Academic year 2023-24

Introduction to String Theory

Lecture notes by Troels Harmark

Contents

1 Classical String Theory

The goal of this chapter is to understand the classical theory of infinitely thin relativistic strings.

1.1 Action principle for the relativistic point particle

In this section we consider the action principle for the relativistic point particle. From the action principle one derives the dynamics of a free relativistic point particle moving in Minkowski space which should be familiar from the theory of special relativity. Our formulations of the action principle will be generalized in Secs. 1.2 and 1.3 to the action principle for the infinitely thin relativistic string.

For later convenience we consider the relativistic point particle in D space-time dimensions.

Notation: Position in D-dimensional space-time is x^μ, $\mu = 0, 1, 2, ..., D-1$. We also write $x^0 = t$ and $\vec{x} = (x^1, x^2, ..., x^{D-1})$.

We consider motion in D-dimensional Minkowski space. What is the action for a relativistic point particle? *It should pick the longest proper time between two events.* Natural candidate is:

$$S = -m \int_{\mathrm{WL}} dl \tag{1.1.1}$$

Here m is mass of the particle and $\int_{\mathrm{WL}} dl$ is the total proper time of the worldline (called "WL") of the particle which starts at the initial event E_i and ends at the final event E_f. One computes $\int_{\mathrm{WL}} dl$ by summing over the proper time for infinitesimal pieces of the worldline. For a given infinitesimal piece of the worldline we have two infinitesimally close events $(t, \vec{x})$ and $(t + dt, \vec{x} + d\vec{x})$ and dl is the proper time between these two events. See Fig. 1 for illustration. In D-dimensional Minkowski space we have

$$dl^2 = dt^2 - d\vec{x}^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - \cdots - (dx^{D-1})^2 = -\eta_{\mu\nu} dx^\mu dx^\nu \tag{1.1.2}$$

Here we have set the velocity of light $c = 1$. Note that in Eq. (1.1.2) we used Einsteins summation convention stating that we sum over repeated indices (which means we have two hidden sums in $-\eta_{\mu\nu} dx^\mu dx^\nu$ over μ and ν). Here $\eta_{\mu\nu}$ is the metric of D-dimensional

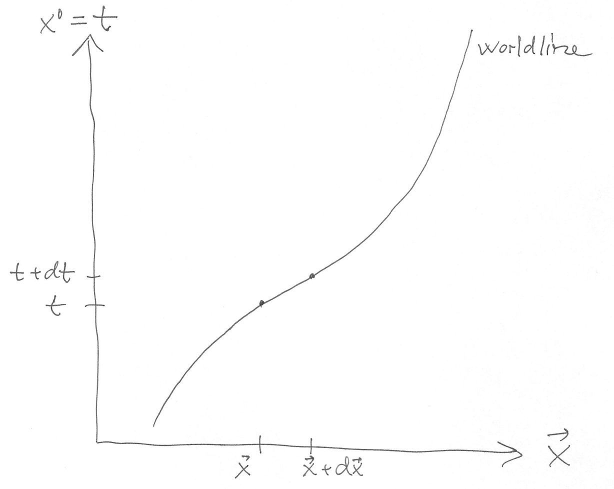

Figure 1: Illustration of an infinitesimal piece of the worldline of a particle.

Minkowski space

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \tag{1.1.3}$$

which is a D by D matrix that only is non-zero in the diagonal. Notice that in the above we are considering a massive particle of mass m. The worldline is thus a timelike path between the two events E_i and E_f which means that $dl^2 > 0$ on the path.

1.1.1 Manifestly physical action

We can choose to make the parametrization $\vec{x} = \vec{x}(t)$ in terms of the time coordinate t. Then

$$dl^2 = dt^2 - \left(\frac{d\vec{x}}{dt}\right)^2 dt^2 \tag{1.1.4}$$

which gives

$$dl = dt\sqrt{1 - \left(\frac{d\vec{x}}{dt}\right)^2} = dt\sqrt{1 - \vec{v}^2} \tag{1.1.5}$$

where we defined the velocity

$$\vec{v} = \frac{d\vec{x}}{dt} \tag{1.1.6}$$

We can also write this as $v^i = \frac{dx^i}{dt}$ for $i = 1, 2, ..., D-1$. The action (1.1.1) then becomes

$$S = -m\int_{t_i}^{t_f} dt\sqrt{1 - \vec{v}^2} \tag{1.1.7}$$

where t_i is the time for the initial event E_i and t_f is the time for the final event E_f.

The Lagrangian for the action (1.1.7) is

$$L = -m\sqrt{1 - \vec{v}^2} \tag{1.1.8}$$

The relativistic momentum for the particle is

$$p_i = \frac{\partial L}{\partial v^i} = \frac{mv^i}{\sqrt{1 - \vec{v}^2}} \tag{1.1.9}$$

with $i = 1, 2, ..., D - 1$. The corresponding Hamiltonian is

$$h = p_i v^i - L = \frac{m}{\sqrt{1 - \vec{v}^2}} = E \tag{1.1.10}$$

which we recognize as the energy E of a relativistic point particle. From the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial v^i} = \frac{\partial L}{\partial x^i} \tag{1.1.11}$$

we find the equations of motion

$$\frac{dp_i}{dt} = 0 \tag{1.1.12}$$

with $i = 1, 2, ..., D - 1$. Thus, we see that the equations of motion corresponds to the conservation of the relativistic momentum.

Advantage of the above formulation Eq. (1.1.7) *of the point particle action:* Degrees of freedom for the point particle are manifest, hence there are no auxiliary variables or Lagrange multipliers.

Disadvantage of the above formulation Eq. (1.1.7) *of the point particle action:* The action (1.1.7) is not manifestly covariant - We have chosen a specific way to parametrize the path of the point particle using the time coordinate t.

1.1.2 Covariant action

We now make the arbitrary parametrization of the worldline path using the parameter τ:

$$\tau \to X^\mu(\tau) \tag{1.1.13}$$

This signifies a map from the worldline, seen as a one-dimensional space parametrized by τ, to what we shall call the *target space*: Namely our D-dimensional space-time which we have chosen to be D-dimensional Minkowski space. See Fig. (2) for illustration.

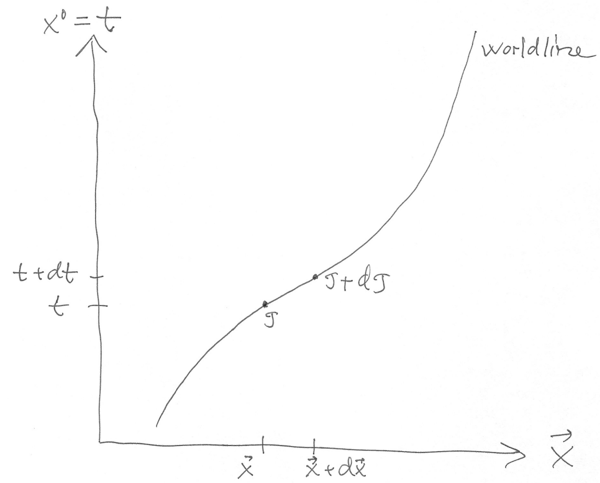

Figure 2: Illustration of an infinitesimal piece of the worldline of a particle.

Thus, a given value τ is mapped to an event $x^\mu = X^\mu(\tau)$ in D-dimensional Minkowski space.[1] We use here capital X to distinguish the map $X^\mu(\tau)$ from an arbitrary point x^μ in the target space (not necessarily on the worldline).

On the worldline we have

$$dX^\mu = \frac{dX^\mu}{d\tau} d\tau \tag{1.1.14}$$

Thus

$$dl^2 = -\eta_{\mu\nu} dX^\mu dX^\nu = -\eta_{\mu\nu} \frac{dX^\mu}{d\tau} \frac{dX^\nu}{d\tau} d\tau^2 = -\eta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu d\tau^2 \tag{1.1.15}$$

where we introduced the notation

$$\dot{X}^\mu = \frac{dX^\mu}{d\tau} \tag{1.1.16}$$

Considering the worldline as a one-dimensional space parametrized by τ we can thus interpret $\eta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu$ as the induced metric on this one-dimensional space.

The action (1.1.1) becomes

$$S = -m \int_{\mathrm{WL}} dl = -m \int d\tau \sqrt{-\eta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu} = -m \int d\tau \sqrt{-\dot{X}^2} \tag{1.1.17}$$

where we introduced the notation

$$\dot{X}^2 = \eta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu \tag{1.1.18}$$

[1] Above we stated that the worldline is a timelike path between the events E_i and E_f. Hence τ should take values $\tau_i \leq \tau \leq \tau_f$ where $x^\mu(\tau_i)$ and $x^\mu(\tau_f)$ are the two events E_i and E_f. For ease of notation we imagine that the path is extended ad infinitum in the following, thus with $\tau_i = -\infty$ and $\tau_f = \infty$.

In this parametrization the action is now manifestly covariant. The corresponding Lagrangian is

$$L = -m\sqrt{-\dot{X}^2} \tag{1.1.19}$$

The conjugate momentum is thus

$$p_\mu = \frac{\partial L}{\partial \dot{X}^\mu} = \frac{m\,\eta_{\mu\nu}\dot{X}^\nu}{\sqrt{-\dot{X}^2}} \tag{1.1.20}$$

The Euler-Lagrange equations

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{X}^\mu} = \frac{\partial L}{\partial X^\mu} \tag{1.1.21}$$

gives the equations of motion

$$\dot{p}_\mu = 0 \tag{1.1.22}$$

The above covariant formulation of the point particle gives the same physics as the formulation of Sec. 1.1.1. To see this one chooses the particular parametrization $\tau = t$ of the worldline. Notice then that $X^0(\tau) = \tau$ which means that X^0 is no longer an independent function. Moreover, $\dot{X}^0 = 1$ so that $-\dot{X}^2 = 1 - \vec{v}^2$ thus the action (1.1.17) indeed reduces to (1.1.7).

Consider now the symmetries of the action (1.1.17):

- Poincaré invariance: the action (1.1.17) is invariant under transformations

  $$X^\mu \to \Lambda^\mu{}_\nu X^\nu + c^\mu \tag{1.1.23}$$

  with $\eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$ where $\Lambda^\mu{}_\nu$ and c^μ are constants. This symmetry combines Lorentz invariance and translational invariance. It is a *global symmetry* on the worldline, in the sense that it does not depend on τ, so it acts the same everywhere on the worldline.

- Reparametrization invariance: the action (1.1.17) is invariant under transformations[2]

  $$\tau \to \tilde{\tau}(\tau) \tag{1.1.24}$$

  under which X^μ transforms as a scalar

  $$X^\mu(\tau) \to \tilde{X}^\mu(\tilde{\tau}) = X^\mu(\tau) \tag{1.1.25}$$

  This is a *local symmetry* on the worldline, as its action depends on where you are on the worldline. One can think of reparametrization invariance as a gauge

[2]Note that we require $d\tilde{\tau}/d\tau > 0$.

freeedom (since gauge transformations are local transformations that do not change the physics). In that sense we can for example think of $X^0(\tau) = \tau$ as a specific gauge choice.

Gauge-fixing and constraints

Comparing the covariant formulation of Sec. 1.1.2 with the manifestly physical formulation of Sec. 1.1.1 we have one extra position, velocity and momentum: X^0, $\dot{X}^0$ and p_0. *But the physics is equivalent.* How is this possible? This is because the extra fields are compensated by the local symmetry on the world-volume in the form of reparametrization invariance (1.1.24)-(1.1.25) that one can think of as a gauge freedom. This means that there are more fields than there are physical degrees of freedom and that if one wants a manifestly physical formulation, where all the fields corresponds to physical degrees of freedom, one has to fix the gauge.

Going from the covariant formulation to the physically manifest formulation (1.1.7) corresponds to the choice of gauge $X^0(\tau) = \tau$ called *static gauge*. This immediately fixes $\dot{X}^0 = 1$ and one can check explicitly that the Lagrangian (1.1.17) reduces to (1.1.7). But how about p_0? What fixes this when going from the covariant formulation to the physically manifest one?

The answer is that one can derive certain covariant identities that we call *constraints* in the covariant formalism. These can be thought of as constraints on the dynamics that effectively reduce the D component of the D-momentum p_μ to $D-1$ free components. We now illustrate this with two examples of such constraints.

The first example is the so-called mass-shell constraint

$$p^2 = p_\mu p^\mu = -m^2 \tag{1.1.26}$$

This identity follows from (1.1.20). We see that it is a covariant equation both in the sense that it is invariant under the reparametrizations (1.1.24)-(1.1.25) but also that it is invariant under Lorentz transformations. Its physical interpretation is that the D-momentum p_μ should have the standard relation to the mass m of the particle.

We see that imposing (1.1.26) precisely gives a constraint on the D components of p_μ, thus giving $D-1$ free components as in the physically manifest theory. If we choose the gauge $X^0(\tau) = \tau$ we find in particular that $p_0^2 = m^2 + \vec{p}^2$ which is the square of the relativistic energy and hence we can fix p_0 in this gauge.

The second example involves the Hamiltonian of the covariant formulation. In the physically manifest formulation of Section 1.1.1 we found the Hamiltonian $h = E$. But this is non-covariant. The Hamiltonian in the covariant formulation is given as

$$H = p_\mu \dot{X}^\mu - L \tag{1.1.27}$$

Inserting (1.1.19) and (1.1.20) we get

$$H = 0 \tag{1.1.28}$$

This identity is known as the Hamiltonian constraint. We see that this is covariant in the same sense as (1.1.26).

Choosing the gauge $X^0 = \tau$ the covariant Hamiltonian H is related to $h = E$ as $H = p_0 \dot{X}^0 + h = p_0 + h = p_0 + E$. Hence the Hamiltonian constraint can be written as $p_0 = -E$ which again is the statement that the time-component of the relativistic D-momentum is equal to the relativistic energy of the particle.

1.1.3 Einbein action

The action (1.1.17) has disadvantages as well:

- The squareroot in (1.1.17) is difficult when quantizing the theory.
- The limit of $m \to 0$ gives $S = 0$ hence it cannot be used for massless particles.

We introduce therefore yet another action for the relativistic point particle:

$$S = \frac{1}{2} \int d\tau (e^{-1} \dot{X}^2 - e\, m^2) \tag{1.1.29}$$

where $e = e(\tau)$ is another field defined on the worldline known as the einbein.[3] Notice that while we added the field $X^0(\tau)$ in the covariant formulation of Sec. 1.1.2 in comparison to the formalism of Sec. (1.1.1) we add here another field as well.

For $m > 0$ the equation of motion for e is $\partial L / \partial e = 0$ which gives

$$e^2 = -\frac{1}{m^2} \eta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu \tag{1.1.30}$$

which shows that e^2 is proportional to the induced metric $\eta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu$ on the worldline. Inserting $e = \frac{1}{m}\sqrt{-\dot{X}^2}$ in the action (1.1.29) we get back the action (1.1.17). Thus, for $m > 0$ the action (1.1.29) is equivalent to (1.1.17). But in the new einbein formalism of the action (1.1.29) we have the advantages

[3] It is the one-dimensional analog of the vierbein which is an alternative formulation of the metric of a four-dimensional space-time.

- No squareroot in the action.
- The limit $m \to 0$ gives a finite action.

In addition the action (1.1.29) has the same symmetries as those of the action (1.1.17) described above, *i.e.* the Poincaré invariance and reparametrization invariance. For the Poincaré invariance $e(\tau)$ is invariant. Instead for the reparametrization invariance (1.1.24) the $X^\mu(\tau)$ fields transforms as scalars (1.1.25) while $e(\tau)$ transforms as

$$e(\tau) \to \tilde{e}(\tilde{\tau}) = \left(\frac{d\tilde{\tau}}{d\tau}\right)^{-1} e(\tau) \tag{1.1.31}$$

Einbein action in flat gauge

In the Einbein formulation of Sec. 1.1.3 we have yet another field $e(\tau)$ on the worldline. We already mentioned in Sec. 1.1.3 that one gets back the covariant action (1.1.17) by imposing the equations of motion on $e(\tau)$. However, there is another way to get rid of the extra field $e(\tau)$ that is reminiscent of what we shall be doing below for the string action. Hence we go through it in detail here.

The action (1.1.29) is invariant under reparametrizations (1.1.24), (1.1.25) and (1.1.31), which we can think of as a gauge transformation. It is clear from the transformation (1.1.31) of $e(\tau)$ that we can pick a gauge in which

$$e(\tau) = 1 \tag{1.1.32}$$

We call this *flat gauge* since it means that the metric on the worldline of the particle is flat. In this gauge, the action (1.1.29) becomes

$$S = \frac{1}{2}\int d\tau(\dot{X}^2 - m^2) \tag{1.1.33}$$

Since we have now fixed $e(\tau) = 1$ we cannot get the equation of motion for $e(\tau)$ from the action (1.1.33). Hence, the equation (1.1.30) has to be imposed as a constraint equation

$$\dot{X}^2 = -m^2 \tag{1.1.34}$$

Clearly the action (1.1.33) has momenta $p_\mu = \eta_{\mu\nu}\dot{X}^\nu$. Hence the constraint equation can alternatively be written as $p^2 = -m^2$ that we recognize as the mass-shell constraint. This provides an alternative covariant formulation of the relativistic point particle action with the same number of physical degrees of freedom as the manifestly physical and covariant formulations in Secs. 1.1.1-1.1.2. The action (1.1.33) has the advantages that it is manifestly covariant with respect to Lorentz transformations, and that the equations of motion are very simple

$$\dot{p}_\mu = 0 \tag{1.1.35}$$

which are easily solved in general as $X^\mu(\tau) = p^\mu \tau + a^\mu$ with a^μ constant. The small price of these advantages is the non-linear constraint (1.1.34) that requires $p^2 = -m^2$.

1.2 The Nambu-Goto action

In this section we introduce our first formulation of the action principle for infinitely thin relativistic strings. We do this in analogy with that of the covariant formulation of the relativistic point particle of Sec. 1.1.2. In that formulation we have the map (1.1.13) from the one-dimensional worldline to the D-dimensional target space. The action then took the form (1.1.7). Note that we consider the string to be infinitely thin which means we do not allow for finite-thickness structure of the string, just as we consider particles to be points in Sec. 1.1. This is not a limitation as we aim to describe fundamental strings.

1.2.1 The map from worldsheet to target space

An infinitely thin relativistic string spans a two-dimensional surface in the D-dimensional space-time. Thus, we can think of such a string as a map

$$(\tau, \sigma) \to X^\mu(\tau, \sigma) \tag{1.2.1}$$

from what we call the *worldsheet*, being a two-dimensional space parametrized by the two coordinates τ and σ, to the *target space*, which here is taken to be D-dimensional Minkowski space.[4] The map $X^\mu(\tau, \sigma)$ is also known as an embedding because we are embedding a two-dimensional space into a higher-dimensional space. The map (1.2.1) is in analogy with the map (1.1.13) for the point particle case. We have illustrated the map in Fig. 3.

Later we shall also use the notation

$$\xi^0 = \tau \ , \quad \xi^1 = \sigma \tag{1.2.2}$$

for the worldsheet coordinates. We shall be thinking of τ as the worldsheet time and correspondingly σ as a spatial direction on the worldsheet. This means concretely that

[4] Note that when we talk about the worldsheet we mean both the two-dimenisonal surface of the string embedded in the target space as well as a two-dimensional space that exists independently of the target space. The latter point of view will be promoted more below.

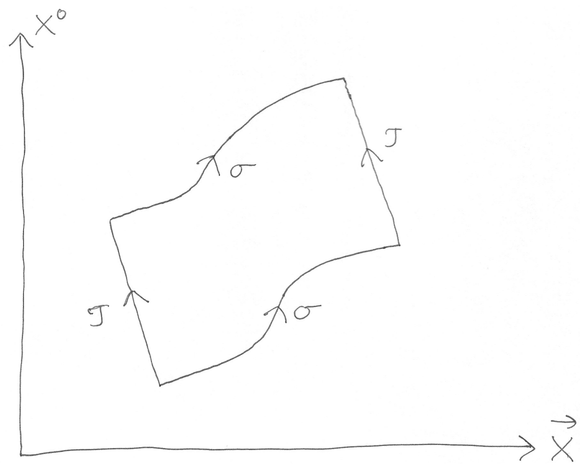

Figure 3: Illustration of the map of the worldsheet to the target space for a string.

for constant σ the map (1.2.1) should map to a time-like curve (or a null curve) while for constant τ it should map to a space-like curve in the target space.

Since τ is considered the worldsheet time we take it to have the range from $-\infty$ to ∞. Instead for the spatial coordinate σ we consider two distinct situations: That $X^\mu(\tau, \sigma)$ is periodic in σ, or that it is not:

- Closed string: A configuration where

$$X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau, \sigma) \tag{1.2.3}$$

  i.e. where X^μ is periodic in σ, is called a *closed string*. See illustration in Fig. 4.

- Open string: A configuration where we require $0 \leq \sigma \leq \pi$ and $X^\mu(\tau, 0) \neq X^\mu(\tau, \pi)$ for at least one μ is called an *open string*. See illustration in Fig. 4.

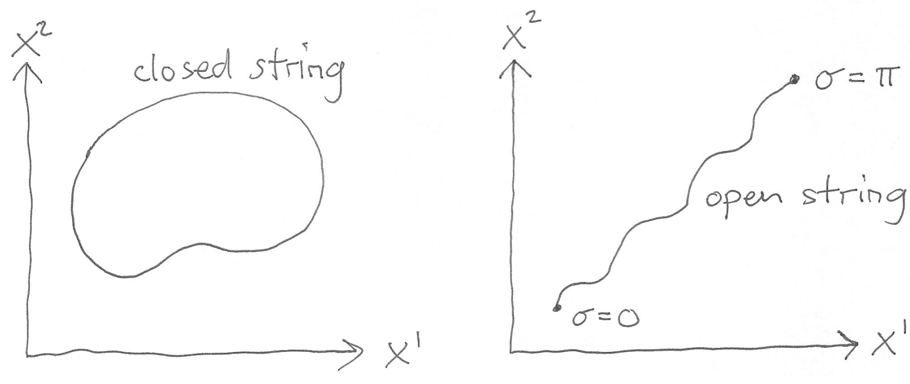

Figure 4: Illustration of a closed and an open string.

We note that one can always pick the period of σ to be 2π for closed strings due to reparametrization invariance of the worldsheet (see below). The same holds for picking the interval $[0, \pi]$ for the range of σ for the open string.

1.2.2 Action principle for strings

Above we have described the geometry of a string. But we need to introduce a dynamical principle. For the relativistic point particle described in Sec. 1.1 the dynamical principle is to extremize the length of the worldline as realized by the action (1.1.1).[5] Naturally, the dynamical principle of the infinitely thin relativistic string should be to extremize the area of the worldsheet. The analogue of the action (1.1.1) is

$$S = -T \int_{\mathrm{WS}} dA \tag{1.2.4}$$

Here dA is the area element for an infinitesimal piece of the worldsheet (abbreviated "WS") and T is the tension of the string. Tension is measured as mass per length.[6]

To compute $\int_{\mathrm{WS}} dA$ we need to consider the metric of space-time restricted to the worldsheet. Differentiating the map (1.2.1) we get

$$dX^\mu = \frac{\partial X^\mu}{\partial \tau} d\tau + \frac{\partial X^\mu}{\partial \sigma} d\sigma = \frac{\partial X^\mu}{\partial \xi^\alpha} d\xi^\alpha \tag{1.2.5}$$

Hence

$$\eta_{\mu\nu} dX^\mu dX^\nu = \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^\alpha} \frac{\partial X^\nu}{\partial \xi^\beta} d\xi^\alpha d\xi^\beta = \gamma_{\alpha\beta} d\xi^\alpha d\xi^\beta \tag{1.2.6}$$

where we defined

$$\gamma_{\alpha\beta} = \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^\alpha} \frac{\partial X^\nu}{\partial \xi^\beta} \tag{1.2.7}$$

and $\alpha, \beta = 0, 1$. We see from this that $\gamma_{\alpha\beta}$ is the induced metric on the worldsheet from the metric $\eta_{\mu\nu}$ on the target space. Considering the infinitesimal square piece of the worldsheet with corners (ξ^0, ξ^1), $(\xi^0 + d\xi^0, \xi^1)$, $(\xi^0 + d\xi^0, \xi^1 + d\xi^1)$ and $(\xi^0, \xi^1 + d\xi^1)$ it has the area

$$dA = d\xi^0 d\xi^1 \sqrt{-\det \gamma} \tag{1.2.8}$$

Here $\det \gamma$ is the determinant of $\gamma_{\alpha\beta}$

$$\det \gamma = \gamma_{00}\gamma_{11} - \gamma_{01}^2 = \dot{X}^2 X'^2 - (\dot{X} \cdot X')^2 \tag{1.2.9}$$

[5] More precisely, it maximizes the proper time.

[6] If one has a closed string with fixed length L then $\int_{\mathrm{WS}} dA = L \int_{\mathrm{WL}} dl$ and $m = TL$ hence the action (1.2.4) reduces to (1.1.1). In Minkowski space one cannot have a string with fixed length but it is possible when one has a compact direction in the target space.

where we introduced the notation

$$\dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau} \ , \quad X'^\mu = \frac{\partial X^\mu}{\partial \sigma} \tag{1.2.10}$$

as well as $\dot{X} \cdot X' = \eta_{\mu\nu} \dot{X}^\mu X'^\nu$.

Thus, we can conclude that the action for infinitely thin relativistic strings is

$$S = -T \int d^2\xi \sqrt{-\det \gamma} = -T \int d^2\xi \sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2} \tag{1.2.11}$$

This is known as the *Nambu-Goto action* for strings. It is the analogue of the action (1.1.17) for the point particle.

In Planck units with $\hbar = c = 1$ mass has unit of inverse length. Hence tension has unit of inverse length squared. One defines therefore

$$T = \frac{1}{2\pi l_s^2} \tag{1.2.12}$$

where l_s is known as the *string length*. Sometimes one also writes $l_s^2 = \alpha'$ (this is "old school" notation). In some sense one can regard l_s as the only parameter of string theory (more on this later).

1.2.3 Symmetries

The Nambu-Goto action (1.2.11) has the following symmetries:

- Poincaré invariance: the action (1.2.11) is invariant under transformations

  $$X^\mu \to \Lambda^\mu{}_\nu X^\nu + c^\mu \tag{1.2.13}$$

  with $\eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$ where $\Lambda^\mu{}_\nu$ and c^μ are constants. This symmetry combines Lorentz invariance and translational invariance. It is a *global symmetry* on the worldsheet, in the sense that it does not depend on τ and σ, so it acts the same everywhere on the worldsheet.

- Reparametrization invariance of the worldsheet: the action (1.2.11) is invariant under the transformations

  $$(\tau, \sigma) \to (\tilde{\tau}(\tau, \sigma), \tilde{\sigma}(\tau, \sigma)) \tag{1.2.14}$$

  One can also write it more compactly as $\xi^\alpha \to \tilde{\xi}^\alpha(\xi)$. This is a *local symmetry* on the worldsheet in that its action depends where you are on the worldsheet.

Just like in the case of the relativistic point particle one can think of the reparametrization invariance of the worldsheet as the "price" one pays for having a covariant formulation of the string. The reparametrization invariance can be thought of as gauge invariance, hence fixing τ and σ in terms of the $X^\mu(\tau,\sigma)$ corresponds to a choice of gauge. Connected to this, the string has two more fields than in a manifestly physical formulation. Later we shall see this explicitly when using the so-called lightcone gauge. The number of physical fields is thus $D-2$ rather than D.[7] When working in the covariant formalism one thus needs two independent constraints to get from D to $D-2$ physical fields. The two constraints are

$$\Pi \cdot X' = 0 \ , \quad \Pi^2 + T^2 X'^2 = 0 \tag{1.2.15}$$

where

$$\Pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} \tag{1.2.16}$$

is the momentum density of the string.

1.3 The Polyakov action

The Nambu-Goto action (1.2.11) for the string has the disadvantage of the squareroot, just like for the covariant action (1.1.17) for the point particle. The squareroot is a problem when quantizing the theory. We got rid of the squareroot for the point particle by introducing the extra field $e(\tau)$ in the action (1.1.29) where $e(\tau)^2$ was seen to be proportional to what one could think of as the metric on the world-line. The action for strings analogous to (1.1.29) is the so-called *Polyakov action*.

1.3.1 Introducing the Polyakov action

The *Polyakov action* for infinitely thin relativistic strings moving in D-dimensional Minkowski space is

$$S_{\rm pol}[g,X] = -\frac{T}{2}\int d^2\xi \sqrt{-g} g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \tag{1.3.1}$$

where we introduced the two by two matrix of fields on the worldsheet $g_{\alpha\beta}(\tau,\sigma)$ with $\alpha,\beta = 0,1$, as well as its inverse matrix $g^{\alpha\beta}(\tau,\sigma)$. Since we require that $g_{\alpha\beta}$ is symmetric $g_{\alpha\beta} = g_{\beta\alpha}$ it corresponds to three extra independent fields on the worldsheet. We introduced also the notation $g = \det(g_{\alpha\beta})$ and the short-hand notation for worldsheet

[7] This is analogous to the point particle case where the number of physical fields is $D-1$ rather than D.

derivatives

$$\partial_\alpha = \frac{\partial}{\partial \xi^\alpha} \tag{1.3.2}$$

We interpret the new field $g_{\alpha\beta}(\tau,\sigma)$ as the metric on the worldsheet. Note that the analogue of the Polyakov action for the point particle is the Einbein action (1.1.29).

We find the equation of motion for $g_{\alpha\beta}$ by varying the action

$$\delta S_{\rm pol} = -\frac{T}{2}\int d^2\xi \left(-\frac{1}{2}\sqrt{-g}\, g_{\alpha\beta} g^{\gamma\delta}\partial_\gamma X\cdot\partial_\delta X + \sqrt{-g}\,\partial_\alpha X\cdot\partial_\beta X \right)\delta g^{\alpha\beta} \tag{1.3.3}$$

We used here $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\alpha\beta}\delta g^{\alpha\beta}$. Thus the equation of motion for $g_{\alpha\beta}$ is given by $\frac{1}{2}g_{\alpha\beta}g^{\gamma\delta}\partial_\gamma X\cdot\partial_\delta X = \partial_\alpha X\cdot\partial_\beta X$. One can check that this is fulfilled if and only if there exists a function $\lambda(\tau,\sigma)$ on the worldsheet such that

$$g_{\alpha\beta} = \lambda(\tau,\sigma)^2\,\gamma_{\alpha\beta} \tag{1.3.4}$$

where $\gamma_{\alpha\beta} = \partial_\alpha X\cdot\partial_\beta X$ is the induced metric on the worldsheet defined in Eq. (1.2.7). Using this one finds $\sqrt{-g}\, g^{\alpha\beta}\partial_\alpha X\cdot\partial_\beta X = \sqrt{-\det\gamma}\,\gamma^{\alpha\beta}\gamma_{\alpha\beta} = 2\sqrt{-\det\gamma}$ hence inserting this in the action (1.3.1) we get back the Nambu-Goto action (1.2.11). Thus, the Polyakov action is classically equivalent to the Nambu-Goto action. We will have to revisit this equivalence again when going to the quantum theory.

There is another interpretation for the variation of the action with respect to $g_{\alpha\beta}$. Interpreting $g_{\alpha\beta}$ as the worldsheet metric one defines the energy-momentum tensor as

$$T_{\alpha\beta} = -\frac{4\pi}{\sqrt{-g}}\frac{\delta S_{\rm pol}}{\delta g^{\alpha\beta}} \tag{1.3.5}$$

This is known as the worldsheet energy-momentum tensor. We compute

$$T_{\alpha\beta} = 2\pi T\Big(\partial_\alpha X\cdot\partial_\beta X - \frac{1}{2}g_{\alpha\beta}g^{\gamma\delta}\partial_\gamma X\cdot\partial_\delta X\Big) \tag{1.3.6}$$

However, the other interpretation of the variation of the action with respect to $g_{\alpha\beta}$ as giving the equation of motion for $g_{\alpha\beta}$ now tells us that this is zero

$$T_{\alpha\beta} = 0 \tag{1.3.7}$$

This becomes a constraint equation for the Polyakov action when one chooses a gauge for $g_{\alpha\beta}$. Thus, just as we learned in the point particle case, whenever we introduce new fields in the formalism all the while we still want to describe the same physics, we get a number of constraints in the theory that ensures that the new fields are restricted in such a way that the same physics comes out. For the classical string theory above, we see that this indeed is the case. Compared to the Nambu-Goto action (1.2.11) we introduced the new fields $g_{\alpha\beta}$ in the Polyakov action (1.3.1). From this action one gets the constraint equation (1.3.7) that is solved by (1.3.4) which when inserted into the Polyakov action (1.3.1) gives back the Nambu-Goto action (1.2.11). We emphasize again that this is an equivalence between actions that at this point is only true at the classical level.

1.3.2 Symmetries

The Polyakov action (1.3.1) has the following symmetries:

- Poincaré invariance: the action (1.3.1) is invariant under transformations

$$X^\mu \to \Lambda^\mu{}_\nu X^\nu + c^\mu \tag{1.3.8}$$

  with $\eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$ where $\Lambda^\mu{}_\nu$ and c^μ are constants. This is a global symmetry on the worldsheet.

- Reparametrization invariance of the worldsheet: the action (1.3.1) is invariant under transformations

$$\xi^\alpha \to \tilde{\xi}^\alpha(\xi) \tag{1.3.9}$$

  Under this transformation the fields $X^\mu(\xi)$ transform as a scalars

$$X^\mu(\xi) \to \tilde{X}^\mu(\tilde{\xi}) = X^\mu(\xi) \tag{1.3.10}$$

  while $g_{\alpha\beta}(\xi)$ transform as a two-dimensional metric

$$g_{\alpha\beta}(\xi) \to \tilde{g}_{\alpha\beta}(\tilde{\xi}) = \frac{\partial\xi^\gamma}{\partial\tilde{\xi}^\alpha}\frac{\partial\xi^\delta}{\partial\tilde{\xi}^\beta}g_{\gamma\delta}(\xi) \tag{1.3.11}$$

  This is a local symmetry on the worldsheet. Note that if we interpret the worldsheet as a two-dimensional space-time with coordinates ξ^α and metric $g_{\alpha\beta}$ then the reparametrization invariance of the worldsheet is nothing but the diffeomorphism invariance of the two-dimensional space-time on which there lives D scalar fields $X^\mu(\xi)$.

- Weyl invariance: the action (1.3.1) is invariant under the so-called Weyl transformations

$$g_{\alpha\beta}(\xi) \to \tilde{g}_{\alpha\beta}(\xi) = \Omega(\xi)^2 g_{\alpha\beta}(\xi) \tag{1.3.12}$$

  while $X^\mu(\xi)$ stays the same and the coordinates ξ^α are not transformed. Thus, Weyl transformations acts only on the metric $g_{\alpha\beta}$. This is another local symmetry on the worldsheet.

We see that the Polyakov action (1.3.1) has the same symmetries of the Nambu-Goto action (1.2.11) - namely the Poincaré invariance and the reparametrization invariance of the worldsheet - but in addition also the Weyl invariance. Weyl transformation of the metric preserves angles but not lengths. Indeed, if we look at two small line elements $d\xi^\alpha$ and $d\hat{\xi}^\alpha$ (one can see them as vectors when taking the zero length limit) then the angle between them is

$$\cos\theta = \frac{g_{\alpha\beta} d\xi^\alpha d\hat{\xi}^\beta}{\sqrt{|g_{\alpha\beta} d\xi^\alpha d\xi^\beta|}\sqrt{|g_{\alpha\beta} d\hat{\xi}^\alpha d\hat{\xi}^\beta|}} \tag{1.3.13}$$

which is seen to be invariant under Weyl transformations (1.3.12).

1.3.3 The Polyakov action in flat gauge

As we now shall see, the diffeomorphism invariance and Weyl invariance of the Polyakov action (1.3.1) exhibited in Sec. 1.3.2 are important since they enable us to make gauge choices for $g_{\alpha\beta}$ that simplifies the action while still keeping the same physics as the Nambu-Goto action (1.2.11). Seen from this point of view the Nambu-Goto action itself corresponds to a gauge choice (1.3.4). The motivation for fixing a different gauge for $g_{\alpha\beta}$ is that the equations of motion as derived from the Nambu-Goto action (1.2.11) are highly non-linear and thus complicated to solve. Furthermore, the non-linearity means that it is an interacting theory which makes it highly difficult to quantize it.

Later we shall see that in going from the classical theory to the quantum theory the symmetries of the classical theory can be broken. This is known as an *anomaly*. It will be important for us to avoid anomalies when quantizing the theory since otherwise we will not preserve the equivalence to the physics of the Nambu-Goto action.

Let us now see the power of diffeomorphism and Weyl invariance. Consider first a general two-dimensional metric

$$g_{\alpha\beta} = \begin{pmatrix} g_{00} & g_{01} \\ g_{01} & g_{11} \end{pmatrix} \tag{1.3.14}$$

The metric only has three independent fields since it is symmetric. One can now use one coordinate transformation to eliminate $g_{01} = 0$ and another to set $g_{00} = -g_{11}$. Thus using diffeomorphism invariance we can - at least locally - get any two-dimensional metric to be of the form

$$g_{\alpha\beta} = e^{2\Phi}\eta_{\alpha\beta} \tag{1.3.15}$$

where $\Phi(\xi)$ is a function of the worldsheet and where we defined the two-dimensional

Minkowski metric

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.3.16}$$

The gauge choice (1.3.15) is known as the *conformal gauge*.

As an interlude we note that we have just argued that any two-dimensional metric can be transformed to be of the form (1.3.15). One says that any two-dimensional metric is locally conformally flat.

We can now employ the Weyl invariance (1.3.12) as well to the worldsheet metric (1.3.15). Choosing $\Omega = e^{-\Phi}$ we get the even simpler worldsheet metric

$$g_{\alpha\beta} = \eta_{\alpha\beta} \tag{1.3.17}$$

corresponding to a two-dimensional Minkowski space. This is known as the *flat gauge* and it is this gauge that we shall be working in extensively in the following.

Choosing the flat gauge (1.3.17), the Polyakov action (1.3.1) simplifies greatly

$$S_{\text{pol}} = -\frac{T}{2} \int d^2\xi \, \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \tag{1.3.18}$$

As we shall exhibit below, this is in fact the action for D free bosons living in two-dimensional Minkowski space. This is in contrast with the Nambu-Goto action (1.2.11) that has highly non-linear equations of motion. Nevertheless, the two actions give the same physics, as long as we in addition to the action (1.3.18) also satisfy the constraint equation (1.3.7).

Indeed, the price of working in the flat gauge (1.3.17) is that the constraint equation (1.3.7) now is non-trivial. Indeed, fixing the gauge (1.3.17) means that (1.3.7) is not satisfied as a equation of motion for $g_{\alpha\beta}$ anymore since $g_{\alpha\beta}$ cannot vary. Also, it is in contrast with the gauge choice (1.3.4) (the gauge giving the Nambu-Goto action (1.2.11)) for which the constraint equation (1.3.7) is satisfied automatically.

Below in Sec. 1.6 we shall see that even though we used both the diffeomorphism and Weyl invariance to obtain the flat gauge (1.3.17) there is still a left over symmetry of the action which will prove to be very important.

Note finally that the point particle analogue of choosing flat gauge for the Polyakov action is the flat gauge (1.1.32) for the Einbein action (1.1.29), resulting in the simpler action (1.1.33). Moreover, the analogue of the constraint (1.3.7) is the mass-shell constraint (1.1.34).

1.4 String solutions of the flat gauge Polyakov action

In this section we find the equations of motion for the Polyakov action in flat gauge (1.3.18) and solve them in terms of mode expansions both in case of closed strings and open strings.

1.4.1 Equations of motion and general solution

The Lagrangian corresponding to the Polyakov action in flat gauge (1.3.18) is

$$L_{\text{pol}} = -\frac{T}{2}\partial_\alpha X^\mu \partial^\alpha X_\mu \tag{1.4.1}$$

From the Euler-Lagrange equations

$$\partial_\alpha \left(\frac{\partial L_{\text{pol}}}{\partial \partial_\alpha X^\mu} \right) = \frac{\partial L_{\text{pol}}}{\partial X^\mu} \tag{1.4.2}$$

we get the equations of motion

$$\partial_\alpha \partial^\alpha X^\mu = 0 \tag{1.4.3}$$

Thus, for each $\mu = 0, 1, ..., D-1$ the field $X^\mu(\xi)$ obeys the free scalar wave equation in two dimensions.

At this point we introduce a new set of coordinates on the worldsheet that will be useful for many purposes, including solving (1.4.3). The *lightcone coordinates* are defined as

$$\xi^\pm = \tau \pm \sigma = \xi^0 \pm \xi^1 \tag{1.4.4}$$

We are in flat gauge thus the metric is (1.3.16). This metric transform to the following in lightcone coordinates

$$\begin{aligned} &\eta_{+-} = \eta_{-+} = -\frac{1}{2}\ , \quad \eta_{--} = \eta_{++} = 0 \\ &\eta^{+-} = \eta^{-+} = -2\ , \quad \eta^{--} = \eta^{++} = 0 \end{aligned} \tag{1.4.5}$$

Instead for the worldsheet derivatives (1.3.9) we have

$$\partial_\pm = \frac{\partial}{\partial \xi^\pm} = \frac{1}{2}(\partial_0 \pm \partial_1) \tag{1.4.6}$$

Using now the lightcone coordinates (1.4.4) the equation of motion (1.4.3) can be written as

$$\partial_- \partial_+ X^\mu = 0 \tag{1.4.7}$$

Thus, we see immediately from this equation that it has the general solution

$$X^\mu(\xi) = X_R^\mu(\xi^-) + X_L^\mu(\xi^+) \tag{1.4.8}$$

for any functions $X_R^\mu(\xi^-)$ and $X_L^\mu(\xi^+)$. Here $X_R^\mu(\xi^-)$ is a right-moving wave while $X_L^\mu(\xi^-)$ is a left-moving wave, as illustrated on Fig. 5.

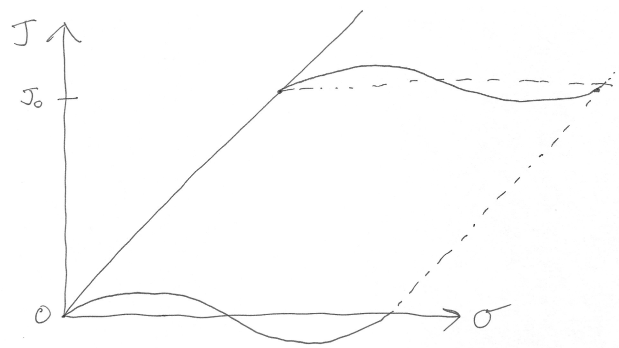

Figure 5: A right-moving wave $X^\mu(\xi) = X_R^\mu(\xi^-)$ on the worldsheet. At $\tau = 0$ we have $\sigma = -\xi^-$ while at $\tau = \tau_0$ we have $\sigma = \tau_0 - \xi^-$. This shows that the wave-profile moves linearly with increasing σ as the worldsheet time τ increases.

1.4.2 Conserved currents

The Polyakov action in flat gauge (1.3.18) is invariant under Poincaré transformations of the target space, as seen in Sec. 1.3.2. Poincaré invariance combine Lorentz invariance and translational invariance of the D-dimensional Minkowski space. These are global symmetries as seen from point of view of the worldsheet of the string. As we shall see below, one can show using the Noether procedure that any given global symmetry leads to a conserved current on the worldsheet. Furthermore, in Secs. 1.4.3 and 1.4.4 we shall see that the conserved currents can give rise to conserved charges on the worldsheet associated to the symmetry.

We begin by reviewing briefly the Noether procedure. Consider first a general action

$$S[X^\mu] = \int d^2\xi \, L(X^\mu, \partial_\alpha X^\mu) \tag{1.4.9}$$

where X^μ depends on ξ^α. Clearly the Polyakov action in flat gauge (1.3.18) is of this form. Assume this action is invariant under the global infinitesimal transformation

$$\delta X^\mu(\xi) = \epsilon M^\mu(X) \tag{1.4.10}$$

where ϵ is a small parameter and $M^\mu(X)$ is some transformation of X^μ. This is assumed to be a symmetry of the action irrespective of whether we impose the equation of motion for X^μ. Since this is a symmetry of the action the change in the Lagrangian must be a total derivative proportional to the small parameter ϵ

$$\delta L = \epsilon\, \partial_\alpha \Omega^\alpha \tag{1.4.11}$$

for some Ω^α. Take now ϵ to be dependent on ξ^α. Then the variation of the Lagrangian gets an extra term proportional to the derivative of ϵ

$$\delta L = \epsilon \partial_\alpha \Omega^\alpha + (\partial_\alpha \epsilon) N^\alpha = \epsilon \partial_\alpha J^\alpha + \partial_\alpha(\epsilon N^\alpha) \tag{1.4.12}$$

where we defined $J^\alpha = \Omega^\alpha - N^\alpha$. Assuming we take $\epsilon(\xi)$ to vanish at the boundaries of the world-sheet of the string the variation of the action is

$$\delta S = \int d^2\xi\, \epsilon(\xi) \partial_\alpha J^\alpha \tag{1.4.13}$$

Assume now that the variation of the action is around an extremum of the action, meaning that the fields $X^\mu(\xi)$ obey the equations of motion. Since $\delta S = 0$ for any infinitesimal variation of the fields (obeying the correct boundary conditions) around a solution of the equations of motion, and since we can take $\epsilon(\xi)$ to be any function, we must have that

$$\partial_\alpha J^\alpha = 0 \tag{1.4.14}$$

Thus, we have derived that J^α is a conserved current when X^μ obeys the equations of motion.

The action (1.3.18) is invariant under infinitesimal translations $\delta X^\mu(\xi) = \epsilon b^\mu$ for constant b^μ. This corresponds to $M^\mu(X) = b^\mu$ in (1.4.10). Using this in the Noether procedure above, we find $J^\alpha = -b^\mu \Pi^\alpha_\mu$ with Π^α_μ being a current given by

$$\Pi^\alpha_\mu = \frac{\partial L}{\partial \partial_\alpha X^\mu} = -T \partial^\alpha X_\mu \tag{1.4.15}$$

The conservation of this current

$$\partial_\alpha \Pi^\alpha_\mu = 0 \tag{1.4.16}$$

is immediately seen by the equations of motion (1.4.3). Of particular importance is the worldsheet time component of the momentum current. We use the notation

$$\Pi_\mu(\tau, \sigma) = \Pi^\tau_\mu(\tau, \sigma) \tag{1.4.17}$$

Notice

$$\Pi_\mu = \frac{\partial L}{\partial \dot{X}^\mu} = T \dot{X}_\mu \tag{1.4.18}$$

Thus, Π_μ is the momentum conjugate of X^μ. Indeed, we call $\Pi_\mu(\tau,\sigma)$ the *momentum density* of the string since if we have an infinitesimal piece of the string going from σ to $\sigma + d\sigma$ then while $X^\mu(\tau,\sigma)$ gives the position of the piece, $\Pi(\tau,\sigma)d\sigma$ gives the momentum of the piece.

The action (1.3.18) is also invariant under infinitesimal Lorentz transformations $M^\mu(X) = \omega^\mu{}_\nu X^\nu$ with $\omega^\mu{}_\nu$ being a constant antisymmetric tensor $\omega_{\mu\nu} = -\omega_{\nu\mu}$. Employing the Noether procedure this gives the conserved current $J^\alpha = \frac{1}{2}\omega^{\mu\nu}\mathcal{J}^\alpha_{\mu\nu}$ where $\mathcal{J}^\alpha_{\mu\nu}$ is the current

$$\mathcal{J}^\alpha_{\mu\nu} = X_\mu \Pi^\alpha_\nu - X_\nu \Pi^\alpha_\mu \tag{1.4.19}$$

The conservation of $\mathcal{J}^\alpha_{\mu\nu}$ on the worldsheet

$$\partial_\alpha \mathcal{J}^\alpha_{\mu\nu} = 0 \tag{1.4.20}$$

easily follows from (1.4.15) and (1.4.16).

1.4.3 Closed string mode expansion

So far we have not imposed boundary conditions on the general solution (1.4.8) of Eq. (1.4.3). We begin this here with the study of the closed and open string. As already noted above, τ is the worldsheet time hence we take it to be unbounded. However, the range of σ depends on whether we consider open or closed strings. We consider here first the closed string case and then turn to the open string case in Sec. 1.4.4.

For closed strings we consider $X^\mu(\tau,\sigma)$ to be a periodic function of σ with period 2π, as written in Eq. (1.2.3). This is illustrated in Fig. 4. We can now write the action (1.3.18) with the range of τ and σ

$$S_{\rm pol} = -\frac{T}{2}\int_{-\infty}^{\infty} d\tau \int_{\sigma=0}^{2\pi} d\sigma\, \partial_\alpha X^\mu \partial^\alpha X_\mu \tag{1.4.21}$$

Varying the action we get

$$\delta S_{\rm pol} = -T\int_{-\infty}^{\infty} d\tau \int_{\sigma=0}^{2\pi} d\sigma \Big(\partial_\alpha(\partial^\alpha X^\mu\, \delta X_\mu) - (\partial_\alpha\partial^\alpha X^\mu)\delta X_\mu\Big) \tag{1.4.22}$$

Assuming $X^\mu(\xi)$ is a solution (1.4.8) we obtain

$$\delta S_{\rm pol} = -T\int_{-\infty}^{\infty} d\tau \Big[X'_\mu \delta X^\mu\Big]_{\sigma=0}^{2\pi} = 0 \tag{1.4.23}$$

which is zero due to the periodicity of $X^\mu(\tau,\sigma)$ in σ.

We now exhibit what the periodicity in σ means for a general solution (1.4.8). The fact that X^μ is periodic in σ gives that also $\partial_+ X^\mu$ and $\partial_- X^\mu$ are periodic in σ with period 2π. Since $\partial_+ X^\mu = \partial_+ X_L^\mu$ and $\partial_- X^\mu = \partial_- X_R^\mu$ this means that

$$\partial_+ X_L^\mu(\xi^+ + 2\pi) = \partial_+ X_L^\mu(\xi^+) \ , \quad \partial_- X_R^\mu(\xi^- + 2\pi) = \partial_- X_R^\mu(\xi^-) \tag{1.4.24}$$

Thus, we can make a Fourier expansion of both $\partial_+ X_L^\mu$ and $\partial_- X_R^\mu$ since they are periodic functions (with period 2π). We write these Fourier expansions as

$$\partial_- X^\mu = \partial_- X_R^\mu = \frac{l_s}{\sqrt{2}} \sum_{n \in \mathbb{Z}} \alpha_n^\mu e^{-in\xi^-} \ , \quad \partial_+ X^\mu = \partial_+ X_L^\mu = \frac{l_s}{\sqrt{2}} \sum_{n \in \mathbb{Z}} \tilde{\alpha}_n^\mu e^{-in\xi^+} \tag{1.4.25}$$

where α_n^μ and $\tilde{\alpha}_n^\mu$ with $n \in \mathbb{Z}$ are the Fourier modes of the Fourier expansion (the factor $l_s/\sqrt{2}$ is chosen for later convenience). Integrating this we get

$$X^\mu(\xi) = x^\mu + \frac{l_s}{\sqrt{2}}(\alpha_0^\mu \xi^- + \tilde{\alpha}_0^\mu \xi^+) + \frac{i l_s}{\sqrt{2}} \sum_{n \neq 0} \frac{1}{n} \Big(\alpha_n^\mu e^{-in\xi^-} + \tilde{\alpha}_n^\mu e^{-in\xi^+} \Big) \tag{1.4.26}$$

However, since $X^\mu(\xi)$ is periodic in σ we get $\alpha_0^\mu = \tilde{\alpha}_0^\mu$. Hence we get what is known as the *mode expansion* for the closed string

$$X^\mu(\xi) = x^\mu + l_s^2 p^\mu \tau + \frac{i l_s}{\sqrt{2}} \sum_{n \neq 0} \frac{1}{n} \Big(\alpha_n^\mu e^{-in\xi^-} + \tilde{\alpha}_n^\mu e^{-in\xi^+} \Big) \tag{1.4.27}$$

where

$$p^\mu = \frac{\sqrt{2}}{l_s} \alpha_0^\mu = \frac{\sqrt{2}}{l_s} \tilde{\alpha}_0^\mu \tag{1.4.28}$$

Using this one finds the following mode expansion for the momentum density (1.4.18) of the closed string

$$\Pi^\mu(\xi) = \frac{1}{2\pi} p^\mu + \frac{1}{2\pi\sqrt{2} l_s} \sum_{n \neq 0} \Big(\alpha_n^\mu e^{-in\xi^-} + \tilde{\alpha}_n^\mu e^{-in\xi^+} \Big) \tag{1.4.29}$$

Eq. (1.4.27), with arbitrary modes x^μ, p^μ, α_n^μ and $\tilde{\alpha}_n^\mu$ for $n \neq 0$, is the most general solution to the equation of motion (1.4.3) with closed string boundary conditions (1.2.3). We note that while this is a solution to (1.4.3) we still need to impose the constraint (1.3.7). We consider this in Sec. 1.5.

Since the field $X^\mu(\tau, \sigma)$ should be real $(X^\mu(\tau, \sigma))^* = X^\mu(\tau, \sigma)$ we need to require

$$(x^\mu)^* = x^\mu \ , \quad (p^\mu)^* = p^\mu \ , \quad (\alpha_n^\mu)^* = \alpha_{-n}^\mu \ , \quad (\tilde{\alpha}_n^\mu)^* = \tilde{\alpha}_{-n}^\mu \tag{1.4.30}$$

for the complex conjugates of the modes.

We now interpret zero modes x^μ and p^μ of the mode expansions (1.4.27) and (1.4.29). First we see that the quantity

$$\int_0^{2\pi} d\sigma\, \Pi_\mu(\tau,\sigma) = p_\mu \tag{1.4.31}$$

is the total momentum of the string. This is a conserved quantity on the worldsheet associated to translation invariance in the target space. One can check this using the conservation of the momentum current (1.4.16)

$$\dot{p}_\mu = \frac{d}{d\tau}\int_0^{2\pi} d\sigma\, \Pi^\tau_\mu = \int_0^{2\pi} d\sigma\, \partial_\tau \Pi^\tau_\mu = -\int_0^{2\pi} d\sigma\, \partial_\sigma \Pi^\sigma_\mu = -\Big[\Pi^\sigma_\mu\Big]_{\sigma=0}^{2\pi} = 0 \tag{1.4.32}$$

The center of mass of the closed string is given by

$$\frac{1}{2\pi}\int_0^{2\pi} d\sigma\, X^\mu(\tau,\sigma) = x^\mu + l_s^2 p^\mu \tau \tag{1.4.33}$$

We see that x^μ is the center of mass position of the string at $\tau = 0$ while the term linear in τ accounts for the free motion of the string center of mass. Thus, the zero-modes x^μ and p_μ characterize the center of mass position and the total momentum of the string. Instead the modes α^μ_n and $\tilde{\alpha}^\mu_n$ for $n \neq 0$ gives the local movements of the string, *i.e.* the vibration modes.

Finally, one has the charges associated with Lorentz invariance of the target space

$$J_{\mu\nu} = \int_0^{2\pi} d\sigma\, \mathcal{J}^\tau_{\mu\nu} = \int_0^{2\pi} d\sigma\, (X_\mu \Pi_\nu - X_\nu \Pi_\mu) \tag{1.4.34}$$

Using a completely analogous argument as in (1.4.32) one can show these charges are conserved

$$\dot{J}_{\mu\nu} = 0 \tag{1.4.35}$$

on the worldsheet.

The interpretation of J_{ij} with $i,j = 1,2,....,D-1$ is the total angular momentum of the closed string in the ij-plane. Thus, the conservation of J_{ij} is the statement that the angular momenta of an isolated system are conserved.

The conservation of J_{0i} with $i = 1,2,....,D-1$ is instead related to the fact that the center of mass motion is linear for an isolated system as seen for the closed string in Eq. (1.4.33). To see this we pick the static gauge $X^0(\tau,\sigma) = a\tau$ where a is a constant. Then we have $\Pi^0 = aT$. Using this we compute

$$J_{0i} = -a\tau p_i + aT\int_0^{2\pi} d\sigma X^i \tag{1.4.36}$$

This gives

$$\frac{1}{2\pi}\int_0^{2\pi} d\sigma X^i = \frac{l_s^2}{a} J_{0i} + l_s^2 p_i \tau \tag{1.4.37}$$

Thus, in the static gauge $J_{0i} = ax^i/l_s^2$.

1.4.4 Open string mode expansion

Open strings have $\sigma \in [0,\pi]$ with two endpoints at $\sigma = 0$ and $\sigma = \pi$. The action (1.3.18) becomes

$$S_{\rm pol} = -\frac{T}{2}\int_{-\infty}^{\infty} d\tau \int_{\sigma=0}^{\sigma=\pi} d\sigma\, \partial_\alpha X^\mu \partial^\alpha X_\mu \tag{1.4.38}$$

Varying the action we get

$$\delta S_{\rm pol} = -T\int_{-\infty}^{\infty} d\tau \int_{\sigma=0}^{\sigma=\pi} d\sigma \Big(\partial_\alpha(\partial^\alpha X_\mu\, \delta X^\mu) - (\partial_\alpha \partial^\alpha X^\mu)\delta X_\mu \Big) \tag{1.4.39}$$

Assuming $X^\mu(\xi)$ is a solution (1.4.8) we obtain

$$\delta S_{\rm pol} = -T\int_{-\infty}^{\infty} d\tau \Big[X'_\mu \delta X^\mu \Big]_{\sigma=0}^{\pi} \tag{1.4.40}$$

Since δX^μ can vary independently for each μ and for each end point one should satisfy $X'_\mu \delta X^\mu = 0$ (here without a sum over μ) for each μ and each end point. This means either $X'_\mu = 0$ or $\delta X^\mu = 0$. Thus, we have two possible boundary conditions for each μ and for each end point. For the end point $\sigma = 0$ and for a given $\mu = 0, 1, ..., D-1$ the two possible boundary conditions are

$$\begin{aligned} &\text{Neumann:} \quad X'^\mu(\tau,0) = 0 \\ &\text{Dirichlet:} \quad \dot{X}^\mu(\tau,0) = 0 \end{aligned} \tag{1.4.41}$$

and the same for $\sigma = \pi$. The Dirichlet boundary condition means that the end point cannot move. Hence for $\mu = 0$ we can only impose the Neumann boundary condition - otherwise the end points would not be able to move on a time-like or null path.[8] Instead for $\mu = 1, 2, ...D-1$ we have four possibilities: NN, ND, DN and DD (*i.e.* ND means that the $\sigma = 0$ end point has the Neumann boundary condition while $\sigma = \pi$ has the Dirichlet boundary condition).

For simplicity we focus in this course on the two possibilities NN and DD, *i.e.* where the two end points have the same boundary condition. Without loss of generality, we can then assume NN conditions for $\mu = 0, 1, ..., p$ and DD for $\mu = p+1, ..., D-1$. For

[8]However, if one goes to the Euclidean section - *i.e.* after a Wick rotation - then one can actually make sense of such a boundary condition in terms of something called a D-instanton.

later conveniece we introduce the notation that the index $a = 0, 1, ..., p$ and the index $I = p + 1, ..., D - 1$. The DD boundary conditions means that

$$X^I(\tau, 0) = c_1^I \ , \quad X^I(\tau, \pi) = c_2^I \tag{1.4.42}$$

where c_1^I and c_2^I are constants. In this way each end point of the open string defines a p-dimensional hyperplane in the $(D-1)$-dimensional Euclidean space. One hyperplane is defined by the equation $x^I = c_1^I$, $I = p+1, ..., D-1$, and the other by $x^I = c_2^I$, $I = p+1, ..., D-1$. Thus, the $p+1$ directions x^a, $a = 0, 1, ..., p$, are longitudinal to the two hyperplanes, while the $D - p - 1$ directions x^I, $I = p+1, ..., D-1$, are transverse to the two hyperplanes, see Fig. 6 for an illustration. The correct string theory interpretation of these p-dimensional hyperplanes are as the so-called Dirichlet p-branes, abbreviated as Dp-branes. In the quantum open string theory one can show that these Dp-branes are dynamical objects which in some sense are made of open string end points. We shall consider quantized open strings and Dp-branes in Sec. 5.

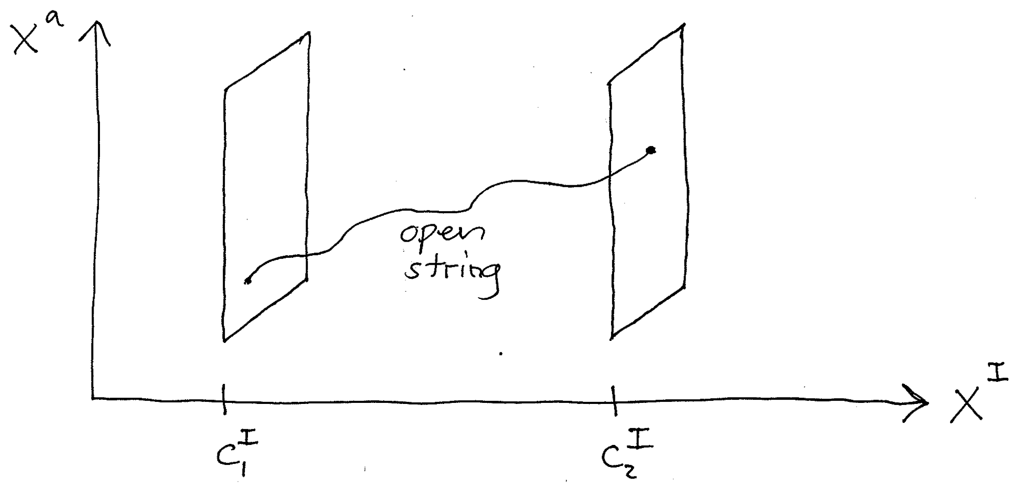

Figure 6: An open string with NN conditions in $p + 1$ directions and DD conditions in the remaining $D - p - 1$ directions ends on two p-dimensional hyperplanes embedded in $(D-1)$-dimensional Euclidean space.

We now turn to the mode expansion for the open string. From the general solution (1.4.8) we know that $\partial_- X^\mu$ only depends on ξ^- while $\partial_+ X^\mu$ only depends on ξ^+. Hence, for use below, we can define the functions

$$f^\mu(\xi^+) = \partial_+ X^\mu(\xi) \ , \quad g^\mu(\xi^-) = \partial_- X^\mu(\xi) \tag{1.4.43}$$

for $\mu = 0, 1, ..., D-1$.

Consider first the $p+1$ directions with NN boundary conditions. For $\sigma = 0$ we have $0 = X'^a(\tau,0) = \partial_+ X^a(\tau,0) - \partial_- X^a(\tau,0)$ so that $\partial_+ X^a(\tau,0) = \partial_- X^a(\tau,0)$. From (1.4.43) we see this means $f^a(\tau) = g^a(\tau)$ for any τ since $\xi^\pm = \tau$ for $\sigma = 0$. Hence f^a and g^a is the same function. Instead for $\sigma = \pi$ we have $0 = X'^a(\tau,\pi) = \partial_+ X^a(\tau,\pi) - \partial_- X^a(\tau,\pi)$ which using (1.4.43) is seen to give $f^a(\tau+\pi) = g^a(\tau-\pi)$ for any τ. Using that $f^a = g^a$ this means that $f^a(\tau+\pi) = f^a(\tau-\pi)$, from which we deduce that $f^a(\tau+2\pi) = f^a(\tau)$. Thus $f^a = g^a$ is a periodic function with period 2π. Using again (1.4.43) we see that this means we can make the following Fourier expansions

$$\partial_- X^a = \frac{l_s}{\sqrt{2}} \sum_{n\in\mathbb{Z}} \alpha_n^a e^{-in\xi^-} \quad , \quad \partial_+ X^a = \frac{l_s}{\sqrt{2}} \sum_{n\in\mathbb{Z}} \alpha_n^a e^{-in\xi^+} \tag{1.4.44}$$

Thus, for the NN open string boundary condition the Fourier modes of the right-moving sector are equal to those of the left-moving sector. Integrating this we get that the mode expansions for the directions with NN boundary conditions are[9]

$$X^a(\tau,\sigma) = x^a + 2l_s^2 p^a \tau + i\sqrt{2} l_s \sum_{n\neq 0} \frac{1}{n} \alpha_n^a e^{-in\tau} \cos(n\sigma) \tag{1.4.45}$$

with $a = 0,1,...,p$. The momentum density (1.4.18) has the mode expansion

$$\Pi^a(\tau,\sigma) = \frac{1}{\pi} p^a + \frac{\sqrt{2}}{2\pi l_s} \sum_{n\neq 0} \alpha_n^a e^{-in\tau} \cos(n\sigma) \tag{1.4.46}$$

Here we have set

$$p^a = \frac{1}{\sqrt{2} l_s} \alpha_0^a \tag{1.4.47}$$

Consider now the $D-p-1$ directions with DD boundary conditions (1.4.42). For $\sigma = 0$ we have $0 = \dot{X}^I(\tau,0) = \partial_+ X^I(\tau,0) + \partial_- X^I(\tau,0)$ so that $\partial_+ X^I(\tau,0) = -\partial_- X^I(\tau,0)$. From (1.4.43) we see this means $f^I(\tau) = -g^I(\tau)$ for any τ since $\xi^\pm = \tau$ for $\sigma = 0$. Instead for $\sigma = \pi$ we have $0 = \dot{X}^I(\tau,\pi) = \partial_+ X^I(\tau,\pi) + \partial_- X^I(\tau,\pi)$ which using (1.4.43) is seen to give $f^I(\tau+\pi) = -g^I(\tau-\pi)$ for any τ. Using that $f^I = -g^I$ this means that $f^I(\tau+\pi) = f^I(\tau-\pi)$, from which we deduce that $f^I(\tau+2\pi) = f^I(\tau)$. Thus $f^I = -g^I$ is a periodic function with period 2π. Using again (1.4.43) we see that this means we can make the following Fourier expansions

$$\partial_- X^I = \frac{l_s}{\sqrt{2}} \sum_{n\in\mathbb{Z}} \alpha_n^I e^{-in\xi^-} \quad , \quad \partial_+ X^I = -\frac{l_s}{\sqrt{2}} \sum_{n\in\mathbb{Z}} \alpha_n^I e^{-in\xi^+} \tag{1.4.48}$$

[9]Since we need that $(X^a(\tau,\sigma))^* = X^a(\tau,\sigma)$ we require $(x^a)^* = x^a$, $(p^a)^* = p^a$ and $(\alpha_n^a)^* = \alpha_{-n}^a$ for the complex conjugates of the modes.

Thus, for the DD open string boundary condition the Fourier modes of the right-moving sector are equal to minus those of the left-moving sector. Integrating this we get that the mode expansions for the directions with DD boundary conditions are[10]

$$X^I(\tau,\sigma) = c_1^I + (c_2^I - c_1^I)\frac{\sigma}{\pi} - \sqrt{2}l_s \sum_{n\neq 0} \frac{1}{n}\alpha_n^I e^{-in\tau}\sin(n\sigma) \tag{1.4.49}$$

with $I = p+1, ..., D-1$ in accordance with (1.4.42). Note that this requires the identification

$$\alpha_0^I = \frac{c_1^I - c_2^I}{\sqrt{2}\pi l_s} \tag{1.4.50}$$

of the zero-mode. The momentum density (1.4.18) has the mode expansion

$$\Pi^I(\tau,\sigma) = \frac{i\sqrt{2}}{2\pi l_s}\sum_{n\neq 0}\alpha_n^I e^{-in\tau}\sin(n\sigma) \tag{1.4.51}$$

For directions with NN boundary conditions the center of mass position and the total momentum is

$$\frac{1}{\pi}\int_0^\pi d\sigma\, X^a(\tau,\sigma) = x^a + 2l_s^2 p^a\tau \ , \quad \int_0^\pi d\sigma\, \Pi_a(\tau,\sigma) = p_a \tag{1.4.52}$$

Thus x^a is the center of mass position of the string at $\tau = 0$. The total momentum p^a is conserved $\dot{p}^a = 0$ since the momentum current Π_a^α is conserved (1.4.16) using the same manipulations as in (1.4.32) and that $[\Pi_a^\sigma]_{\sigma=0}^\pi = 0$. Similarly, the charges associated with Lorentz invariance

$$J_{ab} = \int_0^{2\pi} d\sigma\, \mathcal{J}_{ab}^\tau = \int_0^{2\pi} d\sigma\, (X_a\Pi_b - X_b\Pi_a) \tag{1.4.53}$$

are conserved $\dot{J}_{ab} = 0$ on the worldsheet.

For directions with DD boundary conditions there are no dynamical zero modes, instead the center of mass position and total momentum includes contributions from all α_n^I, $n \neq 0$.

1.5 Energy-momentum constraint in flat gauge

In this section we consider the constraint equation (1.3.7) in the flat gauge (1.3.17). In Sec. 1.4 we found the general solution to the equations of motion (1.4.3) to the Polyakov action in the flat gauge (1.3.18) in the form of mode expansions for $X^\mu(\tau,\sigma)$ both for the

[10] Since we need that $(X^I(\tau,\sigma))^* = X^I(\tau,\sigma)$ we require $(\alpha_n^I)^* = \alpha_{-n}^I$ for the complex conjugates of the modes.

closed and the open string. However, an arbitrary set of modes for the mode expansions do not in general correspond to the physics of the closed and open string since we have not imposed the constraint equation (1.3.7). We consider in this section how to impose the constraint equation (1.3.7) for the closed string mode expansion (1.4.27). We briefly comment on the open string as well.

Consider the closed string with mode expansion (1.4.27). For any values of the modes this solves the equations of motion (1.4.3) to the Polyakov action in the flat gauge (1.3.18). However, the solutions should also satisfy the constraint $T_{\alpha\beta} = 0$ with the worldsheet energy-momentum tensor given by

$$T_{\alpha\beta} = l_s^{-2}\Big(\partial_\alpha X \cdot \partial_\beta X - \frac{1}{2}\eta_{\alpha\beta}\partial_\gamma X \cdot \partial^\gamma X\Big) \tag{1.5.1}$$

In the lightcone coordinates (1.4.4) we find using (1.4.5) and (1.4.6) that $T_{+-} = 0$ identically and therefore $T_{+-} = 0$ does not constrain $X^\mu(\xi)$. We can write this as

$$\eta^{\alpha\beta}T_{\alpha\beta} = 0 \tag{1.5.2}$$

which states that the trace of the energy-momentum tensor is zero, a fact that we return to in Sec. 2.4. The non-trivial components of the worldsheet energy-momentum tensor are thus

$$T_{--} = l_s^{-2}(\partial_- X)^2\ ,\quad T_{++} = l_s^{-2}(\partial_+ X)^2 \tag{1.5.3}$$

Thus, in the flat gauge (1.3.17) the constraints are $T_{--} = 0$ and $T_{++} = 0$.

Notice now that the conservation of the energy-momentum tensor $\partial_\alpha T^{\alpha\beta} = 0$ can be written $\eta^{\alpha\beta}\partial_\alpha T_{\beta\gamma} = 0$ which in lightcone coordinates is equivalent to

$$\partial_+ T_{--} = 0\ ,\quad \partial_- T_{++} = 0 \tag{1.5.4}$$

hence $T_{--} = T_{--}(\xi^-)$ and $T_{++} = T_{++}(\xi^+)$. Since $\partial_\pm X^\mu$ is a function of $\xi^\pm$ we see from (1.5.3) that these equations are satisfied automatically for the mode expansion (1.4.27). Since $\partial_\pm X^\mu$ is a period function of $\xi^\pm$ with period 2π we can make the following mode expansion of the worldsheet energy-momentum tensor

$$T_{--}(\xi^-) = \sum_{n\in\mathbb{Z}} L_n e^{-in\xi^-}\ ,\quad T_{++}(\xi^+) = \sum_{n\in\mathbb{Z}} \tilde{L}_n e^{-in\xi^+} \tag{1.5.5}$$

Using (1.4.25) we compute

$$L_n = \frac{1}{2}\sum_{k\in\mathbb{Z}} \alpha_{n-k}\cdot\alpha_k\ ,\quad \tilde{L}_n = \frac{1}{2}\sum_{k\in\mathbb{Z}} \tilde{\alpha}_{n-k}\cdot\tilde{\alpha}_k \tag{1.5.6}$$

Reality of T_{--} and T_{++} requires

$$(L_n)^* = L_{-n} \ , \quad (\tilde{L}_n)^* = \tilde{L}_{-n} \tag{1.5.7}$$

for $n \in \mathbb{Z}$. The constraints $T_{--} = 0$ and $T_{++} = 0$ is then equivalent to requiring

$$L_n = \tilde{L}_n = 0 \ \text{ for all } \ n \geq 0 \tag{1.5.8}$$

Thus, this tells us how to implement the constraint (1.3.7) on the α_n^μ and $\tilde{\alpha}_n^\mu$ modes of the mode expansion (1.4.27) of the closed string.

Consider the constraints $L_0 = \tilde{L}_0 = 0$. We find

$$L_0 = \frac{l_s^2}{4} p^2 + \sum_{k=1}^{\infty} \alpha_{-k} \cdot \alpha_k \ , \quad \tilde{L}_0 = \frac{l_s^2}{4} p^2 + \sum_{k=1}^{\infty} \tilde{\alpha}_{-k} \cdot \tilde{\alpha}_k \tag{1.5.9}$$

Since the mass-shell condition is $p^2 = -M^2$ with M being the mass of the string the constraints $L_0 = \tilde{L}_0 = 0$ can be written

$$M^2 = \frac{4}{l_s^2} \sum_{k=1}^{\infty} \alpha_{-k} \cdot \alpha_k = \frac{4}{l_s^2} \sum_{k=1}^{\infty} \tilde{\alpha}_{-k} \cdot \tilde{\alpha}_k \tag{1.5.10}$$

Thus, this gives the mass of the string in terms of the excited oscillator modes. We also see that it gives a relation between the right-moving modes α_n^μ and the left-moving modes $\tilde{\alpha}_n^\mu$. We note that getting M^2 from the $L_0 = \tilde{L}_0 = 0$ constraints is somewhat analogous to getting p_0 from the $H = 0$ constraint in the case of the relativistic point particle, see Sec. 1.1.2.

For the open string the constraint (1.3.7) is implemented in a very similar fashion. For both the NN and DD boundary conditions one has that $\partial_+ X^\mu$ can be computed from $\partial_- X^\mu$ thus there is only one set of modes α_n^μ. Thus, if one sets $\tilde{\alpha}_n^\mu = \alpha_n^\mu$ in the above, one finds the correct mode expansion of $T_{\alpha\beta}$ and the correct constraint equations in terms of these modes as well.

1.6 Conformal symmetry

In Sec. 1.3.3 we derived the flat gauge for the Polyakov action (1.3.1). In the flat gauge the Polyakov action takes a particularly simple form (1.3.18). This gauge was found by exploiting both the diffeomorphism invariance as well as the Weyl invariance of the Polyakov action, see Sec. 1.3.2. In this section we shall see that there is still remnant symmetry left over in the flat gauge. This left over symmetry will be seen to take a prominent role both when discussing the constraint (1.3.7) and when quantizing the theory.

We furthermore consider the relation between the conformal symmetry and the modes of the worldsheet energy momentum. This will point to a connection between imposing the constraint (1.3.7) and the conformal symmetry.

1.6.1 Conformal symmetry as remnant symmetry in flat gauge

A *conformal transformation* is a particular kind of coordinate transformation that preserves angles. If we consider this in two dimensions, with coordinates ξ^α and metric $g_{\alpha\beta}$, a general coordinate transformation $\xi^\alpha \to \tilde{\xi}^\alpha$ transforms the metric as follows

$$g_{\alpha\beta} \to \tilde{g}_{\alpha\beta}(\tilde{\xi}) = \frac{\partial \xi^\gamma}{\partial \tilde{\xi}^\alpha} \frac{\partial \xi^\delta}{\partial \tilde{\xi}^\beta} g_{\gamma\delta}(\xi) \tag{1.6.1}$$

Instead a conformal transformation is a coordinate transformation $\xi^\alpha \to \tilde{\xi}^\alpha$ such that

$$g_{\alpha\beta} \to \tilde{g}_{\alpha\beta}(\tilde{\xi}) = e^{2\Phi(\xi)} g_{\alpha\beta}(\xi) \tag{1.6.2}$$

Consider now the Polyakov action in flat gauge (1.3.18). Our starting point is that the worldsheet metric is $\eta_{\alpha\beta}$. If we perform a conformal transformation $\xi^\alpha \to \tilde{\xi}^\alpha$ we get a metric of the form $\tilde{g}_{\alpha\beta}(\tilde{\xi}) = e^{2\Phi(\xi)} \eta_{\alpha\beta}$. However, combining this with a Weyl transformation (1.3.12) with $\Omega(\tilde{\xi}) = e^{-\Phi(\xi)}$ we see that the Weyl transformed metric is $\hat{g}_{\alpha\beta}(\tilde{\xi}) = \Omega^2(\tilde{\xi}) \tilde{g}_{\alpha\beta}(\tilde{\xi}) = \eta_{\alpha\beta}$. This thus brings us back to the flat gauge choice for the worldsheet metric. Conversely, it is also clear that if we had coordinate transformations that is not of the form $\tilde{g}_{\alpha\beta}(\tilde{\xi}) = e^{2\Phi(\xi)} \eta_{\alpha\beta}(\xi)$ then we could not get back to the flat gauge choice. Thus, we conclude that the Polyakov action in flat gauge (1.3.18) is invariant under conformal transformations on the worldsheet, and that this constitutes all the remnant local symmetry on the worldsheet after fixing the flat gauge (1.3.17).

To find the specific form of the conformal transformations we first consider infinitesimal diffeomorphisms on the worldsheet, *i.e.* coordinate transformations of the form

$$\xi^\alpha \to \tilde{\xi}^\alpha = \xi^\alpha - a^\alpha(\xi) \tag{1.6.3}$$

where $a^\alpha(\xi)$ is infinitesimally small. The embedding $X^\mu(\xi)$ and the metric $g_{\alpha\beta}(\xi)$ transforms in general as (1.3.10) and (1.3.11) which in the infinitesimal case become

$$\delta X^\mu(\xi) = a^\alpha \partial_\alpha X^\mu \ , \quad \delta g_{\alpha\beta}(\xi) = a^\gamma \partial_\gamma g_{\alpha\beta} + g_{\alpha\gamma} \partial_\beta a^\gamma + g_{\beta\gamma} \partial_\alpha a^\gamma \tag{1.6.4}$$

where $\delta X^\mu(\xi) = \tilde{X}^\mu(\xi) - X^\mu(\xi)$ and $\delta g_{\alpha\beta}(\xi) = \tilde{g}_{\alpha\beta}(\xi) - g_{\alpha\beta}(\xi)$. Since we start in flat gauge $g_{\alpha\beta} = \eta_{\alpha\beta}$ this reduces to

$$\delta X^\mu(\xi) = a^\alpha \partial_\alpha X^\mu \ , \quad \delta g_{\alpha\beta}(\xi) = \partial_\alpha a_\beta + \partial_\beta a_\alpha \tag{1.6.5}$$

Infinitesimal conformal transformations should be of the form

$$\delta g_{\alpha\beta}(\xi) = 2\Phi(\xi)\eta_{\alpha\beta} \tag{1.6.6}$$

The task is thus to find the $a^\alpha(\xi)$ functions for which the metric transformation is of the form (1.6.6). From (1.6.5) and (1.6.6) we get $\eta^{\alpha\beta}\delta g_{\alpha\beta} = 4\Phi(\xi) = 2\partial_\alpha a^\alpha$ hence $2\Phi(\xi) = \partial_\alpha a^\alpha$. Thus the conformal transformations are those for which $\partial_\alpha a_\beta + \partial_\beta a_\alpha = \partial_\gamma a^\gamma \eta_{\alpha\beta}$. It is not difficult to infer that the necessary and sufficient conditions are

$$\partial_0 a_0 + \partial_1 a_1 = 0 \ , \quad \partial_0 a_1 + \partial_1 a_0 = 0 \tag{1.6.7}$$

We now write this in lightcone coordinates (1.4.4). Define $a^\pm = a^0 \pm a^1 = -a_0 \pm a_1$. Then the conditions (1.6.7) are equivalent to

$$\partial_+ a^- = 0 \ , \quad \partial_- a^+ = 0 \tag{1.6.8}$$

Thus, the infinitesimal conformal transformations are such that $a^- = a^-(\xi^-)$ and $a^+ = a^+(\xi^+)$.

Combining many infinitesimal transformations (1.6.8) one finds that a general conformal transformation is of the form

$$\xi^- \to \tilde{\xi}^-(\xi^-) \ , \quad \xi^+ \to \tilde{\xi}^+(\xi^+) \tag{1.6.9}$$

when starting from the flat gauge. Hence we have derived that the Polyakov action in flat gauge (1.3.18) is invariant under the worldsheet conformal transformations (1.6.9) and that these transformations constitute all the remnant local symmetry after fixing the flat gauge (1.3.17).

1.6.2 Conserved charges

The action (1.3.18) is symmetric under worldsheet translations

$$\xi^\alpha \to \xi^\alpha - c^\alpha \tag{1.6.10}$$

where c^α is constant. For any symmetry of the action one has a corresponding conserved current. In this case it is

$$J_\alpha = T_{\alpha\beta} c^\beta \tag{1.6.11}$$

We see immediately that this is conserved $\partial^\alpha J_\alpha = 0$ by using the energy-momentum conservation $\partial^\alpha T_{\alpha\beta} = 0$. Using this one derives easily that

$$\int d\sigma J_0 \tag{1.6.12}$$

is a conserved charge. In the special case $c^1 = 0$ and $c^0 \neq 0$ we have that (1.6.10) is worldsheet time-translation and the conserved charge (1.6.12) is proportional to the energy, while for $c^0 = 0$ and $c^1 \neq 0$ (1.6.10) is spatial translation and the conserved charge (1.6.12) is the momentum along σ.

Consider instead the infinitesimal conformal transformations

$$\xi^\alpha \to \xi^\alpha - a^\alpha(\xi) \ , \quad \partial_- a^+ = \partial_+ a^- = 0 \tag{1.6.13}$$

Note that worldsheet translations are included in these transformations. A good guess for a corresponding conserved current is thus

$$J_\alpha = T_{\alpha\beta} a^\beta \tag{1.6.14}$$

One can indeed show that this is a conserved current. Note first from (1.4.5) that

$$\partial^\alpha J_\alpha = \eta^{\alpha\beta} \partial_\alpha J_\beta = -2\partial_+ J_- - 2\partial_- J_+ \tag{1.6.15}$$

Instead from (1.6.14) we see

$$J_- = T_{--} a^- \ , \quad J_+ = T_{++} a^+ \tag{1.6.16}$$

using that $T_{+-} = 0$ from (1.5.2). Using now energy-momentum conservations (1.5.4) we see that

$$\partial_+ J_- = 0 \ , \quad \partial_- J_+ = 0 \tag{1.6.17}$$

which from (1.6.15) indeed shows that the current (1.6.14) for conformal transformation is conserved $\partial^\alpha J_\alpha = 0$. Note that since $a^\pm$ are arbitrary functions of $\xi^\pm$ we have that

$$\begin{array}{l}\text{Worldsheet energy momentum conservation} \\ \Leftrightarrow \text{Conservation of current for conformal transformations}\end{array} \tag{1.6.18}$$

Since $T_{\pm\pm}$ is a period function of $\xi^\pm$ with period 2π, we also take $a^\pm$ to be a periodic function of $\xi^\pm$ with period 2π. Hence we can make the Fourier decompositions $a^\pm = \sum_{n\in\mathbb{Z}} A_n^\pm e^{in\xi^\pm}$. For any given function $a^-(\xi^-)$ we can now construct the conserved charge

$$Q[a^-] = \int_0^{2\pi} d\xi^- \, J_- = \int_0^{2\pi} d\xi^- \, T_{--} a^- = \sum_{n\in\mathbb{Z}} A_n^- \int_0^{2\pi} d\xi^- \, T_{--} e^{in\xi^-} \tag{1.6.19}$$

We see that this charge is conserved. Firstly, it does not depend on ξ^-. Secondly, we see from (1.6.17) that it does not depend on ξ^+ either. Inserting now the mode expansion (1.5.5) for T_{--} we get

$$Q[a^-] = \sum_{n\in\mathbb{Z}} 2\pi A_n^- L_n \tag{1.6.20}$$

In particular for $a^- = \frac{1}{2\pi} e^{in\xi^-}$ we get $Q[a^-] = L_n$.[11] Similarly, we can construct the conserved charge $\tilde{Q}[a^+] = \int_0^{2\pi} d\xi^+ \, T_{++} a^+$ which for $a^+ = \frac{1}{2\pi} e^{in\xi^+}$ gives $\tilde{Q}[a^+] = \tilde{L}_n$. Hence we conclude that the conformal symmetry of the Polyakov action in flat gauge gives rise to an infinite set of conserved charges and that this infinite set of charges corresponds precisely to the set of modes L_n and $\tilde{L}_n$ of the worldsheet energy momentum tensor.

1.7 Exercises for Chapter 1

Exercise 1.1. Consider the relativistic point particle moving in the background of a general D-dimensional space-time with metric $G_{\mu\nu}$. Take the action to be

$$S = -m \int d\tau \sqrt{-G_{\mu\nu} \dot{X}^\mu \dot{X}^\nu} \tag{1.7.1}$$

where $\dot{X}^\mu$ is the derivative of X^μ with respect to τ.

- Find the equation of motion by varying the action. Does the fact that the metric can be arbitrary give extra terms?
- The world line is parameterised by τ and this action is invariant under transformations of $\tau \to \tilde{\tau}(\tau)$. Discuss what condition τ should obey in order for τ to be the proper time of the particle.
- Show that the equation of motion for the particle is equivalent to the geodesic equation as known from General Relativity.

Exercise 1.2. We consider 3-dimensional Euclidean space $\mathbb{R}^3$ in Cartesian coordinates hence with metric $ds^2 = \delta_{ij} dx^i dx^j$, $i, j = 1, 2, 3$. In this 3-dimensional space we embed a 2-dimensional surface described by the embedding coordinates $X^i(\xi^1, \xi^2)$ where (ξ^1, ξ^2) are the coordinates parameterising the surface.

- What is the general expression for the induced metric γ_{ab}, $a, b = 1, 2$, on the surface in terms of $X^i(\xi)$ and the metric δ_{ij}? In terms of this induced metric γ_{ab} what is the general formula for the area of the surface?
- Consider a finite cylinder of length L and radius R described as

$$\begin{gathered} X^1(\xi) = R\cos(\xi^1) \; , \quad X^2(\xi) = R\sin(\xi^1) \; , \quad X^3(\xi) = L\xi^2 \\ 0 \le \xi^1 \le 2\pi \; , \quad 0 \le \xi^2 \le 1 \end{gathered} \tag{1.7.2}$$

[11] If one wants to stick to real functions one can use $a^- = \frac{1}{2\pi}\cos(n\xi^-)$ giving $\frac{1}{2}(L_n + L_{-n}^*)$ and $a^- = \frac{1}{2\pi}\sin(n\xi^-)$ giving $-\frac{i}{2}(L_n - L_{-n})$ corresponding to the real and imaginary parts of L_n.

Find the induced metric on the cylinder and use the general formula for the area to compute the area of the cylinder.

- Consider a sphere of radius R described as

$$X^1(\xi) = R\cos(\xi^1)\sin(\xi^2)\ , \quad X^2(\xi) = R\sin(\xi^1)\sin(\xi^2)\ , \quad X^3(\xi) = R\cos(\xi^2)$$
$$0 \leq \xi^1 \leq 2\pi\ , \quad 0 \leq \xi^2 \leq \pi \tag{1.7.3}$$

Find the induced metric on the sphere and use the general formula for the area to compute the area of the sphere.

Exercise 1.3. Consider the Nambu-Goto action (1.2.11).

- Write down the conjugate momentum

$$\Pi_\mu(\tau,\sigma) = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} \tag{1.7.4}$$

in terms of $X^\mu(\tau,\sigma)$.

- Show that $\Pi \cdot X' = 0$ and $\Pi^2 + T^2(X')^2 = 0$. These are constraints.

- Find the canonical Hamiltonian and show this gives another constraint.

Exercise 1.4. Consider the Nambu-Goto action (1.2.11). What are the boundary conditions for the open and closed strings?

Exercise 1.5. Derive that the variation of the square root of the determinant of the metric of a D-dimensional space-time is

$$\delta\sqrt{-\det g} = \frac{1}{2}\sqrt{-\det g}\, g^{\mu\nu}\delta g_{\mu\nu} = -\frac{1}{2}\sqrt{-\det g}\, g_{\mu\nu}\delta g^{\mu\nu} \tag{1.7.5}$$

Hint: Seing $g_{\mu\nu}$ as a D by D matrix we can write $g = \exp(M)$. What is $\det(g)$ then in terms of M?

Exercise 1.6. We consider a classical closed string in a four-dimensional space-time as described by the Polyakov action in flat gauge (1.3.18). We work in the static gauge $X^0(\tau,\sigma) = c\,\tau$ where c is a constant.[12] Consider a closed string that at $\tau = 0$ is in the following configuration

$$X^1 = R\cos\sigma\ , \quad X^2 = R\sin\sigma\ , \quad X^3 = 0\ , \quad 0 \leq \sigma \leq 2\pi$$
$$\dot{X}^1 = \dot{X}^2 = \dot{X}^3 = 0 \tag{1.7.6}$$

[12] One can show that this gauge is always possible using the residual symmetry (1.6.9) of the Polyakov action in flat gauge and the fact that $\partial_\alpha\partial^\alpha X^0 = 0$.

- What happens to the closed string? Hint: You can use the ansatz $X^1(\tau,\sigma) = r(\tau)\cos\sigma$ and $X^2(\tau,\sigma) = r(\tau)\sin\sigma$. Don't forget to also impose the constraints $\dot{X}\cdot X' = 0$ and $\dot{X}^2 + X'^2 = 0$.[13]

- Give a physical explanation for what happens. At what point has the string only "potential energy"? At what point has the string only "kinetic energy"? What is the nature of the "potential energy"?

Exercise 1.7. We consider a classical closed string in a four-dimensional space-time as described by the Polyakov action in flat gauge (1.3.18). Pick the static gauge $X^0(\tau,\sigma) = c\,\tau$ where c is a constant.

- Write the two constraints $\dot{X}\cdot X' = 0$ and $\dot{X}^2 + X'^2 = 0$ in terms of $\vec{X}(\tau,\sigma)$.

- Consider the first constraint $\dot{X}\cdot X' = 0$. What does this say about the motion of the string in the direction longitudinal to the string?

- Consider the second constraint $\dot{X}^2 + X'^2 = 0$ when $\dot{\vec{X}} = 0$. What is the relation between the constant c and the length of the string?

Exercise 1.8. The Polyakov action is diffeomorphism invariant under coordinate transformations of the world sheet coordinates. Show that under the infinitesimal coordinate transformation

$$\xi^\alpha \to \tilde{\xi}^\alpha(\xi) = \xi^\alpha - a^\alpha(\xi) \tag{1.7.7}$$

the target space map $X^\mu(\xi)$ and the world sheet metric $g_{\alpha\beta}(\xi)$ transform as follows

$$\delta X^\mu = a^\alpha \partial_\alpha X^\mu \ , \quad \delta g_{\alpha\beta} = a^\gamma \partial_\gamma g_{\alpha\beta} + g_{\alpha\gamma}\partial_\beta a^\gamma + g_{\beta\gamma}\partial_\alpha a^\gamma \tag{1.7.8}$$

where $\delta X^\mu(\xi) = \tilde{X}^\mu(\xi) - X^\mu(\xi)$ and $\delta g_{\alpha\beta}(\xi) = \tilde{g}_{\alpha\beta}(\xi) - g_{\alpha\beta}(\xi)$.

Exercise 1.9. Consider two-dimensional Minkowski space-time with coordinates $\xi^\alpha = (\xi^0,\xi^1)$ and metric $\eta_{\alpha\beta}d\xi^\alpha d\xi^\beta = -(d\xi^0)^2 + (d\xi^1)^2$. Consider the coordinate transformation $(\xi^0,\xi^1) \to (\xi^+,\xi^-)$ to the lightcone coordinates

$$\xi^\pm = \xi^0 \pm \xi^1 \tag{1.7.9}$$

- Show that in the new coordinates we have the metric

$$\eta_{+-} = \eta_{-+} = -\frac{1}{2} \ , \quad \eta_{++} = \eta_{--} = 0 \tag{1.7.10}$$

[13]These constraints follow from the constraint (1.3.7) using (1.5.1).

- Show that the inverse metric is

$$\eta^{+-} = \eta^{-+} = -2 \ , \quad \eta^{++} = \eta^{--} = 0 \tag{1.7.11}$$

- Show that the derivatives are related as

$$\partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1) \tag{1.7.12}$$

Exercise 1.10. In this exercise we find the conserved currents for the Poincare invariance of the string using the Noether procedure introduced in Sec. 1.4.2. Consider a classical bosonic closed string as described by the Polyakov action (1.3.18) in the flat gauge.

- Using $M^\mu(X) = b^\mu$ in (1.4.10) derive from the Noether procedure that the conserved current for translations is $J^\alpha = -b^\mu \Pi^\alpha_\mu$ with Π^α_μ given by (1.4.15).
- Check that the conservation (1.4.16) of the current Π^α_μ follows from the equations of motion for the string.
- Using $M^\mu(X) = \omega^\mu{}_\nu X^\nu$ with $\omega^\mu{}_\nu$ being a constant antisymmetric tensor $\omega_{\mu\nu} = -\omega_{\nu\mu}$, derive from the Noether procedure that the conserved current for Lorentz transformations is $J^\alpha = \frac{1}{2}\omega^{\mu\nu}\mathcal{J}^\alpha_{\mu\nu}$ where $\mathcal{J}^\alpha_{\mu\nu}$ is the current given by (1.4.19).
- Check that the conservation (1.4.20) of the current $\mathcal{J}^\alpha_{\mu\nu}$ follows from the equations of motion for the string.

Exercise 1.11. This exercise concerns the Fourier mode expansion of periodic functions.

- Consider a function $f(\sigma)$ which is known to be periodic in σ with period 2π. Then we can make the Fourier series expansion in Fourier modes f_n

  $$f(\sigma) = \sum_{n\in\mathbb{Z}} f_n e^{in\sigma} \tag{1.7.13}$$

  Show that we have

  $$f_n = \frac{1}{2\pi}\int_0^{2\pi} d\sigma f(\sigma) e^{-in\sigma} \tag{1.7.14}$$

- Consider the bosonic closed string. The mode expansions of the position field $X^\mu(\tau,\sigma)$ and the momentum density $\Pi_\mu(\tau,\sigma) = T\dot{X}_\mu$ are given in Eqs. (1.4.27) and (1.4.29). Show that one can express the modes of the mode expansion as follows

$$
\begin{aligned}
x^\mu &= \int_0^{2\pi} d\sigma \left(\frac{1}{2\pi} X^\mu(\tau,\sigma) - l_s^2 \tau \Pi^\mu(\tau,\sigma) \right) \ , \quad p^\mu = \int_0^{2\pi} d\sigma \Pi^\mu(\tau,\sigma) \\
\alpha_n^\mu &= e^{in\tau} \left(\frac{l_s}{\sqrt{2}} \int_0^{2\pi} d\sigma \Pi^\mu(\tau,\sigma) e^{-in\sigma} - \frac{in}{2\pi\sqrt{2} l_s} \int_0^{2\pi} d\sigma X^\mu(\tau,\sigma) e^{-in\sigma} \right) \qquad (1.7.15) \\
\tilde{\alpha}_n^\mu &= e^{in\tau} \left(\frac{l_s}{\sqrt{2}} \int_0^{2\pi} d\sigma \Pi^\mu(\tau,\sigma) e^{in\sigma} - \frac{in}{2\pi\sqrt{2} l_s} \int_0^{2\pi} d\sigma X^\mu(\tau,\sigma) e^{in\sigma} \right)
\end{aligned}
$$

2 Covariant Quantization of the Closed String

The goal of this chapter is to quantize the infinitely thin relativistic string that we described classically in Chapter 1. We consider in this chapter only the closed string. The open string will be treated in Chapter 5.

There are two approaches to quantize the string both using the Polyakov action in flat gauge (1.3.18) with constraints (1.3.7) as starting point:

- Covariant quantization: In the *covariant quantization* approach one quantizes all the D fields of $X^\mu(\xi)$ on the same footing. Then one imposes the constraints $T_{--} = 0$ and $T_{++} = 0$ in the quantum theory Fock space. A challenge in this approach is how to deal with the negative norm states - called *ghost states* - which naturally occur. One can show that under certain conditions, including that $D \leq 26$, the ghost states disappear when imposing the constraints. Another issue in the covariant quantization approach is that there is an anomaly in the conformal symmetry algebra. As we shall see in Sec. 2.5 one can remove this anomaly precisely for $D = 26$ and then the theory is fully consistent when imposing the quantum version of the constraints $T_{--} = 0$ and $T_{++} = 0$. To show that this is fully consistent one needs to use the path-integral approach, introduce ghost fields and perform a BRST quantization.

- Lightcone quantization: A rather different approach is *lightcone quantization*. This is reviewed in Appendix A. In this approach one first solves the constraint (1.3.7) in the classical theory thus finding a classical description that is manifestly physical. After this, one quantizes the theory without having to deal with imposing constraints in the quantum theory. The price one needs to pay in this case is that an anomaly in the Lorentz invariance of the target space generically enters in the quantum theory. Amazingly, one can show that this anomaly precisely disappears for $D = 26$.

The infinitely thin relativistic string is known as the *bosonic string* as it only gives rise to bosonic degrees of freedom. In Chapter 6 we discuss instead the quantization of the superstring that also has fermionic degrees of freedom.

2.1 A first look at covariant quantization

Thanks to the fact that the Polyakov action in flat gauge (1.3.18) is a theory of D free scalar fields living on the two-dimensional worldsheet the first step in the covariant quantization procedure is very simple. We have found the mode expansions (1.4.27) for

$X^\mu(\xi)$ and (1.4.29) for its conjugate momentum $\Pi^\mu(\xi)$ (the momentum density of the string). These mode expansions are in terms of the modes x^μ, p^μ as well as α_n^μ and $\tilde{\alpha}_n^\mu$. We can now quantize $X^\mu(\xi)$ by promoting the modes x^μ, p^μ as well as α_n^μ and $\tilde{\alpha}_n^\mu$ to operators and imposing the canonical equal-time commutation relations

$$\begin{aligned}
&[X^\mu(\tau,\sigma),\Pi_\nu(\tau,\sigma')] = i\delta(\sigma-\sigma')\delta^\mu_\nu \\
&[X^\mu(\tau,\sigma),X^\nu(\tau,\sigma')] = [\Pi_\mu(\tau,\sigma),\Pi_\nu(\tau,\sigma')] = 0
\end{aligned} \tag{2.1.1}$$

Moreover, we also impose the reality conditions

$$(X^\mu(\tau,\sigma))^\dagger = X^\mu(\tau,\sigma) \ , \quad (\Pi_\mu(\tau,\sigma))^\dagger = \Pi_\mu(\tau,\sigma) \tag{2.1.2}$$

In doing this, we find that imposing (2.1.1) and (2.1.2) for $X^\mu(\xi)$ and $\Pi^\mu(\xi)$ is equivalent to imposing the following commutations relations on the x^μ, p^μ, α_n^μ and $\tilde{\alpha}_n^\mu$ operators

$$\begin{aligned}
&[x^\mu,p_\nu] = i\delta^\mu_\nu \ , \quad [\alpha^\mu_m,\alpha^\nu_n] = [\tilde{\alpha}^\mu_m,\tilde{\alpha}^\nu_n] = m\delta_{m+n,0}\eta^{\mu\nu} \\
&[x^\mu,x^\nu] = [p_\mu,p_\nu] = 0 \ , \quad [\alpha^\mu_m,\tilde{\alpha}^\nu_n] = 0
\end{aligned} \tag{2.1.3}$$

as well as the reality conditions

$$(x^\mu)^\dagger = x^\mu \ , \quad (p_\mu)^\dagger = p_\mu \ , \quad (\alpha^\mu_n)^\dagger = \alpha^\mu_{-n} \ , \quad (\tilde{\alpha}^\mu_n)^\dagger = \tilde{\alpha}^\mu_{-n} \tag{2.1.4}$$

Clearly, the commutator $[x^\mu,p_\nu] = i\delta^\mu_\nu$ is like the standard commutation relation between position and momentum in quantum mechanics, at least when $\mu,\nu \neq 0$. Considering the commutators for the α_n^μ and $\tilde{\alpha}_n^\mu$ modes we see that defining

$$a^\mu_n = \frac{1}{\sqrt{n}}\alpha^\mu_n \ , \quad (a^\mu_n)^\dagger = \frac{1}{\sqrt{n}}\alpha^\mu_{-n} \tag{2.1.5}$$

for $n > 0$ we get the commutation relation

$$[a^\mu_n,(a^\nu_m)^\dagger] = \delta_{n,m}\eta^{\mu\nu} \tag{2.1.6}$$

Thus, this corresponds to a bunch of harmonic oscillators (at least for $\mu,\nu \neq 0$) with α_n^μ being the annihilation operators and α^μ_{-n} the creation operators for $n > 0$, and similarly for the left-moving sector with the $\tilde{\alpha}_n^\mu$ modes. That we get a bunch of harmonic oscillators when quantizing the $X^\mu(\xi)$ field should be no surprise since that is what one should expect in quantum field theory by quantizing a free field.

Following the idea that α_n^μ and $\tilde{\alpha}_n^\mu$ are annihilation operators while α^μ_{-n} and $\tilde{\alpha}^\mu_{-n}$ are creation operators for $n > 0$, we see that a ground state $|0\rangle$ should obey

$$\alpha^\mu_n|0\rangle = \tilde{\alpha}^\mu_n|0\rangle = 0 \ , \quad n > 0 \tag{2.1.7}$$

However, we have not specified how the zero modes x^μ and p_μ acts on this ground state. This choice in fact gives the ground state a further structure. We choose that a ground state should have a definite D-momentum $p_\mu = k_\mu$, and thus write it as $|0;k\rangle$, hence it is defined by

$$\begin{aligned} p_\mu|0;k\rangle &= k_\mu|0;k\rangle \\ \alpha^\mu_n|0;k\rangle = \tilde{\alpha}^\mu_n|0;k\rangle &= 0\ , \quad n>0 \end{aligned} \tag{2.1.8}$$

with $\mu = 0,1,...,D-1$. We furthermore normalize it as

$$\langle 0;k|0;k\rangle = 1 \tag{2.1.9}$$

Note that this normalization only makes sense if we only act on the ground state with one particular momentum $p_\mu = k_\mu$. This is consistent since we are only working with a single string. However, in some cases one needs to work with several vacua with different momentum eigenvalues. In these cases one should normalize the vacua as $\langle 0;k|0;k'\rangle = c\,\delta^D(k-k')$ with c a suitably chosen constant.

We can now write a basis for the closed string Fock space as

$$\alpha^{\mu_1}_{-m_1}\alpha^{\mu_2}_{-m_2}\cdots\alpha^{\mu_p}_{-m_p}\tilde{\alpha}^{\nu_1}_{-n_1}\tilde{\alpha}^{\nu_2}_{-n_2}\cdots\tilde{\alpha}^{\nu_q}_{-n_q}|0;k\rangle \tag{2.1.10}$$

with $m_i, n_i > 0$. However, notice now that for a given $n>0$ the state

$$|g\rangle = \frac{1}{\sqrt{n}}\alpha^0_{-n}|0;k\rangle \tag{2.1.11}$$

has negative norm

$$\langle g|g\rangle = \frac{1}{n}\langle 0;k|\alpha^0_n\alpha^0_{-n}|0;k\rangle = \frac{1}{n}\langle 0;k|[\alpha^0_n,\alpha^0_{-n}]|0;k\rangle = \langle 0;k|\eta^{00}|0;k\rangle = -1 \tag{2.1.12}$$

Such states are unphysical and are known as *ghost states*. This means that the physical states is only a subset of the states in the closed string Fock space with basis (2.1.10). At the same time, we should also impose the constraints $T_{--} = T_{++} = 0$ in the quantum theory. This also defines for us a subset of the states of the full Fock space that we can call physical. However, what one finds is that the two conditions for physical states actually are related, and that imposing the constraints also solves the problem with ghost states. Looking at the classical theory this relation should not be a surprise. Consider for instance the covariant action of the relativistic point particle: we have included too many field configurations for the particle, and the constraints are what ensures that end up with physically sensible field configurations.

In the classical theory for a closed string the constraints $T_{--} = T_{++} = 0$ can be written as $L_n = \tilde{L}_n = 0$ for $n \in \mathbb{Z}$ with L_n and $\tilde{L}_n$ given in (1.5.6). What is the quantum version of this? One finds that the constraints should be imposed in terms of their expectation values. In particular we require that for any two physical states $|\phi\rangle$ and $|\phi'\rangle$

$$\langle\phi'|L_n|\phi\rangle = \langle\phi'|\tilde{L}_n|\phi\rangle = 0 \quad \text{for} \quad n \neq 0 \tag{2.1.13}$$

This is easily seen to follow from requiring

$$L_n|\phi\rangle = \tilde{L}_n|\phi\rangle = 0 \quad \text{for} \quad n > 0 \tag{2.1.14}$$

for any physical state $|\phi\rangle$ and using that $L_{-n} = L_n^\dagger$. We shall show in Exercise 2.6 that it is not possible to require $L_n|\phi\rangle = 0$ on a state $|\phi\rangle$ for all $n \neq 0$ hence (2.1.14) is the strongest requirement that it is possible to make (for $n \neq 0$).

We should also make a quantum version of the classical constraints $L_0 = \tilde{L}_0 = 0$. However, here we run into what is known as a normal ordering ambiguity. Namely, for $n = 0$ α^μ_{n-k} and α^μ_k do not commute in the quantum version of the classical expression for L_n in Eq. (1.5.6). Indeed, we have $\alpha^\mu_1\alpha^\nu_{-1} = \eta^{\mu\nu} + \alpha^\nu_{-1}\alpha^\mu_1$. For definiteness, we define therefore the quantum versions of the classical expressions (1.5.6) as

$$L_n = \frac{1}{2}\sum_{k\in\mathbb{Z}} :\alpha_{n-k}\cdot\alpha_k: \,, \quad \tilde{L}_n = \frac{1}{2}\sum_{k\in\mathbb{Z}} :\tilde{\alpha}_{n-k}\cdot\tilde{\alpha}_k: \tag{2.1.15}$$

Here we define for any composite operator A not made entirely out of zero modes (*i.e.* x^μ and p^μ) that the normal ordered expression $:A:$ is obtained by moving all creation operators to the left of all annihilation operators, such that $\langle 0;k| :A: |0;k\rangle = 0$. Note that if A is composed of commuting operators one can always write $:A:=A$. For instance, we have $:\alpha_1\cdot\alpha_{-1}:=\alpha_{-1}\cdot\alpha_1$. For the quantum expression (2.1.15) the normal ordering is only non-trivial for $n = 0$ for which we compute

$$L_0 = \frac{1}{4}l_s^2p^2 + \sum_{k=1}^{\infty}\alpha_{-k}\cdot\alpha_k \,, \quad \tilde{L}_0 = \frac{1}{4}l_s^2p^2 + \sum_{k=1}^{\infty}\tilde{\alpha}_{-k}\cdot\tilde{\alpha}_k \tag{2.1.16}$$

However, if we take the classical expression (1.5.6) for L_0 literally as a quantum expression, this is not equal to the quantum definition (2.1.15). Indeed, at each k in the sum one gets a number, adding up to a so-called normal ordering constant. This normal ordering constant relies entirely on how one translates the classical expression into the quantum expression. A priori there is an infinite number of ways to do that, as we have an infinite sum in L_0, all leading to a different normal ordering constant. We parametrize this unknown normal ordering constant by stating that the quantum versions of the classical modes (1.5.6) are $L_n - a\delta_{n,0}$ and $\tilde{L}_n - \tilde{a}\delta_{n,0}$ here with the L_n and $\tilde{L}_n$ quantum operators defined as in (2.1.15), and with a and $\tilde{a}$ being constants. Since the right and left moving sectors are completely identical we have $a = \tilde{a}$ as whatever procedure one uses to obtain these constants would necessarily be identical for the two sectors. The quantum version of the classical constaint $L_0 = 0$ is thus $(L_0 - a)|\phi\rangle = 0$ for any physical state $|\phi\rangle$.

In conclusion, the quantum version of the classical constraint (1.3.7) is to require

$$(L_n - a\delta_{n,0})|\phi\rangle = (\tilde{L}_n - a\delta_{n,0})|\phi\rangle = 0 \quad \text{for} \quad n \geq 0 \tag{2.1.17}$$

for any physical state $|\phi\rangle$ where a so far is an undetermined constant. This defines the set of physical states in the covariantly quantized theory for closed strings.

As such, the constraints (2.1.17) on physical states do not rule out ghost states. However, one can show the following so-called *no ghost theorem*:

$$\begin{array}{l}\text{No physical states are ghost states provided} \\ \text{either } a = 1 \text{ and } D = 26, \text{ or } a \leq 1 \text{ and } D \leq 25.\end{array} \tag{2.1.18}$$

Thus, we can get rid of the ghost states among the physical states defined by (2.1.17) by choosing the above values of the normal ordering constant a and the space-time dimension D of the target space.

Below we shall see that there is stronger condition on a and D by demanding that the conformal symmetry of the string, as seen classically in Sec. 1.6, is without anomalies in the quantum string theory. As we learn in Sec. 2.5 this fixes that $a = 1$ and $D = 26$, which overlaps with the conditions of the no-ghost theorem (2.1.18).

Amazingly, when employing the lightcone quantization described in Appendix A one gets that the Lorentz symmetry of the target space is only without anomaly when $a = 1$ and $D = 26$ which thus reproduce the same critical space-time dimension and normal ordering constant as the covariant quantization procedure.

Assuming that $a = 1$ and $D = 26$ no physical states have negative norm. However, among the physical states we have a particular class of states that we need to address. A physical state (thus obeying (2.1.17)) is called a *spurious state* if it is orthogonal to all physical states, *i.e.* all states obeying (2.1.17). Thus, $|\chi\rangle$ is a spurious state if $|\chi\rangle$ obeys (2.1.17) and if $\langle\chi|\phi\rangle = 0$ for any state $|\phi\rangle$ obeying (2.1.17). In particular we see that $\langle\chi|\chi\rangle = 0$ thus spurious states have zero norm. We cannot make sense of a zero norm state as describing a quantum configuration of the string since a non-zero norm is needed in quantum mechanics for the probability interpretation. Instead, spurious states have a different interpretation. Consider any state $|\phi\rangle$ that is physical (2.1.17) and with non-zero norm $\langle\phi|\phi\rangle > 0$. This corresponds to a certain physical quantum configuration of the string. Add now an arbitrary spurious state $|\chi\rangle$ to get the state $|\phi'\rangle = |\phi\rangle + |\chi\rangle$. Then we claim that the two states $|\phi'\rangle$ and $\phi\rangle$ contains the same physics, *i.e.* that they correspond to the same physical quantum configuration of the string. This defines an equivalence relation for physical states. Spurious states are thus equivalent to 0 which means they do not contain any physical information. On the other hand, adding a spurious state to a given physical state $|\phi\rangle$ with non-zero norm is interpreted as a gauge transformation of $|\phi\rangle$.

An important check of the above is that the norm of a physical state is preserved when adding spurious states. To see this, consider the state $|\phi'\rangle = |\phi\rangle + |\chi\rangle$ with $|\phi\rangle$ physical and $|\chi\rangle$ spurious, then we get $\langle\phi'|\phi'\rangle = \langle\phi|\phi\rangle + \langle\chi|\phi\rangle + \langle\phi|\chi\rangle + \langle\chi|\chi\rangle = \langle\phi|\phi\rangle$.

2.2 Particle spectrum of the closed string

Before taking a closer look on the covariant quantization of closed string theory we first consider one the main physical properties namely the spectrum at low energies. Since we shall see later that we need $a = 1$ and $D = 26$ for the consistency of the quantum theory we assume these values in the following.

2.2.1 Particle spectrum

Our interpretation of a given physical state $|\phi\rangle$ in the closed string Fock space obeying (2.1.17) is that it is a superposition of particle states. Indeed, looking at the constraints (2.1.17) for $n = 0$ it is not hard to see that different physical states in the spectrum of a closed string leads to different values of p^2. Hence the space of physical states of the closed string leads to a spectrum of particles with masses M given by the mass-shell condition $M^2 = -p^2$.

In detail, using (2.1.16) we can write the constraints (2.1.17) for $n = 0$ as

$$M^2|\phi\rangle = \frac{4}{l_s^2}(N-1)|\phi\rangle = \frac{4}{l_s^2}(\tilde{N}-1)|\phi\rangle \tag{2.2.1}$$

on a physical state $|\phi\rangle$ where we inserted the value $a = 1$ and defined the operators

$$N = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n \ , \quad \tilde{N} = \sum_{n=1}^{\infty} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n \tag{2.2.2}$$

Acting with N on the Fock space basis state (2.1.10) one gets the state multiplied with the number $\sum_{i=1}^{p} m_i$. Similarly, the action of $\tilde{N}$ gives the state times $\sum_{j=1}^{q} n_j$. We can infer from this that the eigenvalues of N and $\tilde{N}$ are positive integers, or zero. We see from (2.2.1) that

$$(N - \tilde{N})|\phi\rangle = 0 \tag{2.2.3}$$

This is called the *level-matching* constraint and is an extra constraint that one needs to impose on the closed string Fock space (2.1.10). This provides a non-trivial connection between the right-moving and left-moving sectors. In particular for the basis state (2.1.10) it would impose $\sum_{i=1}^{p} m_i = \sum_{j=1}^{q} n_j$. This number is known as the *level* of the state.

2.2.2 Tachyon

We can now consider the particle spectrum of the closed string. We start with level zero $N = \tilde{N} = 0$. This singles out the ground state

$$|0;k\rangle \tag{2.2.4}$$

of the closed string, as defined by (2.1.8). Imposing the physical state conditions (2.1.17) we see using (2.2.1) that they require

$$M^2 = -k^2 = -\frac{4}{l_s^2} \tag{2.2.5}$$

This is the only condition for $|0;k\rangle$ being a physical state since it is easy to see using (2.1.15) that $L_n|0;k\rangle = \tilde{L}_n|0;k\rangle = 0$ for $n > 0$. A particle with M^2 negative is known as a tachyon, and it has the special property that it moves faster than light since p^2 is positive. The closed string state $|0;k\rangle$ is thus known as the *closed string tachyon*. It is a scalar particle since the state $|0;k\rangle$ only depends on k. Since we have a k dependence in the state $|0;k\rangle$ we can more generally think of the tachyon as a momentum dependent field $T(k)$. That the closed string has a tachyon state is considered a problem for the theory. However, it does not necessarily mean that the closed string predicts scalar particles moving faster than light. A more sensible interpretation is that the closed string ground state is unstable and thus prone to decay to a new ground state. This could be the case if the negative mass squared came from a potential for the tachyon for which the tachyon is sitting on the top of a hill, see Fig. 7. This view point has been shown to be correct in case of the open string. For string theory as a whole, however, it is not considered a major issue as one anyway considers the *bosonic closed string*, which is what the string we describe in this chapter in called, only as a warming up exercise to the superstring. As we shall see in Chapter 6 the tachyon is absent both for the closed and the open superstring.

2.2.3 Massless spectrum

The next step is to consider level one: $N = \tilde{N} = 1$. A basis for states of level one obeying the level-matching condition is provided by the states $\alpha_{-1}^{\mu}\tilde{\alpha}_{-1}^{\nu}|0;k\rangle$. Thus, we can write

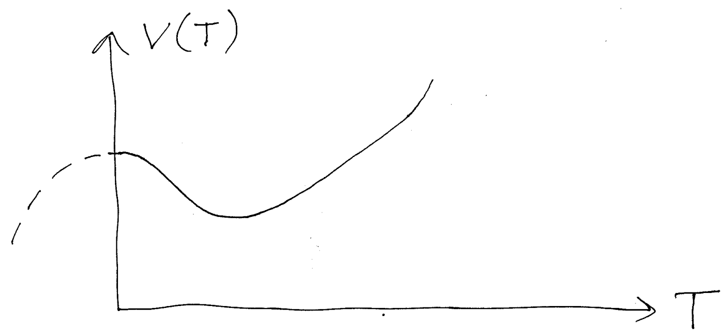

Figure 7: Illustration of the closed string tachyon potential.

a general linear superposition as

$$\zeta_{\mu\nu}(k)\, \alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|0;k\rangle \tag{2.2.6}$$

with coefficients $\zeta_{\mu\nu}(k)$ where we added the index k referring to the specific momentum of the ground state. More generally, we can think of $\zeta_{\mu\nu}(k)$ as a field in momentum space. Imposing the constraints (2.1.17) we find the following conditions

$$M^2 = -k^2 = 0 \ , \quad k^{\mu}\zeta_{\mu\nu}(k) = 0 \ , \quad k^{\nu}\zeta_{\mu\nu}(k) = 0 \tag{2.2.7}$$

The first constraint comes from $n = 0$ in (2.1.17) which is equivalent to (2.2.1). This means that the states correspond to massless particles. The two other constraints comes from $n = 1$ in (2.1.17). Physically they mean that the particles do not have longitudinal modes, in accordance with what one would expect for massless particles.

One can show that a state of the form $k_{\mu}m_{\nu}\, \alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|0;k\rangle$ is spurious assuming $k^2 = 0$ and that m_{μ} obeys $k^{\mu}m_{\mu} = 0$. Similarly, also $m_{\mu}k_{\nu}\, \alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|0;k\rangle$ is spurious if $k^{\mu}m_{\mu} = 0$ and $k^2 = 0$. This has the consequence that the physics of the state (2.2.6) is invariant under the transformations $\zeta_{\mu\nu}(k) \to \zeta_{\mu\nu}(k) + k_{\mu}m_{\nu}$ and $\zeta_{\mu\nu}(k) \to \zeta_{\mu\nu}(k) + m_{\mu}k_{\nu}$ for any m_{μ} with $k^{\mu}m_{\mu} = 0$. Thus, the physics of the state (2.2.6) is invariant under the transformations

$$\zeta_{\mu\nu}(k) \to \zeta_{\mu\nu}(k) + k_{\mu}m_{\nu} + \tilde{m}_{\mu}k_{\nu} \ , \quad k \cdot m = k \cdot \tilde{m} = 0 \tag{2.2.8}$$

These transformations correspond to gauge transformations, as we shall see.

To see what particles $\zeta_{\mu\nu}(k)$ corresponds to we have to break it up in to irreducible representations of the Lorentz group $SO(1,25)$ (more precisely into representations of the little Lorentz group $SO(24)$ since we are considering massless particles). We use that for a given k_{μ} one can find $\bar{k}_{\mu}$ such that

$$\bar{k}^2 = 0 \ , \quad k \cdot \bar{k} = 1 \tag{2.2.9}$$

The three irreducible parts are the symmetric and traceless part $G_{\mu\nu}(k)$, the antisymmetric part $B_{\mu\nu}(k)$, and the trace part $\Phi(k)$, given by

$$\begin{aligned} G_{\mu\nu}(k) &= \frac{1}{2}(\zeta_{\mu\nu}(k) + \zeta_{\nu\mu}(k)) - \frac{1}{24}(\eta_{\mu\nu} - k_\mu \bar{k}_\nu - \bar{k}_\mu k_\nu)\eta^{\rho\sigma}\zeta_{\rho\sigma}(k) \\ B_{\mu\nu}(k) &= \frac{1}{2}(\zeta_{\mu\nu}(k) - \zeta_{\nu\mu}(k)) \ , \quad \Phi(k) = \frac{1}{24}\eta^{\rho\sigma}\zeta_{\rho\sigma}(k) \end{aligned} \tag{2.2.10}$$

such that

$$\zeta_{\mu\nu}(k) = G_{\mu\nu}(k) + B_{\mu\nu}(k) + (\eta_{\mu\nu} - k_\mu \bar{k}_\nu - \bar{k}_\mu k_\nu)\Phi(k) \tag{2.2.11}$$

The field $G_{\mu\nu}(k)$ corresponds in four dimensions to a spin 2 representation of the $SO(1,3)$ Lorentz group. This is interpreted as a graviton field. Apart that it transforms in the right representation of the Lorentz group one can furthermore show that the low energy equations of motion for $G(k)$ on a general background corresponds to the Einstein equations, possibly also with matter fields. We find below the linearized versions of the Einstein equations as well as the diffeomorphism transformations with 26-dimensional Minkowski space as background. We shall discuss general space-times and the derivation of the full non-linear Einstein equations in Chapter 3.

This is one of the major successes of string theory: string theory provides a quantum description of the graviton, the quantum of the gravitational field. This quantum description is consistent (there are no ill-defined states in the theory *etc.*) and we can get out the theory of general relativity in a low energy limit. So far, no other proposal for a quantum theory of gravity has been able to match this success.

To see this explicitly, note that one gets the following position space equations for $G_{\mu\nu}(x)$ from (2.2.7) and (2.2.10)

$$\partial_\mu \partial^\mu G_{\nu\rho} = 0 \ , \quad \partial^\mu G_{\mu\nu} = 0 \ , \quad \eta^{\mu\nu} G_{\mu\nu} = 0 \tag{2.2.12}$$

The second and third equations are gauge conditions on $G_{\mu\nu}(x)$. In this gauge, the first equation corresponds to the linearized vacuum Einstein equations (linearized around 26-dimensional Minkowski space). These equations are known to describe gravitational waves propagating in vacuum, which fits with the fact that one gets a classical gravitational wave from superposing a large number of gravitons.

The transformations (2.2.8) of $\zeta_{\mu\nu}(k)$ become the gauge transformations of $G_{\mu\nu}(k)$

$$G_{\mu\nu}(k) \to G_{\mu\nu}(k) + k_\mu m_\nu + m_\mu k_\nu \ , \quad k \cdot m = 0 \tag{2.2.13}$$

Seen in position space these transformations of $G_{\mu\nu}$ are[14]

$$G_{\mu\nu} \to G_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \ , \quad \partial^\mu \xi_\mu = 0 \ , \quad \partial^\mu \partial_\mu \xi_\nu = 0 \tag{2.2.14}$$

[14]One can go from momentum space to position space by using a Fourier transform. We note that in

which correspond to linearized diffeomorphism transformations that are consistent with the gauge choice given by the second and third equations of (2.2.12).

The antisymmetric field $B_{\mu\nu}$ is known as the *Kalb-Ramond* field. We see from (2.2.8) that it is invariant under the transformations

$$B_{\mu\nu}(k) \to B_{\mu\nu}(k) + k_\mu m_\nu - m_\mu k_\nu \ , \quad k \cdot m = 0 \tag{2.2.15}$$

In position space these become $B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu A_\nu - \partial_\nu A_\mu$ (along with $\partial^\mu A_\mu = 0$ and $\partial^\mu \partial_\mu A_\nu = 0$) which are known as the gauge transformations for a two-form potential with a corresponding gauge invariant three-form field strength $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$. Moreover, from (2.2.7) we get $\partial_\mu \partial^\mu B_{\nu\rho} = 0$ and $\partial^\mu B_{\mu\nu} = 0$ that we can combine to show that $\partial^\mu H_{\mu\nu\rho} = 0$ which is the equation of motion for a source-free three-form field strength. The equation $\partial^\mu B_{\mu\nu} = 0$ can be interpreted as a gauge choice consistent with $\partial^\mu A_\mu = 0$. We shall discuss the interpretation of these findings further in Chapter 3 and Sec. 4.2.3.

Finally we have the field $\Phi(k)$ which is known as the *dilaton* field. We see from its definition in (2.2.10) that $\Phi(k)$ is invariant under the transformations (2.2.8) of $\zeta_{\mu\nu}(k)$. In correspondence with this, $\Phi(k)$ is interpreted to be a scalar field in the 26-dimensional space-time. The conditions (2.2.7) gives that $k^2 = 0$ which in position space becomes

$$\partial^\mu \partial_\mu \Phi = 0 \tag{2.2.16}$$

which is the Klein-Gordon equation for a massless scalar field in the background of 26-dimensional Minkowski space. The interpretation of the dilaton field will be discussed in Chapter 3.

One can continue with considering states with level $N = \tilde{N} \geq 2$. These states correspond to a discrete spectrum of massive particles with $M^2 = 4(N-1)l_s^{-2}$. One can show that for high level number the number of available states at a given level increases exponentially with the level number. This has the interesting consequence that the partition function for a gas of non-interacting strings is not defined above a certain temperature called the Hagedorn temperature.

2.3 Poincaré invariance of the target space

It is a general fact that when quantizing a theory which classically has a conserved charge corresponding to a symmetry, the conserved charge is mapped to a generator of that

going to position space we need to use the more general normalization $\langle 0; k | 0; k' \rangle = c\, \delta^D(k - k')$ for the ground state.

symmetry. For instance, in a classical theory with translation invariance the momentum is conserved. Quantizing the theory, the momentum operator now corresponds to the generator of translations. In this section we shall see how this works for the Poincaré symmetry of the closed string, as described by the Polyakov action in flat gauge (1.3.18). In Sec. 2.4 we turn to the conformal symmetry.

2.3.1 Poincaré symmetry and algebra

Consider D-dimensional Minkowski space with coordinates x^μ and metric $\eta_{\mu\nu}$. This is invariant under Poincaré transformations, consisting of translations and Lorentz transformations. Consider a quantum state $|\phi\rangle$ on D-dimensional Minkowski space with corresponding wave-function $\phi(x) = \langle x|\phi\rangle$ defined on the Minkowski space. The wave function transforms as a scalar $\tilde{\phi}(\tilde{x}) = \phi(x)$ under Poincaré transformations. We map here a general Poincaré transformation into a corresponding operator that acting on the quantum state yields the same result. Making a translation $x^\mu \to x^\mu - b^\mu$ corresponds to acting with the operator

$$\exp(i\, b^\mu p_\mu) \tag{2.3.1}$$

where the translation generators is

$$p_\mu = -i\partial_\mu \tag{2.3.2}$$

To see that this is the case, consider an infinitesimal translation. Since the wave function should transform as $\tilde{\phi}(\tilde{x}) = \phi(x)$ we see that $\phi(x) = \tilde{\phi}(\tilde{x}) = \tilde{\phi}(x - b)$, and hence $\tilde{\phi}(x) = \phi(x + b) = \phi(x) + b^\mu \partial_\mu \phi$. On the other hand, acting with the operator (2.3.1) gives $\tilde{\phi}(x) = e^{ib^\mu p_\mu}\phi(x) = (1 + ib^\mu p_\mu)\phi(x) = \phi(x) + b^\mu \partial_\mu \phi(x)$. In the same way, one can check that making a Lorentz transformation corresponds to acting with the operator

$$\exp\left(-\frac{i}{2}\omega^{\mu\nu} J_{\mu\nu}\right) \tag{2.3.3}$$

with the Lorentz generators given by

$$J_{\mu\nu} = -i(x_\mu \partial_\nu - x_\nu \partial_\mu) \tag{2.3.4}$$

Having now the translation and Lorentz generators (2.3.2) and (2.3.4) we can find their algebra. For instance, to compute $[p_\rho, J_{\mu\nu}]$ we do the following

$$\begin{aligned}
[p_\rho, J_{\mu\nu}]\phi(x) &= (p_\rho J_{\mu\nu} - J_{\mu\nu} p_\rho)\phi(x) = -\partial_\rho(x_\mu \partial_\nu \phi) + \partial_\rho(x_\nu \partial_\mu \phi) + x_\mu \partial_\nu \partial_\rho \phi - x_\nu \partial_\mu \partial_\rho \phi \\
&= -\eta_{\rho\mu}\partial_\nu \phi + \eta_{\rho\nu}\partial_\mu \phi = -i(\eta_{\rho\mu} p_\nu - \eta_{\rho\nu} p_\mu)\phi(x)
\end{aligned} \tag{2.3.5}$$

revealing that $[p_\rho, J_{\mu\nu}] = -i(\eta_{\rho\mu} p_\nu - \eta_{\rho\nu} p_\mu)$. In this way we find the *Poincaré symmetry algebra* for the Poincaré generators

$$\begin{aligned} [p_\mu, p_\nu] = 0 \ , \quad [p_\rho, J_{\mu\nu}] = -i(\eta_{\rho\mu} p_\nu - \eta_{\rho\nu} p_\mu) \\ [J_{\mu\nu}, J_{\rho\sigma}] = -i(\eta_{\nu\rho} J_{\mu\sigma} + \eta_{\mu\sigma} J_{\nu\rho} - \eta_{\mu\rho} J_{\nu\sigma} - \eta_{\nu\sigma} J_{\mu\rho}) \end{aligned} \tag{2.3.6}$$

2.3.2 Poincaré generators of the string

Classically, the Polyakov action in flat gauge is invariant under Poincaré transformations (1.3.8) of the target space, see Sec. 1.3.2, consisting of translations and Lorentz transformation. For the closed string the translations give rise to the conserved charge p_μ of Eq. (1.4.31) which is the total momentum. The Lorentz transformations give the conserved charge $J_{\mu\nu}$ of Eq. (1.4.34). We now consider the quantized theory to see if the corresponding operators fulfil the Poincaré symmetry algebra (2.3.6).

Quantizing the theory we have already established in Sec. 2.1 that the total momentum p_μ as an operator. Moreover, the $[x^\mu, p_\nu] = i\delta^\mu_\nu$ commutator is in accordance with (2.3.2). Instead for $J_{\mu\nu}$ we should quantize the expression (1.4.34). Since it contains a product of fields there is a possible normal ordering constant. To this end, consider the ground state $|0;0\rangle$ corresponding to zero momentum $p_\mu|0;0\rangle = 0$. From a physical point of view this should be a Lorentz invariant state. Hence this means that $J_{\mu\nu}|0;0\rangle = 0$. Thus, we can conclude from this that the quantum operator $J_{\mu\nu}$ is normal ordered

$$J_{\mu\nu} = \int_0^{2\pi} d\sigma \ :(X_\mu \Pi_\nu - X_\nu \Pi_\mu): \tag{2.3.7}$$

We compute

$$J^{\mu\nu} = l^{\mu\nu} + E^{\mu\nu} + \tilde{E}^{\mu\nu} \tag{2.3.8}$$

where

$$\begin{aligned} l^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu \\ E^{\mu\nu} = -i \sum_{k=1}^{\infty} \frac{1}{k} (\alpha^\mu_{-k} \alpha^\nu_k - \alpha^\nu_{-k} \alpha^\mu_k) \ , \quad \tilde{E}^{\mu\nu} = -i \sum_{k=1}^{\infty} \frac{1}{k} (\tilde{\alpha}^\mu_{-k} \tilde{\alpha}^\nu_k - \tilde{\alpha}^\nu_{-k} \tilde{\alpha}^\mu_k) \end{aligned} \tag{2.3.9}$$

It is now a straightforward exercise, using the commutators (2.1.3), to find that p_μ and $J_{\mu\nu}$ for the closed string indeed obey the Poincaré symmetry algebra (2.3.6).

2.4 Conformal invariance on the worldsheet

In Sec. 1.6 we found that the Polyakov action in flat gauge (1.3.18) is invariant under two-dimensional conformal transformations (1.6.9). The conformal invariance on the worldsheet of the closed string was seen in Sec. 1.6.2 to give rise to the conserved charges (1.5.6) which are the modes of the worldsheet energy-momentum tensor. In this section we consider the conformal symmetry of the quantum string and the generators of the conformal symmetry corresponding to the conserved charges (1.5.6) of the classical string.

2.4.1 Two-dimensional conformal symmetry and algebra

Consider a two-dimensional space-time with coordinates ξ^α and Minkowski metric $\eta_{\alpha\beta}$. We consider a quantum state $|\phi\rangle$ on the worldsheet with wave-function $\phi(\xi) = \langle\xi|\phi\rangle$. We established in Sec. 1.6.1 that conformal transformations are of the form (1.6.9). In particular, infinitesimal conformation transformations $\xi^\alpha \to \tilde{\xi}^\alpha = \xi^\alpha - a^\alpha(\xi)$ have $\partial_+ a^- = \partial_- a^+ = 0$. In our application to the string theory worldsheet all fields are periodic in both ξ^- and ξ^- with period 2π hence we impose this on the conformal transformations as well. For an infinitesimal conformal transformation given by $a^\pm(\xi^\pm)$ this means we can make the Fourier expansion

$$a^\pm(\xi^\pm) = \sum_{n\in\mathbb{Z}} A_n^\pm e^{in\xi^\pm} \tag{2.4.1}$$

where $A_n^\pm$ are constant. Turning on a single A_n^- corresponds to the conformal transformation $\xi^- \to \xi^- - A_n^- e^{in\xi^-}$. Since a wave-function $\phi(\xi)$ transforms as $\tilde{\phi}(\tilde{\xi}^-) = \phi(\xi^-)$ we find $\tilde{\phi}(\xi^-) \simeq \phi(\xi^- + A_n^- e^{in\xi^-}) \simeq \phi(\xi^-) + A_n^- e^{in\xi^-}\partial_-\phi$ corresponding to acting with the operator $e^{-iA_n^- D_n^-}$ on $\phi(\xi^-)$ where $D_n^- = ie^{in\xi^-}\partial_-$. Thus, conformal symmetry in two dimensions has the generators

$$D_n^- = ie^{in\xi^-}\partial_- \ , \quad D_n^+ = ie^{in\xi^+}\partial_+ \tag{2.4.2}$$

Thus, we have two infinite sets of generators for the conformal symmetry. Acting with $[D_m^-, D_n^-]$ on $\phi(\xi^-)$ and similarly for the left moving sector, we find the following conformal symmetry algebra

$$[D_m^-, D_n^-] = (m-n)D_{m+n}^- \ , \quad [D_m^+, D_n^+] = (m-n)D_{m+n}^+ \ , \quad [D_m^-, D_n^+] = 0 \tag{2.4.3}$$

for $m, n \in \mathbb{Z}$. This algebra consists of two copies of the so-called *Virasoro algebra*. Thus, both the right-moving and left-moving sectors obey the Virasoro algebra separately, and the sectors commute with each other. The generators $D_n^\pm$ are correspondingly called *Virasoro generators*.

2.4.2 Conformal symmetry generators of the string

We found in Sec. 1.6.2 for the classical description of the closed string that the modes L_n and $\tilde{L}_n$ of the worldsheet energy-momentum tensor are conserved charges due to the conformal symmetry on the worldsheet. As exemplified in Sec. 2.3, the conserved charges due to a symmetry in a classical theory become the generators of the symmetry in the quantum theory. Hence, the quantum operators L_n and $\tilde{L}_n$ defined in (2.1.15) should be generators of the conformal symmetry on the worldsheet of the quantized closed string. Specifically, L_n should be correspond to D_n^- and $\tilde{L}_n$ with D_n^+ of the two-dimensional conformal symmetry algebra (2.4.3) of Sec. 2.4.1. Using now the commutator relations (2.1.3) one can compute the following algebra for L_n and $\tilde{L}_n$

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{D}{12}(m^3-m)\delta_{m+n,0} \\
[\tilde{L}_m, \tilde{L}_n] &= (m-n)\tilde{L}_{m+n} + \frac{D}{12}(m^3-m)\delta_{m+n,0} \\
[L_m, \tilde{L}_n] &= 0
\end{aligned}
\tag{2.4.4}
$$

for $m, n \in \mathbb{Z}$. This algebra consists of two copies of the same algebra and is a particular case of an algebra known as the *centrally extended Virasoro algebra* (see also below in Sec. 2.4.3). The extra part $\frac{D}{12}(m^3-m)\delta_{m+n,0}$ which is absent in the Virasoro algebra is called the *central extension*.

Comparing the algebras (2.4.3) and (2.4.4) we see that (D_m^-, D_n^+) and $(L_m, \tilde{L}_n)$ do not obey the same algebra. The difference is the central extension in (2.4.4). Thus, there is an anomaly in the algebra (2.4.4). This suggests that the conformal symmetry of the quantized closed string is anomalous. Why is this a problem? The answer is that in the classical description the local symmetries of the Polyakov action is what enables one to claim that one gets the same physics as if one used the Nambu-Goto action. Hence it is crucial that we keep the local symmetries that we found in the classical description. We shall see the resolution to this below in Sec. 2.5 but first we shall examine the properties of two-dimensional conformal symmetry a bit further.

2.4.3 Properties of two-dimensional CFTs

Consider a two-dimensional classical field theory. The energy-momentum tensor is defined via the variation of the action

$$
\delta S = -\frac{1}{4\pi}\int d^2\xi\, \sqrt{-g}T_{\alpha\beta}\delta g^{\alpha\beta}
\tag{2.4.5}
$$

For a classical conformally invariant field theory the action should be invariant under conformal transformations $\delta g_{\alpha\beta} = 2\Phi\, g_{\alpha\beta}$. From the above we see that this is only possible if the trace of the energy-momentum tensor is zero

$$T_\alpha{}^\alpha = 0 \tag{2.4.6}$$

Conversely, if (2.4.6) holds for a classical field theory, then it is conformally invariant.

A two-dimensional *conformal field theory* (CFT) is a two-dimensional quantum field theory which is conformally invariant in its classical limit, or, equivalently, obeys (2.4.6). Clearly, the two-dimensional quantum field theory for the closed string that is the main subject of this chapter is an example of a two-dimensional CFT. In particular, we found (1.5.2).

Consider a given two-dimensional CFT. Since any two-dimensional metric is locally conformally flat we can go to coordinates such that the metric is of the form $e^{2\Phi(\xi)}\eta_{\alpha\beta}$. One can then use a conformal transformation in the classical limit to remove the factor $e^{2\Phi(\xi)}$ while at the same time imposing that the theory is periodic along the ξ^1 direction with period 2π. With this, a general conformal transformation is now written as (1.6.9) where $\xi^\pm = \xi^0 \pm \xi^1$ and $\xi^\pm$ are periodic coordinates with period 2π. Moreover, (2.4.6) is equivalent to $T_{+-} = 0$. This means we can define the Fourier modes $\mathcal{L}_n$ and $\tilde{\mathcal{L}}_n$ in terms of the mode expansions of the T_{--} and T_{++} components of the energy-momentum tensor of the CFT (in the classical limit) just like we did in (1.5.5). Quantizing the theory, the algebra of the generators of the right-moving sectors is of the general form

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} + \left(\frac{c}{12}m^3 + km\right)\delta_{m+n,0} \tag{2.4.7}$$

and similarly for the left-moving sector (possibly with different c and k). The right and left-moving sectors commute. The algebra (2.4.7) is called the *centrally extended Virasoro algebra*, now with its central extension $\left(\frac{c}{12}m^3 + km\right)\delta_{m+n,0}$ in the most general form. The generators are called *Virasoro generators*. As explained above in Sec. 2.4.2 a non-zero central extension means that the conformal symmetry algebra is anomalous.

The constant k in (2.4.7) depends on the normal ordering scheme when quantizing the theory. In particular, replacing $\mathcal{L}_n$ with $\mathcal{L}_n + b\delta_{n,0}$ in the algebra (2.4.7) gives an algebra of the same form as (2.4.7), with the same c, but with k replaced by $k - 2b$. Thus, one can in particular make a redefinition of the Virasoro generators that sets $k = -c/12$. This is a natural choice since then the algebra of $\mathcal{L}_{-1}$, $\mathcal{L}_0$ and $\mathcal{L}_1$ is without anomalies. This is a subalgebra of the Virasoro algebra with commutators $[\mathcal{L}_1, \mathcal{L}_{-1}] = 2\mathcal{L}_0$ and $[\mathcal{L}_0, \mathcal{L}_{\pm 1}] = \mp\mathcal{L}_{\pm 1}$. The subalgebra of $\mathcal{L}_0$, $\mathcal{L}_{\pm 1}$, $\tilde{\mathcal{L}}_0$ and $\tilde{\mathcal{L}}_{\pm 1}$ corresponds to the subalgebra of the conformal symmetry that one also finds in more than two dimensions, consisting of translations, rotations and boosts, dilatations and special conformal transformations.

The constant c in (2.4.7) is known as the *central charge* of the two-dimensional CFT. This constant is not sensitive to normal ordering and is considered to be a measure of the number of degrees of freedom (it is connected to the entropy of the CFT via the so-called Cardy formula and is also an important quantity in the so-called c-theorem for renormalization group flows in two dimensions).

Considering the anomalous conformal algebra of the closed string (2.4.4) it corresponds to two copies of the extended Virasoro algebra (2.4.7) with central charge $c = D$. Thus, the central charge is equal to the number of space-time dimensions of the target space which fits well with the interpretation of c as a measure of the number of degrees of freedom.

Classically, conformal invariance is equivalent to the trace of the energy-momentum tensor being zero (2.4.6). Generically, a CFT has the anomalous conformal symmetry algebra (2.4.7) (along with the equivalent algebra for the left-moving sector). Thus, for non-zero central charge c one would expect that the expectation value of the trace of the energy-momentum tensor should be non-zero, in accordance with the anomalous algebra. Indeed, one can derive that the expectation value of the trace of the energy-momentum tensor is

$$\langle \phi | T_\alpha{}^\alpha | \phi \rangle = -\frac{c}{12} R \tag{2.4.8}$$

where c is the central charge of the CFT, as in (2.4.7), and R is the Ricci scalar of the two-dimensional metric. When this expectation value is non-zero it is known as the conformal anomaly.

Note that an alternative definition of a two-dimensional CFT is a two-dimensional quantum field theory for which the expectation value of the trace of the energy-momentum tensor vanishes in flat space. We see that this is in accordance with the above since the conformal anomaly (2.4.8) is zero in flat space.

2.5 Ghost field sector from string path integral

Consider the general Polyakov action $S_{\text{pol}}[g, X]$ of Eq. (1.3.1). This action depends on the metric field g and the string embedding map X. To properly quantize this action we should understand the string theory path integral

$$Z = \int \mathcal{D}g \, \mathcal{D}X \exp(i S_{\text{pol}}[g, X]) \tag{2.5.1}$$

Notice that this path integral includes integration of all two-dimensional metrics $g_{\alpha\beta}$ (given certain global boundary conditions such as topology). So given this, how can we make sense of fixing the gauge to a specific metric? The way to do this is called the Fadeev-Popov method. We briefly outline this method and the resulting ghost field sector in the following. A more detailed version can be found in Appendix B.

We want to gauge-fix the world-sheet metric to the flat gauge (1.3.17) in the path integral (2.5.1). Any metric can be connected the flat metric via a combination of a diffeomorphism $\tilde{\xi}^\alpha(\xi) = \xi^\alpha - a^\alpha(\xi)$ and a Weyl rescaling (1.3.12) with here written as $\Omega(\xi) = e^{\Lambda(\xi)}$. Hence instead of integrating over all metrics $g_{\alpha\beta}(\xi)$ in the path integral (2.5.1) we can integrate over all diffeomorphisms $a^\alpha(\xi)$ and Weyl rescalings $\Lambda(\xi)$. One can think of this as a change of integration variables. However, when one changes integration variable in an integral $\int dy F(y)$ to a new variable $x(y)$ one gets an extra factor dy/dx in the integral $\int dx \frac{dy}{dx} F(y(x))$. In the same way this happens in the path integral, giving an extra factor in the integration measure

$$\mathcal{D}g = \Delta_{\mathrm{FP}}\, \mathcal{D}\Lambda\, \mathcal{D}a \tag{2.5.2}$$

This extra factor Δ_{FP} is called the *Fadeev-Popov determinant*. One can show that it does not depend on Λ and a. Using this together with the fact that the Polyakov action (1.3.1) is invariant under diffeomorphisms and Weyl rescalings, and that the measure $\mathcal{D}X$ is invariant under diffeomorphism, one can now write the path integral as

$$Z = \int \mathcal{D}\Lambda\, \mathcal{D}a\, \mathcal{D}X\, \Delta_{\mathrm{FP}} \exp(iS_{\mathrm{pol}}[\eta, X]) \tag{2.5.3}$$

One sees now that nothing in the integrand depends on $\Lambda(\xi)$ or $a^\alpha(\xi)$. This means that the factor $\int \mathcal{D}\Lambda\, \mathcal{D}a$ decouples from the theory and hence one can remove it from Z without changing the physics. We get now

$$Z = \int \mathcal{D}X\, \Delta_{\mathrm{FP}} \exp(iS_{\mathrm{pol}}[\eta, X]) \tag{2.5.4}$$

This is our gauge fixed path integral, since we now got rid of the integration over the metrics, we fixed the metric to the gauge choice (1.3.17) and we are left only with an integral over the embedding fields $X^\mu(\xi)$. However, we notice also that the correct gauge fixing procedure in the quantum theory includes the Fadeev-Popov determinant factor Δ_{FP}. This factor can be computed using so-called Grassmannian-valued fields that are fields that anti-commute. We shall encounter such fields again in Section 6 when considering the superstring. One finds

$$\Delta_{\mathrm{FP}} = \int \mathcal{D}b\, \mathcal{D}c\, \exp(iS_{\mathrm{gh}}[b, c]) \tag{2.5.5}$$

where

$$S_{\rm gh}[b,c] = \frac{i}{\pi}\int d^2\xi \Big(b_{--}\partial_+ c^- + b_{++}\partial_- c^+ \Big) \tag{2.5.6}$$

The fields b_{--}, b_{++}, c^- and c^+ are all Grassmann-valued fields. They are called *ghost fields* since they transform as bosonic fields under worldsheet diffeomorphisms but are Grassmann-valued like one would have for fermionic fields. Thus, they cannot correspond to physical degrees of freedom. Nevertheless they are important in the quantum theory, as we shall review in the following. Writing the Fadeev-Popov determinant as (2.5.5)-(2.5.6) the gauge-fixed path integral is written as

$$Z = \int \mathcal{D}b\,\mathcal{D}c\,\mathcal{D}X\ \exp\Big(iS_{\rm pol}[\eta,X] + iS_{\rm gh}[b,c] \Big) \tag{2.5.7}$$

One can show that b-c ghost system defined by the action (2.5.6) is a two-dimensional CFT. Indeed, in the quantized theory one can define modes $L_n^{(\rm gh)}$ and $\tilde{L}_n^{(\rm gh)}$ of the energy-momentum tensor corresponding to the action (2.5.6). The algebra of these modes can be computed as

$$\begin{gathered}
[L_m^{(\rm gh)}, L_n^{(\rm gh)}] = (m-n)L_{m+n}^{(\rm gh)} + \frac{1}{12}(-26m^3+2m)\delta_{m+n,0} \\
[\tilde{L}_m^{(\rm gh)}, \tilde{L}_n^{(\rm gh)}] = (m-n)\tilde{L}_{m+n}^{(\rm gh)} + \frac{1}{12}(-26m^3+2m)\delta_{m+n,0} \\
[L_m^{(\rm gh)}, \tilde{L}_n^{(\rm gh)}] = 0
\end{gathered} \tag{2.5.8}$$

This corresponds to two copies of an extended Virasoro algebra, as defined in Sec. 2.4.3. Comparing with (2.4.7) we conclude that the b-c ghost field sector corresponds to a two-dimensional CFT with central charge $c_{bc} = -26$.

Going back to the path integral (2.5.7) we see that in the complete quantum theory of the closed string we should add the contributions from the Polyakov action and the ghost field action to get the complete energy-momentum tensor. Hence the Virasoro generators in the complete quantum theory are

$$\mathcal{L}_n = L_n + L_n^{(\rm gh)} - a\delta_{n,0}\ ,\quad \tilde{\mathcal{L}}_n = \tilde{L}_n + \tilde{L}_n^{(\rm gh)} - a\delta_{n,0} \tag{2.5.9}$$

for $n\in\mathbb{Z}$ where L_n and $\tilde{L}_n$ are the Virasoro generators (2.1.15) for the contribution from the X^μ field. Finally, the term $-a\delta_{n,0}$ allows for a possible normal ordering constant. One computes

$$\begin{aligned}
&[\mathcal{L}_m, \mathcal{L}_n] = [L_m, L_n] + [L_m^{(\rm gh)}, L_n^{(\rm gh)}] \\
&= (m-n)(L_{m+n} + L_{m+n}^{(\rm gh)}) + \left(\frac{D}{12}(m^3-m) + \frac{1}{12}(-26m^3+2m) \right)\delta_{m+n,0}
\end{aligned} \tag{2.5.10}$$

Hence one finds

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} + \frac{1}{12}\Big((D-26)m^3 + (24a+2-D)m\Big)\delta_{m+n,0} \tag{2.5.11}$$

and the equivalent result for the left-moving sector Virasoro generators $\tilde{\mathcal{L}}_n$. Thus, we conclude that for

$$D = 26 \quad \text{and} \quad a = 1 \tag{2.5.12}$$

we get the algebra

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} \ , \quad [\tilde{\mathcal{L}}_m, \tilde{\mathcal{L}}_n] = (m-n)\tilde{\mathcal{L}}_{m+n} \ , \quad [\mathcal{L}_m, \tilde{\mathcal{L}}_n] = 0 \tag{2.5.13}$$

which corresponds to the anomaly-free conformal symmetry algebra (2.4.3). Hence for the critical values (2.5.12) of the dimension D of the target space, and the normal ordering constant a, the covariant quantization of the closed string is consistent when including the ghost field sector.

For the central charge we see that it is additive when combining the X^μ field and the b-c ghost fields

$$c = c_{\rm X} + c_{\rm bc} = D - 26 \tag{2.5.14}$$

which also means that the conformal anomaly (2.4.8) disappears for $D = 26$, as expected.

In Appendix B.5 we review the last step in the covariant quantization with the ghost fields where one looks at the full spectrum of states that come both from the matter sector and the ghost sector. Introducing the so-called BRST charge Q, one can show that demanding that this charge is zero on physical states precisely corresponds to imposing the physical state condition (2.1.17) with $a = 1$ and $D = 26$. Thus, one has a fully consistent quantum theory of the closed string.

2.6 Exercises for Chapter 2

Exercise 2.1. Consider the (bosonic) closed string in the covariant quantization approach of Section 2.1. The mode expansions of the position field $X^\mu(\tau,\sigma)$ and the momentum density $\Pi_\mu(\tau,\sigma) = T\dot{X}_\mu$ are given in Eqs. (1.4.27) and (1.4.29). As found in a previous exercise, the modes written in terms of the $X^\mu(\tau,\sigma)$ and $\Pi^\mu(\tau,\sigma)$ fields are given in Eq. (1.7.15).

- Show that the canonical equal-time commutation relations (2.1.1) imply the commutation relations for the modes (2.1.3). Show then the reverse, that the canonical equal-time commutation relations (2.1.1) follow from the commutation relations for the modes (2.1.3).

- Show that the reality conditions (2.1.2) on $X^\mu(\tau,\sigma)$ and $\Pi_\mu(\tau,\sigma)$ imply the reality conditions on the modes (2.1.4). Show the reverse, that (2.1.2) follows from (2.1.4).

Exercise 2.2. Consider the spectrum of the (bosonic) closed string in the covariant quantization approach of Section 2.1.

- Show that the ground state $|0;k\rangle$ is a physical state provided the conditions (2.2.5).
- Show that the level 1 states Eq. (2.2.6) are physical provided the conditions (2.2.7).
- Find all the states at level $N=\tilde{N}=2$ without imposing the physicality constraint.

Exercise 2.3. Consider the (bosonic) closed string in the covariant quantization approach of Section 2.1 with $a=1$ and $D=26$ (the space-time dimension D will not play a role in the following). Consider a state $|\chi\rangle$ of the form

$$|\chi\rangle = L_{-1}|\eta\rangle \tag{2.6.1}$$

with the state $|\eta\rangle$ obeying

$$L_n|\eta\rangle = 0 \quad \text{and} \quad (\tilde{L}_n - \delta_{n,0})|\eta\rangle = 0 \quad \text{for} \quad n \geq 0 \tag{2.6.2}$$

- Show that the state $|\chi\rangle$ given by (2.6.1) is orthogonal to any given physical state $|\phi\rangle$, *i.e.* that $\langle\chi|\phi\rangle = 0$.
- Show using the algebra (2.4.4) that $|\chi\rangle$ is a physical state.

As a result we have shown that a state $|\chi\rangle$ of the form (2.6.1) is a spurious state for any state $|\eta\rangle$ obeying (2.6.2).

Exercise 2.4. Consider the (bosonic) closed string in the covariant quantization approach of Section 2.1 with $a=1$ and $D=26$ (the space-time dimension D will not play a role in the following). Consider the state

$$|\eta\rangle = \frac{\sqrt{2}}{l_s} m_\mu \tilde{\alpha}^\mu_{-1}|0;k\rangle \tag{2.6.3}$$

with $k^2=0$ and $k^\mu m_\mu = 0$.

- Show that the state (2.6.3) obeys the conditions (2.6.2).
- Using Exercise 2.3, we now know that $|\chi\rangle = L_{-1}|\eta\rangle$, with $|\eta\rangle$ given in (2.6.3), is a spurious state. Show that $|\chi\rangle$ is given explicitly by

  $$|\chi\rangle = k_\mu m_\nu \alpha^\mu_{-1}\tilde{\alpha}^\nu_{-1}|0;k\rangle \tag{2.6.4}$$

  with $k^\mu m_\mu = 0$ and $k^2 = 0$.

States of the form (2.6.4) provides part of the gauge transformation of the massless states, as explained in Sec. 2.2 (one can easily show the other part of the gauge transformation from spurious states by repeating the above arguments of this exercise and Exercise 2.3 with the left- and right-moving sectors interchanged).

Exercise 2.5. Consider the (bosonic) closed string in the covariant quantization approach of Section 2.1. The generators of Lorentz transformations for the covariantly quantized closed string are given by (2.3.7) in terms of the quantum fields $X^\mu(\xi)$ and $\Pi_\mu(\xi)$ for the string.

- Derive Eqs. (2.3.8)-(2.3.9) which expresses the Lorentz generators $J_{\mu\nu}$ in terms of the mode expansion operators x^μ, p_μ, α_n^μ and $\tilde{\alpha}_n^\mu$.
- Show using Eqs. (2.3.8)-(2.3.9) as well as the commutator relations (2.1.3) that $J_{\mu\nu}$ obeys the Lorentz generator part of the Poincaré algebra (2.3.6)

$$[J_{\mu\nu}, J_{\rho\sigma}] = -i(\eta_{\nu\rho}J_{\mu\sigma} + \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\mu\rho}J_{\nu\sigma} - \eta_{\nu\sigma}J_{\mu\rho}) \tag{2.6.5}$$

Exercise 2.6. Consider the (bosonic) closed string in the covariant quantization approach of Sec. 2.1. Let $|\phi\rangle$ be a given state in the closed string Fock space. Assume that $L_n|\phi\rangle = 0$ for all $n \geq 1$. Show using the algebra (2.4.4) that it is not possible to have $L_{-n}|\phi\rangle = 0$ for all $n \geq 1$.

Exercise 2.7. This exercise is about the form of the central extension of the extended Virasoro algebra (2.4.7) for a general two-dimensional CFT. Consider a set of quantum operators $\mathcal{L}_n$ with algebra

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} + g(m)\delta_{m+n,0} \tag{2.6.6}$$

where $g(m)$ is an undetermined function of m.

- Explain why $g(-m) = -g(m)$.
- Using the Jacobi identity

$$[\mathcal{L}_m, [\mathcal{L}_n, \mathcal{L}_k]] + [\mathcal{L}_k, [\mathcal{L}_m, \mathcal{L}_n]] + [\mathcal{L}_n, [\mathcal{L}_k, \mathcal{L}_m]] = 0 \tag{2.6.7}$$

  show that for $m + n + k = 0$ we can derive

$$(n-k)g(m) + (m-n)g(k) + (k-m)g(n) = 0 \tag{2.6.8}$$

- Derive a recursion relation for $g(m+1)$ in terms of $g(m)$ and $g(1)$ using Eq. (2.6.8) for $k=1$ and $n=-m-1$ and using $g(-m)=-g(m)$.

- Show that $g(m)=\frac{c}{12}m^3+km$ satisfies the recursion relation for any values of c and k. Argue that this is the most general solution of the recursion relation. Thus, the algebra (2.6.6) for the $\mathcal{L}_n$ operators must be of the form

$$[\mathcal{L}_m,\mathcal{L}_n]=(m-n)\mathcal{L}_{m+n}+\left(\frac{c}{12}m^3+km\right)\delta_{m+n,0} \tag{2.6.9}$$

  As reviewed in Sec. 2.4.3 an algebra of this form is known as a centrally extended Virasoro algebra (for $c=k=0$ it is instead called a Virasoro algebra).

Exercise 2.8. Consider the (bosonic) closed string in the covariant quantization approach of Section 2.1. We consider in this exercise the algebra for the Virasoro generators L_n and $\tilde{L}_n$ as given in Eq. (2.1.15).

- Show that for $m\neq 0$, $n\neq 0$ and $m+n\neq 0$ we have

$$[L_m,L_n]=(m-n)L_{m+n} \tag{2.6.10}$$

- Show that for $m\neq 0$ we have

$$[L_m,L_{-m}]=2mL_0+b(m) \tag{2.6.11}$$

  where $b(m)$ at this point is an undetermined function of m arising from normal ordering.

- Show that for $m\neq 0$ we have

$$[L_m,L_0]=mL_m \tag{2.6.12}$$

- Explain using the three above results that we have derived

$$[L_m,L_n]=(m-n)L_{m+n}+b(m)\delta_{m+n,0} \tag{2.6.13}$$

- Argue that $[L_m,\tilde{L}_n]=0$ and that the $\tilde{L}_n$ Virasoro generators satisfy the same algebra as the $\tilde{L}_n$ generators with the same function $b(m)$. Conclude from the result of Exercise 2.7 that this means the algebra of the Virasoro generators L_n and $\tilde{L}_n$ must be of the form

$$\begin{aligned}
[L_m,L_n]&=(m-n)L_{m+n}+\left(\frac{c}{12}m^3+km\right)\delta_{m+n,0}\\
[\tilde{L}_m,\tilde{L}_n]&=(m-n)\tilde{L}_{m+n}+\left(\frac{c}{12}m^3+km\right)\delta_{m+n,0}\\
[L_m,\tilde{L}_n]&=0
\end{aligned} \tag{2.6.14}$$

  where c and k are constants.

Exercise 2.9. This exercise is a continuation of Exercise 2.8. We continue with studying the algebra of the Virasoro generators (2.1.15) for the (bosonic) closed string in the covariant quantization approach of Sec. 2.1. Our starting point is the result (2.6.14). We use the state $|0;k\rangle$ defined by Eqs. (2.1.8) and (2.1.9).

- Show that $L_n|0;0\rangle = 0$ for $n \geq -1$.
- Show that $L_{-2}|0;0\rangle = \frac{1}{2}\alpha_{-1}\cdot\alpha_{-1}|0;0\rangle$.
- Compute $\langle 0;0|[L_1, L_{-1}]|0;0\rangle$ and use this to show that $b(1) = 0$.
- Compute $\langle 0;0|[L_2, L_{-2}]|0;0\rangle$ and use this to show that $b(2) = D/2$.
- Use the results for $b(1)$ and $b(2)$ to show that the L_n Virasoro generators obey the extended Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12}(m^3 - m)\delta_{m+n,0} \tag{2.6.15}$$

  and conclude from this that we have derived the algebra (2.4.4).

Exercise 2.10. This exercise considers the graviton part of the massless closed string states discussed in Sec. 2.2. Consider the closed string state (2.2.6) written in terms of $\zeta_{\mu\nu}(k)$ with the conditions (2.2.7) and gauge transformation (2.2.8). The graviton part $G_{\mu\nu}(k)$ defined in (2.2.10) is the symmetric and traceless part of $\zeta_{\mu\nu}(k)$.

- Show using (2.2.10) that the conditions (2.2.7) implies

$$k^2 = 0\ ,\quad k^\mu G_{\mu\nu}(k) = 0 \tag{2.6.16}$$

  for $G_{\mu\nu}(k)$. Discuss that the first equation implies

$$k^2 G_{\mu\nu}(k) = 0 \tag{2.6.17}$$

  since either $G_{\mu\nu}(k) = 0$ or $k^2 = 0$. Show using (2.2.10) that

$$\eta^{\mu\nu} G_{\mu\nu}(k) = 0 \tag{2.6.18}$$

  Moreover, show that the gauge transformation (2.2.8) for $\zeta_{\mu\nu}(k)$ implies the gauge transformation (2.2.13) for $G_{\mu\nu}(k)$.

- Write the Fourier transform of $G_{\mu\nu}(k)$ as

$$G_{\mu\nu}(x) = \frac{1}{(2\pi)^D}\int d^Dk\, G_{\mu\nu}(k) e^{ik\cdot x} \tag{2.6.19}$$

Show that for the Fourier transformed field $G_{\mu\nu}(x)$ defined by (2.6.19) it follows from Eqs. (2.6.16)-(2.6.18) that

$$\partial_\mu \partial^\mu G_{\nu\rho} = 0 \ , \quad \partial^\mu G_{\mu\nu} = 0 \ , \quad \eta^{\mu\nu} G_{\mu\nu} = 0 \tag{2.6.20}$$

Hint for the first identity: Try to act with $\partial_\mu \partial^\mu$ on the right-hand side of (2.6.19) for $G_{\nu\rho}(x)$ and use (2.6.17).

- Show that the gauge transformation (2.2.13) for $G_{\mu\nu}(k)$ implies the gauge transformation (2.2.14) for $G_{\mu\nu}(x)$.

Exercise 2.11. The Virasoro generators L_n and $\tilde{L}_n$ are defined in Eq. (2.1.15). Show that

$$[L_m, \alpha^\mu_n] = -n\alpha^\mu_{m+n} \ , \quad [L_m, \tilde{\alpha}^\mu_n] = 0 \ , \quad [\tilde{L}_m, \tilde{\alpha}^\mu_n] = -n\tilde{\alpha}^\mu_{m+n} \ , \quad [\tilde{L}_m, \alpha^\mu_n] = 0 \tag{2.6.21}$$

Exercise 2.12. In this and the next two exercises we explore the quantum versions of the constraints on physical states in the covariant quantisation of the bosonic string. For simplicity we consider open strings with Neumann boundary conditions in both end points (since then we do not have independent left and right moving sectors). The X^μ field has mode expansion

$$X^\mu(\tau,\sigma) = x^\mu + 2l_s^2 p^\mu \tau + i\sqrt{2} l_s \sum_{n\neq 0} \frac{1}{n} \alpha^\mu_n e^{-in\tau} \cos(n\sigma) \tag{2.6.22}$$

We see that we can think of this as the closed string mode expansion but with the identification $\tilde{\alpha}^\mu_n = \alpha^\mu_n$ and with the total momentum operator $p^\mu = \alpha^\mu_0/(\sqrt{2}l_s)$ (this is a factor of two different compared to the closed string because of the different range of σ for 0 to π). In particular this means that defining the Virasoro generator

$$L_n = \frac{1}{2} \sum_{k\in\mathbb{Z}} : \alpha_{n-k} \cdot \alpha_k : \tag{2.6.23}$$

we find the same algebra (2.6.15). In the following we call a state $|\phi\rangle$ a physical state if it has

$$(L_n - a\delta_{n,0})|\phi\rangle = 0 \ \text{ for } \ n \geq 0 \tag{2.6.24}$$

For use below we remind that the ground state $|0;k\rangle$ is defined as $p_\mu|0;k\rangle = k_\mu|0;k\rangle$, $\alpha^\mu_n|0;k\rangle = 0$ for $n > 0$ and $\langle 0;k|0;k\rangle = 1$.

- Consider a physical state $|\phi\rangle$. Assume $a = 1$ and $D = 26$. Show using the algebra (2.6.15) that $L_n|\phi\rangle \neq 0$ for $n < 0$.

- Show that $(L_n)^\dagger = L_{-n}$. Show this means that for any two physical states $|\phi\rangle$ and $|\phi'\rangle$ we have $\langle\phi'|(L_n - a\delta_{n,0})|\phi\rangle = 0$ for all $n \in \mathbb{Z}$.

- Show that if a state $|\phi\rangle$ obeys $(L_0 - a)|\phi\rangle = L_1|\phi\rangle = L_2|\phi\rangle = 0$ then it is a physical state.

- Show that the ground state $|0; k\rangle$ has $L_n|0; k\rangle = 0$ for $n > 0$. Show that it is a physical state provided $l_s^2 k^2 = a$.

Exercise 2.13. Continuing Exercise 2.12.

- Show that
$$[L_m, \alpha_n^\mu] = -n\alpha_{m+n}^\mu \tag{2.6.25}$$

- Consider the state $|\phi\rangle = q \cdot \alpha_{-1}|0; k\rangle$ where q^μ is a real valued vector. Find the norm $\langle\phi|\phi\rangle$ in terms of q. Write down the conditions on q and k for $|\phi\rangle$ being a physical state.

- We would like to pick a and D to avoid ghost states in the spectrum of physical states (ghost states are states with negative norm). For what values of a is it possible that $|\phi\rangle$ is both a physical state and a ghost state?

Exercise 2.14. Continuing Exercise 2.12 and 2.13. Assume $a = 1$. Consider the state
$$|\phi\rangle = \Big[(\alpha_{-1})^2 + c_1\alpha_0 \cdot \alpha_{-2} + c_2(\alpha_0 \cdot \alpha_{-1})^2\Big]|0; k\rangle \tag{2.6.26}$$

- Find the values of c_1, c_2 and k^2 such that $|\phi\rangle$ is a physical state.

- Find the norm of $|\phi\rangle$. For what values of D is it possible that $|\phi\rangle$ is both a physical state and a ghost state?

3 General Relativity from String Theory

We have seen in Sec. 2.2.3 that the graviton can be consistently described as a mode of the quantum bosonic closed string. This will be further elucidated when we consider the superstring in Chapter 6 in which case we can get rid of the tachyon of bosonic closed string theory while still keeping the graviton mode. However, to establish string theory as a theory of quantum gravity, it is not enough that one can find a quantum description of a spin two particle (although that is already a highly non-trivial problem). One needs also to be able to get the theory of general relativity in a classical limit of the quantum gravity theory. This is what we consider in this Chapter.

The way to get general relativity as a classical limit of string theory is to consider the limit of having a large number of strings, corresponding to having high quantum numbers, and then consider the low energy limit of this. One does this by considering the consistency of having a single string moving in the background of the many strings at low energies, which then results in an effective description given by general relativity in 26 dimensions. As we shall see, the key to all of this is conformal invariance of the string.

3.1 Polyakov action for general background fields

So far we have been considering the quantized bosonic string in a 26-dimensional Minkowski space. Similarly in Chapter 6 we consider the quantized superstring in a 10-dimensional Minkowski space. Thus, in both cases one has D-dimensional Minkowski space as a target space for the string. However, as seen in Sec. 2.2.3 the closed string on 26-dimensional Minkowski space has a massless excitation that corresponds to the graviton, which is the quantum of the gravitational field. Below in Sec. 6.6 we shall see the same for the closed superstring in 10-dimensional Minkowski space. Hence, if we imagine considering a lot of closed (super)strings, giving rise to a lot of gravitons, we should be able to create a gravitational field that differs from the metric of Minkowski space, also in the classical limit. This parallels that one can create a classical electrical field out of photons, if only one has enough of them. These considerations points to the fact that one should be able to describe closed strings moving around in backgrounds other than D-dimensional Minkowski space. And with *background* we mean a classical configuration made of many string excitations for the metric field, as well as the other massless fields that corresponds to massless modes of the closed string. The reason that the classical configuration is described by the massless fields such as the metric is that one is considering a low energy limit.

The goal in the following is to describe the dynamics of a single closed string in the background of many closed strings. We work in the limit in which the energy of the single string is small. In this regime we can describe the background of the many closed strings using the massless fields that they induce.

For simplicity we stick to bosonic strings in the following. The induced background is thus described by the fields $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ, as found in Sec. 2.2.3. We ignore here the closed string tachyon, regarding the results as a precursor to the superstring.

The effective theory for a single bosonic closed string in the low energy limit $E_{\text{string}} \ll l_s^{-1}$ is now

$$
\begin{gathered}
S = S_{GB} + S_\Phi \\
S_{GB} = \frac{1}{4\pi l_s^2}\int d^2\xi \left[\sqrt{g}g^{\alpha\beta}G_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu + \epsilon^{\alpha\beta}B_{\mu\nu}(X)\partial_\alpha X^\mu\partial_\beta X^\nu\right] \\
S_\Phi = \frac{1}{4\pi}\int d^2\xi\sqrt{g}R^{(2)}\,\Phi(X)
\end{gathered}
\tag{3.1.1}
$$

where $\mu, \nu = 0, 1, ..., D-1$, with D being the dimension of the target space (one can also think of D as the number of scalar fields $X^\mu(\xi)$ on the worldsheet). An action of this type is known as a *non-linear sigma-model* and we see that it generalizes the Polyakov action (1.3.1). Several comments are in order here. First, we are working now in two-dimensional Euclidean signature (for later convenience). This means that we have made a Wick rotation $\xi^2 = i\tau = i\xi^0$. Hence our worldsheet coordinates are now (ξ^1, ξ^2), and the sums over α and β above in (3.1.1) are over $\alpha, \beta = 1, 2$. The metric $g_{\alpha\beta}$ is thus a two-dimensional metric with Euclidean signature and hence $g = \det(g_{\alpha\beta})$. Moreover, $R^{(2)}$ in the last term of (3.1.1) denotes the Ricci scalar of $g_{\alpha\beta}$. Note that the two-dimensional ϵ-symbol is given by $\epsilon^{12} = 1$ and $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$.

The Euclidean path integral corresponding to the action (3.1.1) is

$$
Z = \int_M \mathcal{D}g \int \mathcal{D}X\, e^{-S[g,X]} \tag{3.1.2}
$$

given the background fields $G_{\mu\nu}(X)$, $B_{\mu\nu}(X)$ and $\Phi(X)$ that in general can depend on $X^\mu(\xi^1, \xi^2)$. Here M denotes the space of all two-dimensional Riemannian manifolds.[15]

[15] A Riemannian manifold is a manifold equipped with a Euclidean signature metric.

3.2 Dilaton field as the string coupling

Consider a closed string moving in the background of a constant dilaton field $\Phi(X) = \Phi$. The dilaton part of the action (3.1.1) S_Φ is then

$$S_\Phi = \Phi\chi \ , \quad \chi = \frac{1}{4\pi}\int d^2\xi\sqrt{g}R^{(2)} \tag{3.2.1}$$

where χ is a purely geometrical quantity known as the *Euler character*. One can show that χ for a given two-dimensional manifold is an integer that depends solely on the topology

$$\chi = -2(g-1) \tag{3.2.2}$$

where g is the *genus* of the manifold, found as the number of holes (hence $g \geq 0$). In Fig. 8 we have illustrated the types of manifolds one has for genus $g = 0, 1, 2, 3$.

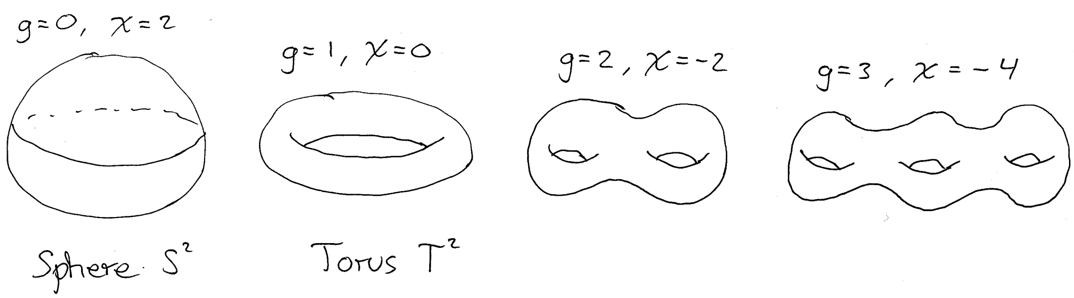

Figure 8: Two-dimensional manifolds with genus $g = 0, 1, 2, 3$ and the corresponding Euler character χ.

From (3.2.1) and (3.2.2) we see now that the path integral can be written as an expansion in the topology of the worldsheet

$$\begin{aligned} Z &= \int_M \mathcal{D}g \int \mathcal{D}X \, e^{-S_{GB}[g,X]-S_\Phi[g,X]} = \sum_{g=0}^{\infty} \int_{M_g} \mathcal{D}g \int \mathcal{D}X \, e^{-S_{GB}[g,X]-S_\Phi[g,X]} \\ &= \sum_{g=0}^{\infty} (e^\Phi)^{2(g-1)} \int_{M_g} \mathcal{D}g \int \mathcal{D}X \, e^{-S_{GB}[g,X]} \end{aligned} \tag{3.2.3}$$

where M_g denotes the space of all two-dimensional Riemannian manifolds with genus g. In particular, when $e^\Phi \ll 1$ we see that the genus zero contribution dominates of the higher geni $g \geq 1$ manifolds. Thus, we can think of e^Φ like a coupling constant. Following this, we define therefore the *string coupling constant*

$$g_s = \langle e^\Phi \rangle \tag{3.2.4}$$

as the expectation value of e^{Φ}. Hence,

$$
\begin{aligned}
Z &= g_s^{-2}\sum_{g=0}^{\infty} g_s^{2g}\int_{M_g}\mathcal{D}g\int\mathcal{D}X\, e^{-S_{GB}[g,X]} \\
&= g_s^{-2}\left[\int_{M_0}\mathcal{D}g\int\mathcal{D}X\, e^{-S_{GB}[g,X]} + g_s^2\int_{M_1}\mathcal{D}g\int\mathcal{D}X\, e^{-S_{GB}[g,X]}\right. \\
&\quad \left. +g_s^4\int_{M_2}\mathcal{D}g\int\mathcal{D}X\, e^{-S_{GB}[g,X]} + \cdots\right]
\end{aligned}
\tag{3.2.5}
$$

In particular for $g_s = 0$ we see that only the spherical topology of the worldsheet contributes. This corresponds to what we have been considering in Chapters 1-6 where we analyzed the Polyakov action in flat gauge (1.3.18) as well as the version of this with global supersymmetry (6.1.1). To see this, note first that $\mathbb{R}^2$ is the Wick rotation of two-dimensional Minkowski space, and use then that the plane $\mathbb{R}^2$ can be mapped to S^2 by a conformal transformation and adding a point at infinity.

For g_s small but non-zero one can perturbatively include the higher genus contributions. This is called string perturbation theory.

3.3 Conformal invariance of the sigma-model

Consider the non-linear sigma-model action (3.1.1). Classically, it has the following local symmetries

- Diffeomorphism invariance: the non-linear sigma-model action (3.1.1) is invariant under worldsheet coordinate transformations

$$\xi^\alpha \to \tilde{\xi}^\alpha(\xi) \tag{3.3.1}$$

  with $\alpha = 1, 2$, along with the transformations

$$X^\mu(\xi) \to \tilde{X}^\mu(\tilde{\xi}) = X^\mu(\xi) \tag{3.3.2}$$

$$g_{\alpha\beta}(\xi) \to \tilde{g}_{\alpha\beta}(\tilde{\xi}) = \frac{\partial\xi^\gamma}{\partial\tilde{\xi}^\alpha}\frac{\partial\xi^\delta}{\partial\tilde{\xi}^\beta}g_{\gamma\delta}(\xi) \tag{3.3.3}$$

- Weyl invariance: for a constant dilaton field $\Phi(X) = \Phi$ the non-linear sigma-model action (3.1.1) is invariant under the Weyl transformations

$$g_{\alpha\beta}(\xi) \to \tilde{g}_{\alpha\beta}(\xi) = \Omega(\xi)^2 g_{\alpha\beta}(\xi) \tag{3.3.4}$$

  under which $X^\mu(\xi)$ and the coordinates ξ^α are not transformed.

Combining these local symmetries one can see that the non-linear sigma-model action (3.1.1) is classically conformally invariant. In detail, choosing any particular gauge for the metric $g_{\alpha\beta}$, one can first perform a conformal transformation, and afterwards a Weyl rescaling to get back the chosen gauge for the metric. This works in particular for the flat gauge $g_{\alpha\beta} = \eta_{\alpha\beta}$ in the same way is explained in Sec. 1.6.1. Thus, for constant dilaton field Φ the non-linear sigma-model action (3.1.1) is classically conformally invariant.

We now impose that also the quantum theory of the non-linear sigma-model action (3.1.1) is conformally invariant. We do this since we view the conformal invariance of the string theory just like the gauge invariance of Yang-Mills theory: it is an underlying symmetry that defines the quantum theory.

However, the non-linear sigma-model action (3.1.1) is a non-linear two-dimensional quantum field theory. Hence, we have to quantize it using perturbation theory. This perturbation theory is defined in terms of the string length scale versus the length scale of the variations of the background fields $G_{\mu\nu}$ and $B_{\mu\nu}$. Indeed, we work in the regime

$$l_s \ll L_{\text{background}} \tag{3.3.5}$$

where $L_{\text{background}}$ is the smallest length scale associated with the background fields $G_{\mu\nu}$ and $B_{\mu\nu}$, *e.g.* it could be the length scale of the geometry that $G_{\mu\nu}$ defines. In this way, we can expand in powers of the string length, as we shall see.

For the two-dimensional quantum field theory with action (3.1.1) one introduces a UV cutoff to regulate divergencies. After renormalization, this means that the physical quantities depends on the scale μ. One can therefore introduce the β-functions

$$\beta^G_{\nu\rho} = \mu \frac{\partial G_{\nu\rho}}{\partial \mu} \ , \quad \beta^B_{\nu\rho} = \mu \frac{\partial B_{\nu\rho}}{\partial \mu} \ , \quad \beta^\Phi = \mu \frac{\partial \Phi}{\partial \mu} \tag{3.3.6}$$

that parametrize the dependence of the background fields $G_{\nu\rho}$, $B_{\nu\rho}$ and Φ on the scale μ. In terms of these β-functions the trace of the worldsheet energy-momentum tensor is

$$T^\alpha{}_\alpha = \frac{\beta^\Phi}{12} R^{(2)} + \frac{1}{2l_s^2} \Big(\beta^G_{\nu\rho} g^{\alpha\beta} + \beta^B_{\nu\rho} \epsilon^{\alpha\beta} \Big) \partial_\alpha X^\nu \partial_\beta X^\rho \tag{3.3.7}$$

where $R^{(2)}$ is the Ricci scalar on the worldsheet. We see now that insisting on conformal invariance in the quantum theory means that we should require that all the β-functions are zero

$$\beta^G_{\nu\rho} = 0 \ , \quad \beta^B_{\nu\rho} = 0 \ , \quad \beta^\Phi = 0 \tag{3.3.8}$$

which means that no new scale will appear in the theory after quantization.

By direct quantum field theory calculations one finds

$$\begin{aligned}\beta^G_{\nu\rho} &= l_s^2\left(R_{\nu\rho} - \tfrac{1}{4}H_{\nu\sigma\kappa}H_\rho{}^{\sigma\kappa} + 2\nabla_\nu\nabla_\rho\Phi\right) + \mathcal{O}(l_s^4)\\ \beta^B_{\nu\rho} &= -\tfrac{1}{2}l_s^2\nabla^\sigma\left[e^{-2\Phi}H_{\nu\rho\sigma}\right] + \mathcal{O}(l_s^4)\\ \beta^\Phi &= D - 26 + \tfrac{3}{2}l_s^2\left[4(\nabla\Phi)^2 - 4\nabla_\rho\nabla^\rho\Phi - R + \tfrac{1}{12}H^2\right] + \mathcal{O}(l_s^4)\end{aligned} \tag{3.3.9}$$

where ∇_ρ is the target space covariant derivative with respect to the metric $G_{\mu\nu}(X)$ and where we defined

$$H_{\nu\rho\sigma} = 3\partial_{[\nu}B_{\rho\sigma]} = \partial_\nu B_{\rho\sigma} + \partial_\rho B_{\sigma\nu} + \partial_\sigma B_{\nu\rho} \tag{3.3.10}$$

and $H^2 = H_{\nu\rho\sigma}H^{\nu\rho\sigma}$. Moreover, $R_{\mu\nu}$ is the Ricci tensor and R the Ricci scalar computed from the D-dimensional metric $G_{\mu\nu}(X)$.

Imposing conformal invariance of the quantum theory (3.3.8) to order l_s^2 we deduce from (3.3.9) that this imply $D = 26$ as well as the following equations for the background fields $G_{\mu\nu}(X)$, $B_{\mu\nu}(X)$ and $\Phi(X)$

$$\begin{aligned}R_{\mu\nu} &= \frac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} - 2\nabla_\mu\nabla_\nu\Phi\\ \nabla_\rho(e^{-2\Phi}H^{\rho\mu\nu}) &= 0\\ 4(\nabla\Phi)^2 - 4\nabla_\rho\nabla^\rho\Phi &= R - \frac{1}{12}H^2\end{aligned} \tag{3.3.11}$$

These equations can be interpreted as equations of motion for the background fields $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ. In the special case for which $B_{\mu\nu} = \Phi = 0$ we see that they reduce to $R_{\mu\nu} = 0$ which is Einsteins equations without matter for a $D = 26$ dimensional space-time. Thus, we have obtained the theory of general relativity as a low energy effective theory from closed string theory. More precisely, putting a single closed string in the background fields generated by many closed strings and demanding conformal invariance of the effective description of the single closed string, general relativity and the equations of motions for the $B_{\mu\nu}$ and Φ matter fields (3.3.11) come out as a requirement.

This means that string theory can describe gravity all the way from an individual graviton (massless mode of a closed string) to the low energy effective theory in the form of general relativity. In other words, we have obtained the theory of general relativity from a classical limit of string theory.

In addition, one can go to higher orders in l_s, computing order by order the β-functions (3.3.9). Since the power of l_s is connected to the number of derivatives, this gives higher-derivative corrections to the equations (3.3.11). These are suppressed in the low energy regime in which we are working. Hence, in conclusion, we can obtain corrections to the equations of general relativity from string theory.

One can show that if one linearizes the equations (3.3.11) one obtains the equations found in Sec. 2.2.3 for $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ in position space, using the gauge freedom to impose suitable gauge choices.[16]

The equations (3.3.11) follows from the action

$$S_{\text{bos}} = \frac{1}{16\pi G_N} \int d^{26}x \sqrt{-G} e^{-2\Phi} \left[R + 4(\nabla\Phi)^2 - \frac{1}{12} H^2 \right] \tag{3.3.12}$$

where G_N is the gravitational coupling. Writing

$$\Phi_{\text{full}} = \Phi_0 + \Phi \tag{3.3.13}$$

where $\Phi_0 = \langle \Phi_{\text{full}} \rangle$ is the expectation value of the dilaton, Φ contains the fluctuations of the dilaton and Φ_{full} is the full dilaton field, we can set

$$g_s = e^{\Phi_0} \tag{3.3.14}$$

We see that this means that $G_N \propto g_s^2$ so perturbation theory of gravity is the same as the topological expansion of strings.

3.4 Exercises for Chapter 3

Exercise 3.1. Show that the non-linear sigma-model action (3.1.1) is invariant under Weyl rescalings (3.3.4) when the dilaton field Φ is constant.

[16]Note that in linearizing $G_{\mu\nu}$ one should write $G_{\mu\nu} = \eta_{\mu\nu} + \tilde{G}_{\mu\nu}$ where $\tilde{G}_{\mu\nu}$ is the metric field we find in Sec. 2.2.3.

4 Branes

In this chapter we introduce the concept of relativistic p-branes. Since 0-branes are particles and 1-branes are strings, we have already encountered examples of branes in the previous chapter. However, in Chapter 5 we shall see that higher-dimensional branes occur in string theory as well, in the form of Dp-branes. Moreover, we introduce the mathematical tool of n-forms that we use to write down the action for a charged p-brane. This is both important to understand that the fundamental string is charged, which is related to the Kalb-Ramond field encountered in Sections 2.2.3 and Chapter 3, and moreover the connection between Ramond-Ramond fields and Dp-branes that will be considered in Chapter 6.

4.1 Action for structureless p-branes

A p-brane is a general term for a dynamical p-dimensional object. A 0-brane is a particle, a 1-brane is a string, a 2-brane is a membrane, and so on. Considering a relativistic p-brane we can generalize the worldline of the particle (1.1.13) and the worldsheet of the string (1.2.1) to a $(p+1)$-dimensional worldvolume of the p-brane. Thus, corresponding to a relativistic p-brane we have a map

$$(\xi^0, \xi^1, ..., \xi^p) \to X^\mu(\xi^0, \xi^1, ..., \xi^p) \tag{4.1.1}$$

from the $(p+1)$-dimensional worldvolume to the D-dimensional target space, which we here pick as D-dimensional Minkowski space.

Considering an infinitely thin p-brane the dynamical principle should generalize the extremization of the worldline of the worldline of the particle, or the area of the worldsheet of the string. Thus, the dynamical principle of an infinitely thin relativistic p-brane is the extremization of the volume of its $(p+1)$-dimensional worldvolume. We can define the induced metric for the worldvolume as

$$\gamma_{ab} = \eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu \tag{4.1.2}$$

for $a, b = 0, 1, ..., p$ with $\partial_a = \partial/\partial\xi^a$. With this, we can write the action as

$$S = -T_p \int d^{p+1}\xi \sqrt{-\det\gamma} \tag{4.1.3}$$

where T_p is the tension of the p-brane (in units of inverse length to the power $p+1$). This is known as the *Dirac action*. This action is the direct generalization of the point particle action (1.1.17) and the Nambu-Goto action for strings (1.2.11) to p-branes.[17] The EOMs can be found by variation of the action to be[18]

$$\partial_a(\sqrt{-\det\gamma}\,\gamma^{ab}\partial_b X^\mu) = 0 \tag{4.1.4}$$

The infinitely thin relativistic p-brane is an object without further structure than the embedding map (4.1.1) and its tension T_p. In Section 5 we shall encounter so-called Dirichlet p-branes in string theory that are made out of open strings, and thus have a further structure than the branes presented here.

4.2 Charged branes

4.2.1 Electrically charged branes

In this section we generalize the infinitely thin relativistic p-brane to include an electric coupling to a background field. This is relevant both for the fundamental string and for the Dirichlet p-branes found in Sec. 5.

Consider the following action for a relativistic point particle

$$S = -m\int d\tau\sqrt{-\dot{X}^2} + q\int d\tau\, A_\mu \dot{X}^\mu \tag{4.2.1}$$

This describes a relativistic point particle with mass m and charge q moving in Minkowski space on the worldline $X^\mu(\tau)$ with a background electromagnetic field given by the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\nu$. Notice that the first term in the action is the same as (1.1.17). Instead the second term gives the coupling to a background electromagnetic field $F_{\mu\nu}$. The EOMs are

$$m\frac{d}{d\tau}\left(\frac{\dot{X}^\mu}{\sqrt{-\dot{X}^2}}\right) = qF^\mu{}_\nu \dot{X}^\nu \tag{4.2.2}$$

We now generalize the action (4.2.1) to higher-dimensional p-branes, including the string. To this end, we first introduce something called n-forms. This is a special type of tensor that we can define on any D-dimensional space-time assuming $0 \leq n \leq D$. We assume below that we are in D-dimensional Minkowski space. An n-form U is a

[17] We remark that if one tries to propose a fundamental theory of infinitely thin p-branes with $p \geq 2$ then one encounters problems in that the spectrum seemingly is continuous. Hence the infinitely thin relativistic string is special in this sense since it gives a discrete spectrum. Despite this, there is evidence that the superstring can be derived from a fundamental membrane theory of the so-called M-theory. How this works is still a mystery.

[18] One can alternatively write them as $D^a D_a X^\mu = 0$ with D_a being the covariant derivative on the worldvolume with respect to the induced metric γ_{ab}.

completely antisymmetric tensor with n lower indices $U_{\mu_1\mu_2\cdots\mu_n}$, *i.e.* one gets a minus by interchanging any two of the indices. Often one uses the notation $U_{(n)}$ for an n-form to emphasize that it has n indices. Note that a zero-form is a scalar field. Given any n-form U we define the so-called exterior derivative dU as an $(n+1)$-form with components

$$(dU)_{\mu_1\mu_2\cdots\mu_{n+1}} = (n+1)\partial_{[\mu_1}U_{\mu_2\cdots\mu_{n+1}]} \tag{4.2.3}$$

The notation $[\cdots]$ means the completely antisymmetrised sum over the indices inside the square paranthesis weighted by the number of terms in the sum. Consider $V_{[\mu_1\cdots\mu_m]}$ for a given tensor $V_{\mu_1\cdots\mu_m}$ with m lower indices. In this case we have m indices and hence we sum over the $m!$ possible permutations of the m indices. When there is an even number of permutations, one puts a plus in front of the term, and when there is an odd number of permutations, one puts a minus. Indeed, we have

$$V_{[\mu\nu]} = \frac{1}{2}(V_{\mu\nu} - V_{\nu\mu}) \ , \quad V_{[\mu\nu\rho]} = \frac{1}{3!}(V_{\mu\nu\rho} - V_{\nu\mu\rho} + V_{\nu\rho\mu} - V_{\rho\nu\mu} + V_{\rho\mu\nu} - V_{\mu\rho\nu}) \tag{4.2.4}$$

and so on.

As examples of the exterior derivative, consider a one-form A_μ, then its exterior derivative is the 2-form

$$(dA)_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{4.2.5}$$

For a two-form $B_{\mu\nu}$ we have

$$(dB)_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} \tag{4.2.6}$$

A very important property of the exterior derivative of an n-form U is that

$$d^2U = d(dU) = 0 \tag{4.2.7}$$

A form U is called *exact* is its the exterior derivative of another form $U = dV$. A form U is called *closed* if its exterior derivative is zero $dU = 0$. Thus, (4.2.7) means that all exact forms are closed. For our purposes here, we can also assume that the converse is true, *i.e.* that all closed forms are exact.[19]

For use in Chapter 6 we also define the wedge product of forms. Given an n-form U and an m-form V we can combine them into a $n+m$ form $U \wedge V$ as

$$(U \wedge V)_{\mu_1\cdots\mu_{n+m}} = \frac{(n+m)!}{n!m!}U_{[\mu_1\cdots\mu_n}V_{\mu_{n+1}\cdots\mu_{n+m}]} \tag{4.2.8}$$

[19] It is not true in general that all closed forms are exact. This is only the case if the manifold in question has sufficiently simple topology, or if one considers a subset of the manifold with a simple topology. In mathematics, one uses this fact to define topological invariants of manifolds via something called De Rham cohomology. For our purposes here it is true since we are in D-dimensional Minkowski space which topologically is like $\mathbb{R}^D$ and hence has a trivial topology.

Finally, for a given n-form U we define its Hodge dual ${}^*U$ as the $(D-n)$-form given by[20]

$$ {}^*U^{\mu_{n+1}\mu_{n+2}\cdots\mu_D} = \frac{1}{n!} U_{\mu_1\mu_2\cdots\mu_n} \epsilon^{\mu_1\mu_2\cdots\mu_D} \tag{4.2.9} $$

where we uplifted the $D-n$ indices of ${}^*U$ with the Minkowski metric and where $\epsilon^{\mu_1\mu_2\cdots\mu_D}$ is defined as the completely antisymmetric object with $\epsilon^{01\cdots(D-1)} = 1$.

Suppose now we are considering an electric p-brane. This brane is assumed to be infinitely thin and relativistic. Such a brane can couple to a higher-dimensional version of Maxwell's electromagnetism. Indeed, consider the $(p+1)$-form A known as the gauge field. The exterior derivative gives a $(p+2)$-form $F = dA$ that we call the field strength. The equivalent of Maxwell's equations without sources are

$$ dF = 0 \ , \quad d^*F = 0 \tag{4.2.10} $$

We see that the first equation in (4.2.10) follows from $F = dA$ and the general property (4.2.7) of the exterior derivative. In component form we can write the equations as

$$ \partial_{[\mu_1} F_{\mu_2\cdots\mu_{p+3}]} = 0 \ , \quad \partial^\mu F_{\mu\nu_1\cdots\nu_{p+1}} = 0 \tag{4.2.11} $$

assuming for the second equation that we are in D-dimensional Minkowski space. One can check that for $p = 0$ the above equations reduces to the source-free Maxwell's equations. The $(p+2)$-form field strength $F = dA$ is invariant under gauge transformations of the $(p+1)$-form gauge field A

$$ A \to A + d\chi \tag{4.2.12} $$

where χ is a p-form.

We are now ready to formulate the action for the electric p-brane. It is

$$ S = -T_p \int d^{p+1}\xi \sqrt{-\det\gamma} + Q_p \int d^{p+1}\xi \frac{1}{(p+1)!} \epsilon^{a_1\cdots a_{p+1}} A_{\mu_1\cdots\mu_{p+1}} \partial_{a_1} X^{\mu_1} \cdots \partial_{a_{p+1}} X^{\mu_{p+1}} \tag{4.2.13} $$

While the first term is the Dirac action (4.1.3) the second term generalizes the coupling term in the action (4.2.1) for the electrically charged relativistic point particle where Q_p is the charge. The EOMs are

$$ T_p \partial_a (\sqrt{-\det\gamma}\, \gamma^{ab} \partial_b X^\mu) = -\frac{Q_p}{(p+1)!} \epsilon^{a_1\cdots a_{p+1}} F^\mu{}_{\nu_1\cdots\nu_{p+1}} \partial_{a_1} X^{\nu_1} \cdots \partial_{a_{p+1}} X^{\nu_{p+1}} \tag{4.2.14} $$

Note that in the presence of electrically charge p-branes the $(p+2)$-form field strength F is sourced. Hence the second equation (4.2.11) is modified to $\partial^\mu F_{\mu\nu_1\cdots\nu_{p+1}} = J_{\nu_1\cdots\nu_{p+1}}$ where $J_{\nu_1\cdots\nu_{p+1}}$ is the $(p+1)$-form current that corresponds to the distribution of the charged branes.

[20]We assume here the D-dimensional Minkowski metric $\eta_{\mu\nu}$. The definition is modified for a general metric.

4.2.2 Magnetically charged branes

A p-brane can also couple magnetically to a $(D-p-2)$-form field strength $\tilde{F}$. The source-free equations of motion for $\tilde{F}$ are

$$d\tilde{F} = 0 \ , \quad d^*\tilde{F} = 0 \tag{4.2.15}$$

Define from $\tilde{F}$ the dual $(p+2)$-form field strength F as

$$F = {}^*\tilde{F} \tag{4.2.16}$$

The equations of motion (4.2.15) for $\tilde{F}$ can now be translated to (4.2.10) for the dual field strength F. The equation $dF = 0$ means that one can find a $(p+1)$-form A such that $F = dA$. Employing this, the action (4.2.13) can now be used to describe how the p-brane couples magnetically to the $(D-p-2)$-form field strength $\tilde{F}$ with Q_p being its magnetic charge.

4.2.3 Fundamental string is charged

We now apply the above formalism that we developed in order to write the down the action (4.2.13) for a charged p-brane to the case of the fundamental bosonic string. This corresponds to setting $p = 1$. In this case the Dirac action (4.1.3) clearly becomes the Nambu-Goto action (1.2.11) which is equivalent to the Polyakov action (1.3.1).

The string can couple electrically to a three-form field strength with a corresponding two-form gauge field. Indeed one can show that the two-form gauge field is the Kalb-Ramond field $B_{\mu\nu}$ that we first encountered in Sec. (2.2.3) where we found it as a massless mode of the closed string. In Sec. 3.1 we considered a closed string in a general background given by a metric, Kalb-Ramond field and dilaton field, corresponding to the action (3.1.1) (for an euclidean worldsheet). Restricting now to a background with Minkowski space $G_{\mu\nu} = \eta_{\mu\nu}$ and zero dilaton $\Phi = 0$ and Wick-rotating to a Lorentzian worldsheet, this becomes

$$S = -\frac{T}{2}\int d^2\xi \sqrt{-g} g^{\alpha\beta} \eta_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu + T \int d^2\xi \, \frac{1}{2} \epsilon^{\alpha\beta} B_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu \tag{4.2.17}$$

We recognize the first term as the Polyakov action (1.3.1). The second term is precisely the second term in (4.2.13) for $p = 1$ with $B_{\mu\nu}$ being the two-form gauge field. Hence we see that the fundamental string is electrically charged. Note that the charge is equal to the tension T of the string.

Interpretating the Kalb-Ramond field $B_{\mu\nu}$ as a two-form gauge field, we know from Sec. 4.2.1 that that the corresponding three-form field strength is the exterior derivative $H = dB$. In components we read from (4.2.1) that

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} \tag{4.2.18}$$

which is the same definition as (3.3.10) in Sec. 3.3. This field strength obeys $dH = 0$ and $d\,{}^*H = 0$ which we can write as

$$\partial_{[\mu} H_{\nu\rho\sigma]} = 0 \ , \quad \partial^\mu H_{\mu\nu\rho} = 0 \tag{4.2.19}$$

These equations are the three-form analog of the source-free Maxwell equations. The first equation follows from (4.2.18). Notice that the second equation in (4.2.19) is the same as we found in Sec. (2.2.3) from imposing the physical state conditions (including some additional gauge conditions). Furthermore, it is the same as the second equation of Eq. (3.3.11) for $G_{\mu\nu} = \eta_{\mu\nu}$ and $\Phi = 0$ found from imposing conformal invariance of the general sigma-model (3.1.1).

As explained in Sec. 4.2.1, one can make gauge transformations of the two-form gauge field B of the form $B \to B + d\chi$ where χ is a one-form. In components, this is

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu \chi_\nu - \partial_\nu \chi_\mu \tag{4.2.20}$$

Since $H = dB$ this means that H is invariant under such gauge transformations.

4.3 Exercises for Chapter 4

Exercise 4.1. Derive the expressions (4.2.5) and (4.2.6) for the exterior derivatives of a one-form A and a two-form B from the general formula for the exterior derivative (4.2.3).

Exercise 4.2. Consider a one-form U in a D-dimensional space-time. Show that $d^2 U = 0$.

Exercise 4.3. In this exercise we derive the EOMs for relativistic p-branes.

- Derive the EOMs (4.1.4) from the Dirac action (4.1.3). [Hint: You can use the identity $\delta\sqrt{-\det\gamma} = \frac{1}{2}\sqrt{-\det\gamma}\gamma^{ab}\delta\gamma_{ab}$.]

- Derive the EOMs (4.2.14) from the action (4.2.13) of a charged relativistic p-brane.

- Show that the EOMs (4.2.14) reduce to (4.2.2) for $p = 0$.

Exercise 4.4. Consider a $(p + 2)$-form field strength F.

- Show that (4.2.10) is equivalent to (4.2.11) in D-dimensional Minkowski space. One can use that

$$\epsilon^{\mu_1\mu_2\cdots\mu_D}\epsilon_{\nu_1\cdots\nu_n\mu_{n+1}\cdots\mu_D} = (D-n)!n!\delta^{\mu_1}_{[\nu_1}\delta^{\mu_2}_{\nu_2}\cdots\delta^{\mu_n}_{\nu_n]} \tag{4.3.1}$$

  where $\epsilon_{\mu_1\cdots\mu_D}$ is completely antisymmetric with $\epsilon_{01\cdots(D-1)} = 1$.

- Consider the case $p = 0$. Argue that (4.2.10) are the source-free Maxwell's equations with F as the field strength and A as the gauge field. Argue furthermore that (4.2.12) corresponds to gauge transformations of A.

5 Quantization of the Open String and D-branes

5.1 Quantization of the open string for a single D-brane

In Sec. 1.4.4 we analyzed the classical solution of an open string that has Neumann boundary conditions at both end points in $p+1$ directions (including the time-direction) and Dirichlet boundary at both end points in the remaining $D-p-1$ directions. As discussed in Sec. 1.4.4 the open string ends on two flat p-dimensional hyperplanes defined by $x^I = c_1^I$ and $x^I = c_2^I$ for $I = p+1, ..., D-1$. In this section we consider a special case of this with only one hyperplane, thus $c_1^I = c_2^I = c^I$. Hence both end points of the open string ends on the same flat p-dimensional hyperplane defined by $x^I = c^I$, $I = p+1, ..., D-1$. Thus, the boundary conditions are

$$X'^a(\tau, 0) = X'^a(\tau, \pi) = 0 \ , \quad X^I(\tau, 0) = X^I(\tau, \pi) = c^I \tag{5.1.1}$$

where $a = 0, 1, ..., p$ and $I = p+1, ..., D-1$. This setup is illustrated in Fig. 9. The mode expansions for $X^\mu(\xi)$ and $\Pi^\mu(\xi)$ are given by Eqs. (1.4.45), (1.4.46), (1.4.49) and(1.4.51) with $c_1^I = c_2^I = c^I$.[21]

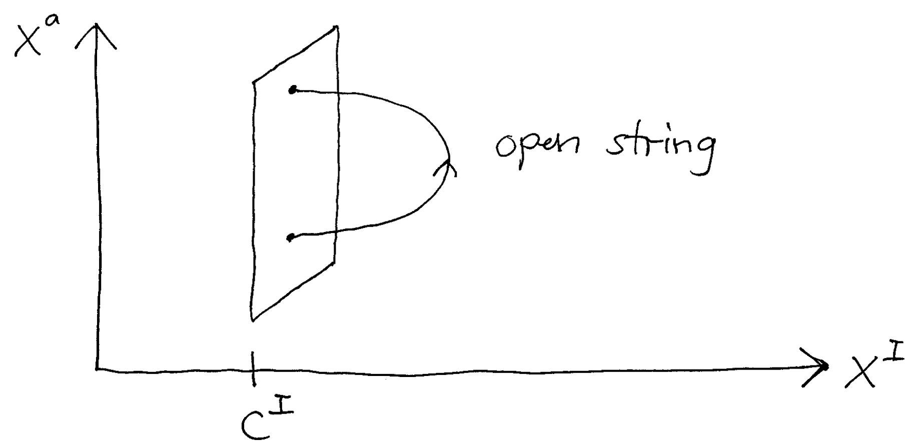

Figure 9: An open string ending on a Dp-brane with its worldvolume parallel to the x^a directions, $a = 0, 1, ..., p$, and located at $x^I = c^I$, $I = p+1, ..., D-1$, in D-dimensional Minkowski space. The arrow on the open string specifies the direction of increasing σ.

The difference between the open string and closed string is a matter of boundary conditions. Thus, considering the constraint equations (1.3.7) for the classical open string one has the same energy momentum tensor components (1.5.3) which again are periodic

[21] Note that the non-zero mode parts of the open string mode expansions can be obtained from the closed string mode expansions (1.4.27) and (1.4.29) by setting $\tilde{\alpha}_n^a = \alpha_n^a$ and $\tilde{\alpha}_n^I = -\alpha_n^I$.

functions. However, as one can infer from (1.4.44) and (1.4.48) the Fourier modes of $T_{--}(\xi^-)$ are equal to those of $T_{++}(\xi^+)$,

$$T_{--} = \sum_{n\in\mathbb{Z}} L_n e^{-in\xi^-} \ , \quad T_{++} = \sum_{n\in\mathbb{Z}} L_n e^{-in\xi^+} \ , \quad L_n = \frac{1}{2}\sum_{k\in\mathbb{Z}} \alpha_{n-k}\cdot\alpha_k \tag{5.1.2}$$

Thus, classically the constraints are $L_n = 0$ for $n \in \mathbb{Z}$.

We can readily apply the covariant quantization method to this setup for the open string. Again the fields $X^\mu(\xi)$ and $\Pi_\mu(\xi)$ are promoted to quantum fields that obey the canonical equal-time commutation relations (2.1.1) and the reality conditions (2.1.2). Correspondingly, the modes x^a, p_a and α_n^μ are promoted to operators with commutation relations

$$\begin{gathered} [x^a, p_b] = i\delta^a_b \ , \quad [x^a, x^b] = [p_a, p_b] = 0 \\ [\alpha^\mu_m, \alpha^\nu_n] = m\delta_{m+n,0}\eta^{\mu\nu} \end{gathered} \tag{5.1.3}$$

with $a, b = 0, 1, ..., p$ and $\mu, \nu = 0, 1, ..., D-1$. The reality conditions are

$$(x^a)^\dagger = x^a \ , \quad (p_a)^\dagger = p_a \ , \quad (\alpha^\mu_n)^\dagger = \alpha^\mu_{-n} \tag{5.1.4}$$

We have the ground state $|0;k\rangle$ defined for a given momentum k_a, $a = 0, 1, ..., p$, by

$$p_a|0;k\rangle = k_a|0;k\rangle \ , \quad \alpha^\mu_n|0;k\rangle = 0 \ \text{ for } \ n > 0 \tag{5.1.5}$$

in line with the interpretation of α^μ_n as annihilation operators and α^μ_{-n} as creation operators for $n > 0$. The open string Fock space has basis

$$\alpha^{\mu_1}_{-n_1}\alpha^{\mu_2}_{-n_2}\cdots\alpha^{\mu_q}_{-n_q}|0;k\rangle \tag{5.1.6}$$

with $n_i > 0$. One can now proceed with the covariant quantization of the open string in completely analogous manner as we did for the closed string. Physical states obey

$$(L_n - a\delta_{n,0})|\phi\rangle \ \text{ for } \ n \geq 0 \tag{5.1.7}$$

where we defined the Virasoro generator

$$L_n = \frac{1}{2}\sum_{k\in\mathbb{Z}} :\alpha_{n-k}\cdot\alpha_k: \tag{5.1.8}$$

One can follow the same steps as for the closed string in Sec. 2, computing the same extended Virasoro algebra as for the right-moving sector of the closed string algebra (2.4.4), thus with the same central extension. Also for the ghost field sector one finds the same extended Virasoro algebra as for the right-moving sector of the clsoed string algebra

(2.5.8). Hence one finds that the conformal symmetry algebra is only anomaly-free for $a = 1$ and $D = 26$, just as for the closed string.[22] With this, we can consider the open string spectrum and interpret it as a spectrum of particles. Setting $a = 1$ and $n = 0$ in (5.1.7) we derive for a physical state $|\phi\rangle$

$$M^2|\phi\rangle = -p_a p^a|\phi\rangle = \frac{1}{l_s^2}(N_\parallel + N_\perp - 1)|\phi\rangle \tag{5.1.9}$$

with

$$N_\parallel = \sum_{k=1}^{\infty} \eta_{ab}\alpha_{-k}^a \alpha_k^b \ , \quad N_\perp = \sum_{k=1}^{\infty} \sum_{I=p+1}^{D-1} \alpha_{-k}^I \alpha_k^I \tag{5.1.10}$$

where we used the mass-shell condition $M^2 = -p_a p^a$ for the total momentum p_a of the open string along the directions with Neumann boundary conditions as well as $\alpha_0^a = \sqrt{2} l_s p^a$. Note that here and in the following we will be using the Einstein summation convention for the indices a and b with sums from 0 to p such that for instance $p_a p^a = \sum_{a=0}^p p_a p^a$.

Consider first $N_\parallel = N_\perp = 0$. This singles out the open string ground state $|0;k\rangle$. According to (5.1.9) this state has $M^2 = -p_a p^a = -l_s^{-2}$. This corresponds to a tachyon and is hence known as the open string tachyon. We shall consider the interpretation of this below.

Consider $N_\parallel = 1$ and $N_\perp = 0$. A general linear superposition is

$$A_a(k)\alpha_{-1}^a|0;k\rangle \tag{5.1.11}$$

The physical state conditions (5.1.7) give

$$k_a k^a = 0 \ , \quad k^a A_a(k) = 0 \tag{5.1.12}$$

Thus it is a massless particle. Note also that the state $k_a \alpha_{-1}^a|0;k\rangle$ is spurious, hence we have the gauge transformations

$$A_a(k) \to A_a(k) + k_a m(k) \tag{5.1.13}$$

where $m(k)$ can depend on k_a. Going to position space by the Fourier transform

$$A_a(x) = \frac{1}{(2\pi)^{p+1}} \int d^{p+1}k \, A_a(k) e^{ik_a x^a} \tag{5.1.14}$$

[22] Note that the two-dimensional conformal symmetry for the open string only consists of one Virasoro algebra since the right-moving and left-moving sectors are linked. One can easily see this by repeating the arguments of Sec. 2.4.1 for open strings.

we see that in position space the field $A_a(x)$ is only supported on the $(p+1)$-dimensional surface with embedding in the 26-dimensional Minkowski space defined by $x^I = c^I$, $I = p+1, ..., 25$. Thus, one can say that the field lives on a $(p+1)$-dimensional space-time that is a subspace of the 26-dimensional Minkowski space. In this sense we see that the open string end points define a $(p+1)$-dimensional subspace with dynamical fields living on it. Below in Sec. 5.2 we interpret this $(p+1)$-dimensional surface as a special case of a worldvolume of a p-brane, the higher-dimensional dynamical objects that we defined in Chapter 4. With the predictions of certain types of particles living on the p-brane it is a special type p-brane called a Dp-brane, where D is for Dirichlet (named after the boundary conditions of the open string). As we shall see, a Dp-brane is defined by the open strings that end on it.

Going back to the $A_a(x)$ field defined by the Fourier transform (5.1.14) we see that the above conditions (5.1.12) become

$$\partial_a \partial^a A_b = 0 \ , \quad \partial^a A_a = 0 \tag{5.1.15}$$

whereas the gauge transformation (5.1.13) can be written

$$A_a \to A_a + \partial_a \chi \ , \quad \partial_a \partial^a \chi = 0 \tag{5.1.16}$$

We see that this is nothing but a $U(1)$ gauge boson in the Lorenz gauge $\partial_a A^a = 0$ living in $(p+1)$-dimensions. Indeed, we can define the corresponding field strength $F_{ab} = \partial_a A_b - \partial_b A_a$. This is clearly invariant under the gauge transformations (5.1.16). Moreover, we see that we write the remaining condition on A_a as $\partial^a F_{ab} = 0$ (in addition to the Lorentz gauge condition). We recognize this as the equations of motion for A_a since it is a source-free Maxwell equation.

Consider instead $N_\perp = 1$ and $N_\parallel = 0$. A general linear superposition is

$$\sum_{I=p+1}^{25} \Phi_I(k) \alpha_{-1}^I |0; k\rangle \tag{5.1.17}$$

The physical state conditions (5.1.7) give $k_a k^a = 0$ thus the state is a superposition of massless particles. Since the indices are transverse to the momentum k_a it is in fact $25-p$ scalar fields $\Phi_I(k)$, one for each value of $I = p+1, ..., 25$. Going to position space

$$\Phi_I(x) = \frac{1}{(2\pi)^{p+1}} \int d^{p+1}k \, \Phi_I(k) e^{ik_a x^a} \tag{5.1.18}$$

we see that for a given $I = p+1, ..., 25$ the field $\Phi_I(x)$ obeys the equation of motion $\partial_a \partial^a \Phi_I = 0$ which is the massless Klein-Gordon equation in $p+1$ dimensions. Below in

Sec. 5.2 these scalar fields will be interpreted in terms of the transverse position of the Dp-brane.

One can proceed with considering the infinite tower of massive states in the spectrum of open strings. As we mainly are interesting in low energy physics of the open strings we will not go through that. However we note below that even in the low energy limit the massive open string states become relevant in that if one consistently integrate them out in a low energy limit one can include their effect as higher order terms in l_s^2 for the above found equations of motion for $A_a(x)$ and $\Phi_I(x)$.

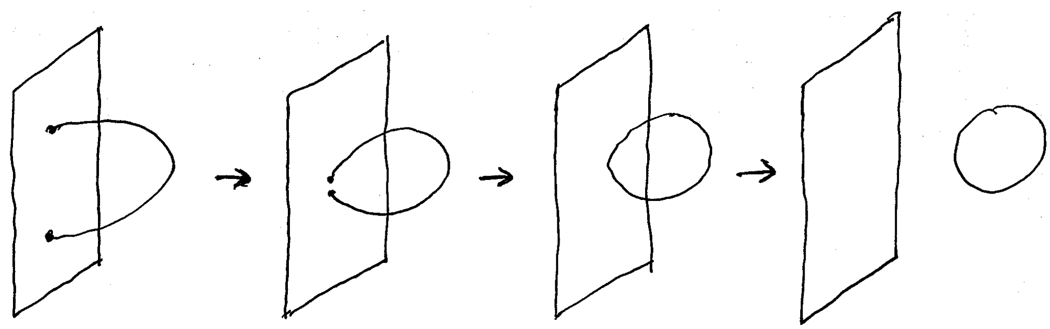

Figure 10: If one has a non-zero string coupling one can have a process where an open string ending on a D-brane can become a closed string which is sent off the D-brane and into the bulk.

One could wonder why closed string theory and open string theory should share the same critical dimension $D = 26$ for the target space. So far, the open string and closed string have seemed two completely separate systems. However, once one has a non-zero string coupling, which leads to the splitting and joining of closed strings, it is also possible with a process in which an open string that ends on a D-brane becomes a closed string by the joining of the two end points, and the closed string subsequently is send off into the bulk of the space-time. This process is illustrated in Figure 10. This process connects the open and closed string thus making them part of the same overall string theory. It furthermore gives a way for the closed string to interact with the D-brane.

Below in Section 5.2 we shall see how the above results for open strings imply the existence of Dp-branes. Building on that one can consider further the interpretation of the open string tachyon. One has done impressive research using open string field theory to reveal that the correct interpretation of the open string tachyon is that one sits at the top of a potential. Rolling down this potential one sheds away an energy that corresponds to the energy of a Dp-brane. The end state of this decay process is interpreted as the closed string vacuum (here meaning 26-dimensional Minkowski space), thus without the Dp-brane present. The open string tachyon is absent for the open superstring, as we shall see in Chapter 6.

5.2 D-brane dynamics

The infinitely thin relativistic p-brane considered in Chapter 4 with action (4.1.3) is an object without further structure than the embedding map (4.1.1). However, in string theory there is evidence for dynamical p-branes that are made out of other more fundamental constituents. In particular, the Dp-branes (short for Dirichlet p-branes) can be seen as being made out of open strings. They have a thickness of order the string length l_s. If we consider string theory at low energies energies $E \ll l_s^{-1}$ we have that a Dp-brane can be seen approximately as infinitely thin, but with the low energy excitations of the open string living on the Dp-brane.

Consider now our findings of Sec. 5.1 in this context. The $(p+1)$-dimensional worldvolume considered in Sec. 5.1 corresponds to the following special choice for the embedding map (4.1.1)

$$X^a(\xi) = \xi^a \quad \text{for} \quad a = 0, 1, ..., p \ , \quad X^I(\xi) = c^I \quad \text{for} \quad I = p+1, ..., D-1 \tag{5.2.1}$$

In this case we get the induced metric $\gamma_{ab} = \eta_{ab}$. The $U(1)$ gauge field $A_a(\xi)$ and $D-p-1$ scalar field $\Phi_I(\xi)$ (renaming here the worldvolume directions to ξ^a) of Sec. 5.1 is thus seen to live on the Dp-brane worldvolume. What action would provide the equations of motion that we found for A_a and Φ_I? Consider the action

$$S_{\mathrm{A}\Phi} = -\frac{1}{g^2} \int d^{p+1}\xi \left(\frac{1}{4} F_{ab} F^{ab} + \frac{1}{2} \sum_{I=p+1}^{D-1} \partial_a \Phi_I \partial^a \Phi_I \right) \tag{5.2.2}$$

with an appropriate normalization $1/g^2$. While this action provides the correct equations of motion for A_a and Φ_I it is not immediately clear how one should generalize this to an action that could work for other embeddings than the flat one (5.2.1), especially since somehow it should generalize the Dirac action (4.1.3). However, one finds that the general action is the so-called Dirac-Born-Infeld (DBI) action

$$S_{\mathrm{DBI}} = -T_{\mathrm{Dp}} \int d^{p+1}\xi \sqrt{-\det(\gamma_{ab} + 2\pi l_s^2 F_{ab})} \tag{5.2.3}$$

where γ_{ab} is the induced metric (4.1.2) and T_{Dp} is the tension of the Dp-brane that can be computed to be

$$T_{\mathrm{Dp}} = \frac{1}{(2\pi)^p g_s l_s^{p+1}} \tag{5.2.4}$$

in terms of the string coupling g_s and string length l_s. In the DBI action (5.2.3) we seem to have lost the $\Phi_I(\xi)$ fields but these are in fact included in this action as well. Namely, one can show that the $\Phi_I(\xi)$ fields can be interpreted as small deviations in the position of the brane around the flat embedding (5.2.1). More precisely, write

$$\begin{aligned} X^a(\xi) &= \xi^a \quad \text{for} \quad a = 0, 1, ..., p \\ X^I(\xi) &= c^I + 2\pi l_s^2 \Phi_I(\xi) \quad \text{for} \quad I = p+1, ..., D-1 \end{aligned} \tag{5.2.5}$$

Inserting this into the DBI action (5.2.3) one can make a low energy expansion in powers of the string length l_s. At order T_{Dp} we see that the DBI action reduces to a constant proportional to the infinite volume of the brane. The next non-zero contribution is at order $T_{\mathrm{Dp}} l_s^4$, giving precisely the action (5.2.2) with normalization $1/g^2 = (2\pi)^2 l_s^4 T_{\mathrm{Dp}}$. To derive the scalar part one uses that $\gamma_{ab} = \eta_{ab} + (2\pi)^2 l_s^4 \sum_{I=p+1}^{D-1} \partial_a \Phi_I \partial_b \Phi_I$.

Analyzing the embedding (5.2.5) we can conclude that the $D-p-1$ massless scalar particles that we found in Sec. 5.1 from the spectrum of the open string actually corresponds to the transverse position of the Dp-brane. This means that the position of the Dp-brane is a dynamical quantity. Since we are working in a regime with a very small, if not zero, string coupling g_s, one can see from (5.2.4) that T_{Dp} is very large, and hence the dynamics of the Dp-brane can be described classically to a good approximation.

We see also that the fields $A_a(\xi)$ and $\Phi_I(\xi)$ enter non-linearly in the action (5.2.3). This effect arises by considering the contribution of all the massive open string modes to the dynamics of the massless fields. This gives rise to the action as an effective theory at low energies for the full tower of open string states.

Considering the $A_a(\xi)$ field in particular we see that while the action (5.2.2) corresponds to a $(p+1)$-dimensional generalization of Maxwell's electromagnetism, the full action (5.2.3) corresponds to a non-linear version of electromagnetism. This type of non-linear electromagnetism theory was first written down by Born and Infeld in the 1930's (their action corresponds to $p+1 = D = 4$ and $\gamma_{ab} = \eta_{ab}$). Note that even if we are at low energies we can still see the non-linear electromagnetism in (5.2.3) by considering $l_s^2 F_{ab}$ to be of order one.

5.3 The open string for multiple D-branes

So far we have only considered the situation in which an open string ends on a single D-brane. We now consider what happens if there are more than one D-brane present in D-dimensional Minkowski space.

Consider the specific situation in which we have two parallel Dp-branes both with flat world-volumes, *i.e.* the first Dp-brane has embedding

$$X^a(\xi) = \xi^a \quad \text{for} \quad a = 0, 1, ..., p \ , \quad X^I(\xi) = c_1^I \quad \text{for} \quad I = p+1, ..., D-1 \tag{5.3.1}$$

and the second Dp-brane has embedding

$$X^a(\xi) = \xi^a \quad \text{for} \quad a = 0, 1, ..., p \ , \quad X^I(\xi) = c_2^I \quad \text{for} \quad I = p+1, ..., D-1 \tag{5.3.2}$$

From point of view of the open string, we now have four possible situations to analyze:

- **11-string**: The open string starts at Dp-brane 1 and ends at Dp-brane 1.
- **12-string**: The open string starts at Dp-brane 1 and ends at Dp-brane 2.
- **21-string**: The open string starts at Dp-brane 2 and ends at Dp-brane 1.
- **22-string**: The open string starts at Dp-brane 2 and ends at Dp-brane 2.

These four possibilities are illustrated in Figure 11. In terms of open string boundary conditions we see that in all four situations the open string has Neumann conditions at both end points in $(p+1)$-directions: $X'^a(\tau, 0) = X'^a(\tau, \pi) = 0$. In the remaining $D-p-1$ directions the open string has Dirichlet boundary conditions. For instance for the 12-string we have $X^I(\tau, 0) = c_1^I$ and $X^I(\tau, \pi) = c_2^I$ with $I = p+1, ..., D-1$. Classically, the 12-string has mode expansions given by (1.4.45), (1.4.46), (1.4.49) and (1.4.51).

Considering the quantum theory for the open string in the presence of the two Dp-branes (5.3.1) and(5.3.2) we see that we have a Fock space for each of the four possibilities. Thus, the full Fock space for the open string becomes a product space

$$F_{11} \otimes F_{12} \otimes F_{21} \otimes F_{22} \tag{5.3.3}$$

where F_{ij} is the Fock space for the ij-string. We introduce now a convenient notation that can capture the full Fock space. First of all, we introduce the ground states $|0; k; ij\rangle$, $i, j = 1, 2$, for the open string. These are the four ground states for the four separate Fock spaces in (5.3.3). The extra indices in the ground state are known as *Chan-Paton indices* for the open string. For the $p+1$ directions with Neumann boundary conditions at both end points we use the mode expansions (1.4.45) and (1.4.46) for $X^a(\xi)$ and $\Pi_a(\xi)$, $a = 0, 1, ..., p$. Instead for the $D-p-1$ directions with Dirichlet boundary conditions at both ends we use the slightly generalized mode expansions

$$\begin{aligned} X^I(\tau, \sigma) &= x^I - \sqrt{2} l_s \alpha_0^I \sigma - \sqrt{2} l_s \sum_{n \neq 0} \frac{1}{n} \alpha_n^I e^{-in\tau} \sin(n\sigma) \\ \Pi_I(\tau, \sigma) &= \frac{i\sqrt{2}}{2\pi l_s} \sum_{n \neq 0} \alpha_n^I e^{-in\tau} \sin(n\sigma) \end{aligned} \tag{5.3.4}$$

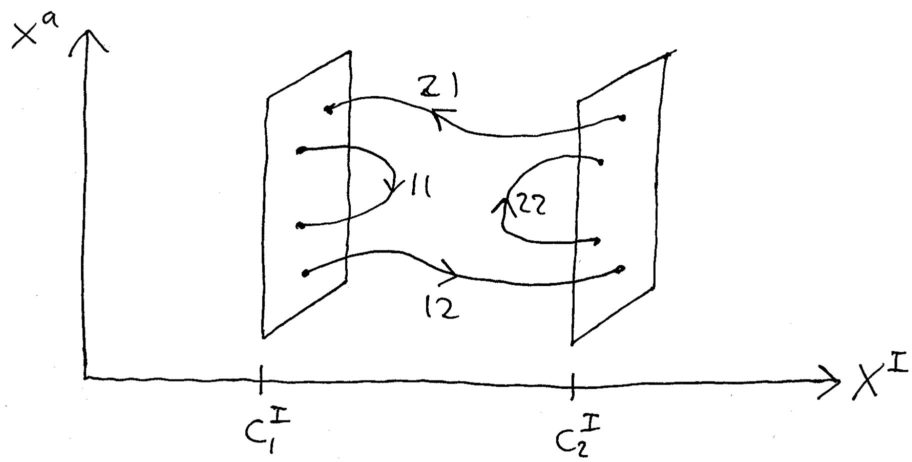

Figure 11: The four possibilities for open strings ending on two parallel Dp-branes with their worldvolumes parallel to the x^a directions, $a = 0, 1, ..., p$, and with their locations in the transverse directions given by $x^I = c_1^I$ and at $x^I = c_2^I$, $I = p+1, ..., D-1$, in D-dimensional Minkowski space. The arrows on the open strings specify the direction of increasing σ.

for $I = p+1, ..., D-1$. For the zero modes x^I and α_0^I we require

$$
\begin{aligned}
&\textbf{11-string}: \quad x^I|0;k;11\rangle = c_1^I|0;k;11\rangle \ , \quad \alpha_0^I|0;k;11\rangle = 0 \\
&\textbf{12-string}: \quad x^I|0;k;12\rangle = c_1^I|0;k;12\rangle \ , \quad \alpha_0^I|0;k;12\rangle = \tfrac{c_1^I - c_2^I}{\sqrt{2}\pi l_s}|0;k;12\rangle \\
&\textbf{21-string}: \quad x^I|0;k;21\rangle = c_2^I|0;k;21\rangle \ , \quad \alpha_0^I|0;k;21\rangle = \tfrac{c_2^I - c_1^I}{\sqrt{2}\pi l_s}|0;k;21\rangle \\
&\textbf{22-string}: \quad x^I|0;k;22\rangle = c_2^I|0;k;22\rangle \ , \quad \alpha_0^I|0;k;22\rangle = 0
\end{aligned}
\tag{5.3.5}
$$

for $I = p+1, ..., D-1$. Instead the other modes are quantized as

$$
\begin{gathered}
[x^a, p_b] = i\delta^a_b \ , \quad [x^a, x^b] = [p_a, p_b] = 0 \\
[\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+n,0}\eta^{\mu\nu}
\end{gathered}
\tag{5.3.6}
$$

with $a, b = 0, 1, ..., p$ and $\mu, \nu = 0, 1, ..., D-1$. The reality conditions are

$$
(x^a)^\dagger = x^a \ , \quad (p_a)^\dagger = p_a \ , \quad (\alpha_n^\mu)^\dagger = \alpha_{-n}^\mu \tag{5.3.7}
$$

The ground state $|0;k;ij\rangle$ with the Chan-Paton indices is defined for a given momentum k_a, $a = 0, 1, ..., p$, by

$$
p_a|0;k;ij\rangle = k_a|0;k;ij\rangle \ , \quad \alpha_n^\mu|0;k;ij\rangle = 0 \ \text{ for } \ n > 0 \tag{5.3.8}
$$

for $i, j = 1, 2$. A basis for the Fock space (5.3.3) is thus

$$
\alpha_{-n_1}^{\mu_1}\alpha_{-n_2}^{\mu_2}\cdots\alpha_{-n_q}^{\mu_q}|0;k;ij\rangle \tag{5.3.9}
$$

with $n_i > 0$ and $i, j = 1, 2$. With this we are setup to quantize the open string in the presence of two Dp-branes (5.3.1) and (5.3.2). As in Sec. 5.1, one implements the constraints as (5.1.7) in terms of the Virasoro generators (5.1.8). This is consistent only for $a = 1$ and $D = 26$. From the $n = 0$ constraint of (5.1.7) we see that

$$M^2|\phi\rangle = -p_a p^a|\phi\rangle = \frac{1}{l_s^2}\left(\frac{1}{2}\sum_{I=p+1}^{D-1}(\alpha_0^I)^2 + N_\parallel + N_\perp - 1\right)|\phi\rangle \tag{5.3.10}$$

with $N_\parallel$ and $N_\perp$ defined as in (5.1.10). Using (5.3.5) we see that the term with α_0^I gives the square of the mass

$$M_{12} = TL = \frac{|\vec{c}_2 - \vec{c}_1|}{2\pi l_s^2} \tag{5.3.11}$$

corresponding to the mass of a classical straight open string stretched between the two Dp-branes when either $ij = 12$ or $ij = 21$ with the length $L = |\vec{c}_2 - \vec{c}_1| = \sqrt{\sum_{I=p+1}^{D-1}(c_2^I - c_1^I)^2}$. In the cases $ij = 11$ and $ij = 22$ that term is instead zero.

Consider now moving the two Dp-brane to the same position $c_2^I = c_1^I$. Then the mass (5.3.11) goes to zero. In this case one gets extra massless particles on the brane from the $N_\parallel + N_\perp = 1$ level. Namely, one obtains massless particles not only from the 11-string and 22-string, as analyzed in Sec. 5.1, but also from the 12-string and 21-string.

Consider $N_\parallel = 1$ and $N_\perp = 0$. A general linear superposition is

$$\sum_{i,j=1}^{2} A_{a,ij}(k)\alpha_{-1}^a|0;k;ij\rangle \tag{5.3.12}$$

Such a state corresponds to a massless particle $k^a k_a = 0$ with $k^a A_{a,ij}(k) = 0$. Seen as a matrix $A_a(k)$ is required to be hermitian due to the reality condition on the open string. Above we saw that a single Dp-brane had a $U(1)$ gauge boson living on it. Now we see that this generalizes to a $U(2)$ gauge boson since the extra massless modes from the 12-string and 21-string corresponds to the enhancement from $U(1)^2$ to $U(2)$. Indeed, a $U(2)$ gauge boson is precisely characterized by being a hermitian 2 by 2 matrix corresponding to the adjoint representation of $U(2)$. Under a global gauge transformation we have the transformation $A_a \to UA_aU^{-1}$ with U a unitary matrix $UU^\dagger = I$. Since A_a is in the adjoint representation the indices i and j of $A_{a,ij}$ transforms in the fundamental and anti-fundamental representations, respectively, *i.e.*

$$A_{a,ij} \to \sum_{i',j'} U_{ii'}U_{jj'}^* A_{a,i'j'} = (UA_aU^\dagger)_{ij} \tag{5.3.13}$$

In this way we see that also the ground state $0;k;ij\rangle$ transforms in the adjoint representation of $U(2)$. Continuing along these lines one can show that the equations for $A_{a,ij}$

corresponds to a Yang-Mills theory with gauge group $U(2)$ living on the two coinciding Dp-branes. Thus, whereas for one Dp-brane we found photons, for two Dp-branes we find gluons. If one takes N coinciding Dp-branes one can show that the above generalizes to a $U(N)$ gauge boson for a Yang-Mills theory with gauge group $U(N)$.

Consider instead $N_\perp = 1$ and $N_\parallel = 0$. A general linear superposition is

$$\sum_{I=p+1}^{D-1} \sum_{i,j=1}^{2} \Phi_{I,ij}(k) \alpha_{-1}^I |0; k; ij\rangle \tag{5.3.14}$$

These states corresponds to massless particles $k^a k_a = 0$. The matrix $\Phi_I(k)$ should be hermitian. For a single Dp-brane, we found for each transverse direction $I = p+1, ..., 25$ in Sec. 5.1 a real scalar field $\Phi_I(k)$ living on the brane that corresponds to a small deviation in the transverse position of the brane. Now instead for two Dp-branes, $\Phi_I(k)$ is a hermitian 2 by 2 matrix. What is the interpretation of this? Clearly, in case Φ_I is diagonal we can interpret $\Phi_{I,11}$ and $\Phi_{I,22}$ as the position of Dp-brane 1 and 2, respectively. However, in the general case, we see that Φ_I must have the interpretation of a generalized type of position in which the position is a matrix. Since two matrices do not commute in general, it means that such a generalized type of position corresponds to a non-commutative geometry. Thus, we find that multiple coinciding Dp-brane can realize a non-commutative geometry.

Taking into account the massive open string states in a low energy limit one should again be able to find an effective action that includes non-linear effects for both the $U(2)$ gauge boson and the 2 by 2 hermitian matrix position field, thus generalizing the DBI action (5.2.3) to the case of two coinciding Dp-branes. At present it is not fully understood what this effective action is.[23]

5.4 Exercises for Chapter 5

Exercise 5.1. Consider the classical open string in the setting of Sec. 1.4.4 (for general c_1^I and c_2^I).

- The worldsheet energy-momentum tensor in flat gauge is (1.5.3). Using this, derive that the worldsheet energy-momentum tensor has the mode expansion given by (5.1.2) in terms of the mode expansion of $X^\mu(\tau, \sigma)$. Note that α_0^a and α_0^I are given by (1.4.47) and (1.4.50).

[23]Unlike for a single Dp-brane one has not been able to derive the effective action from first principles, *i.e.* by integrating out the massive open string degrees of freedom in a low energy regime. However, there is a proposal for it that has passed many test.

- Argue that the constraint (1.3.7) in flat gauge is equivalent to $L_n = 0$ for all $n \in \mathbb{Z}$ for the open string.

Exercise 5.2. In this exercise we consider the conformal symmetry of the open string.

- Let $X^\mu(\xi)$ be any solution (1.4.8) to the equation of motion (1.4.7) (not imposing closed or open string boundary conditions). Consider a general conformal transformation on the worldsheet

$$\xi^\pm \to \tilde{\xi}^\pm(\xi^\pm) \tag{5.4.1}$$

  How do $\partial_- X$ and $\partial_+ X$ transform under such a general transformation?

- Consider now $X^\mu(\xi)$ with the open string boundary conditions of Sec. 1.4.4 imposed. Argue that the conformal transformations that preserve the open string boundary conditions are of the form

$$\xi^- \to \tilde{\xi}^- = u(\xi^-) \ , \quad \xi^+ \to \tilde{\xi}^+ = u(\xi^+) \tag{5.4.2}$$

  for a one-variable function u that is periodic with period 2π.

- In the context of Sec. 2.4.1, deduce from (5.4.2) that the subset of the infinitesimal conformal transformations (2.4.1) that preserve open string boundary conditions have $A_n^+ = A_n^-$. Moreover, show that the infinitesimal conformal transformation

$$\xi^- \to \xi^- - A_n e^{in\xi^-} \ , \quad \xi^+ \to \xi^+ - A_n e^{in\xi^+} \tag{5.4.3}$$

  corresponds to acting on the state with the operator $e^{-iA_n D_n}$ where the generator is given by

$$D_n = ie^{in\xi^-}\partial_- + ie^{in\xi^+}\partial_+ \tag{5.4.4}$$

  Finally, show that the algebra of the generators D_n, $n \in \mathbb{Z}$, is the Virasoro algebra

$$[D_m, D_n] = (m-n)D_{m+n} \tag{5.4.5}$$

  Hence, one can conclude that this is the conformal symmetry algebra that is consistent with open string boundary conditions.

Exercise 5.3. Consider the covariantly quantized open string in the setup of Sec. 5.1.

- Given the Virasoro generators L_n defined in (5.1.8), argue using the results of Exercise 2.7, 2.8 and 2.9 that they have the algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12}(m^3 - m)\delta_{m+n,0} \tag{5.4.6}$$

being the same extended Virasoro algebra as the right-moving sector of the closed string (2.4.4). Note that no new computations should be needed, it should be enough to compare to the Virasoro generators of the right-moving sector of the closed string (however, pay attention to the zero modes).

- The ghost field sector that arises from the gauge fixing of the Polyakov action is the same as found in Sec. 2.5. However, the open string boundary conditions means that the left and right-moving sectors are identified also in the ghost sector. Hence one can show that there is only one set of Virasoro generators $L_n^{(gh)}$, and that they have the algebra

$$[L_m^{(\mathrm{gh})}, L_n^{(\mathrm{gh})}] = (m-n)L_{m+n}^{(\mathrm{gh})} + \frac{1}{12}(-26m^3 + 2m)\delta_{m+n,0} \tag{5.4.7}$$

being the same extended Virasoro algebra as the right-moving sector of the closed string (2.5.8). Writing the total Virasoro generator as $\mathcal{L}_n = L_n + L_n^{(gh)} - a\delta_{n,0}$, argue that one avoids the anomaly in the conformal symmetry for the covariantly quantized open string of Sec. 5.1, if and only if $a = 1$ and $D = 26$.

Exercise 5.4. We consider here the $U(1)$ gauge boson derived from the spectrum of the open string in the setting of Sec. 5.1.

- Derive the general formula (5.1.9) with (5.1.10) for the mass squared spectrum of the open string.

- For $N_\parallel = 1$ and $N_\perp = 0$, derive that the most general state is of the form (5.1.11).

- Show that the physical state conditions (5.1.7) gives the conditions (5.1.12).

- By following the same steps as in Exercise 2.3, one can show in general that a state of the form $|\chi\rangle = L_{-1}|\eta\rangle$ with $L_n|\eta\rangle = 0$ for $n \geq 0$, is spurious (for the covariantly quantized open string in the setting of Sec. 5.1). Use this fact to show that the state $k_a \alpha_{-1}^a|0;k\rangle$ with $k_a k^a = 0$ is spurious (thus using the same strategy as employed in Exercise 2.4).

- Show that the field $A_a(k)$ has the gauge transformation (5.1.13).

- Using the Fourier transform (5.1.14), show that $A_a(x)$ should obey (5.1.15) and have the gauge transformation (5.1.16). Argue along the lines described below Eq. (5.1.16) that $A_a(x)$ is a $U(1)$ gauge boson in Lorentz gauge .

6 Superstring Theory

In this chapter we introduce superstrings. A superstring is a supersymmetric version of the bosonic string that we have been considering in Chapters 1 to 5.

Supersymmetry (SUSY) is a symmetry that can transform bosons into fermions and vice versa. It can either be a global or a local symmetry. Global SUSY - *i.e.* when SUSY is a global symmetry - is relevant for theories defined on a fixed background. It is for example relevant in particle physics since here one is dealing with theories defined in Minkowski space with Poincare invariance. Indeed, when one thinks of extensions of the Poincare symmetry of the S-matrix in particle physics there is a theorem of Haag, Sohnius and Lopuszanski saying that SUSY is the only possible extension of the Poincare symmetry of the S-matrix of a relativistic Quantum Field Theory. Global SUSY is also relevant as an extension of the conformal symmetry of Conformal Field Theories (CFT's). Such theories are known as Superconformal Field Theories (SCFT's). Instead local SUSY - *i.e.* when SUSY is a local symmetry - implies that one has a theory with a dynamical metric. Theories with local SUSY where the bosonic part describes gravity are known as supergravity (SUGRA) theories.

6.1 Action and constraints

In the following we consider closed and open superstrings moving in the background of a D-dimensional Minkowski space with metric $\eta_{\mu\nu}$.

We begin by considering the Polyakov action in flat gauge (1.3.18). We would like to make this into a theory with global SUSY on the two-dimensional world-sheet. This can be achieved by adding a kinetic term with spinors living on the world-sheet resulting in the following action

$$S = -\frac{T}{2}\int d^2\xi\Big(\eta^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu - i\bar{\psi}^\mu\rho^\alpha\partial_\alpha\psi_\mu\Big) \tag{6.1.1}$$

Here we have defined the two-dimensional Dirac matrices

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{6.1.2}$$

which obey[24]

$$\{\rho^\alpha, \rho^\beta\} = -2\eta^{\alpha\beta} \tag{6.1.3}$$

[24]The anticommutator is defined as $\{x, y\} = xy + yx$.

In general, spinors on the worldsheet can be chosen to be real and with two components

$$\chi = \begin{pmatrix} \chi_- \\ \chi_+ \end{pmatrix} \tag{6.1.4}$$

One defines the *bar* of a spinor χ as $\bar{\chi} = \chi^T \rho^0$. The chirality operator is

$$\rho_0 \rho_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{6.1.5}$$

From this we see that the two components of a world-sheet spinor correspond to the two chirality states. Accordingly, for each $\mu = 0, 1, ..., D-1$ one has that ψ^μ is a real world-sheet spinor field on the world-sheet of the form

$$\psi^\mu = \begin{pmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{pmatrix} \tag{6.1.6}$$

Since $\psi^\mu(\xi)$ should correspond to a fermion on the worldsheet it is a *Grassmann-valued field* which means that $\psi^\mu_\pm(\xi)$ is a *Grassmann-valued number* for any given ξ. A Grassmann-valued number is defined as an anticommuting number. For instance for two Grassmann-valued numbers x and y we have $xy = -yx$. Thus, $x^2 = y^2 = 0$. Using the anticommutator we can write $\{x, y\} = 0$. This is in contrast with the ordinary c-numbers, meaning commuting numbers, since any two c-numbers a and b (real, integer, complex, etc.) have $[a, b] = 0$. We remark that combining two Grassmann-valued numbers x and y as xy gives a c-number and that Grassmann-valued numbers commutes with c-numbers.[25]

One can now check that the action (6.1.1) is invariant under the global world-sheet SUSY transformation

$$\delta X^\mu = \bar{\epsilon}\psi^\mu, \quad \delta\psi^\mu = -i\rho^\alpha \partial_\alpha X^\mu \epsilon \tag{6.1.7}$$

where ϵ is a constant real world-sheet spinor (which also implies that it is Grassmann-valued).

Just like one can write down the Polyakov action for the bosonic string for any world-sheet metric $g_{\alpha\beta}$ one can also find an action $S^{(gen)}[g, \zeta, X, \psi]$ generalising (6.1.1) to any world-sheet metric $g_{\alpha\beta}$ (strictly speaking one should use a zweibein field). To do this, one has to supplement the worldsheet metric (zweibein) with its superpartner field ζ_α called

[25] The reason for introducing Grassmann-valued numbers for fermionic fields is that while the classical limit $\hbar \to 0$ of a harmonic oscillator algebra $[a, a^\dagger] = \hbar$ gives commuting numbers $[a, a^\dagger] = 0$, the classical limit $\hbar \to 0$ of a fermionic harmonic oscillator algebra $\{f, f^\dagger\} = \hbar$ gives anticommuting numbers $\{f, f^\dagger\} = 0$. See Exercise 6.4 for more on the fermionic harmonic oscillator.

the gravitino which is a spinor field for each α. Beyond the global symmetry of Poincare invariance this action $S^{(gen)}[g, \zeta, X, \psi]$ has the local symmetries of diffeomorphism invariance, Weyl rescaling invariance and furthermore also local SUSY. Since this action has diffeomorphism and Weyl rescaling invariance we can make the gauge choice of the flat gauge for the metric $g_{\alpha\beta} = \eta_{\alpha\beta}$ and gravitino $\zeta_\alpha = 0$ and the action then reduces to (6.1.1). This also reduces the local SUSY into the global SUSY (6.1.7). Then the remaining local symmetries preserving this gauge makes the theory a two-dimensional theory with super-conformal invariance.

While the equations of motion for $g_{\alpha\beta}$ gives that the worldsheet energy-momentum tensor is zero $T_{\alpha\beta} = 0$, the equations of motion for the gravitino ζ_α gives that the supercurrent is zero J_α. Hence, when choosing the flat gauge $g_{\alpha\beta} = \eta_{\alpha\beta}$ and $\zeta_\alpha = 0$ we need to impose the equations of motions of $g_{\alpha\beta}$ and ζ_α as constraints. Thus, in addition to the equations of motion that one derives from the action (6.1.1) one needs also to impose the constraints

$$T_{\alpha\beta} = 0 \ , \quad J_\alpha = 0 \tag{6.1.8}$$

The energy-momentum tensor (in flat gauge) is

$$T_{\alpha\beta} = \frac{1}{l_s^2}\left[\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}\eta_{\alpha\beta}\eta^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X_\mu - \frac{i}{4}\bar{\psi}^\mu(\rho_\alpha\partial_\beta + \rho_\beta\partial_\alpha - \eta_{\alpha\beta}\rho^\gamma\partial_\gamma)\psi_\mu\right] \tag{6.1.9}$$

In lightcone coordinates $\xi^\pm = \tau \pm \sigma$ we have $T_{+-} = 0$ (since it is conformally invariant) and

$$l_s^2 T_{--} = \partial_- X \cdot \partial_- X + \frac{i}{2}\psi_- \cdot \partial_- \psi_-, \quad l_s^2 T_{++} = \partial_+ X \cdot \partial_+ X + \frac{i}{2}\psi_+ \cdot \partial_+ \psi_+ \tag{6.1.10}$$

The supercurrent (in flat gauge) is

$$J_\alpha = -\frac{1}{2l_s^2}\rho^\beta \rho_\alpha \psi^\mu \partial_\beta X_\mu \tag{6.1.11}$$

In lightcone coordinates we have

$$J_- = \begin{pmatrix} J_{--} \\ 0 \end{pmatrix}, \quad J_+ = \begin{pmatrix} 0 \\ J_{++} \end{pmatrix} \tag{6.1.12}$$

with

$$J_{--} = \frac{1}{l_s^2}\psi_- \cdot \partial_- X, \quad J_{++} = \frac{1}{l_s^2}\psi_+ \cdot \partial_+ X \tag{6.1.13}$$

Thus we can write the constraints as

$$T_{--} = T_{++} = 0 \ , \quad J_{--} = J_{++} = 0 \tag{6.1.14}$$

6.2 Equations of motion and mode expansions

We can write the superstring action (6.1.1) in lightcone coordinates $\xi^\pm = \tau \pm \sigma$ as

$$S = T \int d^2\xi \Big(2\partial_- X \cdot \partial_+ X + i\psi_- \cdot \partial_+ \psi_- + i\psi_+ \cdot \partial_- \psi_+ \Big) \tag{6.2.1}$$

Impose now that σ is defined in the range $0 \leq \sigma \leq \bar{\sigma}$ (below we impose $\bar{\sigma} = 2\pi$ for a closed string and $\bar{\sigma} = \pi$ for an open strings). The variation of the action with respect to ψ^μ then gives

$$\delta S = -2iT \int_{-\infty}^{\infty} d\tau \int_0^{\bar{\sigma}} d\sigma \Big(\partial_+ \psi_- \cdot \delta\psi_- + \partial_- \psi_+ \cdot \delta\psi_+ \Big) + i\frac{T}{2} \int_{-\infty}^{\infty} d\tau \Big[\psi_- \cdot \delta\psi_- - \psi_+ \cdot \delta\psi_+ \Big]_{\sigma=0}^{\bar{\sigma}} \tag{6.2.2}$$

From this we read off the EOM's

$$\partial_+ \psi_-^\mu = 0, \quad \partial_- \psi_+^\mu = 0 \tag{6.2.3}$$

Clearly the EOM's imply that ψ_-^μ is a function only of ξ^- and that ψ_+^μ is a function only of ξ^+.

6.2.1 The closed superstring

We consider here the classical closed superstring theory that leads to the so-called type IIA and IIB string theories when quantized (see Sec. 6.6 for more on this). See instead Sec. 8.1 for a different type of closed superstring theories known as Heterotic string theories.

As we encountered in bosonic string theory a closed string has $X^\mu(\xi)$ being a periodic function of σ with period 2π. This means we should choose $\bar{\sigma} = 2\pi$ in (6.2.2). As in bosonic string theory $X^\mu(\xi)$ has the mode expansion

$$X^\mu(\xi) = x^\mu + l_s^2 p^\mu \tau + \frac{i l_s}{\sqrt{2}} \sum_{n \neq 0} \frac{1}{n} (\alpha_n^\mu e^{-in\xi^-} + \tilde{\alpha}_n^\mu e^{-in\xi^+}) \tag{6.2.4}$$

with $l_s p^\mu = \sqrt{2}\alpha_0^\mu = \sqrt{2}\tilde{\alpha}_0^\mu$.

For the ψ_-^μ and ψ_+^μ fields, we have two different choices for periodicity conditions: The Ramond(R) condition $\psi_\pm^\mu(\tau, \sigma + 2\pi) = \psi_\pm^\mu(\tau, \sigma)$ and the Neveu-Schwarz(NS) condition $\psi_\pm^\mu(\tau, \sigma + 2\pi) = -\psi_\pm^\mu(\tau, \sigma)$. Combining the right and left-moving sectors this means we have four possible periodicity conditions on $\psi^\mu(\tau, \sigma)$: R-R, R-NS, NS-R and NS-NS, meaning that for example R-NS is the case with the R periodicity condition on ψ_-^μ and the NS periodicity condition on ψ_+^μ. It is easy to see that the boundary term in the variation of the action (6.2.2) vanishes in all four cases.

Consider the right-moving sector. With the R condition, we have that ψ_-^μ is periodic in σ with period 2π. This means that ψ_-^μ is a periodic function of ξ^- with period 2π. With the NS condition, we have instead that ψ_-^μ is antiperiodic in σ with period 2π, so ψ_-^μ is a antiperiodic function of ξ^- with period 2π. Thus, for the R and NS conditions we find the mode expansions

$$\text{R:} \;\; \psi_-^\mu(\xi^-) = l_s \sum_{n\in\mathbb{Z}} d_n^\mu e^{-in\xi^-} \;\; , \;\; \text{NS}: \;\; \psi_-^\mu(\xi^-) = l_s \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-ir\xi^-} \tag{6.2.5}$$

where the set $\mathbb{Z}+\frac{1}{2}$ means numbers of the form $n+\frac{1}{2}$ with n being an integer. Turning to the left-moving sector we find in the same way the mode expansions

$$\text{R:} \;\; \psi_+^\mu(\xi^+) = l_s \sum_{n\in\mathbb{Z}} \tilde{d}_n^\mu e^{-in\xi^+} \;\; , \;\; \text{NS}: \;\; \psi_+^\mu(\xi^+) = l_s \sum_{r\in\mathbb{Z}+\frac{1}{2}} \tilde{b}_r^\mu e^{-ir\xi^+} \tag{6.2.6}$$

Regarding the mode expansions of the energy-momentum we have

$$T_{--} = \sum_{n\in\mathbb{Z}} L_n e^{-in\xi^-} \;\; , \;\; T_{++} = \sum_{n\in Z} \tilde{L}_n e^{-in\xi^+} \tag{6.2.7}$$

with $L_n = L_n^{(X)} + L_n^{(\psi)}$ and $\tilde{L}_n = \tilde{L}_n^{(X)} + \tilde{L}_n^{(\psi)}$ where the bosonic part is

$$L_n^{(X)} = \frac{1}{2}\sum_{k\in\mathbb{Z}} \alpha_{n-k}\cdot\alpha_k \;\; , \;\; \tilde{L}_n^{(X)} = \frac{1}{2}\sum_{k\in\mathbb{Z}} \tilde{\alpha}_{n-k}\cdot\tilde{\alpha}_k \tag{6.2.8}$$

and the fermionic part depends on which of the four possible conditions we impose on the fermions. Consider the right-moving sector. For the R and NS conditions on ψ_-^μ we find

$$\text{R:} \;\; L_n^{(\psi)} = \frac{1}{2}\sum_{k\in\mathbb{Z}} \big(k+\frac{1}{2}n\big) d_{-k}\cdot d_{n+k} \;\; , \;\; \text{NS:} \;\; L_n^{(\psi)} = \frac{1}{2}\sum_{r\in\mathbb{Z}+\frac{1}{2}} \big(r+\frac{1}{2}n\big) b_{-r}\cdot b_{n+r} \tag{6.2.9}$$

For the left-moving sectors it works analogously.

Considering the right-moving part J_{--} of the supercurrent we have the following mode expansion in case of the R condition on ψ_-^μ

$$\text{R:} \;\; J_{--} = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} F_n e^{-in\xi^-} \tag{6.2.10}$$

and the following mode expansion in case of the NS condition

$$\text{NS:} \;\; J_{--} = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+\frac{1}{2}} G_r e^{-ir\xi^-} \tag{6.2.11}$$

and analogously for the left-moving sector J_{++} (now with the names $\tilde{F}_n$ and $\tilde{G}_r$ for the modes). We find

$$F_n = \sum_{k\in\mathbb{Z}} \alpha_{-k}\cdot d_{n+k} \;\; , \;\; G_r = \sum_{k\in\mathbb{Z}} \alpha_{-k}\cdot b_{r+k} \tag{6.2.12}$$

with $n \in \mathbb{Z}$ and $r \in \mathbb{Z} + \frac{1}{2}$ and analogously for the left-moving sector.

We can thus formulate the constraints (6.1.14) as

$$\begin{aligned} &\text{R:} \;\; L_n = F_n = 0 \;\; \text{for} \;\; n \in \mathbb{Z} \; , \;\;\; \text{NS:} \;\; L_n = G_r = 0 \;\; \text{for} \;\; n \in \mathbb{Z}, \; r \in \mathbb{Z} + \tfrac{1}{2} \\ &\text{R:} \;\; \tilde{L}_n = \tilde{F}_n = 0 \;\; \text{for} \;\; n \in \mathbb{Z} \; , \;\;\; \text{NS:} \;\; \tilde{L}_n = \tilde{G}_r = 0 \;\; \text{for} \;\; n \in \mathbb{Z}, \; r \in \mathbb{Z} + \tfrac{1}{2} \end{aligned} \tag{6.2.13}$$

with the R and NS conditions for the right and left-moving sectors.

The conserved current Π^α_μ for translation invariance is the same as for the bosonic string (1.4.15) along with the definition of the momentum density (1.4.18) as the world-sheet time-component of the current (1.4.17). The conserved current for Lorentz invariance is

$$\mathcal{J}^\alpha_{\mu\nu} = X_\mu \Pi^\alpha_\nu - X_\nu \Pi^\alpha_\mu - iT\bar{\psi}_\mu \rho^\alpha \psi_\nu \tag{6.2.14}$$

Using the EOMs one finds indeed that $\partial_\alpha \mathcal{J}^\alpha_{\mu\nu} = 0$. The associated conserved charges are

$$J_{\mu\nu} = \int_0^{2\pi} d\sigma \mathcal{J}^\tau_{\mu\nu} = \int_0^{2\pi} d\sigma (X_\mu \Pi_\nu - X_\nu \Pi_\mu - iT\bar{\psi}_\mu \rho^0 \psi_\nu) \tag{6.2.15}$$

generalizing the Lorentz charges (1.4.34) for the bosonic string.

6.2.2 The open superstring

Open superstrings are relevant in the presence of D-branes. Moreover, they are present in the so-called Type I string theory as reviewed in Sec. 8.2.

We take the range of σ to be $0 \leq \sigma \leq \pi$ hence $\bar{\sigma} = \pi$ in the variation of the action (6.2.2). For the bosonic part of the string we have two possible boundary conditions on each of the end points: The Neumann condition $X'^\mu = 0$ and the Dirichlet condition $\dot{X}^\mu = 0$. Consider now the boundary term in (6.2.2). We see the boundary conditions are

$$(\psi_- \cdot \delta\psi_- - \psi_+ \cdot \delta\psi_+)|_{\sigma=0} = 0 \; , \quad (\psi_- \cdot \delta\psi_- - \psi_+ \cdot \delta\psi_+)|_{\sigma=\pi} = 0 \tag{6.2.16}$$

No matter which condition we impose on X^μ at the end points we see that for $\psi^\mu_\pm$ we can satisfy these boundary conditions by relating ψ^μ_+ to ψ^μ_- on the end points, specifically by having them equal up to a sign. Without loss of generality we can choose $\psi^\mu_-(\tau, 0) = \psi^\mu_+(\tau, 0)$ on the $\sigma = 0$ end point.[26] This means we have two possible boundary conditions for the other end points at $\sigma = \pi$: The Ramond (R) condition $\psi^\mu_-(\tau, \pi) = \psi^\mu_+(\tau, \pi)$ and the Neveu-Schwarz (NS) condition $\psi^\mu_-(\tau, \pi) = -\psi^\mu_+(\tau, \pi)$.

[26]Generally one can satisfy the boundary condition at $\sigma = 0$ by $\psi^\mu_-(\tau, 0) = \pm\psi^\mu_+(\tau, 0)$ but since the action is invariant under $\psi^\mu_- \to -\psi^\mu_-$ one can always transform this to $\psi^\mu_-(\tau, 0) = \psi^\mu_+(\tau, 0)$.

With the R condition we have[27] $\psi^\mu_-(\tau) = \psi^\mu_+(\tau)$ and $\psi^\mu_-(\tau-\pi) = \psi^\mu_+(\tau+\pi)$ so ψ^μ_- is a periodic function of ξ^- with period 2π, and we have

$$\text{R:}\quad \psi^\mu_-(\xi^-) = l_s \sum_{n\in\mathbb{Z}} d^\mu_n e^{-in\xi^-} \ , \quad \psi^\mu_+(\xi^+) = l_s \sum_{n\in\mathbb{Z}} d^\mu_n e^{-in\xi^+} \tag{6.2.17}$$

where we used $\psi^\mu_-(\tau,0) = \psi^\mu_+(\tau,0)$ to infer the mode expansion for ψ^μ_+ as well. With the NS condition we have that $\psi^\mu_-(\tau) = \psi^\mu_+(\tau)$ and $\psi^\mu_-(\tau-\pi) = -\psi^\mu_+(\tau+\pi)$ so ψ^μ_- is an antiperiodic function of ξ^- with period 2π, and we have

$$\text{NS:}\quad \psi^\mu_-(\xi^-) = l_s \sum_{r\in\mathbb{Z}+\frac{1}{2}} b^\mu_r e^{-ir\xi^-} \ , \quad \psi^\mu_+(\xi^+) = l_s \sum_{r\in\mathbb{Z}+\frac{1}{2}} b^\mu_r e^{-ir\xi^+} \tag{6.2.18}$$

where we used again $\psi^\mu_-(\tau,0) = \psi^\mu_+(\tau,0)$ to infer the mode expansion for ψ^μ_+ as well.

The mode expansions for the energy-momentum and supercurrent works the same way as for the closed superstring.

6.3 Covariant quantization of the closed superstring

We consider here the covariant quantization of the closed superstring. Note that we consider here only the Type II string theories. See Chapter 8 for other types of superstring theories.

6.3.1 Covariant quantization

We now quantize the closed superstring using the covariant quantization approach. For the bosonic part we should impose the equal-time commutator relation $[X^\mu(\tau,\sigma), \Pi_\nu(\tau,\sigma')] = i\delta(\sigma-\sigma')\delta^\mu_\nu$ which gives the commutator relations

$$[x^\mu, p_\nu] = i\delta^\mu_\nu \ , \quad [\alpha^\mu_m, \alpha^\nu_n] = [\tilde\alpha^\mu_m, \tilde\alpha^\nu_n] = m\delta_{m+n,0}\eta^{\mu\nu} \tag{6.3.1}$$

along with the reality conditions $(x^\mu)^\dagger = x^\mu$, $(p_\mu)^\dagger = p_\mu$, $(\alpha^\mu_n)^\dagger = \alpha^\mu_{-n}$ and $(\tilde\alpha^\mu_n)^\dagger = \tilde\alpha^\mu_{-n}$. For the fermionic part we should impose the equal-time anti-commutator

$$\{\psi^\mu_A(\tau,\sigma), \psi^\nu_B(\tau,\sigma')\} = 2\pi l_s^2 \delta(\sigma-\sigma')\delta_{AB}\eta^{\mu\nu} \tag{6.3.2}$$

with $A,B = +,-$ and $\mu,\nu = 0,1,...,D-1$.[28] The fermionic field commutes with the bosonic fields. Considering the right-moving sector we get for the R and NS periodicity conditions

$$\text{R:}\quad \{d^\mu_m, d^\nu_n\} = \eta^{\mu\nu}\delta_{m+n,0} \ , \quad \text{NS:}\quad \{b^\mu_r, b^\nu_s\} = \eta^{\mu\nu}\delta_{r+s,0} \tag{6.3.3}$$

[27] Here we write ψ^μ_- as a function of one variable since it is only a function of ξ^-. The same thing goes for ψ^μ_+.

[28] the δ_{AB} for $A,B = +,-$ is defined by $\delta_{++} = \delta_{--} = 1$ and $\delta_{+-} = \delta_{-+} = 0$.

with $m, n \in \mathbb{Z}$ and $r, s \in \mathbb{Z} + \frac{1}{2}$. Similarly for the left-moving sector

$$\text{R: } \{\tilde{d}^\mu_m, \tilde{d}^\nu_n\} = \eta^{\mu\nu}\delta_{m+n,0} \ , \quad \text{NS: } \{\tilde{b}^\mu_r, \tilde{b}^\nu_s\} = \eta^{\mu\nu}\delta_{r+s,0} \tag{6.3.4}$$

We also impose the reality condition $(\psi^\mu_A(\tau,\sigma))^\dagger = \psi^\mu_A(\tau,\sigma)$ which gives $(d^\mu_n)^\dagger = d^\mu_{-n}$, $(b^\mu_r)^\dagger = b^\mu_{-r}$, $(\tilde{d}^\mu_n)^\dagger = \tilde{d}^\mu_{-n}$ and $(\tilde{b}^\mu_r)^\dagger = \tilde{b}^\mu_{-r}$.

In Section 2.1 we made a harmonic oscillator interpretation of modes of the closed bosonic string. For the right-moving modes α^μ_n one can write the commutation relation as (2.1.6) by defining (2.1.5), and it works analogously for the left-moving sector. Thus, α^μ_n and $\tilde{\alpha}^\mu_n$ with $n > 0$ are annihilation operators and $\alpha^\mu_n$and $\tilde{\alpha}^\mu_n$ with $n < 0$ are creation operators. This holds for the bosonic modes of the closed superstring as well.

Similarly, we can interpret the fermionic modes in terms of a fermionic harmonic oscillator. A fermionic harmonic oscillator, with f as the annihilation operator and $f^\dagger$ the creation operator, has the algebra

$$\{f, f\} = 0 \ , \quad \{f^\dagger, f^\dagger\} = 0 \ , \quad \{f, f^\dagger\} = 1 \tag{6.3.5}$$

A ground state $|0\rangle$ has $f|0\rangle = 0$ so that the only excited state of this fermionic harmonic oscillator is $f^\dagger|0\rangle$. In analogy with this, we interpret the modes for the R periodicity condition in the right- and left-moving sectors d^μ_n, $\tilde{d}^\mu_n$ operators with $n > 0$ as annihilation operators and those with $n < 0$ as creation operators, and the modes for the NS periodicity condition in the right- and left-moving sectors b^μ_r, $\tilde{b}^\mu_r$ operators with $r > 0$ as annihilation operators and those with $r < 0$ as creation operators.

Thus, if one is in the R-NS sector of the closed superstring, one has the R periodicity condition on the right-moving sector and the NS periodicity condition for the left-moving sector. In this case the fermionic annihilation operators are d^μ_n and $\tilde{b}^\mu_r$ with $n > 0$ and $r > 0$ and the fermionic creation operators are d^μ_{-n} and $\tilde{b}^\mu_{-r}$ with $n > 0$ and $r > 0$. Using these creation operators, as well as the bosonic ones, one can build up the closed superstring Fock space from a given ground state.

Since we now know which operators are creation operators, and which are annihilation operators, the natural next step will be to define the ground state of the superstring. This turns out to be quite involved, as the ground state in some sectors of the superstring is degenerate. We address this in Section 6.5.

6.3.2 Constraints on physical states

Just like in the case of the bosonic string the physical states should be found by imposing a quantum version of the classical constraints (6.2.13). In the quantum theory we have the Virasoro generators

$$L_n = L_n^{(X)} + L_n^{(\psi)} \ , \quad \tilde{L}_n = \tilde{L}_n^{(X)} + \tilde{L}_n^{(\psi)} \tag{6.3.6}$$

with bosonic part given by

$$L_n^{(X)} = \frac{1}{2}\sum_{k\in\mathbb{Z}} : \alpha_{n-k}\cdot\alpha_k : \ , \quad \tilde{L}_n^{(X)} = \frac{1}{2}\sum_{k\in\mathbb{Z}} : \tilde{\alpha}_{n-k}\cdot\tilde{\alpha}_k : \tag{6.3.7}$$

and the fermionic part by

$$\text{R:} \ L_n^{(\psi)} = \frac{1}{2}\sum_{k\in\mathbb{Z}} \big(k+\frac{1}{2}n\big) : d_{-k}\cdot d_{n+k} : , \ \ \text{NS:} \ L_n^{(\psi)} = \frac{1}{2}\sum_{r\in\mathbb{Z}+\frac{1}{2}} \big(r+\frac{1}{2}n\big) : b_{-r}\cdot b_{n+r} : \tag{6.3.8}$$

$$\text{R:} \ \tilde{L}_n^{(\psi)} = \frac{1}{2}\sum_{k\in\mathbb{Z}} \big(k+\frac{1}{2}n\big) : \tilde{d}_{-k}\cdot \tilde{d}_{n+k} : , \ \ \text{NS:} \ \tilde{L}_n^{(\psi)} = \frac{1}{2}\sum_{r\in\mathbb{Z}+\frac{1}{2}} \big(r+\frac{1}{2}n\big) : \tilde{b}_{-r}\cdot \tilde{b}_{n+r} : \tag{6.3.9}$$

in addition to this we also have the generators corresponding to the supercurrent modes

$$\begin{aligned} F_n &= \sum_{k\in\mathbb{Z}} \alpha_{-k}\cdot d_{n+k} \ , \quad G_r = \sum_{k\in\mathbb{Z}} \alpha_{-k}\cdot b_{r+k} \\ \tilde{F}_n &= \sum_{k\in\mathbb{Z}} \tilde{\alpha}_{-k}\cdot \tilde{d}_{n+k} \ , \quad \tilde{G}_r = \sum_{k\in\mathbb{Z}} \tilde{\alpha}_{-k}\cdot \tilde{b}_{r+k} \end{aligned} \tag{6.3.10}$$

for which we do not have to normal order since the operators commute.

The constraints on a physical state $|\phi\rangle$ in terms of the above generators are given by

$$\begin{aligned} &\text{R:} \ \ L_n|\phi\rangle = a_{\mathrm{R}}\delta_{n,0}|\phi\rangle \ \text{ for } \ n\geq 0 \ \text{ and } \ F_n|\phi\rangle = 0 \ \text{ for } \ n\geq 0 \\ &\text{NS:} \ \ L_n|\phi\rangle = a_{\mathrm{NS}}\delta_{n,0}|\phi\rangle \ \text{ for } \ n\geq 0 \ \text{ and } \ G_r|\phi\rangle = 0 \ \text{ for } \ r> 0 \\ &\text{R:} \ \ \tilde{L}_n|\phi\rangle = a_{\mathrm{R}}\delta_{n,0}|\phi\rangle \ \text{ for } \ n\geq 0 \ \text{ and } \ \tilde{F}_n|\phi\rangle = 0 \ \text{ for } \ n\geq 0 \\ &\text{NS:} \ \ \tilde{L}_n|\phi\rangle = a_{\mathrm{NS}}\delta_{n,0}|\phi\rangle \ \text{ for } \ n\geq 0 \ \text{ and } \ \tilde{G}_r|\phi\rangle = 0 \ \text{ for } \ r> 0 \end{aligned} \tag{6.3.11}$$

where we introduced the normal ordering constants a_{R} and a_{NS} for the R and NS boundary conditions on the fermions. For the generators corresponding to the supercurrent no normal ordering constant is needed since the operators involved commutes.

6.3.3 Lorentz generators

Considering the Lorentz generators for the closed superstring they follow by quantization of the Lorentz charges (6.2.15)

$$J_{\mu\nu} = \int_0^{2\pi} d\sigma : (X_\mu\Pi_\nu - X_\nu\Pi_\mu - iT\bar{\psi}_\mu\rho^0\psi_\nu) : \tag{6.3.12}$$

generalizing (2.3.7). In terms of modes this gives

$$J^{\mu\nu} = l^{\mu\nu} + E^{\mu\nu} + \tilde{E}^{\mu\nu} + K^{\mu\nu} + \tilde{K}^{\mu\nu} \tag{6.3.13}$$

with the bosonic part given by (2.3.9) and the fermionic part given by

$$\begin{aligned} &\text{R:} \;\; K^{\mu\nu} = -\frac{i}{2}[d_0^\mu, d_0^\nu] - i\sum_{k=1}^{\infty}(d_{-k}^\mu d_k^\nu - d_{-k}^\nu d_k^\mu) \;, \quad \text{NS:} \;\; K^{\mu\nu} = -i\sum_{k=1/2}^{\infty}(b_{-k}^\mu b_k^\nu - b_{-k}^\nu b_k^\mu) \\ &\text{R:} \;\; \tilde{K}^{\mu\nu} = -\frac{i}{2}[\tilde{d}_0^\mu, \tilde{d}_0^\nu] - i\sum_{k=1}^{\infty}(\tilde{d}_{-k}^\mu \tilde{d}_k^\nu - \tilde{d}_{-k}^\nu \tilde{d}_k^\mu) \;, \quad \text{NS:} \;\; \tilde{K}^{\mu\nu} = -i\sum_{k=1/2}^{\infty}(\tilde{b}_{-k}^\mu \tilde{b}_k^\nu - \tilde{b}_{-k}^\nu \tilde{b}_k^\mu) \end{aligned} \tag{6.3.14}$$

where we indicated the possible periodicity conditions for the right-moving and left-moving sectors.

6.4 Superconformal symmetry

For the classical bosonic string the remnant local symmetry of the Polyakov action in the flat gauge means that it is conformally invariant. For the classical superstring the action (6.1.1) can be shown to be superconformally invariant. The quantized bosonic string in the covariant quantization approach have an apparant anomaly in the Virasoro algebra that can be removed by properly taking into account the $b-c$ ghost fields which originates from gauge fixing the metric in the path integral for the bosonic string to work in the flat gauge. Similarly, we see below that the quantized superstring has an apparant anomaly in what is called the Super-Virasoro algebra that also can be removed by properly taking into account the gauge fixing of the metric in the path integral for the superstring.

First of all, the right-moving and the left-moving sectors are decoupled, meaning that all the symmetry generators for the right-moving sector commute with the ones of the left-moving sector. Furthermore, the algebra of the right and left-moving sectors are exactly the same. Therefore, we work below solely in the right-moving sector.

Consider first the R condition. The generators L_n and F_n have the algebra

$$\begin{aligned} &[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{8}m^3\delta_{m+n,0} \\ &[L_m, F_n] = \Big(\frac{1}{2}m - n\Big)F_{m+n} \;, \quad \{F_m, F_n\} = 2L_{m+n} + \frac{D}{2}m^2\delta_{m+n,0} \end{aligned} \tag{6.4.1}$$

We now analyze this result. Consider first the anti-commutator $\{F_0, F_0\} = 2L_0$. Acting with this on a physical state $|\phi\rangle$ obeying (6.3.11) we see that this anti-commutator relation immediately implies $a_{\rm R} = 0$.

Consider now the full algebra (6.4.1). An algebra of the form

$$\begin{aligned} [\mathcal{L}_m, \mathcal{L}_n] &= (m-n)\mathcal{L}_{m+n} + \frac{1}{12}(cm^3 + qm)\delta_{m+n,0} \\ [\mathcal{L}_m, \mathcal{F}_n] = \Big(\frac{1}{2}m - n\Big)\mathcal{F}_{m+n}\ , &\quad \{\mathcal{F}_m, \mathcal{F}_n\} = 2\mathcal{L}_{m+n} + \frac{1}{12}(4cm^2 + q)\delta_{m+n,0} \end{aligned} \tag{6.4.2}$$

is called a centrally extended Super-Virasoro algebra and the generators $\mathcal{L}_n$ and $\mathcal{F}_n$ are collectively known as Super-Virasoro generators. Here c, q are constants where c is known as the central charge of the algebra. An algebra of this form with $c = q = 0$ is called a Super-Virasoro algebra and this is the algebra one finds for a theory that has super-conformal symmetry also at the quantum level. We see that the algebra (6.4.1) is a particular case of a centrally extended Super-Virasoro algebra for which $c = \frac{3}{2}D$ and $q = 0$ hence this suggests that the superstring theory is not super-conformal at the quantum level. Therefore, we have encountered an apparent anomaly in the super-conformal invariance of the superstring.

Let us focus for a moment on the central charge. This involves only the conformal symmetry of the theory, so only the $[\mathcal{L}_m, \mathcal{L}_n]$ part of the algebra (6.4.2). The bosonic and fermionic parts of the theory are separately classically conformally invariant. For the bosonic part with $\mathcal{L}_m = L_m^{(X)}$ we find the central charge $c = D$ (and $q = -D$). For the fermionic part $\mathcal{L}_m = L_m^{(\psi)}$ we find central charge $c = D/2$ (and $q = D$). Since the bosonic fields commute with the fermionic fields we can find the total central charge simply by adding them, so $c_{\text{total}} = c_X + c_\psi = \frac{3}{2}D$ as found in (6.4.1). To achieve an anomaly-free Super-Conformal symmetry it necessary to have zero central charge. For the bosonic string we found that the gauge fixing of the metric in the path-integral gave an extra part to the theory, namely the $b - c$ ghost system. This has central charge $c = -26$ (and $q = 2$). However, obviously we cannot find a dimension D such that $\frac{3}{2}D = 26$. Hence we must be missing a further part of the theory.

Reconsidering the gauge fixing of the metric in the path-integral for the superstring one finds that there is an additional sector of the theory called the $\beta - \gamma$ ghost system. This is the ghost system for the fermionic part, just as the $b - c$ ghost system is the ghost system for the bosonic part. With the R condition one finds that it has central charge $c = 11$ (and $q = -2$). Adding up the central charges for the four different sectors one finds the total central charge $c_{\text{total}} = c_X + c_\psi + c_{bc} + c_{\beta\gamma} = \frac{3}{2}D - 15$. Hence for an anomaly free conformal symmetry we need $D = 10$. One can furthermore look at the constant q related to the normal ordering constant a_{R}. Adding up the four different sectors one finds $q = 0$. Hence q is zero without the need of adding an additional normal ordering constant to the zeroth Virasoro generator. Hence we find that $a_{\text{R}} = 0$. Thus, we record that for the R condition we have conformal invariance of the quantized superstring provided

$$D = 10 \ , \quad a_{\mathrm{R}} = 0 \tag{6.4.3}$$

We would also like to have superconformal invariance as well. If one combines the $b-c$ ghost system with the $\beta-\gamma$ ghost system one has a classically superconformally invariant theory that when quantized has a centrally extended Super-Virasoro algebra (6.4.2) with $c = -15$ and $q = 0$. Hence combining the Super-Virasoro generators $L_n^{(\mathrm{gh})} = L_n^{(bc)} + L_n^{(\beta\gamma)}$ and $F_n^{(\mathrm{gh})}$ for the two ghost systems with the Super-Virasoro generators L_n and F_n for the bosonic and fermionic fields we find that the combined Super-Virasoro generators $\mathcal{L}_n = L_n + L_n^{(\mathrm{gh})}$ and $\mathcal{F}_n = F_n + F_n^{(\mathrm{gh})}$ obey a Super-Virasoro algebra without anomalies, *i.e.* the algebra (6.4.2) with $c = q = 0$. Thus, the combined theory is super-conformally invariant.

We now turn to the NS condition. The generators L_n and G_r have the algebra

$$\begin{gathered}
[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{8}(m^3 - m)\delta_{m+n,0} \\
[L_m, G_r] = \Big(\frac{1}{2}m - r\Big)G_{m+r} \ , \quad \{G_r, G_s\} = 2L_{r+s} + \frac{D}{2}(r^2 - \frac{1}{4})\delta_{r+s,0}
\end{gathered} \tag{6.4.4}$$

with $m,n \in \mathbb{Z}$ and $r,s \in \mathbb{Z} + \frac{1}{2}$. This is a special case of an algebra of the form

$$\begin{gathered}
[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} + \frac{1}{12}(cm^3 + qm)\delta_{m+n,0} \\
[\mathcal{L}_m, \mathcal{G}_r] = \Big(\frac{1}{2}m - r\Big)\mathcal{G}_{m+r} \ , \quad \{\mathcal{G}_r, \mathcal{G}_s\} = 2\mathcal{L}_{r+s} + \frac{1}{12}(4cr^2 + q)\delta_{r+s,0}
\end{gathered} \tag{6.4.5}$$

which is known as a centrally extended Super-Virasoro algebra with Super-Virasoro generators $\mathcal{L}_n$ and $\mathcal{G}_r$. Clearly the algebra (6.4.4) is a special case of this with $c = \frac{3}{2}D$ and $q = -\frac{3}{2}D$. As for the R condition we have a ghost sector with a $b-c$ and a $\beta-\gamma$ ghost system. The ghost sector has Super-Virasoro generators $L_n^{(\mathrm{gh})} = L_n^{(bc)} + L_n^{(\beta\gamma)}$ and $G_r^{(\mathrm{gh})}$ which obey the algebra (6.4.5) with $c = -15$ and $q = 3$. Notice that setting $\mathcal{L}_n = -a_{\mathrm{NS}}\delta_{n,0}$ and $\mathcal{G}_r = 0$ the generators $\mathcal{L}_n$ and $\mathcal{G}_r$ obey the algebra (6.4.5) with $c = 0$ and $q = 24a_{\mathrm{NS}}$. Therefore, the combined Super-Virasoro generators $\mathcal{L}_n = L_n + L_n^{(\mathrm{gh})} - a_{\mathrm{NS}}\delta_{n,0}$ and $\mathcal{G}_r = G_r + G_r^{(\mathrm{gh})}$ obey the algebra (6.4.5) with $c = \frac{3}{2}D - 15$ and $q = -\frac{3}{2}D + 3 + 24a_{\mathrm{NS}}$. Hence, provided we have

$$D = 10 \ , \quad a_{\mathrm{NS}} = \frac{1}{2} \tag{6.4.6}$$

we find that the combined Super-Virasoro generators obey the Super-Virasoro algebra without anomalies, *i.e.* with $c = q = 0$. Thus, we need the condition (6.4.6) to have super-conformal invariance for the quantum superstring in case of the NS condition.

6.5 Ground states for the closed superstring

In this section we address the problem of the degeneracy of the ground state in the R-NS, NS-R and R-R sectors of the closed superstring. We use in this section that $D = 10$.

6.5.1 Degenerate ground state for R condition

Consider first the right-moving sector with the R periodicity condition. Consider a state $|\phi\rangle$ with $\alpha_n^\mu|\phi\rangle = 0$ and $d_n^\mu|\phi\rangle = 0$ for $n > 0$ as well as $p_\mu|\phi\rangle = k_\mu|\phi\rangle$. Clearly, we can think of this as a ground state for this sector. However, it is not unique. To see this we notice first that the ten operators d_0^μ, $\mu = 0, 1, ..., 9$, anticommutes with all the creation and annihilation operators d_n^ν with $n \neq 0$. Furthermore, the operators d_0^μ have the algebra

$$\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu} \tag{6.5.1}$$

This means that acting with the operators d_0^μ on $|\phi\rangle$ we can find more ground states. Hence we have a degeneracy of the ground state for this sector. To see this degeneracy more explicitly, define the ten operators

$$h_\pm^1 = \frac{i}{\sqrt{2}}(d_0^0 \pm d_0^1) \ , \quad h_\pm^a = \frac{1}{\sqrt{2}}(d_0^{2a-2} \pm i d_0^{2a-1}) \ \text{ for } \ a = 2, 3, 4, 5 \tag{6.5.2}$$

Using the algebra (6.5.1) we find that these ten operators have the algebra

$$\{h_+^a, h_-^b\} = \delta^{ab} \ , \quad \{h_+^a, h_+^b\} = \{h_-^a, h_-^b\} = 0 \tag{6.5.3}$$

for $a, b = 1, 2, ..., 5$. Thus, comparing to (6.3.5) we see that we have five fermionic oscillators. Impose now on a state $|\phi\rangle$ that $\alpha_n^\mu|\phi\rangle = d_n^\mu|\phi\rangle = 0$ for $n > 0$, $p_\mu|\phi\rangle = k_\mu|\phi\rangle$ as well as $h_-^a|\phi\rangle = 0$ for $a = 1, 2, ..., 5$. Then the states

$$(h_+^1)^{n_1}(h_+^2)^{n_2}(h_+^3)^{n_3}(h_+^4)^{n_4}(h_+^5)^{n_5}|\phi\rangle \tag{6.5.4}$$

are non-zero only if $n_a \in \{0, 1\}$ for $a = 1, 2, ..., 5$. This means we have $2^5 = 32$ linearly independent states that all are ground states. Write now a basis for these ground states as

$$|0; k\rangle_\alpha \ \text{ with } \ \alpha = 1, 2, ..., 32 \tag{6.5.5}$$

with the conditions

$$\begin{aligned} \alpha_n^\mu|0; k\rangle_\alpha = d_n^\mu|0; k\rangle_\alpha = 0 \ , \quad n > 0 \\ p_\mu|0; k\rangle_\alpha = k_\mu|0; k\rangle_\alpha \end{aligned} \tag{6.5.6}$$

The action of d_0^μ on $|0;k\rangle_\alpha$ is now a linear combination of the states $|0;k\rangle_\alpha$ hence we can write this linear combination as

$$d_0^\mu|0;k\rangle_\alpha = \frac{1}{\sqrt{2}}(\Gamma^\mu)^\beta{}_\alpha|0;k\rangle_\beta \tag{6.5.7}$$

where a sum over $\beta = 1,2,...,32$ is understood. Then we can think of $(\Gamma^\mu)^\beta{}_\alpha/\sqrt{2}$ as a representation of d_0^μ on the ground states in terms of the ten 32 by 32 matrices Γ^μ, $\mu = 0,1,...,9$, known as the *gamma matrices*.

Applying two d_0^μ operators gives

$$\begin{aligned} 2d_0^\mu d_0^\nu|0;k\rangle_\alpha &= \sqrt{2}d_0^\mu(\Gamma^\nu)^\beta{}_\alpha|0;k\rangle_\beta = (\Gamma^\nu)^\beta{}_\alpha\sqrt{2}d_0^\mu|0;k\rangle_\beta \\ &= (\Gamma^\nu)^\beta{}_\alpha(\Gamma^\mu)^\gamma{}_\beta|0;k\rangle_\gamma = (\Gamma^\mu)^\gamma{}_\beta(\Gamma^\nu)^\beta{}_\alpha|0;k\rangle_\gamma \\ &= (\Gamma^\mu\Gamma^\nu)^\gamma{}_\alpha|0;k\rangle_\gamma \end{aligned} \tag{6.5.8}$$

where in the last line we wrote the product of the matrices Γ^μ and Γ^ν as $(\Gamma^\mu\Gamma^\nu)^\gamma{}_\alpha = (\Gamma^\mu)^\gamma{}_\beta(\Gamma^\nu)^\beta{}_\alpha$. Using this result together with the algebra (6.5.1) one gets that the gamma matrices obey the algebra[29]

$$\{\Gamma^\mu,\Gamma^\nu\} = 2\eta^{\mu\nu}I \tag{6.5.9}$$

for $\mu,\nu = 0,1,...,9$ where I is the 32×32 identity matrix. We recognize this algebra as the Clifford algebra for ten-dimensional Minkowski space for which the gamma matrices Γ^μ provide a representation. One can show that it is possible to impose that these matrices are real and that $(\Gamma^\mu)^T = \Gamma_\mu$ where $\Gamma_\mu = \eta_{\mu\nu}\Gamma^\nu$.

Since the Γ^μ matrices provide a representation of the Clifford algebra (6.5.9) for ten-dimensional Minkowski space the index α on the ground states $|0;k\rangle_\alpha$ is a spinor index, as seen from a ten-dimensional point of view. In other words, while the μ index on $X^\mu(\xi)$ means that $X^\mu(\xi)$ transforms like a vector with respect to ten-dimensional Lorentz transformations, the ground states $|0;k\rangle_\alpha$ instead transform like a spinor.

6.5.2 The ground state for each of the four sectors

We are now ready to write the ground states for the four sectors of the closed superstring. In the NS-NS sector we do not have any degeneracy of the ground state with respect to fermionic zero-modes, since b_r^μ and $\tilde{b}_r^\mu$ modes do not have zero modes. Hence in this case

[29] One gets $(\{\Gamma^\mu,\Gamma^\nu\} - 2\eta^{\mu\nu}I\}^\beta{}_\alpha|0;k\rangle_\beta = 0$ which implies (6.5.9) since the ground states $|0;k\rangle_\alpha$ are linearly independent.

we have the ground state $|0;k\rangle$ defined by

$$\text{NS-NS:} \quad \begin{cases} \alpha_n^\mu|0;k\rangle = \tilde{\alpha}_n^\mu|0;k\rangle = 0 \ , \quad n > 0 \\ b_r^\mu|0;k\rangle = \tilde{b}_r^\mu|0;k\rangle = 0 \ , \quad r > 0 \\ p_\mu|0;k\rangle = k_\mu|0;k\rangle \end{cases} \tag{6.5.10}$$

Since all the raising operators in the NS-NS sector α_{-n}^μ, $\tilde{\alpha}_{-n}^\mu$, $n > 0$, b_{-r}^μ, $\tilde{b}_{-r}^\mu$, $r > 0$ transforms like vectors under ten-dimensional Lorentz transformations, and since $|0;k\rangle$ instead transform like a scalar, all the states in the Fock space of the NS-NS sector are bosonic, as seen from point of view of the ten-dimensional Minkowski space.

For the R-NS sector we have the R periodicity condition in the right-moving sector. Thus, we can use directly the result (6.5.6). Hence we define the 32 ground states $|0;k\rangle_\alpha$ in this sector by

$$\text{R-NS:} \quad \begin{cases} \alpha_n^\mu|0;k\rangle_\alpha = \tilde{\alpha}_n^\mu|0;k\rangle_\alpha = 0 \ , \quad n > 0 \\ d_n^\mu|0;k\rangle_\alpha = 0 \ , \quad n > 0 \ , \quad \tilde{b}_r^\mu|0;k\rangle_\alpha = 0 \ , \quad r > 0 \\ \sqrt{2}\, d_0^\mu|0;k\rangle_\alpha = (\Gamma^\mu)^\beta{}_\alpha|0;k\rangle_\beta \ , \quad p_\mu|0;k\rangle_\alpha = k_\mu|0;k\rangle_\alpha \end{cases} \tag{6.5.11}$$

where the gamma matrices Γ^μ are real 32 by 32 matrices, $(\Gamma^\mu)^T = \Gamma_\mu$ and $\Gamma_\mu = \eta_{\mu\nu}\Gamma^\nu$. Since all the raising operators in the R-NS sector α_{-n}^μ, $\tilde{\alpha}_{-n}^\mu$, d_{-n}^μ, $n > 0$, $\tilde{b}_{-r}^\mu$, $r > 0$, transforms like vectors under ten-dimensional Lorentz transformations, and since $|0;k\rangle$ transforms like a spinor, then all the states in the Fock space of the R-NS sector are fermionic, as seen from point of view of the ten-dimensional Minkowski space.

The NS-R sector ground states is found by exchanging the right- and left-moving sectors of the R-NS sector. Hence, we define the ground states $|0;k\rangle_\alpha$ in this sector by

$$\text{NS-R:} \quad \begin{cases} \alpha_n^\mu|0;k\rangle_\alpha = \tilde{\alpha}_n^\mu|0;k\rangle_\alpha = 0 \ , \quad n > 0 \\ b_r^\mu|0;k\rangle_\alpha = 0 \ , \quad r > 0 \ , \quad \tilde{d}_n^\mu|0;k\rangle_\alpha = 0 \ , \quad n > 0 \\ \sqrt{2}\, \tilde{d}_0^\mu|0;k\rangle_\alpha = (\Gamma^\mu)^\beta{}_\alpha|0;k\rangle_\beta \ , \quad p_\mu|0;k\rangle_\alpha = k_\mu|0;k\rangle_\alpha \end{cases} \tag{6.5.12}$$

Note that we require the representation of the gamma matrices to be the same for the R-NS and NS-R sectors, i.e. they are the same ten 32 by 32 matrices. Clearly, by applying the same reasoning as in the R-NS sector, one has that all the states in the Fock space of the R-NS sector are fermionic, as seen from point of view of the ten-dimensional Minkowski space.

Finally, we have the R-R sector. This is special in that we have the R periodicity condition in both the right- and left-moving sectors. Hence we have a degeneracy with respect to both d_0^μ and $\tilde{d}_0^\nu$. This gives a degeneracy of $32^2 = 1024$ states. We write these ground states as $|0;k\rangle_\alpha{}^\beta$ where both α and β takes values from 1 to 32. This in turn gives rise to two separate representations of the Clifford algebra (6.5.9). We define the R-R sector ground states $|0;k\rangle_\alpha{}^\beta$ as

$$\text{R-R:}\quad \left\{ \begin{array}{c} \alpha_n^\mu|0;k\rangle_\alpha{}^\beta = \tilde{\alpha}_n^\mu|0;k\rangle_\alpha{}^\beta = 0\ ,\quad n>0 \\ d_n^\mu|0;k\rangle_\alpha{}^\beta = \tilde{d}_n^\mu|0;k\rangle_\alpha{}^\beta = 0\ ,\quad n>0 \\ \sqrt{2}\, d_0^\mu|0;k\rangle_\alpha{}^\beta = (\Gamma^\mu)^\gamma{}_\alpha|0;k\rangle_\gamma{}^\beta\ ,\quad \sqrt{2}\,\tilde{d}_0^\mu|0;k\rangle_\alpha{}^\beta = (\Gamma^\mu)^\beta{}_\gamma|0;k\rangle_\alpha{}^\gamma \\ p_\mu|0;k\rangle_\alpha{}^\beta = k_\mu|0;k\rangle_\alpha{}^\beta \end{array}\right. \tag{6.5.13}$$

where we use the same representation of the gamma matrices in the right- and left-moving sectors as the one used in the R-NS and NS-R sectors, i.e. they are the same ten 32 by 32 matrices. Notice that the α index in $|0;k\rangle_\alpha{}^\beta$ refers to the right-moving sector while β refers to the left-moving sector. Using the same reasoning as in (6.5.8) one finds

$$\begin{aligned} 2d_0^\mu d_0^\nu|0;k\rangle_\alpha{}^\beta &= (\Gamma^\mu\Gamma^\nu)^\gamma{}_\alpha|0;k\rangle_\gamma{}^\beta \\ 2\tilde{d}_0^\mu\tilde{d}_0^\nu|0;k\rangle_\alpha{}^\beta &= (\Gamma^\nu\Gamma^\mu)^\beta{}_\gamma|0;k\rangle_\alpha{}^\gamma \end{aligned} \tag{6.5.14}$$

showing that the algebras $\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu}$ and $\{\tilde{d}_0^\mu, \tilde{d}_0^\nu\} = \eta^{\mu\nu}$ are consistent with the gamma matrices obeying the Clifford algebra (6.5.9).

Since the degeneracy of the R-R ground state corresponds to a product of two spinor indices, this means in efffect that the R-R ground state transform like a boson under ten-dimensional Lorentz transformations. This will be confirmed in detail below in Sec. 6.6.2. Since all the raising operators transforms as bosons as well this means that the states in the Fock space of the R-R sector are bosonic, as seen from point of view of the ten-dimensional Minkowski space.

6.5.3 Chirality and spinors

Using the ten gamma matrices Γ^μ we can define an additional 32 by 32 matrix $\Gamma_{11} = \Gamma^{11}$ by

$$\Gamma_{11} = \Gamma_0\Gamma_1\cdots\Gamma_9 \tag{6.5.15}$$

One can check that it follows from the Clifford algebra (6.5.9) that

$$(\Gamma_{11})^2 = I\ ,\quad \{\Gamma_{11},\Gamma_\mu\} = 0 \tag{6.5.16}$$

Since $(\Gamma_{11})^2 = I$ the eigenvalues of Γ_{11} are ± 1. This defines a notion of chirality. Writing $J_{\mu\nu} = \frac{i}{2}[\Gamma_\mu,\Gamma_\nu]$ one can check that the irreducible representation of the Clifford algebra gives rise to a representation of the ten-dimensional Lorentz algebra

$$[J_{\mu\nu}, J_{\rho\sigma}] = -i(\eta_{\nu\rho}J_{\mu\sigma} + \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\mu\rho}J_{\nu\sigma} - \eta_{\nu\sigma}J_{\mu\rho}) \tag{6.5.17}$$

known as the Dirac representation. Since $[\Gamma_{11}, \frac{i}{2}[\Gamma_\mu, \Gamma_\nu]] = 0$ one can split up the Dirac representation into representations according to the eigenvalues of Γ_{11}. This splits up the 32-dimensional reducible representation into two irreducible 16-dimensional representations called Weyl representations of the Lorentz algebra.

For use below we now define an operator version of the Γ_{11} matrix (6.5.15) in the R-NS, NS-R and R-R sectors. When we have the R-periodicity condition in the right-moving sector we define

$$\widehat{\Gamma}_{11} = -2^5 d_0^0 d_0^1 \cdots d_0^9 \tag{6.5.18}$$

Using the algebra (6.5.1) one finds

$$\widehat{\Gamma}_{11}^2 = 1 \ , \quad \{\widehat{\Gamma}_{11}, d_0^\mu\} = 0 \tag{6.5.19}$$

for $\mu = 0, 1, ..., 9$. Similarly, when we have the R periodicity in the left-moving sector we define

$$\widehat{\tilde{\Gamma}}_{11} = -2^5 \tilde{d}_0^0 \tilde{d}_0^1 \cdots \tilde{d}_0^9 \tag{6.5.20}$$

that obeys

$$\widehat{\tilde{\Gamma}}_{11}^2 = 1 \ , \quad \{\widehat{\tilde{\Gamma}}_{11}, \tilde{d}_0^\mu\} = 0 \tag{6.5.21}$$

One can now apply the operators (6.5.18) and (6.5.20) to the ground states defined in the R-NS, NS-R and R-R sectors. This gives

$$\begin{aligned}
&\text{R-NS:} \quad \widehat{\Gamma}_{11}|0;k\rangle_\alpha = (\Gamma_{11})^\beta{}_\alpha |0;k\rangle_\beta \\
&\text{NS-R:} \quad \widehat{\tilde{\Gamma}}_{11}|0;k\rangle_\alpha = (\Gamma_{11})^\beta{}_\alpha |0;k\rangle_\beta \\
&\text{R-R:} \quad \left\{ \begin{array}{l} \widehat{\Gamma}_{11}|0;k\rangle_\alpha{}^\beta = (\Gamma_{11})^\gamma{}_\alpha |0;k\rangle_\gamma{}^\beta \\ \widehat{\tilde{\Gamma}}_{11}|0;k\rangle_\alpha{}^\beta = -(\Gamma_{11})^\beta{}_\gamma |0;k\rangle_\alpha{}^\gamma \end{array} \right.
\end{aligned} \tag{6.5.22}$$

The minus sign in the left-moving sector of the R-R periodicity condition can be derived using (6.5.14).

The 32-dimensional space on which we found an irreducible representation of d_0^μ is the linear space of spinors of ten-dimensional Minkowski space. Thus given a spinor χ, by which we also mean that it is Grassman-valued, it is a vector in the 32-dimensional linear space. Weyl spinors are spinors that obey $\Gamma_{11}\chi = \pm\chi$. This is directly related to the splitup of the 32-dimensional Dirac representation into the two 16-dimensional Weyl representations of the Lorentz algebra. One can furthermore formulate the Majorana condition on a spinor simply as $\chi^* = \chi$, where χ^* means the complex conjugate of χ. Since the matrices Γ_μ are real this means we can impose a spinor to be Majorana-Weyl.

So starting from having spinors defined on a 32-dimensional complex linear space we can restrict them to be defined on a 16-dimensional real linear space.

It is important to be able to combine two spinors into a bosonic object such as a scalar or a vector with regard to the Lorentz transformations. Writing $\bar{\chi} = \chi^T \Gamma^0$ one finds that for two spinors χ and λ the object $\bar{\lambda}\chi$ transforms as a scalar while $\bar{\lambda}\Gamma^\mu\chi$ transform as a vector under Lorentz transformations. It is useful for our considerations below to introduce an index for the spinor χ^α, $\alpha = 1, 2, ..., 32$, as well as $\bar{\chi}_\alpha$, such that we can write $\bar{\lambda}\Gamma^\mu\chi = \bar{\lambda}_\alpha(\Gamma^\mu)^\alpha{}_\beta\chi^\beta$ where $(\Gamma^\mu)^\alpha{}_\beta$ is the matrix entry at row α and column β of Γ^μ.

6.6 Spectrum of the closed superstring

We now consider the spectrum of the closed superstring. Using the constraints (6.3.11) on a physical state with $a_{\rm R} = 0$ and $a_{\rm NS} = \frac{1}{2}$ we find the following expressions for the mass of the superstring states in the right-moving and left-moving sectors

$$\begin{aligned} &\text{R:}\ \ M^2 = \frac{4}{l_s^2}\Big(N^{(\alpha)} + N^{(d)}\Big) \ , \quad \text{NS:}\ \ M^2 = \frac{4}{l_s^2}\Big(N^{(\alpha)} + N^{(b)} - \frac{1}{2}\Big) \\ &\text{R:}\ \ M^2 = \frac{4}{l_s^2}\Big(\tilde{N}^{(\alpha)} + \tilde{N}^{(d)}\Big) \ , \quad \text{NS:}\ \ M^2 = \frac{4}{l_s^2}\Big(\tilde{N}^{(\alpha)} + \tilde{N}^{(b)} - \frac{1}{2}\Big) \end{aligned} \tag{6.6.1}$$

where we defined

$$\begin{aligned} N^{(\alpha)} &= \sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n \ , \quad N^{(d)} = \sum_{n=1}^{\infty} n\, d_{-n}\cdot d_n \ , \quad N^{(b)} = \sum_{r=\frac{1}{2}}^{\infty} r\, b_{-r}\cdot b_r \\ \tilde{N}^{(\alpha)} &= \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n \ , \quad \tilde{N}^{(d)} = \sum_{n=1}^{\infty} n\, \tilde{d}_{-n}\cdot \tilde{d}_n \ , \quad \tilde{N}^{(b)} = \sum_{r=\frac{1}{2}}^{\infty} r\, \tilde{b}_{-r}\cdot \tilde{b}_r \end{aligned} \tag{6.6.2}$$

It is not difficult to see that the eigenvalues of $N^{(\alpha)}$, $\tilde{N}^{(\alpha)}$, $N^{(d)}$ and $\tilde{N}^{(d)}$ are integers that are greater than or equal to zero. Instead the eigenvalues of $N^{(b)}$ and $\tilde{N}^{(b)}$ consist of integers and half-integers that are greater than or equal to zero.

6.6.1 GSO projection

Consider now the R-NS periodicity condition on the fermions. From the equations for the mass of superstring states (6.6.1) we encounter a problem. Clearly the mass as computed in the right-moving and the left-moving sectors should agree. This is the condition of level-matching. But this forces $\tilde{N}^{(b)}$ to be a half-integer. In particular, the vacuum case $\tilde{N}^{(b)} = 0$ is not consistent with level-matching. Thus, in the R-NS sector it seems that part of the spectrum for the NS condition has been projected out. The question is then whether this is true in the NS-NS sector as well? Indeed, this is part of a necessary truncation of the states in the closed superstring Fock space that is implemented by the so-called *GSO projection*.

The GSO projection on the closed superstring Fock space means that only states $|\phi\rangle$ that fulfil the conditions

$$\begin{aligned} &\text{R:}\;\; \widehat{\Gamma}_{11}(-1)^{\sum_{n=1}^{\infty} d_{-n}\cdot d_n}|\phi\rangle = |\phi\rangle \;, \quad \text{NS:}\;\; (-1)^{\sum_{r=\frac{1}{2}}^{\infty} b_{-r}\cdot b_r}|\phi\rangle = -|\phi\rangle \\ &\text{R:}\;\; \widehat{\tilde{\Gamma}}_{11}(-1)^{\sum_{n=1}^{\infty} \tilde{d}_{-n}\cdot \tilde{d}_n}|\phi\rangle = s|\phi\rangle \;, \quad \text{NS:}\;\; (-1)^{\sum_{r=\frac{1}{2}}^{\infty} \tilde{b}_{-r}\cdot \tilde{b}_r}|\phi\rangle = -|\phi\rangle \end{aligned} \tag{6.6.3}$$

are part of the closed superstring Fock space. Here $s = \pm 1$ and we used the operators $\widehat{\Gamma}_{11}$ and $\widehat{\tilde{\Gamma}}_{11}$ defined in Eqs. (6.5.18) and (6.5.20). A state that does not fulfil the conditions (6.6.3) is not in the closed superstring Fock space, and we say that it is *projected out* by the GSO projection.

For the NS periodicity condition the GSO projection in the right-moving sector means that there should be an odd number of raising operators b^μ_{-r} $(r > 0)$ which implies that $N^{(b)}$ is a half-integer, and similarly for the left-moving sector. Thus, the above-mentioned problems of the level-matching conditions in the R-NS and NS-R sector are solved. In connection with this, notice that as consequence of the GSO projection on the NS periodicity condition the NS-NS, R-NS and NS-R ground states are no longer part of the closed superstring Fock space.

For the R periodicity condition the GSO projection works by restricting the helicity of the superstring state according to the helicity operators $\widehat{\Gamma}_{11}$ and $\widehat{\tilde{\Gamma}}_{11}$ defined above. In particular, the ground states (6.5.11), (6.5.12) and (6.5.13) for the R-NS, NS-R and R-R sectors are now restricted to be of a certain helicity. One finds

$$\begin{aligned} \text{R-NS}: &\quad \widehat{\Gamma}_{11}|0;k\rangle_\alpha = |0;k\rangle_\alpha \\ \text{NS-R}: &\quad \widehat{\tilde{\Gamma}}_{11}|0;k\rangle_\alpha = s|0;k\rangle_\alpha \\ \text{R-R}: &\quad \widehat{\Gamma}_{11}|0;k\rangle_\alpha{}^\beta = |0;k\rangle_\alpha{}^\beta \;, \quad \widehat{\tilde{\Gamma}}_{11}|0;k\rangle_\alpha{}^\beta = s|0;k\rangle_\alpha{}^\beta \end{aligned} \tag{6.6.4}$$

Using (6.5.22) one can alternatively write this as

$$\begin{aligned} \text{R-NS}: &\quad (\Gamma_{11})^\beta{}_\alpha|0;k\rangle_\beta = |0;k\rangle_\alpha \\ \text{NS-R}: &\quad (\Gamma_{11})^\beta{}_\alpha|0;k\rangle_\beta = s|0;k\rangle_\alpha \\ \text{R-R}: &\quad (\Gamma_{11})^\gamma{}_\alpha|0;k\rangle_\gamma{}^\beta = |0;k\rangle_\alpha{}^\beta \;, \quad (\Gamma_{11})^\beta{}_\gamma|0;k\rangle_\alpha{}^\gamma = -s|0;k\rangle_\alpha{}^\beta \end{aligned} \tag{6.6.5}$$

Unlike in the NS case, for the R case there is a choice of the overall sign of the projector. By itself the right-moving sector is equivalent whether one chooses $\widehat{\Gamma}_{11}$ or $-\widehat{\Gamma}_{11}$ to be the helicity operator. However, combining the right- and left-moving sectors, the relative sign denoted in (6.6.3) by s between the projectors gives rise to two physically different superstring theories:

$$s = -1 : \quad \text{Type IIA string theory} \ , \quad s = 1 : \quad \text{Type IIB string theory} \tag{6.6.6}$$

Thus, depending on the sign of s we are either studying closed superstrings in Type IIA string theory, or in Type IIB string theory. These two superstring theories have important differences, as we shall see below.

As already mentioned the GSO projection (6.6.3) for the NS condition resolves the issue with level-matching for the R-NS and NS-R sectors. However, there are other reasons that the GSO projection is necessary. A very important reason is that if one studies the one-loop contribution in perturbative closed string theory one finds that the GSO projection is necessary for having a consistent theory. This is very likely connected to a very interested consequence of implementing the GSO projection: That the spectrum of the superstring now becomes supersymmetric. Thus, with the GSO projection not only the world-sheet theory is supersymmetric but also for the ten-dimensional space-time that the superstring lives in is supersymmetric. Moreover, with the GSO projection there are no tachyon states in the superstring spectrum. This also follows from having space-time supersymmetry.

Thus, by imposing the GSO projection one gets supersymmetry in the target space, in addition to the supersymmetric on the world-sheet. This poses the natural question whether one can find a formulation of the superstring in which the target space supersymmetric is manifest, rather than something one derives, imposing the GSO projection. This is indeed possible. One can find a way to formulate superstring theory such that the superstring action has manifest space-time supersymmetry, just as the action (6.1.1) has manifest space-time Poincare symmetry. This way to formulate the superstring is known as the Green-Schwarz formalism. One can show that the Green-Schwarz formulation is equivalent to the above description - known as the RNS formalism - when including the GSO projection. For the Green-Schwarz formulation one is forced to start directly in $D = 10$ dimensions since the space-time supersymmetry is highly dependent on the number of dimensions D. Moreover, it is not clear how to covariantly quantize the superstring in the Green-Schwarz formalism. It is only known how to quantize the Green-Schwarz superstring in the lightcone gauge.

6.6.2 Massless spectrum

We consider now the lowest lying states in the spectrum of the closed superstring. For the NS-NS sector we see that due to the GSO projection (6.6.3) the lowest lying states are of the form

$$\text{NS-NS:}\ \ \zeta_{\mu\nu}(k)\, b^{\mu}_{-\frac{1}{2}}\tilde{b}^{\nu}_{-\frac{1}{2}}|0;k\rangle \tag{6.6.7}$$

The field $\zeta_{\mu\nu}(k)$ splits into three fields transforming in irreducible representations of the ten-dimensional Lorentz group as follows

$$\begin{aligned} \text{Graviton:}\ \ G_{\mu\nu}(k) &= \tfrac{1}{2}(\zeta_{\mu\nu}(k)+\zeta_{\nu\mu}(k)) - \tfrac{1}{8}(\eta_{\mu\nu} - k_\mu \bar{k}_\nu - \bar{k}_\mu k_\nu)\eta^{\rho\sigma}\zeta_{\rho\sigma}(k) \\ \text{Kalb-Ramond:}\ \ B_{\mu\nu}(k) &= \tfrac{1}{2}(\zeta_{\mu\nu}(k)-\zeta_{\nu\mu}(k)) \\ \text{Dilaton:}\ \ \Phi(k) &= \tfrac{1}{8}\eta^{\rho\sigma}\zeta_{\rho\sigma}(k) \end{aligned} \tag{6.6.8}$$

where we used for a given k_μ that one can find $\bar{k}_\mu$ such that

$$\bar{k}^2 = 0\ ,\quad k\cdot \bar{k} = 1 \tag{6.6.9}$$

With this, we have that $\zeta_{\mu\nu}(k) = G_{\mu\nu}(k) + B_{\mu\nu}(k) + (\eta_{\mu\nu} - k_\mu\bar{k}_\nu - \bar{k}_\mu k_\nu)\Phi(k)$. These are all massless $M=0$ according to (6.6.1) thus we have found that the NS-NS sector has the same massless spectrum as that of the closed bosonic string, namely a massless graviton field, a massless Kalb-Ramond 2-form field and a massless scalar called the dilaton. However, unlike the closed bosonic string these massless state are the lowest lying states. In other words: We have managed to get rid of the closed string tachyon by going to superstrings. Notice that the massless NS-NS spectrum does not depend on the sign s.

One can derive from the $G_{\frac{1}{2}}$ and $\tilde{G}_{\frac{1}{2}}$ constraints that $k^\mu\zeta_{\mu\nu}(k)=0$ and $k^\nu\zeta_{\mu\nu}(k)=0$ (see Exercise 6.8). This implies that $G_{\mu\nu}(k)$, $B_{\mu\nu}(k)$ and $\Phi(k)$ fulfil the same equations as in the case of the massless modes of the bosonic closed string (See Sec. 2.2.3) except now in 10 dimensions instead of 26. For instance, one finds (2.2.12) for $G_{\mu\nu}$ in position space. Similarly one can derive the gauge transformations (2.2.8) by finding the appropriate spurious states (see Exercise 6.8), thus leading in particular to the gauge transformation (2.2.14) of $G_{\mu\nu}$ in position space.

For R-NS and NS-R sectors the lowest lying states according to the GSO projection (6.6.3) are

$$\begin{aligned} \text{R-NS:}\ \ & S^{\alpha}_{\mu}(k)\,\tilde{b}^{\mu}_{-\frac{1}{2}}|0;k\rangle_\alpha \\ \text{NS-R:}\ \ & \tilde{S}^{\alpha}_{\mu}(k)\,b^{\mu}_{-\frac{1}{2}}|0;k\rangle_\alpha \end{aligned} \tag{6.6.10}$$

where we sum over both μ and α. These states are massless $M = 0$ according to (6.6.1). We see from (6.6.5) that $\Gamma_{11} S_\mu = S_\mu$ and $\Gamma_{11}\tilde{S}_\mu = s\tilde{S}_\mu$. The two fields $S^\alpha_\mu(k)$ and $\tilde{S}^\alpha_\mu(k)$ each split up into irreducible representations of the Lorentz group which consist of

$$\begin{aligned} \text{Two spinors:} \quad & \chi^\alpha = \tfrac{1}{10}(\Gamma^\mu S_\mu)^\alpha \ , \quad \tilde{\chi}^\alpha = \tfrac{1}{10}(\Gamma^\mu \tilde{S}_\mu)^\alpha \\ \text{Two gravitinos:} \quad & \lambda^\alpha_\mu = S^\alpha_\mu - \tfrac{1}{10}(\Gamma_\mu \Gamma^\rho S_\rho)^\alpha \ , \quad \tilde{\lambda}^\alpha_\mu = \tilde{S}^\alpha_\mu - \tfrac{1}{10}(\Gamma_\mu \Gamma^\rho \tilde{S}_\rho)^\alpha \end{aligned} \tag{6.6.11}$$

We consider the constraints on these fields and their gauge transformations in Exercise 6.9. The two spinors are Majorana-Weyl spinors with $\Gamma_{11}\chi = -\chi$ and $\Gamma_{11}\tilde{\chi} = -s\tilde{\chi}$. Similarly the two gravitinos are Majorana-Weyl with respect to the spinor index $\Gamma_{11}\lambda_\mu = \lambda_\mu$ and $\Gamma_{11}\tilde{\lambda}_\mu = s\tilde{\lambda}_\mu$. We see here an important difference between type IIA and type IIB string theory: For Type IIA string theory the massless spectrum is non-chiral while for Type IIB string theory it is chiral. As one can see from the GSO projection (6.6.3) this is not only true for the massless states but is a general feature for all the states of the Type IIA and IIB string theories. Said more explicitly, Type IIA string theory is non-chiral since for any fermionic chiral state of the superstring we find an otherwise equivalent state of opposite chirality. Instead Type IIB string theory is chiral since for a given fermionic chiral state we can find an otherwise equivalent independent state of the same chirality.

Turning finally to the R-R sector we see from the GSO projection (6.6.3) that the lowest lying states are of the form

$$\text{R-R:} \quad H^\alpha{}_\beta(k)\,|0;k\rangle_\alpha{}^\beta \tag{6.6.12}$$

where we sum over α and β. These states are clearly massless $M = 0$ according to (6.6.1). The chirality conditions on the R-R ground state (6.6.5) give the matrix equations

$$H = \Gamma_{11} H = -sH\Gamma_{11} \tag{6.6.13}$$

We define the antisymmetric product of the Γ^μ matrices as

$$\Gamma^{\mu_1\mu_2\cdots\mu_n} = \Gamma^{[\mu_1}\Gamma^{\mu_2}\cdots\Gamma^{\mu_n]} \tag{6.6.14}$$

One can then show that the matrices $\Gamma^{\mu_1\mu_2\cdots\mu_n}$ from $n=0$ (defined as the identity matrix) to $n=10$ provides a basis for the linear space of 32×32 matrices. Hence we can express the matrix H as a linear combination of these antisymmetric products as follows

$$H^\alpha{}_\beta(k) = \sum_{n=0}^{10} \frac{1}{n!} F_{\mu_1\mu_2\cdots\mu_n}(k)(\Gamma^{\mu_1\mu_2\cdots\mu_n})^\alpha{}_\beta \tag{6.6.15}$$

where the fields $F_{\mu_1\mu_2\cdots\mu_n}(k)$ are completely antisymmetric in the indices. Hence $F_{\mu_1\mu_2\cdots\mu_n}(k)$ is an n-form, as defined in Sec. 4.2.1. These n-form corresponds dividing the field $H^\alpha{}_\beta(k)$ into irreducible representations of the Lorentz group. Using (6.5.16) we notice

$$\Gamma_{11}H = \sum_{n=0}^{10} \frac{1}{n!} F_{\mu_1\mu_2\cdots\mu_n} \Gamma_{11}\Gamma^{\mu_1\mu_2\cdots\mu_n} = \sum_{n=0}^{10} (-1)^n \frac{1}{n!} F_{\mu_1\mu_2\cdots\mu_n} \Gamma^{\mu_1\mu_2\cdots\mu_n}\Gamma_{11} \tag{6.6.16}$$

Using now $\Gamma_{11}H = -sH\Gamma_{11}$ from Eq. (6.6.13) we see that for type IIA string theory with $s=-1$ we can only have n-form fields with n being even while for type IIB string theory with $s=1$ we can only have n-form fields with n being odd.

From the physical state conditions $F_0|\phi\rangle = \tilde{F}_0|\phi\rangle = 0$ in (6.3.11) for the R-R massless states (6.6.12) one gets the matrix equations

$$k_\mu \Gamma^\mu H = k_\mu H \Gamma^\mu = 0 \tag{6.6.17}$$

where we used the properties of the R-R grounds states (6.5.13). One can show that

$$\begin{aligned} \Gamma^\mu \Gamma^{\nu_1\cdots\nu_n} &= \Gamma^{\mu\nu_1\cdots\nu_n} + n\, \eta^{\mu[\nu_1}\Gamma^{\nu_2\cdots\nu_n]} \\ (-1)^n \Gamma^{\nu_1\cdots\nu_n}\Gamma^\mu &= \Gamma^{\mu\nu_1\cdots\nu_n} - n\, \eta^{\mu[\nu_1}\Gamma^{\nu_2\cdots\nu_n]} \end{aligned} \tag{6.6.18}$$

Combining this with (6.6.17) one finds the following equations for $F_{\mu_1\mu_2\cdots\mu_n}(k)$

$$k_{[\mu_1} F_{\mu_2\cdots\mu_{n+1}]}(k) = 0\ , \quad k_\rho F^{\rho\mu_1\cdots\mu_{n-1}}(k) = 0 \tag{6.6.19}$$

In position space this becomes

$$\partial_{[\mu_1} F_{\mu_2\cdots\mu_{n+1}]} = 0\ , \quad \partial_\rho F^{\rho\mu_1\cdots\mu_{n-1}} = 0 \tag{6.6.20}$$

We recognize these equations as the source-free equations of motion for an n-form field strength, as discussed in Sec. 4.2.1. One can alternatively write them using the exterior derivative and the Hodge dual as $dF_{(n)} = 0$ and $d^*F_{(n)} = 0$ where $F_{(n)}$ is a short-hand notation for the n-form $F_{\mu_1\mu_2\cdots\mu_n}$. We call therefore $F_{(n)}$ the *Ramond-Ramond n-form field strength*. Since $dF_{(n)} = 0$ one can find an $(n-1)$-form gauge field $C_{(n-1)}$ such that $F_{(n)} = dC_{(n-1)}$. $C_{(n-1)}$ is called the *Ramond-Ramond $(n-1)$-form gauge field*.[30] We discuss below in Sec. 6.6.3 that the Ramond-Ramond field strengths couples to D-branes.

Using the relation $\Gamma_{11}H = H$ from (6.6.13) one finds that the n-form is equal to the Hodge dual of the $(10-n)$-form up to a sign[31]

$${}^*F_{(10-n)} = \pm F_{(n)} \tag{6.6.21}$$

[30]Notice that for $n=0$ Eqs. (6.6.20) gives that $F_{(0)}$ is a constant which means it does not correspond to a physical degree of freedom.

[31]One can show this by using that $\Gamma_{11}\Gamma^{\mu_1\mu_2\cdots\mu_n} = \pm\frac{1}{(10-n)!}\epsilon^{\mu_1\mu_2\cdots\mu_{10}}\Gamma_{\mu_{n+1}\mu_{n+2}\cdots\mu_{10}}$. One can see this by inserting the definition (6.5.15) of Γ_{11} in the identity.

where the Hodge dual is defined in (4.2.9). This means that for type IIA string theory the 6-form can be written in terms of the 4-form and the 8-form in terms of the 2-form and the 10-form in terms of the 0-form. Hence we only need to use the 0-form, 2-form and 4-form fields. Instead for type IIB string theory we only need to use the 1-form, 3-form and 5-form fields. Notice in particular that the 5-form field is special in that it is self-dual, *i.e.* it is equal to its Hodge dual.

In summary, we have found that in the R-R sector of the type IIA string theory the lowest lying states are massless and they correspond to the 2-form and 4-form Ramond-Ramond field strength $F_{(2)}$ and $F_{(4)}$ with the 1-form and 3-form gauge fields $C_{(1)}$ and $C_{(3)}$. The 2-form and 4-form field strengths are dual to the 6-form and 8-form field strengths $F_{(6)}$ and $F_{(8)}$.

For Type IIB string theory in the R-R sector the lowest lying states are massless and they correspond to the 1-form, 3-form and 5-form Ramond-Ramond field strengths $F_{(1)}$, $F_{(3)}$ and $F_{(5)}$ with the scalar (0-form), 2-form and and 4-form gauge fields $C_{(0)}$, $C_{(2)}$ and $C_{(4)}$. The 1-form and 3-form field strengths are dual to the 7-form and 9-form field strengths $F_{(7)}$ and $F_{(9)}$. Moreover, the five-form Ramond-Ramond field strength is self-dual ${}^*F_{(5)} = F_{(5)}$.

Considering all the massless states for the type IIA and IIB string theories one can see that they fit into two different supersymmetric multiplets (with respect to supersymmetry in ten dimensions). These multiplets are precisely those corresponding to Type IIA and Type IIB supergravity. Thus, we have found for Type IIA (IIB) string theory that the massless states are related by supersymmetry in ten dimensions and that they correspond to the field content of Type IIA (IIB) supergravity.

One can also consider the whole spectrum of type IIA or IIB string theory, including both the massless and massive states. As argued in Sec. 6.5, the NS-NS and R-R sectors give rise to bosonic states while the R-NS and NS-R give rise to fermionic states, as seen from a ten-dimensional point of view. Amazingly, one can show that for each mass-level in the discrete spectrum of the closed superstring the number of fermionic degrees of freedom is equal to the number of bosonic degrees of freedom. Moreover, one can relate them by a supersymmetry transformation. Thus, the spectra of Type IIA and Type IIB string theories are supersymmetric from a ten-dimensional point of view. The easiest way to show this is by using the Green-Schwarz formulation of the superstring mentioned above.

6.6.3 Ramond-Ramond field strengths and D-branes

In the previous section we found the Ramond-Ramond field strengths $F_{\mu_1\mu_2\cdots\mu_n}$ that satisfy the source-free equations of motion (6.6.20) for an n-form field strength. For type IIA string theory, n is required to be even, while for type IIB string theory n is odd. We learned in Sec. 4.2.1 that an $(p+2)$-form field strength can couple electrically to an electrically charged p-brane. So, the questions is, what branes do the Ramond-Ramond field strengths couple to? This is in fact the D-branes, made from open superstrings.

One can show that a Dp-brane in type IIA or IIB string theory is electrically charged with respect to the $(p+2)$-form Ramond-Ramond field strength $F_{(p+2)}$. Specifically, the bosonic part of the action for a Dp-brane is

$$S = S_{\rm DBI} + T_{\rm Dp} \int d^{p+1}\xi \frac{1}{(p+1)!} \epsilon^{a_1 \cdots a_{p+1}} C_{\mu_1 \cdots \mu_{p+1}} \partial_{a_1} X^{\mu_1} \cdots \partial_{a_{p+1}} X^{\mu_{p+1}} \tag{6.6.22}$$

in accordance with (4.2.13) and $S_{\rm DBI}$ is the Dirac-Born-Infeld action of Eq. (5.2.3). The gauge field $C_{(p+1)}$ is $(p+1)$-form Ramond-Ramond gauge field defined by $F_{(p+2)} = dC_{(p+1)}$. This action is valid for ten-dimensional Minkowski space as background, with zero Kalb-Ramond field and dilaton. In the action (6.6.22) we have only written the bosonic part of the action for a Dp-brane. There are fermionic terms as well, making it into a supersymmetric theory. This is discussed in Sec. 6.8.

One says that the Dp-branes in type IIA and IIB string theory carry Ramond-Ramond charge. We see from (6.6.22) that the charges of the Dp-branes are equal to their tensions $Q_p = T_{\rm Dp}$. Connected to the fact that type IIA string theory only has Ramond-Ramond n-form field strength with n even, one can show that there are only Dp-branes in type IIA string theory for even p. Thus, in type IIA string theory one has the D0-brane (also called the D-particle), D2-brane, D4-brane, D6-brane and D8-brane. Following the same line of reasoning, one finds that in type IIB string theory one has the D1-brane (also called the D-string), D3-brane, D5-brane, D7-brane and D9-brane.[32]

6.7 Supergravity from superstring theory

We consider here the low energy effective action of the closed superstring of type IIA and type IIB string theory.

[32] One can also define Dp-branes with p odd in type IIA string theory and p even in type IIB string theory, but they are unstable due to a tachyon in the spectrum of the open superstring. Hence the Dp-branes with p even (odd) in type IIA (IIB) string theory are the ones that are stable and in fact also supersymmetric.

We saw in Chapter 3 that the low energy effective action for the massless fields of bosonic closed string theory is (3.3.12). This follows from demanding conformal invariance of the theory for a single bosonic closed string immersed in a background formed by many bosonic closed strings. One can in principle do the same for the closed superstring.[33] The closed superstring has the following massless fields living in a ten-dimensional target space. The bosonic fields are the graviton field $G_{\mu\nu}$, the Kalb-Ramond field $B_{\mu\nu}$ and the dilaton Φ from the NS-NS sector and the n-form field strengths $F_{(n)}$ from the R-R sector, with n even for type IIA and odd for type IIB. The fermionic fields are the two spinors and two gravitinos from the R-NS and NS-R sectors. As noticed in Sec. 6.6, these fields correspond to the field content of ten-dimensional type IIA (IIB) supergravity for the type IIA (IIB) superstring. Indeed, one can show that type IIA (IIB) supergravity is the low energy effective description of type IIA (IIB) string theory.

Type IIA supergravity has action

$$\begin{aligned} S_{\mathrm{IIA}} = & \frac{1}{16\pi G_N} \int d^{10}x \sqrt{-G} \left[e^{-2\Phi} \Big(R + 4(\nabla\Phi)^2 - \frac{1}{12} H^2 \Big) - \frac{1}{4} F_{(2)}^2 - \frac{1}{2\cdot 4!} F_{(4)}^2 \right] \\ & - \frac{1}{32\pi G_N} \int B_{(2)} \wedge dC_{(3)} \wedge dC_{(3)} + \text{fermionic terms} \end{aligned} \tag{6.7.1}$$

with the field strengths

$$H_{(3)} = dB_{(2)} \ , \quad F_{(2)} = dC_{(1)} \ , \quad F_{(4)} = dC_{(3)} + H_{(3)} \wedge C_{(1)} \tag{6.7.2}$$

and the supergravity coupling constant

$$16\pi G_N = (2\pi)^7 l_s^8 g_s^2 \tag{6.7.3}$$

We note that the last bosonic term with $B_{(2)} \wedge dC_{(3)} \wedge dC_{(3)}$ is written using the wedge product of forms defined in (4.2.8). This term is topological, meaning that it does not depend on the metric.

Type IIB supergravity has action

$$\begin{aligned} S_{\mathrm{IIB}} = & \frac{1}{16\pi G_N} \int d^{10}x \sqrt{-G} \left[e^{-2\Phi} \Big(R + 4(\nabla\Phi)^2 - \frac{1}{12} H^2 \Big) - \frac{1}{2} F_{(1)}^2 - \frac{1}{2\cdot 3!} F_{(3)}^2 - \frac{1}{4\cdot 5!} F_{(5)}^2 \right] \\ & - \frac{1}{32\pi G_N} \int C_{(4)} \wedge H_{(3)} \wedge F_{(3)} + \text{fermionic terms} \end{aligned} \tag{6.7.4}$$

[33] In practise it is not clear how to do it this way since one does not know how to covariantly quantize a superstring in the presence of generic Ramond-Ramond field strengths. However, since one can show that the theory should be supersymmetric in ten dimensions, one can infer the equations of motion of the fields of the R-NS, NS-R and R-R sectors from those of the fields in the NS-NS sector. This determines the low energy effective action uniquely.

with the field strengths

$$H_{(3)} = dB_{(2)} \ , \ \ F_{(1)} = dC_{(0)} \ , \ \ F_{(3)} = dC_{(2)} - C_{(0)} \wedge H_{(3)} \ , \ \ F_{(5)} = dC_{(4)} - C_{(2)} \wedge H_{(3)} \quad (6.7.5)$$

and the supergravity coupling constant

$$16\pi G_N = (2\pi)^7 l_s^8 g_s^2 \quad (6.7.6)$$

6.8 Quantization of the open superstring and D-branes

We consider here the quantization of an open superstring ending on a Dp-brane. We take the Dp-brane to have flat world-volume with $x^0, x^1, ..., x^p$ being the parallel space-time directions to the Dp-brane and $x^{p+1}, ..., x^9$ being the orthogonal directions to the Dp-brane. The position of the Dp-brane in the orthogonal directions is given by $x^I = c^I$, $I = p+1, ..., 9$. The mode expansion of X^μ is

$$\begin{aligned} X^a(\tau,\sigma) &= x^a + 2l_s^2 p^a \tau + i\sqrt{2} l_s \sum_{n\neq 0} \frac{1}{n} \alpha_n^a e^{-in\tau} \cos(n\sigma) \\ X^I(\tau,\sigma) &= c^I - \sqrt{2} l_s \sum_{n\neq 0} \frac{1}{n} \alpha_n^I e^{-in\tau} \sin(n\sigma) \end{aligned} \quad (6.8.1)$$

with $a = 0, 1, ..., p$ and $I = p+1, ..., 9$ where we imposed Neumann conditions on the parallel directions and Dirichlet conditions on the orthogonal directions.

As found in Section 6.2.2 the boundary conditions for the fermionic field ψ^μ are independent of the ones on X^μ thus we can impose either the R condition or the NS condition on ψ^μ which gives the mode expansions (6.2.17) and (6.2.18).

We can now covariantly quantize the open superstring following the covariant quantization of the closed superstring. Essentially the quantization of the open superstring is read off from the quantization of the right-moving sector of the closed superstring. Thus we have the anti-commutator relation (6.3.3) for the modes d_n^μ and b_r^μ. The Super-Virasoro generators are given by the right-moving part of (6.3.6)-(6.3.10) and hence the constraints on the physical states by the right-moving part of (6.3.11). The results on the superconformal symmetry of Section 6.4 are the same as well, since the right and left-moving sectors of the closed superstring can be treated independently. Therefore, we find also for the open superstring that

$$D = 10 \ , \ \ a_{\rm R} = 0 \ , \ \ a_{\rm NS} = \frac{1}{2} \quad (6.8.2)$$

Thus, we record here the constraints on a physical state $|\phi\rangle$ of the open superstring

$$\begin{aligned} &\text{R:}\ \ L_n|\phi\rangle = 0 \ \ \text{for} \ \ n \geq 0 \ \ \text{and} \ \ F_m|\phi\rangle = 0 \ \ \text{for} \ \ m \geq 0 \\ &\text{NS:}\ \ L_n|\phi\rangle = \frac{1}{2}\delta_{n,0}|\phi\rangle \ \ \text{for} \ \ n \geq 0 \ \ \text{and} \ \ G_r|\phi\rangle = 0 \ \ \text{for} \ \ r > 0 \end{aligned} \tag{6.8.3}$$

We find from this the mass formula

$$\text{R:}\ \ M^2 = \frac{1}{l_s^2}\Big(N^{(\alpha)} + N^{(d)} \Big) \ , \quad \text{NS:}\ \ M^2 = \frac{1}{l_s^2}\Big(N^{(\alpha)} + N^{(b)} - \frac{1}{2} \Big) \tag{6.8.4}$$

where we defined

$$N^{(\alpha)} = \sum_{n=1}^{\infty} \eta_{\mu\nu}\alpha^\mu_{-n}\alpha^\nu_n \ , \quad N^{(d)} = \sum_{n=1}^{\infty} n\, \eta_{\mu\nu} d^\mu_{-n} d^\nu_n \ , \quad N^{(b)} = \sum_{r=\frac{1}{2}}^{\infty} r\, \eta_{\mu\nu} b^\mu_{-r} b^\nu_r \tag{6.8.5}$$

where we sum over $\mu,\nu = 0,1,...,9$. The GSO projection on a state $|\phi\rangle$ is

$$\text{R:}\ \ \widehat{\Gamma}_{11}(-1)^{\sum_{n=1}^\infty d_n\cdot d_n}|\phi\rangle = |\phi\rangle \ , \quad \text{NS:}\ \ (-1)^{\sum_{r=\frac{1}{2}}^\infty b_{-r}\cdot b_r}|\phi\rangle = -|\phi\rangle \tag{6.8.6}$$

where $\widehat{\Gamma}_{11}$ is defined as in Eq. (6.5.18).

Starting with an open superstring with the NS condition we see that the ground state $|0;k\rangle$ in the NS sector is defined by

$$\text{NS:}\ \ \begin{cases} \alpha^\mu_n|0;k\rangle = 0 \text{ for } n > 0 \ , \quad b^\mu_r|0;k\rangle = 0 \text{ for } r > 0 \\ p_a|0;k\rangle = k_a|0;k\rangle \end{cases} \tag{6.8.7}$$

The ground state $|0;k\rangle$ is projected out by the GSO projection (6.8.6) hence the lowest lying states in the NS sector are linear combinations of $b^\mu_{-1/2}|0;k\rangle$. We write

$$\text{NS:}\ \ A_a(k)\, b^a_{-1/2}|0;k\rangle \ , \quad \Phi_I(k)\, b^I_{-1/2}|0;k\rangle \tag{6.8.8}$$

where A_a, $a = 0,1,...,p$ is a $U(1)$ gauge field living on the world-volume of the Dp-brane and Φ_I, $I = p+1,...,9$, are $D-p-1$ scalar fields living on the world-volume of the Dp-brane (that they live on the world-volume can be seen from the fact that the momentum k_a is only along the Dp-brane world-volume directions hence the same is true also in position space). We see from (6.8.4) that these fields are massless $M = 0$. We notice that this is the same massless spectrum as that of an open bosonic string ending on a Dp-brane. Indeed, one can show that the $A_a(x)$ field in position space satisfies the equations (5.1.15) due to the physical state conditions and using spurious states one finds the gauge transformation (5.1.16). The Φ_I fields in position space satisfy the massless Klein-Gordon equation on the Dp-brane $\partial_a\partial^a\Phi_I = 0$. Thus, one has exactly the same interpretation of A_a as a $U(1)$ gauge boson living on the Dp-brane and Φ_I as describing a small fluctuation of the transverse position of the Dp-brane, as described by (5.2.5).

Turning to the open superstring with the R condition we see that the ground states are given by

$$\text{R:} \quad \begin{cases} \alpha_n^\mu|0;k\rangle_\alpha = d_n^\mu|0;k\rangle_\alpha = 0 \text{ for } n>0 \\ p_a|0;k\rangle_\alpha = k_a|0;k\rangle_\alpha \\ \sqrt{2}\, d_0^\mu|0;k\rangle_\alpha = (\Gamma^\mu)^\beta{}_\alpha|0;k\rangle_\beta \ , \quad (\Gamma_{11})^\beta{}_\alpha|0;k\rangle_\beta = |0;k\rangle_\alpha \end{cases} \tag{6.8.9}$$

which by construction already fulfil the GSO projection (6.8.6). Hence they correspond to the lowest lying states in this sector and we write a state as

$$\text{R:} \quad \chi^\alpha(k)|0;k\rangle_\alpha \tag{6.8.10}$$

The spinor field $\chi(k)$ is a Majorano-Weyl spinor with $\Gamma_{11}\chi = \chi$. As one can see from (6.8.4) it is a massless spinor field $M = 0$. This is the superpartner of the A_a and Φ_I fields that we found in the NS sector.

If we turn off χ^α the dynamics of the Dp-brane in ten-dimensional Minkowski space (without any of the background fields of type IIA or IIB supergravity turned on) is described by the DBI action 5.2.3. Since χ^a is a superpartner, one should use a supersymmetric version of the DBI action for general χ^a. This action has indeed been found.

Altogether we found that the lowest lying states of the open superstring ending on a Dp-brane are massless and one can furthermore infer that they belong to a supermultiplet. Thus, we again see that the superstring gives rise to supersymmetry in ten dimensions. This is true for the massive states of the open superstring as well.

Generalizing the above to the case of N parallel Dp-branes one finds the same spectrum just with the inclusion of the Chan-Paton factors in the ground states as well as in the fields. Considering the case of N coinciding Dp-branes one finds that the massless states of the open superstring ending on the N Dp-branes have the massless fields $A_a(k)$, Φ_I and $\chi^\alpha(k)$ all being in the adjoint representation of $U(N)$. In particular $A_a(k)$ becomes a $U(N)$ gauge field.

6.9 Exercises for Chapter 6

Exercise 6.1. Consider the classical superstring with action (6.1.1).

- Show that the SUSY transformation (6.1.7) is a symmetry of the action (6.1.1) where ϵ is a (Grassmann-valued) constant spinor.

- Use the Noether procedure of Section 1.4.2 to find the conserved current associated with the supersymmetry of the action. The answer (up to normalization) should be the supercurrent (6.1.11).

Exercise 6.2. In this exercise we play around with the Dirac matrices in two dimensions. We use in the following Eqs. (6.1.3) and (6.1.5). We define the helicity projectors

$$P_\pm = \frac{1 \pm \rho_0 \rho_1}{2} \tag{6.9.1}$$

and we use $\rho^\pm = \rho^0 \pm \rho^1$ and $\rho_\pm = \frac{1}{2}(\rho_0 \pm \rho_1)$.

- Show that

$$P_- \psi^\mu = \begin{pmatrix} \psi_-^\mu \\ 0 \end{pmatrix} \; , \quad P_+ \psi^\mu = \begin{pmatrix} 0 \\ \psi_+^\mu \end{pmatrix} \tag{6.9.2}$$

- Show the identities

$$\rho^\pm = 2\rho^0 P_\mp \; , \quad \rho^- \rho_- = -2P_- \; , \quad \rho^+ \rho_+ = -2P_+ \; , \quad \rho^+ \rho_- = \rho^- \rho_+ = 0 \tag{6.9.3}$$

- Using (6.9.2) and the first identity in (6.9.3) show that the superstring action (6.1.1) can be written in lightcone coordinates as (6.2.1).

- The supercurrent is given by (6.1.11). Show using (6.9.3) that in lightcone coordinates the supercurrent takes the form given by Eqs. (6.1.12) and (6.1.13).

- The energy-momentum tensor for the action (6.1.1) is given by Eq. (6.1.9). Show that in lightcone coordinates it has components given by (6.1.10) and $T_{+-} = T_{-+} = 0$.

Exercise 6.3. Consider the classical superstring with action (6.1.1). In this exercise we generalize the derivation of Section 1.4.2 and Exercise 1.10 of the conserved currents for Poincare invariance to the superstring.

- Consider the infinitesimal translation transformation $\delta X^\mu = \epsilon b^\mu$ and $\delta \psi^\mu = 0$ where ϵ is a small parameter. Derive that the current for translational invariance is

$$J^\alpha = b^\mu \Pi_\mu^\alpha \; , \quad \Pi_\mu^\alpha = T \partial^\alpha X_\mu \tag{6.9.4}$$

- Consider the infinitesimal Lorentz transformation

$$\delta X^\mu = \epsilon \omega^\mu{}_\nu X^\nu \; , \quad \delta \psi^\mu = \epsilon \omega^\mu{}_\nu \psi^\nu \tag{6.9.5}$$

where $\omega_{\mu\nu} = -\omega_{\nu\mu}$. Derive that the current for Lorentz invariance is

$$J_\alpha = \frac{1}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}_\alpha \ , \quad \mathcal{J}^{\mu\nu}_\alpha = -T(X^\mu\partial_\alpha X^\nu - X^\nu\partial_\alpha X^\mu + i\bar{\psi}^\mu\rho_\alpha\psi^\nu) \tag{6.9.6}$$

which is the same as (6.2.14).

Exercise 6.4. Fermionic harmonic oscillators.

- Consider an operator f and its hermitian adjoint $f^\dagger$ with the anti-commutators

$$\{f, f\} = 0 \ , \quad \{f^\dagger, f^\dagger\} = 0 \ , \quad \{f, f^\dagger\} = 1 \tag{6.9.7}$$

Define a ground state $|0\rangle$ as $f|0\rangle = 0$. Let the Hamiltonian be $H = f^\dagger f$. This is called a fermionic harmonic oscillator system. Derive the spectrum of the Hamiltonian.

- Define the three matrices corresponding to the spin-1/2 rotation generators

$$S_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad S_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad S_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.9.8}$$

Show that identifying $f = S_x - iS_y$ and $f^\dagger = S_x + iS_y$ gives the same anti-commutators (6.9.7). Show that one can identify the ground state $|0\rangle$ with the vector

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{6.9.9}$$

corresponding to the spin-down state (with respect to S_z). Finally, show that the Hamiltonian of the fermionic harmonic oscillator can be identified with $H = S_z + \frac{1}{2}$.

Exercise 6.5. Consider the following centrally extended super-Virasoro algebra corresponding to R periodicity condition with the super-Virasoro generators $\mathcal{L}_n$ and $\mathcal{F}_n$

$$\begin{gathered} [\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} + \Big(\frac{c}{12}m^3 + km\Big)\,\delta_{m+n,0} \\ [\mathcal{L}_m, \mathcal{F}_n] = \Big(\frac{1}{2}m - n\Big)\mathcal{F}_{m+n} \ , \quad \{\mathcal{F}_m, \mathcal{F}_n\} = 2\mathcal{L}_{m+n} + f(m)\delta_{m+n,0} \end{gathered} \tag{6.9.10}$$

for $m, n \in \mathbb{Z}$ where c and k are constants and $f(m)$ a function of m. Show using the (graded) Jacobi identity

$$[\mathcal{L}_m, \{\mathcal{F}_n, \mathcal{F}_l\}] - \{\mathcal{F}_l, [\mathcal{L}_m, \mathcal{F}_n]\} + \{\mathcal{F}_n, [\mathcal{F}_l, \mathcal{L}_m]\} = 0 \tag{6.9.11}$$

with $m, n, l \in \mathbb{Z}$ that the function $f(m)$ in the algebra (6.9.10) is fixed in terms of c and k to be

$$f(m) = \frac{c}{3}m^2 + k \tag{6.9.12}$$

Exercise 6.6. Consider the covariantly quantized closed superstring of Sec. 6.3. Take the spinor field $\psi^\mu(\tau,\sigma)$ to have the R-NS periodicity condition. Using the mode expansions written in (6.2.5) and (6.2.6), derive from the equal-time anticommutator (6.3.2) for $\psi^\mu(\tau,\sigma)$ that the right-moving modes have anticommutator $\{d^\mu_m, d^\nu_n\} = \eta^{\mu\nu}\delta_{m+n,0}$ and the left-moving modes have anticommutator $\{\tilde{b}^\mu_r, \tilde{b}^\nu_s\} = \eta^{\mu\nu}\delta_{r+s,0}$.[34]

Exercise 6.7. We consider here the degeneracy of the ground states for the R periodicity condition and the related 32-dimensional representation of the Clifford algebra.

- The ten operators $h^a_\pm$ are defined by (6.5.2) as linear combinations of the ten d^μ_0 zero-mode operators for the R periodicity condition (in the right-moving sector). Using the algebra (6.5.1) for the d^μ_0 operators derive the algebra (6.5.3) for the $h^a_\pm$ operators. Using this algebra, argue that (6.5.4) gives a basis of 32 linearly independent ground states.[35]

- Go through the derivation (6.5.8) that follows from (6.5.7). Use this to show that the 32 by 32 matrices Γ^μ obey the Clifford algebra (6.5.9) given that the operators d^μ_0 obey the algebra (6.5.1).

- Show the result (6.5.14) for the R-R sector ground states.

- Defining the operators $\widehat{\Gamma}_{11}$ and $\widehat{\tilde{\Gamma}}_{11}$ by Eqs. (6.5.18) and (6.5.20) show that for the R-R sector ground states $|0;k\rangle_\alpha{}^\beta$ one gets the representation written in (6.5.22).

Exercise 6.8. Consider the massless states (6.6.7) in the NS-NS sector.

- Show that the $G_{\frac{1}{2}}$ and $\tilde{G}_{\frac{1}{2}}$ constraints give the conditions $k^\mu\zeta_{\mu\nu}(k) = k^\nu\zeta_{\mu\nu}(k) = 0$.

- Consider a state $|\eta\rangle$ that obeys

$$\begin{aligned} L_n|\eta\rangle = 0\ , \quad (\tilde{L}_n - \tfrac{1}{2}\delta_{n,0})|\eta\rangle = 0 \quad \text{for} \quad n \geq 0 \\ G_r|\eta\rangle = \tilde{G}_r|\eta\rangle = 0 \quad \text{for} \quad r > 0 \end{aligned} \tag{6.9.13}$$

  Show that $|\chi\rangle = G_{-\frac{1}{2}}|\eta\rangle$ is a spurious state.[36]

[34] Hint: See Exercise 1.11 for how to get a specific Fourier mode of a periodic function by integrating over it. How does one do it for an anti-periodic function?

[35] Hint: It is easier to derive the algebra for $h^a_\pm$ if one defines $D^0 = id^0_0$ and $D^\mu = d^\mu_0$ for $\mu = 1,2,...,9$ since $\{D^\mu, D^\nu\} = \delta^{\mu\nu}$ and $\sqrt{2}h^a_\pm = D^{2a-2} \pm iD^{2a-1}$.

[36] Hint: Use the centrally extended Super-Virasoro algebra (6.4.4) to show that it is a physical state.

- Show that the state
$$|\eta\rangle = \frac{\sqrt{2}}{l_s} m_\mu \tilde{b}^\mu_{-\frac{1}{2}} |0;k\rangle \tag{6.9.14}$$
obeys the conditions (6.9.13) if we demand $k_\mu m^\mu = 0$. Use this to argue that
$$|\chi\rangle = k_\mu m_\nu b^\mu_{-\frac{1}{2}} \tilde{b}^\nu_{-\frac{1}{2}} |0;k\rangle \tag{6.9.15}$$
is a spurious state provided $k_\mu m^\mu = 0$. Hence we have found a gauge transformation of NS-NS massless states (6.6.7).

Exercise 6.9. Consider the massless states (6.6.10) in the R-NS sector.

- Show that it follows from the $\tilde{G}_{\frac{1}{2}}$ constraint that
$$k^\mu S^\alpha_\mu(k) = 0 \tag{6.9.16}$$

- Show that it follows from the F_0 constraint that
$$k_\nu (\Gamma^\nu)^\alpha{}_\beta S^\beta_\mu(k) = 0 \tag{6.9.17}$$

- Use Eqs. (6.9.16) and (6.9.17) to derive the massless Dirac equation for $\chi^\alpha(k)$ (in momentum space)
$$k^\mu (\Gamma_\mu)^\alpha{}_\beta \chi^\beta(k) = 0 \tag{6.9.18}$$

- Show that the following state in the R-NS sector
$$\tilde{G}_{-\frac{1}{2}} |0;k\rangle_\alpha \tag{6.9.19}$$
is spurious [Hint: Use the centrally extended Super-Virasoro algebra to show that it is a physical state.]. How does the state look if you use the explicit expression for $\tilde{G}_{-\frac{1}{2}}$?

- Use the spurious state just found to show that $S^\alpha_\mu(k)$ is invariant under the gauge transformation
$$S^\alpha_\mu(k) \to S^\alpha_\mu(k) + i k_\mu \eta^\alpha(k) \tag{6.9.20}$$
This implies that in position one has the gauge transformation
$$S^\alpha_\mu \to S^\alpha_\mu + \partial_\mu \eta^\alpha \tag{6.9.21}$$
where $\partial^\mu \partial_\mu \eta^\alpha = 0$. With some work, one can show that this imply a gauge transformation $\lambda^\alpha_\mu \to \lambda^\alpha_\mu + \partial_\mu \tilde{\eta}^\alpha$ with $\Gamma^\mu \partial_\mu \tilde{\eta}^\alpha = 0$ and $\partial^\mu \partial_\mu \tilde{\eta}^\alpha = 0$ and a gauge transformation $\chi^\alpha \to \chi^\alpha + \Gamma^\mu \partial_\mu \hat{\eta}^\alpha$ with $\partial^\mu \partial_\mu \hat{\eta}^\alpha = 0$.

Exercise 6.10. Consider the massless states (6.6.12) of the R-R sector.

- Derive from the chirality conditions (6.6.5) on the R-R ground states the relations (6.6.13) for the 32 by 32 matrix H.
- Show that Eq. (6.6.17) follows from the physical state conditions $F_0|\phi\rangle = \tilde{F}_0|\phi\rangle = 0$.
- Assume that only for a single n one has a non-zero $F_{(n)}$ field. Using (6.6.18), show that (6.6.19) follows from Eq. (6.6.17).[37]

[37] With a bit of work, one can also show it when all $F_{(n)}$ fields are turned on.

7 T-duality of Type IIA/B String Theory

7.1 Closed superstring on $\mathbb{R}^{1,8} \times S^1$

We have seen in Sec. 3 that for a general background the two-dimensional quantum field theory for a single string is interacting. Thus, we need to do perturbation theory in l_s to solve the theory order by order in a low energy expansion.[38] Instead for the Minkowski space background $\mathbb{R}^{1,9}$ the two-dimensional quantum field theory is a free theory. This means we can solve it exactly. However, there are also other backgrounds in which the two-dimensional quantum field theory is free. An example of this is provided by the type IIA/B closed string theory in the background $G_{\mu\nu} = \eta_{\mu\nu}$, $B_{\mu\nu} = 0$, $\Phi = 0$ and $F_{(n)} = 0$ but with one of the directions being periodic

$$x^9 \equiv x^9 + 2\pi R \tag{7.1.1}$$

Thus, we are considering a closed superstring in the background of nine-dimensional Minkowski space times a circle $\mathbb{R}^{1,8} \times S^1$.

Consider first the bosonic part of the superstring. Since the x^9 direction is periodic we can generalize the periodicity condition of the closed string in this direction as follows

$$X^9(\tau, \sigma + 2\pi) = X^9(\tau, \sigma) - 2\pi m R \ , \quad m \in \mathbb{Z} \tag{7.1.2}$$

The integer m is the winding number of the string since it counts how many times the closed string is wound around the circle in the x^9 direction, as illustrated on Fig. 12. The integer m is more commonly known as the *winding mode* of the string.

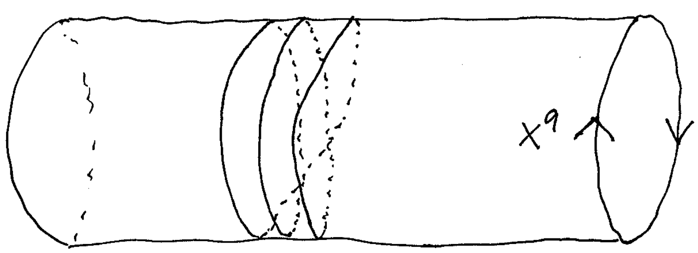

Figure 12: Illustration of the winding number m of the string in the x^9 direction for $m = 3$.

[38]In some cases it is possible to solve the theory to all orders using integrability of the string action.

From (1.4.26) we have the general mode expansions of $X^9(\xi)$ and $\Pi^9(\xi)$

$$X^9(\xi) = x^9 + \frac{l_s}{\sqrt{2}}(\alpha_0^9 + \tilde{\alpha}_0^9)\tau - \frac{l_s}{\sqrt{2}}(\alpha_0^9 - \tilde{\alpha}_0^9)\sigma + \frac{il_s}{\sqrt{2}}\sum_{n\neq 0}\frac{1}{n}\Big(\alpha_n^9 e^{-in\xi^-} + \tilde{\alpha}_n^9 e^{-in\xi^+}\Big)$$
$$\Pi^9(\xi) = \frac{1}{2\pi\sqrt{2}l_s}(\alpha_0^9 + \tilde{\alpha}_0^9) + \frac{1}{2\pi\sqrt{2}l_s}\sum_{n\neq 0}\Big(\alpha_n^9 e^{-in\xi^-} + \tilde{\alpha}_n^9 e^{-in\xi^+}\Big) \tag{7.1.3}$$

Imposing the periodicity boundary condition (7.1.2) we get

$$\alpha_0^9 - \tilde{\alpha}_0^9 = \frac{\sqrt{2}}{l_s}mR \tag{7.1.4}$$

The total momentum along the x^9 direction is $p_9 = \int_0^{2\pi} d\sigma \Pi_9 = (\alpha_0^9 + \tilde{\alpha}_0^9)/(\sqrt{2}l_s)$. Since the we need the translation operator $e^{ip_9 x^9}$ to be single-valued, we get that the total momentum is quantized $p_9 = k/R$ with $k \in \mathbb{Z}$. Hence,

$$\alpha_0^9 + \tilde{\alpha}_0^9 = \sqrt{2}l_s\frac{k}{R} \tag{7.1.5}$$

Solving for α_0^9 and $\tilde{\alpha}_0^9$ we get

$$\alpha_0^9 = \frac{l_s}{\sqrt{2}}\Big(\frac{k}{R} + \frac{mR}{l_s^2}\Big)\ ,\quad \tilde{\alpha}_0^9 = \frac{l_s}{\sqrt{2}}\Big(\frac{k}{R} - \frac{mR}{l_s^2}\Big) \tag{7.1.6}$$

Considering instead the fermionic part with the worldsheet spinor $\psi^\mu(\xi)$ the periodicity of the x^9 direction does not change anything compared to having ten-dimensional Minkowski space as the target space. Thus, nearly everything that we have derived in Secs. 6.1-6.3 carries over to the closed superstring on $\mathbb{R}^{1,8} \times S^1$. Only the formulas that depend on the zero modes α_0^9 and $\tilde{\alpha}_0^9$ are affected. In particular, the whole covariant quantization procedure carries over to this case as well. Therefore, in the following we use the Einstein summation convention for the target space indices $\mu, \nu = 0, 1, ..., 9$ to include the x^9 direction. The same goes for formulas like $\alpha_{-k} \cdot \alpha_k$ which includes the sum over $\alpha_{-k}^9 \alpha_k^9$.

Consider the covariantly quantized superstring with target space $\mathbb{R}^{1,8} \times S^1$. Since we are considering a space-time with a compact direction one should consider the mass M of the string states as measured in the nine-dimensional Minkowski space $\mathbb{R}^{1,8}$. Hence,

$$M^2 = -p_\mu p^\mu + (p_9)^2 \tag{7.1.7}$$

where we sum over $\mu = 0, 1, ..., 9$ (as explained in paragraph above). Considering the right- and left-moving sectors with R and NS conditions, we find

$$\begin{aligned}
\text{R:}\ \ M^2|\phi\rangle &= \left[\frac{4}{l_s^2}(N^{(\alpha)}+N^{(d)}) + \left(\frac{k}{R}+\frac{mR}{l_s^2}\right)^2\right]|\phi\rangle \\
\text{NS:}\ \ M^2|\phi\rangle &= \left[\frac{4}{l_s^2}\left(N^{(\alpha)}+N^{(b)}-\frac{1}{2}\right) + \left(\frac{k}{R}+\frac{mR}{l_s^2}\right)^2\right]|\phi\rangle \\
\text{R:}\ \ M^2|\phi\rangle &= \left[\frac{4}{l_s^2}(\tilde{N}^{(\alpha)}+\tilde{N}^{(d)}) + \left(\frac{k}{R}-\frac{mR}{l_s^2}\right)^2\right]|\phi\rangle \\
\text{NS:}\ \ M^2|\phi\rangle &= \left[\frac{4}{l_s^2}\left(\tilde{N}^{(\alpha)}+\tilde{N}^{(b)}-\frac{1}{2}\right) + \left(\frac{k}{R}-\frac{mR}{l_s^2}\right)^2\right]|\phi\rangle
\end{aligned} \tag{7.1.8}$$

where we used the definitions (6.6.2). From these formulas one can derive the spectrum of the superstring on $\mathbb{R}^{1,8}\times S^1$.

7.2 T-duality

T-duality for a superstring on $\mathbb{R}^{1,8}\times S^1$ is a symmetry that exchanges momentum with winding modes on the circle, *i.e.* $k\leftrightarrow m$. For this to be possible we need M^2 to be invariant under the symmetry, hence $\left(\frac{k}{R}\pm\frac{mR}{l_s^2}\right)^2$ should be invariant. This is only possible if we exchange $R\leftrightarrow l_s^2/R$ under the T-duality. Thus, the T-duality transformation sends

$$k\to m\ ,\quad m\to k\ ,\quad R\to\frac{l_s^2}{R} \tag{7.2.1}$$

We see that this corresponds to $\alpha_0^9\to\alpha_0^9$ and $\tilde{\alpha}_0^9\to-\tilde{\alpha}_0^9$. Since a given physical state fulfilling the constraints (6.3.11) should also be physical after the T-duality transformation we need to send $L_n\to L_n$ and $\tilde{L}_n\to\tilde{L}_n$. From the bosonic part (6.3.7) we see that this requires that

$$\alpha_n^9\to\alpha_n^9\ ,\quad \tilde{\alpha}_n^9\to-\tilde{\alpha}_n^9 \tag{7.2.2}$$

for all $n\in\mathbb{Z}$ under the T-duality transformation. Considering the constraints (6.3.11) involving the supercurrent modes (6.3.10) we find that we should send

$$d_n^9\to d_n^9\ ,\quad \tilde{d}_n^9\to-\tilde{d}_n^9\ ,\quad b_r^9\to b_r^9\ ,\quad \tilde{b}_r^9\to-\tilde{b}_r^9 \tag{7.2.3}$$

for all $n\in\mathbb{Z}$ and $r\in\mathbb{Z}+\frac{1}{2}$. We see that this in particular sends $\widehat{\Gamma}_{11}\to\widehat{\Gamma}_{11}$ and $\tilde{\widehat{\Gamma}}_{11}\to-\tilde{\widehat{\Gamma}}_{11}$. Thus, looking at the GSO projection (6.6.3) we find that T-duality sends

$$s\to-s \tag{7.2.4}$$

Hence T-duality maps type IIA string theory to type IIB string theory, and vice versa.

In conclusion, we have derived that

$$\begin{gathered}\text{Type IIA string theory on } \mathbb{R}^{1,8} \times S^1 \text{ with radius } R \\ \text{is dual to} \\ \text{Type IIB string theory on } \mathbb{R}^{1,8} \times S^1 \text{ with radius } l_s^2/R\end{gathered} \tag{7.2.5}$$

and that under this duality the winding modes and momentum modes on the circle are exchanged (7.2.1). Note that the T-duality transformation does not affect the string length l_s and string coupling g_s. This duality gives an exact map between the two string theories (7.2.1)-(7.2.4). Thus, any physics that we describe in one of the string theories can be mapped to the other.

A very important conclusion follows from the above T-duality (7.2.5): the space-time geometry of the target space is not a fundamental property in string theory. For instance, we can relate type IIA string theory on $\mathbb{R}^{1,8} \times S^1$ with R very large $R \gg l_s$ such that it is approximately ten-dimensional Minkowski space $\mathbb{R}^{1,9}$. Then on the type IIB string theory side the target space is approximately nine-dimensional Minkowski space $\mathbb{R}^{1,8}$ since the S^1 has an extremely small radius. So which of these very different target spaces are the right space-time? They both are!

One can also study T-duality for open superstrings. Then one learns that a Dp-brane in one string theory can be mapped either to a D$(p-1)$-brane or a D$(p+1)$-brane in the other string theory, depending on whether the circle is along the Dp-brane worldvolume, or not.

8 Landscape of Superstring Theories

In Chapter 6 we constructed two distinct superstring theories: The type IIA and IIB string theories. These are closed superstring theories that include D-branes on which open superstrings ends. However, one can show that it is possible to construct other superstring theories as well, namely two so-called Heterotic string theories and the type I string theory. Indeed, one has found five consistent superstring theories:

- Type IIA string theory
- Type IIB string theory
- $SO(32)$ Heterotic string theory
- $E_8 \times E_8$ Heterotic string theory
- Type I string theory

So why five superstring theories? As we shall review below, this turns out to be the wrong question to ask, since we shall see below that they are all part of one and the same theory, just in different disguises.

8.1 Heterotic string theories

To construct Heterotic string theories we exploit the fact that the right-moving and left-moving sectors of closed bosonic string and superstrings are decoupled from each other, apart from the level-matching condition. A Heterotic string theory is made by combining the right-moving sector of a closed superstring with the left-moving sector of a bosonic string. Classically, this is straightforward to do. But when quantizing the Heterotic string, one has a problem: For the right-moving sector the space-time dimension should be $D = 10$ whereas for the left-moving sector the space-time dimension should be $D = 26$. From point of view of the two-dimensional world-sheet theory, this is not a problem, since one simply specify that the right-moving sector consists of 10 right-moving scalar fields and 10 right-moving spinor fields, and that the left-moving sector consists of 26 left-moving scalar fields. But seen from the target-space point of view, how can we both live in ten and twenty-six dimensions at the same time?

To make this a consistent string theory, also from point of view of the target space, one compactifies the 26 dimensions on a 16-dimensional compact space. Hence, while the target space of the right movers are ten-dimensional Minkowski space $\mathbb{R}^{1,9}$, the target space of the left-movers are ten-dimensional Minkowski space times a 16-dimensional torus $\mathbb{R}^{1,9} \times T^{16}$. A 16-dimensional torus correponds to having 16 periodic directions, not necessarily at right angles with respect to each other. Therefore, one has in general the symmetry group $U(1)^{16}$. However, for special tori this symmetry group can be enhanced to a non-abelian Lie group of rank 16. One finds that the Lie groups of rank 16 which do not give rise to anomalies in the string theory when quantizing are the groups $SO(32)$ and $E_8 \times E_8$. Therefore, only these two Heterotic string theories are consistent. Note furthermore that one should impose the GSO projection (6.6.3) on the right-moving sector and that this makes the two Heterotic string theories space-time supersymmetric. The bosonic part of the massless spectrum consists of the graviton $G_{\mu\nu}$, Kalb-Ramond $B_{\mu\nu}$ and dilaton field Φ. There are no Ramond-Ramond fields and hence no D-branes in the theory.

8.2 Type I string theory

Consider type IIB string theory. In this string theory we find D9-branes. We now consider a setup where we put N D9-branes in type IIB string theory. D9-branes are space-filling branes which means that we do not break the ten-dimensional Poincare symmetry by putting them in the background of ten-dimensional Minkowski space. Now we have a theory with both open and closed superstrings where the open superstring have Chan-Paton factors corresponding to the adjoint representation of $U(N)$. Clearly the open strings ending on the D9-branes have Neumann boundary conditions along all the directions.

However, one finds that in general that the non-abelian gauge group $U(N)$ gives rise to anomalies. These anomalies can be cancelled first by putting a so-called orientifold 9-plane that basically makes the open superstrings lose their orientation by identifying σ with $\pi - \sigma$. This changes the $U(N)$ gauge group into an $SO(2N)$ gauge group. When one then puts $N = 16$ the theory is free of anomalies from the non-abelian group. Hence this gives rise to an additional superstring theory called Type I string theory with $SO(32)$ symmetry.

Thus Type I string theory comes from putting 16 D9-branes and one orientifold 9-plane in Type IIB string theory. Type I string theory is a superstring theory since it is supersymmetric seen from a ten-dimensional point of view. In particular the massless states of type I string theory are in a supermultiplet that is the same as that of type I supergravity with gauge group $SO(32)$.

The bosonic massless fields in Type I string theory are the graviton $G_{\mu\nu}$, the dilaton ϕ

and the Ramond-Ramond 3-form field strength $F_{(3)}$. This means one has D1-branes and D5-branes in the theory. The reason that one has less massless fields than compared to the type IIB string theory is that some of the fields are projected out by the orientifold plane.

8.3 T-duality, S-duality and unification of the five string theories

In Section 7.2 we saw that type IIA and type IIB string theory are dual to each other through the so-called T-duality (7.2.5). This means that any phenomena in type IIA string theory can be translated to type IIB string theory, and vice versa.[39] Thus, knowing only one superstring theory one can describe the other in full.

In addition to the T-duality (7.2.5) between type IIA and IIB string theory one has other dualities as well. For one thing, one has more T-dualities. For instance, one has a T-duality that maps $E_8 \times E_8$ Heterotic string theory on $\mathbb{R}^{1,8} \times S^1$ to $SO(32)$ Heterotic string theory on $\mathbb{R}^{1,8} \times S^1$.

Moreover, T-duality is not the only type of duality that one has in string theory. S-duality is a completely different type of duality that in general maps a strongly coupled $g_s \gg 1$ string theory to another theory (in most cases weakly coupled).

For type IIB string theory with string coupling g_s and string length l_s one finds that it is mapped to type IIB string theory with new string coupling $\hat{g}_s = 1/g_s$ and new string length $\hat{l}_s = \sqrt{g_s} l_s$. Hence strongly coupled IIB string theory with $g_s \gg 1$ is dual to weakly coupled type IIB string theory with $\hat{g}_s \ll 1$.

For type IIA string theory with string coupling g_s and string length l_s one finds that it is mapped to the so-called M-theory on the 11-dimensional space-time $\mathbb{R}^{1,9} \times S^1$ where the circle has radius $R = g_s l_s$ and Planck length $l_p = g_s^{1/3} l_s$. Hence $g_s \gg 1$ implies $R \gg l_p$ which means that one gets eleven-dimensional Minkowski space if one takes $g_s \to \infty$.

What is M-theory? The fundamental description of M-theory is not known. However, it is known that the low energy effective description of M-theory is eleven-dimensional supergravity theory. This supergravity theory is special since 11 is the highest possible space-time dimension for a supergravity theory, and since there is only one possible su-

[39] One could object that this duality relies on having the two theories compactified on a circle. However, having type IIA on a very large circle is approximately the same as having it in ten-dimensional Minkowski space. The winding modes then become arbitrarily heavy compared to states with a finite or zero mass in the spectrum. This can then be described exactly by type IIB on a vanishingly small circle with very light winding modes.

pergravity theory in eleven dimensions, hence it is a unique theory. This suggests that M-theory, what ever it is, is also a unique theory.

Also for the Heterotic string theories and the type I string theory we have an S-duality map. The $SO(32)$ Heterotic string theory with string coupling g_s is mapped to Type I string theory with string coupling $\hat{g}_s = 1/g_s$.

Finally, the $E_8 \times E_8$ Heterotic string theory with string coupling g_s and string length l_s is mapped to M-theory on the eleven-dimensional space-time $\mathbb{R}^{1,9} \times [0, \pi R]$ with $R = g_s l_s$ and Planck length $l_p = g_s^{1/3} l_s$. Hence $g_s \gg 1$ implies $R \gg l_p$. Note that $\mathbb{R}^{1,9} \times [0, \pi R]$ is a manifold with two boundaries, and the two E_8 groups are associated to supersymmetric gauge theories that live on each boundary.

Using the above described T- and S-dualities, we can connect all of the five string theories and M-theory. This is illustrated in Fig. 13. Thus, what we learn from this is that we should not think of the five string theories as separate theories, but as particular cases of a larger theory that connects the five of them.

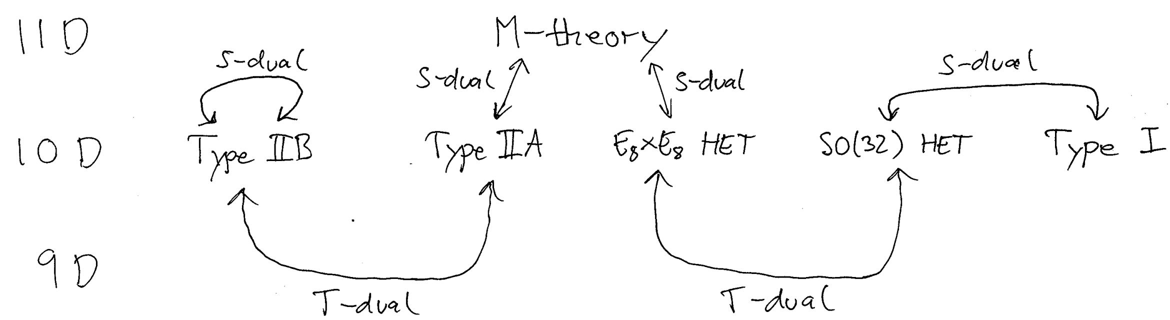

Figure 13: Illustration of the T-dualities and S-dualities that connects the five weakly coupled string theories and M-theory.

More generally, if one thinks more abstractly about the space of theories of non-perturbative string theory, one has that the five weakly coupled superstring theories are special points of a more general theory, and so is 11-dimensional supergravity, as illustrated on Fig. 14. One has in addition found many other interesting special points that one can describe.

In modern terms, one describes the theory space illustrated in Fig. 14 as a landscape of string theories. Included in this landscape are all the string theories that one gets by compactifying to lower-dimensional space-times, including four dimensions.

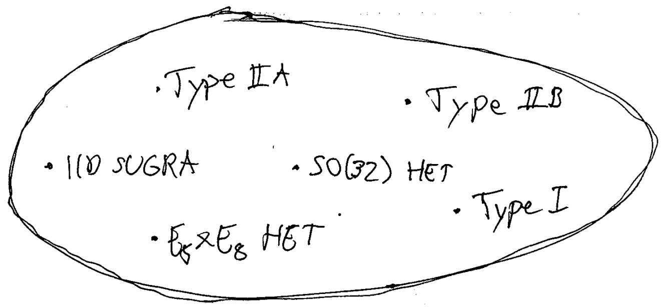

Figure 14: The landscape of theories, including the five weakly coupled string theories and 11-dimensional supergravity as special points.

A Lightcone Quantization of the Closed String

The idea of lightcone quantization is that one uses the remaining conformal symmetry (1.6.9) of the Polyakov action in flat gauge (1.3.18) to make a nice gauge choice that breaks explicit Lorentz invariance of the target space. In this gauge one can solve the constraint (1.3.7) explicitly, thus revealing a classical theory for the string that is manifestly physical. One then proceeds to quantize this theory, avoiding completely the problems with ghost states and with imposing constraints in the old covariant approach of Sec. 2.1. However, the price one has to pay, as we shall see, is that breaking the explicit Lorentz invariance of the classical theory makes it possible that an anomaly occurs for the Lorentz invariance in the quantum theory.

A.1 Classical string in lightcone gauge

Our starting point is the Polyakov action in flat gauge (1.3.18). This action has D-dimensional Minkowski space has D-dimensional Minkowski space as its target space. Clearly the Lorentz invariance is explicitly realized in the action (1.3.18) as it is invariant under $X^\mu \to \Lambda^\mu{}_\nu X^\nu$ with $\eta_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$. Introduce now lightcone coordinates for the target space

$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^{D-1}) \qquad \text{(A.1.1)}$$

such that the D-dimensional Minkowski space has coordinates (x^+, x^-, x^i), $i = 1, 2, ..., D-2$. This gives the target space metric (and its inverse)

$$\begin{aligned} \eta_{+-} = \eta_{-+} = \eta^{+-} = \eta^{-+} = -1 \\ \eta_{ij} = \eta^{ij} = \delta_{ij} \quad , \quad i, j = 1, 2, ..., D-2 \end{aligned} \qquad \text{(A.1.2)}$$

with all other components being zero.

We found in Sec. 1.6 that the Polyakov action in flat gauge (1.3.18) is invariant under the conformal transformations (1.6.9). We can write these transformations as

$$\tau \to \tilde{\tau}(\tau,\sigma) = \frac{1}{2}(\tilde{\xi}^+(\xi^+) + \tilde{\xi}^-(\xi)^-) \ , \quad \sigma \to \tilde{\sigma}(\tau,\sigma) = \frac{1}{2}(\tilde{\xi}^+(\xi^+) - \tilde{\xi}^-(\xi)^-) \tag{A.1.3}$$

This means that the remnant conformal symmetry of (1.3.18) gives us the freedom to choose the new coordinate $\tilde{\tau}(\tau,\sigma)$ equal to any function that satisfies the wave equation $\partial_-\partial_+\tilde{\tau} = 0$ as long as our choice is consistent with having $\tilde{\sigma}(\tau,\sigma)$ a periodic coordinate. Given such a choice $\tilde{\sigma}(\tau,\sigma)$ is fixed up to a constant.[40] Since $X^+(\xi)$ has the equation of motion $\partial_-\partial_+X^+ = 0$ we see from the above argument that we can make the gauge choice

$$\tilde{\tau}(\xi) = \frac{1}{l_s^2 p^+}(X^+(\xi) - x^+) \tag{A.1.4}$$

where x^+ is the center of mass for $\tau = 0$ and p^+ is the total momentum for the string in the x^+ direction. Hence choosing this gauge means that we have

$$X^+(\xi) = x^+ + l_s^2 p^+ \tau \tag{A.1.5}$$

This is called the *lightcone gauge*. Note we renamed $\tilde{\xi}^\alpha$ as ξ^α and $\tilde{\tau}$ as τ in (A.1.5). Comparing to the mode expansion (1.4.27) without gauge fixing we see that the lightcone gauge (A.1.5) effectively means that we have set $\alpha_n^+ = \tilde{\alpha}_n^+ = 0$ for all $n \neq 0$. We note that with this gauge choice we used up all the remaining local symmetry of the Polyakov action in flat gauge (1.3.18) (apart from a trivial translation in σ). Thus, since we used up all the local symmetry on the worldsheet it means that the gauge-fixed theory should be manifestly physical. This is indeed what we shall see below.

Consider now the constraints $T_{--} = 0$ and $T_{++} = 0$. The constraint $T_{--} = 0$ means $\eta_{\mu\nu}\partial_- X^\mu \partial_- X^\nu = 0$. Using (A.1.2) we see that this gives $\sum_{i=1}^{D-2} \partial_- X^i \partial_- X^i - 2\partial_- X^+ \partial_- X^- = 0$. Using now the lightcone gauge (A.1.5) we see this gives

$$\partial_- X^- = \frac{1}{l_s^2 p^+} \sum_{i=1}^{D-2} (\partial_- X^i)^2 \tag{A.1.6}$$

Inserting the mode expansion (1.4.27) for X^- and X^i in this we get

$$\alpha_n^- = \frac{1}{\sqrt{2} l_s p^+} \sum_{k\in\mathbb{Z}} \sum_{i=1}^{D-2} \alpha_{n-k}^i \alpha_k^i \tag{A.1.7}$$

[40] This statement can be seen as follows. If we know $\tilde{\tau}(\xi)$ for all ξ, it means we know $\tilde{\tau}(\xi^-,\xi^+)$ for all ξ^- and ξ^+. We have $\tilde{\xi}^+(\xi^+) - \tilde{\xi}^+(0) + \tilde{\xi}^-(\xi)^- - \tilde{\xi}^-(0) = 2\tilde{\tau}(\xi^-,\xi^+) - 2\tilde{\tau}(0,0)$. Hence from $\tilde{\tau}(\xi^-,\xi^+)$ we can find both $\tilde{\xi}^+(\xi^+) - \tilde{\xi}^+(0)$ and $\tilde{\xi}^-(\xi)^- - \tilde{\xi}^-(0)$. Thus, we find $2\tilde{\sigma}(\xi^-,\xi^+)$ up to the constant $\tilde{\xi}^+(0) - \tilde{\xi}^-(0)$.

Similarly we get for the left-moving sector

$$\partial_+ X^- = \frac{1}{l_s^2 p^+} \sum_{i=1}^{D-2} (\partial_+ X^i)^2 \ , \quad \tilde{\alpha}_n^- = \frac{1}{\sqrt{2} l_s p^+} \sum_{k \in \mathbb{Z}} \sum_{i=1}^{D-2} \tilde{\alpha}_{n-k}^i \tilde{\alpha}_k^i \tag{A.1.8}$$

With this, we have accomplished to write $X^\mu(\xi)$ solely in terms of modes α_n^i and $\tilde{\alpha}_n^i$, with $n \in \mathbb{Z}$ and $i = 1, 2, ..., D-2$, and the two zero-modes x^+ and p^+. The interpretation is here that in the lightcone gauge (A.1.5) only the *transverse directions* $X^i(\xi)$, $i = 1, 2, ..., D-2$, corresponds to physical degrees of freedom. In conclusion, we used the remaining local symmetry (1.6.9) on the world-sheet to fix the gauge (A.1.5) and with this our description of the closed string is manifestly only in terms of physical fields.[41]

The above is perfectly analogous to what happens in the case of the relativistic point particle. In the covariant formalism of Sec. 1.1.2 $X^0(\tau)$ can be any function such that the equations of motion (1.1.22) and the constraint (1.1.28) are obeyed. However, choosing the static gauge $X^0(\tau) = \tau$ and solving the constraint we get the gauge fixed description of Sec. 1.1.1 which is manifestly physical.

A.2 Quantized string in lightcone gauge

So far we have been describing the lightcone gauge-fixed theory classically. We now turn to the quantized theory. We quantize the theory by quantizing the transverse fields $X^i(\xi)$, $i = 1, 2, ..., D-2$. This is done by promoting all the transverse modes α_n^i and $\tilde{\alpha}_n^i$ to operators and impose the canonical equal-time commutation relations

$$\begin{gathered} [X^j(\tau,\sigma), \Pi_k(\tau,\sigma')] = i\delta(\sigma-\sigma')\delta_k^j \\ [X^j(\tau,\sigma), X^k(\tau,\sigma')] = [\Pi_j(\tau,\sigma), \Pi_k(\tau,\sigma')] = 0 \end{gathered} \tag{A.2.1}$$

for $j, k = 1, 2, ..., D-2$, along with the reality conditions

$$(X^j(\tau,\sigma))^\dagger = X^j(\tau,\sigma) \ , \quad (\Pi_j(\tau,\sigma))^\dagger = \Pi_j(\tau,\sigma) \tag{A.2.2}$$

We find that this is equivalent to imposing the following commutators for the transverse modes

$$\begin{gathered} [x^j, p_k] = i\delta_k^j \ , \quad [\alpha_m^j, \alpha_n^k] = [\tilde{\alpha}_m^j, \tilde{\alpha}_n^k] = m\delta_{m+n,0}\delta^{jk} \\ [x^j, x^k] = [p_j, p_k] = 0 \ , \quad [\alpha_m^j, \tilde{\alpha}_n^k] = 0 \end{gathered} \tag{A.2.3}$$

as well as the reality conditions

$$(x^j)^\dagger = x^j \ , \quad (p_j)^\dagger = p_j \ , \quad (\alpha_n^j)^\dagger = \alpha_{-n}^j \ , \quad (\tilde{\alpha}_n^j)^\dagger = \tilde{\alpha}_{-n}^j \tag{A.2.4}$$

[41]Note however that the constraint $\alpha_0^- = \tilde{\alpha}_0^-$ coming from (1.4.28) is not solved with the above. This will have to be dealt with later on when discussing the spectrum of the theory.

with $j,k = 1,2,...,D-2$. Superficially, the above quantization might look very similar to what we found in the covariant approach of Sec. 2.1. The differences are crucial, however. Defining

$$a_n^j = \frac{1}{\sqrt{n}}\alpha_n^j \ , \quad (a_n^j)^\dagger = \frac{1}{\sqrt{n}}\alpha_{-n}^j \tag{A.2.5}$$

for $n > 0$ we get the commutation relation

$$[a_n^j, (a_m^k)^\dagger] = \delta_{n,m}\delta^{jk} \tag{A.2.6}$$

We can again interpret this as a bunch of harmonic oscillators. However, the right-hand side of (A.2.6) is now positive definite. Thus, we have completely removed the problem of ghost states that one has to deal with in the covariant approach of Sec. 2.1.

In line with the interpretation of α_n^i and $\tilde{\alpha}_n^i$ as annihilation operators and α_{-n}^i and $\tilde{\alpha}_{-n}^i$ as creating operators for $n > 0$, we define the ground state $|0;k\rangle = |0;k_i;p^+\rangle$ as

$$\begin{aligned} p_j|0;k\rangle &= k_j|0;k\rangle \\ \alpha_n^j|0;k\rangle &= \tilde{\alpha}_n^j|0;k\rangle = 0 \ , \quad n > 0 \end{aligned} \tag{A.2.7}$$

for $j = 1,2,...,D-2$, as well as demanding that $|0;k\rangle$ is in a eigenstate of p^+ (the operator) with eigenvalue p^+. We use the normalization $\langle 0;k|0;k\rangle = 1$ just as in Sec. 2.1. A basis for the closed string Fock space is

$$\alpha_{-n_1}^{j_1}\alpha_{-n_2}^{j_2}\cdots\alpha_{-n_p}^{j_p}\tilde{\alpha}_{-m_1}^{k_1}\tilde{\alpha}_{-m_2}^{k_2}\cdots\tilde{\alpha}_{-m_q}^{k_p}|0;k\rangle \tag{A.2.8}$$

with $n_i, m_i > 0$. We see that all states in this Fock space have positive norm.

However, we should also quantize the classical expressions Eqs. (A.1.6)-(A.1.8) for $X^-(\xi)$ in terms of the transverse modes. Here we again encounter a possible normal ordering constant when quantizing the expressions for α_0^- and $\tilde{\alpha}_0^-$ in terms of the transverse modes α_n^i and $\tilde{\alpha}_n^i$. In line with our considerations of Sec. 2.1 we introduce the normal ordering constant a to parametrize the ambiguity in the quantization prescription and we write

$$\begin{aligned} \alpha_n^- &= \frac{1}{\sqrt{2}l_s p^+}\left(\sum_{k\in\mathbb{Z}}\sum_{i=1}^{D-2} :\alpha_{n-k}^i\alpha_k^i: -2a\delta_{n,0}\right) \\ \tilde{\alpha}_n^- &= \frac{1}{\sqrt{2}l_s p^+}\left(\sum_{k\in\mathbb{Z}}\sum_{i=1}^{D-2} :\tilde{\alpha}_{n-k}^i\tilde{\alpha}_k^i: -2a\delta_{n,0}\right) \end{aligned} \tag{A.2.9}$$

Note that at this point it is not obvious that the constant a in the above is related to the constant a introduced in Sec. 2.1. However, we shall see below that this actually is the case. We also note that we picked the same constant for both the right- and the left-moving sectors as the procedure to find these normal ordering constants will be identical for the two sectors.

Considering in particular the zero-modes α_0^- and $\tilde{\alpha}_0^-$ we get from (1.4.28) that $l_s p^-/\sqrt{2} = \alpha_0^- = \tilde{\alpha}_0^-$. Hence using (A.2.9) we find

$$2p^+p^- = \sum_{i=1}^{D-2} p_i^2 + \frac{4}{l_s^2}(N-a) = \sum_{i=1}^{D-2} p_i^2 + \frac{4}{l_s^2}(\tilde{N}-a) \tag{A.2.10}$$

with N and $\tilde{N}$ defined as

$$N = \sum_{i=1}^{D-2}\sum_{k=1}^{\infty} \alpha_{-k}^i \alpha_k^i \ , \quad \tilde{N} = \sum_{i=1}^{D-2}\sum_{k=1}^{\infty} \tilde{\alpha}_{-k}^i \tilde{\alpha}_k^i \tag{A.2.11}$$

and where we used (1.4.28) for the transverse directions as well. We see now that for consistency we have to impose the constraint

$$(N - \tilde{N})|\phi\rangle = 0 \tag{A.2.12}$$

for any physical state $|\phi\rangle$ in the quantum theory of the closed string. This is called the *level-matching* constraint. It is imposed directly on the closed string Fock space, with basis (A.2.8). With this we have identified all the physical states of the closed string.

Thus, using the lightcone gauge (A.1.5) we have succesfully found a manifestly physical quantum description of the closed string. However, when going from a classical to a quantum theory, one should always beware of symmetries that are not explicitly realized in the theory. Indeed, one might encounter an anomaly, which means a new contribution in the quantum theory that is absent in the classical theory but which can spoil the symmetry. In this case we notice that in the classical theory in lightcone gauge the Lorentz invariance is not explicitly realized. Of course, in the classical theory we know that the Lorentz invariance is still there. Instead in the quantum theory it is not obvious.

In the covariant quantization approach of Chapter 2 it is shown explicitly that the quantum theory is Lorentz invariant. Indeed, this can be seen from the fact that the Lorentz generators (2.3.7) obey the Lorentz symmetry algebra

$$[J^{\mu\nu}, J^{\rho\sigma}] = -i(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho}) \tag{A.2.13}$$

as part of the Poincaré algebra (2.3.6). As we have shown in Sec. 2.3.1 this algebra is a direct consequence of the Lorentz invariance of the theory. Conversely, any theory that satisfy the Lorentz algebra is Lorentz invariant.

Going back to the lightcone quantization of the closed string the Lorentz algebra (A.2.13) should still hold, otherwise it would not be a Lorentz invariant theory. Consider now the subset of generators

$$J^{j-} = \int_0^{2\pi} d\sigma : (X^j \Pi^- - X^- \Pi^j) : \tag{A.2.14}$$

with $j = 1, 2, ..., D-2$. After a substantial amount of very tedious algebra, one finds

$$[J^{j-}, J^{k-}] = \frac{2}{(p^+)^2} \sum_{n=1}^{\infty} \Delta_n (\alpha^j_{-n}\alpha^k_n - \alpha^k_{-n}\alpha^j_n + \tilde{\alpha}^j_{-n}\tilde{\alpha}^k_n - \tilde{\alpha}^k_{-n}\tilde{\alpha}^j_n) \tag{A.2.15}$$

with

$$\Delta_n = \left(\frac{D-2}{24} - 1\right) n + \left(a - \frac{D-2}{24}\right)\frac{1}{n} \tag{A.2.16}$$

Instead the Lorentz algebra (A.2.13) dictates $[J^{j-}, J^{k-}] = 0$. Thus, we can conclude that the lightcone quantized closed string only has target space Lorentz invariance provided

$$a = 1 \quad \text{and} \quad D = 26 \tag{A.2.17}$$

since only in that case is $\Delta_n = 0$ for all $n \geq 1$.

Conversely, we see that while the classical closed string in the lightcone gauge has Lorentz symmetry, this symmetry is generically broken when going to the quantum theory (apart from the special case (A.2.17)). This is an example of an *anomaly*, a symmetry that exists in the classical theory but which is broken by a new term that only appears in the quantum theory.

A.3 Exercises for Appendix A

Exercise A.1. In this exercise we consider the closed bosonic string in the lightcone quantization approach. This means that the constraints impose that α^-_n and $\tilde{\alpha}^-_n$ are given by (A.2.9). For the right-moving sector define the operators

$$A_n = \frac{l_s}{\sqrt{2}} p^+ \alpha^-_n \tag{A.3.1}$$

with $n \in \mathbb{Z}$.

- Using Exercise 2.7 and 2.8 argue that the algebra of the A_n operators is a centrally extended Virasoro algebra

$$[A_m, A_n] = (m-n)A_{m+n} + (a_1 m + a_2 m^3)\delta_{m+n,0} \tag{A.3.2}$$

where a_1 and a_2 are undetermined constants. Hint: Look first at the set of operators $A_n + a\delta_{n,0}$ and argue they obey a centrally extended Virasoro algebra (no computations needed for this).

- We use here the ground state with zero transverse momentum written as $|0;0;p^+\rangle$. It obeys $p_j|0;0;p^+\rangle = 0$, $\alpha_n^j|0;0;p^+\rangle = 0$ for $n > 0$ and $\langle 0;0;p^+|0;0;p^+\rangle = 1$ with $j = 1,2,...,D-2$. Using Exercise 2.9 as a guide, argue that (no computations needed for this)

$$\begin{aligned} &A_n|0;0;p^+\rangle = 0 \ \text{ for } \ n \geq 1 \ , \quad A_0|0;0;p^+\rangle = -a|0;0;p^+\rangle \\ &A_{-1}|0;0;p^+\rangle = 0 \ , \quad A_{-2}|0;0;p^+\rangle = \frac{1}{2}\sum_{j=1}^{D-2} \alpha_{-1}^j \alpha_{-1}^j |0;0;p^+\rangle \end{aligned} \tag{A.3.3}$$

- Using again Exercise 2.9 as a guide, show using (A.3.3) that $a_1 + a_2 = 2a$ and $2a_1 + 8a_2 = 4a + \frac{1}{2}(D-2)$ which means the centrally extended Virasoro algebra of the A_n operators is

$$[A_m, A_n] = (m-n)A_{m+n} + \left[\frac{D-2}{12}(m^3 - m) + 2am\right]\delta_{m+n,0} \tag{A.3.4}$$

Exercise A.2. In this exercise we consider the closed bosonic string in the lightcone quantization approach. This means that the constraints impose that α_n^- and $\tilde{\alpha}_n^-$ are given by (A.2.9). In particular since $l_s p^- = \sqrt{2}\alpha_0^-$ we find

$$p^- = \frac{1}{2p^+}\sum_{j=1}^{D-2}(p^j)^2 + \frac{2}{l_s^2 p^+}\left(\sum_{n=1}^{\infty}\sum_{j=1}^{D-2}\alpha_{-n}^j\alpha_n^j - a\right) \tag{A.3.5}$$

The goal of this exercise is to compute the commutator $[J^{j-}, J^{k-}]$ for $j,k = 1,2,...,D-2$. Here J^{j-} can be read from (2.3.8)-(2.3.9) to be

$$\begin{aligned} &J^{j-} = l^{j-} + E^{j-} + \tilde{E}^{j-} \ , \quad l^{j-} = x^j p^- - x^- p^j \\ &E^{j-} = -i\sum_{n=1}^{\infty}\frac{1}{n}\left(\alpha_{-n}^j\alpha_n^- - \alpha_{-n}^-\alpha_n^j\right) \ , \quad \tilde{E}^{j-} = -i\sum_{n=1}^{\infty}\frac{1}{n}\left(\tilde{\alpha}_{-n}^j\tilde{\alpha}_n^- - \tilde{\alpha}_{-n}^-\tilde{\alpha}_n^j\right) \end{aligned} \tag{A.3.6}$$

Since α_n^- and specifically p^- are given in terms of the other modes we cannot impose the commutator relations that involve α_n^- and p^-. Those should be derived. However, for all other possible commutator relations one can read them off from (2.1.3). Thus we are assuming the following commutators

$$\begin{aligned} &[x^j, x^k] = [x^-, x^j] = [x^+, x^j] = [x^+, x^-] = 0 \ , \quad [p^j, p^k] = [p^+, p^j] = 0 \\ &[x^j, p^+] = [x^+, p^j] = [x^-, p^j] = 0 \ , \quad [x^j, p^k] = i\delta^{jk} \ , \quad [x^-, p^+] = -i \end{aligned} \tag{A.3.7}$$

as well as

$$
\begin{aligned}
&[\alpha^j_m, \alpha^k_n] = [\tilde{\alpha}^j_m, \tilde{\alpha}^k_n] = m\delta_{m+n,0}\delta^{jk} \ , \quad [\alpha^j_m, \tilde{\alpha}^k_n] = 0 \\
&[x^j, \alpha^k_m] = [x^j, \tilde{\alpha}^k_m] = 0 \quad \text{for} \quad m \neq 0 \\
&[x^+, \alpha^j_m] = [x^-, \alpha^j_m] = [p^j, \alpha^k_m] = [p^+, \alpha^j_m] = 0 \\
&[x^+, \tilde{\alpha}^j_m] = [x^-, \tilde{\alpha}^j_m] = [p^j, \tilde{\alpha}^k_m] = [p^+, \tilde{\alpha}^j_m] = 0
\end{aligned}
\tag{A.3.8}
$$

for $j, k = 1, 2, ..., D-2$. We note that p^+ commutes with all operators except x^-. Therefore, in the computations below we can effectively regard p^+ as a number in all the commutators and expressions where x^- is not present.

- Show that

$$
[p^j, p^-] = [p^+, p^-] = 0 \ , \quad [x^-, p^-] = \frac{i}{p^+}p^- \ , \quad [x^j, p^-] = \frac{i}{p^+}p^j \tag{A.3.9}
$$

  Hint for the third one: Show first that $[x^-, p^+p^-] = 0$. Use this and (A.3.7) to show that

$$
[l^{j-}, l^{k-}] = 0 \tag{A.3.10}
$$

- Show that

$$
\begin{gathered}
[x^j, \alpha^-_n] = \frac{i}{p^+}\alpha^j_n \ , \quad [x^-, \alpha^-_n] = \frac{i}{p^+}\alpha^-_n \ , \quad [p^j, \alpha^-_n] = 0 \\
[p^-, \alpha^-_n] = -\frac{2}{l_s^2 p^+} n\alpha^-_n \ , \quad [\alpha^j_m, \alpha^-_n] = \frac{\sqrt{2}}{l_s p^+} m\alpha^j_{n+m} \ , \quad [\alpha^j_m, p^-] = \frac{2}{l_s^2 p^+} m\alpha^j_m
\end{gathered}
\tag{A.3.11}
$$

  Hint for the second one: First show that $[x^-, p^+\alpha^-_n] = 0$. Hint for the fourth one: Use the algebra (A.3.4). Use these commutators along with (A.3.8) to show

$$
[p^j, E^{k-}] = [p^-, E^{j-}] = 0 \ , \quad [x^j, E^{k-}] = -\frac{i}{p^+}E^{jk} \ , \quad [x^-, E^{j-}] = \frac{i}{p^+}E^{j-} \tag{A.3.12}
$$

- (To be continued...)

B Ghost Field Sector from Gauge-Fixing of the Quantum String

We have seen in Sec. 2.4 that the conformal algebra that arises when covariantly quantizing the closed string seemingly is anomalous. The anomaly being that the central charge of the two copies of the extended Virasoro algebra (2.4.4) is D rather than zero, as it should be according to (2.4.3). As mentioned above, it is important that the conformal symmetry of string theory as formulated by the Polyakov action in flat gauge (1.3.18) is not anomalous since the presence of the conformal symmetry ensures that the Polyakov action in flat gauge describes the same physics as the Nambu-Goto action (1.2.11). However, it turns out, as we shall see below, that by understanding better the covariant quantization of the Polyakov action we can in fact remove this anomaly under specific circumstances, thus getting a consistent quantum theory for the closed string.

What we shall see in the following is that the source of the problem with the anomaly in the conformal algebra is that we have not properly implemented the gauge of flat gauge (1.3.17) for the worldsheet metric in the quantum theory. While one can easily implement a gauge choice in a classical theory, it is less obvious in the quantum theory since one should include not only the classical configuration in the path integral, but sum over all possible configurations, using a measure that gives a weight for each path.

B.1 Gauge fixing the string path integral

Consider the general Polyakov action (1.3.1). This action depends on the metric field g and the string embedding map X. To properly quantize this action we should understand the path integral

$$Z = \int \mathcal{D}g\, \mathcal{D}X \exp(iS_{\rm pol}[g, X]) \tag{B.1.1}$$

Notice that this path integral includes integration of all two-dimensional metrics $g_{\alpha\beta}$ (given certain global boundary conditions such as topology). So given this, how can we make sense of fixing the gauge to a specific metric? The way to do this is called the Fadeev-Popov method. We write here $\hat{g}_{\alpha\beta}$ for the specific metric we want to implement as our gauge choice (thus $\hat{g}_{\alpha\beta} = \eta_{\alpha\beta}$ for flat gauge). Using a diffeomorphism and a Weyl rescaling we can transform $\hat{g}$ into any two-dimensional metric g, at least locally. We write this statement as

$$g = \hat{g}^{(\Lambda,a)} \tag{B.1.2}$$

Here the notation $\hat{g}^{(\Lambda,a)}$ denotes the metric that we obtain after a Weyl-rescaling (1.3.12) parameterized by $\Omega(\xi) = e^{\Lambda(\xi)}$ and subsequently a coordinate transformation (1.3.11) parametrized by $\tilde{\xi}^\alpha(\xi) = \xi^\alpha - a^\alpha(\xi)$. Thus, in this sense the physics of g and $\hat{g}$ are the same, they merely correspond to two different gauge choices and we can relate them by gauge transformations (in terms of Weyl-rescalings and diffeomorphism transformations).

We would now like to rewrite the path-integral over metrics g into a path integral over gauge transformations parametrized by Λ and a. Thus, instead of integrating over all metrics g we want to integrate over all Weyl rescalings Λ and diffeomorphisms a. However, in general when one changes from one set of integration variables to another one gets a Jacobian determinant. Consider the simple case of integrating the function $F(y)$ of the n variables $y = (y_1, y_2, ..., y_n)$. Write this integral as $\int d^n y\, F(y)$. If we now introduce a new set of n integration variables $x = (x_1, x_2, ..., x_n)$ specified by the n functions $y_i(x_1, x_2, ..., x_n)$, $i = 1, 2, ..., n$, one finds that

$$\int d^n y\, F(y) = \int d^n x\, J(x) F(y(x)) \tag{B.1.3}$$

where $J(x) = \det(\frac{\partial y_i}{\partial x_j})$ is the Jacobian given as the determinant of the n by n matrix with entries $\frac{\partial y_i}{\partial x_j}$. Analogously to this, we can write

$$\mathcal{D}g = \Delta_{\mathrm{FP}}[\Lambda, a]\, \mathcal{D}\Lambda\, \mathcal{D}a \tag{B.1.4}$$

where the Jacobian of the transformation is given by

$$\Delta_{\mathrm{FP}}[\Lambda, a] = \det\left(\frac{\partial \hat{g}^{(\Lambda,a)}}{\partial(\Lambda, a)}\right) \tag{B.1.5}$$

This is known as the *Fadeev-Popov determinant*. With this, the string path integral (B.1.1) becomes

$$Z = \int \mathcal{D}\Lambda\, \mathcal{D}a\, \mathcal{D}X\, \Delta_{\mathrm{FP}}[\Lambda, a] \exp(iS_{\mathrm{pol}}[\hat{g}^{(\Lambda,a)}, X]) \tag{B.1.6}$$

Under a diffeomorphism transformation a the measure $\mathcal{D}X$ is invariant, *i.e.* $\mathcal{D}X \to \mathcal{D}X^{(a)}$. Using this, we get

$$Z = \int \mathcal{D}\Lambda\, \mathcal{D}a\, \mathcal{D}X\, \Delta_{\mathrm{FP}}[\Lambda, a] \exp(iS_{\mathrm{pol}}[\hat{g}^{(\Lambda,a)}, X^{(a)}]) \tag{B.1.7}$$

From the diffeomorphism invariance and Weyl rescaling invariance of the Polyakov action (1.3.1) we have

$$S_{\mathrm{pol}}[\hat{g}^{(\Lambda,a)}, X^{(a)}] = S_{\mathrm{pol}}[\hat{g}, X] \tag{B.1.8}$$

Notice that the right-hand side is the Polyakov action gauge fixed to the metric $\hat{g}$. One can also show that the Fadeev-Popov determinant is gauge invariant, which means that

$$\Delta_{\mathrm{FP}}[\Lambda, a] = \Delta_{\mathrm{FP}}[0, 0] \tag{B.1.9}$$

Hence we can write the path integral as

$$Z = \int \mathcal{D}\Lambda\, \mathcal{D}a\, \mathcal{D}X\, \Delta_{\mathrm{FP}}[0, 0] \exp(iS_{\mathrm{pol}}[\hat{g}, X]) \tag{B.1.10}$$

However, notice now that nothing in the integrand depends on the gauge variables (Λ, a). This means that Z is proportional to the factor $V = \int \mathcal{D}\Lambda\, \mathcal{D}a$. V is the (infinite) volume of the symmetry group of diffeomorphisms and Weyl transformations. Hence V decouples from the theory since it is just an irrelevant overall normalization factor that we can remove from Z without changing the physics. We get now

$$Z = \int \mathcal{D}X\, \Delta_{\mathrm{FP}}[0,0] \exp(iS_{\mathrm{pol}}[\hat{g}, X]) \tag{B.1.11}$$

This is our gauge fixed path integral, since we now got rid of the integration over the metrics, we fixed the metric to the gauge choice $\hat{g}$ and we are left only with an integral over the embedding fields $X^\mu(\xi)$. However, we notice also that the correct gauge fixing procedure in the quantum theory includes the Fadeev-Popov determinant factor

$$\Delta_{\mathrm{FP}}[0,0] = \det \left(\frac{\partial \hat{g}^{(\Lambda,a)}}{\partial(\Lambda, a)} \right) \Bigg|_{\Lambda=a=0} \tag{B.1.12}$$

This we shall compute in the following.

B.2 Ghost field sector from Fadeev-Popov determinant

Let x_i and y_i for $i = 1, 2, ..., n$ be $2n$ Grassmann-valued variables,

$$\{x_i, x_j\} = \{y_i, y_j\} = \{x_i, y_i\} = 0 \quad \text{for} \quad i, j = 1, 2, ..., n \tag{B.2.1}$$

Then one has the integral identity

$$\int d^n x\, d^n y\, \exp\left(\sum_{i,j=1}^{n} x_i A_{ij} y_j \right) = \det A \tag{B.2.2}$$

where A_{ij} is an n by n real antisymmetric matrix. We now want to exploit this integral identity to compute the Fadeev-Popov determinant (B.1.12). We see that the analogue of A_{ij} should be $\frac{\partial \hat{g}^{(\Lambda,a)}}{\partial(\Lambda,a)}\Big|_{\Lambda=a=0}$. In this analogy the sums over i, j become integrals over the worldsheet, the variables x_i and y_i become Grassmann-valued fields and the matrix A_{ij} becomes an operator acting on fields.

Consider now (Λ, a) near $(0, 0)$. We have

$$\delta g_{\alpha\beta} = \Lambda \hat{g}_{\alpha\beta} + \nabla_\alpha a_\beta + \nabla_\beta a_\alpha \tag{B.2.3}$$

where ∇_α is the covariant derivative on the worldsheet. From this we see that

$$\frac{\partial \hat{g}^{(\Lambda,a)}_{\alpha\beta}}{\partial \Lambda} \Bigg|_{\Lambda=0} = \hat{g}_{\alpha\beta} \ , \quad \frac{\partial \hat{g}^{(\Lambda,a)}_{\alpha\beta}}{\partial a^\gamma} \Bigg|_{a=0} = \hat{g}_{\alpha\gamma} \nabla_\beta + \hat{g}_{\beta\gamma} \nabla_\alpha \tag{B.2.4}$$

From this we see that the analogy of $\sum_{i,j=1}^{n} x_i A_{ij} y_j$ is the integral

$$\int d^2\xi \sqrt{-\hat{g}}\, b^{\alpha\beta}(\tilde{\Lambda}\hat{g}_{\alpha\beta} + \hat{g}_{\alpha\gamma}\nabla_\beta c^\gamma + \hat{g}_{\beta\gamma}\nabla_\alpha c^\gamma) \tag{B.2.5}$$

Here x_i becomes the Grassmann-valued field $b^{\alpha\beta}(\xi)$ that connects with the metric part of $\frac{\partial \hat{g}^{(\Lambda,a)}}{\partial(\Lambda,a)}\Big|_{\Lambda=a=0}$. Note that since $\hat{g}_{\alpha\beta}$ is symmetric also $b^{\alpha\beta}$ should be, *i.e.* $b^{\alpha\beta} = b^{\beta\alpha}$. Similarly, y_j becomes the Grassmann-valued fields $(\tilde{\Lambda}(\xi), c^\gamma(\xi))$ that connects with the gauge transformation fields (Λ, a) of $\frac{\partial \hat{g}^{(\Lambda,a)}}{\partial(\Lambda,a)}\Big|_{\Lambda=a=0}$. We can then write (B.1.12) as

$$\Delta_{\rm FP}[0,0] = \int \mathcal{D}b\,\mathcal{D}c\,\mathcal{D}\tilde{\Lambda}\,\exp\left(\frac{1}{4\pi}\int d^2\xi\sqrt{-\hat{g}}\,b^{\alpha\beta}(\tilde{\Lambda}\hat{g}_{\alpha\beta} + \hat{g}_{\alpha\gamma}\nabla_\beta c^\gamma + \hat{g}_{\beta\gamma}\nabla_\alpha c^\gamma)\right) \tag{B.2.6}$$

Notice that $\tilde{\Lambda}$ acts as an Lagrange multiplier since integrating over $\tilde{\Lambda}$ simply restricts $b^{\alpha\beta}\hat{g}_{\alpha\beta} = 0$. Thus, we obtain

$$\Delta_{\rm FP}[0,0] = \int \mathcal{D}b\,\mathcal{D}c\,\exp\left(\frac{1}{2\pi}\int d^2\xi\sqrt{-\hat{g}}\,b^{\alpha\beta}\hat{g}_{\alpha\gamma}\nabla_\beta c^\gamma\right) \tag{B.2.7}$$

where we now require $b^{\alpha\beta}\hat{g}_{\alpha\beta} = 0$. For convenience we shall work below with the field $b_{\alpha\beta} = \hat{g}_{\alpha\gamma}\hat{g}_{\beta\delta}b^{\gamma\delta}$. Taking this into account, we can conclude that the Fadeev-Popov determinant (B.1.12) can be computed as the path integral

$$\Delta_{\rm FP}[0,0] = \int \mathcal{D}b\,\mathcal{D}c\,\exp(iS_{\rm gh}[b,c]) \tag{B.2.8}$$

with

$$S_{\rm gh}[b,c] = -\frac{i}{2\pi}\int d^2\xi\sqrt{-\hat{g}}\,\hat{g}^{\alpha\beta}b_{\alpha\gamma}\nabla_\beta c^\gamma \tag{B.2.9}$$

where the field $b_{\alpha\beta}$ is symmetric and traceless,

$$\hat{g}^{\alpha\beta}b_{\alpha\beta} = 0\ , \quad b_{\alpha\beta} = b_{\beta\alpha} \tag{B.2.10}$$

Both $b_{\alpha\beta}(\xi)$ and $c^\gamma(\xi)$ are Grassmann-valued fields. In this sense they resemble fermionic fields. However, at the same time they transform like bosons on the worldsheet, *i.e.* like particles with integer spins. Thus their are neither like bosons, nor like fermions. For this reason they cannot correspond to actual physical modes of the theory and they are known as *ghost fields*.

We can now write the gauge fixed path integral (B.1.11) as

$$Z = \int \mathcal{D}b\,\mathcal{D}c\,\mathcal{D}X\,\exp\Big(iS_{\rm pol}[\hat{g},X] + iS_{\rm gh}[b,c]\Big) \tag{B.2.11}$$

from which we can conclude that the gauge fixed path integral includes the ghost field sector in addition to the gauge fixed Polyakov action which becomes (1.3.18) in the flat gauge $\hat{g}_{\alpha\beta} = \eta_{\alpha\beta}$. Above in Sec. 2.1 we only considered the contribution from the gauge fixed Polyakov action. Now we can see that to get the correct full quantum theory we should also include the above derived ghost field sector.

B.3 Extended Virasoro algebra of the ghost field sector

We consider now the newly derived ghost field sector in the flat gauge $\hat{g}_{\alpha\beta} = \eta_{\alpha\beta}$. We use the worldsheet lightcone coordinates (1.4.4) in the following. From (B.2.10) we see using (1.4.5) that $b_{+-} = b_{-+} = 0$. Thus the action for the ghost fields reduces to

$$S_{\text{gh}}[b,c] = \frac{i}{\pi} \int d^2\xi \Big(b_{--}\partial_+ c^- + b_{++}\partial_- c^+ \Big) \tag{B.3.1}$$

This gives the equations of motion

$$\partial_+ b_{--} = \partial_+ c^- = 0 \ , \quad \partial_- b_{++} = \partial_- c^+ = 0 \tag{B.3.2}$$

We consider here a closed string, hence $b_{\alpha\beta}(\tau,\sigma)$ and $c^\alpha(\tau,\sigma)$ are periodic in σ with period 2π. Thus, we have the mode expansions

$$b_{--}(\tau,\sigma) = \sum_{n\in\mathbb{Z}} b_n e^{-in\xi^-} \ , \quad b_{++}(\tau,\sigma) = \sum_{n\in\mathbb{Z}} \tilde{b}_n e^{-in\xi^+} \tag{B.3.3}$$

$$c^-(\tau,\sigma) = \sum_{n\in\mathbb{Z}} c_n e^{-in\xi^-} \ , \quad c^+(\tau,\sigma) = \sum_{n\in\mathbb{Z}} \tilde{c}_n e^{-in\xi^+} \tag{B.3.4}$$

The worldsheet energy-momentum tensor for the ghost fields in the flat gauge has components

$$T^{(\text{gh})}_{--} = 2i\, b_{--}\partial_- c^- + i(\partial_- b_{--})c^- \ , \quad T^{(\text{gh})}_{++} = 2i\, b_{++}\partial_+ c^+ + i(\partial_+ b_{++})c^+ \tag{B.3.5}$$

as well as $T^{(\text{gh})}_{+-} = T^{(\text{gh})}_{-+} = 0$. This means that the trace of the energy-momentum tensor is zero $\eta^{\alpha\beta}T^{(\text{gh})}_{\alpha\beta} = 0$ which we learned in Sec. 2.4.3 means that it is a conformally invariant theory. Write the mode expansion of the energy-momentum tensor as

$$T^{(\text{gh})}_{--} = \sum_{n\in\mathbb{Z}} L^{(\text{gh})}_n e^{-in\xi^-} \ , \quad T^{(\text{gh})}_{++} = \sum_{n\in\mathbb{Z}} \tilde{L}^{(\text{gh})}_n e^{-in\xi^+} \tag{B.3.6}$$

with

$$L^{(\text{gh})}_n = \sum_{k\in\mathbb{Z}} (n-k) b_{n+k} c_{-k} \ , \quad \tilde{L}^{(\text{gh})}_n = \sum_{k\in\mathbb{Z}} (n-k) \tilde{b}_{n+k} \tilde{c}_{-k} \tag{B.3.7}$$

By the same logic as in Sec. 1.6.2 these modes can be thought of as conserved charges due to the conformal invariance of the theory.

So far we have been considering classical field theory for the ghost fields. We now quantize them. We use the canonical anti-commutators

$$\{b_{++}(\tau,\sigma), c^+(\tau,\sigma')\} = 2\pi\delta(\sigma-\sigma') \ , \quad \{b_{--}(\tau,\sigma), c^-(\tau,\sigma')\} = 2\pi\delta(\sigma-\sigma') \tag{B.3.8}$$

This corresponds to the following non-zero anti-commutators for the modes

$$\{b_m, c_n\} = \delta_{m+n,0} \ , \quad \{\tilde{b}_m, \tilde{c}_n\} = \delta_{m+n,0} \tag{B.3.9}$$

One can interpret these anti-commutator relations in terms of so-called fermionic harmonic oscillators (see Chapter 6 for more on fermionic harmonic oscillators). More precisely, for a given $n > 0$ the operators c_n, $\tilde{c}_n$, b_n and $\tilde{b}_n$ are interpreted as annihilation operators and b_{-n}, $\tilde{b}_{-n}$, c_{-n} and $\tilde{c}_{-n}$ are interpreted as creating operators for fermionic harmonic oscillators. The interpretation for the zero-modes is considered in Appendix B.5. For the modes of the ghost field energy-momentum tensor (B.3.7) we use the normal ordering prescription (we allow for a normal ordering constant below)

$$L_n^{(\mathrm{gh})} = \sum_{k\in\mathbb{Z}} (n-k) : b_{n+k} c_{-k} : \ , \quad \tilde{L}_n^{(\mathrm{gh})} = \sum_{k\in\mathbb{Z}} (n-k) : \tilde{b}_{n+k} \tilde{c}_{-k} : \tag{B.3.10}$$

With this we are ready to compute the algebra for the generators $L_n^{(\mathrm{gh})}$ and $\tilde{L}_n^{(\mathrm{gh})}$. After a complicated computation one gets

$$\begin{gathered}
[L_m^{(\mathrm{gh})}, L_n^{(\mathrm{gh})}] = (m-n) L_{m+n}^{(\mathrm{gh})} + \frac{1}{12}(-26m^3 + 2m)\delta_{m+n,0} \\
[\tilde{L}_m^{(\mathrm{gh})}, \tilde{L}_n^{(\mathrm{gh})}] = (m-n) \tilde{L}_{m+n}^{(\mathrm{gh})} + \frac{1}{12}(-26m^3 + 2m)\delta_{m+n,0} \\
[L_m^{(\mathrm{gh})}, \tilde{L}_n^{(\mathrm{gh})}] = 0
\end{gathered} \tag{B.3.11}$$

This corresponds to two copies of an extended Virasoro algebra, as defined in Sec. 2.4.3. Comparing with (2.4.7) we conclude that the b-c ghost field sector corresponds to a two-dimensional CFT with central charge $c_{bc} = -26$.

B.4 Conformal symmetry of quantized closed string without anomaly

Going back to the path integral (B.2.11) we see that in the complete quantum theory of the closed string we should add the contributions from the Polyakov action and the ghost field action to get the complete energy-momentum tensor. Hence the Virasoro generators in the complete quantum theory are

$$\mathcal{L}_n = L_n + L_n^{(\mathrm{gh})} - a\delta_{n,0} \ , \quad \tilde{\mathcal{L}}_n = \tilde{L}_n + \tilde{L}_n^{(\mathrm{gh})} - a\delta_{n,0} \tag{B.4.1}$$

for $n \in \mathbb{Z}$ where L_n and $\tilde{L}_n$ are the Virasoro generators (2.1.15) for the contribution from the X^μ field as described by the Polyakov action in flat gauge, while $L_n^{(\mathrm{gh})}$ and $\tilde{L}_n^{(\mathrm{gh})}$

defined in (B.3.10) are the Virasoro generators for the ghost field sector. Finally, the term $-a\delta_{n,0}$ allows for a possible normal ordering constant.

We compute

$$
\begin{aligned}
[\mathcal{L}_m, \mathcal{L}_n] &= [L_m, L_n] + [L_m^{(\mathrm{gh})}, L_n^{(\mathrm{gh})}] \\
&= (m-n)(L_{m+n} + L_{m+n}^{(\mathrm{gh})}) + \left(\frac{D}{12}(m^3 - m) + \frac{1}{12}(-26m^3 + 2m) \right) \delta_{m+n,0}
\end{aligned}
\tag{B.4.2}
$$

Hence we find

$$
[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} + \frac{1}{12}\Big((D-26)m^3 + (24a + 2 - D)m \Big) \delta_{m+n,0} \tag{B.4.3}
$$

and the equivalent result for the left-moving sector Virasoro generators $\tilde{\mathcal{L}}_n$. Thus, we conclude that for

$$
D = 26 \quad \text{and} \quad a = 1 \tag{B.4.4}
$$

we get the algebra

$$
[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} \ , \quad [\tilde{\mathcal{L}}_m, \tilde{\mathcal{L}}_n] = (m-n)\tilde{\mathcal{L}}_{m+n} \ , \quad [\mathcal{L}_m, \tilde{\mathcal{L}}_n] = 0 \tag{B.4.5}
$$

which corresponds to the anomaly-free conformal symmetry algebra (2.4.3). Hence for the critical values (B.4.4) of the dimension D of the target space, and the normal ordering constant a, the covariant quantization of the closed string is consistent when including the ghost field sector.

For the central charge we see that it is additive when combining the X^μ field and the b-c ghost fields

$$
c = c_{\mathrm{X}} + c_{\mathrm{bc}} = D - 26 \tag{B.4.6}
$$

which also means that the conformal anomaly (2.4.8) disappears for $D = 26$, as expected.

B.5 BRST quantization

Consider the full quantum theory of the closed string as given by the path integral (B.2.11) for any space-time dimension D. With the introduction of the ghost fields $b_{\alpha\beta}$ and c^α in addition to the matter fields X^μ what are the physical states of the theory? We need to find a consistent way to define what is meant by a state without ghost field parts. Moreover, whereas in Sec. 2.1 we used the Virasoro generators $L_n^{(X)}$ and $\tilde{L}_n^{(X)}$ only for the matter fields X^μ, but now we have the full Virasoro generators (B.4.1). Finally, we should somehow be able to get the approach of Sec. 2.1 out of demanding a state to be physical in the full quantum theory including the ghost field sector. The answers to all the above questions and issues is provided by employing the very powerful method of BRST quantization to the full quantum theory of the closed string. BRST quantization is a general method to quantize theories with constraint that one can also apply for instance on Quantum Electrodynamics (QED).

In the following we shall describe briefly how BRST quantization is applied to the quantum theory of the closed string and how it can determine the physical spectrum of the theory. We shall keep the space-time dimension D and the normal ordering constant a in (B.4.1) general for now.

We begin by introducing the operator

$$Q = \sum_{n\in\mathbb{Z}} : \Big(L_{-n} + \frac{1}{2} L_{-n}^{(\mathrm{gh})} - a\delta_{n,0} \Big) c_n : + \sum_{n\in\mathbb{Z}} : \Big(\tilde{L}_{-n} + \frac{1}{2} \tilde{L}_{-n}^{(\mathrm{gh})} - a\delta_{n,0} \Big) \tilde{c}_n : \tag{B.5.1}$$

known as the *BRST operator*. In addition to the Virasoro generators for the matter fields and ghost fields this also includes the modes of the c^α field defined in (B.3.4). We notice that Q is a Grassmann-valued operator. One can show that this operator has the property

$$Q = Q^\dagger \tag{B.5.2}$$

We claim now that Q is the generator of a global fermionic symmetry of the gauge-fixed action $S_{\mathrm{pol}}[\eta, X] + S_{\mathrm{gh}}[b, c]$. The transformations that Q generates can be written compactly as $\delta Y = [\lambda Q, Y]$ where λ is a Grassmann-valued constant and Y can be any of the fields $X^\mu(\xi)$, $b_{\alpha\beta}(\xi)$ or $c^\alpha(\xi)$. Thus, for any given Grassmann-valued constant λ the expression (B.6.11) provides a specific transformation of the fields $X^\mu(\xi)$, $b_{\alpha\beta}(\xi)$ and $c^\alpha(\xi)$ of the closed string. These transformations are known as *BRST transformations*.

For a translation generator $P = -i\partial/\partial x$ we have $[P, P] = 0$. In analogy with this we would like to have that

$$0 = [\lambda_1 Q, \lambda_2 Q] = \lambda_1 \lambda_2 \{Q, Q\} = \lambda_1 \lambda_2 Q^2 \tag{B.5.3}$$

for any two Grassmann-valued constants λ_1 and λ_2. Thus, we would like that the BRST operators has the property

$$Q^2 = 0 \tag{B.5.4}$$

Starting from the definition (B.5.1) one can compute

$$Q^2 = \frac{1}{2} \sum_{m,n\in\mathbb{Z}} \big([\mathcal{L}_m, \mathcal{L}_n] - (m-n)\mathcal{L}_{m+n} \big) c_m c_n + \frac{1}{2} \sum_{m,n\in\mathbb{Z}} \big([\tilde{\mathcal{L}}_m, \tilde{\mathcal{L}}_n] - (m-n)\tilde{\mathcal{L}}_{m+n} \big) \tilde{c}_m \tilde{c}_n \tag{B.5.5}$$

here written in terms of the complete Virasoro generators (B.4.1). Hence, using the result (B.4.3) we see that $Q^2 = 0$ if and only if $D = 26$ and $a = 1$. Thus, $Q^2 = 0$ is equivalent to demanding an anomaly-free algebra for the conformal symmetry as we discussed in Sec. B.4. In conclusion, we can avoid anomalies in the conformal symmetry precisely when the BRST quantization is consistent, and vice versa.

Since Q is a symmetry generator we would like that physical states are invariant under this symmetry. Hence we demand that a physical state $|\phi\rangle$ should obey

$$Q|\phi\rangle = 0 \tag{B.5.6}$$

States $|\phi\rangle$ with $Q|\phi\rangle = 0$ are known as *BRST invariant states*. Notice now that if $|\phi\rangle$ is a BRST invariant state then also $|\phi\rangle + Q|\psi\rangle$ is a BRST invariant state since

$$Q(|\phi\rangle + Q|\psi\rangle) = Q|\phi\rangle + Q^2|\psi\rangle = 0 \tag{B.5.7}$$

We use this fact to define an equivalence relation between BRST invariant states, *i.e.* that $|\phi\rangle$ and $|\phi'\rangle = |\phi\rangle + Q|\psi\rangle$ are equivalent. Or said in more physical terms, we say that that $|\phi\rangle$ and $|\phi'\rangle = |\phi\rangle + Q|\psi\rangle$ corresponds to the same physical quantum configuration. In particular, one can easily check that $|\phi\rangle$ and $|\phi'\rangle$ have the same norm

$$\langle\phi'|\phi'\rangle = (\langle\phi| + \langle\psi|Q^\dagger)(|\phi\rangle + Q|\psi\rangle) = \langle\phi|\phi\rangle \tag{B.5.8}$$

where we used both (B.5.2) and (B.5.4). States of the form $|\chi\rangle = Q|\psi\rangle$ for some state $|\psi\rangle$ are known as *spurious states*. Spurious states are automatically BRST invariant since $Q|\chi\rangle = Q^2|\psi\rangle = 0$ and they have zero norm $\langle\chi|\chi\rangle = \langle\psi|Q^\dagger Q|\psi\rangle = 0$. Notice moreover that any given spurious state $|\chi\rangle = Q|\psi\rangle$ is orthogonal to all BRST invariant states. To see this let $|\phi\rangle$ be BRST invariant then $\langle\chi|\phi\rangle = \langle\psi|Q^\dagger|\phi\rangle = \langle\psi|Q|\phi\rangle = 0$.

However, there is one remaining subtlety that prevents us from identifying the physical states of the theory. The issue is that the solutions to $Q|\phi\rangle = 0$ can be divided into disjoint sets called BRST cohomology classes.[42] Thus, to identify the physical states within the set of BRST invariant states one needs to specify the correct cohomology class. It turns out that the correct cohomology class is given by states $|\phi\rangle$ that fulfil

$$Q|\phi\rangle = 0\ ,\quad c_n|\phi\rangle = \tilde{c}_n|\phi\rangle = b_n|\phi\rangle = \tilde{b}_n|\phi\rangle = 0 \ \text{ for }\ n > 0\ ,\quad b_0|\phi\rangle = \tilde{b}_0|\phi\rangle = 0 \tag{B.5.9}$$

Thus, we define the physical states of the quantum theory of the closed string as states $|\phi\rangle$ that obeys the conditions (B.5.9). Note that the condition for the non-zero modes of

[42]We know from mathematics that any non-trivial operator $\mathcal{O}$ that squares to zero $\mathcal{O}^2$ gives rise to a cohomology with distinct cohomology classes.

the ghost fields in (B.5.9) fit intuitively with demanding that there are no ghost field part in the physical spectrum of states. Thus, the non-trivial part of the conditions (B.5.9) is the zero-mode conditions $b_0|\phi\rangle = \tilde{b}_0|\phi\rangle = 0$.[43]

Consider now a physical state $|\phi\rangle$, thus meaning a BRST invariant states that obeys the conditions (B.5.9). One can easily check that

$$: L^{(\mathrm{gh})}_{-m} c_m : |\phi\rangle = -\sum_{k\in\mathbb{Z}} (m+k) : b_{-m+k} c_{-k} c_m : |\phi\rangle = 0 \tag{B.5.10}$$

for all $m \in \mathbb{Z}$. Moreover, one finds that $L_{-m} c_m |\phi\rangle = 0$ for $m > 0$. Thus, we see now that the condition of BRST invariance $Q|\phi\rangle = 0$ becomes

$$\begin{aligned} 0 = Q|\phi\rangle &= \sum_{m\in\mathbb{Z}} \Big(L_{-m} c_m - \delta_{m,0} c_m + \tilde{L}_{-m} \tilde{c}_m - \delta_{m,0} \tilde{c}_m \Big) |\phi\rangle \\ &= \Big(c_0 (L_0 - 1) + \sum_{n=1}^{\infty} c_{-n} L_n + \tilde{c}_0 (\tilde{L}_0 - 1) + \sum_{n=1}^{\infty} \tilde{c}_{-n} \tilde{L}_n \Big) |\phi\rangle \end{aligned} \tag{B.5.11}$$

Thus, from this we conclude that a physical state $|\phi\rangle$ according to (B.5.9) is physical if the non-ghost-field part of it obeys the physical state condition (2.1.17) that we found in Sec. 2.1. Hence, in other words, we have shown that for $D = 26$ and $a = 1$ the covariant quantization approach of Sec. 2.1 is indeed consistent. In particular, note that the spurious states as defined above corresponds to the spurious states defined in Sec. 2.1.

B.6 Exercises for Appendix B

Exercise B.1. In the classical theory for the ghost sector, consider the energy-momentum tensor (B.3.5) expanded in Fourier modes (B.3.6). Derive the classical expression (B.3.7) for the Fourier modes in terms of the mode expansions (B.3.3)-(B.3.4) of $b_{\alpha\beta}$ and c^α.

Exercise B.2. In this exercise we consider the centrally extended Virasoro algebra for the ghost fields $b_{\alpha\beta}$ and c^γ for the closed bosonic string. The Virasoro generators for the ghost fields are given in (B.3.10). Hint: Use below that for any three operators A, B and C one has the identities $[AB, C] = A\{B, C\} - \{A, C\}B$ and $[A, BC] = \{A, B\}C - B\{A, C\}$.

- Following the same line of arguments as in Exercise 2.8 show that the algebra for the right-moving sector is of the form

$$[L^{(\mathrm{gh})}_m, L^{(\mathrm{gh})}_n] = (m-n) L^{(\mathrm{gh})}_{m+n} + f(m)\delta_{m+n,0} \tag{B.6.1}$$

[43]Since $\{b_0, c_0\} = \{\tilde{b}_0, \tilde{c}_0\} = 1$ one can interpret b_0 and $\tilde{b}_0$ as annihilation operators and c_0 and $\tilde{c}_0$ as creation operators for two fermionic harmonic oscillators.

Argue using Exercise 2.7 (no computations needed) that $f(m)$ is of the form

$$f(m) = f_1 m + f_2 m^3 \tag{B.6.2}$$

Hence the algebra for the $L_n^{(\mathrm{gh})}$ generators is a centrally extended Virasoro algebra.

- Consider a state $|\phi\rangle$ with $\langle\phi|\phi\rangle = 1$ and

$$b_m|\phi\rangle = c_m|\phi\rangle = 0 \quad \text{for} \quad m > 0 \tag{B.6.3}$$

This means that $\langle\phi|b_{-m} = \langle\phi|c_{-m} = 0$ for $m > 0$. Show the following identities

$$\begin{aligned} L_{-1}^{(\mathrm{gh})}|\phi\rangle &= -(b_{-1}c_0 + 2b_0c_{-1})|\phi\rangle \\ \langle\phi|L_1^{(\mathrm{gh})} &= \langle\phi|(2b_0c_1 + b_1c_0) \\ L_{-2}^{(\mathrm{gh})}|\phi\rangle &= -(2b_{-2}c_0 + 3b_{-1}c_{-1} + 4b_0c_{-2})|\phi\rangle \\ \langle\phi|L_2^{(\mathrm{gh})} &= \langle\phi|(2b_2c_0 + 3b_1c_1 + 4b_0c_2) \end{aligned} \tag{B.6.4}$$

Use these identities to show

$$\langle\phi|[L_1^{(\mathrm{gh})}, L_{-1}^{(\mathrm{gh})}]|\phi\rangle = -2 \ , \quad \langle\phi|[L_2^{(\mathrm{gh})}, L_{-2}^{(\mathrm{gh})}]|\phi\rangle = -17 \tag{B.6.5}$$

and that this implies $f(1) = -2$ and $f(2) = -17$. Argue from this that the algebra for the $L_n^{(\mathrm{gh})}$ generators is

$$[L_m^{(\mathrm{gh})}, L_n^{(\mathrm{gh})}] = (m-n)L_{m+n}^{(\mathrm{gh})} + \frac{1}{12}(-26m^3 + 2m)\delta_{m+n,0} \tag{B.6.6}$$

Exercise B.3. The BRST operator Q for the closed bosonic string is given by (B.5.1). Consider a state $|\phi\rangle$ obeying

$$b_n|\phi\rangle = c_n|\phi\rangle = \tilde{b}_n|\phi\rangle = \tilde{c}_n|\phi\rangle = 0 \quad \text{for} \quad n > 0 \quad \text{and} \quad b_0|\phi\rangle = \tilde{b}_0|\phi\rangle = 0 \tag{B.6.7}$$

Show that the condition that $|\phi\rangle$ is BRST invariant

$$Q|\phi\rangle = 0 \tag{B.6.8}$$

is equivalent to imposing the conditions

$$(L_n - a\delta_{n,0})|\phi\rangle = (\tilde{L}_n - a\delta_{n,0})|\phi\rangle = 0 \quad \text{for} \quad n \geq 0 \tag{B.6.9}$$

which we recognize as the physical state conditions (2.1.17) found in Sec. 2.1.

Exercise B.4. The gauge fixed Polyakov action in flat gauge including the ghost action is

$$S[X,b,c] = \frac{1}{\pi}\int d^2\sigma \left[\frac{1}{l_s^2}\partial_- X \cdot \partial_+ X + i(b_{--}\partial_+ c^- + b_{++}\partial_- c^+)\right] \tag{B.6.10}$$

Consider the transformation

$$\begin{gathered} \delta X^\mu = -i(\lambda c^- \partial_- X^\mu + \lambda c^+ \partial_+ X^\mu) \\ \delta c^- = -i\lambda c^- \partial_- c^- \ , \quad \delta c^+ = -i\lambda c^+ \partial_+ c^+ \\ \delta b_{--} = \lambda(T_{--}^{(X)} + T_{--}^{(\mathrm{gh})}) \ , \quad \delta b_{++} = \lambda(T_{++}^{(X)} + T_{++}^{(\mathrm{gh})}) \end{gathered} \tag{B.6.11}$$

where

$$T_{\pm\pm}^{(X)} = \frac{1}{l_s^2}(\partial_\pm X)^2 \ , \quad T_{\pm\pm}^{(\mathrm{gh})} = 2ib_{\pm\pm}\partial_\pm c^\pm + i(\partial_\pm b_{\pm\pm})c^\pm \tag{B.6.12}$$

and where λ is a Grassmann-valued constant. Show that this transformation is a symmetry of the action (B.6.10). This is the classical version of the BRST symmetry.

Exercise B.5. The BRST operator Q for the (bosonic) closed string is (B.5.1). We want to show in the following that this is the symmetry generator corresponding to the classical symmetry transformation (B.6.11) in the sense that a field Y on the world-sheet is transformed as

$$\delta Y = [\lambda Q, Y] \tag{B.6.13}$$

where λ is a Grassman-valued constant. Since (B.6.11) is a classical transformation it is sufficient to show this correspondence while ignoring all normal ordering issues (since normal ordering constants go to zero in the classical limit).

- Show using the mode expansions of X^μ and $c^\pm$ (Hint: Use $[L_m, \alpha_n^\mu] = -n\alpha_{m+n}^\mu$)

  $$[\lambda \sum_{n\in\mathbb{Z}}(L_{-n}c_n + \tilde{L}_{-n}\tilde{c}_n), X^\mu] = -i(\lambda c^- \partial_- X^\mu + \lambda c^+ \partial_+ X^\mu) \tag{B.6.14}$$

  This gives the δX^μ part of the transformation (B.6.11).

- Show the identity

  $$[\lambda \frac{1}{2}\sum_{n,k\in\mathbb{Z}}(-n-k)b_{-n+k}c_{-k}c_n, c^-] = -i\lambda c^- \partial_- c^- \tag{B.6.15}$$

  This gives the δc^- part (and by analogy also the δc^+ part) of the transformation (B.6.11).

- Show the identities

$$
\begin{aligned}
&[\lambda \sum_{n\in\mathbb{Z}} L_{-n} c_n, b_{--}] = \lambda T^{(X)}_{--} \\
&[\lambda \frac{1}{2} \sum_{n,k\in\mathbb{Z}} (-n-k) b_{-n+k} c_{-k} c_n, b_{--}] = \lambda T^{(\mathrm{gh})}_{--}
\end{aligned}
\tag{B.6.16}
$$

where $T^{(X)}_{--}$ and $T^{(\mathrm{gh})}_{--}$ are given in (B.6.12). This gives the δb_{--} part (and by analogy also the δb_{++} part) of the transformation (B.6.11).