{2}T_{\sigma\mu} - \frac{1}{2}V^{\lambda}\left(T_{\sigma}T_{\lambda\mu} + T_{\mu}T_{\lambda\sigma}\right), \tag{4.31a}$$

$$\overset{(\tilde{C})}{\nabla}_{\sigma}\Pi^{\mu\nu} = V^{(\mu}\Pi^{\nu)\rho}T_{\lambda\rho}\left(\delta^{\lambda}_{\sigma} - T_{\sigma}V^{\lambda}\right), \tag{4.31b}$$

$$\overset{(\tilde{C})}{\nabla}_{\mu}\Pi^{\mu\nu} = V^{\mu}\Pi^{\nu\rho}T_{\mu\rho}, \tag{4.31c}$$

$$\overset{(\tilde{C})}{\nabla}_{\sigma}\mathcal{K}_{\mu\nu} = \partial_{\sigma}\mathcal{K}_{\mu\nu} - \left(\Pi^{\rho\lambda}\left(\partial_{(\sigma}\Pi_{\mu)\lambda} - \frac{1}{2}\partial_{\lambda}\Pi_{\sigma\mu}\right) - \Pi^{\rho\lambda}T_{\mu}\mathcal{K}_{\sigma\lambda}\right)\mathcal{K}_{\rho\nu}$$

$$- \left(\Pi^{\rho\lambda} \left(\partial_{(\sigma}\Pi_{\nu)\lambda} - \frac{1}{2}\partial_\lambda \Pi_{\sigma\nu} \right) - \Pi^{\rho\lambda} T_\nu \mathcal{K}_{\sigma\lambda} \right) \mathcal{K}_{\mu\rho}. \tag{4.31d}$$

Lastly, we can further show an identity relating the covariant derivative of an arbitrary vector X^μ to a total derivative, using the vielbein determinant (4.28) and the trace of the connection (4.27):

$$\begin{aligned} \partial_\mu(EX^\mu) &= \partial_\mu E X^\mu + E\partial_\mu X^\mu \\ &= E\left(\overset{(\tilde{C})}{\nabla}_\mu X^\mu + T_\mu X^\mu K \right), \end{aligned} \tag{4.32}$$

where we has used:

$$\begin{aligned} \overset{(\tilde{C})}{\nabla}_\mu X^\mu &= \partial_\mu X^\mu + \tilde{C}^\mu_{\mu\sigma} X^\sigma \\ &= \partial_\mu X^\mu + \left(\frac{1}{E}\partial_\sigma E - T_\sigma \mathcal{K} \right) X^\sigma. \end{aligned} \tag{4.33}$$

Later in this thesis, we will make use of (4.32) repeatedly to simplify covariant derivatives, taking advantage of the fact that total derivatives will vanish due to boundary conditions.

4.5. Curvature: Ricci tensor and scalar

Taking the Ricci tensor (2.12b) for the Levi-Civita connection and inserting the decomposition of the Levi Civita connection (4.25), we find

$$R_{\mu\nu} = 2\,\partial_{[\rho}\Gamma^\rho_{\mu]\nu} + 2\,\Gamma^\rho_{[\rho|\sigma|}\Gamma^\sigma_{\mu]\nu} \tag{4.34}$$

$$\begin{aligned} &= 2\Bigg[\partial_{[\rho} \left(c^{-2}\overset{(-2)}{C}{}^\rho_{\mu]\nu} + \tilde{C}^\rho_{\mu]\nu} + S^\rho_{\mu]\nu} + c^2 \overset{(2)}{C}{}^\rho_{\mu]\nu} \right) \\ &\quad + \left(c^{-2}\overset{(-2)}{C}{}^\rho_{[\rho|\sigma|} + \tilde{C}^\rho_{[\rho|\sigma|} + S^\rho_{[\rho|\sigma|} + c^2\overset{(2)}{C}{}^\rho_{[\rho|\sigma|} \right)\left(c^{-2}\overset{(-2)}{C}{}^\sigma_{\mu]\nu} + \tilde{C}^\sigma_{\mu]\nu} + S^\sigma_{\mu]\nu} + c^2\overset{(2)}{C}{}^\sigma_{\mu]\nu} \right) \Bigg] \\ &= 2\Bigg[c^{-4}\overset{(-2)}{C}{}^\rho_{[\rho|\sigma|}\overset{(-2)}{C}{}^\sigma_{\mu]\nu} + c^{-2}\left(\partial_{[\rho}\overset{(-2)}{C}{}^\rho_{\mu]\nu} + \overset{(-2)}{C}{}^\rho_{[\rho|\sigma|}\left(\tilde{C}^\sigma_{\mu]\nu} + S^\sigma_{\mu]\nu} \right) + \left(\tilde{C}^\rho_{[\rho|\sigma|} + S^\rho_{[\rho|\sigma|} \right)\overset{(-2)}{C}{}^\sigma_{\mu]\nu} \right) \\ &\quad + \partial_{[\rho}\left(\tilde{C}^\rho_{\mu]\nu} + S^\rho_{\mu]\nu} \right) + \overset{(-2)}{C}{}^\rho_{[\rho|\sigma|}\overset{(2)}{C}{}^\sigma_{\mu]\nu} + \left(\tilde{C}^\rho_{[\rho|\sigma|} + S^\rho_{[\rho|\sigma|} \right)\left(\tilde{C}^\sigma_{\mu]\nu} + S^\sigma_{\mu]\nu} \right) + \overset{(2)}{C}{}^\rho_{[\rho|\sigma|}\overset{(-2)}{C}{}^\sigma_{\mu]\nu} \\ &\quad + c^2\left(\partial_{[\rho}\overset{(2)}{C}{}^\rho_{\mu]\nu} + \left(\tilde{C}^\rho_{[\rho|\sigma|} + S^\rho_{[\rho|\sigma|} \right)\overset{(2)}{C}{}^\sigma_{\mu]\nu} + \overset{(2)}{C}{}^\rho_{[\rho|\sigma|}\left(\tilde{C}^\sigma_{\mu]\nu} + S^\sigma_{\mu]\nu} \right) \right) + c^4\overset{(2)}{C}{}^\rho_{[\rho|\sigma|}\overset{(2)}{C}{}^\sigma_{\mu]\nu} \Bigg], \end{aligned} \tag{4.35}$$

where, again, we split it up into orders of c:

$$R_{\mu\nu} = c^{-4}\overset{(-4)}{R}_{\mu\nu} + c^{-2}\overset{(-2)}{R}_{\mu\nu} + \overset{(0)}{R}_{\mu\nu} + c^2\overset{(2)}{R}_{\mu\nu} + c^4\overset{(4)}{R}_{\mu\nu}. \tag{4.36}$$

Here, terms like:

$$\overset{(-2)}{C}{}^\rho_{\rho\mu} = -V^\rho \mathcal{K}_{\rho\mu} = 0, \tag{4.37a}$$

$$S^{\rho}_{\sigma\mu}\overset{(-2)}{C}{}^{\sigma}_{\nu\lambda} = -\Pi^{\rho\alpha}T_\mu\mathcal{K}_{\sigma\alpha}V^\sigma\mathcal{K}_{\nu\lambda} = 0, \tag{4.37b}$$

$$S^{\rho}_{\mu\sigma}S^{\sigma}_{\nu\lambda} = \Pi^{\rho\alpha}T_\sigma\mathcal{K}_{\mu\alpha}\Pi^{\sigma\beta}T_\lambda\mathcal{K}_{\nu\beta} = 0, \tag{4.37c}$$

$$\overset{(2)}{C}{}^{\rho}_{\rho\sigma}\overset{(2)}{C}{}^{\sigma}_{\mu\nu} = \frac{1}{4}\Pi^{\rho\lambda}\left(T_\rho T_{\lambda\sigma} + T_\sigma T_{\lambda\rho}\right)\Pi^{\sigma\gamma}\left(T_\mu T_{\gamma\nu} + T_\nu T_{\gamma\mu}\right) = 0, \tag{4.37d}$$

$$\overset{(2)}{C}{}^{\rho}_{\rho\sigma}S^{\sigma}_{\mu\nu} = \frac{1}{2}\Pi^{\rho\lambda}\left(T_\rho T_{\lambda\sigma} + T_\sigma T_{\lambda\rho}\right)\Pi^{\sigma\alpha}T_\nu\mathcal{K}_{\mu\alpha} = 0, \tag{4.37e}$$

$$S^{\rho}_{\lambda\sigma}\overset{(2)}{C}{}^{\sigma}_{\mu\nu} = \frac{1}{2}\Pi^{\rho\alpha}T_\sigma\mathcal{K}_{\lambda\alpha}\Pi^{\sigma\beta}\left(T_\mu T_{\beta\nu} + T_\nu T_{\beta\mu}\right) = 0, \tag{4.37f}$$

vanish due to $\mathcal{K}_{\mu\nu}$ being purely spatial, see (4.17), and due to the identity $T_\mu\Pi^{\mu\nu} = 0$, (4.5a). Terms containing $\overset{(2)}{C}{}^{\rho}_{\rho\nu}$ vanish in the following way:

$$\begin{aligned}
\overset{(2)}{C}{}^{\rho}_{\rho\nu} &= \frac{1}{2}\Pi^{\rho\sigma}\Big(\underbrace{T_\rho \quad T_{\sigma\nu}}_{=0} + T_\nu T_{\sigma\rho}\Big) \\
&= \Pi^{\rho\sigma}T_\nu\left(\partial_\sigma T_\rho - \partial_\rho T_\sigma\right) \\
&= T_\nu\left(-T_\rho\partial_\sigma\Pi^{\rho\sigma} + T_\sigma\partial_\rho\Pi^{\rho\sigma}\right) \\
&= T_\nu\left(-T_\rho\partial_\sigma\Pi^{\rho\sigma} + T_\rho\partial_\sigma\Pi^{\sigma\rho}\right) = 0,
\end{aligned} \tag{4.38}$$

due to the symmetry of $\Pi^{\mu\nu}$. Thus, we identify (see Appendix A.2 for derivation):

$$\overset{(-4)}{R}_{\mu\nu} = 0, \tag{4.39a}$$

$$\overset{(-2)}{R}_{\mu\nu} = -V^\rho\overset{(\check{C})}{\nabla}_\rho\mathcal{K}_{\mu\nu} + \mathcal{K}\mathcal{K}_{\mu\nu}, \tag{4.39b}$$

$$\overset{(0)}{R}_{\mu\nu} = \overset{(\check{C})}{R}_{\mu\nu} + 2\overset{(\check{C})}{\nabla}_{[\rho}S^{\rho}_{\mu]\nu} + 2\tilde{C}^{\sigma}_{[\rho\mu]}S^{\rho}_{\sigma\nu} - 2\overset{(-2)}{C}{}^{\rho}_{\sigma(\mu}\overset{(2)}{C}{}^{\sigma}_{\nu)\rho}, \tag{4.39c}$$

$$\overset{(2)}{R}_{\mu\nu} = \frac{1}{2}\overset{(\check{C})}{\nabla}_\rho\left(\Pi^{\rho\sigma}\left(T_\mu T_{\sigma\nu} + T_\nu T_{\sigma\mu}\right)\right), \tag{4.39d}$$

$$\overset{(4)}{R}_{\mu\nu} = -\frac{1}{4}\Pi^{\rho\lambda}T_\mu T_{\lambda\sigma}\Pi^{\sigma\gamma}T_\nu T_{\gamma\rho}, \tag{4.39e}$$

where we have defined the Ricci tensor for the PUL connection as

$$\overset{(\check{C})}{R}_{\mu\nu} = 2\,\partial_{[\rho}\tilde{C}^{\rho}_{\mu]\nu} + 2\,\tilde{C}^{\rho}_{[\rho|\sigma|}\tilde{C}^{\sigma}_{\mu]\nu}. \tag{4.40}$$

Now for the Ricci scalar, we begin by inserting (4.4b):

$$R = g^{\mu\nu}R_{\mu\nu} = \left(-\frac{1}{c^2}V^\mu V^\nu + \Pi^{\mu\nu}\right)R_{\mu\nu} \tag{4.41}$$

$$\begin{aligned}
&= -\frac{1}{c^4}V^\mu V^\nu\overset{(-2)}{R}_{\mu\nu} + \frac{1}{c^2}\left(\Pi^{\mu\nu}\overset{(-2)}{R}_{\mu\nu} - V^\mu V^\nu\overset{(0)}{R}_{\mu\nu}\right) \\
&\quad + \Pi^{\mu\nu}\overset{(0)}{R}_{\mu\nu} - V^\mu V^\nu\overset{(2)}{R}_{\mu\nu} + c^2\left(\Pi^{\mu\nu}\overset{(2)}{R}_{\mu\nu} - V^\mu V^\nu\overset{(4)}{R}_{\mu\nu}\right) + \cancel{c^4\Pi^{\mu\nu}\overset{(4)}{R}_{\mu\nu}},
\end{aligned} \tag{4.42}$$

where $\overset{(-4)}{R}_{\mu\nu}$ has previously been shown to vanish and the c^4 term can be shown to vanish as well:

$$\Pi^{\mu\nu}\overset{(4)}{R}_{\mu\nu} = -\frac{1}{4}\Pi^{\mu\nu}\Pi^{\rho\alpha}T_\mu T_{\alpha\sigma}\Pi^{\sigma\beta}T_\nu T_{\beta\rho} = 0. \tag{4.43}$$

Grouping together all the $R_{\mu\nu}$ terms of the same order and considering them separately, we find (once again, see Appendix A.2 for derivation):

$$\left(\frac{1}{c^2}\Pi^{\mu\nu} - \frac{1}{c^4}V^\mu V^\nu\right)\overset{(-2)}{R}_{\mu\nu} = -\frac{1}{c^2}\frac{1}{E}\partial_\rho(E\mathcal{K}V^\rho), \tag{4.44a}$$

$$\left(\Pi^{\mu\nu} - \frac{1}{c^2}V^\mu V^\nu\right)\overset{(0)}{R}_{\mu\nu} = \Pi^{\mu\nu}\overset{(\check{C})}{R}_{\mu\nu} + \frac{1}{c^2}\left(-\frac{1}{E}\partial_\mu(E\mathcal{K}V^\mu) + \mathcal{K}^{\sigma\gamma}\mathcal{K}_{\sigma\gamma}\right), \tag{4.44b}$$

$$\left(c^2\Pi^{\mu\nu} - V^\mu V^\nu\right)\overset{(2)}{R}_{\mu\nu} = \frac{c^2}{2}\Pi^{\mu\nu}\Pi^{\rho\sigma}T_{\rho\mu}T_{\sigma\nu} - \frac{1}{E}\partial_\rho\big(EV^\mu\Pi^{\rho\sigma}T_{\mu\sigma}\big), \tag{4.44c}$$

$$-c^2V^\mu V^\nu\overset{(4)}{R}_{\mu\nu} = -\frac{c^2}{4}\Pi^{\rho\sigma}\Pi^{\lambda\gamma}T_{\sigma\lambda}T_{\rho\gamma}. \tag{4.44d}$$

Here, (4.32) has been used in various places as well as the following expression:

$$\begin{aligned}[\overset{(\check{C})}{\nabla}_\mu, \overset{(\check{C})}{\nabla}_\nu]V^\nu &= -\overset{(\check{C})}{R}_{\mu\sigma}V^\sigma - 2\,\tilde{C}^\sigma_{[\mu\nu]}\overset{(\check{C})}{\nabla}_\sigma V^\nu = 0 \\ &\Rightarrow \overset{(\check{C})}{R}_{\mu\sigma}V^\sigma = 0.\end{aligned} \tag{4.45}$$

The Ricci scalar is thus:

$$\begin{aligned}R = \frac{1}{c^2}\Big(&-\frac{2}{E}\partial_\rho(E\mathcal{K}V^\rho) - \mathcal{K}^2 + \mathcal{K}^{\mu\nu}\mathcal{K}_{\mu\nu}\Big) \\ &+ \Pi^{\mu\nu}\overset{(\check{C})}{R}_{\mu\nu} - \frac{1}{E}\partial_\rho\big(EV^\mu\Pi^{\rho\sigma}T_{\mu\sigma}\big) + \frac{c^2}{4}\Pi^{\mu\nu}\Pi^{\rho\sigma}T_{\mu\rho}T_{\nu\sigma}.\end{aligned} \tag{4.46}$$

The leading-order term contains only the extrinsic curvature and can therefore be interpreted as a kinetic term and at next-to-leading-order we have a curvature term. Having derived the Ricci scalar, we can now move onto constructing the Einstein-Hilbert action.

4.6. PUL Einstein-Hilbert action

The Einstein-Hilbert action, as defined in (2.11), can now be expanded using the previously derived PUL form of the Ricci scalar (4.46):

$$\begin{aligned}S_{\text{EH}} &= \frac{c^3}{16\pi G}\int_M R\sqrt{-g}\,\mathrm{d}^{d+1}x \\ &= \frac{1}{16\pi G}\int_M \Bigg[c^2\Big(\mathcal{K}^{\mu\nu}\mathcal{K}_{\mu\nu} - \mathcal{K}^2 - \frac{2}{E}\partial_\rho(E\mathcal{K}V^\rho)\Big) \\ &\qquad + c^4\Big(\Pi^{\mu\nu}\overset{(\check{C})}{R}_{\mu\nu} - \frac{1}{E}\partial_\rho\big(EV^\mu\Pi^{\rho\sigma}T_{\mu\sigma}\big)\Big) + \frac{c^6}{4}\Pi^{\mu\nu}\Pi^{\rho\sigma}T_{\mu\rho}T_{\nu\sigma}\Bigg]E\,\mathrm{d}^{d+1}x.\end{aligned} \tag{4.47}$$

Here the PUL expansion of the metric (4.4a) has been inserted into $\sqrt{-g} = \sqrt{-\det[g_{\mu\nu}]}$:

$$\begin{aligned}\sqrt{-g} &= \sqrt{-\det[-c^2T_\mu T_\nu + \Pi_{\mu\nu}]} \\ &= c\sqrt{\det\big[\eta_{AB}(T_\mu, E_\mu{}^a)(T_\nu, E_\nu{}^a)\big]}\end{aligned}$$

$$= c\sqrt{E^2} = cE, \tag{4.48}$$

and E is, again, the vielbein determinant. The factoring out of the $-c^2$ can be done in the following way:

$$\begin{aligned} -\det[-c^2 T_\mu T_\nu + \Pi_{\mu\nu}] &= \epsilon^{\mu_0\mu_1\cdots}\left(-c^2 T_{\mu_0} T_0 + \underbrace{\Pi_{\mu_0 0}}_{=0}\right)\left(-c^2 T_{\mu_1}\underbrace{T_1}_{=0} + \Pi_{\mu_1 1}\right)\cdots \\ &= \epsilon^{\mu_0\mu_1\cdots}\left(-c^2 T_{\mu_0} T_0 \Pi_{\mu_1 1}\Pi_{\mu_2 2}\cdots\right) \\ &= c^2 \det[T_\mu T_\nu + \Pi_{\mu\nu}], \end{aligned} \tag{4.49}$$

where on has used the fact that T_μ has no spatial part and $\Pi_{\mu\nu}$ no temporal part.

The total derivatives, boundary terms, will now vanish due to boundary conditions, and thus we arrive at the following PUL action:

$$S_{\mathrm{EH}} = \frac{1}{16\pi G}\int_M \left[c^2\left(\mathcal{K}^{\mu\nu}\mathcal{K}_{\mu\nu} - \mathcal{K}^2\right) + c^4 \Pi^{\mu\nu} \overset{(\check{C})}{R}_{\mu\nu} + \frac{c^6}{4}\Pi^{\mu\nu}\Pi^{\rho\sigma}T_{\mu\rho}T_{\nu\sigma}\right] E\, \mathrm{d}^{d+1}x. \tag{4.50}$$

One wants the leading-order term of the action (and the Lagrangian) to have no preceding orders of c and thus defines $S = c^\Delta \tilde{S}$, where c^Δ are then all the pre-leading orders of c that can be pulled outside of the integral in the action:

$$S_{\mathrm{EH}} = c^\Delta\left(\overset{(\Delta)}{S}_{\mathrm{LO}} + c^N \overset{(\Delta+N)}{S}_{\mathrm{NLO}} + \mathcal{O}(c^{2N})\right), \tag{4.51}$$

where $N = 1$ for the expansion in all orders of c and $N = 2$ for the expansion in even orders of c^2.

5. Carrollian expansion of general relativity

This section will consist of a review of the derivation of the equations of motion for the leading order of the Carroll expansion, as done before in [8], as well as a derivation of the full next-to-leading-order equations of motion of the theory. Until now, this has only been done to truncated order and is therefore a novel result. In this thesis we will only consider vacuum solutions, for solutions including a cosmological constant as well as other solutions see [8, 47]. Moreover, we consider both the case of the c and c^2 expansion for the NLO theory.

5.1. Carroll expansion

As mentioned in the previous section, when taking the Carroll limit of the PUL variables they reduce to their leading order and the PUL connection becomes the Carroll connection. Now that we have derived the PUL action, we first have to return to the topic of expanding in all powers of c versus c^2 before deriving the equations of motion of the theory. The reason being that the expansion one chooses affects how the theory behaves at higher orders. The LO equations will be unaffected by this choice but depending on whether we choose to expand in all powers of c or only in even powers, the NLO term of the PUL action (4.50) will be preceded by an extra order of either c or c^2. We will start by presenting the full expansion in all powers of c, followed by the even powers expansion.

5.1.1. Expanding in all powers of c

The expansion for all powers in c, even and odd, will be referred to as the c expansion throughout the rest of this thesis. The c expansion will be useful for the cases where the assumption of analyticity in c^2 breaks down. The vielbeine are then as defined in (4.10) with $N = 1$. We take the contravariant vielbeine V^μ and $E^\mu{}_a$ to be the defining ones, i.e. the higher order terms for all the other vielbeine will be defined in terms of the higher order terms of the two along with all the leading order terms. Here we will show the derivation for the expansion of T_μ and then referring to Appendix B for the remaining vielbeine.

Expanding the first two of the completeness/orthogonality relations (4.5a)

$$\begin{aligned} V^\mu T_\mu = v^\mu_{(0)}\tau^{(0)}_\mu + c\left(v^\mu_{(1)}\tau^{(0)}_\mu + v^\mu_{(0)}\tau^{(1)}_\mu\right) + c^2\left(v^\mu_{(2)}\tau^{(0)}_\mu + v^\mu_{(0)}\tau^{(2)}_\mu + v^\mu_{(1)}\tau^{(1)}_\mu\right) \\ + \mathcal{O}\left(c^3\right) = -1, \end{aligned} \tag{5.1a}$$

$$\begin{aligned} E^\mu{}_a T_\mu = e_{(0)}{}^\mu{}_a\tau^{(0)}_\mu + c\left(e_{(0)}{}^\mu{}_a\tau^{(1)}_\mu + e_{(1)}{}^\mu{}_a\tau^{(0)}_\mu\right) \\ + c^2\left(e_{(0)}{}^\mu{}_a\tau^{(2)}_\mu + e_{(2)}{}^\mu{}_a\tau^{(0)}_\mu + e_{(1)}{}^\mu{}_a\tau^{(1)}_\mu\right) + \mathcal{O}\left(c^3\right) = 0, \end{aligned} \tag{5.1b}$$

and demanding that they hold at leading order, such that $\tau^{(0)}_\mu v^\mu_{(0)} = -1$ and $e_{(0)}{}^\mu{}_a\tau^{(0)}_\mu = 0$, we find constraint equations for $J \geq 1$:

$$\sum_{j=0}^{J}\tau^{(j)}_\mu v^\mu_{(J-j)} = 0, \qquad \sum_{j=0}^{J}\tau^{(j)}_\mu e_{(J-j)}{}^\mu{}_a = 0 \tag{5.2}$$

with J representing the order of c in the expansion. Taking these two equations and contracting them with $\tau^{(0)}_\mu$ and $e^{(0)}{}_\mu{}^a$ respectively, we have

$$\tau^{(0)}_\mu \sum_{j=0}^{J} \tau^{(j)}_\rho v^\rho_{(J-j)} = \tau^{(0)}_\mu \sum_{j=0}^{J-1} \tau^{(j)}_\rho v^\rho_{(J-j)} + \tau^{(J)}_\rho \tau^{(0)}_\mu v^\rho_{(0)} = 0, \tag{5.3a}$$

$$e^{(0)}{}_\mu{}^a \sum_{j=0}^{J} \tau^{(j)}_\rho e_{(J-j)}{}^\rho{}_a = e^{(0)}{}_\mu{}^a \sum_{j=0}^{J-1} \tau^{(j)}_\rho e_{(J-j)}{}^\rho{}_a + \tau^{(J)}_\rho \underbrace{e^{(0)}{}_\mu{}^a e_{(0)}{}^\rho{}_a}_{\delta^\rho_\mu + v^\rho_{(0)}\tau^{(0)}_\mu} = 0, \tag{5.3b}$$

where we then combine the two using the common term appearing in both, giving:

$$\tau^{(J)}_\mu = \tau^{(0)}_\mu \sum_{j=0}^{J-1} \tau^{(j)}_\rho v^\rho_{(J-j)} - e^{(0)}{}_\mu{}^a \sum_{j=0}^{J-1} \tau^{(j)}_\rho e_{(J-j)}{}^\rho{}_a. \tag{5.4}$$

This equation can then be solved iteratively for each order of the expansion, for the derivation for all the other vielbeine we refer to Appendix B. For $J = 1$ we have

$$\tau^{(1)}_\mu = \tau^{(0)}_\mu \tau^{(0)}_\rho v^\rho_{(1)} - e^{(0)}{}_\mu{}^a \tau^{(0)}_\rho e_{(1)}{}^\rho{}_a, \tag{5.5a}$$

$$e^{(1)}{}_\mu{}^a = \tau^{(0)}_\mu e^{(0)}{}_\rho{}^a v^\rho_{(1)} - e^{(0)}{}_\mu{}^b e^{(0)}{}_\rho{}^a e_{(1)}{}^\rho{}_b, \tag{5.5b}$$

$$h^{\mu\nu}_{(1)} = \delta^{ab}\Big(e_{(0)}{}^\mu{}_a e_{(1)}{}^\nu{}_b + e_{(1)}{}^\mu{}_a e_{(0)}{}^\nu{}_b\Big), \tag{5.5c}$$

$$h^{(1)}_{\mu\nu} = 2\,\delta_{ab} e^{(0)}{}_\rho{}^b e^{(0)}{}_{(\mu}{}^a \Big(\tau^{(0)}_{\nu)} v^\rho_{(1)} - e^{(0)}{}_{\nu)}{}^c e_{(1)}{}^\rho{}_c\Big), \tag{5.5d}$$

and then for $J = 2$:

$$\begin{aligned}
\tau^{(2)}_\mu &= \tau^{(0)}_\rho \Big(\tau^{(0)}_\mu v^\rho_{(2)} - e^{(0)}{}_\mu{}^a e_{(2)}{}^\rho{}_a\Big) \\
&\quad + \Big(\tau^{(0)}_\rho \tau^{(0)}_\sigma v^\sigma_{(1)} - e^{(0)}{}_\rho{}^a \tau^{(0)}_\sigma e_{(1)}{}^\sigma{}_a\Big)\Big(\tau^{(0)}_\mu v^\rho_{(1)} - e^{(0)}{}_\mu{}^a e_{(1)}{}^\rho{}_a\Big),
\end{aligned} \tag{5.6a}$$

$$\begin{aligned}
e^{(2)}{}_\mu{}^a &= e^{(0)}{}_\rho{}^a \Big(\tau^{(0)}_\mu v^\rho_{(2)} - e^{(0)}{}_\mu{}^b e_{(2)}{}^\rho{}_b\Big) \\
&\quad + \Big(\tau^{(0)}_\rho e^{(0)}{}_\sigma{}^a v^\sigma_{(1)} - e^{(0)}{}_\rho{}^b e^{(0)}{}_\sigma{}^a e_{(1)}{}^\sigma{}_b\Big)\Big(\tau^{(0)}_\mu v^\rho_{(1)} + e^{(0)}{}_\mu{}^b e_{(1)}{}^\rho{}_b\Big),
\end{aligned} \tag{5.6b}$$

$$h^{\mu\nu}_{(2)} = \delta^{ab}\Big(e_{(0)}{}^\mu{}_a e_{(2)}{}^\nu{}_b + e_{(1)}{}^\mu{}_a e_{(1)}{}^\nu{}_b + e_{(2)}{}^\mu{}_a e_{(0)}{}^\nu{}_b\Big), \tag{5.6c}$$

$$\begin{aligned}
h^{(2)}_{\mu\nu} &= 2\,\delta_{ab} v^\rho_{(2)} e^{(0)}{}_\rho{}^a \tau^{(0)}_{(\mu} e^{(0)}{}_{\nu)}{}^b - 2\,\delta_{ab} e_{(2)}{}^\rho{}_c e^{(0)}{}_\rho{}^b e^{(0)}{}_{(\mu}{}^a e^{(0)}{}_{\nu)}{}^c \\
&\quad + \delta_{ab} v^\rho_{(1)} \tau^{(0)}_\mu \Big(\tau^{(0)}_\rho e^{(0)}{}_\sigma{}^a v^\sigma_{(1)} \tau^{(0)}_\nu e^{(0)}{}_\alpha{}^b v^\alpha_{(1)} - e^{(0)}{}_\rho{}^c e^{(0)}{}_\sigma{}^a e_{(1)}{}^\sigma{}_c \tau^{(0)}_\nu e^{(0)}{}_\alpha{}^b v^\alpha_{(1)} \\
&\qquad\qquad - \tau^{(0)}_\rho e^{(0)}{}_\sigma{}^a v^\sigma_{(1)} e^{(0)}{}_\nu{}^d e^{(0)}{}_\alpha{}^b e_{(1)}{}^\alpha{}_d \\
&\qquad\qquad + e^{(0)}{}_\rho{}^c e^{(0)}{}_\sigma{}^a e_{(1)}{}^\sigma{}_c e^{(0)}{}_\nu{}^d e^{(0)}{}_\alpha{}^b e_{(1)}{}^\alpha{}_d\Big) \\
&\quad + \delta_{ab} v^\rho_{(1)} \Big(\tau^{(0)}_\nu \tau^{(0)}_\sigma v^\sigma_{(1)} - e^{(0)}{}_\nu{}^c \tau^{(0)}_\sigma e_{(1)}{}^\sigma{}_c\Big) e^{(0)}{}_\mu{}^a e^{(0)}{}_\rho{}^b \\
&\quad - \delta_{ab} e^{(0)}{}_\mu{}^c \Big(\tau^{(0)}_\nu e^{(0)}{}_\rho{}^a v^\rho_{(1)} \tau^{(0)}_\sigma e^{(0)}{}_\alpha{}^b v^\alpha_{(1)} - e^{(0)}{}_\nu{}^d e^{(0)}{}_\rho{}^a e_{(1)}{}^\rho{}_d \tau^{(0)}_\sigma e^{(0)}{}_\alpha{}^b v^\alpha_{(1)}
\end{aligned}$$

$$
\begin{aligned}
&- \tau_\nu^{(0)} e^{(0)}{}_\rho{}^a v_{(1)}^\rho e^{(0)}{}_\sigma{}^e e^{(0)}{}_\alpha{}^b e_{(1)}{}^\alpha{}_e \\
&+ e^{(0)}{}_\nu{}^d e^{(0)}{}_\rho{}^a e_{(1)}{}^\rho{}_d e^{(0)}{}_\sigma{}^e e^{(0)}{}_\alpha{}^b e_{(1)}{}^\alpha{}_e \Big) e_{(1)}{}^\sigma{}_c \\
&- \delta_{ab} e^{(0)}{}_\mu{}^a e^{(0)}{}_\rho{}^b \delta_{cd} \delta^{ef} e_{(1)}{}^\rho{}_f e^{(0)}{}_\nu{}^c e^{(0)}{}_\sigma{}^d e_{(1)}{}^\sigma{}_e \\
&- \delta_{ab} e^{(0)}{}_\mu{}^a e^{(0)}{}_\rho{}^b \delta_{cd} \delta^{ef} e_{(1)}{}^\rho{}_f \Big(\tau_\nu^{(0)} e^{(0)}{}_\alpha{}^c v_{(1)}^\alpha - e^{(0)}{}_\nu{}^g e^{(0)}{}_\alpha{}^c e_{(1)}{}^\alpha{}_g \Big) e^{(0)}{}_\beta{}^d e_{(1)}{}^\beta{}_e .
\end{aligned} \tag{5.6d}
$$

Now for the general expansion of the metric (4.4a)

$$
\begin{aligned}
g_{\mu\nu} &= -c^2 T_\mu T_\nu + \Pi_{\mu\nu} \\
&= -c^2 \tau_\mu \tau_\nu + h_{\mu\nu} + c^N h_{\mu\nu}^{(1)} + c^{2N} h_{\mu\nu}^{(2)},
\end{aligned} \tag{5.7}
$$

we can write for the c expansion:

$$
g_{\mu\nu} = h_{\mu\nu} + c h_{\mu\nu}^{(1)} + c^2 \Big(h_{\mu\nu}^{(2)} - \tau_\mu \tau_\nu \Big) + \mathcal{O}\Big(c^3\Big), \tag{5.8}
$$

which will be useful later on.

5.1.2. Only even powers of c

Here, one will assume that the PUL parameters are analytic in c^2 and can thus be expanded using only even powers of c, thus setting $N = 2$. We will call this the c^2 expansion. This assumption will hold in many cases and is convenient due to simplicity. For V^μ and $E^\mu{}_a$ we define the expansions as

$$
V^\mu = v^\mu + c^2 M^\mu + \mathcal{O}(c^4) \quad \text{and} \quad E^\mu{}_a = e^\mu{}_a + c^2 \pi^\mu{}_a + \mathcal{O}(c^4), \tag{5.9}
$$

where the LO and NLO parameters have been defined in the following manner:

$$
v_{(0)}^\mu = v^\mu, \qquad v_{(1)}^\mu = M^\mu, \tag{5.10a}
$$

$$
e_{(0)}{}^\mu{}_a = e^\mu{}_a, \qquad e_{(1)}{}^\mu{}_a = \pi^\mu{}_a. \tag{5.10b}
$$

For the leading order of all the other parameters, we drop the (0) index in the same manner. Inserting (5.10) into (5.5) we find the expansions to be:

$$
V^\mu = v^\mu + c^2 M^\mu + \mathcal{O}(c^4), \tag{5.11a}
$$

$$
T_\mu = \tau_\mu + c^2 \Big(\tau_\mu \tau_\nu M^\nu - e_\mu{}^a \tau_\nu \pi^\nu{}_a \Big) + \mathcal{O}(c^4), \tag{5.11b}
$$

$$
E^\mu{}_a = e^\mu{}_a + c^2 \pi^\mu{}_a + \mathcal{O}(c^4), \tag{5.11c}
$$

$$
E_\mu{}^a = e_\mu{}^a + c^2 \Big(\tau_\mu M^\nu e_\nu{}^a - e_\mu{}^b e_\nu{}^a \pi^\nu{}_b \Big) + \mathcal{O}(c^4), \tag{5.11d}
$$

$$
\Pi^{\mu\nu} = h^{\mu\nu} + c^2 \Phi^{\mu\nu} + \mathcal{O}(c^4), \tag{5.11e}
$$

$$
\Pi_{\mu\nu} = h_{\mu\nu} + 2\, c^2 \delta_{ab} e_{(\mu}{}^a \Big(\tau_{\nu)} M^\rho e_\rho{}^b - e_{\nu)}{}^c e_\rho{}^b \pi^\rho{}_c \Big) + \mathcal{O}(c^4), \tag{5.11f}
$$

We can further verify that inserting (5.11) into (4.5b) we find:

$$
\tau_\mu v^\mu = -1, \quad \tau_\mu h^{\mu\nu} = 0, \quad h_{\mu\nu} v^\nu = 0, \quad e_\mu{}^A e^\nu{}_A = \delta_\mu^\nu = -\tau_\mu v^\nu + h_{\mu\rho} h^{\rho\nu}. \tag{5.12}
$$

Lastly, the metric $g_{\mu\nu}$ in the c^2 expansion becomes:

$$
g_{\mu\nu} = h_{\mu\nu} + c^2 \Big(h_{\mu\nu}^{(1)} - \tau_\mu \tau_\nu \Big) + \mathcal{O}\Big(c^4\Big), \tag{5.13}
$$

that will, again, be used later in this thesis.

5.2. Leading-order theory

In order to explore the leading order of the theory, one does not have to specify whether one has chosen the c or c^2 expansion, since the leading order variables will be the same for both expansions. Here we will choose to drop the (0) labels on the leading order variables and we stay with that choice throughout the rest of this thesis. By defining the leading order of the extrinsic curvature as $K_{\mu\nu} = -\frac{1}{2}\pounds_v h_{\mu\nu}$ and the vielbein determinant as $e = \det[(\tau_\mu, e_\mu{}^a)]$, we can write the leading-order term of the Einstein-Hilbert action (4.50), thus effectively taking the Carroll limit:

$$\overset{(2)}{S}_{\mathrm{LO}} = \frac{1}{16\pi G}\int_M \left[K^{\mu\nu}K_{\mu\nu} - K^2\right] e\, \mathrm{d}^{d+1}x. \tag{5.14}$$

As a reminder, the label (2) above the LO action represents that there are two powers of c multiplying that term.

Before tackling the variation of the action, we first present some variational identities.

$$\begin{aligned}
\delta\tau_\mu &= -\tau_\gamma h_{\mu\rho}\delta h^{\rho\gamma} + \tau_\gamma\tau_\mu\delta v^\gamma, && (5.15a)\\
\delta h_{\mu\nu} &= -h_{\mu\rho}h_{\nu\lambda}\delta h^{\rho\lambda} + 2\, h_{\lambda(\mu}\tau_{\nu)}\delta v^\lambda, && (5.15b)\\
\delta K_{\mu\nu} &= \left(-\tau_\lambda K_{\mu\nu} + K_{\lambda(\nu}\tau_{\mu)} - \frac{1}{2}h_{\lambda\mu}v^\sigma\tau_{\sigma\nu} - \frac{1}{2}h_{\lambda\nu}v^\sigma\tau_{\sigma\mu}\right)\delta v^\lambda - h_{\sigma(\nu}\tilde{\nabla}_{\mu)}\delta v^\sigma\\
&\quad - v^\sigma h_{\alpha(\mu}\tau_{\nu)}\tilde{\nabla}_\sigma\delta v^\alpha - K_{\lambda(\nu}h_{\mu)\rho}\delta h^{\rho\lambda} + \frac{1}{2}h_{\mu\rho}h_{\nu\lambda}v^\sigma\tilde{\nabla}_\sigma\delta h^{\rho\lambda}, && (5.15c)\\
\delta K &= -(v^\sigma\tau_{\sigma\lambda} + \tau_\lambda K)\delta v^\lambda - \left(\delta^\mu_\lambda + v^\mu\tau_\lambda\right)\tilde{\nabla}_\mu\delta v^\lambda + \frac{1}{2}h_{\rho\lambda}v^\sigma\tilde{\nabla}_\sigma\delta h^{\rho\lambda}, && (5.15d)\\
\delta\left(K^{\mu\nu}K_{\mu\nu}\right) &= -\left(2\,\tau_\lambda K^{\mu\nu}K_{\mu\nu} + K^\nu{}_\lambda v^\sigma\tau_{\sigma\nu}\right)\delta v^\lambda - 2\,K^\mu{}_\lambda\tilde{\nabla}_\mu\delta v^\lambda + K_{\rho\lambda}v^\sigma\tilde{\nabla}_\sigma\delta h^{\rho\lambda}, && (5.15e)\\
\delta e &= e\tau_\lambda\delta v^\lambda - e\frac{1}{2}h_{\rho\lambda}\delta h^{\rho\lambda}. && (5.15f)
\end{aligned}$$

Here we have made the choice of having all the variations being defined in terms of δv^μ and $\delta h^{\mu\nu}$. The derivation of the identities can be found in Appendix C.1. For covariant derivatives of the variations, we use (4.32) along with the vanishing of boundary terms to show

$$\begin{aligned}
\int_M eX\tilde{\nabla}_\mu\delta x^\mu\,\mathrm{d}^{d+1}x &= \int_M e\left[\tilde{\nabla}_\mu(X\delta x^\mu) - \tilde{\nabla}_\mu X\delta x^\mu\right]\mathrm{d}^{d+1}x\\
&= \int_M \left[\underbrace{\partial_\mu(eX\delta x^\mu)}_{=0} - e\tau_\mu X\delta x^\mu - e\tilde{\nabla}_\mu X\delta x^\mu\right]\mathrm{d}^{d+1}x.
\end{aligned} \tag{5.16}$$

This will also hold if one replaces X with an object of arbitrary tensor indices that then contract with the index of the varied variable and the index of the derivative, such as $\partial_\mu\left(X^{\mu\rho}Y_{\rho\nu}\delta x^\nu\right)$, as well as for multiple indices in the varied variable.

Now that we have found the variational identities (5.15d) - (5.15f), we can return to the LO action (5.14) and vary it:

$$\delta\overset{(2)}{S}_{\mathrm{LO}} = \frac{1}{16\pi G}\int_M \left[\delta\left(K^{\mu\nu}K_{\mu\nu} - K^2\right)e + \left(K^{\mu\nu}K_{\mu\nu} - K^2\right)\delta e\right]\mathrm{d}^{d+1}x$$

$$
\begin{aligned}
&= \frac{1}{16\pi G}\int_M \Big[e\big(\delta\big(K^{\mu\nu}K_{\mu\nu}\big) - 2\,K\delta K\big) + \Big(K^{\mu\nu}K_{\mu\nu} - K^2\Big)\delta e\Big]\,\mathrm{d}^{d+1}x \\
&= \frac{1}{16\pi G}\int_M \Bigg[e\Bigg(-2\Big(K^{\mu\nu}K_{\mu\nu}\tau_\lambda + K^\mu{}_\lambda v^\sigma\tau_{\sigma\mu}\Big)\delta v^\lambda - 2\,K^\mu{}_\lambda\tilde{\nabla}_\mu\delta v^\lambda + K_{\rho\lambda}v^\sigma\tilde{\nabla}_\sigma\delta h^{\rho\lambda} \\
&\qquad\qquad - 2\,K\Big(-(v^\sigma\tau_{\sigma\lambda} + \tau_\lambda K)\delta v^\lambda - h^{\mu\sigma}h_{\sigma\lambda}\tilde{\nabla}_\mu\delta v^\lambda + \frac{1}{2}h_{\rho\lambda}v^\sigma\tilde{\nabla}_\sigma\delta h^{\rho\lambda}\Big)\Bigg) \\
&\qquad\quad + \Big(K^{\mu\nu}K_{\mu\nu} - K^2\Big)e\Big(\tau_\lambda\delta v^\lambda - \frac{1}{2}h_{\rho\lambda}\delta h^{\rho\lambda}\Big)\Bigg]\,\mathrm{d}^{d+1}x \\
&= \frac{1}{16\pi G}\int_M e\Bigg[-2\Big(-\frac{1}{2}\Big(K^{\mu\nu}K_{\mu\nu} - K^2\Big)\tau_\lambda + v^\sigma\tau_{\sigma\mu}\Big(K^\mu{}_\lambda - \delta^\mu_\lambda K\Big)\Big)\delta v^\lambda \\
&\qquad - \frac{1}{2}\Big(K^{\mu\nu}K_{\mu\nu} - K^2\Big)h_{\rho\lambda}\delta h^{\rho\lambda} \\
&\qquad + \big(K_{\rho\lambda} - Kh_{\rho\lambda}\big)\Big(-2\,h^{\mu\rho}\tilde{\nabla}_\mu\delta v^\lambda + v^\sigma\tilde{\nabla}_\sigma\delta h^{\rho\lambda}\Big)\Bigg]\,\mathrm{d}^{d+1}x. \qquad (5.17)
\end{aligned}
$$

Then the identity (4.32), more specifically the procedure in (5.16), allows us to rewrite the last term as

$$
\begin{aligned}
&\int_M e\Big[\big(K_{\rho\lambda} - Kh_{\rho\lambda}\big)\Big(-2\,h^{\sigma\rho}\tilde{\nabla}_\sigma\delta v^\lambda + v^\sigma\tilde{\nabla}_\sigma\delta h^{\rho\lambda}\Big)\Big]\,\mathrm{d}^{d+1}x \\
&\quad = \int_M e\Big[\tilde{\nabla}_\sigma\Big(\big(K_{\rho\lambda} - Kh_{\rho\lambda}\big)\Big(-2\,h^{\sigma\rho}\delta v^\lambda + v^\sigma\delta h^{\rho\lambda}\Big)\Big) \\
&\qquad + \tilde{\nabla}_\sigma\big(K_{\rho\lambda} - Kh_{\rho\lambda}\big)\Big(2\,h^{\sigma\rho}\delta v^\lambda - v^\sigma\delta h^{\mu\lambda}\Big) \\
&\qquad + 2\,\big(K_{\rho\lambda} - Kh_{\rho\lambda}\big)\tilde{\nabla}_\sigma h^{\sigma\rho}\delta v^\lambda\Big]\,\mathrm{d}^{d+1}x \\
&\quad = \int_M \Big[\partial_\sigma\Big(e\big(K_{\rho\lambda} - Kh_{\rho\lambda}\big)\Big(-2\,h^{\sigma\rho}\delta v^\lambda + v^\sigma\delta h^{\rho\lambda}\Big)\Big) \\
&\qquad - eK\tau_\sigma\big(K_{\rho\lambda} - Kh_{\rho\lambda}\big)\Big(-2\underbrace{h^{\sigma\rho}}_{=0}\delta v^\lambda + v^\sigma\delta h^{\rho\lambda}\Big) \\
&\qquad + e\tilde{\nabla}_\sigma\big(K_{\rho\lambda} - Kh_{\rho\lambda}\big)\Big(2\,h^{\sigma\rho}\delta v^\lambda - v^\sigma\delta h^{\mu\lambda}\Big) \\
&\qquad + 2\,\big(K_{\rho\lambda} - Kh_{\rho\lambda}\big)\tilde{\nabla}_\sigma h^{\sigma\rho}\delta v^\lambda\Big]\,\mathrm{d}^{d+1}x \\
&\quad \approx \int_M e\Big[K\big(K_{\rho\lambda} - Kh_{\rho\lambda}\big)\delta h^{\rho\lambda} + \tilde{\nabla}_\sigma\big(K_{\rho\lambda} - Kh_{\rho\lambda}\big)\Big(2\,h^{\sigma\rho}\delta v^\lambda - v^\sigma\delta h^{\mu\lambda}\Big) \\
&\qquad + 2\,\big(K_{\rho\lambda} - Kh_{\rho\lambda}\big)v^\sigma h^{\rho\mu}\tau_{\sigma\tau}\delta v^\lambda\Big]\,\mathrm{d}^{d+1}x, \qquad (5.18)
\end{aligned}
$$

and the variation of the action thus becomes:

$$
\begin{aligned}
\delta\overset{(2)}{S}_{\mathrm{LO}} = \frac{1}{16\pi G}\int_M e\Bigg[&-2\Big(-\frac{1}{2}\Big(K^{\mu\nu}K_{\mu\nu} - K^2\Big)\tau_\lambda + \cancel{v^\sigma\tau_{\sigma\mu}\Big(K^\mu{}_\lambda - \delta^\mu_\lambda K\Big)}\Big)\delta v^\lambda \\
&- \frac{1}{2}\Big(K^{\mu\nu}K_{\mu\nu} - K^2\Big)h_{\rho\lambda}\delta h^{\rho\lambda} + K\big(K_{\rho\lambda} - Kh_{\rho\lambda}\big)\delta h^{\rho\lambda}
\end{aligned}
$$

$$
\begin{aligned}
&+\tilde{\nabla}_\sigma\left(K_{\rho\lambda}-Kh_{\rho\lambda}\right)\left(2\,h^{\sigma\rho}\delta v^\lambda - v^\sigma\delta h^{\mu\lambda}\right)\\
&+\cancel{\left(K_{\rho\lambda}-Kh_{\rho\lambda}\right)v^\sigma h^{\rho\mu}\tau_{\sigma\mu}\delta v^\lambda}\Bigg]\,\mathrm{d}^{d+1}x
\end{aligned}
$$

$$
\begin{aligned}
=\frac{1}{16\pi G}\int_M e\Bigg[&2\Big(-\frac{1}{2}\Big(K^{\mu\nu}K_{\mu\nu}-K^2\Big)\tau_\lambda + h^{\sigma\rho}\tilde{\nabla}_\sigma\left(K_{\rho\lambda}-Kh_{\rho\lambda}\right)\Big)\delta v^\lambda\\
&+\Big(-\frac{1}{2}\Big(K^{\mu\nu}K_{\mu\nu}-K^2\Big)h_{\rho\lambda} - v^\sigma\tilde{\nabla}_\sigma\left(K_{\rho\lambda}-Kh_{\rho\lambda}\right)\\
&\quad+K\left(K_{\rho\lambda}-Kh_{\rho\lambda}\right)\Big)\delta h^{\rho\lambda}\Bigg]\,\mathrm{d}^{d+1}x. \qquad (5.19)
\end{aligned}
$$

We have now finally arrived at the LO equations of motion

$$\overset{(2)}{G}{}^{v}_{\mu} = -\frac{1}{2}\Big(K^{\rho\sigma}K_{\rho\sigma}-K^2\Big)\tau_\mu + h^{\nu\rho}\tilde{\nabla}_\rho\left(K_{\mu\nu}-Kh_{\mu\nu}\right), \qquad (5.20a)$$

$$\overset{(2)}{G}{}^{h}_{\mu\nu} = -\frac{1}{2}\Big(K^{\rho\sigma}K_{\rho\sigma}-K^2\Big)h_{\mu\nu} + K\left(K_{\mu\nu}-Kh_{\mu\nu}\right) - v^\rho\tilde{\nabla}_\rho\left(K_{\mu\nu}-Kh_{\mu\nu}\right), \qquad (5.20b)$$

for the variation of the LO action:

$$\delta\overset{(2)}{S}_{\mathrm{LO}} = \frac{1}{8\pi G}\int_M e\left[\overset{(2)}{G}{}^{v}_{\mu}\delta v^\mu + \frac{1}{2}\overset{(2)}{G}{}^{h}_{\mu\nu}\delta h^{\mu\nu}\right]\mathrm{d}^{d+1}x. \qquad (5.21)$$

Furthermore, on comparison with the general form of the variation

$$\delta\overset{(2)}{S}_{\mathrm{LO}} = \frac{1}{8\pi G}\int_M e\left[\frac{\delta\overset{(2)}{\mathcal{L}}_{\mathrm{LO}}}{\delta v^\mu}\delta v^\mu + \frac{\delta\overset{(2)}{\mathcal{L}}_{\mathrm{LO}}}{\delta h^{\mu\nu}}\delta h^{\mu\nu}\right]\mathrm{d}^{d+1}x, \qquad (5.22)$$

we find that

$$\overset{(2)}{G}{}^{v}_{\mu} = \frac{\delta\overset{(2)}{\mathcal{L}}_{\mathrm{LO}}}{\delta v^\mu}, \qquad \frac{1}{2}\overset{(2)}{G}{}^{h}_{\mu\nu} = \frac{\delta\overset{(2)}{\mathcal{L}}_{\mathrm{LO}}}{\delta h^{\mu\nu}}. \qquad (5.23)$$

Comparing these results to non-relativistic (or Galilean) gravity, where the LO EOMs only serve as constraints on the theory by foiliating the spacetime into time slices [78], we see that here we already have interesting dynamics at the leading-order. Similar dynamics first appears at NLO and NNLO in NR gravity, thus showing that Carrollian gravity is considerably more difficult to calculate to higher orders.

5.3. Next-to-leading-order theory

What we consider here to be the next-to-leading-order hinges on which expansion parameter one chooses, c versus c^2. Following an equivalent treatment for non-relativistic gravity from [64], we begin by showing a few characteristics of Lagrangians consisting of fields that are expanded in orders of c^N as such:

$$\phi^I = \phi^I_{(0)} + c^N\phi^I_{(1)} + c^{2N}\phi^I_{(2)} + \mathcal{O}(c^{3N}), \qquad (5.24a)$$

$$\partial_\mu\phi^I = \partial_\mu\phi^I_{(0)} + c^N\partial_\mu\phi^I_{(1)} + c^{2N}\partial_\mu\phi^I_{(2)} + \mathcal{O}(c^{3N}), \qquad (5.24b)$$

which then allows us to specify later whether $N = 1$ or $N = 2$. As for the action (4.50) we define for the Lagrangian $\mathcal{L} = c^\Delta \tilde{\mathcal{L}}$, with c^Δ the pre-leading orders. It is evident from the action (4.50) that we have $\Delta = 2$ and thus:

$$\mathcal{L} = c^2\left(\overset{(2)}{\mathcal{L}}_{\text{LO}} + c^N \overset{(2+N)}{\mathcal{L}}_{\text{NLO}} + O(c^{2N})\right). \tag{5.25}$$

The Lagrangian is in general a function of the field ϕ^I and its derivative $\partial_\mu\phi^I$, all of which can be considered to be a function of c^N. Thus, we define a total derivative

$$\frac{\mathrm{d}}{\mathrm{d}c^N} = \frac{\partial}{\partial c^N} + \frac{\partial\phi^I}{\partial c^N}\frac{\partial}{\partial\phi^I} + \frac{\partial(\partial_\mu\phi^I)}{\partial c^N}\frac{\partial}{\partial(\partial_\mu\phi^I)}, \tag{5.26}$$

which is used to define the Taylor expansion:

$$\tilde{\mathcal{L}}(c^N) = \tilde{\mathcal{L}}(0) + c^N\tilde{\mathcal{L}}'(0) + \frac{c^{2N}}{2}\tilde{\mathcal{L}}''(0) + O(c^{3N}). \tag{5.27}$$

This in turn shows that:

$$\overset{(2)}{\mathcal{L}}_{\text{LO}} = \tilde{\mathcal{L}}(0) = \tilde{\mathcal{L}}(\phi^I_{(0)}, \partial_\mu\phi^I_{(0)}; c^N = 0) \tag{5.28}$$

$$\begin{aligned}
\overset{(2+N)}{\mathcal{L}}_{\text{NLO}} = \tilde{\mathcal{L}}'(0) &= \left(\frac{\partial\tilde{\mathcal{L}}}{\partial c^N} + \frac{\partial\phi^I}{\partial c^N}\frac{\partial\tilde{\mathcal{L}}}{\partial\phi^I} + \frac{\partial(\partial_\mu\phi^I)}{\partial c^N}\frac{\partial\tilde{\mathcal{L}}}{\partial(\partial_\mu\phi^I)}\right)\Bigg|_{c^N=0} \\
&= \frac{\partial\tilde{\mathcal{L}}}{\partial c^N}\bigg|_{c^N=0} + \frac{\partial\phi^I}{\partial c^N}\frac{\partial\tilde{\mathcal{L}}}{\partial\phi^I}\bigg|_{c^N=0} + \frac{\partial(\partial_\mu\phi^I)}{\partial c^N}\frac{\partial\tilde{\mathcal{L}}}{\partial(\partial_\mu\phi^I)}\bigg|_{c^N=0} \\
&= \frac{\partial\tilde{\mathcal{L}}}{\partial c^N}\bigg|_{c^N=0} + \phi^I_{(1)}\frac{\partial\overset{(2)}{\mathcal{L}}_{\text{LO}}}{\partial\phi^I_{(0)}} + \partial_\mu\phi^I_{(1)}\frac{\partial\overset{(2)}{\mathcal{L}}_{\text{LO}}}{\partial(\partial_\mu\phi^I_{(0)})} \\
&= \frac{\partial\tilde{\mathcal{L}}}{\partial c^N}\bigg|_{c^N=0} + \phi^I_{(1)}\frac{\partial\overset{(2)}{\mathcal{L}}_{\text{LO}}}{\partial\phi^I_{(0)}} + \underbrace{\partial_\mu\left(\phi^I_{(1)}\frac{\partial\overset{(2)}{\mathcal{L}}_{\text{LO}}}{\partial(\partial_\mu\phi^I_{(0)})}\bigg|_{\delta M}\right)}_{=0} \\
&\quad - \phi^I_{(1)}\partial_\mu\left(\frac{\partial\overset{(2)}{\mathcal{L}}_{\text{LO}}}{\partial(\partial_\mu\phi^I_{(0)})}\right) \\
&= \frac{\partial\tilde{\mathcal{L}}}{\partial c^N}\bigg|_{c^N=0} + \phi^I_{(1)}\frac{\delta\overset{(2)}{\mathcal{L}}_{\text{LO}}}{\delta\phi^I_{(0)}},
\end{aligned} \tag{5.29}$$

where $\frac{\delta}{\delta\phi^I_{(0)}}$ is the Euler-Lagrange derivative and terms two and three in the third line are found in the following way:

$$\begin{aligned}
\frac{\partial\phi^I}{\partial c^N}\frac{\partial\tilde{\mathcal{L}}}{\partial\phi^I}\bigg|_{c^N=0} &= \left[\left(\phi^I_{(1)} + 2\,c^N\phi^I_{(2)} + O(c^{2N})\right)\frac{\partial\tilde{\mathcal{L}}}{\partial\left(\phi^I_{(0)} + c^N\phi^I_{(1)} + O(c^{2N})\right)}\right]_{c^N=0} \\
&= \phi^I_{(1)}\frac{\partial\tilde{\mathcal{L}}(0)}{\partial\phi^I_{(0)}} = \phi^I_{(1)}\frac{\partial\overset{(2)}{\mathcal{L}}_{\text{LO}}}{\partial\phi^I_{(0)}}.
\end{aligned} \tag{5.30}$$

For the Einstein-Hilbert action we have chosen the defining fields to be V^μ and $\Pi^{\mu\nu}$ with their Carroll expansion given as

$$V^\mu = v^\mu + c^N v^\mu_{(1)} + O(c^{2N}), \tag{5.31a}$$
$$\Pi^{\mu\nu} = h^{\mu\nu} + c^N h^{\mu\nu}_{(1)} + O(c^{2N}). \tag{5.31b}$$

Thus, in order to obtain the NLO Lagrangian, we simply use (5.29):

$$\begin{aligned}\overset{(2+N)}{\mathcal{L}}_{\text{NLO}} &= \frac{\partial \tilde{\mathcal{L}}}{\partial c^N}\bigg|_{c^N=0} + v^\mu_{(1)} \frac{\delta \overset{(2)}{\mathcal{L}}_{\text{LO}}}{\delta v^\mu} + h^{\mu\nu}_{(1)} \frac{\delta \overset{(2)}{\mathcal{L}}_{\text{LO}}}{\delta h^{\mu\nu}} \\ &= \frac{\partial \tilde{\mathcal{L}}}{\partial c^N}\bigg|_{c^N=0} + v^\mu_{(1)} \overset{(2)}{G}{}^v_\mu + \frac{1}{2} h^{\mu\nu}_{(1)} \overset{(2)}{G}{}^h_{\mu\nu},\end{aligned} \tag{5.32}$$

where in the last line, (5.23) has been inserted. It is evident that the LO equations of motion are encoded in the NLO equations of motion. This is repeated at every level in the expansion and thus every higher order action will contain all the dynamics of the lower order. The variation of the NLO action will now become:

$$\begin{aligned}\delta \overset{(2+N)}{S}_{\text{NLO}} = \frac{1}{8\pi G}\int_M e \Bigg[\delta\left(\frac{\partial \tilde{\mathcal{L}}}{\partial c^N}\bigg|_{c^N=0}\right) + \delta v^\mu_{(1)} \overset{(2)}{G}{}^v_\mu + \frac{1}{2}\delta h^{\mu\nu}_{(1)} \overset{(2)}{G}{}^h_{\mu\nu} \\ + v^\mu_{(1)} \delta \overset{(2)}{G}{}^v_\mu + \frac{1}{2} h^{\mu\nu}_{(1)} \delta \overset{(2)}{G}{}^h_{\mu\nu} + \overset{(2+N)}{\mathcal{L}}_{\text{NLO}} \frac{1}{e}\delta e \Bigg] \mathrm{d}^{d+1}.\end{aligned} \tag{5.33}$$

We will now assemble all the ingredients to be able to derive the variation of the NLO action for both the c and c^2 expansion. Starting with the first term of (5.33) we begin by noting that from (4.50) it is clear that the stripped Lagrangian $\tilde{\mathcal{L}}$ only contains even powers of c. Thus when considering the first term of (5.32) and setting $N = 1$, the term vanishes. However, if we take $N = 2$, we are left with

$$\frac{\partial \tilde{\mathcal{L}}}{\partial c^2}\bigg|_{c^2=0} = \frac{1}{2} h^{\mu\nu} \tilde{R}_{\mu\nu}, \tag{5.34}$$

where we have defined $\overset{(\tilde{C})}{R}_{\mu\nu}\big|_{c^2=0} = \tilde{R}_{\mu\nu}$.

In order to calculate the variation of (5.34) we first show:

$$\tilde{R}_{\mu\nu} = 2\,\partial_{[\rho}\tilde{\Gamma}^\rho_{\mu]\nu} + 2\,\tilde{\Gamma}^\rho_{[\rho|\sigma|}\tilde{\Gamma}^\sigma_{\mu]\nu}, \tag{5.35a}$$
$$\delta\tilde{R}_{\mu\nu} = 2\Big(\partial_{[\rho}\delta\tilde{\Gamma}^\rho_{\mu]\nu} + \delta\tilde{\Gamma}^\rho_{[\rho|\sigma|}\tilde{\Gamma}^\sigma_{\mu]\nu} + \tilde{\Gamma}^\rho_{[\rho|\sigma|}\delta\tilde{\Gamma}^\sigma_{\mu]\nu}\Big), \tag{5.35b}$$

which when used with

$$\begin{aligned}2\,\tilde{\nabla}_{[\rho}\delta\tilde{\Gamma}^\rho_{\mu]\nu} &= 2\Big(\partial_{[\rho}\delta\tilde{\Gamma}^\rho_{\mu]\nu} + \tilde{\Gamma}^\rho_{[\rho|\sigma|}\delta\tilde{\Gamma}^\sigma_{\mu]\nu} - \tilde{\Gamma}^\sigma_{[\rho\mu]}\delta\tilde{\Gamma}^\rho_{\sigma\nu} - \tilde{\Gamma}^\sigma_{[\rho|\nu|}\delta\tilde{\Gamma}^\rho_{\mu]\sigma}\Big) \\ \Rightarrow \quad \delta\tilde{R}_{\mu\nu} &= 2\Big(\tilde{\nabla}_{[\rho}\delta\tilde{\Gamma}^\rho_{\mu]\nu} - \tilde{\Gamma}^\sigma_{[\mu\rho]}\delta\tilde{\Gamma}^\rho_{\sigma\nu}\Big),\end{aligned} \tag{5.36}$$

allows us to write:

$$\delta\big(h^{\mu\nu}\tilde{R}_{\mu\nu}\big) = \delta h^{\mu\nu}\tilde{R}_{\mu\nu} + h^{\mu\nu}\delta\tilde{R}_{\mu\nu}$$

$$
\begin{aligned}
&= \delta h^{\mu\nu}\tilde{R}_{\mu\nu} + 2\,h^{\mu\nu}\Big(\tilde{\nabla}_{[\rho}\delta\tilde{\Gamma}^{\rho}_{\mu]\nu} - \tilde{\Gamma}^{\sigma}_{[\mu\rho]}\delta\tilde{\Gamma}^{\rho}_{\sigma\nu}\Big) \\
&= \delta h^{\mu\nu}\tilde{R}_{\mu\nu} + \tilde{\nabla}_{\rho}\Big(h^{\mu\nu}\delta\tilde{\Gamma}^{\rho}_{\mu\nu}\Big) - \tilde{\nabla}_{\rho}h^{\mu\nu}\delta\tilde{\Gamma}^{\rho}_{\mu\nu} \\
&\quad - \tilde{\nabla}_{\mu}\Big(h^{\mu\nu}\delta\tilde{\Gamma}^{\rho}_{\rho\nu}\Big) + \tilde{\nabla}_{\mu}h^{\mu\nu}\delta\tilde{\Gamma}^{\rho}_{\rho\nu} - 2\,h^{\mu\nu}h^{\sigma\gamma}\tau_{[\mu}K_{\rho]\gamma}\delta\tilde{\Gamma}^{\rho}_{\sigma\nu}.
\end{aligned} \tag{5.37}
$$

Two of the terms with the covariant derivatives are then rewritten using (4.32)

$$
\int_M e\Big[\tilde{\nabla}_{\rho}\Big(h^{\mu\nu}\delta\tilde{\Gamma}^{\rho}_{\mu\nu}\Big)\Big]\,\mathrm{d}^{d+1}x = \int_M \Big[\underbrace{\partial_{\rho}\Big(eh^{\mu\nu}\delta\tilde{\Gamma}^{\rho}_{\mu\nu}\Big)}_{=0} - e\tau_{\rho}Kh^{\mu\nu}\delta\tilde{\Gamma}^{\rho}_{\mu\nu}\Big]\,\mathrm{d}^{d+1}x, \tag{5.38a}
$$

$$
-\int_M e\Big[\tilde{\nabla}_{\mu}\Big(h^{\mu\nu}\delta\tilde{\Gamma}^{\rho}_{\rho\nu}\Big)\Big]\,\mathrm{d}^{d+1}x = -\int_M \Big[\underbrace{\partial_{\mu}\Big(eh^{\mu\nu}\delta\tilde{\Gamma}^{\rho}_{\rho\nu}\Big)}_{=0} - eK\delta\tilde{\Gamma}^{\rho}_{\rho\nu}\underbrace{\tau_{\mu}h^{\mu\nu}}_{=0}\Big]\,\mathrm{d}^{d+1}x \tag{5.38b}
$$

which gives:

$$
\begin{aligned}
\delta\left(\frac{\partial\tilde{\mathcal{L}}}{\partial c^2}\bigg|_{c^2=0}\right) &= \frac{1}{2}\Big(\delta h^{\mu\nu}\tilde{R}_{\mu\nu} + \tau_{\rho}\delta\tilde{\Gamma}^{\rho}_{\mu\nu}(K^{\mu\nu} - Kh^{\mu\nu}) \\
&\qquad - v^{\mu}h^{\nu\gamma}\tau_{\lambda\gamma}\Big(\delta^{\lambda}_{\rho} - \tau_{\rho}v^{\lambda}\Big)\delta\tilde{\Gamma}^{\rho}_{(\mu\nu)} + v^{\lambda}h^{\nu\gamma}\tau_{\lambda\gamma}\delta\tilde{\Gamma}^{\rho}_{\rho\nu}\Big),
\end{aligned} \tag{5.39}
$$

where we have used (4.31b) and (4.31c). Here we encounter terms with $\delta\tilde{\Gamma}$ that require lengthy calculations. Looking at (5.20a) and (5.20b) it is clear that we will be varying covariant derivatives which also lead to similar terms. Thus, we refer to Appendix C.2 for detailed derivations of such terms. This results in (5.39) becoming

$$
\begin{aligned}
\delta\left(\frac{\partial\tilde{\mathcal{L}}}{\partial c^2}\bigg|_{c^2=0}\right) &= \frac{1}{2}\bigg[\frac{1}{2}v^{\eta}\tau_{\eta\gamma}h^{\nu\gamma}\Big(\delta^{\rho}_{\lambda} + v^{\rho}\tau_{\lambda}\Big)\tau_{\nu\rho}\delta v^{\lambda} + h^{\nu\gamma}\tau_{\rho\gamma}\tilde{\nabla}_{\nu}\delta v^{\rho} \\
&\qquad + \bigg(\tilde{R}_{\rho\lambda} + 2\,Kv^{\gamma}\tau_{\rho}\tau_{\gamma\lambda} + \Big(\frac{1}{2}\delta^{\eta}_{\rho} - v^{\eta}\tau_{\rho}\Big)\tau_{\eta\gamma}K^{\gamma}{}_{\lambda}\bigg)\delta h^{\rho\lambda} \\
&\qquad + \bigg(K\Big(\delta^{\mu}_{\rho} + v^{\mu}\tau_{\rho}\Big)\tau_{\lambda} - K^{\mu}{}_{\rho}\tau_{\lambda} - \frac{1}{2}v^{\eta}\tau_{\eta\gamma}h^{\mu\gamma}h_{\rho\lambda}\bigg)\tilde{\nabla}_{\mu}\delta h^{\rho\lambda}\bigg] \\
&\approx \frac{1}{2}\bigg[\bigg(\frac{1}{2}v^{\lambda}\tau_{\lambda\gamma}h^{\nu\gamma}\Big(\delta^{\rho}_{\mu} + v^{\rho}\tau_{\mu}\Big)\tau_{\nu\rho} - \tilde{\nabla}_{\nu}\big(h^{\nu\gamma}\tau_{\mu\gamma}\big)\bigg)\delta v^{\mu} \\
&\qquad + \bigg(\tilde{R}_{\mu\nu} + 2\,Kv^{\gamma}\tau_{\mu}\tau_{\gamma\nu} + \Big(\frac{1}{2}\delta^{\eta}_{\mu} - v^{\eta}\tau_{\mu}\Big)\tau_{\eta\gamma}K^{\gamma}{}_{\nu} \\
&\qquad - \tilde{\nabla}_{\rho}\bigg(K\Big(\delta^{\rho}_{\mu} + v^{\rho}\tau_{\mu}\Big)\tau_{\nu} - K^{\rho}{}_{\mu}\tau_{\nu} - \frac{1}{2}v^{\eta}\tau_{\eta\gamma}h^{\rho\gamma}h_{\mu\nu}\bigg)\bigg)\delta h^{\mu\nu}\bigg].
\end{aligned} \tag{5.40}
$$

From Appendix C.2 we also find the fourth term from (5.33):

$$
\begin{aligned}
v^{\mu}_{(1)}\delta\overset{(2)}{G}{}^{v}_{\mu} \approx v^{\mu}_{(1)}\bigg[\bigg(&2\,\tau_{\mu}K^{\rho\gamma}K_{\rho\gamma}\tau_{\lambda} + K^{\sigma}{}_{\alpha}\Big(\frac{1}{2}\delta^{\alpha}_{\lambda}\tau_{\mu\sigma} + v^{\gamma}\tau_{\gamma\sigma}\Big(2\,\delta^{\alpha}_{\lambda}\tau_{\mu} + 3\,\delta^{\alpha}_{\mu}\tau_{\lambda}\Big)\Big) \\
&+ \frac{1}{2}K\big(\tau_{\lambda\mu} - \tau_{\mu}v^{\sigma}\tau_{\sigma\lambda} - v^{\gamma}\tau_{\lambda}\tau_{\gamma\mu} + K\tau_{\mu}\tau_{\lambda}\big) - 2\,\tilde{\nabla}_{\rho}K^{\rho}{}_{\lambda}\tau_{\mu}
\end{aligned}
$$

$$
\begin{aligned}
&+ v^\sigma \tilde{\nabla}_\rho \tau_{\sigma\gamma}\Big(-\frac{1}{2}\delta^\gamma_\lambda\Big(\delta^\rho_\mu - v^\rho\tau_\mu\Big) + \delta^\gamma_\mu\Big(\delta^\rho_\lambda + v^\rho\tau_\lambda\Big) - \frac{1}{2}h^{\rho\gamma}h_{\lambda\mu}\Big)\Big)\delta v^\lambda \\
&+ \Bigg(\Big(\frac{3}{2}K^{\gamma\sigma}K_{\gamma\sigma} - \frac{1}{2}K^2 + v^\sigma\tilde{\nabla}_\sigma K\Big)\tau_\lambda h_{\mu\rho} + \frac{1}{2}\Big(K^2 - v^\sigma\tilde{\nabla}_\sigma K\Big)\tau_\mu h_{\rho\lambda} \\
&\quad + \Big(2\,K^\gamma{}_\rho K_{\lambda\gamma} - \frac{3}{2}KK_{\rho\lambda} + v^\sigma\tilde{\nabla}_\sigma K_{\rho\lambda}\Big)\tau_\mu - \Big(K_{\mu\lambda}K + v^\sigma\tilde{\nabla}_\sigma K_{\mu\lambda}\Big)\tau_\rho \\
&\quad + \frac{1}{2}v^\sigma\tau_{\sigma\gamma}K^\gamma{}_\lambda h_{\mu\rho} - \frac{1}{2}v^\sigma\tau_{\sigma\rho}K_{\lambda\mu} + v^\sigma\tau_{\sigma\mu}\Big(K_{\rho\lambda} - \frac{1}{2}h_{\rho\lambda}K\Big) \\
&\quad + 2\,K^\gamma{}_\rho K_{\mu\gamma}\tau_\lambda + \frac{1}{2}\tilde{\nabla}_\sigma K^\sigma{}_\mu h_{\rho\lambda} - \frac{1}{2}\tilde{\nabla}_\mu K_{\rho\lambda}\Bigg)\delta h^{\rho\lambda}\Bigg] \\
&+ \tilde{\nabla}_\sigma v^\mu_{(1)}\Bigg[\Bigg(K^\sigma{}_\gamma\Big(\delta^\gamma_\mu\tau_\lambda - \frac{5}{2}\delta^\gamma_\lambda\tau_\mu\Big) + K\Big(\delta^\sigma_\lambda\tau_\mu - \delta^\sigma_\mu\tau_\lambda\Big) + \frac{3}{4}v^\sigma\tau_{\gamma\mu}(\delta^\gamma_\lambda + v^\gamma\tau_\lambda) \\
&\quad + v^\rho\tau_{\rho\gamma}\Big(-\frac{5}{2}\delta^\sigma_\mu\delta^\gamma_\lambda - \frac{9}{4}\delta^\gamma_\lambda v^\sigma\tau_\mu + \delta^\sigma_\lambda\delta^\gamma_\mu - \frac{1}{2}h^{\sigma\gamma}h_{\lambda\mu}\Big)\Bigg)\delta v^\lambda \\
&\quad + \Bigg(\frac{1}{2}K^\sigma{}_\gamma\Big(\delta^\gamma_\lambda h_{\mu\rho} + \delta^\gamma_\mu h_{\rho\lambda}\Big) - \frac{1}{2}v^\sigma\tau_\mu K h_{\rho\lambda} \\
&\qquad - \frac{1}{2}K_{\lambda\gamma}\Big(\delta^\gamma_\mu\Big(\delta^\sigma_\rho + v^\sigma\tau_\rho\Big) + \delta^\gamma_\rho\Big(\delta^\sigma_\mu - 2\,v^\sigma\tau_\mu\Big)\Big)\Bigg)\delta h^{\rho\lambda}\Bigg] \\
&+ \tilde{\nabla}_\rho\tilde{\nabla}_\sigma v^\mu_{(1)}\Bigg[\frac{1}{2}\Big(\delta^\sigma_\lambda\delta^\rho_\mu + \delta^\sigma_\lambda v^\rho\tau_\mu + v^\sigma\tau_\lambda\delta^\rho_\mu + v^\sigma\tau_\lambda v^\rho\tau_\mu\Big) - \frac{1}{2}h^{\sigma\rho}h_{\lambda\mu}\Bigg]\delta v^\lambda, \qquad (5.41)
\end{aligned}
$$

and the fifth term of (5.33) to be

$$
\begin{aligned}
\frac{1}{2}h^{\mu\nu}_{(1)}\delta\overset{(2)}{G}{}^h_{\mu\nu} \approx \frac{1}{2}h^{\mu\nu}_{(1)}\Bigg[&\Bigg(K^{\sigma\gamma}K_{\sigma\gamma}(h_{\mu\nu}\tau_\lambda - h_{\lambda\mu}\tau_\nu) - K(K_{\mu\nu}\tau_\lambda + 2\,\tau_\mu K_{\lambda\nu}) \\
&\quad + K^\rho{}_\gamma\Big(\delta^\gamma_\lambda\tau_\mu K_{\nu\rho} + v^\sigma\tau_{\sigma\rho}\Big(\frac{1}{2}\delta^\gamma_\lambda h_{\mu\nu} + \delta^\gamma_\nu h_{\mu\lambda}\Big)\Big) \\
&\quad - K^2(h_{\mu\nu}\tau_\lambda + h_{\lambda\mu}\tau_\nu) - \tilde{\nabla}_\sigma K^\sigma{}_\gamma(\delta^\gamma_\lambda h_{\mu\nu} + \delta^\gamma_\nu h_{\mu\lambda}) \\
&\quad + \tilde{\nabla}_\sigma K_{\lambda\nu}\Big(\delta^\gamma_\mu v^\sigma\tau_\lambda - \Big(3\,\delta^\sigma_\mu + v^\sigma\tau_\mu\Big)\Big) \\
&\quad + \tilde{\nabla}_\sigma K\Big(2\,\delta^\sigma_\lambda h_{\mu\nu} + 2\,v^\sigma h_{\lambda\mu}\tau_\nu + v^\sigma h_{\mu\nu}\tau_\lambda\Big)\Bigg)\delta v^\lambda \\
&+ \Bigg(\frac{1}{2}\Big(K^{\sigma\gamma}K_{\sigma\gamma} + K^2 - 2\,v^\sigma\tilde{\nabla}_\sigma K\Big)h_{\mu\rho}h_{\nu\lambda} + \frac{1}{2}\Big(K^2 - v^\sigma\tilde{\nabla}_\sigma K\Big)h_{\mu\nu}h_{\rho\lambda} \\
&\quad + \Big(K^\sigma{}_\lambda K_{\sigma\nu} - KK_{\lambda\nu} + v^\sigma\tilde{\nabla}_\sigma K_{\lambda\nu}\Big)h_{\mu\rho} \\
&\quad + \frac{1}{2}\Big(v^\sigma\tilde{\nabla}_\sigma K_{\rho\lambda} - KK_{\rho\lambda}\Big)h_{\mu\nu}
\end{aligned}
$$

$$
\begin{aligned}
&\qquad - K_{\lambda\mu}K_{\nu\rho} + \frac{1}{2}\Big(KK_{\mu\nu} - v^\sigma\tilde{\nabla}_\sigma K_{\mu\nu}\Big)h_{\rho\lambda}\Big)\delta h^{\rho\lambda}\Bigg] \\
&+ \frac{1}{2}\tilde{\nabla}_\sigma h^{\mu\nu}_{(1)}\Bigg[\Bigg(K\big(\delta^\sigma_\lambda + 2\,v^\sigma\tau_\lambda\big)h_{\mu\nu} + K_{\gamma\nu}\Big(\delta^\gamma_\mu\delta^\sigma_\lambda - 3\,\delta^\gamma_\lambda\delta^\sigma_\mu\Big) \\
&\qquad - K^\sigma{}_\gamma\big(\delta^\gamma_\lambda h_{\mu\nu} + \delta^\gamma_\nu h_{\mu\lambda}\big)\Bigg)\delta v^\lambda \\
&\qquad + \frac{1}{2}v^\sigma\Bigg(K\big(h_{\mu\rho}h_{\nu\lambda} - 2\,h_{\mu\nu}h_{\rho\lambda}\big) + h_{\mu\nu}K_{\rho\lambda} - K_{\mu\nu}h_{\rho\lambda}\Bigg)\delta h^{\rho\lambda}\Bigg] \\
&+ \frac{1}{2}v^\gamma\tilde{\nabla}_\gamma\tilde{\nabla}_\sigma h^{\mu\nu}_{(1)}\Bigg[\Bigg(\delta^\sigma_\mu h_{\lambda\nu} - h_{\mu\nu}\big(\delta^\sigma_\lambda + v^\sigma\tau_\lambda\big) + v^\sigma h_{\lambda\mu}\tau_\nu\Bigg)\delta v^\lambda \\
&\qquad + \frac{1}{2}v^\sigma\Bigg(h_{\mu\rho}h_{\nu\lambda} + h_{\mu\nu}h_{\rho\lambda}\Bigg)\delta h^{\rho\lambda}\Bigg].
\end{aligned}
\tag{5.42}
$$

Finally, the last term of (5.33) is

$$
\begin{aligned}
\overset{(2+N)}{\mathcal{L}}_{\mathrm{NLO}}\frac{1}{e}\delta e &= \left(\frac{\partial\tilde{\mathcal{L}}}{\partial c^N}\bigg|_{c^N=0} + v^\mu_{(1)}\overset{(2)}{G}{}^v_\mu + \frac{1}{2}h^{\mu\nu}_{(1)}\overset{(2)}{G}{}^h_{\mu\nu}\right)\left(\tau_\rho\delta v^\rho - \frac{1}{2}h_{\rho\lambda}\delta h^{\rho\lambda}\right) \\
&= \Bigg[\frac{\partial\tilde{\mathcal{L}}}{\partial c^N}\bigg|_{c^N=0} + v^\mu_{(1)}\Bigg(-\frac{1}{2}\Big(K^{\gamma\eta}K_{\gamma\eta} - K^2\Big)\tau_\mu + h^{\nu\gamma}\tilde{\nabla}_\gamma\big(K_{\mu\nu} - Kh_{\mu\nu}\big)\Bigg) \\
&\quad + \frac{1}{2}h^{\mu\nu}_{(1)}\Bigg(-\frac{1}{2}\Big(K^{\gamma\eta}K_{\gamma\eta} - K^2\Big)h_{\mu\nu} + K\big(K_{\mu\nu} - Kh_{\mu\nu}\big) \\
&\qquad - v^\gamma\tilde{\nabla}_\gamma\big(K_{\mu\nu} - Kh_{\mu\nu}\big)\Bigg)\Bigg]\left(\tau_\lambda\delta v^\lambda - \frac{1}{2}h_{\rho\lambda}\delta h^{\rho\lambda}\right).
\end{aligned}
\tag{5.43}
$$

Having assembled all the ingredients, we are now finally ready to present the NLO equations for both expansions.

5.3.1. NLO all powers of c

For the c expansion, i.e. $N = 1$, we have the varied action

$$
\delta\overset{(3)}{S}_{\mathrm{NLO}} = \frac{1}{8\pi G}\int_M e\left[\overset{(3)}{G}{}^v_\mu\delta v^\mu + \frac{1}{2}\overset{(3)}{G}{}^h_{\mu\nu}\delta h^{\mu\nu} + \overset{(3)}{G}{}^{v_{(1)}}_\mu\delta v^\mu_{(1)} + \frac{1}{2}\overset{(3)}{G}{}^{h_{(1)}}_{\mu\nu}\delta h^{\mu\nu}_{(1)}\right]\mathrm{d}^{d+1}. \tag{5.44}
$$

The equations of motion are thus

$$
\overset{(3)}{G}{}^{v_{(1)}}_\mu = \overset{(2)}{G}{}^v_\mu, \qquad \overset{(3)}{G}{}^{h_{(1)}}_{\mu\nu} = \overset{(2)}{G}{}^h_{\mu\nu}, \tag{5.45}
$$

repeating the EOMs from the LO just as expected, along with:

$$
\overset{(3)}{G}{}^v_\lambda = \Bigg[v^\mu_{(1)}\Bigg(2\,\tau_\mu K^{\rho\gamma}K_{\rho\gamma}\tau_\lambda + K^\sigma{}_\alpha\left(\frac{1}{2}\delta^\alpha_\lambda\tau_{\mu\sigma} + v^\gamma\tau_{\gamma\sigma}\Big(2\,\delta^\alpha_\lambda\tau_\mu + 3\,\delta^\alpha_\mu\tau_\lambda\Big)\right)
$$

$$
\begin{aligned}
&+ \frac{1}{2}K\left(\tau_{\lambda\mu} - \tau_\mu v^\sigma \tau_{\sigma\lambda} - v^\gamma \tau_\lambda \tau_{\gamma\mu} + K\tau_\mu\tau_\lambda\right) - 2\,\tilde{\nabla}_\rho K^\rho{}_\lambda \tau_\mu \\
&+ v^\sigma \tilde{\nabla}_\rho \tau_{\sigma\gamma} \Big(-\frac{1}{2}\delta^\gamma_\lambda \big(\delta^\rho_\mu - v^\rho \tau_\mu\big) + \delta^\gamma_\mu \big(\delta^\rho_\lambda + v^\rho\tau_\lambda\big) - \frac{1}{2} h^{\rho\gamma} h_{\lambda\mu} \Big) \\
&+ \tau_\lambda \Big(-\frac{1}{2}\big(K^{\gamma\eta}K_{\gamma\eta} - K^2\big)\tau_\mu + h^{\nu\gamma}\tilde{\nabla}_\gamma \left(K_{\mu\nu} - K h_{\mu\nu}\right)\Big)\Big) \\
&+ \frac{1}{2} h^{\mu\nu}_{(1)} \Big(K^{\sigma\gamma}K_{\sigma\gamma}\left(h_{\mu\nu}\tau_\lambda - h_{\lambda\mu}\tau_\nu\right) + K^\rho{}_\gamma \Big(\delta^\gamma_\lambda \tau_\mu K_{\nu\rho} + v^\sigma \tau_{\sigma\rho}\Big(\frac{1}{2}\delta^\gamma_\lambda h_{\mu\nu} + \delta^\gamma_\nu h_{\mu\lambda}\Big)\Big) \\
&\quad - K\left(K_{\mu\nu}\tau_\lambda + 2\,\tau_\mu K_{\lambda\nu}\right) - K^2\left(h_{\mu\nu}\tau_\lambda + h_{\lambda\mu}\tau_\nu\right) - \tilde{\nabla}_\sigma K^\sigma{}_\gamma \left(\delta^\gamma_\lambda h_{\mu\nu} + \delta^\gamma_\nu h_{\mu\lambda}\right) \\
&\quad + \tilde{\nabla}_\sigma K_{\lambda\nu} \Big(\delta^\gamma_\mu v^\sigma \tau_\lambda - \big(3\,\delta^\sigma_\mu + v^\sigma\tau_\mu\big)\Big) \\
&\quad + \tilde{\nabla}_\sigma K \Big(2\,\delta^\sigma_\lambda h_{\mu\nu} + 2\,v^\sigma h_{\lambda\mu}\tau_\nu + v^\sigma h_{\mu\nu}\tau_\lambda\Big) \\
&\quad + \tau_\lambda \Big(-\frac{1}{2}\big(K^{\gamma\eta}K_{\gamma\eta} - K^2\big) h_{\mu\nu} + K\left(K_{\mu\nu} - K h_{\mu\nu}\right) \\
&\qquad\qquad - v^\gamma \tilde{\nabla}_\gamma \left(K_{\mu\nu} - K h_{\mu\nu}\right)\Big)\Big) \\
&+ \tilde{\nabla}_\sigma v^\mu_{(1)} \Big(K^\sigma{}_\gamma \Big(\delta^\gamma_\mu \tau_\lambda - \frac{5}{2}\delta^\gamma_\lambda \tau_\mu\Big) + K\big(\delta^\sigma_\lambda \tau_\mu - \delta^\sigma_\mu\tau_\lambda\big) + \frac{3}{4} v^\sigma \tau_{\gamma\mu}\left(\delta^\gamma_\lambda + v^\gamma\tau_\lambda\right) \\
&\qquad + v^\rho \tau_{\rho\gamma}\Big(-\frac{5}{2}\delta^\sigma_\mu \delta^\gamma_\lambda - \frac{9}{4}\delta^\gamma_\lambda v^\sigma \tau_\mu + \delta^\sigma_\lambda \delta^\gamma_\mu - \frac{1}{2}h^{\sigma\gamma}h_{\lambda\mu}\Big)\Big) \\
&+ \tilde{\nabla}_\rho \tilde{\nabla}_\sigma v^\mu_{(1)} \Big(\frac{1}{2}\Big(\delta^\sigma_\lambda \delta^\rho_\mu + \delta^\sigma_\lambda v^\rho \tau_\mu + v^\sigma \tau_\lambda \delta^\rho_\mu + v^\sigma\tau_\lambda v^\rho \tau_\mu\Big) - \frac{1}{2}h^{\sigma\rho}h_{\lambda\mu}\Big) \\
&+ \frac{1}{2}\tilde{\nabla}_\sigma h^{\mu\nu}_{(1)} \Big(K\left(\delta^\sigma_\lambda + 2\,v^\sigma\tau_\lambda\right) h_{\mu\nu} + K_{\gamma\nu}\Big(\delta^\gamma_\mu \delta^\sigma_\lambda - 3\,\delta^\gamma_\lambda\delta^\sigma_\mu\Big) - K^\sigma{}_\gamma\left(\delta^\gamma_\lambda h_{\mu\nu} + \delta^\gamma_\nu h_{\mu\lambda}\right)\Big) \\
&+ \frac{1}{2} v^\gamma \tilde{\nabla}_\gamma \tilde{\nabla}_\sigma h^{\mu\nu}_{(1)} \Big(\delta^\sigma_\mu h_{\lambda\nu} - h_{\mu\nu}\left(\delta^\sigma_\lambda + v^\sigma\tau_\lambda\right) + v^\sigma h_{\lambda\mu}\tau_\nu\Big)\Big], \qquad (5.46)
\end{aligned}
$$

and

$$
\begin{aligned}
\overset{(3)}{G}{}^h_{(\rho\lambda)} = \Big[& v^\mu_{(1)} \Big(\Big(3\,K^{\gamma\sigma}K_{\gamma\sigma} - K^2 + v^\sigma\tilde{\nabla}_\sigma K\Big)\tau_{(\lambda}h_{\rho)\mu} + \Big(K^2 - v^\sigma \tilde{\nabla}_\sigma K\Big)\tau_\mu h_{\rho\lambda} \\
&\quad + \Big(4\,K^\gamma{}_{(\rho}K_{\lambda)\gamma} - 3\,KK_{\rho\lambda} + 2\,v^\sigma\tilde{\nabla}_\sigma K_{\rho\lambda}\Big)\tau_\mu + 2\Big(-K_{\mu(\lambda}K - v^\sigma\tilde{\nabla}_\sigma K_{\mu(\lambda}\Big)\tau_{\rho)} \\
&\quad + v^\sigma \tau_{\sigma\gamma} K^\gamma{}_{(\lambda} h_{\rho)\mu} - v^\sigma \tau_{\sigma(\rho}K_{\lambda)\mu} + v^\sigma\tau_{\sigma\mu}\left(2\,K_{\rho\lambda} - h_{\rho\lambda}K\right) \\
&\quad + 4\,K^\gamma{}_{(\rho}\tau_{\lambda)}K_{\mu\gamma} + \tilde{\nabla}_\sigma K^\sigma{}_\mu h_{\rho\lambda} - \tilde{\nabla}_\mu K_{\rho\lambda} \\
&\quad + h_{\rho\lambda}\Big(\frac{1}{2}\Big(K^{\gamma\eta}K_{\gamma\eta} - K^2\Big)\tau_\mu - h^{\nu\gamma}\tilde{\nabla}_\gamma\left(K_{\mu\nu} - Kh_{\mu\nu}\right)\Big)\Big) \\
& + h^{\mu\nu}_{(1)} \Big(\frac{1}{2}\Big(K^{\sigma\gamma}K_{\sigma\gamma} + K^2 - 2\,v^\sigma\tilde{\nabla}_\sigma K\Big) h_{\mu(\rho}h_{\lambda)\nu} + \frac{1}{2}\Big(K^2 - v^\sigma\tilde{\nabla}_\sigma K\Big)h_{\mu\nu}h_{\rho\lambda} \\
&\quad + \Big(K_{\sigma\nu}K^\sigma{}_{(\lambda} - KK_{\nu(\lambda} + v^\sigma\tilde{\nabla}_\sigma K_{\nu(\lambda}\Big) h_{\rho)\mu} - K_{\nu(\rho}K_{\lambda)\mu} \\
&\quad + \frac{1}{2}\Big(v^\sigma\tilde{\nabla}_\sigma K_{\rho\lambda} - KK_{\rho\lambda}\Big)h_{\mu\nu} + \frac{1}{2}\Big(KK_{\mu\nu} - v^\sigma\tilde{\nabla}_\sigma K_{\mu\nu}\Big)h_{\rho\lambda}
\end{aligned}
$$

$$
\begin{aligned}
&\qquad + \frac{1}{2} h_{\rho\lambda}\left(\frac{1}{2}\left(K^{\gamma\eta}K_{\gamma\eta} - K^2\right)h_{\mu\nu} - K\left(K_{\mu\nu} - K h_{\mu\nu}\right) + v^{\gamma}\tilde{\nabla}_{\gamma}\left(K_{\mu\nu} - K h_{\mu\nu}\right)\right)\Bigg)\\
&+ \tilde{\nabla}_{\sigma} v^{\mu}_{(1)}\Bigg(K^{\sigma}{}_{\gamma}\left(\delta^{\gamma}_{(\lambda} h_{\rho)\mu} + \delta^{\gamma}_{\mu} h_{\rho\lambda}\right) - v^{\sigma}\tau_{\mu} K h_{\rho\lambda}\\
&\qquad\qquad - K_{\gamma(\lambda}\left(\left(\left(\delta^{\sigma}_{\rho)} + \tau_{\rho)} v^{\sigma}\right)\delta^{\gamma}_{\mu} + \delta^{\gamma}_{\rho)}\left(\delta^{\sigma}_{\mu} - 2\, v^{\sigma}\tau_{\mu}\right)\right)\right)\Bigg)\\
&+ \frac{1}{2}\tilde{\nabla}_{\sigma} h^{\mu\nu}_{(1)} v^{\sigma}\left(K\left(h_{\mu(\rho} h_{\lambda)\nu} - 2\, h_{\mu\nu} h_{\rho\lambda}\right) + h_{\mu\nu} K_{\rho\lambda} - K_{\mu\nu} h_{\rho\lambda}\right)\\
&+ \frac{1}{2} v^{\gamma} v^{\sigma}\tilde{\nabla}_{\gamma}\tilde{\nabla}_{\sigma} h^{\mu\nu}_{(1)}\left(h_{\mu(\rho} h_{\lambda)\nu} + h_{\mu\nu} h_{\rho\lambda}\right)\Bigg].
\end{aligned}
\tag{5.47}
$$

The last two are therefore the new equations of motion that appear at NLO and are considerably more complicated than the LO EOMs.

5.3.2. NLO even powers of c

For the c^2 expansion, i.e. $N = 2$, we have the varied action

$$
\delta \overset{(4)}{S}_{\text{NLO}} = \frac{1}{8\pi G}\int_M e\left[\overset{(4)}{G}{}^{v}_{\mu}\delta v^{\mu} + \frac{1}{2}\overset{(4)}{G}{}^{h}_{\mu\nu}\delta h^{\mu\nu} + \overset{(4)}{G}{}^{v_{(1)}}_{\mu}\delta v^{\mu}_{(1)} + \frac{1}{2}\overset{(4)}{G}{}^{h_{(1)}}_{\mu\nu}\delta h^{\mu\nu}_{(1)}\right]\mathrm{d}^{d+1},
\tag{5.48}
$$

which has the following equations of motion:

$$
\overset{(4)}{G}{}^{v_{(1)}}_{\mu} = \overset{(2)}{G}{}^{v}_{\mu}, \qquad \overset{(4)}{G}{}^{h_{(1)}}_{\mu\nu} = \overset{(2)}{G}{}^{h}_{\mu\nu},
\tag{5.49}
$$

again repeating the dynamics of the LO theory, along with:

$$
\begin{aligned}
\overset{(4)}{G}{}^{v}_{\lambda} = \Bigg[& v^{\mu}_{(1)}\Bigg(2\,\tau_{\mu} K^{\rho\gamma} K_{\rho\gamma}\tau_{\lambda} + K^{\sigma}{}_{\alpha}\left(\frac{1}{2}\delta^{\alpha}_{\lambda}\tau_{\mu\sigma} + v^{\gamma}\tau_{\gamma\sigma}\left(2\,\delta^{\alpha}_{\lambda}\tau_{\mu} + 3\,\delta^{\alpha}_{\mu}\tau_{\lambda}\right)\right)\\
&\quad + \frac{1}{2} K\left(\tau_{\lambda\mu} - \tau_{\mu} v^{\sigma}\tau_{\sigma\lambda} - v^{\gamma}\tau_{\lambda}\tau_{\gamma\mu} + K\tau_{\mu}\tau_{\lambda}\right) - 2\,\tilde{\nabla}_{\rho} K^{\rho}{}_{\lambda}\tau_{\mu}\\
&\quad + v^{\sigma}\tilde{\nabla}_{\rho}\tau_{\sigma\gamma}\left(-\frac{1}{2}\delta^{\gamma}_{\lambda}\left(\delta^{\rho}_{\mu} - v^{\rho}\tau_{\mu}\right) + \delta^{\gamma}_{\mu}\left(\delta^{\rho}_{\lambda} + v^{\rho}\tau_{\lambda}\right) - \frac{1}{2} h^{\rho\gamma} h_{\lambda\mu}\right)\\
&\quad + \tau_{\lambda}\left(-\frac{1}{2}\left(K^{\gamma\eta}K_{\gamma\eta} - K^2\right)\tau_{\mu} + h^{\nu\gamma}\tilde{\nabla}_{\gamma}\left(K_{\mu\nu} - K h_{\mu\nu}\right)\right)\Bigg)\\
+& \frac{1}{2} h^{\mu\nu}_{(1)}\Bigg(K^{\sigma\gamma} K_{\sigma\gamma}\left(h_{\mu\nu}\tau_{\lambda} - h_{\lambda\mu}\tau_{\nu}\right) + K^{\rho}{}_{\gamma}\left(\delta^{\gamma}_{\lambda}\tau_{\mu} K_{\nu\rho} + v^{\sigma}\tau_{\sigma\rho}\left(\frac{1}{2}\delta^{\gamma}_{\lambda} h_{\mu\nu} + \delta^{\gamma}_{\nu} h_{\mu\lambda}\right)\right)\\
&\quad - K\left(K_{\mu\nu}\tau_{\lambda} + 2\,\tau_{\mu} K_{\lambda\nu}\right) - K^2\left(h_{\mu\nu}\tau_{\lambda} + h_{\lambda\mu}\tau_{\nu}\right) - \tilde{\nabla}_{\sigma} K^{\sigma}{}_{\gamma}\left(\delta^{\gamma}_{\lambda} h_{\mu\nu} + \delta^{\gamma}_{\nu} h_{\mu\lambda}\right)\\
&\quad + \tilde{\nabla}_{\sigma} K_{\lambda\nu}\left(\delta^{\gamma}_{\mu} v^{\sigma}\tau_{\lambda} - \left(3\,\delta^{\sigma}_{\mu} + v^{\sigma}\tau_{\mu}\right)\right)\\
&\quad + \tilde{\nabla}_{\sigma} K\left(2\,\delta^{\sigma}_{\lambda} h_{\mu\nu} + 2\, v^{\sigma} h_{\lambda\mu}\tau_{\nu} + v^{\sigma} h_{\mu\nu}\tau_{\lambda}\right)\\
&\quad + \tau_{\lambda}\Bigg(-\frac{1}{2}\left(K^{\gamma\eta}K_{\gamma\eta} - K^2\right)h_{\mu\nu} + K\left(K_{\mu\nu} - K h_{\mu\nu}\right)\\
&\qquad\qquad - v^{\gamma}\tilde{\nabla}_{\gamma}\left(K_{\mu\nu} - K h_{\mu\nu}\right)\Bigg)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+\tilde{\nabla}_\sigma v^\mu_{(1)}\bigg(K^\sigma{}_\gamma\Big(\delta^\gamma_\mu\tau_\lambda-\frac{5}{2}\delta^\gamma_\lambda\tau_\mu\Big)+K\Big(\delta^\sigma_\lambda\tau_\mu-\delta^\sigma_\mu\tau_\lambda\Big)+\frac{3}{4}v^\sigma\tau_{\gamma\mu}\big(\delta^\gamma_\lambda+v^\gamma\tau_\lambda\big)\\
&\qquad\qquad+v^\rho\tau_{\rho\gamma}\Big(-\frac{5}{2}\delta^\sigma_\mu\delta^\gamma_\lambda-\frac{9}{4}\delta^\gamma_\lambda v^\sigma\tau_\mu+\delta^\sigma_\lambda\delta^\gamma_\mu-\frac{1}{2}h^{\sigma\gamma}h_{\lambda\mu}\Big)\bigg)\\
&+\tilde{\nabla}_\rho\tilde{\nabla}_\sigma v^\mu_{(1)}\bigg(\frac{1}{2}\Big(\delta^\sigma_\lambda\delta^\rho_\mu+\delta^\sigma_\lambda v^\rho\tau_\mu+v^\sigma\tau_\lambda\delta^\rho_\mu+v^\sigma\tau_\lambda v^\rho\tau_\mu\Big)-\frac{1}{2}h^{\sigma\rho}h_{\lambda\mu}\bigg)\\
&+\frac{1}{2}\tilde{\nabla}_\sigma h^{\mu\nu}_{(1)}\bigg(K\big(\delta^\sigma_\lambda+2\,v^\sigma\tau_\lambda\big)h_{\mu\nu}+K_{\gamma\nu}\Big(\delta^\gamma_\mu\delta^\sigma_\lambda-3\,\delta^\gamma_\lambda\delta^\sigma_\mu\Big)-K^\sigma{}_\gamma\big(\delta^\gamma_\lambda h_{\mu\nu}+\delta^\gamma_\nu h_{\mu\lambda}\big)\bigg)\\
&+\frac{1}{2}v^\gamma\tilde{\nabla}_\gamma\tilde{\nabla}_\sigma h^{\mu\nu}_{(1)}\Big(\delta^\sigma_\mu h_{\lambda\nu}-h_{\mu\nu}\big(\delta^\sigma_\lambda+v^\sigma\tau_\lambda\big)+v^\sigma h_{\lambda\mu}\tau_\nu\Big)\\
&+\frac{1}{4}v^\mu\tau_{\mu\gamma}h^{\sigma\gamma}\Big(\delta^\rho_\lambda+v^\rho\tau_\lambda\Big)\tau_{\sigma\rho}-\frac{1}{2}\tilde{\nabla}_\rho\big(h^{\rho\gamma}\tau_{\lambda\gamma}\big)+\frac{1}{2}h^{\mu\nu}\tilde{R}_{\mu\nu}\tau_\lambda\bigg],
\end{aligned}
\tag{5.50}
$$

and

$$
\begin{aligned}
\overset{(4)}{G}{}^h_{(\rho\lambda)}=\bigg[&v^\mu_{(1)}\bigg(\Big(3\,K^{\gamma\sigma}K_{\gamma\sigma}-K^2+v^\sigma\tilde{\nabla}_\sigma K\Big)\tau_{(\lambda}h_{\rho)\mu}+\Big(K^2-v^\sigma\tilde{\nabla}_\sigma K\Big)\tau_\mu h_{\rho\lambda}\\
&\quad+\Big(4\,K^\gamma{}_{(\rho}K_{\lambda)\gamma}-3\,KK_{\rho\lambda}+2\,v^\sigma\tilde{\nabla}_\sigma K_{\rho\lambda}\Big)\tau_\mu+2\Big(-K_{\mu(\lambda}K-v^\sigma\tilde{\nabla}_\sigma K_{\mu(\lambda}\Big)\tau_{\rho)}\\
&\quad+v^\sigma\tau_{\sigma\gamma}K^\gamma{}_{(\lambda}h_{\rho)\mu}-v^\sigma\tau_{\sigma(\rho}K_{\lambda)\mu}+v^\sigma\tau_{\sigma\mu}\big(2\,K_{\rho\lambda}-h_{\rho\lambda}K\big)\\
&\quad+4\,K^\gamma{}_{(\rho}\tau_{\lambda)}K_{\mu\gamma}+\tilde{\nabla}_\sigma K^\sigma{}_\mu h_{\rho\lambda}-\tilde{\nabla}_\mu K_{\rho\lambda}\\
&\quad+h_{\rho\lambda}\Big(\frac{1}{2}\Big(K^{\gamma\eta}K_{\gamma\eta}-K^2\Big)\tau_\mu-h^{\nu\gamma}\tilde{\nabla}_\gamma\big(K_{\mu\nu}-Kh_{\mu\nu}\big)\Big)\bigg)\\
&+h^{\mu\nu}_{(1)}\bigg(\frac{1}{2}\Big(K^{\sigma\gamma}K_{\sigma\gamma}+K^2-2\,v^\sigma\tilde{\nabla}_\sigma K\Big)h_{\mu(\rho}h_{\lambda)\nu}+\frac{1}{2}\Big(K^2-v^\sigma\tilde{\nabla}_\sigma K\Big)h_{\mu\nu}h_{\rho\lambda}\\
&\quad+\Big(K_{\sigma\nu}K^\sigma{}_{(\lambda}-KK_{\nu(\lambda}+v^\sigma\tilde{\nabla}_\sigma K_{\nu(\lambda}\Big)h_{\rho)\mu}-K_{\nu(\rho}K_{\lambda)\mu}\\
&\quad+\frac{1}{2}\Big(v^\sigma\tilde{\nabla}_\sigma K_{\rho\lambda}-KK_{\rho\lambda}\Big)h_{\mu\nu}+\frac{1}{2}\Big(KK_{\mu\nu}-v^\sigma\tilde{\nabla}_\sigma K_{\mu\nu}\Big)h_{\rho\lambda}\\
&\quad+\frac{1}{2}h_{\rho\lambda}\Big(\frac{1}{2}\Big(K^{\gamma\eta}K_{\gamma\eta}-K^2\Big)h_{\mu\nu}-K\big(K_{\mu\nu}-Kh_{\mu\nu}\big)+v^\gamma\tilde{\nabla}_\gamma\big(K_{\mu\nu}-Kh_{\mu\nu}\big)\Big)\bigg)\\
&+\tilde{\nabla}_\sigma v^\mu_{(1)}\bigg(K^\sigma{}_\gamma\Big(\delta^\gamma_{(\lambda}h_{\rho)\mu}+\delta^\gamma_\mu h_{\rho\lambda}\Big)-v^\sigma\tau_\mu Kh_{\rho\lambda}\\
&\qquad\qquad-K_{\gamma(\lambda}\Big(\Big(\delta^\sigma_{\rho)}+\tau_{\rho)}v^\sigma\Big)\delta^\gamma_\mu+\delta^\gamma_{\rho)}\Big(\delta^\sigma_\mu-2\,v^\sigma\tau_\mu\Big)\Big)\bigg)\\
&+\frac{1}{2}\tilde{\nabla}_\sigma h^{\mu\nu}_{(1)}v^\sigma\Big(K\big(h_{\mu(\rho}h_{\lambda)\nu}-2\,h_{\mu\nu}h_{\rho\lambda}\big)+h_{\mu\nu}K_{\rho\lambda}-K_{\mu\nu}h_{\rho\lambda}\Big)\\
&+\frac{1}{2}v^\gamma v^\sigma\tilde{\nabla}_\gamma\tilde{\nabla}_\sigma h^{\mu\nu}_{(1)}\Big(h_{\mu(\rho}h_{\lambda)\nu}+h_{\mu\nu}h_{\rho\lambda}\Big)+\tilde{R}_{\rho\lambda}+v^\sigma\tau_\rho\tau_{\sigma\gamma}\big(2\,\delta^\gamma_\lambda K-K^\gamma{}_\lambda\big)\\
&-\frac{1}{2}h^{\mu\nu}\tilde{R}_{\mu\nu}h_{\rho\lambda}-\tilde{\nabla}_\sigma\Big(K\Big(\delta^\sigma_\rho+v^\sigma\tau_\rho\Big)\tau_\lambda-K^\sigma{}_\rho\tau_\lambda-\frac{1}{2}v^\eta\tau_{\eta\gamma}h^{\sigma\gamma}h_{\rho\lambda}\Big)\bigg].
\end{aligned}
\tag{5.51}
$$

On comparison with the c expansion, the EOMs are equivalent up to a few extra terms that appear in the c^2 expansion. Among these are terms containing curvature, $\tilde{R}_{\mu\nu}$, which first appear at NNLO for the c expansion. The appearance of curvature terms means that the theory allows for massive solutions, see [8].

5.3.3. Electric and magnetic sector

Akin to the treatment of the Carrollian field theory presented in Section 3.4, we will now define the leading-order of the Carrollian geometry to be the electric sector of the theory. The NLO equations for the c^2 have previously only been derived to truncated order, i.e. with $v^{\mu}_{(0)} = 0$ and $h^{\mu\nu}_{(0)} = 0$. This simplifies the equations immensely, leading to:

$$\overset{(4)}{G}{}^{v}_{\lambda}\Big|_{v^{\mu}_{(0)}=h^{\mu\nu}_{(0)}=0} = \frac{1}{4}v^{\mu}\tau_{\mu\gamma}h^{\sigma\gamma}\left(\delta^{\rho}_{\lambda} + v^{\rho}\tau_{\lambda}\right)\tau_{\sigma\rho} - \frac{1}{2}\tilde{\nabla}_{\rho}\left(h^{\rho\gamma}\tau_{\lambda\gamma}\right) + \frac{1}{2}h^{\mu\nu}\tilde{R}_{\mu\nu}\tau_{\lambda}, \tag{5.52a}$$

$$\begin{aligned}\overset{(4)}{G}{}^{h}_{\rho\lambda}\Big|_{v^{\mu}_{(0)}=h^{\mu\nu}_{(0)}=0} &= \tilde{R}_{\rho\lambda} + v^{\sigma}\tau_{\rho}\tau_{\sigma\gamma}\left(2\,\delta^{\gamma}_{\lambda}K - K^{\gamma}{}_{\lambda}\right) - \frac{1}{2}h^{\mu\nu}\tilde{R}_{\mu\nu}h_{\rho\lambda} \\ &\quad - \tilde{\nabla}_{\sigma}\left(K\left(\delta^{\sigma}_{\rho} + v^{\sigma}\tau_{\rho}\right)\tau_{\lambda} - K^{\sigma}{}_{\rho}\tau_{\lambda} - \frac{1}{2}v^{\eta}\tau_{\eta\gamma}h^{\sigma\gamma}h_{\rho\lambda}\right).\end{aligned} \tag{5.52b}$$

This truncation is equivalent to the one described in Section 3.4 for the scalar Carrollian field theory, where we set the LO of the theory to be zero via Lagrange multipliers and thus make the NLO part of the theory Carroll invariant. Here, however, we set the higher order fields $v^{\mu}_{(0)}$ and $h^{\mu\nu}_{(0)}$ to zero which also results in the vanishing of the leading order. Hence, one calls this truncated NLO theory the magnetic Carroll theory. Truncating the NLO for the c expansion is a trivial effort since it makes both $\overset{(3)}{G}{}^{v}_{\lambda}$ and $\overset{(3)}{G}{}^{h}_{\rho\lambda}$ vanish.

6. LO and NLO EOMs and solutions

Having derived the equations of motion for the leading-order theory and both expansions of the next-to-leading-order theory we now study these further based on approaches from [8] and [47]. We first present the LO EOMs, find projections of those and solutions of said projections. The NLO EOMs and its projections are merely presented for completeness. Lastly we will Carroll expand the Schwarzschild and Kerr metrics for both the electric and magnetic sector.

6.1. LO equations of motion

Starting with the LO EOMs (5.20), we first project out the temporal part by contracting them with v^μ

$$v^\mu \overset{(2)}{G}{}^v_\mu = \frac{1}{2}\left(K^{\rho\sigma}K_{\rho\sigma} - K^2\right) + h^{\nu\rho}\tilde{\nabla}_\rho\Big(\underbrace{v^\mu\left(K_{\mu\nu} - Kh_{\mu\nu}\right)}_{=0}\Big) = 0, \tag{6.1a}$$

$$v^\mu \overset{(2)}{G}{}^h_{\mu\nu} = -\frac{1}{2}\left(K^{\rho\sigma}K_{\rho\sigma} - K^2\right)\underbrace{v^\mu h_{\mu\nu}}_{=0} + K\underbrace{v^\mu\left(K_{\mu\nu} - Kh_{\mu\nu}\right)}_{=0} - v^\rho\tilde{\nabla}_\rho\Big(\underbrace{v^\mu\left(K_{\mu\nu} - Kh_{\mu\nu}\right)}_{=0}\Big) = 0, \tag{6.1b}$$

and then by contracting with $h^{\mu\nu}$ we project out the spatial part of the equations

$$h^{\mu\nu} \overset{(2)}{G}{}^v_\mu = \frac{1}{2}\left(K^{\rho\sigma}K_{\rho\sigma} - K^2\right)\underbrace{h^{\mu\nu}\tau_\mu}_{=0} + h^{\mu\nu}h^{\nu\rho}\tilde{\nabla}_\rho\left(K_{\mu\nu} - Kh_{\mu\nu}\right) = 0, \tag{6.2a}$$

$$\begin{aligned}
h^{\mu\nu} \overset{(2)}{G}{}^h_{\mu\nu} &= -\frac{1}{2}\left(K^{\rho\sigma}K_{\rho\sigma} - K^2\right)\underbrace{h^{\mu\nu}h_{\mu\nu}}_{=0} + K\Big(h^{\mu\nu}K_{\mu\nu} - K\underbrace{h^{\mu\nu}h_{\mu\nu}}_{=0}\Big) \\
&\quad - h^{\mu\nu}v^\rho\tilde{\nabla}_\rho\Big(K_{\mu\nu} - K\underbrace{h_{\mu\nu}}_{=0}\Big) \\
&= K^2 - h^{\mu\nu}\pounds_v K_{\mu\nu} + 4\,h^{\mu\nu}v^\rho\tilde{\Gamma}^\lambda_{[\rho\mu]}K_{\lambda\nu} \\
&= K^2 - h^{\mu\nu}\pounds_v K_{\mu\nu} - 2\,h^{\mu\nu}h^{\lambda\gamma}K_{\mu\gamma}K_{\lambda\nu} \\
&= h^{\mu\nu}\left(-\pounds_v K_{\mu\nu} + KK_{\mu\nu} - 2\,K_\mu{}^\lambda K_{\lambda\nu}\right) = 0,
\end{aligned} \tag{6.2b}$$

where we have used (2.5) and (4.26). We arrive at the following equations:

$$K^{\rho\sigma}K_{\rho\sigma} - K^2 = 0, \tag{6.3a}$$

$$h^{\nu\rho}\tilde{\nabla}_\rho\left(K_{\mu\nu} - Kh_{\mu\nu}\right) = 0, \tag{6.3b}$$

$$KK_{\mu\nu} - 2\,K_\mu{}^\lambda K_{\lambda\nu} = \pounds_v K_{\mu\nu}, \tag{6.3c}$$

where we can now interpret the first two of the equations as constraint equations and the third one as a time evolution equation. This is reminiscent of the 3+1 decomposition of the Einstein equation where a split in time and space is introduced in order to carry out a Hamiltonian analysis [62].

For further studying solutions to the EOMs we have just derived, it can be practical to restrict to a spatial hypersurface. We start by defining some properties of spatial hypersurfaces on a Carroll manifold. Spatial covectors are defined as

$$v^\mu X_\mu = 0, \tag{6.4}$$

however, a spatial vector $\tau_\mu X^\mu$ is not boost-invariant. Thus one has to specify to a particular boost frame in order to define a spatial hypersurface and from that one can generalize frame-independent quantities. Choosing $x^\mu = (t, x^i)$ as the coordinates, we define a lapse function α such that

$$v^\mu \partial_\mu = \frac{1}{\alpha}\partial_t, \tag{6.5a}$$

$$\tau_\mu \, \mathrm{d}x^\mu = -\alpha \, \mathrm{d}t + b_i \, \mathrm{d}x^i, \tag{6.5b}$$

and we can further choose a boost-frame where $b_i = 0$ which yields

$$v^\mu \partial_\mu = \frac{1}{\alpha}\partial_t, \tag{6.6a}$$

$$\tau_\mu \, \mathrm{d}x^\mu = -\alpha \, \mathrm{d}t, \tag{6.6b}$$

$$h_{\mu\nu} \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu = h_{ij} \, \mathrm{d}x^i \, \mathrm{d}x^j, \tag{6.6c}$$

$$h^{\mu\nu}\partial_\mu\partial_\nu = h^{ij}\partial_i\partial_j, \tag{6.6d}$$

where we have defined the Riemannian metric h_{ij} on the spatial hypersurfaces. This particular boost-frame will now be used to study various Carroll solutions.

6.1.1. Solution to the evolution equation

Using the parameterization as defined above, we write the extrinsic curvature and the Lie derivative as:

$$\begin{aligned}
K_{ij} &= -\frac{1}{2}\pounds_v h_{ij} = -\frac{1}{2}\left(v^\mu\partial_\mu h_{ij} + h_{\mu j}\partial_i v^\mu + h_{i\mu}\partial_j v^\mu\right) \\
&= -\frac{1}{2}\left(\frac{1}{\alpha}\partial_t h_{ij} + \underbrace{h_{tj}}_{=0}\,\partial_i\frac{1}{\alpha} + \underbrace{h_{ti}}_{=0}\,\partial_j\frac{1}{\alpha}\right) \\
&= -\frac{e^{-B/2}}{2}\dot{h}_{ij},
\end{aligned} \tag{6.7}$$

$$\begin{aligned}
\pounds_v K_{ij} &= -v^\mu\partial_\mu\left(\frac{e^{-B/2}}{2}\dot{h}_{ij}\right) - \frac{e^{-B/2}}{2}\underbrace{\left(\dot{h}_{\mu j}\partial_i v^\mu + \dot{h}_{i\mu}\partial_j v^\mu\right)}_{=0} \\
&= -\frac{e^{-B}}{2}\left(-\frac{\dot{B}}{2}\dot{h}_{ij} + \ddot{h}_{ij}\right),
\end{aligned} \tag{6.8}$$

where the lapse function has been defined to be $\alpha = e^{B/2}$ and the over-dots represent a t derivative. The terms in the second and the fourth line vanish since h has no t component. Inserting these into (6.3c):

$$\pounds_v K_{\mu\nu} = K K_{\mu\nu} - 2\,K_\mu{}^\lambda K_{\lambda\nu}, \tag{6.9a}$$

$$\Rightarrow\ -\frac{e^{-B}}{2}\left(-\frac{\dot{B}}{2}\dot{h}_{ij}+\ddot{h}_{ij}\right)=\frac{e^{-B}}{4}h^{kl}\dot{h}_{kl}\dot{h}_{ij}-2\,\frac{e^{-B}}{4}h^{kl}\dot{h}_{il}\dot{h}_{kj}, \tag{6.9b}$$

gives the evolution equation in the form of an ordinary differential equation:

$$\ddot{h}_{ij}+\frac{1}{2}\dot{h}_{ij}\left(h^{kl}\dot{h}_{kl}-\dot{B}\right)-h^{kl}\dot{h}_{il}\dot{h}_{kj}=0, \tag{6.10}$$

which is then further simplified by choosing $\dot{B}=h^{ij}\dot{h}_{ij}$:

$$\ddot{h}_{ij}-h^{kl}\dot{h}_{il}\dot{h}_{kj}=0. \tag{6.11}$$

Dropping the indices for now and noting that h^{kl} is just h^{-1}, the ODE can be solved via substitution:

$$\begin{aligned} u(t)=\dot{h}(t),\quad \dot{u}(t)=\frac{\mathrm{d}u}{\mathrm{d}h}\frac{\mathrm{d}h}{\mathrm{d}t}=u\frac{\mathrm{d}u}{\mathrm{d}h},\\ \Rightarrow\quad u\frac{\mathrm{d}u}{\mathrm{d}h}-h^{-1}u^2=u\left(\frac{\mathrm{d}u}{\mathrm{d}h}-uh^{-1}\right)=0, \end{aligned} \tag{6.12}$$

whose nontrivial solution is:

$$\begin{aligned} u^{-1}\,\mathrm{d}u-h^{-1}\,\mathrm{d}h=0\quad &\Rightarrow\quad u=c_1h=\frac{\mathrm{d}h}{\mathrm{d}t},\\ \Rightarrow\quad h^{-1}\,\mathrm{d}h=c_1\,\mathrm{d}t\quad &\Rightarrow\quad h(t)=c_2e^{c_1t}. \end{aligned} \tag{6.13}$$

Setting initial conditions as

$$h_{ij}(t=0)=h_{(0)ij}=c_2,\qquad K_{ij}(t=0)=K_{(0)ij}, \tag{6.14}$$

we find

$$K_{(0)ij}=-\frac{e^{-B/2}}{2}h_{(0)ij}c_1\quad\Rightarrow\quad c_1=-2\,e^{B/2}K_{(0)ij}h^{jk}_{(0)}, \tag{6.15}$$

and the solution is thus

$$h_{ij}(t)=h_{(0)ik}e^{-2te^{B/2}h^{kl}_{(0)}K_{(0)lj}}. \tag{6.16}$$

Comparing our solution here with the one presented in [8] we encounter an extra factor of $e^{e^{B/2}}$ in our solution. This is most likely an error resulting from setting $B=0$ for a specific case and forgetting to include the factor in the more general case. This solution will be used later in this section to show that the electric Carroll limit of the Schwarzschild metric is a solution of the evolution equation.

6.2. NLO equations of motion

Moving on to the NLO EOMs, there we have four equations to be contracted with v^μ and $h^{\mu\nu}$ but two of them are the same as the LO EOMS, (5.45) and (5.49), and therefore we only need to consider the other two but for both expansions. Beginning with the c expansion we have:

$$v^\lambda\overset{(3)}{G}{}^v_\lambda=\Bigg[v^\mu_{(1)}\Bigg(-\frac{3}{2}\,\tau_\mu K^{\rho\gamma}K_{\rho\gamma}-3\,K^\sigma{}_\alpha v^\gamma\tau_{\gamma\sigma}\delta^\alpha_\mu+Kv^\lambda\tau_{\lambda\mu}$$

$$
\begin{aligned}
&\qquad - K^2\tau_\mu - h^{\nu\gamma}\tilde{\nabla}_\gamma\big(K_{\mu\nu} - Kh_{\mu\nu}\big)\Big)\\
&+ \tilde{\nabla}_\sigma v^\mu_{(1)}\Big(K\Big(v^\sigma\tau_\mu + \delta^\sigma_\mu\Big) - K^\sigma{}_\mu + v^\rho\tau_{\rho\mu}v^\sigma\Big) + \frac{1}{2}\tilde{\nabla}_\sigma h^{\mu\nu}_{(1)}\Big(K_{\mu\nu}v^\sigma - Kv^\sigma h_{\mu\nu}\Big)\\
&+ \frac{1}{2}h^{\mu\nu}_{(1)}\Big(-\frac{1}{2}K^{\sigma\gamma}K_{\sigma\gamma}h_{\mu\nu} + KK_{\mu\nu} + \frac{3}{2}K^2h_{\mu\nu} + v^\gamma\tilde{\nabla}_\gamma K_{\mu\nu}\Big)\Big] = 0, \qquad (6.17a)
\end{aligned}
$$

$$
\begin{aligned}
v^\lambda \overset{(3)}{G}{}^h_{\rho\lambda} = \Big[&v^\mu_{(1)}\Big(-\Big(3\,K^{\gamma\sigma}K_{\gamma\sigma} - K^2 + v^\sigma\tilde{\nabla}_\sigma K\Big)h_{\mu\rho} - 4\,K^\gamma{}_\rho K_{\mu\gamma}\Big)\\
&+ h^{\mu\nu}_{(1)}\Big(-K\big(K_{\mu\nu} - Kh_{\mu\nu}\big) + v^\gamma\tilde{\nabla}_\gamma\big(K_{\mu\nu} - Kh_{\mu\nu}\big)\Big)\Big] = 0, \qquad (6.17b)
\end{aligned}
$$

as well as

$$
\begin{aligned}
h^{\kappa\lambda}\overset{(3)}{G}{}^v_\lambda = \Big[&v^\mu_{(1)}\Big(K^{\sigma\kappa}\Big(\frac{1}{2}\tau_{\mu\sigma} + 2\,v^\gamma\tau_{\gamma\sigma}\tau_\mu\Big) + \frac{1}{2}K\Big(h^{\kappa\lambda}\tau_{\lambda\mu} - h^{\kappa\lambda}\tau_\mu v^\sigma\tau_{\sigma\lambda}\Big) - 2\,h^{\kappa\lambda}\tilde{\nabla}_\rho K^\rho{}_\lambda\tau_\mu\\
&\qquad + v^\sigma\tilde{\nabla}_\rho\tau_{\sigma\gamma}\Big(-\frac{1}{2}h^{\kappa\gamma}\Big(\delta^\rho_\mu - v^\rho\tau_\mu\Big) + \delta^\gamma_\mu h^{\kappa\rho} - \frac{1}{2}h^{\rho\gamma}\Big(\delta^\kappa_\mu + v^\kappa\tau_\mu\Big)\Big)\Big)\\
&+ \frac{1}{2}h^{\mu\nu}_{(1)}\Big(-K^{\sigma\gamma}K_{\sigma\gamma}\Big(\delta^\kappa_\mu + v^\kappa\tau_\mu\Big)\tau_\nu + K^{\rho\gamma}\Big(\tau_\mu K_{\nu\rho} + \frac{1}{2}v^\sigma\tau_{\sigma\rho}h_{\mu\nu}\Big)\\
&\qquad - 2\,h^{\kappa\lambda}K\tau_\mu K_{\lambda\nu} - \tilde{\nabla}_\sigma K^\sigma{}_\gamma\Big(h^{\kappa\gamma}h_{\mu\nu} + \delta^\gamma_\nu\Big(\delta^\kappa_\mu + v^\kappa\tau_\mu\Big)\Big)\\
&\qquad - K^2\Big(\delta^\kappa_\mu + v^\kappa\tau_\mu\Big)\tau_\nu + h^{\kappa\lambda}\tilde{\nabla}_\sigma K_{\lambda\nu}\Big(\delta^\gamma_\mu v^\sigma\tau_\lambda - \Big(3\,\delta^\sigma_\mu + v^\sigma\tau_\mu\Big)\Big)\\
&\qquad + K^\rho{}_\nu v^\sigma\tau_{\sigma\rho}\Big(\delta^\kappa_\mu + v^\kappa\tau_\mu\Big) + \tilde{\nabla}_\sigma K\Big(2\,h^{\kappa\sigma}h_{\mu\nu} + 2\,v^\sigma\Big(\delta^\kappa_\mu + v^\kappa\tau_\mu\Big)\tau_\nu\Big)\Big)\\
&+ \tilde{\nabla}_\sigma v^\mu_{(1)}\Big(-\frac{5}{2}K^{\sigma\kappa}\tau_\mu + Kh^{\kappa\sigma}\tau_\mu + \frac{3}{4}v^\sigma\tau_{\gamma\mu}h^{\kappa\gamma}\\
&\qquad + v^\rho\tau_{\rho\gamma}\Big(-\frac{5}{2}\delta^\sigma_\mu h^{\kappa\gamma} - \frac{9}{4}h^{\kappa\gamma}v^\sigma\tau_\mu + h^{\kappa\sigma}\delta^\gamma_\mu - \frac{1}{2}h^{\sigma\gamma}\Big(\delta^\kappa_\mu + v^\kappa\tau_\mu\Big)\Big)\Big)\\
&+ \tilde{\nabla}_\rho\tilde{\nabla}_\sigma v^\mu_{(1)}\Big(\frac{1}{2}\Big(h^{\kappa\sigma}\delta^\rho_\mu + h^{\kappa\sigma}v^\rho\tau_\mu\Big) - \frac{1}{2}h^{\sigma\rho}\Big(\delta^\kappa_\mu + v^\kappa\tau_\mu\Big)\Big)\\
&+ \frac{1}{2}\tilde{\nabla}_\sigma h^{\mu\nu}_{(1)}\Big(Kh^{\kappa\sigma}h_{\mu\nu} + K_{\gamma\nu}\Big(\delta^\gamma_\mu h^{\kappa\sigma} - 3\,h^{\kappa\gamma}\delta^\sigma_\mu\Big)\\
&\qquad - K^\sigma{}_\gamma\Big(h^{\kappa\gamma}h_{\mu\nu} + \delta^\gamma_\nu\Big(\delta^\kappa_\mu + v^\kappa\tau_\mu\Big)\Big)\Big)\\
&+ \frac{1}{2}v^\gamma\tilde{\nabla}_\gamma\tilde{\nabla}_\sigma h^{\mu\nu}_{(1)}\Big(\delta^\sigma_\mu(\delta^\kappa_\nu + v^\kappa\tau_\nu) - h_{\mu\nu}h^{\kappa\sigma} + v^\sigma\Big(\delta^\kappa_\mu + v^\kappa\tau_\mu\Big)\tau_\nu\Big)\Big] = 0, \qquad (6.17c)
\end{aligned}
$$

$$
\begin{aligned}
h^{\rho\lambda}\overset{(3)}{G}{}^h_{\rho\lambda} = \Big[&v^\mu_{(1)}\Big(\Big(4\,K^{\gamma\lambda}K_{\lambda\gamma} - 3\,K^2 + 2\,h^{\rho\lambda}v^\sigma\tilde{\nabla}_\sigma K_{\rho\lambda}\Big)\tau_\mu + 2\,Kv^\sigma\tau_{\sigma\mu} - h^{\rho\lambda}\tilde{\nabla}_\mu K_{\rho\lambda}\Big)\\
&+ h^{\mu\nu}_{(1)}\Big(\frac{1}{2}K^{\sigma\gamma}K_{\sigma\gamma}h_{\mu\nu} - v^\sigma\tilde{\nabla}_\sigma Kh_{\mu\nu} - KK_{\mu\nu} + v^\sigma\tilde{\nabla}_\sigma K_{\mu\nu} + \frac{1}{2}h^{\rho\lambda}v^\sigma\tilde{\nabla}_\sigma K_{\rho\lambda}h_{\mu\nu}\Big)
\end{aligned}
$$

$$+\tilde{\nabla}_\sigma v^\mu_{(1)} K\Big(\delta^\sigma_\mu - 2\,v^\sigma\tau_\mu\Big) + \tilde{\nabla}_\sigma h^{\mu\nu}_{(1)} v^\sigma K h_{\mu\nu} + \frac{1}{2} v^\gamma v^\sigma \tilde{\nabla}_\gamma \tilde{\nabla}_\sigma h^{\mu\nu}_{(1)} h_{\mu\nu}\Bigg] = 0. \quad (6.17d)$$

For the c^2 expansion we get

$$\begin{aligned} v^\lambda \overset{(4)}{G}{}^v_\lambda = \Bigg[& v^\mu_{(1)}\Bigg(-\frac{3}{2}\,\tau_\mu K^{\rho\gamma}K_{\rho\gamma} - 3\,K^\sigma{}_\alpha v^\gamma \tau_{\gamma\sigma}\delta^\alpha_\mu + K v^\lambda \tau_{\lambda\mu} \\ & - K^2\tau_\mu - h^{\nu\gamma}\tilde{\nabla}_\gamma\big(K_{\mu\nu} - K h_{\mu\nu}\big)\Bigg) \\ & + \frac{1}{2} h^{\mu\nu}_{(1)}\Bigg(-\frac{1}{2}K^{\sigma\gamma}K_{\sigma\gamma}h_{\mu\nu} + K K_{\mu\nu} + \frac{3}{2}K^2 h_{\mu\nu} + v^\gamma\tilde{\nabla}_\gamma K_{\mu\nu}\Bigg) \\ & + \tilde{\nabla}_\sigma v^\mu_{(1)}\Bigg(-K^\sigma{}_\mu + K\Big(v^\sigma\tau_\mu + \delta^\sigma_\mu\Big) + v^\rho\tau_{\rho\mu}v^\sigma\Bigg) \\ & + \frac{1}{2}\tilde{\nabla}_\sigma h^{\mu\nu}_{(1)}\Bigg(-K v^\sigma h_{\mu\nu} + K_{\mu\nu}v^\sigma\Bigg) \\ & - \frac{1}{2} v^\lambda \tilde{\nabla}_\rho\big(h^{\rho\gamma}\tau_{\lambda\gamma}\big) - \frac{1}{2} h^{\mu\nu}\tilde{R}_{\mu\nu}\Bigg] = 0, \end{aligned} \quad (6.18a)$$

$$\begin{aligned} v^\lambda \overset{(4)}{G}{}^h_{\rho\lambda} = \Bigg[& v^\mu_{(1)}\Bigg(-\Big(3\,K^{\gamma\sigma}K_{\gamma\sigma} - K^2 + v^\sigma\tilde{\nabla}_\sigma K\Big)h_{\mu\rho} - 4\,K^\gamma{}_\rho K_{\mu\gamma}\Bigg) \\ & + h^{\mu\nu}_{(1)}\Bigg(-K\big(K_{\mu\nu} - K h_{\mu\nu}\big) + v^\gamma\tilde{\nabla}_\gamma\big(K_{\mu\nu} - K h_{\mu\nu}\big)\Bigg) \\ & + v^\lambda \tilde{R}_{\rho\lambda} + \tilde{\nabla}_\sigma\Bigg(K\Big(\delta^\sigma_\rho + v^\sigma\tau_\rho\Big) - K^\sigma{}_\rho\Bigg)\Bigg] = 0, \end{aligned} \quad (6.18b)$$

along with

$$\begin{aligned} h^{\kappa\lambda} \overset{(4)}{G}{}^v_\lambda = \Bigg[& v^\mu_{(1)}\Bigg(K^{\sigma\kappa}\Bigg(\frac{1}{2}\tau_{\mu\sigma} + 2\,v^\gamma\tau_{\gamma\sigma}\tau_\mu\Bigg) + \frac{1}{2}K\Big(h^{\kappa\lambda}\tau_{\lambda\mu} - h^{\kappa\lambda}\tau_\mu v^\sigma\tau_{\sigma\lambda}\Big) - 2\,h^{\kappa\lambda}\tilde{\nabla}_\rho K^\rho{}_\lambda\tau_\mu \\ & \quad + v^\sigma\tilde{\nabla}_\rho\tau_{\sigma\gamma}\Bigg(-\frac{1}{2}h^{\kappa\gamma}\Big(\delta^\rho_\mu - v^\rho\tau_\mu\Big) + \delta^\gamma_\mu h^{\kappa\rho} - \frac{1}{2}h^{\rho\gamma}\Big(\delta^\kappa_\mu + v^\kappa\tau_\mu\Big)\Bigg)\Bigg) \\ & + \frac{1}{2}h^{\mu\nu}_{(1)}\Bigg(-K^{\sigma\gamma}K_{\sigma\gamma}\Big(\delta^\kappa_\mu + v^\kappa\tau_\mu\Big)\tau_\nu + K^{\rho\gamma}\Bigg(\tau_\mu K_{\nu\rho} + \frac{1}{2}v^\sigma\tau_{\sigma\rho}h_{\mu\nu}\Bigg) \\ & \quad - 2\,h^{\kappa\lambda}K\tau_\mu K_{\lambda\nu} - \tilde{\nabla}_\sigma K^\sigma{}_\gamma\Big(h^{\kappa\gamma}h_{\mu\nu} + \delta^\gamma_\nu\Big(\delta^\kappa_\mu + v^\kappa\tau_\mu\Big)\Big) \\ & \quad - K^2\Big(\delta^\kappa_\mu + v^\kappa\tau_\mu\Big)\tau_\nu + h^{\kappa\lambda}\tilde{\nabla}_\sigma K_{\lambda\nu}\Big(\delta^\gamma_\mu v^\sigma\tau_\lambda - \Big(3\,\delta^\sigma_\mu + v^\sigma\tau_\mu\Big)\Big) \\ & \quad + K^\rho{}_\nu v^\sigma\tau_{\sigma\rho}\Big(\delta^\kappa_\mu + v^\kappa\tau_\mu\Big) + \tilde{\nabla}_\sigma K\Big(2\,h^{\kappa\sigma}h_{\mu\nu} + 2\,v^\sigma\Big(\delta^\kappa_\mu + v^\kappa\tau_\mu\Big)\tau_\nu\Big)\Bigg) \\ & + \tilde{\nabla}_\sigma v^\mu_{(1)}\Bigg(-\frac{5}{2}K^{\sigma\kappa}\tau_\mu + K h^{\kappa\sigma}\tau_\mu + \frac{3}{4}v^\sigma\tau_{\gamma\mu}h^{\kappa\gamma} \\ & \quad + v^\rho\tau_{\rho\gamma}\Bigg(-\frac{5}{2}\delta^\sigma_\mu h^{\kappa\gamma} - \frac{9}{4}h^{\kappa\gamma}v^\sigma\tau_\mu + h^{\kappa\sigma}\delta^\gamma_\mu - \frac{1}{2}h^{\sigma\gamma}\Big(\delta^\kappa_\mu + v^\kappa\tau_\mu\Big)\Bigg)\Bigg) \end{aligned}$$

$$
\begin{aligned}
&+ \tilde{\nabla}_\rho \tilde{\nabla}_\sigma v^\mu_{(1)} \left(\frac{1}{2} \left(h^{\kappa\sigma} \delta^\rho_\mu + h^{\kappa\sigma} v^\rho \tau_\mu \right) - \frac{1}{2} h^{\sigma\rho} \left(\delta^\kappa_\mu + v^\kappa \tau_\mu \right) \right) \\
&+ \frac{1}{2} \tilde{\nabla}_\sigma h^{\mu\nu}_{(1)} \Big(K h^{\kappa\sigma} h_{\mu\nu} + K_{\gamma\nu} \left(\delta^\gamma_\mu h^{\kappa\sigma} - 3\, h^{\kappa\gamma} \delta^\sigma_\mu \right) \\
&\qquad\qquad - K^\sigma{}_\gamma \left(h^{\kappa\gamma} h_{\mu\nu} + \delta^\gamma_\nu \left(\delta^\kappa_\mu + v^\kappa \tau_\mu \right) \right) \Big) \\
&+ \frac{1}{2} v^\gamma \tilde{\nabla}_\gamma \tilde{\nabla}_\sigma h^{\mu\nu}_{(1)} \left(\delta^\sigma_\mu (\delta^\kappa_\nu + v^\kappa \tau_\nu) - h_{\mu\nu} h^{\kappa\sigma} + v^\sigma \left(\delta^\kappa_\mu + v^\kappa \tau_\mu \right) \tau_\nu \right) \\
&+ \frac{1}{4} v^\mu \tau_{\mu\gamma} h^{\sigma\gamma} h^{\kappa\rho} \tau_{\sigma\rho} - \frac{1}{2} h^{\kappa\lambda} \tilde{\nabla}_\rho \left(h^{\rho\gamma} \tau_{\lambda\gamma} \right) \Bigg] = 0,
\end{aligned}
\tag{6.18c}
$$

$$
\begin{aligned}
h^{\rho\lambda} \overset{(4)}{G}{}^h_{\rho\lambda} = \Bigg[& v^\mu_{(1)} \left(\left(4\, K^{\gamma\lambda} K_{\lambda\gamma} - 3\, K^2 + 2\, h^{\rho\lambda} v^\sigma \tilde{\nabla}_\sigma K_{\rho\lambda} \right) \tau_\mu + 2\, K v^\sigma \tau_{\sigma\mu} - h^{\rho\lambda} \tilde{\nabla}_\mu K_{\rho\lambda} \right) \\
&+ h^{\mu\nu}_{(1)} \left(\frac{1}{2} K^{\sigma\gamma} K_{\sigma\gamma} h_{\mu\nu} - v^\sigma \tilde{\nabla}_\sigma K h_{\mu\nu} - K K_{\mu\nu} + v^\sigma \tilde{\nabla}_\sigma K_{\mu\nu} + \frac{1}{2} h^{\rho\lambda} v^\sigma \tilde{\nabla}_\sigma K_{\rho\lambda} h_{\mu\nu} \right) \\
&+ \tilde{\nabla}_\sigma v^\mu_{(1)} K \left(\delta^\sigma_\mu - 2\, v^\sigma \tau_\mu \right) + \tilde{\nabla}_\sigma h^{\mu\nu}_{(1)} v^\sigma K h_{\mu\nu} + \frac{1}{2} v^\gamma v^\sigma \tilde{\nabla}_\gamma \tilde{\nabla}_\sigma h^{\mu\nu}_{(1)} h_{\mu\nu} \\
&+ h^{\rho\lambda} \tilde{R}_{\rho\lambda} - h^{\rho\lambda} \tilde{\nabla}_\sigma \left(K \left(\delta^\sigma_\rho + v^\sigma \tau_\rho \right) \tau_\lambda - K^\sigma{}_\rho \tau_\lambda - \frac{1}{2} v^\eta \tau_{\eta\gamma} h^{\sigma\gamma} h_{\rho\lambda} \right) \Bigg] = 0.
\end{aligned}
\tag{6.18d}
$$

Here we won't show any further solutions to the equations, only stating them for completeness.

6.3. Carrollian spacetimes

In this section we will present the Schwarzschild and the Kerr metrics in a Carroll expansion. In studying these Carroll expansions of GR solutions, we find different expansions than those considered in Section 5. They correspond to the so-called electric and magnetic limit as described in [47], where the electric and magnetic limit of the Schwarzschild metric is studied. These limits yield Carrollian solutions that one can then compare with the electric and magnetic sectors of the Carroll expansion. In taking these limits what one essentially does is to choose which parameters to keep fixed in the limit and rescale other parameters appropriately to accommodate for that. For the Kerr solution we will draw inspiration from the method used in [79] for deriving the non-relativistic limit of the Kerr metric.

6.3.1. Schwarzschild metric

The Schwarzschild metric can be written as

$$
\begin{aligned}
g_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu &= -c^2 \left(1 - \frac{r_s}{r} \right) \mathrm{d}t^2 + \frac{1}{1 - \frac{r_s}{r}}\, \mathrm{d}r^2 + r^2\, \mathrm{d}\Omega^2 \\
&= -c^2 \left(1 - \frac{2\, M G_N}{c^2 r} \right) \mathrm{d}t^2 + \frac{1}{1 - \frac{2\, M G_N}{c^2 r}}\, \mathrm{d}r^2 + r^2\, \mathrm{d}\Omega^2,
\end{aligned}
\tag{6.19}
$$

where $r_s = 2MG_N/c^2$ is the Schwarzschild radius, G_N being the Newtonian gravitational constant and M the characteristic mass, and $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$.

Electric limit

For taking the electric limit we will keep the combination MG_N fixed, which is often referred to as the black hole energy. Using the relation $E = Mc^2$ it is evident that we need to rescale G_N as such:

$$MG_N = \frac{E}{c^2}G_N = EG_C^{(el)}, \qquad G_C^{(el)} = \frac{G_N}{c^2}, \tag{6.20}$$

where the subscript C indicates that it is the gravitational constant for the Carroll limit and the superscript (el) indicates the electric sector of said limit. Since we keep both E and $G_C^{(el)}$ fixed, the limit describes a region of strong gravity within a black hole. Inserting this into the Schwarzschild metric above and taking the limit $c \longrightarrow 0$ we have:

$$g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \frac{2\,EG_C^{(el)}}{r}\,\mathrm{d}t^2 - c^2\frac{r}{2\,EG_C^{(el)}}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2. \tag{6.21}$$

We can relate this to Carroll quantities by defining:

$$H = \frac{1}{2\,EG_C^{(el)}}, \quad \tau = \frac{1}{\sqrt{2\,EG_C^{(el)}}}r^{3/2} = r^{3/2}\sqrt{H}, \tag{6.22}$$

and then the metric becomes

$$\begin{aligned} g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu &= \frac{1/H}{\left(\tau/\sqrt{H}\right)^{2/3}}\,\mathrm{d}t^2 - c^2\frac{1}{1/H}\left(\frac{2\,\mathrm{d}\tau}{3\sqrt{H}}\right)^2 + \left(\tau/\sqrt{H}\right)^{4/3}\mathrm{d}\Omega^2 \\ &= -\frac{4}{9}c^2\,\mathrm{d}\tau^2 + \frac{1}{(H\tau)^{2/3}}\,\mathrm{d}t^2 + \frac{\tau^{4/3}}{H^{2/3}}\,\mathrm{d}\Omega^2. \end{aligned} \tag{6.23}$$

In order to compare with the Carroll expanded metric, we have to choose which expansion to consider. The Schwarzschild metric has been shown to be compatible with the c^2 expansion and therefore, that would be the natural choice. On comparison with (5.13) we identify:

$$\tau_\tau\,\mathrm{d}\tau = \frac{2}{3}\,\mathrm{d}\tau, \quad v^\tau\partial_\tau = -\frac{3}{2}\partial_\tau, \quad h_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \frac{1}{(H\tau)^{2/3}}\,\mathrm{d}t^2 + \frac{\tau^{4/3}}{H^{2/3}}\,\mathrm{d}\Omega^2, \tag{6.24}$$

which are the leading order Carrollian vielbeine. Here an important choice has been made of only considering the $\tau_\mu\tau_\nu$ part of the c^2 term in the expansion. This corresponds to either setting $h^{(1)}_{\mu\nu}$ to zero or absorbing it into the $\tau_\mu\tau_\nu$ term. If we would instead choose to compare with the general metric in the c expansion, we also have to set $h^{(1)}_{\mu\nu} = 0$ due to the Schwarzschild solution not containing any factors of c and then one also ignores $h^{(2)}_{\mu\nu}$ in the same way as one did for $h^{(1)}_{\mu\nu}$ before.

We can show that this is a solution of the evolution equation (6.3c), as mentioned in [47], by identifying the purely spatial metric as

$$h_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j = \frac{\tau^{4/3}}{H^{2/3}}\,\mathrm{d}\Omega^2 \tag{6.25}$$

$$= \frac{\tau^{4/3}}{H^{2/3}}\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j, \quad x^2+y^2+z^2=1. \tag{6.26}$$

and the extrinsic curvature

$$\begin{aligned}K_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu &= -\frac{1}{2}\pounds_v h_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -\frac{1}{2}v^\sigma\partial_\sigma h_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\\ &= -\frac{1}{2}\left(-\frac{3}{2}\right)\partial_\tau h_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\\ &= \frac{3}{4}\left(-\frac{2}{3}\frac{1}{H^{2/3}\tau^{5/3}}\,\mathrm{d}t^2+\frac{4}{3}\frac{\tau^{1/3}}{H^{2/3}}\,\mathrm{d}\Omega^2\right)\\ &= -\frac{1}{2\,H^{2/3}\tau^{5/3}}\,\mathrm{d}t^2+\frac{\tau^{1/3}}{H^{2/3}}\,\mathrm{d}\Omega^2,\end{aligned} \tag{6.27}$$

with purely spatial part as

$$K_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j = \frac{\tau^{1/3}}{H^{2/3}}\,\mathrm{d}\Omega^2 \tag{6.28a}$$

$$= \frac{\tau^{1/3}}{H^{2/3}}\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j, \quad x^2+y^2+z^2=1. \tag{6.28b}$$

Choosing

$$\tau^{4/3}(t) = e^{-2te^{B/2}} \quad\Rightarrow\quad \tau(0)=1, \tag{6.29}$$

we then find:

$$h_{(0)ij} = \frac{1}{H^{2/3}}\delta_{ij}, \quad K_{(0)ij} = \frac{1}{H^{2/3}}\delta_{ij} \quad\Rightarrow\quad e^{h^{kl}_{(0)}K_{(0)lj}} = e^{\delta^k_j}, \tag{6.30}$$

which, when inserted into (6.16), gives

$$\begin{aligned}h_{ij}(t) &= \frac{1}{H^{2/3}}\delta_{ik}\tau^{4/3}e^{\delta^k_j}\\ &= \frac{\tau^{4/3}}{H^{2/3}}\delta_{ij}.\end{aligned} \tag{6.31}$$

We have thus confirmed that the electric Carroll limit of the Schwarzschild metric is indeed a solution of the evolution equation (6.3c) and is therefore compatible with the LO of the Carroll expansion of GR.

Magnetic limit

Now for the magnetic limit of the Schwarzschild metric we would like to keep the Schwarzschild radius r_s fixed. Again with $E = Mc^2$ we then show

$$r_s = \frac{2\,MG_N}{c^2} = \frac{2\,EG_N}{c^4} = 2\,EG_C^{(m)}, \qquad G_C^{(m)} = \frac{G_N}{c^4}, \tag{6.32}$$

where the superscript (m) implies the magnetic sector. We then insert the expression into the Schwarzschild metric and take the Carroll limit to arrive at:

$$g_{\mu\nu}\,\mathrm{d}x^{\mu}\,\mathrm{d}x^{\nu} = -c^2\left(1-\frac{2\,EG_C^{(m)}}{r}\right)\mathrm{d}t^2 + \frac{1}{1-\frac{2\,EG_C^{(m)}}{r}}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2. \tag{6.33}$$

We now identify

$$\tau_t\,\mathrm{d}t = \sqrt{1-\frac{2\,EG_C^{(m)}}{r}}\,\mathrm{d}t, \quad v^t\partial_t = -\sqrt{\frac{r}{r-2\,EG_C^{(m)}}}\partial_t, \tag{6.34a}$$

$$h_{\mu\nu}\,\mathrm{d}x^{\mu}\,\mathrm{d}x^{\nu} = \frac{1}{1-\frac{2\,EG_C^{(m)}}{r}}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2. \tag{6.34b}$$

Notice that since $h_{\mu\nu}$ is static the Lie-derivative with respect to v is zero, i.e. $K_{\mu\nu} = 0$, which makes it fit naturally within the framework of the truncation of the NLO of the Carroll expansion, discussed in Section 5.3.3. The vanishing of $K_{\mu\nu}$ also tells us that the metric describes a constant time slice of a Schwarzschild black hole [47].

6.3.2. Kerr metric

Another important metric to consider is the Kerr metric for rotating black holes. We follow a similar approach as done for non-relativistic GR in [79] and use oblate spherical coordinates

$$x = \sqrt{R^2+a^2}\sin\Theta\cos\phi, \tag{6.35a}$$

$$y = \sqrt{R^2+a^2}\sin\Theta\sin\phi, \tag{6.35b}$$

$$z = R\cos\Theta. \tag{6.35c}$$

R and Θ are here variables that will reduce to be r, θ at leading order. The Kerr metric then becomes:

$$\begin{aligned} g_{\mu\nu}\,\mathrm{d}x^{\mu}\,\mathrm{d}x^{\nu} = &-\left(1-\frac{r_sR}{\Sigma}\right)c^2\,\mathrm{d}t^2 + \frac{\Sigma}{R^2+a^2}\left(1+\frac{r_sR}{\Delta}\right)\mathrm{d}R^2 + \Sigma\,\mathrm{d}\Theta^2 \\ &+ \sin^2\Theta\left(R^2+a^2+\frac{r_sR}{\Sigma}a^2\sin^2\Theta\right)\mathrm{d}\phi^2 - c\frac{2ar_sR}{\Sigma}\sin^2\Theta\,\mathrm{d}t\,\mathrm{d}\phi, \end{aligned} \tag{6.36}$$

where we have defined

$$\Sigma = R^2 + a^2\cos^2\Theta, \tag{6.37a}$$

$$a = \frac{J}{cM}, \tag{6.37b}$$

$$\Delta = R^2 + a^2 - r_sR, \tag{6.37c}$$

with J being the angular momentum.

Electric limit

We choose the same electric limit here as before, i.e. we want to keep MG_N fixed and define $G_C^{(el)} = G_N/c^2$. The Kerr bound describes how self-gravitating compact objects, where the gravitational binding energy dominates the total energy, have to respect the following relation [80]:

$$Jc \leq G_N M^2. \tag{6.38}$$

This bound is then used to find the rescaling of the angular momentum in the following way:

$$\frac{Jc}{G_N M^2} = \frac{Jc}{G_C^{(el)} c^2 \frac{E^2}{c^4}} = \frac{J_C^{(el)}}{G_C^{(el)} E^2} \leq 1, \qquad J_C^{(el)} = Jc^3, \tag{6.39}$$

which further gives

$$a = \frac{J_C^{(el)}}{Ec^2}, \qquad r_s = \frac{2\,EG_C^{(el)}}{c^2}. \tag{6.40}$$

Inserting these into the Kerr metric and expanding all variables to expose all factors of c, we arrive at:

$$\begin{aligned} g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = &-\left(1 - \frac{2\,EG_C^{(el)}R}{R^2c^2 + \frac{\left(J_C^{(el)}\right)^2}{E^2c^2}\cos^2\Theta}\right)c^2\,\mathrm{d}t^2 + \frac{R^2c^2 + \frac{\left(J_C^{(el)}\right)^2}{E^2c^2}\cos^2\Theta}{R^2c^2 + \frac{\left(J_C^{(el)}\right)^2}{E^2c^2} - 2\,EG_C^{(el)}R}\,\mathrm{d}R^2 \\ &+ \left(R^2 + \frac{\left(J_C^{(el)}\right)^2}{E^2c^4}\cos^2\Theta\right)\mathrm{d}\Theta^2 - c\frac{4\,J_C^{(el)}G_C^{(el)}R}{R^2c^4 + \frac{\left(J_C^{(el)}\right)^2}{E^2}\cos^2\Theta}\sin^2\Theta\,\mathrm{d}t\,\mathrm{d}\phi \\ &+ \sin^2\Theta\left(R^2 + \frac{\left(J_C^{(el)}\right)^2}{E^2c^4} + \frac{2\,EG_C^{(el)}\left(J_C^{(el)}\right)^2R}{R^2E^2c^6 + \left(J_C^{(el)}\right)^2c^2\cos^2\Theta}\sin^2\Theta\right)\mathrm{d}\phi^2. \end{aligned} \tag{6.41}$$

Expanding to order c^2 and assuming that the variables R and Θ approach their leading order terms, $R \to r$ and $\Theta \to \theta$, under the Carroll limit we have:

$$\begin{aligned} g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = &-c^2\,\mathrm{d}t^2 + \cos^2\theta\left(1 + c^2\frac{2\,G_C^{(el)}E^3r}{\left(J_C^{(el)}\right)^2}\right)\mathrm{d}r^2 \\ &+ \left(r^2 + \frac{\left(J_C^{(el)}\right)^2}{E^2c^4}\cos^2\theta\right)\mathrm{d}\theta^2 - c\frac{4\,G_C^{(el)}E^2r}{J_C^{(el)}\cos^2\theta}\sin^2\theta\,\mathrm{d}t\,\mathrm{d}\phi \\ &+ \sin^2\theta\left(r^2 + \frac{\left(J_C^{(el)}\right)^2}{E^2c^4} + \frac{2\,G_C^{(el)}Er\sin^4\theta}{c^2\cos^2\theta} + c^2\frac{2\,G_C^{(el)}E^3r^3\sin^2\theta}{\left(J_C^{(el)}\right)^2\cos^2\theta}\right)\mathrm{d}\phi^2. \end{aligned} \tag{6.42}$$

where we have kept higher order terms in c that would normally vanish under the Carroll limit. This is done for illustrative purposes.

Note the single factor of c preceding the second term in the second line. This singular term of c can not be removed by any rescaling without obtaining odd powers of c elsewhere. As it turns out, this happens to be a problem for the assumption of only even powers in the Carroll expansion, hence why we introduced the c expansion in Section 5.1.1. A bigger problem, however, is that of the super-leading orders of c appearing in the expansion. These will essentially dominate over all other terms and the limit $c \to 0$ should therefore leave us with a metric

$$g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \frac{\left(J_C^{(el)}\right)^2}{E^2c^4}\left(\cos^2\theta\,\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right), \tag{6.43}$$

that has shrunk down to two dimensions and does not contain any interesting dynamics.

Moreover, taking the limit $J_C^{(el)} \to 0$ of (6.42) we find

$$\begin{aligned} g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -c^2\,\mathrm{d}t^2 + \cos^2\theta\left(1 + c^2\frac{2\,G_C^{(el)}E^3r}{\left(J_C^{(el)}\right)^2}\right)\mathrm{d}r^2 - c\frac{4\,G_C^{(el)}E^2r}{J_C^{(el)}\cos^2\theta}\sin^2\theta\,\mathrm{d}t\,\mathrm{d}\phi \\ + r^2\,\mathrm{d}\theta^2 + \sin^2\theta\left(r^2 + \frac{2\,G_C^{(el)}Er\sin^4\theta}{c^2\cos^2\theta} + c^2\frac{2\,G_C^{(el)}E^3r^3\sin^2\theta}{\left(J_C^{(el)}\right)^2\cos^2\theta}\right)\mathrm{d}\phi^2, \end{aligned} \tag{6.44}$$

and for (6.43) we have $g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \to 0$. Either of those should give us the electric Schwarzschild metric (6.21) but alas they do not, further confirming that something needs to be reconsidered for taking the electric Carroll limit of the Kerr metric. Due to this failure it does not make sense to identify any of the terms with the Carrollian vielbeine.

Magnetic limit

Considering the same magnetic limit as before for the Kerr metric, we find the rescaling of J to be:

$$\frac{Jc}{G_N M^2} = \frac{Jc}{G_C^{(m)}c^4\frac{E^2}{c^4}} = \frac{J_C^{(m)}}{G_C^{(m)}E^2} \leq 1, \qquad J_C^{(m)} = Jc, \tag{6.45}$$

and thus:

$$a = \frac{J_C^{(m)}}{E}, \qquad r_s = 2\,G_C^{(m)}E. \tag{6.46}$$

We now write the Kerr metric as:

$$g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -\left(1 - \frac{2\,E^3G_C^{(m)}R}{R^2E^2 + \left(J_C^{(m)}\right)^2\cos^2\Theta}\right)c^2\,\mathrm{d}t^2 - c\frac{4\,E^2G_C^{(m)}J_C^{(m)}R\sin^2\Theta}{R^2E^2 + \left(J_C^{(m)}\right)^2\cos^2\Theta}\,\mathrm{d}t\,\mathrm{d}\phi$$

$$+\frac{E^2R^2+\left(J_C^{(m)}\right)^2\cos^2\Theta}{\left(J_C^{(m)}\right)^2-2\,E^3RG_C^{(m)}+E^2R^2}\,\mathrm{d}R^2+\left(R^2+\frac{\left(J_C^{(m)}\right)^2\cos^2\Theta}{E^2}\right)\mathrm{d}\Theta^2$$
$$+\sin^2\Theta\left(\frac{\left(J_C^{(m)}\right)^2}{E^2}+R^2+\frac{2\,EG_C^{(m)}\left(J_C^{(m)}\right)^2R\sin^2\Theta}{E^2R^2+\left(J_C^{(m)}\right)^2\cos^2\Theta}\right)\mathrm{d}\phi^2, \tag{6.47}$$

expanding to order c^2 and taking the limit $c \to 0$ (where we only substitute r, θ for R, Θ since no explicit terms of c vanish) we have:

$$g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -\left(1-\frac{2\,E^3G_C^{(m)}r}{r^2E^2+\left(J_C^{(m)}\right)^2\cos^2\theta}\right)c^2\,\mathrm{d}t^2 - c\,\frac{4\,E^2G_C^{(m)}J_C^{(m)}r\sin^2\theta}{r^2E^2+\left(J_C^{(m)}\right)^2\cos^2\theta}\,\mathrm{d}t\,\mathrm{d}\phi$$
$$+\frac{E^2r^2+\left(J_C^{(m)}\right)^2\cos^2\theta}{\left(J_C^{(m)}\right)^2-2\,E^3rG_C^{(m)}+E^2r^2}\,\mathrm{d}r^2+\left(r^2+\frac{\left(J_C^{(m)}\right)^2\cos^2\theta}{E^2}\right)\mathrm{d}\theta^2$$
$$+\sin^2\theta\left(\frac{\left(J_C^{(m)}\right)^2}{E^2}+r^2+\frac{2\,EG_C^{(m)}\left(J_C^{(m)}\right)^2r\sin^2\theta}{E^2r^2+\left(J_C^{(m)}\right)^2\cos^2\theta}\right)\mathrm{d}\phi^2. \tag{6.48}$$

Further taking the limit $J \to 0$ we find:

$$g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -\left(1-\frac{2\,EG_C^{(m)}}{r}\right)c^2\,\mathrm{d}t^2+\frac{r}{r-2\,EG_C^{(m)}}\,\mathrm{d}r^2+r^2\,\mathrm{d}\theta^2+r^2\sin^2\theta\,\mathrm{d}\phi^2, \tag{6.49}$$

which matches exactly the magnetic Schwarzschild metric. Here we don't run into problems from the singular power of c or from super-leading terms of c. The $J \to 0$ limit gets rid of the former but the latter simply does not appear in the magnetic limit. The singular power of c can be incorporated into the theory via the c expansion and thus the magnetic limit of the Kerr metric is established. Hence, we can identify:

$$h_{\mu\nu}\,\mathrm{d}x^\mu x^\nu = \frac{E^2r^2+\left(J_C^{(m)}\right)^2\cos^2\theta}{\left(J_C^{(m)}\right)^2-2\,E^3rG_C^{(m)}+E^2r^2}\,\mathrm{d}r^2+\left(r^2+\frac{\left(J_C^{(m)}\right)^2\cos^2\theta}{E^2}\right)\mathrm{d}\theta^2$$
$$+\sin^2\theta\left(\frac{\left(J_C^{(m)}\right)^2}{E^2}+r^2+\frac{2\,EG_C^{(m)}\left(J_C^{(m)}\right)^2r\sin^2\theta}{E^2r^2+\left(J_C^{(m)}\right)^2\cos^2\theta}\right)\mathrm{d}\phi^2. \tag{6.50}$$

As for the electric Carrollian Schwarzschild solution, this metric is static and thus we have vanishing extrinsic curvature $K_{\mu\nu} = 0$, just as wanted for a magnetic theory. Because we have a term of order c in the expansion of the Kerr metric, we have to compare with the c expansion of the general metric (5.8). There, we see that the term at order c should be $h^{(1)}_{\mu\nu}$:

$$h^{(1)}_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -\frac{4\,E^2 G^{(m)}_C J^{(m)}_C r\sin^2\theta}{r^2E^2 + \left(J^{(m)}_C\right)^2\cos^2\theta}\,\mathrm{d}t\,\mathrm{d}\phi. \tag{6.51}$$

Comparing the $\mathrm{d}t^2$ and $\mathrm{d}r^2$ terms of the magnetic Carrollian Schwarzschild to the Kerr, it is tempting to write the factor preceding $-c^2\,\mathrm{d}t^2$ of the latter in the following way:

$$\begin{aligned}
1 - \frac{2\,E^3 G^{(m)}_C r}{r^2E^2 + \left(J^{(m)}_C\right)^2\cos^2\theta} &= \frac{\left(J^{(m)}_C\right)^2\cos^2\theta - 2\,E^3 r G^{(m)}_C + E^2r^2}{r^2E^2 + \left(J^{(m)}_C\right)^2\cos^2\theta}\\
&= \frac{\left(J^{(m)}_C\right)^2 - 2\,E^3 r G^{(m)}_C + E^2r^2}{r^2E^2 + \left(J^{(m)}_C\right)^2\cos^2\theta} - \frac{\left(J^{(m)}_C\right)^2\sin^2\theta}{r^2E^2 + \left(J^{(m)}_C\right)^2\cos^2\theta}.
\end{aligned} \tag{6.52}$$

This highlights how the terms preceding $\mathrm{d}t^2$ and $\mathrm{d}r^2$ in the Schwarzschild metric are the inverse of each other. Furthermore, this allows us to identify:

$$\tau_t\,\mathrm{d}t = \frac{J^{(m)}_C\sin\theta}{\sqrt{r^2E^2 + \left(J^{(m)}_C\right)^2\cos^2\theta}}\,\mathrm{d}t, \quad v^t\partial_t = -\frac{\sqrt{r^2E^2 + \left(J^{(m)}_C\right)^2\cos^2\theta}}{J^{(m)}_C\sin\theta}\partial_t, \tag{6.53a}$$

$$h^{(2)}_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -\frac{\left(J^{(m)}_C\right)^2 - 2\,E^3 r G^{(m)}_C + E^2r^2}{r^2E^2 + \left(J^{(m)}_C\right)^2\cos^2\theta}\,\mathrm{d}t^2, \tag{6.53b}$$

once again from comparison with (5.8).

6.3.3. Comparison of solutions to Carroll EOMs

The Carrollian Schwarzschild and Kerr solutions that have been presented here were derived by taking the Carroll limit of solutions to general relativity. Earlier in Sections 5.2 and 5.3 we derived equations of motion for the LO and NLO of the Carroll theory, the former we consider the electric theory and the latter, when truncated (as discussed in Section 5.3.3), is the magnetic theory. In order to verify that the Carroll solutions are consistent with our theory, one could ensure that the vielbeine identified from the solutions are solutions to the corresponding EOMs as well. An example of that would be to take the vielbeine identified for the magnetic Carrollian Schwarzschild metric (6.34) and insert into the truncated NLO EOMs (5.52) and verify that they are indeed solutions of those equations. For all Carroll solutions found in this manner, i.e. by taking the Carroll limit of GR solutions, this should be the case. However, it is important to note that the converse is not necessarily true, there exist many solutions to Carrollian geometry that are not solutions to GR.

7. Conclusion and outlook

In this thesis we have examined Carrollian geometry, its derivation and solutions. The main result is the derivation of the full NLO for the Carroll expansion as well as the derivation of the Carrollian Kerr metric for the magnetic theory. Moreover, the failure of the electric Carrollian Kerr metric, along with the realization that the c^2 expansion is not sufficient for either sector of the Carrollian Kerr metric, are important results in their own right. The latter was the motivation behind considering the Carroll expansion in all powers of c, which has not been done before, and applying that to the NLO. Other results of this thesis include verifying that the Carroll expansion of Schwarzschild black holes can be derived from general vacuum solutions to the evolution equation of the LO theory.

The derivation of the full NLO of Carrollian geometry builds upon the foundation laid in [8] for deriving the LO and the truncated NLO. On comparison with those, the NLO derivation turned out to be quite complex and the resulting equations of motion do not simplify to any significant degree. Furthermore, the inclusion of the c expansion resulted in an extra layer of complexity. The full NLO theory has various important applications and utility. Among those would be to explore sub-leading corrections to the Beliniski-Khalatnikov-Lifshitz limit of general relativity that describes dynamics near a singularity [81], which Carrollian gravity is able to describe. Another possibility is that of expanding on the work of [82] where one considers the Palatini formulation of gravity. In the Palatini formulation a lot of calculations tend to simplify and thus it is a worthy effort to examine whether that approach can be taken in deriving higher orders for the Carroll expansion. The full NLO theory allows one to verify if this Palatini approach is valid. Additionally, the full NLO EOMs allow one to revisit the construction of massive solutions at NLO as seen in [8, 38] as well as building further on NLO solutions including a cosmological constant [8].

The explicit verification of the claim in [47] that the electric Carrollian Schwarzschild solutions are solutions to the LO evolution equation is an important step. This confirms that the electric limit chosen for Carroll expanding the Schwarzschild metric is consistent with the LO of the general Carroll expansion. Now that the Carroll expansion of the Kerr metric has been established for the magnetic sector, the next logical step would be to verify that it is also a solution of the truncated NLO EOMs, i.e. the magnetic theory. It would thus be interesting to take the vielbeine we identified from the magnetic Carrollian Kerr metric and check if they solve the EOMs of the magnetic theory. If successful, this could confirm that the magnetic Carrollian Kerr metric is indeed correct, or it could give some hints as to why the electric Carrollian Kerr metric does not work. Moreover, it could be that one needs to choose another way of taking the electric and magnetic limit in order to Carroll expand the Kerr metric and, therefore, it is important to do such consistency checks. Additional work on Carrollian spacetimes could include exploring other metrics via the Carroll expansion as well as further considering the coupling of matter to Carrollian geometry.

The evolution equation of the LO allows one to analytically evolve non-trivial initial data in time. Having derived the full NLO EOMs, one could begin to consider a "post-ultra-relativistic" expansion of GR where one perturbs around the LO Carroll theory. However, given the complexity of the NLO EOMs this could prove to be a technically difficult direction, though it would nonetheless be an interesting research direction.

A. PUL identities

This section shows explicit calculations of some identities from Section 4.

A.1. Connection identities

From Section 4.4 we show the derivation of the identities (4.31):

$$
\begin{aligned}
\overset{(\tilde{C})}{\nabla}_\sigma T_\mu &= \partial_\sigma T_\mu - \tilde{C}^{\rho}_{\sigma\mu} T_\rho \\
&= \partial_\sigma T_\mu - \left(-V^\rho \partial_{(\sigma} T_{\mu)} - V^\rho T_{(\sigma} \pounds_V T_{\mu)}\right) T_\rho \\
&= \partial_\sigma T_\mu - \partial_{(\sigma} T_{\mu)} - T_{(\sigma}\left(V^\lambda \partial_\lambda T_{\mu)} + T_\lambda \partial_{\mu)} V^\lambda\right) \\
&= \partial_{[\sigma} T_{\mu]} - V^\lambda \left(T_{(\sigma} \partial_{|\lambda|} T_{\mu)} - T_{(\sigma} \partial_{\mu)} T_\lambda\right) \\
&= \frac{1}{2} T_{\sigma\mu} - \frac{1}{2} V^\lambda \left(T_\sigma T_{\lambda\mu} + T_\mu T_{\lambda\sigma}\right),
\end{aligned} \tag{A.1a}
$$

$$
\begin{aligned}
\overset{(\tilde{C})}{\nabla}_\sigma \Pi^{\mu\nu} &= \partial_\sigma \Pi^{\mu\nu} + \tilde{C}^{\mu}_{\sigma\rho} \Pi^{\rho\nu} + \tilde{C}^{\nu}_{\sigma\rho} \Pi^{\mu\rho} \\
&= \partial_\sigma \Pi^{\mu\nu} + 2\Bigg(- V^{(\mu}\left(\partial_{(\sigma} T_{\rho)} + T_{(\sigma} \pounds_V T_{\rho)}\right) \\
&\qquad\qquad + \frac{1}{2}\Pi^{(\mu|\lambda|}\Big[\partial_\sigma \Pi_{\rho\lambda} + \partial_\rho \Pi_{\lambda\sigma} - \partial_\lambda \Pi_{\sigma\rho} - 2 \underbrace{T_\rho}_{=0} \mathcal{K}_{\sigma\lambda}\Big]\Bigg)\Pi^{\nu)\rho} \\
&= \partial_\sigma \Pi^{\mu\nu} - 2\, V^{(\mu}\Pi^{\nu)\rho}\left(\partial_{(\sigma} T_{\rho)} + \frac{1}{2} T_\sigma\left(V^\lambda \partial_\lambda T_\rho + T_\lambda \partial_\rho V^\lambda\right)\right) \\
&\quad + \Pi^{\lambda(\mu}\Big[- \Pi_{\rho\lambda}\partial_\sigma \Pi^{\nu)\rho} + V^{\nu)}\partial_\sigma T_\lambda + \underbrace{T_\lambda}_{=0} \partial_\sigma V^{\nu)} + \Pi^{\nu)\rho}\left(\partial_\rho \Pi_{\lambda\sigma} - \partial_\lambda \Pi_{\sigma\rho}\right)\Big] \\
&= \partial_\sigma \Pi^{\mu\nu} - 2\, V^{(\mu}\Pi^{\nu)\rho}\left(\partial_{(\sigma} T_{\rho)} + T_\sigma V^\lambda \partial_{[\lambda} T_{\rho]}\right) \\
&\quad + \Big[- \left(\delta^{(\mu}_\rho + T_\rho V^{(\mu}\right)\partial_\sigma \Pi^{\nu)\rho} + \Pi^{\lambda(\mu} V^{\nu)} \partial_\sigma T_\lambda + \underbrace{\Pi^{\lambda(\mu}\Pi^{\nu)\rho}\partial_{[\rho}\Pi_{\lambda]\sigma}}_{=0}\Big] \\
&= \cancel{\partial_\sigma \Pi^{\mu\nu}} - 2\, V^{(\mu}\Pi^{\nu)\rho}\left(\partial_{(\sigma} T_{\rho)} - T_\sigma V^\lambda \partial_{[\rho} T_{\lambda]}\right) \\
&\quad - \cancel{\partial_\sigma \Pi^{\mu\nu}} + 2\, V^{(\mu}\Pi^{\nu)\rho}\partial_\sigma T_\rho \\
&= -V^{(\mu}\Pi^{\nu)\rho}\left(T_{\rho\sigma} - T_\sigma V^\lambda T_{\rho\lambda}\right) \\
&= V^{(\mu}\Pi^{\nu)\rho} T_{\lambda\rho}\left(\delta^\lambda_\sigma - T_\sigma V^\lambda\right),
\end{aligned} \tag{A.1b}
$$

$$
\begin{aligned}
\overset{(\tilde{C})}{\nabla}_\mu \Pi^{\mu\nu} &= \frac{1}{2}\Big(V^\mu \Pi^{\nu\rho} T_{\mu\rho} + V^\nu \underbrace{\Pi^{\mu\rho} T_{\mu\rho}}_{=0} + V^\lambda \Pi^{\nu\rho} T_{\lambda\rho} - V^\nu \underbrace{\Pi^{\mu\rho} T_\mu}_{=0} V^\lambda T_{\lambda\rho}\Big) \\
&= V^\mu \Pi^{\nu\rho} T_{\mu\rho},
\end{aligned} \tag{A.1c}
$$

$$
\begin{aligned}
\overset{(\tilde{C})}{\nabla}_\sigma \mathcal{K}_{\mu\nu} &= \partial_\sigma \mathcal{K}_{\mu\nu} - \tilde{C}^{\rho}_{\sigma\mu} \mathcal{K}_{\rho\nu} - \tilde{C}^{\rho}_{\sigma\nu} \mathcal{K}_{\mu\rho} \\
&= \partial_\sigma \mathcal{K}_{\mu\nu} - \left(\Pi^{\rho\lambda}\left(\partial_{(\sigma}\Pi_{\mu)\lambda} - \frac{1}{2}\partial_\lambda \Pi_{\sigma\mu}\right) - \Pi^{\rho\lambda} T_\mu \mathcal{K}_{\sigma\lambda}\right)\mathcal{K}_{\rho\nu}
\end{aligned}
$$

$$- \left(\Pi^{\rho\lambda} \left(\partial_{(\sigma} \Pi_{\nu)\lambda} - \frac{1}{2} \partial_\lambda \Pi_{\sigma\nu} \right) - \Pi^{\rho\lambda} T_\nu \mathcal{K}_{\sigma\lambda} \right) \mathcal{K}_{\mu\rho}. \tag{A.1d}$$

A.2. Curvature identities

In Section 4.5, we present some identities for Carroll curvature which will be derived here. Starting with the curvature terms stated in (4.39):

$$\overset{(-4)}{R}_{\mu\nu} = -\overset{(-2)}{C}{}^\rho_{\mu\sigma} \overset{(-2)}{C}{}^\sigma_{\rho\nu} = -V^\rho \underbrace{\mathcal{K}_{\mu\sigma} V^\sigma}_{=0} \mathcal{K}_{\rho\nu} = 0, \tag{A.2a}$$

$$\begin{aligned} \overset{(-2)}{R}_{\mu\nu} &= \partial_\rho \overset{(-2)}{C}{}^\rho_{\mu\nu} - \tilde{C}^\sigma_{\rho\nu} \overset{(-2)}{C}{}^\rho_{\mu\sigma} + \tilde{C}^\rho_{\rho\sigma} \overset{(-2)}{C}{}^\sigma_{\mu\nu} - \tilde{C}^\sigma_{\mu\rho} \overset{(-2)}{C}{}^\rho_{\sigma\nu} + 2\, S^\rho_{[\rho|\sigma|} \overset{(-2)}{C}{}^\sigma_{\mu]\nu} \\ &= \overset{(\tilde{C})}{\nabla}_\rho \overset{(-2)}{C}{}^\rho_{\mu\nu} - 2\, \tilde{C}^\sigma_{[\mu\rho]} \overset{(-2)}{C}{}^\rho_{\sigma\nu} + 2\, S^\rho_{[\rho|\sigma|} \overset{(-2)}{C}{}^\sigma_{\mu]\nu} \\ &= -V^\rho \overset{(\tilde{C})}{\nabla}_\rho \mathcal{K}_{\mu\nu} - 2\, \Pi^{\sigma\lambda} T_{[\rho} \mathcal{K}_{\mu]\lambda} V^\rho \mathcal{K}_{\sigma\nu} - 2\, \Pi^{\rho\lambda} T_\sigma \mathcal{K}_{[\rho|\lambda|} V^\sigma \mathcal{K}_{\mu]\nu} \\ &= -V^\rho \overset{(\tilde{C})}{\nabla}_\rho \mathcal{K}_{\mu\nu} + \cancel{\Pi^{\sigma\lambda} \mathcal{K}_{\mu\lambda} \mathcal{K}_{\sigma\nu}} + \Pi^{\rho\lambda} \mathcal{K}_{\rho\lambda} \mathcal{K}_{\mu\nu} - \cancel{\Pi^{\rho\lambda} \mathcal{K}_{\mu\lambda} \mathcal{K}_{\rho\nu}} \\ &= -V^\rho \overset{(\tilde{C})}{\nabla}_\rho \mathcal{K}_{\mu\nu} + \mathcal{K} \mathcal{K}_{\mu\nu}, \end{aligned} \tag{A.2b}$$

$$\begin{aligned} \overset{(0)}{R}_{\mu\nu} &= \overset{(\tilde{C})}{R}_{\mu\nu} + 2\, \partial_{[\rho} S^\rho_{\mu]\nu} - \overset{(-2)}{C}{}^\rho_{\mu\sigma} \overset{(2)}{C}{}^\sigma_{\rho\nu} + 2\, \tilde{C}^\rho_{[\rho|\sigma|} S^\sigma_{\mu]\nu} + 2\, S^\rho_{[\rho|\sigma|} \tilde{C}^\sigma_{\mu]\nu} - \overset{(2)}{C}{}^\rho_{\mu\sigma} \overset{(-2)}{C}{}^\sigma_{\rho\nu} \\ &= \overset{(\tilde{C})}{R}_{\mu\nu} + 2 \overset{(\tilde{C})}{\nabla}_{[\rho} S^\rho_{\mu]\nu} + 2\, \tilde{C}^\sigma_{[\rho\mu]} S^\rho_{\sigma\nu} - 2\, \overset{(-2)}{C}{}^\rho_{\sigma(\mu} \overset{(2)}{C}{}^\sigma_{\nu)\rho}, \end{aligned} \tag{A.2c}$$

$$\begin{aligned} \overset{(2)}{R}_{\mu\nu} &= 2\, \partial_{[\rho} \overset{(2)}{C}{}^\rho_{\mu]\nu} + \tilde{C}^\rho_{\rho\sigma} \overset{(2)}{C}{}^\sigma_{\mu\nu} - \tilde{C}^\sigma_{\mu\rho} \overset{(2)}{C}{}^\rho_{\sigma\nu} - \tilde{C}^\sigma_{\rho\nu} \overset{(2)}{C}{}^\rho_{\mu\sigma} - \overset{(2)}{C}{}^\rho_{\mu\sigma} S^\sigma_{\rho\nu} \\ &= \overset{(\tilde{C})}{\nabla}_\rho \overset{(2)}{C}{}^\rho_{\mu\nu} + 2\, \tilde{C}^\sigma_{[\rho\mu]} \overset{(2)}{C}{}^\rho_{\sigma\nu} - \partial_\mu \underbrace{\overset{(2)}{C}{}^\rho_{\rho\nu}}_{=0} - \overset{(2)}{C}{}^\rho_{\mu\sigma} S^\sigma_{\rho\nu} \\ &= \overset{(\tilde{C})}{\nabla}_\rho \overset{(2)}{C}{}^\rho_{\mu\nu} - \Pi^{\sigma\lambda} T_{[\mu} \mathcal{K}_{\rho]\lambda} \Pi^{\rho\gamma} \Big(\underbrace{T_\sigma}_{=0} T_{\gamma\nu} + T_\nu T_{\gamma\sigma} \Big) \\ &\quad - \frac{1}{2} \Pi^{\rho\gamma} \Big(T_\mu T_{\gamma\sigma} + \underbrace{T_\sigma}_{=0} T_{\gamma\mu} \Big) \Pi^{\sigma\lambda} T_\nu \mathcal{K}_{\rho\lambda} \\ &= \frac{1}{2} \overset{(\tilde{C})}{\nabla}_\rho \left(\Pi^{\rho\sigma} \left(T_\mu T_{\sigma\nu} + T_\nu T_{\sigma\mu} \right) \right) - T_\mu T_\nu \underbrace{\Pi^{\sigma\lambda} \Pi^{\rho\gamma} \mathcal{K}_{\rho\lambda} T_{\gamma\sigma}}_{=0}, \end{aligned} \tag{A.2d}$$

$$\begin{aligned} \overset{(4)}{R}_{\mu\nu} &= -\overset{(2)}{C}{}^\rho_{\mu\sigma} \overset{(2)}{C}{}^\sigma_{\rho\nu} = -\frac{1}{4} \Pi^{\rho\lambda} \Big(T_\mu T_{\lambda\sigma} + \underbrace{T_\sigma}_{=0} T_{\lambda\mu} \Big) \Pi^{\sigma\gamma} \Big(\underbrace{T_\rho}_{=0} T_{\gamma\nu} + T_\nu T_{\gamma\rho} \Big) \\ &= -\frac{1}{4} \Pi^{\rho\lambda} T_\mu T_{\lambda\sigma} \Pi^{\sigma\gamma} T_\nu T_{\gamma\rho}. \end{aligned} \tag{A.2e}$$

We then move onto (4.44)

$$\left(\frac{1}{c^2} \Pi^{\mu\nu} - \frac{1}{c^4} V^\mu V^\nu \right) \overset{(-2)}{R}_{\mu\nu} = \left(\frac{1}{c^2} \Pi^{\mu\nu} - \frac{1}{c^4} V^\mu V^\nu \right) \left(-V^\rho \overset{(\tilde{C})}{\nabla}_\rho \mathcal{K}_{\mu\nu} + \mathcal{K} \mathcal{K}_{\mu\nu} \right)$$

$$
\begin{aligned}
&= -\frac{1}{c^2}\left(\Pi^{\mu\nu}V^\rho \overset{(\tilde{C})}{\nabla}_\rho \mathcal{K}_{\mu\nu} - \mathcal{K}^2\right) + \frac{1}{c^4}\underbrace{V^\mu V^\nu V^\rho \overset{(\tilde{C})}{\nabla}_\rho \mathcal{K}_{\mu\nu}}_{=0} \\
&= -\frac{1}{c^2}\left(\overset{(\tilde{C})}{\nabla}_\rho\left(\Pi^{\mu\nu}V^\rho \mathcal{K}_{\mu\nu}\right) - \overset{(\tilde{C})}{\nabla}_\rho(\Pi^{\mu\nu}V^\rho)\mathcal{K}_{\mu\nu} - \mathcal{K}^2\right) \\
&= -\frac{1}{c^2}\left(\frac{1}{E}\partial_\rho(E\mathcal{K}V^\rho) - \cancel{\mathcal{K}^2 T_\rho V^\rho} - V^\rho \mathcal{K}_{\mu\nu}\overset{(\tilde{C})}{\nabla}_\rho \Pi^{\mu\nu} - \cancel{\mathcal{K}^2}\right) \\
&= -\frac{1}{c^2}\left(\frac{1}{E}\partial_\rho(E\mathcal{K}V^\rho)\right. \\
&\qquad \left. + V^\rho \underbrace{\mathcal{K}_{\mu\nu}V^{(\mu}}_{=0}\Pi^{\nu)\sigma}T_{\sigma\lambda}\left(\delta^\lambda_\rho - T_\rho V^\lambda\right)\Pi^{\mu\nu}\right), \qquad \text{(A.3a)}
\end{aligned}
$$

$$
\begin{aligned}
\left(\Pi^{\mu\nu} - \frac{1}{c^2}V^\mu V^\nu\right)\overset{(0)}{R}_{\mu\nu} &= \left(\Pi^{\mu\nu} - \frac{1}{c^2}V^\mu V^\nu\right)\left(\overset{(\tilde{C})}{R}_{\mu\nu} + 2\,\overset{(\tilde{C})}{\nabla}_{[\rho}S^\rho{}_{\mu]\nu} + 2\,\tilde{C}^\sigma_{[\rho\mu]}S^\rho{}_{\sigma\nu}\right. \\
&\qquad \left. - 2\,\overset{(-2)}{C}{}^\rho{}_{\sigma(\mu}\overset{(2)}{C}{}^\sigma{}_{\nu)\rho}\right) \\
&= \left(\Pi^{\mu\nu} - \frac{1}{c^2}V^\mu V^\nu\right)\left(\overset{(\tilde{C})}{R}_{\mu\nu} + 2\,\overset{(\tilde{C})}{\nabla}_{[\rho}\left(\Pi^{\rho\lambda}T_{|\nu|}\mathcal{K}_{\mu]\lambda}\right)\right. \\
&\qquad - 2\,\Pi^{\sigma\lambda}T_{[\mu}\mathcal{K}_{\rho]\lambda}\Pi^{\rho\gamma}T_\nu \mathcal{K}_{\sigma\gamma} \\
&\qquad \left. + V^\rho \mathcal{K}_{\sigma\mu}\Pi^{\sigma\lambda}\left(T_\nu T_{\lambda\rho} + T_\rho T_{\lambda\nu}\right)\right) \\
&= \Pi^{\mu\nu}\left(\overset{(\tilde{C})}{R}_{\mu\nu} + 2\,\Pi^{\rho\lambda}\mathcal{K}_{\lambda[\mu}\overset{(\tilde{C})}{\nabla}_{\rho]}T_\nu - \mathcal{K}_{\sigma\mu}\Pi^{\sigma\lambda}T_{\lambda\nu}\right) \\
&\quad - \frac{1}{c^2}\left(\overset{(\tilde{C})}{\nabla}_\rho\left(V^\nu \Pi^{\rho\lambda}T_\nu \underbrace{V^\mu \mathcal{K}_{\mu\lambda}}_{=0}\right) - \overset{(\tilde{C})}{\nabla}_\mu\left(V^\mu V^\nu \Pi^{\rho\lambda}T_\nu \mathcal{K}_{\rho\lambda}\right)\right) \\
&\quad + \frac{1}{c^2}\,\Pi^{\sigma\lambda}\mathcal{K}_{\rho\lambda}\Pi^{\rho\gamma}\mathcal{K}_{\sigma\gamma} \\
&= \Pi^{\mu\nu}\left(\overset{(\tilde{C})}{R}_{\mu\nu} + \Pi^{\rho\lambda}\mathcal{K}_{\lambda\mu}\left(\cancel{T_{\rho\nu}} - V^\gamma\left(\underbrace{T_\rho}_{=0}T_{\gamma\nu} + \underbrace{T_\nu}_{=0}T_{\gamma\rho}\right)\right)\right. \\
&\qquad - \Pi^{\rho\lambda}\mathcal{K}_{\lambda\rho}\left(\underbrace{T_{\mu\nu}}_{=0} - V^\gamma\left(\underbrace{T_\mu}_{=0}T_{\gamma\nu} + \underbrace{T_\nu}_{=0}T_{\gamma\mu}\right)\right) \\
&\qquad \left. - \cancel{\mathcal{K}_{\sigma\mu}\Pi^{\sigma\lambda}T_{\lambda\nu}}\right) \\
&\quad - \frac{1}{c^2}\overset{(\tilde{C})}{\nabla}_\mu\left(V^\mu \Pi^{\rho\lambda}\mathcal{K}_{\rho\lambda}\right) + \frac{1}{c^2}\,\mathcal{K}^{\sigma\gamma}\mathcal{K}_{\sigma\gamma} \\
&= \Pi^{\mu\nu}\overset{(\tilde{C})}{R}_{\mu\nu} + \frac{1}{c^2}\left(-\frac{1}{E}\partial_\mu(E\mathcal{K}V^\mu) + \cancel{T_\mu V^\mu}\cancel{\mathcal{K}^2} + \mathcal{K}^{\sigma\gamma}\mathcal{K}_{\sigma\gamma}\right), \qquad \text{(A.3b)}
\end{aligned}
$$

$$
\left(c^2\Pi^{\mu\nu} - V^\mu V^\nu\right)\overset{(2)}{R}_{\mu\nu} = \left(c^2\Pi^{\mu\nu} - V^\mu V^\nu\right)\overset{(\tilde{C})}{\nabla}_\rho\left(\frac{1}{2}\Pi^{\rho\sigma}\left(T_\mu T_{\sigma\nu} + T_\nu T_{\sigma\mu}\right)\right)
$$

$$
\begin{aligned}
&= c^2\Bigg(\underbrace{\Pi^{\mu\nu}T_\mu}_{=0} T_{\sigma\nu}\overset{(\check{C})}{\nabla}_\rho\Pi^{\rho\sigma} + \Pi^{\mu\nu}\Pi^{\rho\sigma}\overset{(\check{C})}{\nabla}_\rho\big(T_\mu T_{\sigma\nu}\big)\Bigg) \\
&\quad - \overset{(\check{C})}{\nabla}_\rho\big(V^\mu\Pi^{\rho\sigma}T_{\mu\sigma}\big) \\
&= \frac{c^2}{2}\Pi^{\mu\nu}\Pi^{\rho\sigma}\Bigg(T_{\rho\mu} - V^\lambda\Big(\underbrace{T_\rho}_{=0} T_{\lambda\mu} + \underbrace{T_\mu}_{=0} T_{\lambda\rho}\Big)\Bigg)T_{\sigma\nu} \\
&\quad - \Bigg(\frac{1}{E}\partial_\rho\big(EV^\mu\Pi^{\rho\sigma}T_{\mu\sigma}\big) - \mathcal{K}V^\mu \underbrace{T_\rho\Pi^{\rho\sigma}}_{=0} T_{\mu\sigma}\Bigg) \\
&= \frac{c^2}{2}\Pi^{\mu\nu}\Pi^{\rho\sigma}T_{\rho\mu}T_{\sigma\nu} - \frac{1}{E}\partial_\rho\big(EV^\mu\Pi^{\rho\sigma}T_{\mu\sigma}\big), \qquad \text{(A.3c)}
\end{aligned}
$$

$$
\begin{aligned}
-c^2V^\mu V^\nu\overset{(4)}{R}_{\mu\nu} &= -\frac{c^2}{4}V^\mu V^\nu\Pi^{\rho\sigma}T_\mu T_{\sigma\lambda}\Pi^{\lambda\gamma}T_\nu T_{\rho\gamma} \\
&= -\frac{c^2}{4}\Pi^{\rho\sigma}\Pi^{\lambda\gamma}T_{\sigma\lambda}T_{\rho\gamma}. \qquad \text{(A.3d)}
\end{aligned}
$$

B. Expansion of vielbeine in all powers of c

Here we will show the derivation of the expansion for all powers of c for the remaining vielbeine from Section 5.1.1. We use the definitions from (4.10) with $N = 1$. Beginning with writing the higher order terms of $\Pi^{\mu\nu}$ in the following manner:

$$\begin{aligned}
\Pi^{\mu\nu} &= \delta^{ab} E^{\mu}{}_{a} E^{\nu}{}_{b} \\
&= \delta^{ab}\left(e_{(0)}{}^{\mu}{}_{a} + \sum_{j=1}^{\infty} c^{j} e_{(j)}{}^{\mu}{}_{a}\right)\left(e_{(0)}{}^{\nu}{}_{b} + \sum_{j=1}^{\infty} c^{j} e_{(j)}{}^{\nu}{}_{b}\right) \\
&= \delta^{ab}\Big[e_{(0)}{}^{\mu}{}_{a} e_{(0)}{}^{\nu}{}_{b} + c\left(e_{(0)}{}^{\mu}{}_{a} e_{(1)}{}^{\nu}{}_{b} + e_{(1)}{}^{\mu}{}_{a} e_{(0)}{}^{\nu}{}_{b}\right) \\
&\qquad + c^{2}\left(e_{(0)}{}^{\mu}{}_{a} e_{(2)}{}^{\nu}{}_{b} + e_{(1)}{}^{\mu}{}_{a} e_{(1)}{}^{\nu}{}_{b} + e_{(2)}{}^{\mu}{}_{a} e_{(0)}{}^{\nu}{}_{b}\right) + \mathcal{O}\left(c^{3}\right)\Big],
\end{aligned} \tag{B.1}$$

where we can just read off the higher order vielbeine $h^{\mu\nu}_{(j)}$ for $j \geq 1$. Now we write out the completeness/orthogonality relations up to order c^2:

$$\begin{aligned}
V^{\mu} T_{\mu} &= v^{\mu}_{(0)} \tau^{(0)}_{\mu} + c\left(v^{\mu}_{(1)} \tau^{(0)}_{\mu} + v^{\mu}_{(0)} \tau^{(1)}_{\mu}\right) \\
&\quad + c^{2}\left(v^{\mu}_{(2)} \tau^{(0)}_{\mu} + v^{\mu}_{(0)} \tau^{(2)}_{\mu} + v^{\mu}_{(1)} \tau^{(1)}_{\mu}\right) + \mathcal{O}\left(c^{3}\right) = -1,
\end{aligned} \tag{B.2a}$$

$$\begin{aligned}
\Pi^{\mu\rho} \Pi_{\rho\nu} &= h^{\mu\rho}_{(0)} h^{(0)}_{\rho\nu} + c\left(h^{\mu\rho}_{(0)} h^{(1)}_{\rho\nu} + h^{\mu\rho}_{(1)} h^{(0)}_{\rho\nu}\right) \\
&\quad + c^{2}\left(h^{\mu\rho}_{(0)} h^{(2)}_{\rho\nu} + h^{\mu\rho}_{(2)} h^{(0)}_{\rho\nu} + h^{\mu\rho}_{(1)} h^{(1)}_{\rho\nu}\right) + \mathcal{O}\left(c^{3}\right) \\
&= \delta^{\mu}_{\nu} + V^{\mu} T_{\nu} \\
&= \delta^{\mu}_{\nu} + v^{\mu}_{(0)} \tau^{(0)}_{\nu} + c\left(v^{\mu}_{(1)} \tau^{(0)}_{\nu} + v^{\mu}_{(0)} \tau^{(1)}_{\nu}\right) \\
&\quad + c^{2}\left(v^{\mu}_{(2)} \tau^{(0)}_{\nu} + v^{\mu}_{(0)} \tau^{(2)}_{\nu} + v^{\mu}_{(1)} \tau^{(1)}_{\nu}\right) + \mathcal{O}\left(c^{3}\right),
\end{aligned} \tag{B.2b}$$

$$\begin{aligned}
E^{\mu}{}_{a} T_{\mu} &= e_{(0)}{}^{\mu}{}_{a} \tau^{(0)}_{\mu} + c\left(e_{(0)}{}^{\mu}{}_{a} \tau^{(1)}_{\mu} + e_{(1)}{}^{\mu}{}_{a} \tau^{(0)}_{\mu}\right) \\
&\quad + c^{2}\left(e_{(0)}{}^{\mu}{}_{a} \tau^{(2)}_{\mu} + e_{(2)}{}^{\mu}{}_{a} \tau^{(0)}_{\mu} + e_{(1)}{}^{\mu}{}_{a} \tau^{(1)}_{\mu}\right) + \mathcal{O}\left(c^{3}\right) = 0,
\end{aligned} \tag{B.2c}$$

$$\begin{aligned}
E^{\mu}{}_{a} E_{\mu}{}^{b} &= e_{(0)}{}^{\mu}{}_{a} e^{(0)}{}_{\mu}{}^{b} + c\left(e_{(0)}{}^{\mu}{}_{a} e^{(1)}{}_{\mu}{}^{b} + e_{(1)}{}^{\mu}{}_{a} e^{(0)}{}_{\mu}{}^{b}\right) \\
&\quad + c^{2}\left(e_{(0)}{}^{\mu}{}_{a} e^{(2)}{}_{\mu}{}^{b} + e_{(2)}{}^{\mu}{}_{a} e^{(0)}{}_{\mu}{}^{b} + e_{(1)}{}^{\mu}{}_{a} e^{(1)}{}_{\mu}{}^{b}\right) + \mathcal{O}\left(c^{3}\right) \\
&= \delta^{b}_{a} + c\left(e_{(0)}{}^{\mu}{}_{a} e^{(1)}{}_{\mu}{}^{b} + e_{(1)}{}^{\mu}{}_{a} e^{(0)}{}_{\mu}{}^{b}\right) \\
&\quad + c^{2}\left(e_{(0)}{}^{\mu}{}_{a} e^{(2)}{}_{\mu}{}^{b} + e_{(2)}{}^{\mu}{}_{a} e^{(0)}{}_{\mu}{}^{b} + e_{(1)}{}^{\mu}{}_{a} e^{(1)}{}_{\mu}{}^{b}\right) + \mathcal{O}\left(c^{3}\right),
\end{aligned} \tag{B.2d}$$

where we have imposed that they hold at leading order, i.e. $\tau^{(0)}_{\mu} v^{\mu}_{(0)} = -1$, $h^{\mu\rho}_{(0)} h^{(0)}_{\rho\nu} = \delta^{\mu}_{\nu} + v^{\mu}_{(0)} \tau^{(0)}_{\nu}$ and $e_{(0)}{}^{\mu}{}_{a} e^{(0)}{}_{\mu}{}^{b} = \delta^{b}_{a}$. Then, we want to find constraint equations for $J \geq 1$:

$$\sum_{j=0}^{J} \tau^{(j)}_{\mu} v^{\mu}_{(J-j)} = 0, \qquad \sum_{j=0}^{J} \tau^{(j)}_{\mu} e_{(J-j)}{}^{\mu}{}_{a} = 0, \tag{B.3a}$$

$$\sum_{j=0}^{J} e^{(j)}{}_{\mu}{}^{a} v^{\mu}_{(J-j)} = 0, \qquad \sum_{j=0}^{J} e^{(j)}{}_{\mu}{}^{a} e_{(J-j)}{}^{\mu}{}_{b} = 0 \tag{B.3b}$$

$$\sum_{j=0}^{J} \tau^{(j)}_{\mu} h^{\mu\nu}_{(J-j)} = 0, \qquad \sum_{j=0}^{J} v^{\mu}_{(J-j)} h^{(j)}_{\mu\nu} = 0 \tag{B.3c}$$

$$\sum_{j=0}^{J} h^{(j)}_{\rho\nu} h^{\mu\rho}_{(J-j)} - \sum_{j=0}^{J} \tau^{(j)}_{\nu} v^{\mu}_{(J-j)} = 0, \tag{B.3d}$$

where J is the order in c. We can now rewrite and contract the equations

$$\tau^{(0)}_{\mu} \sum_{j=0}^{J} \tau^{(j)}_{\rho} v^{\rho}_{(J-j)} = \tau^{(0)}_{\mu} \sum_{j=0}^{J-1} \tau^{(j)}_{\rho} v^{\rho}_{(J-j)} + \tau^{(J)}_{\rho} \tau^{(0)}_{\mu} v^{\rho}_{(0)} = 0, \tag{B.4a}$$

$$e^{(0)}{}_{\mu}{}^{a} \sum_{j=0}^{J} \tau^{(j)}_{\rho} e_{(J-j)}{}^{\rho}{}_{a} = e^{(0)}{}_{\mu}{}^{a} \sum_{j=0}^{J-1} \tau^{(j)}_{\rho} e_{(J-j)}{}^{\rho}{}_{a} + \tau^{(J)}_{\rho} \underbrace{e^{(0)}{}_{\mu}{}^{a} e_{(0)}{}^{\rho}{}_{a}}_{\delta^{\rho}_{\mu} + v^{\rho}_{(0)} \tau^{(0)}_{\mu}} = 0, \tag{B.4b}$$

$$\tau^{(0)}_{\alpha} \sum_{j=0}^{J} v^{\mu}_{(J-j)} h^{(j)}_{\mu\nu} = \tau^{(0)}_{\alpha} v^{\mu}_{(0)} h^{(J)}_{\mu\nu} + \tau^{(0)}_{\alpha} \sum_{j=0}^{J-1} v^{\mu}_{(J-j)} h^{(j)}_{\mu\nu} = 0, \tag{B.4c}$$

$$\begin{aligned} h^{(0)}_{\alpha\nu} \sum_{j=0}^{J} h^{(j)}_{\mu\rho} h^{\rho\nu}_{(J-j)} - h^{(0)}_{\alpha\nu} \sum_{j=0}^{J} \tau^{(j)}_{\mu} v^{\nu}_{(J-j)} &= h^{(J)}_{\mu\rho} \underbrace{h^{(0)}_{\alpha\nu} h^{\rho\nu}_{(0)}}_{\delta^{\rho}_{\alpha} + v^{\rho}_{(0)} \tau^{(0)}_{\alpha}} + h^{(0)}_{\alpha\nu} \sum_{j=0}^{J-1} h^{(j)}_{\mu\rho} h^{\rho\nu}_{(J-j)} \\ &\quad - \underbrace{h^{(0)}_{\alpha\nu} v^{\nu}_{(0)}}_{=0} \tau^{J}_{\mu} - h^{(0)}_{\alpha\nu} \sum_{j=0}^{J-1} \tau^{(j)}_{\mu} v^{\nu}_{(J-j)} = 0, \end{aligned} \tag{B.4d}$$

and combine them to arrive at

$$\tau^{(J)}_{\mu} = \tau^{(0)}_{\mu} \sum_{j=0}^{J-1} \tau^{(j)}_{\rho} v^{\rho}_{(J-n)} - e^{(0)}{}_{\mu}{}^{a} \sum_{j=0}^{J-1} \tau^{(j)}_{\rho} e_{(J-j)}{}^{\rho}{}_{a}, \tag{B.5a}$$

$$e^{(J)}{}_{\mu}{}^{a} = \tau^{(0)}_{\mu} \sum_{j=0}^{J-1} e^{(j)}{}_{\rho}{}^{a} v^{\rho}_{(J-j)} - e^{(0)}{}_{\mu}{}^{b} \sum_{j=0}^{J-1} e^{(j)}{}_{\rho}{}^{a} e_{(J-j)}{}^{\rho}{}_{b}, \tag{B.5b}$$

$$\begin{aligned} h^{(J)}_{\mu\nu} &= \tau^{(0)}_{\mu} \sum_{j=0}^{J-1} v^{\rho}_{(J-j)} h^{(j)}_{\rho\nu} + h^{(0)}_{\mu\rho} \sum_{j=0}^{J-1} \tau^{(j)}_{\nu} v^{\rho}_{(J-j)} - h^{(0)}_{\mu\rho} \sum_{j=0}^{J-1} h^{(j)}_{\nu\sigma} h^{\sigma\rho}_{(J-j)} \\ &= \delta_{ab} \left(\tau^{(0)}_{\mu} \sum_{j=0}^{J-1} v^{\rho}_{(J-j)} e^{(j)}{}_{\rho}{}^{a} e^{(j)}{}_{\nu}{}^{b} \right. \\ &\qquad \left. + e^{(0)}{}_{\mu}{}^{a} e^{(0)}{}_{\rho}{}^{b} \sum_{j=0}^{J-1} \left(\tau^{(j)}_{\nu} v^{\rho}_{(J-j)} - \delta_{cd} e^{(j)}{}_{\nu}{}^{c} e^{(j)}{}_{\sigma}{}^{d} h^{\sigma\rho}_{(J-j)} \right) \right), \end{aligned} \tag{B.5c}$$

where we have expressed the higher orders of the vielbeine in terms of the lower ones.

Solving the previous equations along with (B.1) iteratively, we start with $J = 1$ to find

$$
\begin{aligned}
\tau^{(1)}_\mu &= \tau^{(0)}_\mu \tau^{(0)}_\rho v^\rho_{(1)} - e^{(0)}{}_\mu{}^a \tau^{(0)}_\rho e_{(1)}{}^\rho{}_a, && \text{(B.6a)}\\
e^{(1)}{}_\mu{}^a &= \tau^{(0)}_\mu e^{(0)}{}_\rho{}^a v^\rho_{(1)} - e^{(0)}{}_\mu{}^b e^{(0)}{}_\rho{}^a e_{(1)}{}^\rho{}_b, && \text{(B.6b)}\\
h^{\mu\nu}_{(1)} &= \delta^{ab}\Big(e_{(0)}{}^\mu{}_a e_{(1)}{}^\nu{}_b + e_{(1)}{}^\mu{}_a e_{(0)}{}^\nu{}_b\Big), && \text{(B.6c)}\\
h^{(1)}_{\mu\nu} &= \delta_{ab}\Big[\tau^{(0)}_\mu v^\rho_{(1)} e^{(0)}{}_\rho{}^a e^{(0)}{}_\nu{}^b + e^{(0)}{}_\mu{}^a e^{(0)}{}_\rho{}^b \tau^{(0)}_\nu v^\rho_{(1)} \\
&\qquad - e^{(0)}{}_\mu{}^a e^{(0)}{}_\rho{}^b \delta_{cd} e^{(0)}{}_\nu{}^c e^{(0)}{}_\sigma{}^d \delta^{ef}\Big(e_{(0)}{}^\sigma{}_e e_{(1)}{}^\rho{}_f + e_{(1)}{}^\sigma{}_e e_{(0)}{}^\rho{}_f\Big)\Big] \\
&= 2\,\delta_{ab} e^{(0)}{}_{(\mu}{}^a \tau^{(0)}_{\nu)} v^\rho_{(1)} e^{(0)}{}_\rho{}^b - \delta_{ab} e^{(0)}{}_\mu{}^a e^{(0)}{}_\rho{}^b e^{(0)}{}_\nu{}^c e_{(1)}{}^\rho{}_c \\
&\quad - \delta_{cd} e^{(0)}{}_\mu{}^a e^{(0)}{}_\nu{}^c e^{(0)}{}_\sigma{}^d e_{(1)}{}^\sigma{}_a \\
&= 2\,\delta_{ab} e^{(0)}{}_\rho{}^b e^{(0)}{}_{(\mu}{}^a \Big(\tau^{(0)}_{\nu)} v^\rho_{(1)} - e^{(0)}{}_{\nu)}{}^c e_{(1)}{}^\rho{}_c\Big), && \text{(B.6d)}
\end{aligned}
$$

and then for $J = 2$:

$$
\begin{aligned}
\tau^{(2)}_\mu &= \tau^{(0)}_\mu\Big(\tau^{(0)}_\rho v^\rho_{(2)} + \tau^{(1)}_\rho v^\rho_{(1)}\Big) - e^{(0)}{}_\mu{}^a\Big(\tau^{(0)}_\rho e_{(2)}{}^\rho{}_a + \tau^{(1)}_\rho e_{(1)}{}^\rho{}_a\Big) \\
&= \tau^{(0)}_\rho\Big(\tau^{(0)}_\mu v^\rho_{(2)} - e^{(0)}{}_\mu{}^a e_{(2)}{}^\rho{}_a\Big) \\
&\quad + \Big(\tau^{(0)}_\rho \tau^{(0)}_\sigma v^\sigma_{(1)} - e^{(0)}{}_\rho{}^a \tau^{(0)}_\sigma e_{(1)}{}^\sigma{}_a\Big)\Big(\tau^{(0)}_\mu v^\rho_{(1)} - e^{(0)}{}_\mu{}^a e_{(1)}{}^\rho{}_a\Big), && \text{(B.7a)}\\
e^{(2)}{}_\mu{}^a &= \tau^{(0)}_\mu\Big(e^{(0)}{}_\rho{}^a v^\rho_{(2)} + e^{(1)}{}_\rho{}^a v^\rho_{(1)}\Big) - e^{(0)}{}_\mu{}^b\Big(e^{(0)}{}_\rho{}^a e_{(2)}{}^\rho{}_b + e^{(1)}{}_\rho{}^a e_{(1)}{}^\rho{}_b\Big) \\
&= e^{(0)}{}_\rho{}^a\Big(\tau^{(0)}_\mu v^\rho_{(2)} - e^{(0)}{}_\mu{}^b e_{(2)}{}^\rho{}_b\Big) \\
&\quad + \Big(\tau^{(0)}_\rho e^{(0)}{}_\sigma{}^a v^\sigma_{(1)} - e^{(0)}{}_\rho{}^b e^{(0)}{}_\sigma{}^a e_{(1)}{}^\sigma{}_b\Big)\Big(\tau^{(0)}_\mu v^\rho_{(1)} + e^{(0)}{}_\mu{}^b e_{(1)}{}^\rho{}_b\Big), && \text{(B.7b)}\\
h^{\mu\nu}_{(2)} &= \delta^{ab}\Big(e_{(0)}{}^\mu{}_a e_{(2)}{}^\nu{}_b + e_{(1)}{}^\mu{}_a e_{(1)}{}^\nu{}_b + e_{(2)}{}^\mu{}_a e_{(0)}{}^\nu{}_b\Big), && \text{(B.7c)}\\
h^{(2)}_{\mu\nu} &= \delta_{ab}\Big[\tau^{(0)}_\mu\Big(v^\rho_{(2)} e^{(0)}{}_\rho{}^a e^{(0)}{}_\nu{}^b + v^\rho_{(1)} e^{(1)}{}_\rho{}^a e^{(1)}{}_\nu{}^b\Big) + e^{(0)}{}_\mu{}^a e^{(0)}{}_\rho{}^b\Big(\tau^{(0)}_\nu v^\rho_{(2)} + \tau^{(1)}_\nu v^\rho_{(1)}\Big) \\
&\qquad - e^{(0)}{}_\mu{}^a e^{(0)}{}_\rho{}^b \delta_{cd}\Big(e^{(0)}{}_\nu{}^c e^{(0)}{}_\sigma{}^d \delta^{ef}\Big(e_{(0)}{}^\sigma{}_e e_{(2)}{}^\rho{}_f + e_{(1)}{}^\sigma{}_e e_{(1)}{}^\rho{}_f + e_{(2)}{}^\sigma{}_e e_{(0)}{}^\rho{}_f\Big) \\
&\qquad\qquad + e^{(1)}{}_\nu{}^c e^{(1)}{}_\sigma{}^d \delta^{ef}\Big(e_{(0)}{}^\sigma{}_e e_{(1)}{}^\rho{}_f + e_{(1)}{}^\sigma{}_e e_{(0)}{}^\rho{}_f\Big)\Big)\Big] \\
&= 2\,\delta_{ab} v^\rho_{(2)} e^{(0)}{}_\rho{}^a \tau^{(0)}_{(\mu} e^{(0)}{}_{\nu)}{}^b + \delta_{ab} v^\rho_{(1)}\Big(\tau^{(0)}_\mu e^{(1)}{}_\rho{}^a e^{(1)}{}_\nu{}^b + \tau^{(1)}_\nu e^{(0)}{}_\mu{}^a e^{(0)}{}_\rho{}^b\Big) \\
&\quad - 2\,\delta_{ab} e_{(2)}{}^\rho{}_c e^{(0)}{}_\rho{}^b e^{(0)}{}_{(\mu}{}^a e^{(0)}{}_{\nu)}{}^c - \delta_{ab} e^{(0)}{}_\mu{}^c e^{(1)}{}_\nu{}^a e^{(1)}{}_\sigma{}^b e_{(1)}{}^\sigma{}_c \\
&\quad - \delta_{ab} e^{(0)}{}_\mu{}^a e^{(0)}{}_\rho{}^b \delta_{cd} \delta^{ef} e_{(1)}{}^\rho{}_f\Big(e^{(0)}{}_\nu{}^c e^{(0)}{}_\sigma{}^d e_{(1)}{}^\sigma{}_e - e^{(1)}{}_\nu{}^c e^{(1)}{}_\sigma{}^d e_{(0)}{}^\sigma{}_e\Big) \\
&= 2\,\delta_{ab} v^\rho_{(2)} e^{(0)}{}_\rho{}^a \tau^{(0)}_{(\mu} e^{(0)}{}_{\nu)}{}^b - 2\,\delta_{ab} e_{(2)}{}^\rho{}_c e^{(0)}{}_\rho{}^b e^{(0)}{}_{(\mu}{}^a e^{(0)}{}_{\nu)}{}^c \\
&\quad + \delta_{ab} v^\rho_{(1)} \tau^{(0)}_\mu\Big(\tau^{(0)}_\rho e^{(0)}{}_\sigma{}^a v^\sigma_{(1)} \tau^{(0)}_\nu e^{(0)}{}_\alpha{}^b v^\alpha_{(1)} - e^{(0)}{}_\rho{}^c e^{(0)}{}_\sigma{}^a e_{(1)}{}^\sigma{}_c \tau^{(0)}_\nu e^{(0)}{}_\alpha{}^b v^\alpha_{(1)}
\end{aligned}
$$

$$
\begin{aligned}
&\qquad - \tau^{(0)}_{\rho} e_{(0)}{}_{\sigma}{}^{a} v^{\sigma}_{(1)} e_{(0)}{}_{\nu}{}^{d} e_{(0)}{}_{\alpha}{}^{b} e_{(1)}{}^{\alpha}{}_{d} + e_{(0)}{}_{\rho}{}^{c} e_{(0)}{}_{\sigma}{}^{a} e_{(1)}{}^{\sigma}{}_{c} e_{(0)}{}_{\nu}{}^{d} e_{(0)}{}_{\alpha}{}^{b} e_{(1)}{}^{\alpha}{}_{d} \Big) \\
&+ \delta_{ab} v^{\rho}_{(1)} \Big(\tau^{(0)}_{\nu} \tau^{(0)}_{\sigma} v^{\sigma}_{(1)} - e_{(0)}{}_{\nu}{}^{c} \tau^{(0)}_{\sigma} e_{(1)}{}^{\sigma}{}_{c} \Big) e_{(0)}{}_{\mu}{}^{a} e_{(0)}{}_{\rho}{}^{b} \\
&- \delta_{ab} e_{(0)}{}_{\mu}{}^{c} \Big(\tau^{(0)}_{\nu} e_{(0)}{}_{\rho}{}^{a} v^{\rho}_{(1)} \tau^{(0)}_{\sigma} e_{(0)}{}_{\alpha}{}^{b} v^{\alpha}_{(1)} - e_{(0)}{}_{\nu}{}^{d} e_{(0)}{}_{\rho}{}^{a} e_{(1)}{}^{\rho}{}_{d} \tau^{(0)}_{\sigma} e_{(0)}{}_{\alpha}{}^{b} v^{\alpha}_{(1)} \\
&\qquad - \tau^{(0)}_{\nu} e_{(0)}{}_{\rho}{}^{a} v^{\rho}_{(1)} e_{(0)}{}_{\sigma}{}^{e} e_{(0)}{}_{\alpha}{}^{b} e_{(1)}{}^{\alpha}{}_{e} \\
&\qquad + e_{(0)}{}_{\nu}{}^{d} e_{(0)}{}_{\rho}{}^{a} e_{(1)}{}^{\rho}{}_{d} e_{(0)}{}_{\sigma}{}^{e} e_{(0)}{}_{\alpha}{}^{b} e_{(1)}{}^{\alpha}{}_{e} \Big) e_{(1)}{}^{\sigma}{}_{c} \\
&- \delta_{ab} e_{(0)}{}_{\mu}{}^{a} e_{(0)}{}_{\rho}{}^{b} \delta_{cd} \delta^{ef} e_{(1)}{}^{\rho}{}_{f} e_{(0)}{}_{\nu}{}^{c} e_{(0)}{}_{\sigma}{}^{d} e_{(1)}{}^{\sigma}{}_{e} \\
&- \delta_{ab} e_{(0)}{}_{\mu}{}^{a} e_{(0)}{}_{\rho}{}^{b} \delta_{cd} \delta^{ef} e_{(1)}{}^{\rho}{}_{f} \Big(\tau^{(0)}_{\nu} e_{(0)}{}_{\alpha}{}^{c} v^{\alpha}_{(1)} - e_{(0)}{}_{\nu}{}^{g} e_{(0)}{}_{\alpha}{}^{c} e_{(1)}{}^{\alpha}{}_{g} \Big) e_{(0)}{}_{\beta}{}^{d} e_{(1)}{}^{\beta}{}_{e} \\
= {}& 2\, \delta_{ab} v^{\rho}_{(2)} e_{(0)}{}_{\rho}{}^{a} \tau^{(0)}_{(\mu} e_{(0)}{}_{\nu)}{}^{b} - 2\, \delta_{ab} e_{(2)}{}^{\rho}{}_{c} e_{(0)}{}_{\rho}{}^{b} e_{(0)}{}_{(\mu}{}^{a} e_{(0)}{}_{\nu)}{}^{c} \\
&+ \delta_{ab} v^{\rho}_{(1)} \Big[\tau^{(0)}_{\mu} e_{(0)}{}_{\sigma}{}^{a} e_{(0)}{}_{\alpha}{}^{b} \Big(\Big(\tau^{(0)}_{\nu} v^{\alpha}_{(1)} - e_{(0)}{}_{\nu}{}^{d} e_{(1)}{}^{\alpha}{}_{d} \Big) \Big(\tau^{(0)}_{\rho} v^{\sigma}_{(1)} - e_{(0)}{}_{\rho}{}^{c} e_{(1)}{}^{\sigma}{}_{c} \Big) \Big) \\
&\qquad + \tau^{(0)}_{\sigma} e_{(0)}{}_{\mu}{}^{a} e_{(0)}{}_{\rho}{}^{b} \Big(\tau^{(0)}_{\nu} v^{\sigma}_{(1)} - e_{(0)}{}_{\nu}{}^{c} e_{(1)}{}^{\sigma}{}_{c} \Big) \\
&\qquad - e_{(0)}{}_{\mu}{}^{c} \Big(\tau^{(0)}_{\nu} e_{(0)}{}_{\rho}{}^{a} \tau^{(0)}_{\sigma} e_{(0)}{}_{\alpha}{}^{b} v^{\alpha}_{(1)} - e_{(0)}{}_{\nu}{}^{d} e_{(0)}{}_{\alpha}{}^{a} e_{(1)}{}^{\alpha}{}_{d} \tau^{(0)}_{\sigma} e_{(0)}{}_{\rho}{}^{b} \\
&\qquad\qquad - \tau^{(0)}_{\nu} e_{(0)}{}_{\rho}{}^{a} e_{(0)}{}_{\sigma}{}^{e} e_{(0)}{}_{\alpha}{}^{b} e_{(1)}{}^{\alpha}{}_{e} \Big) e_{(1)}{}^{\sigma}{}_{c} \Big] \\
&- \delta_{ab} e_{(0)}{}_{\mu}{}^{a} e_{(0)}{}_{\rho}{}^{b} \delta_{cd} \delta^{ef} e_{(1)}{}^{\rho}{}_{f} e_{(0)}{}_{\nu}{}^{c} e_{(0)}{}_{\sigma}{}^{d} e_{(1)}{}^{\sigma}{}_{e} \\
&- \delta_{ab} e_{(0)}{}_{\mu}{}^{c} e_{(0)}{}_{\nu}{}^{d} e_{(0)}{}_{\rho}{}^{a} e_{(1)}{}^{\rho}{}_{d} e_{(0)}{}_{\sigma}{}^{e} e_{(0)}{}_{\alpha}{}^{b} e_{(1)}{}^{\alpha}{}_{e} e_{(1)}{}^{\sigma}{}_{c} \\
&- \delta_{ab} e_{(0)}{}_{\mu}{}^{a} e_{(0)}{}_{\rho}{}^{b} \delta_{cd} \delta^{ef} e_{(1)}{}^{\rho}{}_{f} \Big(\tau^{(0)}_{\nu} e_{(0)}{}_{\alpha}{}^{c} v^{\alpha}_{(1)} - e_{(0)}{}_{\nu}{}^{g} e_{(0)}{}_{\alpha}{}^{c} e_{(1)}{}^{\alpha}{}_{g} \Big) e_{(0)}{}_{\beta}{}^{d} e_{(1)}{}^{\beta}{}_{e} .
\end{aligned}
\tag{B.7d}
$$

C. Variational calculus

C.1. LO variational identities

In the following are detailed derivations of the variational identities presented in Section 5.2. We begin with one of the relations from (5.12)

$$\delta^\lambda_\mu = h_{\mu\rho}h^{\rho\lambda} - \tau_\mu v^\lambda, \tag{C.1}$$

and vary it:

$$\delta\left(\delta^\lambda_\mu\right) = \delta h_{\mu\rho}h^{\rho\lambda} + h_{\mu\rho}\delta h^{\rho\lambda} - \delta\tau_\mu v^\lambda - \tau_\mu \delta v^\lambda = 0. \tag{C.2}$$

We can then use the expression above to isolate both $\delta\tau_\mu$ or $\delta h_{\mu\nu}$:

$$\begin{aligned} -\tau_\lambda\delta\tau_\mu v^\lambda &= -\tau_\lambda\Big(\delta h_{\mu\rho}\underbrace{h^{\rho\lambda}}_{=0} + h_{\mu\rho}\delta h^{\rho\lambda} - \tau_\mu\delta v^\lambda\Big) \\ \Rightarrow \delta\tau_\mu &= -\tau_\lambda h_{\mu\rho}\delta h^{\rho\lambda} + \tau_\lambda\tau_\mu\delta v^\lambda, \qquad (C.3) \\ h_{\lambda\nu}\delta h_{\mu\rho}h^{\rho\lambda} &= -h_{\lambda\nu}\Big(h_{\mu\rho}\delta h^{\rho\lambda} - \delta\tau_\mu\underbrace{v^\lambda}_{=0} - \tau_\mu\delta v^\lambda\Big) \\ &= \left(\delta^\rho_\nu + v^\rho\tau_\nu\right)\delta h_{\mu\rho} \\ \Rightarrow \delta h_{\mu\nu} &= -h_{\mu\rho}h_{\nu\lambda}\delta h^{\rho\lambda} + 2\, h_{\lambda(\mu}\tau_{\nu)}\delta v^\lambda. \qquad (C.4) \end{aligned}$$

For e, we have:

$$\begin{aligned} \delta e &= \delta \det[(\tau_\mu, e_\mu{}^a)] \\ &= \det[(\tau_\mu, e_\mu{}^a)]\mathrm{Tr}\left[(v^\mu, e^\mu{}_a)\delta(\tau_\nu, e_\nu{}^a)\right] \\ &= e\left(-v^\mu\delta\tau_\mu + e^\mu{}_a\delta e_\mu{}^a\right) \\ &= e\left(-v^\mu\delta\tau_\mu + \frac{1}{2}h^{\mu\nu}\delta h_{\mu\nu}\right) \\ &= e\left(\tau_\mu\delta v^\mu + \frac{1}{2}h^{\mu\nu}\Big(-h_{\mu\rho}h_{\nu\lambda}\delta h^{\rho\lambda} + 2\, h_{\lambda(\mu}\underbrace{\tau_{\nu)}}_{=0}\delta v^\lambda\Big)\right) \\ &= e\left(\tau_\mu\delta v^\mu - \frac{1}{2}h_{\rho\lambda}\delta h^{\rho\lambda}\right), \end{aligned} \tag{C.5}$$

where we have used the expression for $\delta h_{\mu\nu}$ as well as (4.29).

For $\delta K_{\mu\nu}$ we show:

$$\begin{aligned} \delta K_{\mu\nu} &= -\frac{1}{2}\delta\left(\pounds_v h_{\mu\nu}\right) \\ &= -\frac{1}{2}\left(\pounds_{\delta v}h_{\mu\nu} + \pounds_v\delta h_{\mu\nu}\right) \\ &= -\frac{1}{2}\Big(\delta v^\sigma\underbrace{\tilde{\nabla}_\sigma h_{\mu\nu}}_{=0} + h_{\sigma\nu}\tilde{\nabla}_\mu\delta v^\sigma + h_{\mu\sigma}\tilde{\nabla}_\nu\delta v^\sigma + 2\,\delta v^\sigma\tilde{\Gamma}^\lambda_{[\sigma\mu]}h_{\lambda\nu} + 2\,\delta v^\sigma\tilde{\Gamma}^\lambda_{[\sigma\nu]}h_{\mu\lambda} \end{aligned}$$

$$
\begin{aligned}
&+ v^\sigma \tilde{\nabla}_\sigma \delta h_{\mu\nu} + \delta h_{\sigma\nu} \underbrace{\tilde{\nabla}_\mu v^\sigma}_{=0} + \delta h_{\mu\sigma} \underbrace{\tilde{\nabla}_\nu v^\sigma}_{=0} + 2\, v^\sigma \tilde{\Gamma}^\lambda_{[\sigma\mu]} \delta h_{\lambda\nu} + 2\, v^\sigma \tilde{\Gamma}^\lambda_{[\sigma\nu]} \delta h_{\mu\lambda} \Big) \\
&= -h_{\sigma(\nu} \tilde{\nabla}_{\mu)} \delta v^\sigma - h^{\lambda\gamma} \tau_\sigma K_{\gamma(\nu} h_{\mu)\lambda} \delta v^\sigma - \frac{1}{2} v^\sigma \tilde{\nabla}_\sigma \delta h_{\mu\nu} - h^{\lambda\gamma} \underbrace{v^\sigma \tau_\sigma}_{-1} K_{\gamma(\nu} \delta h_{\mu)\lambda} \\
&= -h_{\sigma(\nu} \tilde{\nabla}_{\mu)} \delta v^\sigma - \tau_\sigma K_{\mu\nu} \delta v^\sigma + h^{\lambda\gamma} K_{\gamma(\nu} \Big(- h_{\mu)\alpha} h_{\lambda\beta} \delta h^{\alpha\beta} + \underbrace{h_{\beta\mu)} \tau_\lambda}_{=0} \delta v^\beta + h_{\beta\lambda} \tau_{\mu)} \delta v^\beta \Big) \\
&\quad - \frac{1}{2} v^\sigma \Big(-h_{\mu\beta} h_{\nu\alpha} \tilde{\nabla}_\sigma \delta h^{\beta\alpha} + 2\, h_{\alpha(\mu} \tilde{\nabla}_\sigma \tau_{\nu)} \delta v^\alpha + 2\, h_{\alpha(\mu} \tau_{\nu)} \tilde{\nabla}_\sigma \delta v^\alpha \Big) \\
&= \Big(-\tau_\lambda K_{\mu\nu} + K_{\lambda(\nu} \tau_{\mu)} - \frac{1}{2} h_{\lambda\mu} v^\sigma \tau_{\sigma\nu} - \frac{1}{2} h_{\lambda\nu} v^\sigma \tau_{\sigma\mu} \Big) \delta v^\lambda - h_{\sigma(\nu} \tilde{\nabla}_{\mu)} \delta v^\sigma \\
&\quad - v^\sigma h_{\alpha(\mu} \tau_{\nu)} \tilde{\nabla}_\sigma \delta v^\alpha - K_{\lambda(\nu} h_{\mu)\rho} \delta h^{\rho\lambda} + \frac{1}{2} h_{\mu\rho} h_{\nu\lambda} v^\sigma \tilde{\nabla}_\sigma \delta h^{\rho\lambda},
\end{aligned} \tag{C.6}
$$

where after the second equality we have used the definition of the Lie derivative for non-zero torsion (2.5), after the third equality we have used (4.26):

$$
\tilde{\Gamma}^\lambda_{[\sigma\mu]} = h^{\lambda\gamma} \tau_{[\sigma} K_{\mu]\gamma}, \tag{C.7}
$$

and after the fourth equality we have inserted $\delta h_{\mu\nu}$. We has also used (4.31a) to simplify the term:

$$
\begin{aligned}
-h_{\lambda\mu} v^\sigma \tilde{\nabla}_\sigma \tau_\nu \delta v^\lambda &= -\frac{1}{2} h_{\lambda\mu} v^\sigma \Big(\tau_{\sigma\nu} - v^\gamma \Big(\tau_\sigma \tau_{\gamma\nu} + \tau_\nu \underbrace{\tau_{\gamma\sigma}}_{=0} \Big) \Big) \delta v^\lambda \\
&= -h_{\lambda\mu} v^\sigma \tau_{\sigma\nu} \delta v^\lambda.
\end{aligned} \tag{C.8}
$$

Moving on to δK, we have:

$$
\begin{aligned}
\delta K &= \delta h^{\mu\nu} K_{\mu\nu} + h^{\mu\nu} \delta K_{\mu\nu} \\
&= \delta h^{\mu\nu} K_{\mu\nu} + h^{\mu\nu} \Big(\Big(-\tau_\lambda K_{\mu\nu} + K_{\lambda(\nu} \underbrace{\tau_{\mu)}}_{=0} - \frac{1}{2} h_{\lambda\mu} v^\sigma \tau_{\sigma\nu} - \frac{1}{2} h_{\lambda\nu} v^\sigma \tau_{\sigma\mu} \Big) \delta v^\lambda \\
&\qquad - h_{\sigma(\nu} \tilde{\nabla}_{\mu)} \delta v^\sigma - \underbrace{v^\sigma h_{\alpha(\mu} \tau_{\nu)}}_{=0} \tilde{\nabla}_\sigma \delta v^\alpha \\
&\qquad - K_{\lambda(\nu} h_{\mu)\rho} \delta h^{\rho\lambda} + \frac{1}{2} h_{\mu\rho} h_{\nu\lambda} v^\sigma \tilde{\nabla}_\sigma \delta h^{\rho\lambda} \Big) \\
&= \cancel{\delta h^{\mu\nu} K_{\mu\nu}} - (\tau_\lambda K + v^\sigma \tau_{\sigma\lambda}) \delta v^\lambda - h^{\mu\nu} h_{\nu\sigma} \tilde{\nabla}_\mu \delta v^\sigma - \cancel{K_{\lambda\rho} \delta h^{\rho\lambda}} + \frac{1}{2} h_{\rho\lambda} v^\sigma \tilde{\nabla}_\sigma \delta h^{\rho\lambda},
\end{aligned} \tag{C.9}
$$

where we have used the previously derived $\delta K_{\mu\nu}$. Lastly, we use it once again for showing:

$$
\begin{aligned}
\delta\big(K^{\mu\nu} K_{\mu\nu}\big) &= 2\Big(K^\mu{}_\sigma K_{\mu\nu} \delta h^{\nu\sigma} + K^{\mu\nu} \delta K_{\mu\nu} \Big) \\
&= \cancel{2\, K^\mu{}_\sigma K_{\mu\nu} \delta h^{\nu\sigma}}
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\,K^{\mu\nu}\Big(\Big(-\tau_\lambda K_{\mu\nu} + K_{\lambda(\nu}\underbrace{\tau_{\mu)}}_{=0} - \frac{1}{2}h_{\lambda\mu}v^\sigma\tau_{\sigma\nu} - \frac{1}{2}h_{\lambda\nu}v^\sigma\tau_{\sigma\mu}\Big)\delta v^\lambda \\
&\qquad - h_{\sigma(\nu}\tilde{\nabla}_{\mu)}\delta v^\sigma - v^\sigma h_{\alpha(\mu}\underbrace{\tau_{\nu)}}_{=0}\tilde{\nabla}_\sigma\delta v^\alpha \\
&\qquad - \cancel{K_{\lambda(\nu}h_{\mu)\rho}\delta h^{\rho\lambda}} + \frac{1}{2}h_{\mu\rho}h_{\nu\lambda}v^\sigma\tilde{\nabla}_\sigma\delta h^{\rho\lambda}\Big) \\
&= -\big(2\,\tau_\lambda K^{\mu\nu}K_{\mu\nu} + K^{\nu}{}_{\lambda}v^\sigma\tau_{\sigma\nu}\big)\delta v^\lambda - 2\,K^{\mu}{}_{\sigma}\tilde{\nabla}_\mu\delta v^\sigma + K_{\rho\lambda}v^\sigma\tilde{\nabla}_\sigma\delta h^{\rho\lambda}.
\end{aligned}
\tag{C.10}
$$

C.2. NLO variational identities

For the variation of terms with covariant derivatives resulting in terms with $\delta\tilde{\Gamma}$, as encountered in Section 5.3, one can use some clever tricks and identities to simplify the calculations. We begin, as for the LO theory, by showing variational identities that will be useful later on:

$$
\begin{aligned}
\delta\tau_{\mu\nu} &= -\big(h_{\rho[\nu}\big(\tau_{\mu]\lambda} - v^\gamma\big(\tau_{\mu]}\tau_{\gamma\lambda} + \tau_{|\lambda}\tau_{\gamma|\mu]}\big)\big) + 4\,\tau_\rho\tau_{[\mu}K_{\nu]\lambda}\big)\delta h^{\rho\lambda} \\
&\quad + \big(\tau_{[\nu}\big(\tau_{\mu]\lambda} - v^\gamma\big(\tau_{\mu]}\tau_{\gamma\lambda} + \tau_{|\lambda}\tau_{\gamma|\mu]}\big)\big) + \tau_\lambda\tau_{\mu\nu}\big)\delta v^\lambda \\
&\quad - 2\,\tau_\lambda h_{\rho[\nu}\tilde{\nabla}_{\mu]}\delta h^{\rho\lambda} + 2\,\tau_\lambda\tau_{[\nu}\tilde{\nabla}_{\mu]}\delta v^\lambda,
\end{aligned}
\tag{C.11}
$$

$$
\delta\tilde{\Gamma}^\rho_{\nu\gamma}v^\gamma = -\tilde{\nabla}_\nu\delta v^\rho,
\tag{C.12}
$$

$$
\begin{aligned}
(K^{\mu\nu} - Kh^{\mu\nu})\delta\tilde{\Gamma}^\rho_{\mu\nu}\tau_\rho &= \Big(Kv^\lambda\tau_\rho\tau_{\lambda\gamma} - \frac{1}{2}K_\rho{}^{\nu}v^\lambda\tau_\gamma\tau_{\lambda\nu}\Big)\delta h^{\rho\gamma} \\
&\quad + \Big(K\Big(\delta^\mu_\rho + v^\mu\tau_\rho\Big)\tau_\gamma - K^{\mu}{}_{\rho}\tau_\gamma\Big)\tilde{\nabla}_\mu\delta h^{\rho\gamma},
\end{aligned}
\tag{C.13}
$$

$$
v^\mu h^{\nu\gamma}\tau_{\eta\gamma}\delta\tilde{\Gamma}^\rho_{(\nu\mu)} = h^{\nu\sigma}\tau_{\lambda\sigma}\Big(-\frac{1}{2}\Big(K_{\nu\eta}\delta h^{\rho\eta} + K_\nu{}^{\rho}\tau_\mu\delta v^\mu + h^{\rho\eta}\delta K_{\nu\eta}\Big) - \tilde{\nabla}_\nu\delta v^\rho\Big),
\tag{C.14}
$$

$$
\begin{aligned}
h^{\nu\gamma}\delta\tilde{\Gamma}^\rho_{\rho\nu} &= h^{\nu\gamma}\Big(\frac{1}{2}\Big(\delta^\rho_\lambda + v^\rho\tau_\lambda\Big)\tau_{\nu\rho}\delta v^\lambda - \frac{1}{2}h_{\rho\lambda}\tilde{\nabla}_\nu\delta h^{\rho\lambda} + \tau_\rho\tilde{\nabla}_\nu\delta v^\rho\Big) \\
&\quad + K\tau_\lambda\Big(\delta^\gamma_\rho + v^\gamma\tau_\rho\Big)\delta h^{\rho\lambda},
\end{aligned}
\tag{C.15}
$$

$$
\begin{aligned}
h^{\rho\nu}\delta\Big(\tilde{\nabla}_\rho K_{\mu\nu}\Big) &= h^{\rho\nu}\Big(\tilde{\nabla}_\rho\delta K_{\mu\nu} - \delta\tilde{\Gamma}^\lambda_{\rho\mu}K_{\lambda\nu} - \delta\tilde{\Gamma}^\lambda_{\rho\nu}K_{\mu\lambda}\Big) \\
&= h^{\rho\nu}\tilde{\nabla}_\rho\delta K_{\mu\nu} + K^{\rho}{}_{\lambda}\tau_\mu\tilde{\nabla}_\rho\delta v^\lambda \\
&\quad - \Big(\frac{1}{2}K^{\rho}{}_{\lambda}\tau_{\mu\rho} - 2\,K^{\rho\gamma}K_{\rho\gamma}\tau_\mu\tau_\lambda \\
&\qquad - K^{\gamma}{}_{\rho}\tau_\mu v^\sigma\tau_{\sigma\gamma} - 2\,K^{\sigma}{}_{\mu}v^\gamma\tau_\lambda\tau_{\gamma\sigma}\delta v^\lambda\Big)\delta v^\lambda \\
&\quad - \Big(K^{\sigma\gamma}K_{\sigma\gamma}\tau_\lambda h_{\mu\rho} - K^{\gamma}{}_{\rho}K_{\mu\gamma}\tau_\lambda \\
&\qquad - 2\,K^{\gamma}{}_{\rho}\tau_\mu K_{\lambda\gamma} + K_{\mu\lambda}K\tau_\rho - 2\,K^{\sigma}{}_{\mu}K_{\sigma\lambda}\tau_\rho\Big)\delta h^{\rho\lambda} \\
&\quad + \Big(-\frac{1}{2}K^{\sigma}{}_{\mu}h_{\rho\lambda} + K_{\mu\lambda}v^\sigma\tau_\rho + \delta^\sigma_\rho K_{\mu\lambda} + \frac{1}{2}K_{\rho\lambda}\Big(\delta^\sigma_\mu - v^\sigma\tau_\mu\Big)\Big)\tilde{\nabla}_\sigma\delta h^{\rho\lambda},
\end{aligned}
\tag{C.16}
$$

$$\begin{aligned} v^\rho \delta\Big(\tilde\nabla_\rho K_{\mu\nu}\Big) &= v^\rho \tilde\nabla_\rho \delta K_{\mu\nu} - \Big(K^\rho{}_\lambda \tau_{(\mu}K_{\nu)\rho} + v^\sigma \tau_{\sigma\gamma} K^\gamma{}_{(\nu} h_{\mu)\lambda} + v^\sigma \tau_{\sigma(\mu}K_{\nu)\lambda}\Big)\delta v^\lambda \\ &\quad - \big(K^\sigma{}_\lambda K_{\sigma(\nu}h_{\mu)\rho} - K_{\lambda(\mu}K_{\nu)\rho}\big)\delta h^{\rho\lambda} - 3\,K_{\lambda(\nu}\tilde\nabla_{\mu)}\delta v^\lambda \\ &\quad - \Big(K^\sigma{}_{(\nu}h_{\mu)\lambda} + v^\sigma \tau_{(\mu}K_{\nu)\lambda}\Big)\tilde\nabla_\sigma \delta v^\lambda + K_{\lambda(\nu}h_{\mu)\rho} v^\sigma \tilde\nabla_\sigma \delta h^{\rho\lambda}, \end{aligned} \tag{C.17}$$

$$h^{\nu\rho}\delta\Big(h_{\mu\nu}\tilde\nabla_\rho K\Big) = \big(\delta^\sigma_\lambda + v^\sigma\tau_\lambda\big)\Big(\tau_\mu \delta v^\lambda - h_{\mu\rho}\delta h^{\rho\lambda}\Big)\tilde\nabla_\sigma K + \Big(\delta^\rho_\mu + v^\rho \tau_\mu\Big)\tilde\nabla_\rho \delta K, \tag{C.18}$$

$$v^\rho \delta\Big(h_{\mu\nu}\tilde\nabla_\rho K\Big) = v^\sigma\Big(2\,h_{\lambda(\mu}\tau_{\nu)}\delta v^\lambda - h_{\mu\rho}h_{\nu\lambda}\delta h^{\rho\lambda}\Big)\tilde\nabla_\sigma K + v^\rho h_{\mu\nu}\tilde\nabla_\rho \delta K. \tag{C.19}$$

To show $\delta\tau_{\mu\nu}$ we begin by noting that

$$\begin{aligned} \tilde\nabla_{[\mu}\tau_{\nu]} &= \partial_{[\mu}\tau_{\nu]} - 2\,\tilde\Gamma^\lambda_{[\mu\nu]}\tau_\lambda \\ &= \frac{1}{2}\tau_{\mu\nu} - \underbrace{2h^{\lambda\gamma}\tau_{[\mu}K_{\nu]\gamma}\tau_\lambda}_{=0}, \end{aligned} \tag{C.20}$$

such that we can now write

$$\begin{aligned} \delta\tau_{\mu\nu} &= 2\,\tilde\nabla_{[\mu}\delta\tau_{\nu]} - 4\,\delta\tilde\Gamma^\lambda_{[\mu\nu]}\tau_\lambda \\ &= 2\,\tilde\nabla_{[\mu}\Big(-\tau_\lambda h_{\nu]\rho}\delta h^{\rho\lambda} + \tau_\lambda \tau_{\nu]}\delta v^\lambda\Big) - 4\,\tau_\lambda \tau_{[\mu}K_{\nu]\gamma}\delta h^{\lambda\gamma} \\ &= -2\,\Big(h_{\rho[\nu}\tilde\nabla_{\mu]}\tau_\lambda + 2\,\tau_\rho \tau_{[\mu}K_{\nu]\lambda}\Big)\delta h^{\rho\lambda} - 2\,\tau_\lambda h_{\rho[\nu}\tilde\nabla_{\mu]}\delta h^{\rho\lambda} \\ &\quad + \Big(2\,\tau_{[\nu}\tilde\nabla_{\mu]}\tau_\lambda + \tau_\lambda\tau_{\mu\nu}\Big)\delta v^\lambda + 2\,\tau_\lambda \tau_{[\nu}\tilde\nabla_{\mu]}\delta v^\lambda \\ &= -\big(h_{\rho[\nu}\big(\tau_{\mu]\lambda} - v^\gamma\big(\tau_{\mu]}\tau_{\gamma\lambda} + \tau_{|\lambda}\tau_{\gamma|\mu]}\big)\big) + 4\,\tau_\rho\tau_{[\mu}K_{\nu]\lambda}\big)\delta h^{\rho\lambda} - 2\,\tau_\lambda h_{\rho[\nu}\tilde\nabla_{\mu]}\delta h^{\rho\lambda} \\ &\quad + \big(\tau_{[\nu}\big(\tau_{\mu]\lambda} - v^\gamma\big(\tau_{\mu]}\tau_{\gamma\lambda} + \tau_{|\lambda}\tau_{\gamma|\mu]}\big)\big) + \tau_\lambda\tau_{\mu\nu}\big)\delta v^\lambda + 2\,\tau_\lambda\tau_{[\nu}\tilde\nabla_{\mu]}\delta v^\lambda. \end{aligned} \tag{C.21}$$

One of the key methods for finding all the $\delta\tilde\Gamma$ is to find vanishing covariant derivatives, varying them and then isolating the relevant term. This can be seen quite simply for $\delta\tilde\Gamma^\rho_{\nu\gamma}v^\gamma$:

$$\begin{aligned} \delta\Big(\tilde\nabla_\nu v^\rho\Big) &= \tilde\nabla_\nu \delta v^\rho + \delta\tilde\Gamma^\rho_{\nu\gamma}v^\gamma = 0 \\ \Rightarrow \delta\tilde\Gamma^\rho_{\nu\gamma}v^\gamma &= -\tilde\nabla_\nu \delta v^\rho. \end{aligned} \tag{C.22}$$

In the same manner we take the trace of (4.31a) with $h^{\mu\nu}$:

$$h^{\mu\nu}\tilde\nabla_\mu \tau_\nu = h^{\mu\nu}\frac{1}{2}\Big(\tau_{\mu\nu} - v^\lambda\big(\tau_\mu\tau_{\lambda\nu} + \tau_\nu\tau_{\lambda\mu}\big)\Big) = 0, \tag{C.23}$$

and see that it vanishes and thus we find:

$$\begin{aligned} \delta\Big(h^{\mu\nu}\tilde\nabla_\mu\tau_\nu\Big) &= \delta h^{\mu\nu}\tilde\nabla_\mu\tau_\nu + h^{\mu\nu}\Big(\tilde\nabla_\mu\delta\tau_\nu - \delta\tilde\Gamma^\sigma_{\mu\nu}\tau_\sigma\Big) = 0 \\ \Rightarrow\ h^{\mu\nu}\delta\tilde\Gamma^\sigma_{\mu\nu}\tau_\sigma &= \delta h^{\mu\nu}\tilde\nabla_\mu\tau_\nu + h^{\mu\nu}\tilde\nabla_\mu\delta\tau_\nu \\ &= -\delta h^{\mu\nu}v^\lambda\tau_\mu\tau_{\lambda\nu} + h^{\mu\nu}\tilde\nabla_\mu\delta\tau_\nu. \end{aligned} \tag{C.24}$$

Moreover, we show

$$h^{\mu\nu}\tilde\nabla_\mu\delta\tau_\nu = -h^{\mu\nu}h_{\nu\rho}\Big(\tilde\nabla_\mu\tau_\gamma\delta h^{\rho\gamma} + \tau_\gamma\tilde\nabla_\mu\delta h^{\rho\gamma}\Big)$$

$$
\begin{aligned}
&= -\left(\delta^\mu_\rho + v^\mu\tau_\rho\right)\left(\frac{1}{2}\left(\tau_{\mu\gamma} - v^\lambda\left(\tau_\mu\tau_{\lambda\gamma} + \tau_\gamma\tau_{\lambda\mu}\right)\right)\delta h^{\rho\gamma} + \tau_\gamma\tilde{\nabla}_\mu\delta h^{\rho\gamma}\right)\\
&= -\frac{1}{2}\left(-v^\lambda\left(\cancel{\tau_\rho\tau_{\lambda\gamma}} + \cancel{\tau_\gamma\tau_{\lambda\rho}}\right) + \cancel{v^\mu\tau_\rho\tau_{\mu\gamma}} - v^\lambda\left(-\cancel{\tau_\rho\tau_{\lambda\gamma}} + \underbrace{v^\mu\tau_\rho\tau_\gamma\tau_{\lambda\mu}}_{=0}\right)\right)\delta h^{\rho\gamma}\\
&\quad -\left(\delta^\mu_\rho + v^\mu\tau_\rho\right)\tau_\gamma\tilde{\nabla}_\mu\delta h^{\rho\gamma}, \qquad \text{(C.25)}
\end{aligned}
$$

which then gives

$$
h^{\mu\nu}\delta\tilde{\Gamma}^\sigma_{\mu\nu}\tau_\sigma = -\delta h^{\mu\nu}v^\lambda\tau_\mu\tau_{\lambda\nu} - \left(\delta^\mu_\rho + v^\mu\tau_\rho\right)\tau_\gamma\tilde{\nabla}_\mu\delta h^{\rho\gamma}. \qquad \text{(C.26)}
$$

For contracting (4.31a) $K^{\mu\nu}$ we get:

$$
\begin{aligned}
\delta\left(K^{\mu\nu}\tilde{\nabla}_\mu\tau_\nu\right) &= \delta K^{\mu\nu}\tilde{\nabla}_\mu\tau_\nu + K^{\mu\nu}\left(\tilde{\nabla}_\mu\delta\tau_\nu - \delta\tilde{\Gamma}^\sigma_{\mu\nu}\tau_\sigma\right) = 0\\
\Rightarrow\ K^{\mu\nu}\delta\tilde{\Gamma}^\sigma_{\mu\nu}\tau_\sigma &= \delta K^{\mu\nu}\tilde{\nabla}_\mu\tau_\nu + K^{\mu\nu}\tilde{\nabla}_\mu\delta\tau_\nu\\
&= -\delta K^{\mu\nu}v^\lambda\tau_\mu\tau_{\lambda\nu} + K^{\mu\nu}\tilde{\nabla}_\mu\delta\tau_\nu\\
&= K^{\mu\nu}v^\lambda\delta\tau_\mu\tau_{\lambda\nu} + K^{\mu\nu}\tilde{\nabla}_\mu\delta\tau_\nu\\
&= -K_\rho{}^\nu v^\lambda\tau_\gamma\tau_{\lambda\nu}\delta h^{\rho\gamma} + K^{\mu\nu}\tilde{\nabla}_\mu\delta\tau_\nu, \qquad \text{(C.27)}
\end{aligned}
$$

and

$$
\begin{aligned}
K^{\mu\nu}\tilde{\nabla}_\mu\delta\tau_\nu &= -K^{\mu\nu}h_{\nu\rho}\left(\tilde{\nabla}_\mu\tau_\gamma\delta h^{\rho\gamma} + \tau_\gamma\tilde{\nabla}_\mu\delta h^{\rho\gamma}\right)\\
&= -K^{\mu\nu}h_{\nu\rho}\left(\frac{1}{2}\left(\underbrace{\tau_{\mu\gamma}}_{=0} - v^\lambda\left(\underbrace{\tau_\mu\tau_{\lambda\gamma}}_{=0} + \tau_\gamma\tau_{\lambda\mu}\right)\right)\delta h^{\rho\gamma} + \tau_\gamma\tilde{\nabla}_\mu\delta h^{\rho\gamma}\right)\\
&= K^\mu{}_\rho\left(\frac{1}{2}v^\lambda\tau_\gamma\tau_{\lambda\mu}\delta h^{\rho\gamma} - \tau_\gamma\tilde{\nabla}_\mu\delta h^{\rho\gamma}\right), \qquad \text{(C.28)}
\end{aligned}
$$

giving

$$
K^{\mu\nu}\delta\tilde{\Gamma}^\sigma_{\mu\nu}\tau_\sigma = -K_\rho{}^\nu v^\lambda\tau_\gamma\tau_{\lambda\nu}\delta h^{\rho\gamma} + K^\mu{}_\rho\left(\frac{1}{2}v^\lambda\tau_\gamma\tau_{\lambda\mu}\delta h^{\rho\gamma} - \tau_\gamma\tilde{\nabla}_\mu\delta h^{\rho\gamma}\right). \qquad \text{(C.29)}
$$

This allows us to replace the second term of (5.39) like so:

$$
\begin{aligned}
(K^{\mu\nu} - Kh^{\mu\nu})\delta\tilde{\Gamma}^\rho_{\mu\nu}\tau_\rho &= \left(Kv^\lambda\tau_\rho\tau_{\lambda\gamma} - \frac{1}{2}K_\rho{}^\nu v^\lambda\tau_\gamma\tau_{\lambda\nu}\right)\delta h^{\rho\gamma}\\
&\quad + \left(K\left(\delta^\mu_\rho + v^\mu\tau_\rho\right)\tau_\gamma - K^\mu{}_\rho\tau_\gamma\right)\tilde{\nabla}_\mu\delta h^{\rho\gamma}. \qquad \text{(C.30)}
\end{aligned}
$$

For $v^\mu h^{\nu\sigma}\delta\tilde{\Gamma}^\rho_{(\mu\nu)}$ we use that any tensor can be decomposed into a symmetric and antisymmetric part and find:

$$
\begin{aligned}
v^\mu h^{\nu\sigma}\tau_{\lambda\sigma}\delta\tilde{\Gamma}^\rho_{(\nu\mu)} &= v^\mu h^{\nu\sigma}\tau_{\lambda\sigma}\left(-\delta\tilde{\Gamma}^\rho_{[\nu\mu]} + \delta\tilde{\Gamma}^\rho_{\nu\mu}\right)\\
&= h^{\nu\sigma}\tau_{\lambda\sigma}\left(-v^\mu\delta\left(h^{\rho\eta}\tau_{[\nu}K_{\mu]\eta}\right) - \tilde{\nabla}_\nu\delta v^\rho\right)
\end{aligned}
$$

$$
\begin{aligned}
&= h^{\nu\sigma}\tau_{\lambda\sigma}\Big(-\frac{1}{2}\Big(K_{\nu\eta}\delta h^{\rho\eta} - h^{\rho\eta}K_{\nu\eta}v^{\mu}\delta\tau_{\mu} \\
&\qquad + \underbrace{h^{\rho\eta}\tau_{\nu}v^{\mu}}_{=0}\delta K_{\mu\eta} + h^{\rho\eta}\delta K_{\nu\eta}\Big) - \tilde{\nabla}_{\nu}\delta v^{\rho}\Big) \\
&= h^{\nu\sigma}\tau_{\lambda\sigma}\Big(-\frac{1}{2}\Big(K_{\nu\eta}\delta h^{\rho\eta} + K_{\nu}{}^{\rho}\tau_{\mu}\delta v^{\mu} + h^{\rho\eta}\delta K_{\nu\eta}\Big) - \tilde{\nabla}_{\nu}\delta v^{\rho}\Big),
\end{aligned}
\tag{C.31}
$$

where we has used the expression for the torsion (C.7) and (C.12). Now, for $h^{\nu\eta}\delta\tilde{\Gamma}^{\rho}_{\rho\nu}$, we start by showing

$$
\begin{aligned}
\delta\Big(h_{\rho\alpha}\tilde{\nabla}_{\nu}h^{\rho\alpha}\Big) &= \delta h_{\rho\alpha}\tilde{\nabla}_{\nu}h^{\rho\alpha} + h_{\rho\alpha}\tilde{\nabla}_{\nu}\delta h^{\rho\alpha} + h_{\rho\alpha}\Big(\delta\tilde{\Gamma}^{\rho}_{\nu\gamma}h^{\gamma\alpha} + \delta\tilde{\Gamma}^{\alpha}_{\nu\gamma}h^{\rho\gamma}\Big) \\
&= \delta h_{\rho\alpha}\tilde{\nabla}_{\nu}h^{\rho\alpha} + h_{\rho\alpha}\tilde{\nabla}_{\nu}\delta h^{\rho\alpha} + 2\,\delta\tilde{\Gamma}^{\rho}_{\nu\rho} + 2\,\delta\tilde{\Gamma}^{\rho}_{\nu\gamma}v^{\gamma}\tau_{\rho} \\
&= \Big(-\underbrace{h_{\rho\sigma}h_{\alpha\gamma}}_{=0}\delta h^{\sigma\gamma} + 2\,h_{\gamma(\rho}\tau_{\alpha)}\delta v^{\gamma}\Big)v^{(\rho}h^{\alpha)\sigma}\tau_{\lambda\sigma}\Big(\delta^{\lambda}_{\nu} - \tau_{\nu}v^{\lambda}\Big) \\
&\quad + h_{\rho\alpha}\tilde{\nabla}_{\nu}\delta h^{\rho\alpha} + 2\,\delta\tilde{\Gamma}^{\rho}_{\nu\rho} - 2\,\tau_{\rho}\tilde{\nabla}_{\nu}\delta v^{\rho} \\
&= -\Big(\Big(\tau_{\nu\sigma} + v^{\lambda}\tau_{\nu}\tau_{\lambda\sigma}\Big)\delta v^{\sigma} + v^{\sigma}\tau_{\gamma}\tau_{\nu\sigma}\delta v^{\gamma}\Big) \\
&\quad + h_{\rho\alpha}\tilde{\nabla}_{\nu}\delta h^{\rho\alpha} + 2\,\delta\tilde{\Gamma}^{\rho}_{\nu\rho} - 2\,\tau_{\rho}\tilde{\nabla}_{\nu}\delta v^{\rho} \\
\Rightarrow \delta\tilde{\Gamma}^{\rho}_{\nu\rho} &= \frac{1}{2}\Big(\Big(\tau_{\nu\sigma} - v^{\lambda}\tau_{\nu}\tau_{\lambda\sigma}\Big)\delta v^{\sigma} + v^{\sigma}\tau_{\gamma}\tau_{\nu\sigma}\delta v^{\gamma}\Big) - \frac{1}{2}h_{\rho\alpha}\tilde{\nabla}_{\nu}\delta h^{\rho\alpha} + \tau_{\rho}\tilde{\nabla}_{\nu}\delta v^{\rho},
\end{aligned}
\tag{C.32}
$$

where we have used (C.12), (5.15b) and (4.31b). Then splitting $h^{\nu\eta}\delta\tilde{\Gamma}^{\rho}_{\rho\nu}$ into its symmetric and antisymmetric components and inserting the previous result, we have:

$$
\begin{aligned}
h^{\nu\eta}\delta\tilde{\Gamma}^{\rho}_{\rho\nu} &= h^{\nu\eta}\Big(\delta\tilde{\Gamma}^{\rho}_{\nu\rho} - 2\,\delta\tilde{\Gamma}^{\rho}_{[\nu\rho]}\Big) \\
&= h^{\nu\eta}\Big(\frac{1}{2}\Big(\Big(\tau_{\nu\sigma} - v^{\lambda}\underbrace{\tau_{\nu}}_{=0}\tau_{\lambda\sigma}\Big)\delta v^{\sigma} + v^{\sigma}\tau_{\gamma}\tau_{\nu\sigma}\delta v^{\gamma}\Big) \\
&\qquad - \frac{1}{2}h_{\rho\lambda}\tilde{\nabla}_{\nu}\delta h^{\rho\lambda} + \tau_{\rho}\tilde{\nabla}_{\nu}\delta v^{\rho} - 2\,\delta\big(h^{\rho\gamma}\tau_{[\nu}K_{\rho]\gamma}\big)\Big) \\
&= h^{\nu\eta}\Big(\frac{1}{2}\Big(\delta^{\sigma}_{\gamma} + v^{\sigma}\tau_{\gamma}\Big)\tau_{\nu\sigma}\delta v^{\gamma} - \frac{1}{2}h_{\rho\lambda}\tilde{\nabla}_{\nu}\delta h^{\rho\lambda} + \tau_{\rho}\tilde{\nabla}_{\nu}\delta v^{\rho}\Big) \\
&\quad + K\tau_{\lambda}\Big(\delta^{\eta}_{\rho} + v^{\eta}\tau_{\rho}\Big)\delta h^{\rho\lambda},
\end{aligned}
\tag{C.33}
$$

where (5.15a) has been inserted. Here, we have derived the necessary identities to assemble (5.39).

Moving on to $h^{\rho\nu}\delta\Big(\tilde{\nabla}_{\rho}K_{\mu\nu}\Big)$ we have

$$
h^{\rho\nu}\delta\Big(\tilde{\nabla}_{\rho}K_{\mu\nu}\Big) = h^{\rho\nu}\Big(\tilde{\nabla}_{\rho}\delta K_{\mu\nu} - \delta\tilde{\Gamma}^{\lambda}_{\rho\mu}K_{\lambda\nu} - \delta\tilde{\Gamma}^{\lambda}_{\rho\nu}K_{\mu\lambda}\Big).
\tag{C.34}
$$

First considering the second term, we can show by symmetry/antisymmetry arguments

$$
h^{\rho\nu}K_{\lambda\nu}\delta\tilde{\Gamma}^{\lambda}_{\rho\mu} = h^{\rho\nu}K_{\lambda\nu}\Big(\delta\tilde{\Gamma}^{\lambda}_{\mu\rho} - 2\,\delta\tilde{\Gamma}^{\lambda}_{[\mu\rho]}\Big)
$$

$$
\begin{aligned}
&= h^{\rho\nu} K_{\lambda\nu}\left(\delta\tilde{\Gamma}^{\lambda}_{\mu\rho} - 2\,\delta\left(h^{\lambda\gamma}\tau_{[\mu}K_{\rho]\gamma}\right)\right) \\
&= h^{\rho\nu} K_{\lambda\nu}\left(\delta\tilde{\Gamma}^{\lambda}_{\mu\rho} - \delta h^{\lambda\gamma}\tau_{\mu}K_{\rho\gamma} - h^{\lambda\gamma}\delta\tau_{\mu}K_{\rho\gamma} - h^{\lambda\gamma}\tau_{\mu}\delta K_{\rho\gamma} + h^{\lambda\gamma}\delta\tau_{\rho}K_{\mu\gamma}\right),
\end{aligned} \tag{C.35}
$$

and then the first term in the above equation is found as such:

$$
\begin{aligned}
K_{\lambda\nu}\delta\left(\tilde{\nabla}_{\mu}h^{\lambda\nu}\right) &= K_{\lambda\nu}\left(\tilde{\nabla}_{\mu}\delta h^{\lambda\nu} + \delta\tilde{\Gamma}^{\lambda}_{\mu\rho}h^{\rho\nu} + \delta\tilde{\Gamma}^{\nu}_{\mu\rho}h^{\lambda\rho}\right) \\
&= K_{\lambda\nu}\left(\tilde{\nabla}_{\mu}\delta h^{\lambda\nu} + 2\,\delta\tilde{\Gamma}^{\lambda}_{\mu\rho}h^{\rho\nu}\right) \\
&= K_{\lambda\nu}\delta\left(v^{(\lambda}h^{\nu)\sigma}\tau_{\gamma\sigma}\left(\delta^{\gamma}_{\mu} - v^{\gamma}\tau_{\sigma}\right)\right) \\
&= K_{\lambda\nu}\delta v^{\lambda}h^{\nu\sigma}\tau_{\gamma\sigma}\left(\delta^{\gamma}_{\mu} - v^{\gamma}\tau_{\sigma}\right) \\
\Rightarrow h^{\rho\nu}K_{\lambda\nu}\delta\tilde{\Gamma}^{\lambda}_{\mu\rho} &= \frac{1}{2}K_{\lambda\nu}\left(\delta v^{\lambda}h^{\nu\sigma}\tau_{\gamma\sigma}\left(\delta^{\gamma}_{\mu} - v^{\gamma}\tau_{\sigma}\right) - \tilde{\nabla}_{\mu}\delta h^{\lambda\nu}\right),
\end{aligned} \tag{C.36}
$$

where we have inserted (4.31b) in the third line, thus giving:

$$
\begin{aligned}
h^{\rho\nu}K_{\lambda\nu}\delta\tilde{\Gamma}^{\lambda}_{\rho\mu} = {}& \frac{1}{2}K_{\lambda\nu}\left(\delta v^{\lambda}h^{\nu\sigma}\tau_{\gamma\sigma}\left(\delta^{\gamma}_{\mu} - v^{\gamma}\underbrace{\tau_{\sigma}}_{=0}\right) - \tilde{\nabla}_{\mu}\delta h^{\lambda\nu}\right) \\
&+ h^{\rho\nu}K_{\lambda\nu}\Bigg(- \delta h^{\lambda\gamma}\tau_{\mu}K_{\rho\gamma} - h^{\lambda\gamma}K_{\rho\gamma}\left(-\tau_{\sigma}h_{\mu\rho}\delta h^{\rho\sigma} + \tau_{\sigma}\tau_{\mu}\delta v^{\sigma}\right) \\
&\qquad + h^{\lambda\gamma}K_{\mu\gamma}\Big(- \tau_{\sigma}h_{\rho\beta}\delta h^{\beta\sigma} + \tau_{\sigma}\underbrace{\tau_{\rho}}_{=0}\delta v^{\sigma}\Big)\Bigg) \\
&+ K^{\gamma\rho}\tau_{\mu}\Bigg(\Big(- \tau_{\lambda}K_{\rho\gamma} + \underbrace{K_{\lambda(\gamma}\tau_{\rho)}}_{=0} - \frac{1}{2}h_{\lambda\rho}v^{\sigma}\tau_{\sigma\gamma} - \frac{1}{2}h_{\lambda\gamma}v^{\sigma}\tau_{\sigma\rho}\Big)\delta v^{\lambda} \\
&\qquad - h_{\sigma(\gamma}\tilde{\nabla}_{\rho)}\delta v^{\sigma} - v^{\sigma}\underbrace{h_{\alpha(\rho}\tau_{\gamma)}}_{=0}\tilde{\nabla}_{\sigma}\delta v^{\alpha} \\
&\qquad - K_{\lambda(\gamma}h_{\rho)\beta}\delta h^{\beta\lambda} + \frac{1}{2}h_{\rho\beta}h_{\gamma\lambda}v^{\sigma}\tilde{\nabla}_{\sigma}\delta h^{\beta\lambda}\Bigg) \\
= {}& \left(\frac{1}{2}K^{\rho}{}_{\lambda}\tau_{\mu\rho} - 2\,K^{\rho\gamma}K_{\rho\gamma}\tau_{\mu}\tau_{\lambda} - K^{\gamma}{}_{\rho}\tau_{\mu}v^{\sigma}\tau_{\sigma\gamma}\right)\delta v^{\lambda} \\
&+ \left(K^{\sigma\gamma}K_{\sigma\gamma}\tau_{\lambda}h_{\mu\rho} - K^{\gamma}{}_{\rho}K_{\mu\gamma}\tau_{\lambda} - 2\,K^{\gamma}{}_{\rho}\tau_{\mu}K_{\lambda\gamma}\right)\delta h^{\rho\lambda} \\
&- K^{\rho}{}_{\lambda}\tau_{\mu}\tilde{\nabla}_{\rho}\delta v^{\lambda} - \frac{1}{2}K_{\rho\lambda}\left(\delta^{\sigma}_{\mu} - v^{\sigma}\tau_{\mu}\right)\tilde{\nabla}_{\sigma}\delta h^{\rho\lambda}.
\end{aligned} \tag{C.37}
$$

For the third term of (C.34) we show:

$$
\begin{aligned}
K_{\mu\lambda}\delta\left(\tilde{\nabla}_{\rho}h^{\rho\lambda}\right) &= K_{\mu\lambda}\left(\tilde{\nabla}_{\rho}\delta h^{\rho\lambda} + \delta\tilde{\Gamma}^{\rho}_{\rho\nu}h^{\nu\lambda} + \delta\tilde{\Gamma}^{\lambda}_{\rho\nu}h^{\rho\nu}\right) \\
&= K_{\mu\lambda}\Bigg(\tilde{\nabla}_{\rho}\delta h^{\rho\lambda} + h^{\nu\lambda}\left(\frac{1}{2}\left(\delta^{\sigma}_{\gamma} + v^{\sigma}\tau_{\gamma}\right)\tau_{\nu\sigma}\delta v^{\gamma} - \frac{1}{2}h_{\rho\gamma}\tilde{\nabla}_{\nu}\delta h^{\rho\gamma}\right.
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad + \tau_\rho \tilde{\nabla}_\nu \delta v^\rho \Big) + K\tau_\gamma \delta h^{\lambda\gamma} + \delta\tilde{\Gamma}^\lambda_{\rho\nu} h^{\rho\nu} \Big) \\
&= K_{\mu\lambda} \Big(\delta v^\rho h^{\lambda\sigma} \tau_{\rho\sigma} + v^\rho \delta h^{\lambda\sigma} \tau_{\rho\sigma} + v^\rho h^{\lambda\sigma} \delta\tau_{\rho\sigma} \Big) \\
&= K_{\mu\lambda} \Bigg[\delta v^\rho h^{\lambda\sigma} \tau_{\rho\sigma} + v^\rho \delta h^{\lambda\sigma} \tau_{\rho\sigma} \\
&\qquad + v^\rho h^{\lambda\sigma} \Bigg(- \Big(h_{\alpha[\sigma} \big(\tau_{\rho]\beta} - v^\gamma \big(\tau_{\rho]} \tau_{\gamma\beta} + \tau_{|\beta} \tau_{\gamma|\rho]} \big) \big) \\
&\qquad\qquad + 4\,\tau_\alpha \tau_{[\rho} K_{\sigma]\beta} \Big) \delta h^{\alpha\beta} \\
&\qquad\qquad + \big(\tau_{[\sigma} \big(\tau_{\rho]\beta} - v^\gamma \big(\tau_{\rho]} \tau_{\gamma\beta} + \tau_{|\beta} \tau_{\gamma|\rho]} \big) \big) + \tau_\beta \tau_{\rho\sigma} \big) \delta v^\beta \\
&\qquad\qquad - 2\,\tau_\beta h_{\alpha[\sigma} \tilde{\nabla}_{\rho]} \delta h^{\alpha\beta} + 2\,\tau_\beta \tau_{[\sigma} \tilde{\nabla}_{\rho]} \delta v^\beta \Bigg) \Bigg] \\
&= K_{\mu\lambda} \Bigg[\underbrace{\delta v^\rho h^{\lambda\sigma} \tau_{\rho\sigma}}_{-1/2\delta v^\rho h^{\lambda\sigma} \tau_{\rho\sigma}} + \cancel{v^\rho \delta h^{\lambda\sigma}} \tau_{\rho\sigma} + \Big(- \cancel{\delta^\lambda_\alpha v^\rho} \tau_{\rho\beta} - 2\, h^{\lambda\sigma} \tau_\alpha K_{\sigma\beta} \Big) \delta h^{\alpha\beta} \\
&\qquad + h^{\lambda\sigma} \Big(\cancel{\frac{1}{2}} \tau_{\sigma\beta} + \frac{3}{2} v^\gamma \tau_\beta \tau_{\gamma\sigma} \Big) \delta v^\beta - v^\rho \tau_\beta \tilde{\nabla}_\rho \delta h^{\lambda\beta} + h^{\lambda\sigma} \tau_\beta \tilde{\nabla}_\sigma \delta v^\beta \Bigg] \\
&= K_{\mu\lambda} \Bigg[\frac{1}{2} h^{\lambda\sigma} \big(\tau_{\sigma\rho} + 3\, v^\gamma \tau_\rho \tau_{\gamma\sigma} \big) \delta v^\rho - 2\, h^{\lambda\sigma} \tau_\alpha K_{\sigma\beta} \delta h^{\alpha\beta} \\
&\qquad - v^\rho \tau_\beta \tilde{\nabla}_\rho \delta h^{\lambda\beta} + h^{\lambda\sigma} \tau_\beta \tilde{\nabla}_\sigma \delta v^\beta \Bigg] \\
\Rightarrow K_{\mu\lambda} \delta\tilde{\Gamma}^\lambda_{\rho\nu} h^{\rho\nu} &= K_{\mu\lambda} \Bigg[\frac{1}{2} h^{\lambda\sigma} \Big(\cancel{\tau_{\sigma\rho}} + \cancelto{4}{3}\, v^\gamma \tau_\rho \tau_{\gamma\sigma} \Big) \delta v^\rho - 2\, h^{\lambda\sigma} \tau_\alpha K_{\sigma\beta} \delta h^{\alpha\beta} - K\tau_\gamma \delta h^{\lambda\gamma} \\
&\qquad - v^\rho \tau_\beta \tilde{\nabla}_\rho \delta h^{\lambda\beta} + h^{\lambda\sigma} \tau_\beta \tilde{\nabla}_\sigma \delta v^\beta - \tilde{\nabla}_\rho \delta h^{\rho\lambda} \\
&\qquad - h^{\nu\lambda} \Big(\frac{1}{2} \Big(\cancel{\delta^\sigma_\gamma} + \cancel{v^\sigma \tau_\gamma} \Big) \tau_{\nu\sigma} \delta v^\gamma - \frac{1}{2} h_{\rho\gamma} \tilde{\nabla}_\nu \delta h^{\rho\gamma} + \tau_\rho \tilde{\nabla}_\nu \delta v^\rho \Big) \Bigg] \\
&= 2\, K^\sigma{}_\mu v^\gamma \tau_\lambda \tau_{\gamma\sigma} \delta v^\lambda + \Big(-K_{\mu\lambda} K - 2\, K^\sigma{}_\mu K_{\sigma\lambda} \Big) \tau_\rho \delta h^{\rho\lambda} \\
&\quad + \Big(\frac{1}{2} K^\sigma{}_\mu h_{\rho\lambda} - K_{\mu\lambda} v^\sigma \tau_\rho - \delta^\sigma_\rho K_{\mu\lambda} \Big) \tilde{\nabla}_\sigma \delta h^{\rho\lambda},
\end{aligned}
\tag{C.38}
$$

where we have inserted (C.33), (4.31c) and the expression for $\delta\tau_{\mu\nu}$. Gathering it all together, we find for (C.34):

$$
\begin{aligned}
h^{\rho\nu} \delta \Big(\tilde{\nabla}_\rho K_{\mu\nu} \Big) &= h^{\rho\nu} \Big(\tilde{\nabla}_\rho \delta K_{\mu\nu} - \delta\tilde{\Gamma}^\lambda_{\rho\mu} K_{\lambda\nu} - \delta\tilde{\Gamma}^\lambda_{\rho\nu} K_{\mu\lambda} \Big) \\
&= h^{\rho\nu} \tilde{\nabla}_\rho \delta K_{\mu\nu} + K^\rho{}_\lambda \tau_\mu \tilde{\nabla}_\rho \delta v^\lambda \\
&\quad - \Big(\frac{1}{2} K^\rho{}_\lambda \tau_{\mu\rho} - 2\, K^{\rho\gamma} K_{\rho\gamma} \tau_\mu \tau_\lambda - K^\gamma{}_\rho \tau_\mu v^\sigma \tau_{\sigma\gamma} - 2\, K^\sigma{}_\mu v^\gamma \tau_\lambda \tau_{\gamma\sigma} \delta v^\lambda \Big) \delta v^\lambda
\end{aligned}
$$

$$
\begin{aligned}
&- \Big(K^{\sigma\gamma} K_{\sigma\gamma} \tau_\lambda h_{\mu\rho} - K^{\gamma}{}_{\rho} K_{\mu\gamma} \tau_\lambda \\
&\qquad - 2\, K^{\gamma}{}_{\rho} \tau_\mu K_{\lambda\gamma} + K_{\mu\lambda} K \tau_\rho - 2\, K^{\sigma}{}_{\mu} K_{\sigma\lambda} \tau_\rho \Big) \delta h^{\rho\lambda} \\
&+ \Big(-\frac{1}{2} K^{\sigma}{}_{\mu} h_{\rho\lambda} + K_{\mu\lambda} v^\sigma \tau_\rho + \delta^\sigma_\rho K_{\mu\lambda} + \frac{1}{2} K_{\rho\lambda} \Big(\delta^\sigma_\mu - v^\sigma \tau_\mu \Big) \Big) \tilde{\nabla}_\sigma \delta h^{\rho\lambda}.
\end{aligned} \tag{C.39}
$$

For $v^\rho \delta\Big(\tilde{\nabla}_\rho K_{\mu\nu}\Big)$ we begin by expanding the term

$$
v^\rho \delta\Big(\tilde{\nabla}_\rho K_{\mu\nu}\Big) = v^\rho \Big(\tilde{\nabla}_\rho \delta K_{\mu\nu} - 2\, \delta \tilde{\Gamma}^{\lambda}_{\rho(\mu} K_{\nu)\lambda} \Big). \tag{C.40}
$$

Then, we take care of the second term in the following manner:

$$
\begin{aligned}
\delta\Big(\tilde{\nabla}_\mu v^\eta\Big) &= \tilde{\nabla}_\mu \delta v^\eta - \delta \tilde{\Gamma}^{\eta}_{\mu\rho} v^\rho \\
&= \tilde{\nabla}_\mu \delta v^\eta - \delta \tilde{\Gamma}^{\eta}_{\rho\mu} v^\rho + 2\, \delta \tilde{\Gamma}^{\eta}_{[\rho\mu]} v^\rho \\
&= \tilde{\nabla}_\mu \delta v^\eta - \delta \tilde{\Gamma}^{\eta}_{\rho\mu} v^\rho + 2\, \delta \big(h^{\eta\gamma} \tau_{[\rho} K_{\mu]\gamma} \big) v^\rho \\
&= \tilde{\nabla}_\mu \delta v^\eta - \delta \tilde{\Gamma}^{\eta}_{\rho\mu} v^\rho - \delta h^{\eta\gamma} K_{\mu\gamma} + h^{\eta\gamma} v^\rho K_{\mu\gamma} \delta\tau_\rho - h^{\eta\gamma} \big(v^\rho \tau_\mu \delta K_{\rho\gamma} + \delta K_{\mu\gamma} \big) \\
&= \tilde{\nabla}_\mu \delta v^\eta - \delta \tilde{\Gamma}^{\eta}_{\rho\mu} v^\rho - K_{\mu\gamma} \delta h^{\eta\gamma} + h^{\eta\gamma} \big(\cancel{\tau_\mu K_{\rho\gamma}} - \cancel{K_{\mu\gamma} \tau_\rho} \big) \delta v^\rho \\
&\quad + \Big(\cancel{\tau_\lambda K^{\eta}{}_{\mu}} \underbrace{- \frac{1}{2} K^{\eta}{}_{\lambda} \tau_\mu}_{+\frac{1}{2} K^{\eta}{}_{\lambda} \tau_\mu} + \frac{1}{2} h^{\eta\gamma} h_{\lambda\mu} v^\sigma \tau_{\sigma\gamma} + \frac{1}{2} \Big(\delta^\eta_\lambda + v^\eta \tau_\lambda \Big) v^\sigma \tau_{\sigma\mu} \Big) \delta v^\lambda \\
&\quad + \frac{1}{2} \Big(\delta^\eta_\lambda + v^\eta \tau_\lambda \Big) \tilde{\nabla}_\mu \delta v^\lambda + \frac{1}{2} h^{\eta\gamma} h_{\sigma\mu} \tilde{\nabla}_\gamma \delta v^\sigma \\
&\quad + \frac{1}{2} \Big(\delta^\eta_\lambda + v^\eta \tau_\lambda \Big) v^\sigma \tau_\mu \tilde{\nabla}_\sigma \delta v^\lambda + \frac{1}{2} K^{\eta}{}_{\lambda} h_{\mu\rho} \delta h^{\rho\lambda} \\
&\quad + \frac{1}{2} \Big(\delta^\eta_\rho + v^\eta \tau_\rho \Big) K_{\lambda\mu} \delta h^{\rho\lambda} - \frac{1}{2} \Big(\delta^\eta_\lambda + v^\eta \tau_\lambda \Big) h_{\mu\rho} v^\sigma \tilde{\nabla}_\sigma \delta h^{\rho\lambda} = 0 \\
\Rightarrow 2\, \delta \tilde{\Gamma}^{\eta}_{\rho(\mu} K_{\nu)\eta} v^\rho &= \Big(K^{\rho}{}_{\lambda} \tau_{(\mu} K_{\nu)\rho} + v^\sigma \tau_{\sigma\gamma} K^{\gamma}{}_{(\nu} h_{\mu)\lambda} + v^\sigma \tau_{\sigma(\mu} K_{\nu)\lambda} \Big) \delta v^\lambda \\
&\quad + \big(K^{\sigma}{}_{\lambda} K_{\sigma(\nu} h_{\mu)\rho} - K_{\lambda(\mu} K_{\nu)\rho} \big) \delta h^{\rho\lambda} + 3\, K_{\lambda(\nu} \tilde{\nabla}_{\mu)} \delta v^\lambda \\
&\quad + \Big(K^{\sigma}{}_{(\nu} h_{\mu)\lambda} + v^\sigma \tau_{(\mu} K_{\nu)\lambda} \Big) \tilde{\nabla}_\sigma \delta v^\lambda - K_{\lambda(\nu} h_{\mu)\rho} v^\sigma \tilde{\nabla}_\sigma \delta h^{\rho\lambda}.
\end{aligned} \tag{C.41}
$$

Now for $h^{\nu\rho} \delta\Big(h_{\mu\nu} \tilde{\nabla}_\rho\Big) K$ and $v^\rho \delta\Big(h_{\mu\nu} \tilde{\nabla}_\rho K\Big)$ we have:

$$
\begin{aligned}
h^{\nu\rho} \delta\Big(h_{\mu\nu} \tilde{\nabla}_\rho K\Big) &= h^{\nu\rho} \Big(\delta h_{\mu\nu} \tilde{\nabla}_\rho K + h_{\mu\nu} \tilde{\nabla}_\rho \delta K \Big) \\
&= h^{\nu\rho} \Big(\Big(-h_{\mu\gamma} h_{\nu\lambda} \delta h^{\lambda\gamma} + 2\, h_{\lambda(\mu} \tau_{\nu)} \delta v^\lambda \Big) \tilde{\nabla}_\rho K + h_{\mu\nu} \tilde{\nabla}_\rho \delta K \Big) \\
&= \big(\delta^\sigma_\lambda + v^\sigma \tau_\lambda \big) \Big(\tau_\mu \delta v^\lambda - h_{\mu\rho} \delta h^{\lambda\rho} \Big) \tilde{\nabla}_\sigma K + \Big(\delta^\rho_\mu + v^\rho \tau_\mu \Big) \tilde{\nabla}_\rho \delta K,
\end{aligned} \tag{C.42}
$$

and

$$
\begin{aligned}
v^\rho \delta\Big(h_{\mu\nu} \tilde{\nabla}_\rho K\Big) &= v^\rho \Big(\delta h_{\mu\nu} \tilde{\nabla}_\rho K + h_{\mu\nu} \tilde{\nabla}_\rho \delta K \Big) \\
&= v^\rho \Big(-h_{\mu\gamma} h_{\nu\lambda} \delta h^{\lambda\gamma} + 2\, h_{\lambda(\mu} \tau_{\nu)} \delta v^\lambda \Big) \tilde{\nabla}_\rho K + v^\rho h_{\mu\nu} \tilde{\nabla}_\rho \delta K.
\end{aligned} \tag{C.43}
$$

Having derived the variation of all the variables we can then start combining them. We start by finding the variation of terms four and five in (5.33) which will appear in either expansion of the NLO Lagrangian. Thus, we have for the fourth term:

$$
\begin{aligned}
v^{\mu}_{(1)}\delta \overset{(2)}{G}{}^{v}_{\mu} = v^{\mu}_{(1)}\Bigg[& -\frac{1}{2}\tau_\mu\big(\delta\big(K^{\gamma\sigma}K_{\gamma\sigma}\big)-2\,K\delta K\big)-\frac{1}{2}\Big(K^{\gamma\sigma}K_{\gamma\sigma}-K^2\Big)\delta\tau_\mu \\
& +\tilde\nabla_\rho\big(K_{\mu\nu}-Kh_{\mu\nu}\big)\delta h^{\nu\rho}+h^{\nu\rho}\delta\Big(\tilde\nabla_\rho\big(K_{\mu\nu}-Kh_{\mu\nu}\big)\Big)\Bigg] \\
= v^{\mu}_{(1)}\Bigg[& -\frac{1}{2}\tau_\mu\Big(-\big(2\,\tau_\lambda K^{\rho\sigma}K_{\rho\sigma}+K^{\sigma}{}_{\lambda}v^{\gamma}\tau_{\gamma\sigma}\big)\delta v^\lambda-2\,K^{\rho}{}_{\lambda}\tilde\nabla_\rho\delta v^\lambda+K_{\rho\lambda}v^\sigma\tilde\nabla_\sigma\delta h^{\rho\lambda}\Big) \\
& +\tau_\mu K\bigg(-(v^\sigma\tau_{\sigma\lambda}+\tau_\lambda K)\delta v^\lambda-\big(\delta^\sigma_\lambda+v^\sigma\tau_\lambda\big)\tilde\nabla_\sigma\delta v^\lambda+\frac{1}{2}h_{\rho\lambda}v^\sigma\tilde\nabla_\sigma\delta h^{\rho\lambda}\bigg) \\
& -\frac{1}{2}\Big(K^{\gamma\sigma}K_{\gamma\sigma}-K^2\Big)\Big(-\tau_\lambda h_{\mu\rho}\delta h^{\rho\lambda}+\tau_\lambda\tau_\mu\delta v^\lambda\Big)+\tilde\nabla_\rho\big(K_{\mu\nu}-Kh_{\mu\nu}\big)\delta h^{\nu\rho} \\
& +h^{\rho\nu}\tilde\nabla_\rho\delta K_{\mu\nu}+K^{\rho}{}_{\lambda}\tau_\mu\tilde\nabla_\rho\delta v^\lambda \\
& -\bigg(\frac{1}{2}K^{\rho}{}_{\lambda}\tau_{\mu\rho}-2\,K^{\rho\gamma}K_{\rho\gamma}\tau_\mu\tau_\lambda-K^{\gamma}{}_{\lambda}\tau_\mu v^\sigma\tau_{\sigma\gamma}-2\,K^{\sigma}{}_{\mu}v^\gamma\tau_\lambda\tau_{\gamma\sigma}\delta v^\lambda\bigg)\delta v^\lambda \\
& -\Big(K^{\sigma\gamma}K_{\sigma\gamma}\tau_\lambda h_{\mu\rho}-K^{\gamma}{}_{\rho}K_{\mu\gamma}\tau_\lambda \\
& \qquad -2\,K^{\gamma}{}_{\rho}\tau_\mu K_{\lambda\gamma}+K_{\mu\lambda}K\tau_\rho-2\,K^{\sigma}{}_{\mu}K_{\sigma\lambda}\tau_\rho\Big)\delta h^{\rho\lambda} \\
& +\bigg(-\frac{1}{2}K^{\sigma}{}_{\mu}h_{\rho\lambda}+K_{\mu\lambda}v^\sigma\tau_\rho+\delta^\sigma_\rho K_{\mu\lambda}+\frac{1}{2}K_{\rho\lambda}\Big(\delta^\sigma_\mu-v^\sigma\tau_\mu\Big)\bigg)\tilde\nabla_\sigma\delta h^{\rho\lambda} \\
& -\big(\delta^\sigma_\lambda+v^\sigma\tau_\lambda\big)\Big(\tau_\mu\delta v^\lambda-h_{\mu\rho}\delta h^{\rho\lambda}\Big)\tilde\nabla_\sigma K-\Big(\delta^\rho_\mu+v^\rho\tau_\mu\Big)\tilde\nabla_\rho\delta K\Bigg] \\
= v^{\mu}_{(1)}\Bigg[& \Bigg(\tau_\mu\bigg(\frac{3}{2}K^{\sigma}{}_{\lambda}v^\gamma\tau_{\gamma\sigma}-Kv^\sigma\tau_{\sigma\lambda}+\frac{1}{2}K^2\tau_\lambda-\big(\delta^\sigma_\lambda+v^\sigma\tau_\lambda\big)\tilde\nabla_\sigma K+2\,K^{\rho\gamma}K_{\rho\gamma}\tau_\lambda\bigg) \\
& \qquad -\frac{1}{2}K^{\rho}{}_{\lambda}\tau_{\mu\rho}+2\,K^{\sigma}{}_{\mu}v^\gamma\tau_\lambda\tau_{\gamma\sigma}\Bigg)\delta v^\lambda \\
& +\bigg(2\,K^{\rho}{}_{\lambda}\tau_\mu-\tau_\mu K\Big(\delta^\rho_\lambda+v^\rho\tau_\lambda\Big)\bigg)\tilde\nabla_\rho\delta v^\lambda \\
& +\bigg(\frac{3}{2}K^{\gamma\sigma}K_{\gamma\sigma}\tau_\lambda h_{\mu\rho}-\frac{1}{2}K^2\tau_\lambda h_{\mu\rho}+\tilde\nabla_\rho K_{\mu\lambda}+2\,K^{\gamma}{}_{\rho}K_{\mu\gamma}\tau_\lambda \\
& \qquad +2\,K^{\gamma}{}_{\rho}\tau_\mu K_{\lambda\gamma}-K_{\mu\lambda}K\tau_\rho+v^\sigma\tau_\lambda h_{\mu\rho}\tilde\nabla_\sigma K\bigg)\delta h^{\rho\lambda} \\
& +\bigg(v^\sigma\Big(2\,K_{\lambda[\mu}\tau_{\rho]}+\frac{1}{2}\tau_\mu Kh_{\rho\lambda}\Big)-\frac{1}{2}K^{\sigma}{}_{\mu}h_{\rho\lambda}+\delta^\sigma_\rho K_{\mu\lambda}+\frac{1}{2}K_{\rho\lambda}\delta^\sigma_\mu\bigg)\tilde\nabla_\sigma\delta h^{\rho\lambda} \\
& \qquad +h^{\rho\nu}\tilde\nabla_\rho\delta K_{\mu\nu}-\Big(\delta^\rho_\mu+v^\rho\tau_\mu\Big)\tilde\nabla_\rho\delta K\Bigg].
\end{aligned}
\tag{C.44}
$$

Using again (5.16) for the covariant derivatives of $\delta K_{\mu\nu}$ and δK:

$$
\begin{aligned}
&\int_M e\, v^{\mu}_{(1)}\Big[h^{\rho\nu}\tilde{\nabla}_{\rho}\delta K_{\mu\nu} - \Big(\delta^{\rho}_{\mu} + v^{\rho}\tau_{\mu}\Big)\tilde{\nabla}_{\rho}\delta K\Big]\,\mathrm{d}^{d+1}x \\
&\quad = \int_M \Big[\partial_{\rho}\Big(e v^{\mu}_{(1)}\Big(h^{\rho\nu}\delta K_{\mu\nu} - \Big(\delta^{\rho}_{\mu} + v^{\rho}\tau_{\mu}\Big)\delta K\Big)\Big) \\
&\qquad\qquad - eK\,\underbrace{\tau_{\rho}\Big(v^{\mu}_{(1)}\Big(h^{\rho\nu}\delta K_{\mu\nu} - \Big(\delta^{\rho}_{\mu} + v^{\rho}\tau_{\mu}\Big)\delta K\Big)\Big)}_{=0} \\
&\qquad\qquad - e\tilde{\nabla}_{\rho}\Big(v^{\mu}_{(1)}h^{\rho\nu}\Big)\delta K_{\mu\nu} + e\tilde{\nabla}_{\rho}\Big(v^{\mu}_{(1)}\Big(\delta^{\rho}_{\mu} + v^{\rho}\tau_{\mu}\Big)\Big)\delta K\Big]\,\mathrm{d}^{d+1}x \\
&\quad \approx \int_M e\Big[-\tilde{\nabla}_{\sigma}\Big(v^{\mu}_{(1)}h^{\sigma\nu}\Big)\Big(\Big(-\tau_{\lambda}K_{\mu\nu} + K_{\lambda(\nu}\tau_{\mu)} - \frac{1}{2}h_{\lambda\mu}v^{\rho}\tau_{\rho\nu} - \frac{1}{2}h_{\lambda\nu}v^{\rho}\tau_{\rho\mu}\Big)\delta v^{\lambda} \\
&\qquad\qquad - h_{\lambda(\nu}\tilde{\nabla}_{\mu)}\delta v^{\lambda} - v^{\rho}h_{\lambda(\mu}\tau_{\nu)}\tilde{\nabla}_{\rho}\delta v^{\lambda} \\
&\qquad\qquad - K_{\lambda(\nu}h_{\mu)\rho}\delta h^{\rho\lambda} + \cancel{\frac{1}{2}h_{\mu\rho}h_{\nu\lambda}v^{\gamma}\tilde{\nabla}_{\gamma}\delta h^{\rho\lambda}}\Big) \\
&\qquad + \tilde{\nabla}_{\sigma}\Big(v^{\mu}_{(1)}h^{\sigma\nu}\Big)h_{\nu\mu}\Big(-\big(v^{\rho}\tau_{\rho\lambda} + \tau_{\lambda}K\big)\delta v^{\lambda} \\
&\qquad\qquad - \Big(\delta^{\rho}_{\lambda} + v^{\rho}\tau_{\lambda}\Big)\tilde{\nabla}_{\rho}\delta v^{\lambda} + \cancel{\frac{1}{2}h_{\rho\lambda}v^{\gamma}\tilde{\nabla}_{\gamma}\delta h^{\rho\lambda}}\Big)\Big]\,\mathrm{d}^{d+1}x \\
&\quad \approx \int_M e\Big[\tilde{\nabla}_{\sigma}v^{\mu}_{(1)}\Big(\Big(\tau_{\lambda}K^{\sigma}{}_{\mu} - \frac{1}{2}K^{\sigma}{}_{\lambda}\tau_{\mu} + h^{\sigma\nu}\Big(\frac{1}{2}h_{\lambda\mu}v^{\rho}\tau_{\rho\nu} + \frac{1}{2}h_{\lambda\nu}v^{\rho}\tau_{\rho\mu} \\
&\qquad\qquad - h_{\nu\mu}\big(v^{\rho}\tau_{\rho\lambda} + \tau_{\lambda}K\big)\Big)\Big)\delta v^{\lambda} \\
&\qquad\qquad + \frac{1}{2}\big(K^{\sigma}{}_{\lambda}h_{\mu\rho} + h^{\sigma\nu}K_{\lambda\mu}h_{\nu\rho}\big)\delta h^{\rho\lambda} \\
&\qquad\qquad + \Big(\frac{1}{2}h^{\sigma\nu}h_{\lambda\nu}\Big(\delta^{\rho}_{\mu} + v^{\rho}\tau_{\mu}\Big) + \frac{1}{2}h^{\sigma\rho}h_{\lambda\mu} \\
&\qquad\qquad - h^{\sigma\nu}h_{\nu\mu}\Big(\delta^{\rho}_{\lambda} + v^{\rho}\tau_{\lambda}\Big)\Big)\tilde{\nabla}_{\rho}\delta v^{\lambda}\Big) \\
&\qquad + v^{\sigma}\tau_{\sigma\gamma}v^{\mu}_{(1)}\Big(\Big(K^{\gamma}{}_{\mu}\tau_{\lambda} - \frac{1}{2}K^{\gamma}{}_{\lambda}\tau_{\mu} + \frac{1}{2}h^{\nu\gamma}h_{\lambda\mu}v^{\rho}\tau_{\rho\nu} \\
&\qquad\qquad + \frac{1}{2}\delta^{\gamma}_{\lambda}v^{\rho}\tau_{\rho\mu} - \delta^{\gamma}_{\mu}\big(v^{\rho}\tau_{\rho\lambda} + \tau_{\lambda}K\big)\Big)\delta v^{\lambda} \\
&\qquad\qquad + \frac{1}{2}\Big(K^{\gamma}{}_{\lambda}h_{\mu\rho} + \delta^{\gamma}_{\rho}K_{\lambda\mu}\Big)\delta h^{\rho\lambda} \\
&\qquad\qquad + \Big(\frac{1}{2}\delta^{\gamma}_{\lambda}\Big(\delta^{\rho}_{\mu} + v^{\rho}\tau_{\mu}\Big) + \frac{1}{2}h^{\rho\gamma}h_{\lambda\mu} \\
&\qquad\qquad - \delta^{\gamma}_{\mu}\Big(\delta^{\rho}_{\lambda} + v^{\rho}\tau_{\lambda}\Big)\Big)\tilde{\nabla}_{\rho}\delta v^{\lambda}\Big)\Big]\,\mathrm{d}^{d+1}x,
\end{aligned}
\tag{C.45}
$$

and then doing the same for all the $\tilde{\nabla}_\rho \delta v^\lambda$ and $\tilde{\nabla}_\sigma \delta h^{\rho\lambda}$ terms:

$$\begin{aligned}
&\int_M e\Bigg[v^\mu_{(1)}\Big(2\,K^\rho{}_\lambda\tau_\mu - \tau_\mu K\Big(\delta^\rho_\lambda + v^\rho\tau_\lambda\Big) \\
&\qquad + v^\sigma\tau_{\sigma\gamma}\Big(\frac{1}{2}\delta^\gamma_\lambda\Big(\delta^\rho_\mu + v^\rho\tau_\mu\Big) + \frac{1}{2}h^{\rho\gamma}h_{\lambda\mu} - \delta^\gamma_\mu\Big(\delta^\rho_\lambda + v^\rho\tau_\lambda\Big)\Big)\Big)\tilde{\nabla}_\rho\delta v^\lambda \\
&\quad + \tilde{\nabla}_\sigma v^\mu_{(1)}\Big(\frac{1}{2}h^{\sigma\nu}h_{\lambda\nu}\Big(\delta^\rho_\mu + v^\rho\tau_\mu\Big) + \frac{1}{2}h^{\sigma\rho}h_{\lambda\mu} - h^{\sigma\nu}h_{\nu\mu}\Big(\delta^\rho_\lambda + v^\rho\tau_\lambda\Big)\Big)\tilde{\nabla}_\rho\delta v^\lambda\Bigg]\,\mathrm{d}^{d+1}x \\
&= \int_M \Bigg[\partial_\rho\Bigg(e v^\mu_{(1)}\Big(2\,K^\rho{}_\lambda\tau_\mu - \tau_\mu K\Big(\delta^\rho_\lambda + v^\rho\tau_\lambda\Big) \\
&\qquad + v^\sigma\tau_{\sigma\gamma}\Big(\frac{1}{2}\delta^\gamma_\lambda\Big(\delta^\rho_\mu + v^\rho\tau_\mu\Big) + \frac{1}{2}h^{\rho\gamma}h_{\lambda\mu} - \delta^\gamma_\mu\Big(\delta^\rho_\lambda + v^\rho\tau_\lambda\Big)\Big)\Big)\delta v^\lambda \\
&\qquad + e\tilde{\nabla}_\sigma v^\mu_{(1)}\Big(\frac{1}{2}h^{\sigma\nu}h_{\lambda\nu}\Big(\delta^\rho_\mu + v^\rho\tau_\mu\Big) \\
&\qquad\qquad + \frac{1}{2}h^{\sigma\rho}h_{\lambda\mu} - h^{\sigma\nu}h_{\nu\mu}\Big(\delta^\rho_\lambda + v^\rho\tau_\lambda\Big)\Big)\delta v^\lambda\Bigg) \\
&\qquad - eK\tau_\rho\Bigg(v^\mu_{(1)}\Big(2\underbrace{K^\rho{}_\lambda}_{=0}\tau_\mu - \tau_\mu K\underbrace{\Big(\delta^\rho_\lambda + v^\rho\tau_\lambda\Big)}_{=0} \\
&\qquad\qquad + v^\sigma\tau_{\sigma\gamma}\Big(\frac{1}{2}\delta^\gamma_\lambda\underbrace{\Big(\delta^\rho_\mu + v^\rho\tau_\mu\Big)}_{=0} \\
&\qquad\qquad\qquad + \frac{1}{2}\underbrace{h^{\rho\gamma}}_{=0}h_{\lambda\mu} - \delta^\gamma_\mu\underbrace{\Big(\delta^\rho_\lambda + v^\rho\tau_\lambda\Big)}_{=0}\Big)\Big) \\
&\qquad\qquad + \tilde{\nabla}_\sigma v^\mu_{(1)}\Big(\frac{1}{2}h^{\sigma\nu}h_{\lambda\nu}\underbrace{\Big(\delta^\rho_\mu + v^\rho\tau_\mu\Big)}_{=0} \\
&\qquad\qquad\qquad + \frac{1}{2}\underbrace{h^{\sigma\rho}}_{=0}h_{\lambda\mu} - h^{\sigma\nu}h_{\nu\mu}\underbrace{\Big(\delta^\rho_\lambda + v^\rho\tau_\lambda\Big)}_{=0}\Big)\Bigg)\delta v^\lambda \\
&\qquad - e\tilde{\nabla}_\rho\Bigg(v^\mu_{(1)}\Big(2\,K^\rho{}_\lambda\tau_\mu - \tau_\mu K\Big(\delta^\rho_\lambda + v^\rho\tau_\lambda\Big) \\
&\qquad\qquad + v^\sigma\tau_{\sigma\gamma}\Big(\frac{1}{2}\delta^\gamma_\lambda\Big(\delta^\rho_\mu + v^\rho\tau_\mu\Big) + \frac{1}{2}h^{\rho\gamma}h_{\lambda\mu} - \delta^\gamma_\mu\Big(\delta^\rho_\lambda + v^\rho\tau_\lambda\Big)\Big)\Big) \\
&\qquad\qquad + \tilde{\nabla}_\sigma v^\mu_{(1)}\Big(\frac{1}{2}h^{\sigma\nu}h_{\lambda\nu}\Big(\delta^\rho_\mu + v^\rho\tau_\mu\Big) \\
&\qquad\qquad\qquad + \frac{1}{2}h^{\sigma\rho}h_{\lambda\mu} - h^{\sigma\nu}h_{\nu\mu}\Big(\delta^\rho_\lambda + v^\rho\tau_\lambda\Big)\Big)\Bigg)\delta v^\lambda\Bigg]\,\mathrm{d}^{d+1}x
\end{aligned}$$

$$
\begin{aligned}
\approx \int_M e\Bigg[\Bigg(&-v^\mu_{(1)}\Big(2\,\tilde\nabla_\rho K^\rho{}_\lambda\tau_\mu - K^\rho{}_\lambda\left(\tau_{\mu\rho}+v^\gamma\tau_\mu\tau_{\gamma\rho}\right) - \tau_\mu\tilde\nabla_\rho K\Big(\delta^\rho_\lambda + v^\rho\tau_\lambda\Big)\\
&\qquad - \frac{1}{2}K\tau_{\lambda\mu} - \frac{1}{2}Kv^\gamma\left(\tau_\lambda\tau_{\gamma\mu}+\tau_\mu\tau_{\gamma\lambda}\right) + \frac{1}{2}v^\sigma\tilde\nabla_\rho\tau_{\sigma\lambda}\Big(\delta^\rho_\mu + v^\rho\tau_\mu\Big)\\
&\qquad + \frac{1}{2}v^\sigma\tilde\nabla_\rho\tau_{\sigma\gamma}h^{\rho\gamma}h_{\lambda\mu} - v^\sigma\tilde\nabla_\rho\tau_{\sigma\mu}\Big(\delta^\rho_\lambda + v^\rho\tau_\lambda\Big)\\
&\qquad + v^\sigma v^\rho\Big(\frac{1}{2}\tau_{\sigma\lambda}\tau_{\rho\mu} + \frac{1}{2}h^{\gamma\alpha}h_{\lambda\mu}\tau_{\sigma\gamma}\tau_{\rho\alpha} - \tau_{\sigma\mu}\tau_{\rho\lambda}\Big)\Big)\\
&-\tilde\nabla_\sigma v^\mu_{(1)}\Big(2\,K^\rho{}_\lambda\tau_\mu - \tau_\mu K\Big(\delta^\rho_\lambda + v^\rho\tau_\lambda\Big) + v^\rho h^{\sigma\gamma}h_{\lambda\mu}\tau_{\rho\gamma} + \frac{3}{4}v^\sigma\tau_{\mu\lambda}\\
&\qquad + \frac{5}{4}v^\gamma v^\sigma\tau_{\gamma\lambda}\tau_\mu - \frac{1}{4}v^\gamma v^\sigma\tau_{\gamma\mu}\tau_\lambda - \frac{1}{2}v^\rho\delta^\sigma_\lambda\tau_{\rho\mu} + \frac{3}{2}v^\rho\delta^\sigma_\mu\tau_{\rho\lambda}\Big)\\
&-\tilde\nabla_\rho\tilde\nabla_\sigma v^\mu_{(1)}\Big(\frac{1}{2}h^{\sigma\nu}h_{\lambda\nu}\Big(\delta^\rho_\mu + v^\rho\tau_\mu\Big)\\
&\qquad + \frac{1}{2}h^{\sigma\rho}h_{\lambda\mu} - h^{\sigma\nu}h_{\nu\mu}\Big(\delta^\rho_\lambda + v^\rho\tau_\lambda\Big)\Big)\Bigg)\delta v^\lambda\Bigg]\,\mathrm{d}^{d+1}x, \qquad \text{(C.46)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_M e\Bigg[v^\mu_{(1)}\Big(v^\sigma\Big(-\tau_\mu K_{\rho\lambda} + \frac{1}{2}\tau_\mu K h_{\rho\lambda} + K_{\mu\lambda}\tau_\rho\Big)\\
&\qquad\qquad - \frac{1}{2}K^\sigma{}_\mu h_{\rho\lambda} + \delta^\sigma_\rho K_{\mu\lambda} + \frac{1}{2}K_{\rho\lambda}\delta^\sigma_\mu\Big)\tilde\nabla_\sigma\delta h^{\rho\lambda}\Bigg]\,\mathrm{d}^{d+1}x\\
&= \int_M\Bigg[\partial_\sigma\Big(ev^\mu_{(1)}\Big(v^\sigma\Big(-\tau_\mu K_{\rho\lambda} + \frac{1}{2}\tau_\mu K h_{\rho\lambda} + K_{\mu\lambda}\tau_\rho\Big)\\
&\qquad\qquad - \frac{1}{2}K^\sigma{}_\mu h_{\rho\lambda} + \delta^\sigma_\rho K_{\mu\lambda} + \frac{1}{2}K_{\rho\lambda}\delta^\sigma_\mu\Big)\delta h^{\rho\lambda}\Big)\\
&\qquad - eKv^\mu_{(1)}\tau_\sigma\Big(v^\sigma\Big(-\tau_\mu K_{\rho\lambda} + \frac{1}{2}\tau_\mu K h_{\rho\lambda} + K_{\mu\lambda}\tau_\rho\Big)\\
&\qquad\qquad - \frac{1}{2}\underbrace{K^\sigma{}_\mu}_{=0} h_{\rho\lambda} + \delta^\sigma_\rho K_{\mu\lambda} + \frac{1}{2}K_{\rho\lambda}\delta^\sigma_\mu\Big)\delta h^{\rho\lambda}\\
&\qquad - e\tilde\nabla_\sigma\Big(v^\mu_{(1)}\Big(v^\sigma\Big(2\,K_{\lambda[\mu}\tau_{\rho]} + \frac{1}{2}\tau_\mu K h_{\rho\lambda}\Big)\\
&\qquad\qquad - \frac{1}{2}K^\sigma{}_\mu h_{\rho\lambda} + \delta^\sigma_\rho K_{\mu\lambda} + \frac{1}{2}K_{\rho\lambda}\delta^\sigma_\mu\Big)\Big)\delta h^{\rho\lambda}\Bigg]\,\mathrm{d}^{d+1}x\\
&\approx \int_M e\Bigg[v^\mu_{(1)}\Big(-\frac{3}{2}KK_{\rho\lambda}\tau_\mu + \frac{1}{2}\tau_\mu K^2 h_{\rho\lambda} + \frac{1}{2}\tilde\nabla_\sigma K^\sigma{}_\mu h_{\rho\lambda} - \tilde\nabla_\rho K_{\mu\lambda}\\
&\qquad - v^\sigma\tilde\nabla_\sigma\Big(-\tau_\mu K_{\rho\lambda} + \frac{1}{2}\tau_\mu K h_{\rho\lambda} + K_{\mu\lambda}\tau_\rho\Big) - \frac{1}{2}\tilde\nabla_\mu K_{\rho\lambda}\Big)\delta h^{\rho\lambda}
\end{aligned}
$$

$$- \tilde{\nabla}_\sigma v^\mu_{(1)} \bigg(v^\sigma \Big(2\, K_{\lambda[\mu}\tau_{\rho]} + \frac{1}{2}\tau_\mu K h_{\rho\lambda} \Big) - \frac{1}{2} K^\sigma{}_\mu h_{\rho\lambda} + \delta^\sigma_\rho K_{\mu\lambda} + \frac{1}{2} K_{\rho\lambda}\delta^\sigma_\mu \bigg) \delta h^{\rho\lambda} \bigg] \, \mathrm{d}^{d+1}x. \qquad \text{(C.47)}$$

Putting all of this together we arrive at:

$$\begin{aligned}
v^\mu_{(1)} \delta \overset{(2)}{G}{}^v_\mu \approx v^\mu_{(1)} \Bigg[\Bigg(& 2\,\tau_\mu K^{\rho\gamma} K_{\rho\gamma}\tau_\lambda + K^\sigma{}_\alpha \bigg(\frac{1}{2}\delta^\alpha_\lambda \tau_{\mu\sigma} + v^\gamma \tau_{\gamma\sigma}\Big(2\,\delta^\alpha_\lambda \tau_\mu + 3\,\delta^\alpha_\mu \tau_\lambda \Big)\bigg) \\
& + \frac{1}{2} K \big(\tau_{\lambda\mu} - \tau_\mu v^\sigma \tau_{\sigma\lambda} - v^\gamma \tau_\lambda \tau_{\gamma\mu} + K \tau_\mu \tau_\lambda \big) - 2\,\tilde{\nabla}_\rho K^\rho{}_\lambda \tau_\mu \\
& + v^\sigma \tilde{\nabla}_\rho \tau_{\sigma\gamma} \bigg(-\frac{1}{2}\delta^\gamma_\lambda \Big(\delta^\rho_\mu - v^\rho \tau_\mu \Big) + \delta^\gamma_\mu \Big(\delta^\rho_\lambda + v^\rho \tau_\lambda \Big) - \frac{1}{2} h^{\rho\gamma} h_{\lambda\mu} \bigg)\Bigg) \delta v^\lambda \\
+ \Bigg(& \bigg(\frac{3}{2} K^{\gamma\sigma} K_{\gamma\sigma} - \frac{1}{2} K^2 + v^\sigma \tilde{\nabla}_\sigma K \bigg) \tau_\lambda h_{\mu\rho} + \frac{1}{2}\Big(K^2 - v^\sigma \tilde{\nabla}_\sigma K \Big) \tau_\mu h_{\rho\lambda} \\
& + \bigg(2\, K^\gamma{}_\rho K_{\lambda\gamma} - \frac{3}{2} K K_{\rho\lambda} + v^\sigma \tilde{\nabla}_\sigma K_{\rho\lambda} \bigg) \tau_\mu - \Big(K_{\mu\lambda} K + v^\sigma \tilde{\nabla}_\sigma K_{\mu\lambda} \Big) \tau_\rho \\
& + \frac{1}{2} v^\sigma \tau_{\sigma\gamma} K^\gamma{}_\lambda h_{\mu\rho} - \frac{1}{2} v^\sigma \tau_{\sigma\rho} K_{\lambda\mu} + v^\sigma \tau_{\sigma\mu} \Big(K_{\rho\lambda} - \frac{1}{2} h_{\rho\lambda} K \Big) \\
& + 2\, K^\gamma{}_\rho K_{\mu\gamma} \tau_\lambda + \frac{1}{2} \tilde{\nabla}_\sigma K^\sigma{}_\mu h_{\rho\lambda} - \frac{1}{2} \tilde{\nabla}_\mu K_{\rho\lambda} \Bigg) \delta h^{\rho\lambda} \Bigg] \\
+ \tilde{\nabla}_\sigma v^\mu_{(1)} \Bigg[\Bigg(& K^\sigma{}_\gamma \Big(\delta^\gamma_\mu \tau_\lambda - \frac{5}{2} \delta^\gamma_\lambda \tau_\mu \Big) + K \Big(\delta^\sigma_\lambda \tau_\mu - \delta^\sigma_\mu \tau_\lambda \Big) + \frac{3}{4} v^\sigma \tau_{\gamma\mu} \big(\delta^\gamma_\lambda + v^\gamma \tau_\lambda \big) \\
& + v^\rho \tau_{\rho\gamma} \bigg(-\frac{5}{2} \delta^\sigma_\mu \delta^\gamma_\lambda - \frac{9}{4} \delta^\gamma_\lambda v^\sigma \tau_\mu + \delta^\sigma_\lambda \delta^\gamma_\mu - \frac{1}{2} h^{\sigma\gamma} h_{\lambda\mu} \bigg) \Bigg) \delta v^\lambda \\
+ \Bigg(& \frac{1}{2} K^\sigma{}_\gamma \Big(\delta^\gamma_\lambda h_{\mu\rho} + \delta^\gamma_\mu h_{\rho\lambda} \Big) - \frac{1}{2} v^\sigma \tau_\mu K h_{\rho\lambda} \\
& - \frac{1}{2} K_{\lambda\gamma} \bigg(\delta^\gamma_\mu \Big(\delta^\sigma_\rho + v^\sigma \tau_\rho \Big) + \delta^\gamma_\rho \Big(\delta^\sigma_\mu - 2\, v^\sigma \tau_\mu \Big) \bigg) \Bigg) \delta h^{\rho\lambda} \Bigg] \\
+ \tilde{\nabla}_\rho \tilde{\nabla}_\sigma v^\mu_{(1)} \Bigg[& \frac{1}{2} \Big(\delta^\sigma_\lambda \delta^\rho_\mu + \delta^\sigma_\lambda v^\rho \tau_\mu + v^\sigma \tau_\lambda \delta^\rho_\mu + v^\sigma \tau_\lambda v^\rho \tau_\mu \Big) - \frac{1}{2} h^{\sigma\rho} h_{\lambda\mu} \Bigg] \delta v^\lambda. \qquad \text{(C.48)}
\end{aligned}$$

Now for the fifth term of (5.33) we repeat in a similar manner:

$$\begin{aligned}
\frac{1}{2} h^{\mu\nu}_{(1)} \delta \overset{(2)}{G}{}^h_{\mu\nu} = \frac{1}{2} h^{\mu\nu}_{(1)} \bigg[& -\frac{1}{2} h_{\mu\nu} \delta \big(K^{\rho\sigma} K_{\rho\sigma} \big) - \frac{1}{2} \Big(K^{\rho\sigma} K_{\rho\sigma} + K^2 \Big) \delta h_{\mu\nu} + K_{\mu\nu} \delta K \\
& + K \delta K_{\mu\nu} - \tilde{\nabla}_\lambda \big(K_{\mu\nu} - K h_{\mu\nu} \big) \delta v^\lambda - v^\lambda \delta \Big(\tilde{\nabla}_\lambda \big(K_{\mu\nu} - K h_{\mu\nu} \big) \Big) \bigg] \\
= \frac{1}{2} h^{\mu\nu}_{(1)} \bigg[& -\frac{1}{2} h_{\mu\nu} \Big(-\big(2\,\tau_\lambda K^{\sigma\gamma} K_{\sigma\gamma} + K^\gamma{}_\lambda v^\sigma \tau_{\sigma\gamma} \big) \delta v^\lambda
\end{aligned}$$

$$
\begin{aligned}
&\qquad - 2\, K^{\sigma}{}_{\lambda} \tilde{\nabla}_{\sigma} \delta v^{\lambda} + K_{\rho\lambda} v^{\sigma} \tilde{\nabla}_{\sigma} \delta h^{\rho\lambda} \Big) - \tilde{\nabla}_{\lambda} \big(K_{\mu\nu} - K h_{\mu\nu} \big) \delta v^{\lambda} \\
& - \frac{1}{2} \Big(K^{\sigma\gamma} K_{\sigma\gamma} + K^{2} \Big) \Big(- h_{\mu\rho} h_{\nu\lambda} \delta h^{\rho\lambda} + 2\, h_{\lambda(\mu} \tau_{\nu)} \delta v^{\lambda} \Big) \\
& + K_{\mu\nu} \Bigg(- (v^{\sigma} \tau_{\sigma\lambda} + \tau_{\lambda} K) \delta v^{\lambda} \\
&\qquad - \big(\delta^{\sigma}_{\lambda} + v^{\sigma} \tau_{\lambda} \big) \tilde{\nabla}_{\sigma} \delta v^{\lambda} + \frac{1}{2} h_{\rho\lambda} v^{\sigma} \tilde{\nabla}_{\sigma} \delta h^{\rho\lambda} \Bigg) \\
& + K \Bigg(\Bigg(- \tau_{\lambda} K_{\mu\nu} + K_{\lambda(\nu} \tau_{\mu)} - \frac{1}{2} h_{\lambda\mu} v^{\sigma} \tau_{\sigma\nu} - \frac{1}{2} h_{\lambda\nu} v^{\sigma} \tau_{\sigma\mu} \Bigg) \delta v^{\lambda} \\
&\qquad - h_{\sigma(\nu} \tilde{\nabla}_{\mu)} \delta v^{\sigma} - v^{\sigma} h_{\alpha(\mu} \tau_{\nu)} \tilde{\nabla}_{\sigma} \delta v^{\alpha} \\
&\qquad - K_{\lambda(\nu} h_{\mu)\rho} \delta h^{\rho\lambda} + \frac{1}{2} h_{\mu\rho} h_{\nu\lambda} v^{\sigma} \tilde{\nabla}_{\sigma} \delta h^{\rho\lambda} \Bigg) \\
& - v^{\rho} \tilde{\nabla}_{\rho} \delta K_{\mu\nu} + \Big(K^{\rho}{}_{\lambda} \tau_{(\mu} K_{\nu)\rho} + v^{\sigma} \tau_{\sigma\gamma} K^{\gamma}{}_{(\nu} h_{\mu)\lambda} + v^{\sigma} \tau_{\sigma(\mu} K_{\nu)\lambda} \Big) \delta v^{\lambda} \\
& + \big(K^{\sigma}{}_{\lambda} K_{\sigma(\nu} h_{\mu)\rho} - K_{\lambda(\mu} K_{\nu)\rho} \big) \delta h^{\rho\lambda} + 3\, K_{\lambda(\nu} \tilde{\nabla}_{\mu)} \delta v^{\lambda} \\
& + \Big(K^{\sigma}{}_{(\nu} h_{\mu)\lambda} + v^{\sigma} \tau_{(\mu} K_{\nu)\lambda} \Big) \tilde{\nabla}_{\sigma} \delta v^{\lambda} - K_{\lambda(\nu} h_{\mu)\rho} v^{\sigma} \tilde{\nabla}_{\sigma} \delta h^{\rho\lambda} \\
& + v^{\sigma} \Big(2\, h_{\lambda(\mu} \tau_{\nu)} \delta v^{\lambda} - h_{\mu\rho} h_{\nu\lambda} \delta h^{\rho\lambda} \Big) \tilde{\nabla}_{\sigma} K + v^{\rho} h_{\mu\nu} \tilde{\nabla}_{\rho} \delta K \Bigg] \\
= \frac{1}{2} h^{\mu\nu}_{(1)} \Bigg[& \Bigg(h_{\mu\nu} \tau_{\lambda} K^{\sigma\gamma} K_{\sigma\gamma} + \frac{1}{2} h_{\mu\nu} K^{\gamma}{}_{\lambda} v^{\sigma} \tau_{\sigma\gamma} - K^{\sigma\gamma} K_{\sigma\gamma} h_{\lambda\mu} \tau_{\nu} \\
&\quad - K^{2} h_{\lambda\mu} \tau_{\nu} - K_{\mu\nu} v^{\sigma} \tau_{\sigma\lambda} - 2\, K_{\mu\nu} \tau_{\lambda} K + K K_{\lambda\nu} \tau_{\mu} \\
&\quad - K h_{\lambda\mu} v^{\sigma} \tau_{\sigma\nu} - \tilde{\nabla}_{\lambda} K_{\mu\nu} + h_{\mu\nu} \tilde{\nabla}_{\lambda} K + K^{\rho}{}_{\lambda} \tau_{\mu} K_{\nu\rho} \\
&\quad + v^{\sigma} \tau_{\sigma\gamma} K^{\gamma}{}_{\nu} h_{\mu\lambda} + v^{\sigma} \tau_{\sigma\mu} K_{\nu\lambda} + 2\, v^{\sigma} \tilde{\nabla}_{\sigma} K h_{\lambda\mu} \tau_{\nu} \Bigg) \delta v^{\lambda} \\
& + \Bigg(\frac{1}{2} \Big(K^{\sigma\gamma} K_{\sigma\gamma} + K^{2} \Big) h_{\mu\rho} h_{\nu\lambda} - K K_{\lambda\nu} h_{\mu\rho} + K^{\sigma}{}_{\lambda} K_{\sigma\nu} h_{\mu\rho} \\
&\qquad - K_{\lambda\mu} K_{\nu\rho} - v^{\sigma} h_{\mu\rho} h_{\nu\lambda} \tilde{\nabla}_{\sigma} K \Bigg) \delta h^{\rho\lambda} \\
& - v^{\rho} \tilde{\nabla}_{\rho} \delta K_{\mu\nu} + v^{\rho} h_{\mu\nu} \tilde{\nabla}_{\rho} \delta K \\
& + \Bigg(h_{\mu\nu} K^{\sigma}{}_{\lambda} - \delta^{\sigma}_{\lambda} K_{\mu\nu} - v^{\sigma} \tau_{\lambda} K_{\mu\nu} - \delta^{\sigma}_{\mu} K h_{\lambda\nu} - K v^{\sigma} h_{\lambda\mu} \tau_{\nu} \\
&\qquad + 3\, \delta^{\sigma}_{\mu} K_{\lambda\nu} + K^{\sigma}{}_{\nu} h_{\mu\lambda} + v^{\sigma} \tau_{\mu} K_{\nu\lambda} \Bigg) \tilde{\nabla}_{\sigma} \delta v^{\lambda} \\
& + \Bigg(- \frac{1}{2} h_{\mu\nu} K_{\rho\lambda} v^{\sigma} + \frac{1}{2} K_{\mu\nu} h_{\rho\lambda} v^{\sigma} \\
&\qquad + \frac{1}{2} K h_{\mu\rho} h_{\nu\lambda} v^{\sigma} - K_{\lambda\nu} h_{\mu\rho} v^{\sigma} \Bigg) \tilde{\nabla}_{\sigma} \delta h^{\rho\lambda} \Bigg],
\end{aligned}
\tag{C.49}
$$

where once again we use (5.16) for the $\tilde{\nabla}_\rho \delta K_{\mu\nu}$ and $\tilde{\nabla}_\rho \delta K$ terms:

$$
\begin{aligned}
&\int_M \frac{e}{2}\left[h_{(1)}^{\mu\nu} v^\rho \left(-\tilde{\nabla}_\rho \delta K_{\mu\nu} + h_{\mu\nu}\tilde{\nabla}_\rho \delta K\right)\right] \mathrm{d}^{d+1}x \\
&\quad = \int_M \frac{1}{2}\Bigg[\partial_\rho\left(e h_{(1)}^{\mu\nu} v^\rho \left(-\delta K_{\mu\nu} + h_{\mu\nu}\delta K\right)\right) - eK\tau_\rho\left(h_{(1)}^{\mu\nu} v^\rho\left(-\delta K_{\mu\nu} + h_{\mu\nu}\delta K\right)\right) \\
&\qquad\qquad - e\tilde{\nabla}_\rho\left(h_{(1)}^{\mu\nu} v^\rho\right)\left(-\delta K_{\mu\nu} + h_{\mu\nu}\delta K\right)\Bigg] \mathrm{d}^{d+1}x \\
&\quad \approx \int_M \frac{e}{2}\Bigg[\left(v^\rho \tilde{\nabla}_\rho h_{(1)}^{\mu\nu} - K h_{(1)}^{\mu\nu}\right) \\
&\qquad\qquad \times \Bigg(- K_{\lambda\nu} h_{\mu\rho}\delta h^{\rho\lambda} + \left(\frac{1}{2}h_{\mu\rho}h_{\nu\lambda}v^\sigma - \frac{1}{2}h_{\mu\nu}h_{\rho\lambda}v^\sigma\right)\tilde{\nabla}_\sigma \delta h^{\rho\lambda} \\
&\qquad\qquad\quad + \left(-\tau_\lambda K_{\mu\nu} + K_{\lambda\nu}\tau_\mu - h_{\lambda\mu}v^\sigma\tau_{\sigma\nu} + h_{\mu\nu}v^\sigma\tau_{\sigma\lambda} + h_{\mu\nu}\tau_\lambda K\right)\delta v^\lambda \\
&\qquad\qquad\quad + \left(-\delta_\mu^\sigma h_{\lambda\nu} + h_{\mu\nu}\left(\delta_\lambda^\sigma + v^\sigma\tau_\lambda\right) - v^\sigma h_{\lambda\mu}\tau_\nu\right)\tilde{\nabla}_\sigma \delta v^\lambda\Bigg)\Bigg] \mathrm{d}^{d+1}x.
\end{aligned}
\tag{C.50}
$$

Then for $\tilde{\nabla}_\sigma \delta v^\lambda$ and $\tilde{\nabla}_\sigma \delta h^{\rho\lambda}$ we have:

$$
\begin{aligned}
&\int_M \frac{e}{2}\Bigg[\Bigg(h_{(1)}^{\mu\nu}\Big(h_{\mu\nu}K^\sigma{}_\lambda - \delta_\lambda^\sigma K_{\mu\nu} - v^\sigma\tau_\lambda K_{\mu\nu} + 3\,\delta_\mu^\sigma K_{\lambda\nu} \\
&\qquad\qquad + K^\sigma{}_\nu h_{\mu\lambda} + v^\sigma\tau_\mu K_{\nu\lambda} - K h_{\mu\nu}\left(\delta_\lambda^\sigma + v^\sigma\tau_\lambda\right)\Big) \\
&\qquad + v^\rho\tilde{\nabla}_\rho h_{(1)}^{\mu\nu}\left(-\delta_\mu^\sigma h_{\lambda\nu} + h_{\mu\nu}\left(\delta_\lambda^\sigma + v^\sigma\tau_\lambda\right) - v^\sigma h_{\lambda\mu}\tau_\nu\right)\Bigg)\tilde{\nabla}_\sigma\delta v^\lambda\Bigg] \\
&\quad = \int_M \frac{1}{2}\Bigg[\partial_\sigma\Bigg(e\Bigg(h_{(1)}^{\mu\nu}\Big(h_{\mu\nu}K^\sigma{}_\lambda - \delta_\lambda^\sigma K_{\mu\nu} - v^\sigma\tau_\lambda K_{\mu\nu} + 3\,\delta_\mu^\sigma K_{\lambda\nu} \\
&\qquad\qquad\qquad + K^\sigma{}_\nu h_{\mu\lambda} + v^\sigma\tau_\mu K_{\nu\lambda} - K h_{\mu\nu}\left(\delta_\lambda^\sigma + v^\sigma\tau_\lambda\right)\Big) \\
&\qquad\qquad + v^\rho\tilde{\nabla}_\rho h_{(1)}^{\mu\nu}\left(-\delta_\mu^\sigma h_{\lambda\nu} + h_{\mu\nu}\left(\delta_\lambda^\sigma + v^\sigma\tau_\lambda\right) - v^\sigma h_{\lambda\mu}\tau_\nu\right)\Bigg)\delta v^\lambda\Bigg) \\
&\qquad - eK\tau_\sigma\Bigg(h_{(1)}^{\mu\nu}\Big(h_{\mu\nu}\underbrace{K^\sigma{}_\lambda}_{=0} - \underbrace{\delta_\lambda^\sigma K_{\mu\nu} - v^\sigma\tau_\lambda}_{=0} K_{\mu\nu} + \overset{2}{\not{3}}\,\delta_\mu^\sigma K_{\lambda\nu} \\
&\qquad\qquad + \underbrace{K^\sigma{}_\nu}_{=0} h_{\mu\lambda} + \cancel{v^\sigma\tau_\mu K_{\nu\lambda}} - K h_{\mu\nu}\underbrace{\left(\delta_\lambda^\sigma + v^\sigma\tau_\lambda\right)}_{=0}\Big) \\
&\qquad\qquad + v^\rho\tilde{\nabla}_\rho h_{(1)}^{\mu\nu}\Big(- \delta_\mu^\sigma h_{\lambda\nu} + h_{\mu\nu}\underbrace{\left(\delta_\lambda^\sigma + v^\sigma\tau_\lambda\right)}_{=0} - v^\sigma h_{\lambda\mu}\tau_\nu\Big)\Bigg)\delta v^\lambda
\end{aligned}
$$

$$
\begin{aligned}
&- e\tilde{\nabla}_\sigma\bigg(h^{\mu\nu}_{(1)}\Big(h_{\mu\nu}K^\sigma{}_\lambda - \delta^\sigma_\lambda K_{\mu\nu} - v^\sigma\tau_\lambda K_{\mu\nu} + 3\,\delta^\sigma_\mu K_{\lambda\nu} \\
&\qquad + K^\sigma{}_\nu h_{\mu\lambda} + v^\sigma\tau_\mu K_{\nu\lambda} - K h_{\mu\nu}\big(\delta^\sigma_\lambda + v^\sigma\tau_\lambda\big)\Big) \\
&\qquad + v^\rho\tilde{\nabla}_\rho h^{\mu\nu}_{(1)}\Big(-\delta^\sigma_\mu h_{\lambda\nu} + h_{\mu\nu}\big(\delta^\sigma_\lambda + v^\sigma\tau_\lambda\big) - v^\sigma h_{\lambda\mu}\tau_\nu\Big)\bigg)\delta v^\lambda\Bigg] \\
\approx \int_M \frac{e}{2}\Bigg[&\bigg(-h^{\mu\nu}_{(1)}\Big(h_{\mu\nu}\tilde{\nabla}_\sigma K^\sigma{}_\lambda - \tilde{\nabla}_\lambda K_{\mu\nu} - v^\sigma\tau_{\sigma\lambda}K_{\mu\nu} - v^\sigma\tau_\lambda\tilde{\nabla}_\sigma K_{\mu\nu} + 3\,\tilde{\nabla}_\mu K_{\lambda\nu} \\
&\qquad + \tilde{\nabla}_\sigma K^\sigma{}_\nu h_{\mu\lambda} + v^\sigma\tau_{\sigma\mu}K_{\nu\lambda} + v^\sigma\tau_\mu\tilde{\nabla}_\sigma K_{\nu\lambda} - h_{\mu\nu}\tilde{\nabla}_\lambda K \\
&\qquad - h_{\mu\nu}v^\sigma\tilde{\nabla}_\sigma K\tau_\lambda - h_{\mu\nu}Kv^\sigma\tau_{\sigma\lambda} + 2\,K\tau_\mu K_{\lambda\nu}\Big) \\
&- \tilde{\nabla}_\sigma h^{\mu\nu}_{(1)}\Big(h_{\mu\nu}K^\sigma{}_\lambda - \delta^\sigma_\lambda K_{\mu\nu} - v^\sigma\tau_\lambda K_{\mu\nu} + 3\,\delta^\sigma_\mu K_{\lambda\nu} + K^\sigma{}_\nu h_{\mu\lambda} \\
&\qquad + v^\sigma\tau_\mu K_{\nu\lambda} - K h_{\mu\nu}\big(\delta^\sigma_\lambda + v^\sigma\tau_\lambda\big) + v^\sigma h_{\mu\nu}v^\rho\tau_{\rho\lambda} - v^\sigma v^\rho h_{\lambda\mu}\tau_{\rho\nu}\Big) \\
&- v^\rho\tilde{\nabla}_\sigma\tilde{\nabla}_\rho h^{\mu\nu}_{(1)}\Big(-\delta^\sigma_\mu h_{\lambda\nu} + h_{\mu\nu}\big(\delta^\sigma_\lambda + v^\sigma\tau_\lambda\big) - v^\sigma h_{\lambda\mu}\tau_\nu\Big)\bigg)\delta v^\lambda\Bigg],
\end{aligned}
\tag{C.51}
$$

and

$$
\begin{aligned}
&\int_M \frac{e}{2}\Bigg[\bigg(h^{\mu\nu}_{(1)}\Big(-\frac{1}{2}h_{\mu\nu}K_{\rho\lambda}v^\sigma + \frac{1}{2}K_{\mu\nu}h_{\rho\lambda}v^\sigma - K_{\lambda\nu}h_{\mu\rho}v^\sigma + \frac{1}{2}Kh_{\mu\nu}h_{\rho\lambda}v^\sigma\Big) \\
&\qquad + v^\gamma\tilde{\nabla}_\gamma h^{\mu\nu}_{(1)}\Big(\frac{1}{2}h_{\mu\rho}h_{\nu\lambda}v^\sigma - \frac{1}{2}h_{\mu\nu}h_{\rho\lambda}v^\sigma\Big)\bigg)\tilde{\nabla}_\sigma\delta h^{\rho\lambda}\Bigg] \\
&= \int_M \frac{1}{2}\Bigg[\partial_\sigma\Bigg(e\bigg(h^{\mu\nu}_{(1)}\Big(-\frac{1}{2}h_{\mu\nu}K_{\rho\lambda}v^\sigma + \frac{1}{2}K_{\mu\nu}h_{\rho\lambda}v^\sigma - K_{\lambda\nu}h_{\mu\rho}v^\sigma + \frac{1}{2}Kh_{\mu\nu}h_{\rho\lambda}v^\sigma\Big) \\
&\qquad + v^\gamma\tilde{\nabla}_\gamma h^{\mu\nu}_{(1)}\Big(\frac{1}{2}h_{\mu\rho}h_{\nu\lambda}v^\sigma - \frac{1}{2}h_{\mu\nu}h_{\rho\lambda}v^\sigma\Big)\bigg)\delta h^{\rho\lambda}\Bigg) \\
&\qquad + eK\bigg(h^{\mu\nu}_{(1)}\Big(-\frac{1}{2}h_{\mu\nu}K_{\rho\lambda} + \frac{1}{2}K_{\mu\nu}h_{\rho\lambda} - K_{\lambda\nu}h_{\mu\rho} + \frac{1}{2}Kh_{\mu\nu}h_{\rho\lambda}\Big) \\
&\qquad + v^\gamma\tilde{\nabla}_\gamma h^{\mu\nu}_{(1)}\Big(\frac{1}{2}h_{\mu\rho}h_{\nu\lambda} - \frac{1}{2}h_{\mu\nu}h_{\rho\lambda}\Big)\bigg)\delta h^{\rho\lambda} \\
&\qquad - e\tilde{\nabla}_\sigma\bigg(h^{\mu\nu}_{(1)}\Big(-\frac{1}{2}h_{\mu\nu}K_{\rho\lambda}v^\sigma + \frac{1}{2}K_{\mu\nu}h_{\rho\lambda}v^\sigma - K_{\lambda\nu}h_{\mu\rho}v^\sigma + \frac{1}{2}Kh_{\mu\nu}h_{\rho\lambda}v^\sigma\Big) \\
&\qquad + v^\gamma\tilde{\nabla}_\gamma h^{\mu\nu}_{(1)}\Big(\frac{1}{2}h_{\mu\rho}h_{\nu\lambda}v^\sigma - \frac{1}{2}h_{\mu\nu}h_{\rho\lambda}v^\sigma\Big)\bigg)\delta h^{\rho\lambda}\Bigg] \\
&\approx \int_M \frac{e}{2}\Bigg[\bigg(h^{\mu\nu}_{(1)}\Big(-\frac{1}{2}Kh_{\mu\nu}K_{\rho\lambda} + \frac{1}{2}KK_{\mu\nu}h_{\rho\lambda} - KK_{\lambda\nu}h_{\mu\rho} \\
&\qquad + \frac{1}{2}K^2h_{\mu\nu}h_{\rho\lambda} + \frac{1}{2}h_{\mu\nu}v^\sigma\tilde{\nabla}_\sigma K_{\rho\lambda} - \frac{1}{2}v^\sigma\tilde{\nabla}_\sigma K_{\mu\nu}h_{\rho\lambda}
\end{aligned}
$$

$$
\begin{aligned}
&+ v^\sigma \tilde{\nabla}_\sigma K_{\lambda\nu} h_{\mu\rho} - \frac{1}{2} v^\sigma \tilde{\nabla}_\sigma K h_{\mu\nu} h_{\rho\lambda} \Big) \\
&+ v^\sigma \tilde{\nabla}_\sigma h^{\mu\nu}_{(1)} \Big(\frac{1}{2} K h_{\mu\rho} h_{\nu\lambda} - K h_{\mu\nu} h_{\rho\lambda} + \frac{1}{2} h_{\mu\nu} K_{\rho\lambda} - \frac{1}{2} K_{\mu\nu} h_{\rho\lambda} + K_{\lambda\nu} h_{\mu\rho} \Big) \\
&- v^\sigma v^\gamma \tilde{\nabla}_\sigma \tilde{\nabla}_\gamma h^{\mu\nu}_{(1)} \Big(\frac{1}{2} h_{\mu\rho} h_{\nu\lambda} - \frac{1}{2} h_{\mu\nu} h_{\rho\lambda} \Big) \Big) \delta h^{\rho\lambda} \Big]. \qquad \text{(C.52)}
\end{aligned}
$$

Putting this all together we have

$$
\begin{aligned}
\frac{1}{2} h^{\mu\nu}_{(1)} \delta \overset{(2)}{G}{}^h_{\mu\nu} \approx \frac{1}{2} h^{\mu\nu}_{(1)} \Bigg[\Bigg(& K^{\sigma\gamma} K_{\sigma\gamma} \big(h_{\mu\nu} \tau_\lambda - h_{\lambda\mu} \tau_\nu \big) - K \big(K_{\mu\nu} \tau_\lambda + 2\, \tau_\mu K_{\lambda\nu} \big) \\
& + K^\rho{}_\gamma \Big(\delta^\gamma_\lambda \tau_\mu K_{\nu\rho} + v^\sigma \tau_{\sigma\rho} \Big(\frac{1}{2} \delta^\gamma_\lambda h_{\mu\nu} + \delta^\gamma_\nu h_{\mu\lambda} \Big) \Big) \\
& - K^2 \big(h_{\mu\nu} \tau_\lambda + h_{\lambda\mu} \tau_\nu \big) - \tilde{\nabla}_\sigma K^\sigma{}_\gamma \big(\delta^\gamma_\lambda h_{\mu\nu} + \delta^\gamma_\nu h_{\mu\lambda} \big) \\
& + \tilde{\nabla}_\sigma K_{\lambda\nu} \Big(\delta^\gamma_\mu v^\sigma \tau_\lambda - \Big(3\, \delta^\sigma_\mu + v^\sigma \tau_\mu \Big) \Big) \\
& + \tilde{\nabla}_\sigma K \Big(2\, \delta^\sigma_\lambda h_{\mu\nu} + 2\, v^\sigma h_{\lambda\mu} \tau_\nu + v^\sigma h_{\mu\nu} \tau_\lambda \Big) \Bigg) \delta v^\lambda \\
+ \Bigg(& \frac{1}{2} \Big(K^{\sigma\gamma} K_{\sigma\gamma} + K^2 - 2\, v^\sigma \tilde{\nabla}_\sigma K \Big) h_{\mu\rho} h_{\nu\lambda} \\
& + \Big(K^\sigma{}_\lambda K_{\sigma\nu} - K K_{\lambda\nu} + v^\sigma \tilde{\nabla}_\sigma K_{\lambda\nu} \Big) h_{\mu\rho} - K_{\lambda\mu} K_{\nu\rho} \\
& + \frac{1}{2} \Big(K^2 - v^\sigma \tilde{\nabla}_\sigma K \Big) h_{\mu\nu} h_{\rho\lambda} + \frac{1}{2} \Big(v^\sigma \tilde{\nabla}_\sigma K_{\rho\lambda} - K K_{\rho\lambda} \Big) h_{\mu\nu} \\
& + \frac{1}{2} \Big(K K_{\mu\nu} - v^\sigma \tilde{\nabla}_\sigma K_{\mu\nu} \Big) h_{\rho\lambda} \Bigg) \delta h^{\rho\lambda} \Bigg] \\
+ \frac{1}{2} \tilde{\nabla}_\sigma h^{\mu\nu}_{(1)} \Bigg[\Bigg(& K \big(\delta^\sigma_\lambda + 2\, v^\sigma \tau_\lambda \big) h_{\mu\nu} \\
& + K_{\gamma\nu} \Big(\delta^\gamma_\mu \delta^\sigma_\lambda - 3\, \delta^\gamma_\lambda \delta^\sigma_\mu \Big) - K^\sigma{}_\gamma \big(\delta^\gamma_\lambda h_{\mu\nu} + \delta^\gamma_\nu h_{\mu\lambda} \big) \Bigg) \delta v^\lambda \\
& + \frac{1}{2} v^\sigma \Big(K \big(h_{\mu\rho} h_{\nu\lambda} - 2\, h_{\mu\nu} h_{\rho\lambda} \big) + h_{\mu\nu} K_{\rho\lambda} - K_{\mu\nu} h_{\rho\lambda} \Big) \delta h^{\rho\lambda} \Bigg] \\
+ \frac{1}{2} v^\gamma \tilde{\nabla}_\gamma \tilde{\nabla}_\sigma h^{\mu\nu}_{(1)} \Bigg[\Bigg(& \delta^\sigma_\mu h_{\lambda\nu} - h_{\mu\nu} \big(\delta^\sigma_\lambda + v^\sigma \tau_\lambda \big) + v^\sigma h_{\lambda\mu} \tau_\nu \Bigg) \delta v^\lambda \\
& + \frac{1}{2} v^\sigma \Big(h_{\mu\rho} h_{\nu\lambda} + h_{\mu\nu} h_{\rho\lambda} \Big) \delta h^{\rho\lambda} \Bigg]. \qquad \text{(C.53)}
\end{aligned}
$$

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