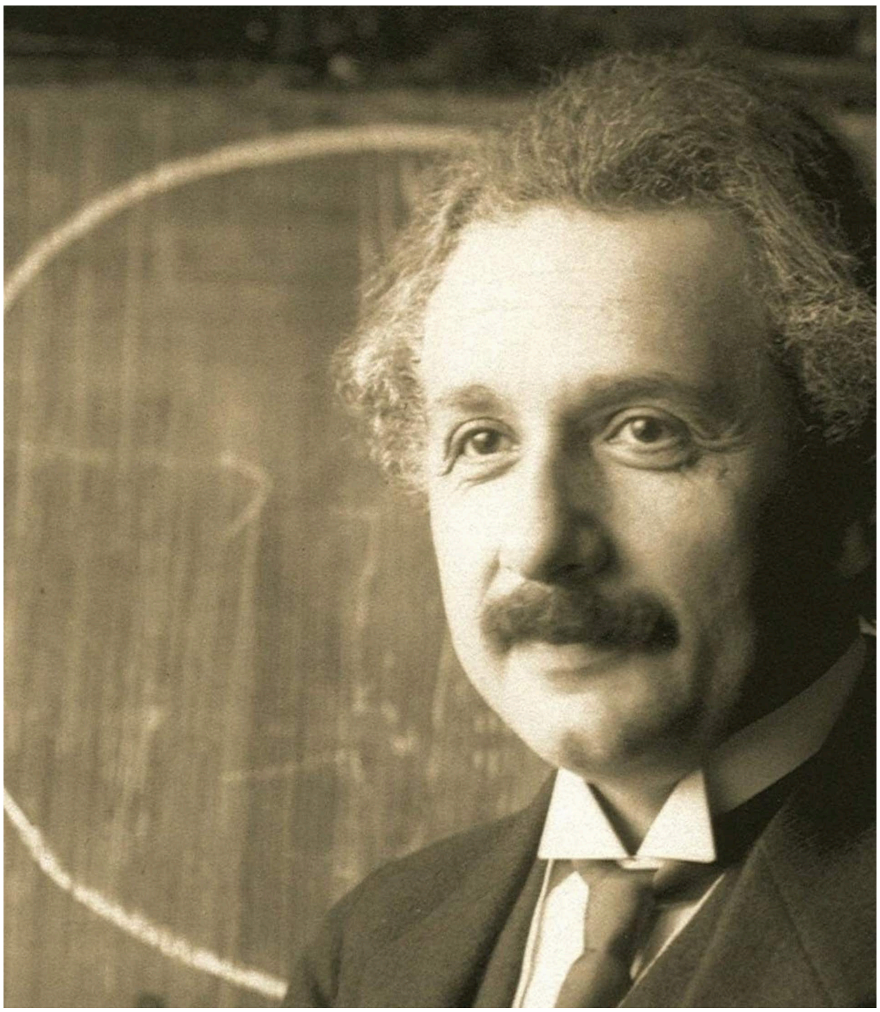
General Relativity and Cosmology

Lecture notes by Troels Harmark



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Contents

1 Introduction to General Relativity

The theory of General Relativity is arguably the most beautiful theory of physics. It encompasses all of classical physics. It is incredibly profound in its consequences, and in its insights about the meaning of space, time and gravity. At the same time, once one understands the underlying mathematics, its formulation is simple and natural. Among the insights of General Relativity are:

- Space and time are not separate entities, but instead inseparable parts of a space-time geometry.
- Gravity is not a force, as Newton originally thought, even if his formulation of the force of gravity could be argued to be the single most succesful insight in the history of science. Instead gravity is a manifestation of the curvature of the space-time geometry.
- Everything, all matter and energy, can source gravity, in the sense that any matter and energy can make the space-time geometry curve.
- The space-time can interact with itself. This means that General Relativity is a non-linear theory, unlike Newtons laws of mechanics and gravity that all are linear.
- Space-time geometry can bend so much that a black hole is created.
- Ripples in the space-time geometry can propagate with the speed of light as a wave. These ripples are called gravitational waves and they can carry energy with them.
- The evolution of the whole universe can be understood in terms of a dynamical space-time geometry.

The theory of General Relativity replaces Newtons laws of mechanics as well as Newtons law of gravity, dating back to the second half of the 17th century. Then in 1905 Einstein understood that Newtons laws of mechanics are not consistent with Maxwells laws of electromagnetism. Based on this, he proposed the theory of Special Relativity, that replaced Newtons laws of mechanics with a new theory, valid in situations where one can neglect gravity. The theory of Special Relativity reduces to Newtons laws of mechanics for small velocities, but when velocities approaches the speed of light, Special Relativity gives the correct description. In the years after publishing the theory of Special Relativity Einstein, with some help and input from other physicists and mathematicians, searched for a theory that could extend the theory of Special Relativity to include Newtons law of gravity as well. Remarkably, this took only about ten years, and resulted in Einstein publishing the theory of General Relativity in 1916 [1].

The theory of General Relativity gives a framework for all of macroscopic physics (also known as classical physics), thus for distances ranging from at least a micrometer and up to the size of the universe. For small enough distances, one needs instead to take quantum mechanics, and possibly quantum field theory, into account. For even smaller distances, so small we have not yet seen them in experiments, quantum effects presumably mix with gravitational interactions, and one needs a theory of quantum gravity to replace the theory of General Relativity.

These lecture notes have the following chapters:

- **Chapter 1:** The theory of General Relativity is introduced. First in Section 1.1 needed aspects of the theory of Special Relativity are reviewed. In Section 1.2 the Equivalence Principle is discussed. This is one of the cornerstones in the new insights into the nature of gravity and its connection to space and time. In Section 1.3 the notions of the metric and space-time are introduced. In Section 1.4 the principle of General Covariance is explained, and the mathematics of tensors is introduced. In Section 1.5, the curvature of space-time is introduced. Finally, in Section 1.6, Einsteins equations are introduced. To formulate Einsteins equations one needs the concept of curvature introduced in the previous section, as well as the energy-momentum tensor for matter and energy.

- **Chapter 2:** Here we derive our first example of a metric for a curved space-time, namely the Schwarzschild metric. This metric can be used to describe the metric of stars and planets, as well as black holes. We then discuss the geodesics in the Schwarzschild metric and how this affects the motion of planets in the solar system, as well as how light is bend around a massive object.

- **Chapter 3:** In this chapter we introduce black holes. We first discuss the Schwarzschild black hole, which is spherically symmetric and static. We then turn to the Kerr black hole which is rotating, thus being our second example of a curved space-time. Finally, we discuss the laws of black hole mechanics which are of interest for many modern studies of black holes.

- **Chapter 4:** In this chapter we study cosmology, *i.e.* the evolution of the universe as a whole, using General Relativity.

- **Chapter 5:** Finally, in the last chapter we consider the weak field limit of gravity, where Einsteins equations become linear. We use this to consider gravitational waves.

1.1 Special Relativity

Before turning to the theory of General Relativity, we briefly consider the theory of Special Relativity. The theory of Special Relativity applies to physical systems for which gravitational interactions can be neglected.

1.1.1 Inertial Systems

In the absence of gravity, one can find special coordinate systems called *Inertial Systems* in which particles that are not subject to external forces either will move in a straight line, or not move at all. We use Cartesian coordinates x, y and z for the three spatial direction and t for the time.

In Newtonian physics, which can be used for velocities much smaller than the speed of light c, time and space can be separated. This means for instance that two observers in two different Inertial Systems will agree that time intervals and lengths are always the same. Instead, in the theory of Special Relativity, time and space can be transformed into each other, and two observers in two different Inertial Systems will in general measure different time intervals and lengths. Thus, it makes sense to introduce a new notation for the space and time coordinates that unify space and time

$$x^0 = c\,t \ , \quad x^1 = x \ , \quad x^2 = y \ , \quad x^3 = z \, , \tag{1.1.1}$$

where

$$c = 2.998 \cdot 10^8 \,\mathrm{m/s} \, , \tag{1.1.2}$$

is the speed of light. In this way all four coordinates x^μ, $\mu = 0, 1, 2, 3$, have the dimension of length, meaning that one for example can measure time intervals in terms of meters. From now on we will denote collectively all four coordinates simply as x^μ. Here μ is called an *index* (plural: *indices*). Since we from now on will measure time in units of length we set the speed of light to one

$$c = 1 \, , \tag{1.1.3}$$

which means that

$$x^0 = t \ , \quad x^1 = x \ , \quad x^2 = y \ , \quad x^3 = z \, , \tag{1.1.4}$$

thus making it explicit that time and space are measured in the same units. Unless explicitly noted, we assume $c = 1$ in the rest of the lecture notes.

According to the theory of Special Relativity, starting with a given Inertial System x^μ one can get a new Inertial System $\tilde{x}^\mu$ by boosts, rotations and translations, and any combinations thereof. Conversely, any two given Inertial Systems x^μ and $\tilde{x}^\mu$ must be related by a combination of boosts, rotations and translations. Any combination of rotations and boosts is known as a *Lorentz transformation*.

A translation can be written

$$\tilde{x}^\mu = x^\mu + a^\mu \,, \tag{1.1.5}$$

where a^μ, $\mu = 0, 1, 2, 3$, is constant. An example of a rotation is

$$\tilde{x}^0 = x^0 \ , \quad \tilde{x}^1 = \cos\theta \, x^1 + \sin\theta \, x^2 \ , \quad \tilde{x}^2 = -\sin\theta \, x^1 + \cos\theta \, x^2 \ , \quad \tilde{x}^3 = x^3 \,, \tag{1.1.6}$$

where θ is constant, corresponding to rotating with angle θ in the 12-plane. An example of a boost is

$$\tilde{x}^0 = \gamma(x^0 - v\, x^1) \ , \quad \tilde{x}^1 = \gamma(x^1 - v\, x^0) \ , \quad \tilde{x}^2 = x^2 \ , \quad \tilde{x}^3 = x^3 \ , \quad \gamma = \frac{1}{\sqrt{1 - v^2}} \,, \tag{1.1.7}$$

corresponding to a boost along the x^1 axis with relative speed v. Both transformations (1.1.6) and (1.1.7) are examples of Lorentz transformations.

1.1.2 Minkowski space

An *event* is a particular point in time and space. *Minkowski space* is defined as the collection of events, in the absence of gravity - *i.e.* the situation that is described by the theory of Special Relativity. A given Inertial System x^μ provides a coordinate parametrization of events. Hence x^μ gives a particular coordinate system for Minkowski space. It follows from the above that two different Inertial Systems x^μ and $\tilde{x}^\mu$ provide two different coordinate systems for Minkowski space.

We see that Minkowski space is four-dimensional as it is parametrized by four coordinates. Since we have one time and three spatial coordinates, one says that Minkowski space is a four-dimensional *space-time*. This is the first example of a space-time that we encounter in this course. Later we shall encounter several other space-times.

Consider two events p_1 and p_2 in Minkowski space. In an Inertial System x^μ the two events are parametrized by $x^\mu_{(1)}$ and $x^\mu_{(2)}$, respectively. Define the difference between the two events

$$\Delta x^\mu = x^\mu_{(2)} - x^\mu_{(1)} \,. \tag{1.1.8}$$

According to the theory of Special Relativity, the quantity

$$\Delta s^2 = -(\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2 , \tag{1.1.9}$$

is the same for all Inertial Systems. Before considering this further, we now introduce a different way of writing this statement. Define $\eta_{\mu\nu}$ with $\mu, \nu = 0, 1, 2, 3$ by

$$\begin{aligned} &\eta_{00} = -1 , \quad \eta_{11} = \eta_{22} = \eta_{33} = 1 , \\ &\eta_{\mu\nu} = 0 \quad \text{for} \quad \mu \neq \nu . \end{aligned} \tag{1.1.10}$$

One can view $\eta_{\mu\nu}$ as a four by four matrix

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{1.1.11}$$

Using this we can write Eq. (1.1.9) as

$$\Delta s^2 = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu . \tag{1.1.12}$$

We now introduce a new notation called the *Einstein summation convention* that will simplify formulas like (1.1.12). This notation means that one does not write explicitly the sums over repeated indices. In Eq. (1.1.12) there are two sums over repeated indices, namely the one over $\mu = 0, 1, 2, 3$ and the one over $\nu = 0, 1, 2, 3$. Thus, with the Einstein summation convention we can write (1.1.12) as

$$\Delta s^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu . \tag{1.1.13}$$

Given a different Inertial System $\tilde{x}^\mu$, where the two events are parametrized by $\tilde{x}^\mu_{(1)}$ and $\tilde{x}^\mu_{(2)}$, respectively, the fact that (1.1.13) is the same for all Inertial Systems means

$$\eta_{\mu\nu} \Delta x^\mu \Delta x^\nu = \Delta s^2 = \eta_{\mu\nu} \Delta \tilde{x}^\mu \Delta \tilde{x}^\nu , \tag{1.1.14}$$

where $\Delta \tilde{x}^\mu = \tilde{x}^\mu_{(2)} - \tilde{x}^\mu_{(1)}$. Another way to state this is that (1.1.13) is invariant under all Lorentz transformations (*i.e.* any combination of rotations and boosts) and translations.

The quantity Δs^2 in (1.1.13) generalizes the square of the distance between points in Euclidean space to Minkowski space. Since Minkowski space involves time as well, Δs^2 can be negative, zero or positive, unlike in Euclidean space where the distance between two distinct points always is positive. We now consider the physical meaning of this.

Consider two distinct events p_1 and p_2 in Minkowski space, parametrized by $x^\mu_{(1)}$ and $x^\mu_{(2)}$ in the Inertial System x^μ. We have now three possibilities

- $\Delta s^2 > 0$. In this case we say that the two events p_1 and p_2 are *space-like separated*. One can find an Inertial System $\tilde{x}^\mu$ in which the two events happen simultaneously $\Delta\tilde{x}^0 = 0$. This means that they are causally disconnected (since nothing can travel faster than the speed of light). The quantity $\sqrt{\Delta s^2} = \sqrt{\eta_{\mu\nu}\Delta x^\mu \Delta x^\nu}$ is the *proper distance* between the events.

- $\Delta s^2 < 0$. In this case we say that the two events p_1 and p_2 are *time-like separated*. One can find an Inertial System $\tilde{x}^\mu$ in which the two events happen at the same spatial point $\Delta\tilde{x}^1 = \Delta\tilde{x}^2 = \Delta\tilde{x}^3 = 0$. Thus, the two events can be causally connected to each other (*e.g.* if $\Delta\tilde{x}^0 > 0$ the event p_1 can influence the event p_2). The quantity $\sqrt{-\Delta s^2} = \sqrt{-\eta_{\mu\nu}\Delta x^\mu \Delta x^\nu}$ is the *proper time* between the events.

- $\Delta s^2 = 0$. In this case we say that the two events p_1 and p_2 are *null separated* (alternatively one says *light-like separated*). This means that one can reach one event from the other by travelling at the speed of light. Thus also in this case the events can be causally connected.

One can illustrate the above statements about Δs^2 in a *Lightcone diagram*, see Figure 1. Seen from point of view of the point p_1 this diagram illustrates how the geometry of Minkowski space is separated into events that are space-like, time-like and null separated from the p_1 event, assuming for simplicity that $\Delta x^2 = \Delta x^3 = 0$.

1.1.3 Unaccelerated motion and proper time

Consider two infinitesimally separated events in an Inertial System x^μ for Minkowski space

$$x^\mu \quad \text{and} \quad x^\mu + dx^\mu \,. \tag{1.1.15}$$

Then we have the invariant infinitesimal quantity called the *line-element*

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \,. \tag{1.1.16}$$

We have three possibilities

$$\begin{aligned} ds^2 > 0&: \ \text{space-like separated events}\,, \\ ds^2 < 0&: \ \text{time-like separated events}\,, \\ ds^2 = 0&: \ \text{null separated events}\,. \end{aligned} \tag{1.1.17}$$

Consider $ds^2 < 0$. Then the infinitesimal proper time $d\tau$ between the two events is

$$d\tau^2 = -ds^2 = -\eta_{\mu\nu} dx^\mu dx^\nu \,. \tag{1.1.18}$$

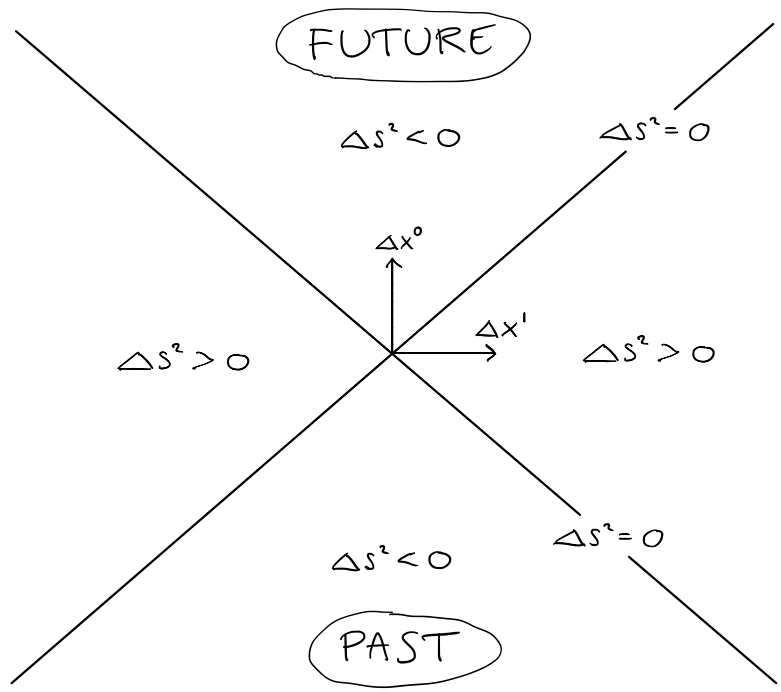

Figure 1: Lightcone diagram for Minkowski space with $\Delta x^2 = \Delta x^3 = 0$.

Consider now a *time-like curve* in Minkowski space, meaning a curve for which each infinitesimal piece of the curve is time-like according to (1.1.16). Using (1.1.18) we define the relativistic velocity

$$u^\mu = \frac{dx^\mu}{d\tau} , \tag{1.1.19}$$

also known as the four-velocity for each point on the curve. One has

$$u^0 = \gamma \ , \quad u^i = \gamma v^i \ , \quad \gamma = \frac{1}{\sqrt{1-v^2}} , \tag{1.1.20}$$

where $i = 1, 2, 3$ and $v^i = dx^i/dx^0$. One can furthermore define the relativistic acceleration

$$a^\mu = \frac{du^\mu}{d\tau} = \frac{d^2 x^\mu}{d\tau^2} . \tag{1.1.21}$$

Particles with non-zero rest mass follow time-like curves. If there are no external forces on the particle, then the relativistic acceleration (1.1.21) of the particle is zero $a^\mu = 0$. Hence the equation for the curve must be given by $x^\mu = u^\mu \tau + b^\mu$ with u^μ and b^μ constant which means the particle is moving along a straight line in Minkowski space.

Straight line maximize the proper time

One can characterize the path with zero acceleration in a different way, as we now shall see. Suppose we are given two time-like separated events p_1 and p_2 (with p_2 being in the future lightcone of p_1). Write these events as $x^\mu_{(1)}$ and $x^\mu_{(2)}$ in the Inertial System x^μ. Consider a time-like curve $x^\mu(\lambda)$ parametrized by λ such that one goes from $x^\mu_{(1)}$ to $x^\mu_{(2)}$ as the parameter λ goes from λ_1 to λ_2, *i.e.* $x^\mu(\lambda_1) = x^\mu_{(1)}$ and $x^\mu(\lambda_2) = x^\mu_{(2)}$. The parameter λ could for instance be chosen to be the time-coordinate x^0 or the proper time on the curve, but any other parametrization works as well, as long as one goes forward in time (*e.g.* coordinate-time or proper time) on the curve when λ increases. See Figure 2 for an illustration.

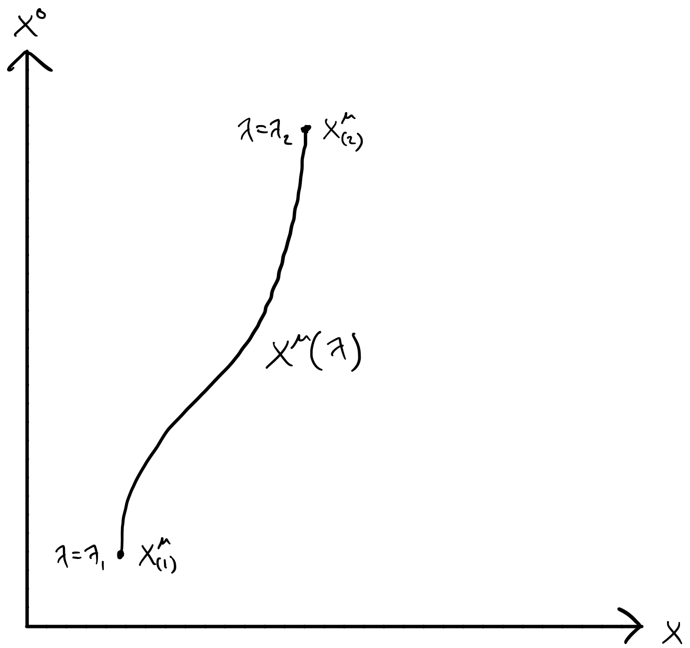

Figure 2: Illustration of time-like curve $x^\mu(\lambda)$ in Minkowski space between two time-like separated events $x^\mu_{(1)}$ and $x^\mu_{(2)}$, here with $x^2(\lambda) = x^3(\lambda) = 0$ for simplicity.

For a given curve $x^\mu(\lambda)$ between p_1 and p_2 one can compute the proper time along the curve as

$$\Delta\tau = \int_{\lambda_1}^{\lambda_2} d\lambda \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,. \tag{1.1.22}$$

This follows from (1.1.18). One can now show the following:

$$\begin{aligned}
&\text{The time-like curve between } p_1 \text{ and } p_2 \text{ that maximize } \Delta\tau \\
&\Leftrightarrow \text{The time-like curve between } p_1 \text{ and } p_2 \text{ with } a^\mu = \tfrac{d^2x^\mu}{d\tau^2} = 0 \\
&\Leftrightarrow \text{Straight line between } p_1 \text{ and } p_2
\end{aligned} \tag{1.1.23}$$

Thus, the time-like curve between p_1 and p_2 that has the longest proper time is the straight line. We will not prove this statement here, since we shall prove it more generally in Section 1.3.

In Euclidean space, one has that the shortest curve between two points is given by a straight line. One can similarly show that the shortest curve between two space-like separated events in Minkowski space is a straight line, where one measures the length of a curve by integrating up the proper distance between infinitesimal events on the path corresponding to the right hand side of (1.1.22) without the minus inside the square-root. Instead for two time-like separate points the straight line corresponds to the one with maximal proper time. One way to understand this difference is by noticing that ds^2 changes sign when going from space-like to time-like curves. Thus, making ds^2 as small as possible, which in the space-like case corresponds to the minimal proper length, corresponds in the time-like case instead to the maximal proper time since $ds^2 = -d\tau^2$.

Twin paradox and maximization of proper time

One can illustrate the fact that one maximizes the proper time by not accelerating with the example of the so-called *Twin paradox*.[1] Our version of it is as follows. Twin A stays at home at Earth while Twin B travels to Mars and returns. This is illustrated in Figure 3.

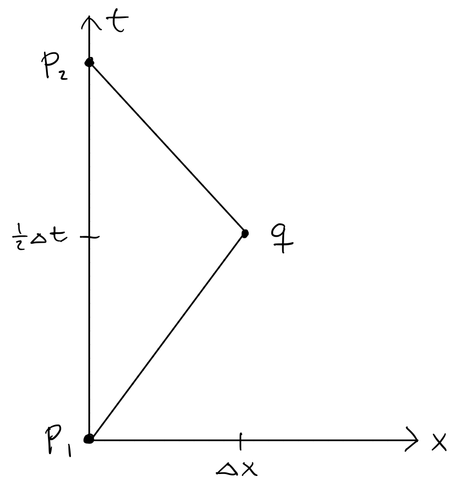

Figure 3: Illustration of the paths of the twins in our version of the Twin paradox.

We have three events. Event p_1 is that Twin B leaves Twin A and starts the spaceship.

[1] Note that the Twin paradox is not an actual paradox. The supposed paradox is that one can try to follow the Inertial Systems of both twins, and then naively both would have larger proper time than the other (which would have been paradoxical if true). However, the resolution is that only one of the twins is in the same Inertial System throughout the trip. Thus, while Twin A is never accelerating, Twin B is accelerating at some point during his trip.

Event q is that Twin B arrives to Mars. Event p_2 is that twin B arrives to see twin A on Earth. Since Twin A is not moving we can choose an Inertial System in which Twin A is at a constant position. We assume for simplicity that Twin B is traveling at approximately constant speed v in this coordinate system. Hence we can parametrize the events as

$$p_1 : \ (t,x) = (0,0) \ , \quad q : \ (t,x) = (\frac{1}{2}\Delta t, \Delta x) \ , \quad p_2 : \ (t,x) = (\Delta t, 0) \, , \tag{1.1.24}$$

with $t = x^0$ and $x = x^1$. Clearly, $\Delta x = v\frac{1}{2}\Delta t$. The proper time of Twin A is $\Delta\tau_A = \Delta t$ while the proper time of Twin B is

$$\Delta\tau_B = \sqrt{1-v^2}\Delta t = \sqrt{1-v^2}\Delta\tau_A \, . \tag{1.1.25}$$

This follows from using that the proper time from p_1 to q is

$$\sqrt{-\eta_{\mu\nu}\Delta x^\mu \Delta x^\nu} = \sqrt{\frac{1}{4}\Delta t^2 - \Delta x^2} = \frac{1}{2}\sqrt{1-v^2}\Delta t \, , \tag{1.1.26}$$

and that this is the same as the proper time from q to p_2. Thus, we see that one always has

$$\Delta\tau_B < \Delta\tau_A \, , \tag{1.1.27}$$

which means Twin B is younger than Twin A once they are together again. In other words, Twin A has a longer proper time than that of Twin B, in accordance with the general statement (1.1.23).

One notices that the greater speed Twin B travels with, the smaller the ration $\Delta\tau_B/\Delta\tau_A$ would be. Indeed, this is another argument for why a straight line cannot correspond to the minimum proper time between two time-like separated events. Because, if we have two given time-like separated events p_1 and p_2, then we can find a time-like curve that gets arbitrarily close to the curve of a light ray that starts at the event p_1, travels to event q where it is reflected, and goes to the event p_2. A light ray always has zero proper time, since it travels with the speed of light which means that $ds^2 = 0$ along the curve of the light ray. Thus, one can approximate the curve of a lightray by travelling close to the speed of light. We illustrated an example of this in Figure 4. This means that one cannot minimize the proper time of time-like curves that goes between two time-like separated events since one can always find a curve that gets closer to zero proper time, but it is not possible to find a time-like curve that has exactly zero proper time.

1.1.4 Motion at speed of light

In Special Relativity massless particles travel at the speed of light, hence with $ds^2 = 0$ along the line. Writing $x^\mu(\lambda)$ as the curve of the massless particle in an Inertial System

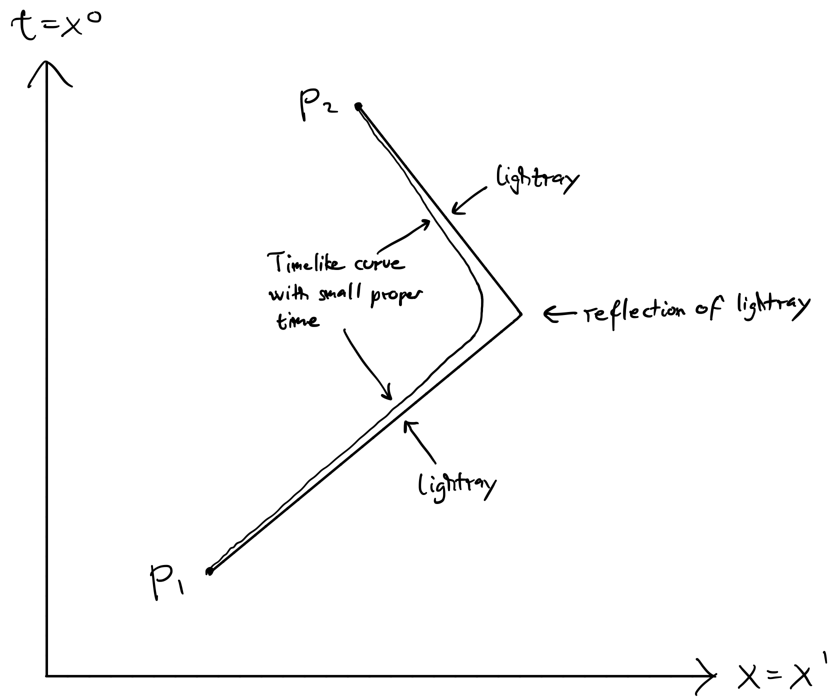

Figure 4: Illustration of how one can minimize the proper time between two time-like separated events.

x^μ we can formulate this as

$$\frac{d^2x^\mu}{d\lambda^2} = 0 \ , \quad \eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0 \, . \tag{1.1.28}$$

The general solution of these conditions is $x^\mu(\lambda) = k^\mu\lambda + b^\mu$ with k^μ and b^μ constant and $\eta_{\mu\nu}k^\mu k^\nu = 0$.

One can change the parametrization $\zeta = \zeta(\lambda)$ so that the curve instead is parametrized as $x^\mu(\zeta)$. However, if we demand that $d^2x^\mu/d\zeta^2 = 0$ then we see that ζ can only depend linearly on λ, *i.e.* it is of the form $\zeta = c_1\lambda + c_2$. ζ and λ are known as *affine parameters*.

In general $dx^\mu/d\lambda$ is proportional to the relativistic momentum p^μ of the massless particle. Hence, since both $k^\mu = dx^\mu/d\lambda$ and p^μ are constant one can use the freedom of parametrization to choose the affine parameter λ such that

$$p^\mu = \frac{dx^\mu}{d\lambda} \, . \tag{1.1.29}$$

1.1.5 Dynamics of particles

Consider a particle with rest mass m and relativistic velocity $u^\mu = dx^\mu/d\tau$ (1.1.19). We assume that the rest mass is conserved. We define the relativistic momentum vector as

$$p^\mu = mu^\mu \,. \tag{1.1.30}$$

Here $E = p^0$ is the energy and $\vec{p} = (p^1, p^2, p^3)$ its momentum. Since $\eta_{\mu\nu}u^\mu u^\nu = -1$ we have $\eta_{\mu\nu}p^\mu p^\nu = -m^2$ and hence

$$E^2 - \vec{p}^2 = m^2 \,. \tag{1.1.31}$$

For a massless particle one gets that $E^2 = \vec{p}^2$.

It is a common misconception that accelerated motion cannot be treated within the framework of Special Relativity. In fact it can, as long as the gravitational interaction can be neglected. The relativistic external force on a particle is

$$\mathcal{F}^\mu = \frac{dp^\mu}{d\tau} \,, \tag{1.1.32}$$

where p^μ is the relativistic momentum (or four-momentum) of the particle. Hence we get a relativistic generalization of Newtons second law

$$\mathcal{F}^\mu = ma^\mu \,. \tag{1.1.33}$$

One finds that $\mathcal{F}^0 = \gamma \vec{F} \cdot \vec{v}$ and $\mathcal{F}^i = \gamma F^i$ for $i = 1, 2, 3$ where $\vec{F} = (F^1, F^2, F^3)$ is the external force of Newton that obeys Newtons second law in the form $\vec{F} = \frac{d\vec{p}}{dt}$ with $\vec{p} = \gamma m\vec{v}$ and $x^0 = t$.

1.1.6 Maxwells equations

The theory of Special Relativity states that the laws of physics are the same in any Inertial System. Here we consider an example of this in the form of Maxwell equations for electromagnetism. Furthermore, the Lorentz force provides a useful illustration of accelerated motion in Special Relativity.

Maxwell equations for electromagnetism in vacuum are

$$\vec{\nabla} \cdot \vec{E} = \rho_e \,, \quad \vec{\nabla} \times \vec{B} = \vec{J} + \frac{\partial \vec{E}}{\partial t} \,, \tag{1.1.34}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \,, \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \,, \tag{1.1.35}$$

where ρ_e is the charge density and $\vec{J}$ is the current density. Eqs. (1.1.35) is equivalent to the statement that there exists a scalar potential ϕ_e and a vector potential $\vec{A}$ such that

$$\vec{E} = -\vec{\nabla}\phi_e - \frac{\partial \vec{A}}{\partial t} \ , \quad \vec{B} = \vec{\nabla} \times \vec{A} \, . \tag{1.1.36}$$

We define the *electromagnetic field strength* $F_{\mu\nu}$ by

$$F_{0i} = -E^i \ , \quad F_{23} = B^1 \ , \quad F_{31} = B^2 \ , \quad F_{12} = B^3 \, , \tag{1.1.37}$$

and by demanding that it is antisymmetric

$$F_{\nu\mu} = -F_{\mu\nu} \, . \tag{1.1.38}$$

Define furthermore the relativistic vector potential A_μ so that (A_1, A_2, A_3) is equal to the above vector potential $\vec{A} = (A^1, A^2, A^3)$ and $A_0 = -\phi_e$. Then we can write the Maxwell equations (1.1.36) as

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \, . \tag{1.1.39}$$

Introduce now also the electromagnetic field strength with upper indices $F^{\mu\nu}$ by

$$F^{0i} = E^i \ , \quad F^{23} = B^1 \ , \quad F^{31} = B^2 \ , \quad F^{12} = B^3 \, , \tag{1.1.40}$$

and by demanding that it is antisymmetric

$$F^{\nu\mu} = -F^{\mu\nu} \, . \tag{1.1.41}$$

Note that the electromagnetic field strengths with lower and upper indices are related by

$$F_{\mu\nu} = \eta_{\mu\rho}\eta_{\nu\sigma}F^{\rho\sigma} \, . \tag{1.1.42}$$

Defining the relativistic current density J^μ so that (J^1, J^2, J^3) is the above current density $\vec{J}$ and $J^0 = \rho_e$, we can write the Maxwell equations (1.1.34) as

$$\frac{\partial F^{\mu\nu}}{\partial x^\mu} = -J^\nu \, . \tag{1.1.43}$$

Under a boost coordinate transformation like (1.1.7) the electric field $\vec{E}$ and the magnetic field $\vec{B}$ mixes together, which means that the four equations in (1.1.34)-(1.1.35) do not preserve their form. For instance, one can start with $\vec{B} = 0$ and $\vec{E}$ non-zero, and then generate a $\vec{B}$ by a boost. Instead the four equations in (1.1.34)-(1.1.35) do preserve their form under rotations and translations like (1.1.6) and (1.1.5). Considering instead the reformulated equations (1.1.39) and (1.1.43) they have the same form in all Inertial

Systems, thus the form of these equations do not change under any Lorentz transformation or translation. One says that Eqs. (1.1.39) and (1.1.43) are *covariant* with respect to Lorentz transformations and translations. This is an example of the general statement in the theory of Special Relativity that the laws of physics are the same in all Inertial Systems. We shall come back to the concept of covariance in the context of General Relativity in Section 1.4.

One can also formulate the Lorentz force $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$ in a natural way in Special Relativity. One finds that for a particle of charge q the relativistic Lorentz force is

$$\mathcal{F}^\mu = -q\eta_{\nu\rho}u^\nu F^{\rho\mu} . \tag{1.1.44}$$

If the particle has rest mass m and is only subject to electromagnetic forces, we find by combining it with (1.1.33) that the relativistic acceleration of the particle is

$$a^\mu = -\frac{q}{m}\eta_{\nu\rho}u^\nu F^{\rho\mu} . \tag{1.1.45}$$

Given q/m and the electromagnetic field strength $F_{\mu\nu}$, one can now use this to find the accelerated motion of a particle in Minkowski space. This is an explicit illustration of the fact that one can treat accelerated motion in Special Relativity.

1.2 Equivalence Principle

The theory of Special Relativity generalizes Newton mechanics so that one can describe velocities close to the speed of light. Since Newton mechanics can be used to describe gravity, one could wonder why is it not possible to describe gravity within the framework of Special Relativity? The answer lies within the Equivalence Principle that follows from the universality of gravity. As consequence of the Equivalence Principle, at least in Einsteins strong version of it, one finds that neither Newtonian mechanics for small velocities, nor Special Relativity, is sufficient to describe gravitational physics. In particular, one finds that a gravitational field affects space and time, so that distances and time intervals can vary depending on where and when you measure them.

1.2.1 Equivalence Principle in Newtonian mechanics

Consider gravity according to Newton. Newtons second law

$$\vec{F} = m_i\vec{a} , \tag{1.2.1}$$

states that the external force on a particle $\vec{F}$ is equal to its inertial mass m_i times its acceleration $\vec{a}$. The force on a particle in a gravitational field is

$$\vec{F}_g = -m_g \vec{\nabla}\phi\,, \tag{1.2.2}$$

where m_g is the gravitational mass and ϕ is the gravitational potential. Hence if the only external force on the particle is from the gravitational field we get

$$m_i \vec{a} = -m_g \vec{\nabla}\phi\,. \tag{1.2.3}$$

Galilei and Newton discovered that the inertial mass and the gravitational mass are one and the same

$$m_i = m_g\,. \tag{1.2.4}$$

Hence, as consequence we have

$$\vec{a} = -\vec{\nabla}\phi\,. \tag{1.2.5}$$

From this we see that the response of matter to gravitation is universal:

> Every object falls at the same rate in a gravitational field, independently of the composition of the object.

This is known as the *Weak Equivalence Principle*. It was first discovered by Galilei.

The Weak Equivalence Principle shows already that the gravitational force works very different from other forces. For instance, a particle only affected by the Lorentz force in an electromagnetic field has

$$m_i \vec{a} = q\vec{E} + q\vec{v} \times \vec{B}\,. \tag{1.2.6}$$

In this case we know very well that m_i and q can vary independently of each other.

While Newtons theory of mechanics is able to include the Weak Equivalence Principle by setting $m_i = m_g$, there is no explanation of why it should be true. If one thinks about this, it poses a deep mystery: why should two physical quantities, the inertial mass m_i, and the gravitational mass m_g, be equal for all matter in the universe? If Newtons theory poses the whole truth about mechanics and gravity, this would require an extreme level of fine-tuning. We shall see below in Section 1.2.2 how the theory of General Relativity solves this mystery.

Consider the consequences of the Weak Equivalence Principle in the following thought experiment. We imagine an observer inside an elevator so that she is not able to see what is outside the elevator. This means she is not able to distinguish a gravitational field giving a constant acceleration $-\vec{\nabla}\phi$ from a constant acceleration $\vec{a}$ of the elevator by observing the motion of freely falling objects.

For instance, if she observes that freely falling objects do not accelerate inside the elevator, she cannot tell from this observation whether it is because the elevator and herself are freely falling in a gravitational field, or whether she is far away from any sources of gravity and hence is not accelerating at all.

A more precise version of the Weak Equivalence Principle, that takes into account the possibility of non-constant gravitational fields, is

> **Weak Equivalence Principle:** The motion of freely falling particles are the same in a gravitational field and a uniformly accelerating system assuming a small enough region and small enough time duration.

Here *uniformly accelerating system* refers to an observer in Newtonian mechanics or in Special Relativity that has a constant acceleration with respect to an Inertial System. The above statement assumes that one adjusts the constant acceleration to the particular location (and time) in a given gravitational field.

1.2.2 Einsteins Equivalence Principle

The idea of Einstein is to promote the Weak Equivalence Principle to include all physical measurements, not just the motion of free falling particles. Hence he formulated what we call *Einsteins Equivalence Principle*:

> **Einsteins Equivalence Principle (EEP):** In small enough regions, and small enough time durations, the laws of physics reduce to those of Special Relativity. Hence, it is impossible to detect the existence of a gravitational field by means of local experiments.

This principle has large consequences for our understanding of gravity and mechanics. The acceleration of a point particle is a local statement since it is a statement that one can define in an arbitrarily small region and time duration. Suppose now that an observer is accelerating with the point particle so that in the system of the observer, the point particle is not accelerating. According to EEP, the physics of the point particle in the system of the observer can be formulated in terms of the theory of Special Relativity which means without gravity. Hence if the observer finds that there are no non-gravitational external forces on the particle then the particle is not accelerating according to the theory of Special Relativity (or Newtonian mechanics if the velocity of the particle is small). Thus, one concludes:

> **Consequence of EEP:** A freely falling particle in a gravitational field (*i.e.* not subject to any non-gravitational forces) is not accelerating.

At first sight, this seems obviously wrong. But that is because we have learned about the concept of acceleration as something that should be measured relative to an Inertial System in Newtonian mechanics. Thus, as consequence of EEP, the Newtonian understanding of acceleration and Inertial Systems is flawed in the presence of a gravitational field.

Imagine we are observing a man standing still in a room, and he throws a ball up in the air so it first flies up and then falls down and lands on the floor. According to Newton, what we are seeing is that the gravitational force makes the ball accelerate downwards towards the floor. Instead the man is standing still, hence he is not accelerating with respect to the room which (at least approximately) can be regarded as an Inertial System. However, according to Einstein, and EEP, this is not the right understanding. When the man has thrown the ball, it is freely falling and hence not accelerating. Instead it is the man that is accelerating, because as consequence of EEP one cannot distinguish locally between being in an accelerated system and in a gravitational field. Indeed, the acceleration of the man is due to the fact that he is stopped by the floor from falling freely. Thus, EEP completely interchanges who is accelerating and who is not, in this situation.

An important lesson that we have learned from the above thought experiment is:

> **Consequence of EEP:** The acceleration of a particle is measured relative to the motion of freely falling particles.

Thus, instead of defining acceleration in comparison to Inertial Systems as in Newtonian mechanics, we should define acceleration in comparison to freely falling particles, meaning particles that are not subject to any non-gravitational external forces.

Furthermore, we conclude that since freely falling particles are not seen as accelerating then gravity cannot be a force since a force is something that leads to acceleration. We will revisit this insight in Section 1.4.4. But we can already anticipate how this solves the mystery of why $m_i = m_g$ in Newtons theory of mechanics. Since there is no gravitational force, one does not need to introduce the coupling m_g between the gravitational field and the corresponding force, as Newton thought. Thus, there is actually no such thing as a gravitational mass m_g and therefore no need of fine-tuning.

1.2.3 Gravitational redshift and blueshift

We now present a thought experiment that shows light is redshifted or blueshifted when propagating in a gravitational field, as consequence of EEP. The idea is to compare two seemingly different physical situations that actually are the same according to EEP, and draw the consequences of this.

First situation: Two rockets

The following situation will be considered purely within the framework of Newtonian mechanics without the presence of gravity. Consider first two rockets, far away from any source of gravity such that gravitational effects can be neglected. We assume that both rockets have constant acceleration a in the same direction and that they have a constant distance L between them. We assume that the speed of the rockets is small compared to the speed of light. See Figure 5 for an illustration.

Figure 5: Illustration of two rockets with acceleration a and distance L.

Let now Rocket 1 send a light signal with wavelength λ_1 towards Rocket 2 at the time $t = t_1$. As illustrated in Figure 5, the light signal is sent in the same direction as the rockets are accelerating. The light signal then reaches Rocket 2 at the time

$$t = t_2 = t_1 + \frac{L}{c}\,, \tag{1.2.7}$$

where c is the speed of light that we are temporarily reinstating in this section. However, at time t_2 the rockets have gained the extra speed

$$\Delta v = a(t_2 - t_1) = \frac{aL}{c}\,, \tag{1.2.8}$$

due to the acceleration. Using the standard Newtonian theory of Doppler-shift, this means that the wave-length of the light is subject to a redshift[2] in the wave-length

$$\Delta\lambda = \frac{\Delta v}{c}\lambda_1 = \frac{aL}{c^2}\lambda_1 \,, \tag{1.2.9}$$

where $\lambda_2 = \lambda_1 + \Delta\lambda$ is the wave-length of the light when received at time t_2 by Rocket 2.

If instead one sends a light signal from Rocket 2 to Rocket 1, one gets a blue shift of the same magnitude as (1.2.9), *i.e.* with $\lambda_1 = \lambda_2 - \Delta\lambda$ being the wave-length received by Rocket 1.

Second situation: Tower in a gravitational field

Consider a tower of height L. We assume that the tower is placed on the surface of the Earth, hence with a uniform gravitational acceleration a_g. A light signal is sent from the ground (denoted 1) with wave-length λ_1 towards the top of the tower (denoted 2). We illustrated the tower in Figure 6.

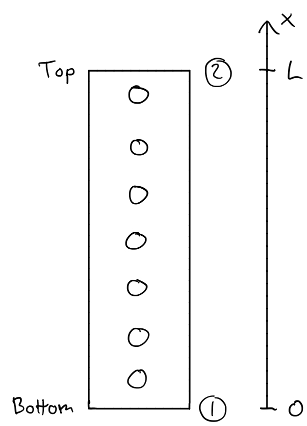

Figure 6: Illustration of a tower of height L.

According to EEP, if we set $a = a_g$ there is no difference between the two situations for the observer receiving the light signal either inside Rocket 2, or on the top of the tower (assuming the observer receiving the light signal is only able to see that light signal and nothing else - and hence she does not know whether she is in a rocket, or on the top of a tower). Hence, according to EEP the observer should receive the light signal with

[2]Note here that a *redshift* means that the frequency of light decreases and the wave-length increases, which for visible light means that it is shifted towards the red part of the spectrum. Instead a *blueshift* means that the frequency of light increases and the wave-length decreases, which for visible light means that it is shifted towards the blue part of the spectrum.

wave-length $\lambda_2 = \lambda_1 + \Delta\lambda$ where

$$\Delta\lambda = \frac{a_g L}{c^2}\lambda_1 \,. \tag{1.2.10}$$

Hence we conclude that light is redshifted when sending it from the bottom to the top of the tower. We assume here that the velocities of the emitter and the receiver of the light are much smaller than the speed of light.

Similarly, one can infer that light is blueshifted with the amount (1.2.10) if one emits a light signal from the top of the tower that is received at the bottom of the tower.

One can understand further the connection to the rocket example by thinking about Einstein's definition of acceleration. For Einstein, a freely falling particle falling down from the tower is not accelerating. Hence the emitter at the ground as well as the receiver at the top of the tower are accelerating with the acceleration a_g upwards, *i.e.* away from the ground, since the acceleration is measured relative to freely falling particles. Thus, using Einstein's concept for acceleration in a gravitational field, we see that we get exactly the same scenario as the two rockets, namely an emitter and a receiver of a light signal, both accelerating with the same acceleration and a constant distance between them.

One can write the formula for the gravitational red- and blueshift using Newtons gravitational potential. Put a coordinate system with the x-axis going upwards and parallel to the tower. Using $\vec{a}_g = -\vec{\nabla}\phi$ we find $-a_g = (\vec{a}_g)_x = -(\vec{\nabla}\phi)_x = -\partial\phi/\partial x$ where $a_g = |\vec{a}_g|$. We find then $\phi = a_g x + \text{constant}$ since a_g can be assumed to be approximately constant. Write now $\Delta\phi = \phi_{\text{receiver}} - \phi_{\text{emitter}}$ and $\Delta\lambda = \lambda_{\text{receive}} - \lambda_{\text{emitter}}$, then we get from (1.2.10):[3]

[3]Note that the above arguments leading to (1.2.11) should be regarded with some reservation and care. It uses thought-experiment and a general principle to show a new physical phenomena. But is it not an exact derivation. First of all, the result (1.2.11) is more general than the setting in which we derived it since it is generally valid when gravity is weak and therefore well-described by Newtonian gravity. Secondly, one ignores that the system has the finite size L and the EEP cannot be applied exactly to the full system, only approximately. Instead one should apply the EEP to the two events of emitting and receiving the light signal. However, as discussed in Section 1.3.5 this is a negligible correction.

Gravitational redshift and blueshift: If one emits light with wave-length λ in a gravitational potential ϕ, and it is received with wave-length $\lambda + \Delta\lambda$ in a gravitational potential $\phi + \Delta\phi$, then the shift of the wave-length is given by

$$\frac{\Delta\lambda}{\lambda} = \frac{1}{c^2}\Delta\phi\,. \tag{1.2.11}$$

Thus, light is redshifted (blueshifted) if the gravitational potential is greater (smaller) at its reception than at its emission. We assume here that the velocities of the emitter and the receiver of the light are much smaller than the speed of light.

Experimental test

In 1959 Pound and Rebka verified the gravitational redshift and blueshift of light by placing two samples of Iron emitting gamma rays at the roof and at the basement of the Jefferson Physical Laboratory building at Harvard University [2]. The difference in height is 22.5 meter. They then measured the blueshifted gamma rays received in the basement and the redshifted gamma rays received at the roof. The results fitted with the formula (1.2.10) to a 10% accuracy. Nowadays the effect has been confirmed with an accuracy of 0.01%.

1.2.4 Gravitational time dilation

Consider again sending a light signal with wave-length λ_1 from the bottom of the tower towards the top of the tower, received at the top of the tower with wave-length λ_2, as illustrated on Figure 6. We now attempt to analyze the light signal from a Newtonian perspective. At the bottom of the tower, when it is sent out, the light signal has period

$$T_1 = \frac{\lambda_1}{c}\,. \tag{1.2.12}$$

When it is received at the top of the tower, it has period

$$T_2 = \frac{\lambda_2}{c}\,. \tag{1.2.13}$$

Consider the propagation of the beginning and the end of a period of a light signal as illustrated on Figure 7.[4] Clearly, one sees from Figure 7 that since the beginning and

[4]A simple definition of the beginning and the end of a period of a light signal is to consider the electromagnetic fields of the electromagnetic wave at a given moment in time as the beginning, and the end as the next moment in time that one has the same configuration for the electromagnetic fields.

the end of the light signal both propagates with speed c and they both have to traverse the distance L, the beginning and the end of the light signal must take the same time to travel from the bottom to the top of the tower. Therefore, the time duration between the beginning and the end of the signal must be equal at the bottom and the top of the tower. Thus, one concluces $T_2 = T_1$.

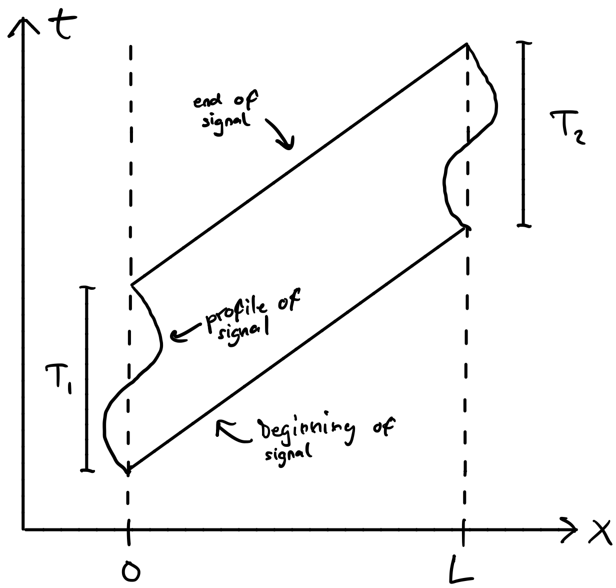

Figure 7: Illustration of the propagation of the beginning and end of a period of a light signal assuming Newtonian concepts of space and time.

However, we have just found above that EEP implies $T_2 \neq T_1$ due to the gravitational redshift effect. How is this possible? The answer is that Newton made certain assumptions about the nature of space and time when formulating his theory of mechanics. In particular, Newton assumed that there is a common time that both applies to the bottom and the roof of the tower. But we now see that this cannot be the case, as consequence of EEP.

In fact, the gravitational redshift can alternatively be seen as a consequence of the time at the bottom of the tower running slower than at the top of the tower. Indeed, we get from (1.2.10)[5]

$$T_2 = \frac{\lambda_2}{c} = \frac{\lambda_1 + \Delta\lambda}{c} > \frac{\lambda_1}{c} = T_1 \,. \tag{1.2.14}$$

Notice that using (1.2.11)

$$\frac{T_2}{T_1} = 1 + \frac{\Delta\lambda}{\lambda_1} = 1 + \frac{\Delta\phi}{c^2} \,. \tag{1.2.15}$$

[5] $T_2 > T_1$ means that the time at the top of the tower runs more quickly since it is larger.

One can infer from this that the closer one is to a source of gravity, the slower time will run since ϕ increases as one moves away from the source. This phenomena is known as *gravitational time-dilation* which we can succinctly formulate as:

Time runs slower in a stronger gravitational field.

More precisely, if two observers are placed in a gravitational field that can be described by Newtons law of gravity, and they are moving with speeds much lower than the speed of light, then the clock of the observer that is in the stronger gravitational field will run slower than that of the other observer.

Note that already the theory of Special Relativity means that one has to abandon the centuries old assumptions behind Newtonian mechanics: that time is an universal quantity that all observers can agree on. Now we see that this is even true for velocities much smaller than the speed of light, since a gravitational field can affect how one measures time. Thus, our concept of time is modified with respect to Newtonian mechanics both in the case of high velocities and in the presence of gravitational fields.

GPS satellites

The effect of gravitational time dilation is very important for the accuracy of the GPS positioning system measuring the positions on the Earth using a GPS receiver. This is because time in the GPS satellites runs more quickly than time at the surface of the Earth.

The satellites are 20 000 km above the surface of the Earth and they move with a relative speed of 14 000 km/hour. On board are atomic clocks that have an accuracy of a nanosecond. The time-dilation effect of Special Relativity means that the atomic clocks should go 7 microseconds slower for every 24 hours. Instead the gravitational time-dilation mentioned above gives that the atomic clocks aboard the satellites should go 45 microseconds faster for every 24 hours. Thus, the net effect is that the atomic clocks goes 38 microseconds faster for every 24 hours. This fits with what the GPS satellites are measuring.

Note finally that the accuracy of the GPS positioning system requires an accuracy for time measurements of 20 to 30 nanoseconds. Hence, one could not measure positions on the Earth accurately without taking into account the effect of gravitational time-dilation.

1.3 General Space-Times

We shall see in this section that the consequence of the gravitational time-dilation effect that we found in Section 1.2.4 is that we are not living in Minkowski space. Hence Special Relativity does not describe physics when gravity is present. To understand this better, we first take a lightning tour of geometry in Section 1.3.1, and how it is described by line-elements and metrics. This will enable us to introduce general space-time geometries in Section 1.3.2 and the geometric concept of a geodesics in Section 1.3.3 which describes freely falling motion in General Relativity. In Section 1.3.4 we shall connect this to the Newtonian equation for a particle in a gravitational field by considering the Newton limit of General Relativity. Finally in Section 1.3.5 we will give a geometric interpretation of Einsteins Equivalence Principle with the concept of Local Inertial Systems.

1.3.1 A lightning introduction to geometry

The main inventors of geometry are Pythagoras and his followers, around 2500 years ago, and Euclid, around 2300 years ago. They invented what today is known as *Euclidean geometry*.

Pythagoras and his followers observed that for a right-angled triangle as depicted in Figure 8 the square of the hypotenuse is given by $L^2 = x^2 + y^2$, a statement known as *Pythagoras' theorem*. This is a corner stone of Euclidean geometry, in this case in two dimensions.

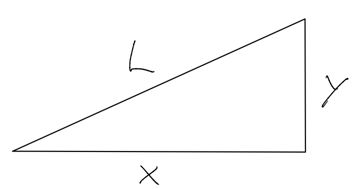

Figure 8: A right-angled triangle.

One can use Pythagoras' theorem to write down relations for triangles without right-angles as well. In Figure 9 we have depicted a triangle with a general angle θ and sides of lengths x, y and L, which for $\theta = \pi/2$ would reduce to a right-angle triangle. One can now compute the length L by using Pythagoras' theorem for the projected right-angle triangle drawn on the Figure, giving

$$L^2 = (x - \cos\theta\, y)^2 + (\sin\theta\, y)^2 = x^2 + y^2 - 2\cos\theta\, x\, y\,, \qquad (1.3.1)$$

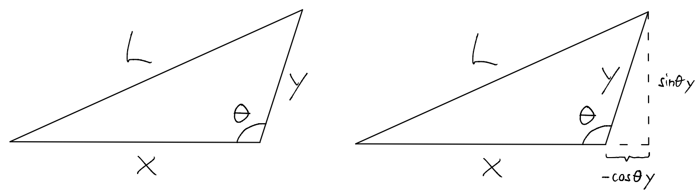

Figure 9: Triangle with general angle θ.

which generalizes Pythagoras' theorem. In this way we see that the cross term $x\,y$ in the above computation of L^2 arises due to having an angle different from $\theta = \pi/2$.

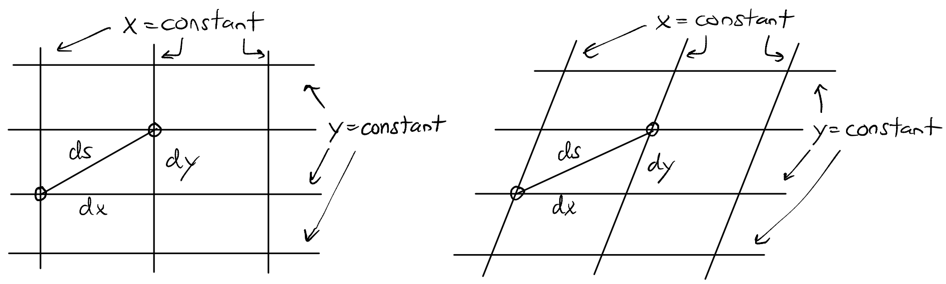

Figure 10: Coordinate grids to measure distance between two points.

So how does this relate to modern geometry? Imagine for a given geometry we want to measure the distance between two nearby points. Then we can lay in a coordinate system (grid) on the geometry to mark where the two points are. We take one point to be at (x, y) while the other point is at $(x + dx, y + dy)$. This is drawn in Figure 10, where ds is the distance between the two points, here taken to be infinitesimally close, for reasons explained below. Thus, dx and dy are the (infinitesimal) differences in the coordinates as illustrated with the coordinate grids in Figure 10 where the lines marks when the coordinates have constant values.

Consider first the left part of Figure 10. Supposing dx and dy are distances between the lines in the coordinate grids, then Pythagoras directly gives $ds^2 = dx^2+dy^2$. However, now we open for the possibility that the difference in coordinate values are not directly related to the difference in distance. So we need a way to translate a difference in coordinates to a distance. This is done with the concept of a *metric*. Concretely, we write

$$ds^2 = g_{xx}dx^2 + g_{yy}dy^2 \tag{1.3.2}$$

This expression means that the distance along the x-direction is measured as $\sqrt{g_{xx}}dx$ while the distance along the y-direction is $\sqrt{g_{yy}}dy$. Given this, (1.3.2) again expresses

Pythagoras' theorem. Thus, the metric components g_{xx} and g_{yy} can be used to translate coordinate differences to distances.

On the right part of Figure 10 we have the more general situation for which the coordinate grid does not have right angles between the two directions. Then we invoke one more component of the metric g_{xy} to write

$$ds^2 = g_{xx}dx^2 + g_{yy}dy^2 + 2g_{xy}dx\,dy\,, \tag{1.3.3}$$

Thus, an angle different from $\pi/2$ again corresponds to a cross term $g_{xy}dx\,dy$, and the angle can be found from $g_{xy}/\sqrt{g_{xx}g_{yy}}$. The formula (1.3.3) is called a *line-element* and gives a general relation between the distance between two points ds, the coordinate system (x, y) and the metric in a two-dimensional geometry.

If the metric components g_{xx}, g_{yy} and g_{xy} do not depend on the coordinates then we are still in two-dimensional Euclidean geometry. However, the mathematician Bernhard Riemann invented curved space geometry almost two centuries years ago by allowing the metric components to depend on the coordinates. Thus, the line-element is then

$$ds^2 = g_{xx}(x, y)dx^2 + g_{yy}(x, y)dy^2 + 2g_{xy}(x, y)dx\,dy\,, \tag{1.3.4}$$

This means that the measurements of distances and angles can vary as one moves around in the geometry. This is why it is important that we use the line-element only to measure distances between infinitesimally separated points, so that the metric components are approximately constant when using the relation (1.3.4). Thus, for a general two-dimensional curved space geometry the line-element only works for infinitesimally separated points. Another way to state this is that curved space geometry is anchored in Euclidean geometry, since for very small distances the metric components are always approximately constant, and then one can use Pythagoras' theorem to measure the distance between two points.

With the above, one can now specify complete two-dimensional geometries by writing down how distances between infinitesimally separated points are measured in a specific coordinate system. Two-dimensional Euclidean space can be written as the line-element

$$ds^2 = dx^2 + dy^2 \tag{1.3.5}$$

meaning that for any two infinitesimally separated points we always have the same relation between ds, dx and dy. A central property of a geometry is that the coordinates are not important. This we touched upon above, where the same infinitesimal distance between two points in Euclidean space (1.3.5) is described by many different line elements (1.3.2) and (1.3.3) (assuming g_{xx}, g_{yy} and g_{xy} are constant) in relation to Figure 10. In fact, one can describe the same geometry in infinitely many different coordinate systems. For the geometry (1.3.5) we can for example also use polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ with line-element $ds^2 = dr^2 + r^2 d\theta^2$. Again, this is the exact same geometry, so with the same distance between two given points, but just described in another coordinate system. We will have more to say about this in the next section.

Another example of a two-dimensional geometry is the sphere with metric

$$ds^2 = a^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{1.3.6}$$

where the coordinates now are the spherical angles θ and ϕ, with ranges $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$, and a is the radius of the sphere. We notice that $g_{\theta\theta} = a^2$, $g_{\phi\phi} = a^2 \sin^2\theta$ and $g_{\theta\phi} = 0$. One can obtain this metric by embedding the sphere in three-dimensional Euclidean space, see Exercise 1.6. The two-dimensional sphere is an example of a curved geometry,[6] and hence a non-Euclidean geometry. This is also known as *Riemannian geometry* after Bernhard Riemann.[7]

We can generalize line-elements and metrics to three-dimensional geometries as

$$ds^2 = \sum_{i,j=1}^{3} g_{ij} dx^i dx^j\,, \tag{1.3.7}$$

where we wrote the coordinates as $x^i = (x^1, x^2, x^3)$. Here g_{ij} can be seen as a three-by-three metric

$$g_{ij} = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}. \tag{1.3.8}$$

However, one imposes the important requirement that it is symmetric

$$g_{ij} = g_{ji}\,, \tag{1.3.9}$$

since cross terms $dx^i dx^j$ in the above line-elements do not care about the order of dx^i and dx^j. A special case is when $g_{ii} = 1$ and $g_{ij} = 0$ for $i \neq j$ giving the line-element

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \tag{1.3.10}$$

which is the metric for three-dimensional Euclidean space in Cartesian coordinates.

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[6]The curvature of the sphere is computed in Exercise 1.18.

[7]Note that for many geometries one cannot expect a single coordinate system to cover the whole geometry. This means one needs to make a patchwork of coordinate systems that works for every part of the geometry, plus the coordinate transformations between the coordinates where more than one set of coordinates are valid. This is a subtlety that will be important when we consider the space-time geometry of the Schwarzschild black hole.

1.3.2 Metric of a space-time

With the above lightning introduction to spatial geometry, we can now return to the concept of a space-time that we already introduced in Sections 1.1.2 and 1.1.3. In Special Relativity, physics is described in Minkowski space. Consider an Inertial System x^μ, and two infinitesimally separated events x^μ and $x^\mu + dx^\mu$. Then we introduced in Eq. (1.1.16) the line-element

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \tag{1.3.11}$$

that gives back either the proper time or the proper distance between the two events, depending on whether they are time-like or space-like separated, or it is simply zero if they are null separated. Comparing to (1.3.5) we see that we extended Pythagoras' theorem to include a time-direction, but with the important difference that one should include a relative sign between time and space.

The line-element (1.3.11) is our first example of a *four-dimensional space-time geometry*. The example is limited to the particular space-time called Minkowski space and furthermore in Inertial System coordinates, as described in Sections 1.1.2 and 1.1.3.

So, the question is now: Can we describe all of physics within the framework of Minkowski space or do we need to generalize to more non-trivial space-time geometries? The answer to this question is found by considering the gravitational time-dilation effect described Section 1.2.4. Let us consider the bottom of the tower of Figure 6 to be at the point $(x^1, x^2, x^3) = (0, 0, 0)$ while the top of the tower is at the point $(x^1, x^2, x^3) = (L, 0, 0)$ in Minkowski space with line-element (1.3.11). When measuring the period T_1 at the bottom of the tower we can regard it as a difference between two events with $dx^1 = dx^2 = dx^3 = 0$, hence $T_1 = dx^0$. This also holds for measuring the period T_2 at the top of the tower, hence $T_2 = dx^0$. Furthermore, since both the beginning and end of the light signal propagates at the speed of light $c = 1$ in Minkowski space, the difference dx^0 is the same at emission as at the detection. Thus, we get that $T_1 = T_2 = dx^0$ from (1.3.11). However, this contradict the effect of gravitational time-dilation inferred on the basis of Einsteins Equivalence Principle in Section 1.2.4. Therefore, we conclude that when gravity is turned on, one can no longer describe physics using Minkowski space.

To account for gravitational time-dilation, we need therefore to consider more general space-time geometries than Minkowski space. Analogously to the spatial geometries of Section 1.3.1, this can be done by allowing for more general line elements, *i.e.* more general relations between coordinates and proper distances/times.

To write down a line-element for a general space-time, we should first pick a coordinate system that parameterize it. We write here a given set of space-time coordinates for the space-time as x^μ, $\mu = 0, 1, 2, 3$. In terms of the coordinate system we can write the line-element of the space-time for two infinitesimally separated events x^μ and $x^\mu + dx^\mu$ as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \,, \tag{1.3.12}$$

where $g_{\mu\nu}(x)$ for each value of $\mu, \nu = 0, 1, 2, 3$ is a function of the space-time coordinates x^μ. The object $g_{\mu\nu}(x)$ is known as the *metric* of the space-time. We require the metric to be symmetric in its indices

$$g_{\mu\nu}(x) = g_{\nu\mu}(x) \,, \tag{1.3.13}$$

since any antisymmetric part would not contribute to the line-element. It is sometimes useful to view $g_{\mu\nu}(x)$ as a four by four symmetric matrix

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{01} & g_{11} & g_{12} & g_{13} \\ g_{02} & g_{12} & g_{22} & g_{23} \\ g_{03} & g_{13} & g_{23} & g_{33} \end{pmatrix} , \tag{1.3.14}$$

where all the ten different entries are functions of the space-time coordinates x^μ. The line-element defines whether two events x^μ and $x^\mu + dx^\mu$ are space-like, time-like or null separated

$$\begin{aligned} ds^2 > 0&: \text{ space-like separated events} \,, \\ ds^2 < 0&: \text{ time-like separated events} \,, \\ ds^2 = 0&: \text{ null separated events} \,. \end{aligned} \tag{1.3.15}$$

This in turn defines which events can causally affect each other, and hence part of the causal structure of the space-time. However, the line-element does not define whether one event lies in the future or the past of another; that one has to specify in addition to the line-element.

Minkowski space is a special case of a space-time geometry. Minkowski space is the space-time for which one can find coordinate systems x^μ, also is known as Inertial Systems, such that the metric is

$$g_{\mu\nu}(x) = \eta_{\mu\nu} \,, \tag{1.3.16}$$

for all x^μ. Hence $\eta_{\mu\nu}$ is called the *Minkowski metric*. It is important to note that there are other coordinate systems for Minkowski space as well where $\eta_{\mu\nu}$ is not the metric, see for instance Exercises 1.6, 1.7 and 1.8 where we consider spherical, polar, and rigidly rotating coordinates for Minkowski space .

So can we accommodate the effect of gravitational time-dilation of Section 1.2.4 within this more general framework of four-dimensional space-time geometries? We shall come back to this question in the end of Section 1.3.4.

Coordinate transformations

A space-time is a four-dimensional geometry. That it is a geometry means that we do not want its properties to depend on what coordinate systems we choose. Specifically, while the space-time geometry depends on the line-element, it should not depend on what coordinates we use to write down the line-element. Hence, we require that the line-element ds^2 should be invariant under any coordinate transformation.

Consider a given coordinate transformation

$$x^\mu \to \tilde{x}^\mu(x)\,. \tag{1.3.17}$$

That ds^2 is invariant means

$$ds^2(x) = d\tilde{s}^2(\tilde{x})\,, \tag{1.3.18}$$

where the LHS is the line-element in the x^μ coordinates, while the RHS (RHS) is the line-element in the $\tilde{x}^\mu$ coordinates. Thus, since $d\tilde{s}^2 = \tilde{g}_{\alpha\beta} d\tilde{x}^\alpha d\tilde{x}^\beta$ we demand

$$g_{\mu\nu}(x) dx^\mu dx^\nu = \tilde{g}_{\alpha\beta}(\tilde{x}) d\tilde{x}^\alpha d\tilde{x}^\beta\,. \tag{1.3.19}$$

Under the coordinate transformation (1.3.17) we have

$$d\tilde{x}^\alpha = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} dx^\mu\,, \tag{1.3.20}$$

where $\partial\tilde{x}^\alpha/\partial x^\mu$ is a partial derivative of $\tilde{x}^\alpha(x)$. This follows from dx^μ and $d\tilde{x}^\alpha$ being infinitesimal. Moreover, we can invert this relation as

$$dx^\mu = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} d\tilde{x}^\alpha\,, \tag{1.3.21}$$

where $\partial x^\mu/\partial\tilde{x}^\alpha$ is a partial derivative of the inverse coordinate transformation $x^\mu(\tilde{x})$. Using (1.3.21) with (1.3.19) we find

$$g_{\mu\nu}(x) \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial x^\nu}{\partial \tilde{x}^\beta} d\tilde{x}^\alpha d\tilde{x}^\beta = \tilde{g}_{\alpha\beta}(\tilde{x}) d\tilde{x}^\alpha d\tilde{x}^\beta\,. \tag{1.3.22}$$

This relation should hold for any $d\tilde{x}^\alpha$. Hence we conclude:

Transformation of the metric: Under a coordinate transformation $x^\mu \to \tilde{x}^\mu(x)$ the metric should transform as

$$\tilde{g}_{\alpha\beta}(\tilde{x}) = g_{\mu\nu}(x) \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial x^\nu}{\partial \tilde{x}^\beta}\,. \tag{1.3.23}$$

The transformation law of the metric (1.3.23) is an important part of the *theory of General Relativity* that we now have started describing. It is a necessary requirement for the line-element (1.3.12) to be describing a space-time geometry, since geometry is something that one should be able to give meaning to in a coordinate-invariant manner. Demanding that our space-time can be described as a geometry is a central assumption of the theory of General Relativity.

Inverse metric

In a given coordinate system x^μ with metric $g_{\mu\nu}(x)$ we can define the so-called *inverse metric*

$$g^{\mu\nu}(x)\,, \tag{1.3.24}$$

as the inverse four by four matrix of the metric in each point (event) of the space-time. Hence, we demand

$$g^{\mu\nu}(x)g_{\nu\rho}(x) = g_{\rho\nu}(x)g^{\nu\mu}(x) = \delta^\mu_\rho\,, \tag{1.3.25}$$

everywhere in the space-time. We use here the *Kronecker delta* which we define as

$$\delta^\mu_\nu = \begin{cases} 1 & \text{if} \quad \mu = \nu\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{1.3.26}$$

One can show that it follows that the inverse metric is symmetric

$$g^{\mu\nu}(x) = g^{\nu\mu}(x)\,. \tag{1.3.27}$$

The inverse metric does not have any direct physical interpretation in General Relativity. However, it is a highly useful object in General Relativity for writing down various equations and for performing computations.

From the transformation law of the metric (1.3.23) one finds the following transformation law for the inverse metric:

Transformation of the inverse metric: Under a coordinate transformation $x^\mu \to \tilde{x}^\mu(x)$ the inverse metric should transform as

$$\tilde{g}^{\alpha\beta}(\tilde{x}) = g^{\mu\nu}(x)\frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\frac{\partial \tilde{x}^\beta}{\partial x^\nu}\,. \tag{1.3.28}$$

One can easily check the validity of this transformation law. First one can see from combining (1.3.20) and (1.3.21) that

$$\frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\frac{\partial x^\mu}{\partial \tilde{x}^\beta} = \delta^\alpha_\beta\,, \quad \frac{\partial x^\mu}{\partial \tilde{x}^\alpha}\frac{\partial \tilde{x}^\alpha}{\partial x^\nu} = \delta^\mu_\nu\,. \tag{1.3.29}$$

Using this, we compute

$$\tilde{g}^{\alpha\beta}\tilde{g}_{\beta\gamma} = g^{\mu\nu}\frac{\partial\tilde{x}^\alpha}{\partial x^\mu}\frac{\partial\tilde{x}^\beta}{\partial x^\nu}g_{\rho\sigma}\frac{\partial x^\rho}{\partial\tilde{x}^\beta}\frac{\partial x^\sigma}{\partial\tilde{x}^\gamma} = g^{\mu\nu}g_{\rho\sigma}\frac{\partial\tilde{x}^\alpha}{\partial x^\mu}\frac{\partial x^\sigma}{\partial\tilde{x}^\gamma}\delta^\rho_\nu = \delta^\mu_\sigma\frac{\partial\tilde{x}^\alpha}{\partial x^\mu}\frac{\partial x^\sigma}{\partial\tilde{x}^\gamma} = \delta^\alpha_\gamma\,. \tag{1.3.30}$$

Thus, we see that the transformation law (1.3.28) ensures that the inverse metric is also the inverse of the metric after the coordinate transformation.

Note finally that in the special case of Minkowski space and using an Inertial System, the inverse metric reduces to the inverse Minkowski metric $g^{\mu\nu} = \eta^{\mu\nu}$ which is defined as

$$\begin{aligned} &\eta^{00} = -1\,, \quad \eta^{11} = \eta^{22} = \eta^{33} = 1\,, \\ &\eta^{\mu\nu} = 0 \;\;\text{for}\;\; \mu \neq \nu\,. \end{aligned} \tag{1.3.31}$$

Using (1.1.10) one can check explicitly that

$$\eta_{\mu\nu}\eta^{\nu\rho} = \eta^{\rho\nu}\eta_{\nu\mu} = \delta^\rho_\mu\,. \tag{1.3.32}$$

Thus, the Minkowski metric $\eta_{\mu\nu}$ and its inverse $\eta^{\mu\nu}$ fulfil the relation (1.3.25).

1.3.3 Geodesics

Without gravity, space-time geometry is described by Minkowski space. In this case one has that a freely falling particle, that in this case means a particle not subject to any external forces, is moving along a straight line. Turning on gravity, one should consider a general space-time with line-element (1.3.12). A freely falling particle is defined as a particle that is not subject to any non-gravitational forces. The question we address in this section is: what is the motion of a freely falling particle in a general space-time?

To address this, we consider two time-like separated events p_1 and p_2 in the space-time. We use a coordinate system x^μ in which the two events are $x^\mu_{(1)}$ and $x^\mu_{(2)}$, respectively. Consider then a time-like curve between the two events parametrized as

$$x^\mu(\lambda)\,, \tag{1.3.33}$$

so that $x^\mu(\lambda)$ goes from $x^\mu_{(1)}$ to $x^\mu_{(2)}$ as λ goes from λ_1 to λ_2. See Figure 11 for an illustration.

For the infinitesimal piece of the curve from $x^\mu(\lambda)$ to $x^\mu(\lambda + d\lambda)$ we have the proper time $d\tau$ given by

$$d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu = -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}d\lambda^2\,. \tag{1.3.34}$$

Hence the proper time for the whole curve from $x^\mu_{(1)}$ to $x^\mu_{(2)}$ is by integrating this, giving

$$\Delta\tau = \int_{\lambda_1}^{\lambda_2} d\lambda\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,. \tag{1.3.35}$$

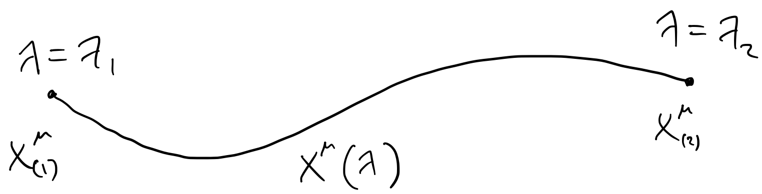

Figure 11: Illustration of time-like curve $x^\mu(\lambda)$ between two time-like separated events $x^\mu_{(1)}$ and $x^\mu_{(2)}$.

In Minkowski space, one has that the unaccelerated path is the path that maximize the proper time, as stated in (1.1.23). We promote now this to a general principle for all space-times:

A freely falling particle in a general space-time will follow a curve that maximizes the proper time.

Thus, among all the time-like curves from $x^\mu_{(1)}$ to $x^\mu_{(2)}$, a freely falling particle will follow a curve that maximizes the proper time (1.3.35). Such a curve is called a *geodesic*.

The concept of a geodesic is known from Riemannian geometry (*i.e.* geometry without time directions) that we considered in Section 1.3.1. In this case a geodesic between two points is a curve that *minimizes* the length between the two points. In the case of the plane with line-element (1.3.5) (*i.e.* the two-dimensional Euclidean space) the geodesics are straight lines. Instead in the case of the sphere with line-element (1.3.6) the geodesics lie along the great circles.

When considering space-time geometry, the closest analogue to geodesics in Riemannian geometry is a geodesic between two space-like separated events. Such a *space-like geodesic* is a curve that *minimizes* the proper length between the two events (measured by integrating up the squareroot of the line-element). Instead, for two time-like separated events, as considered in the special case of Minkowski space in Section 1.1.3, the sign difference in $d\tau^2 = -ds^2$ means that the time-like geodesic between the events is naturally defined as a curve that *maximizes* the proper time.

Geodesic equation

We now find the equations describing geodesics, *i.e.* the motion of freely falling particles in a general space-time.

Consider the same setup as above with a time-like curve $x^\mu(\lambda)$ that goes from $x^\mu_{(1)}$ to $x^\mu_{(2)}$. We write again the proper time (1.3.35)

$$\Delta\tau[x^\mu(\lambda)] = \int_{\lambda_1}^{\lambda_2} d\lambda \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,. \tag{1.3.36}$$

Here we added the dependence of $\Delta\tau$ on the curve $x^\mu(\lambda)$ on the LHS to make this explicit.

Given a time-like curve $x^\mu(\lambda)$ we can consider another curve $x^\mu(\lambda)+\delta x^\mu(\lambda)$ that lies infinitesimally close, meaning that the difference between the curves $\delta x^\mu(\lambda)$ is infinitesimally small. Since both curves start and end at the same events, we have $\delta x^\mu(\lambda_1) = \delta x^\mu(\lambda_2) = 0$. We can then compute the difference in the proper time (1.3.36)

$$\delta\Delta\tau = \Delta\tau[x^\mu(\lambda) + \delta x^\mu(\lambda)] - \Delta\tau[x^\mu(\lambda)]\,. \tag{1.3.37}$$

When $x^\mu(\lambda)$ is a geodesic it maximizes the proper time $\Delta\tau$. Hence any curve that lies infinitesimally close to it should have $\delta\Delta\tau = 0$ to first order in $\delta x^\mu(\lambda)$.[8]

To find a geodesic between $x^\mu_{(1)}$ to $x^\mu_{(2)}$, we need thus to find a curve $x^\mu(\lambda)$ that extremises the proper time such that $\delta\Delta\tau = 0$. To achieve this, one can go ahead and directly perform the variation of (1.3.36) with respect to the infinitesimal difference $\delta x^\mu(\lambda)$. However, we shall instead use a well-known result in analytical mechanics. To this end, one notices that the RHS of (1.3.36) can be viewed as an action corresponding to the integral of the Lagrangian

$$L(x^\mu, \tfrac{dx^\mu}{d\lambda}) = \sqrt{-g_{\mu\nu}(x)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,, \tag{1.3.38}$$

over the curve $x^\mu(\lambda)$ from $x^\mu_{(1)}$ to $x^\mu_{(2)}$, where x^μ can be viewed as four coordinates and $dx^\mu/d\lambda$ as four velocities. It is a well-known result in analytical mechanics that the extremum of the action, and hence an extremum of the proper time $\Delta\tau$, is found when the curve $x^\mu(\lambda)$ satisfy the Euler-Lagrange equations

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \frac{dx^\sigma}{d\lambda}}\right) = \frac{\partial L}{\partial x^\sigma}\,, \tag{1.3.39}$$

where we use the index σ for later convenience. We compute

$$\frac{\partial L}{\partial x^\sigma} = -\frac{1}{2L}\frac{\partial g_{\nu\rho}}{\partial x^\sigma}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda}\,, \quad \frac{\partial L}{\partial \frac{dx^\sigma}{d\lambda}} = -\frac{1}{L} g_{\nu\sigma}\frac{dx^\nu}{d\lambda}\,. \tag{1.3.40}$$

[8] This is analogous to the statement that if a curve $h(u)$ has a maximum at $u = u_0$ then the first derivative of $h(u)$ is zero at $u = u_0$.

Thus, (1.3.39) gives

$$\frac{d}{d\lambda}\left(\frac{1}{L}g_{\nu\sigma}\frac{dx^\nu}{d\lambda}\right) = \frac{1}{2L}\frac{\partial g_{\nu\rho}}{\partial x^\sigma}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda}\,, \tag{1.3.41}$$

which is four equations since σ is a free index. This is true for any parametrization of the curve $x^\mu(\lambda)$ that extremises (1.3.36). Thus, it holds in particular when we use the proper time τ to parametrize the curve, such that $\lambda = \tau$. However, the proper time is special as $d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu$, hence $L = 1$ for this particular parametrization. Therefore, (1.3.41) gives

$$\frac{d}{d\tau}\left(g_{\nu\sigma}\frac{dx^\nu}{d\tau}\right) = \frac{1}{2}\frac{\partial g_{\nu\rho}}{\partial x^\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} \tag{1.3.42}$$

Writing out the derivative with respect to τ, this becomes

$$g_{\nu\sigma}\frac{d^2x^\nu}{d\tau^2} + \left(\frac{\partial g_{\nu\sigma}}{\partial x^\rho} - \frac{1}{2}\frac{\partial g_{\nu\rho}}{\partial x^\sigma}\right)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0 \tag{1.3.43}$$

Contracting now with $g^{\mu\sigma}$, we get

$$\frac{d^2x^\mu}{d\tau^2} + \frac{1}{2}g^{\mu\sigma}\left(2\frac{\partial g_{\nu\sigma}}{\partial x^\rho} - \frac{\partial g_{\nu\rho}}{\partial x^\sigma}\right)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0\,. \tag{1.3.44}$$

Thus, we can conclude that any curve that extremises (1.3.36) should obey (1.3.44). Therefore, we have shown:

Geodesic equation: The *geodesic equation* that describes the motion of a freely falling particle in a general space-time, with metric $g_{\mu\nu}(x)$ and coordinate system x^μ, is

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0\,, \tag{1.3.45}$$

where $\Gamma^\mu_{\nu\rho}$ is the *Christoffel symbol* defined in terms of the metric by

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}\left(\frac{\partial g_{\nu\sigma}}{\partial x^\rho} + \frac{\partial g_{\rho\sigma}}{\partial x^\nu} - \frac{\partial g_{\nu\rho}}{\partial x^\sigma}\right)\,. \tag{1.3.46}$$

Note that in Eqs. (1.3.45)-(1.3.46) we have symmetrized the expression in the second term of (1.3.44). This is because only the symmetric part can contribute. Hence the Christoffel Symbol is symmetric in its lower indices

$$\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\rho\nu}\,. \tag{1.3.47}$$

In the special case of Minkowski space, and choosing an Inertial System, the metric is $g_{\mu\nu} = \eta_{\mu\nu}$. This is immediately seen to give $\Gamma^\mu_{\nu\rho} = 0$. Hence the geodesic equation

(1.3.45) reduces to $d^2x^\mu/d\tau^2 = 0$. Thus, as expected, the geodesics of Minkowski space are straight lines.

We have shown above that a time-like curve that maximizes the proper time should be a geodesic curve, *i.e.* it should obey the geodesic equation (1.3.45), since a maximum is an extremum. However, in principle the condition $\delta\Delta\tau = 0$ holds for any extremum of $\Delta\tau$, and not just for a maximum. Thus, one could ask if the reverse is also true: could the time-like geodesic curve be an extremum without being a maximum? The answer is given in Section 9.3 of [3]: for a time-like geodesic curve to be a maximum of the proper time, a necessary and sufficient condition is that the geodesic does not contain so-called *conjugate events*. Two events in a space-time are called conjugate if there are more than one geodesic curve between them that extremize the proper time. For Riemannian geometry (*i.e.* without a time-direction) an example of two conjugate points are the north and south pole of a sphere (1.3.6) since any curve between the poles with ϕ constant is a geodesic.

In accordance with the above, it is easy to see that one cannot find time-like geodesic curves that minimize the proper time. The reason for this is that one can always connect two time-like separated events by a combination of null curves as illustrated in Minkowski space as illustrated in Figure 4 of Section 1.1.4. A null curve has zero proper time. Thus, if one has a time-like curve that lies very close to the null curve, it should have very small proper time, as also illustrated in Figure 4. Therefore, one can find time-like curves with arbitrarily small proper time in this way. On the other hand, the proper time of a time-like curve can never be zero. Thus, $\Delta\tau$ can be made arbitrarily close to zero but never zero and hence one cannot find a time-like curve that corresponds to a minimum of $\Delta\tau$.

1.3.4 Newton limit of geodesic equation

To understand better the meaning of metrics and geodesics in relation to gravity we apply now the results on geodesics of Section 1.3.3 to the so-called *Newton limit* in which General Relativity should reduce to the Newton theory of gravity.

Suppose we consider a space-time and a massive particle in that space-time following a time-like curve. Then the Newton limit of General Relativity requires that we can find a coordinate system x^μ such that the following conditions are met:

- **Gravity is weak.** The metric $g_{\mu\nu}$ in the coordinate system x^μ is close to the metric of Minkowski space in an Inertial System:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)\,, \tag{1.3.48}$$

where $h_{\mu\nu}(x)$ is small

$$|h_{\mu\nu}(x)| \ll 1\,. \tag{1.3.49}$$

- **Independence of time.** The metric $g_{\mu\nu}$ in the coordinate system x^μ is independent of time

$$\frac{\partial g_{\mu\nu}}{\partial t} = 0\,, \tag{1.3.50}$$

where $t = x^0$.

- **Small velocities.** The velocity of the particle in the coordinate system x^μ is small compared to the speed of light

$$\left|\frac{dx^i}{d\tau}\right| \ll 1 \quad \text{for} \quad i = 1, 2, 3\,. \tag{1.3.51}$$

Consider now a freely falling particle in a space-time, and assume the above requirements of the Newton limit are met in the coordinate system x^μ. Using (1.3.51) the geodesic equation (1.3.45) to leading order becomes

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2 = 0\,. \tag{1.3.52}$$

Using (1.3.48)-(1.3.49) we find to leading order in $h_{\mu\nu}$

$$\Gamma^\mu_{00} = -\frac{1}{2}\eta^{\mu\sigma}\frac{\partial h_{00}}{\partial x^\sigma}\,. \tag{1.3.53}$$

Because of (1.3.50) we see that $\Gamma^0_{00} = 0$ hence we get $d^2t/d\tau^2 = 0$ from (1.3.52) with $\mu = 0$. This means $dt/d\tau$ is a constant. Thus, for $\mu = i = 1, 2, 3$ we find

$$\frac{d^2x^i}{dt^2} = \left(\frac{d\tau}{dt}\right)^2\frac{d^2x^i}{d\tau^2} = -\Gamma^i_{00} = \frac{1}{2}\frac{\partial h_{00}}{\partial x^i}\,. \tag{1.3.54}$$

We can write this in 3-vector notation as

$$\frac{d^2\vec{x}}{dt^2} = \frac{1}{2}\vec{\nabla}h_{00}\,. \tag{1.3.55}$$

We want to compare this to the Newtonian equation (1.2.5) for a freely falling particle in a gravitational field that we write here as

$$\frac{d^2\vec{x}}{dt^2} = -\vec{\nabla}\phi\,, \tag{1.3.56}$$

where ϕ is the Newtonian gravitational potential. Comparing this to (1.3.55) we see that we should identify

$$h_{00} = -2\phi \,. \tag{1.3.57}$$

Here one could add a constant but we choose this to be zero by imposing the boundary condition that ϕ goes to zero when one is far away from any source of gravity.

Thus, in the Newton limit the 00-component of the metric is

$$g_{00} = -1 - 2\phi \,. \tag{1.3.58}$$

This result actually shows how allowing for more general space-time geometries, as introduced in Section 1.3.2, enables us to describe the phenomenon of gravitational time dilation, which in fact was the motivation to introduce the general space-time geometries. To see this explicitly, we imagine again the tower in a coordinate system x^μ so that the bottom of the tower is at $(x^1, x^2, x^3) = (0,0,0)$ and the top at $(x^1, x^2, x^3) = (L,0,0)$. Since the gravitational field around Earth can be well described by Newtonian gravity, which means the tower is in a weak gravitational that does not depend on time, and since the tower is not moving in the coordinate system, we can assume the Newton limit. Thus, the conditions to apply the formula (1.3.58) are met. At the bottom of the tower we now measure the period T_1 of the emitted light signal as the proper time

$$(T_1)^2 = (1 + 2\phi_1)(dx^0)^2 \tag{1.3.59}$$

while at the top the period T_2 of the received light signal is the proper time

$$(T_2)^2 = (1 + 2\phi_2)(dx^0)^2 \tag{1.3.60}$$

A pertinent question to answer is why dx^0 is the same when the light signal is sent and when it is received. This is not immediately obvious since the metric is only approximately the Minkowski metric as in Eqs. (1.3.48)-(1.3.49). However, here we invoke the condition that the metric is time-independent (1.3.50). Denote the coordinate time that a light signal takes to travel from the bottom to the top of the tower as Δx^0. The time Δx^0 depends on the metric along the light signal path since light should propagate along a null curve. However, since the metric is not changing with time, Δx^0 is the same for all light signals that are sent from the bottom to the top of the tower. This implies that the period dx^0 in coordinate time remains the same, since it can be seen as the difference between the propagation of the start of the light signal and the end of the light signal, that always takes the same coordinate time Δx^0 to travel from bottom to top. Thus, given this, we get by taking the squareroot of the ratio of (1.3.59) and (1.3.60)

$$\frac{T_2}{T_1} = \sqrt{\frac{1+2\phi_2}{1+2\phi_1}} \simeq 1 + \phi_2 - \phi_1 = 1 + \Delta\phi \tag{1.3.61}$$

which gives the formula (1.2.15) for gravitational time-dilation. This demonstrates that we were right to introduce more general space-time geometries to account for this effect.

More generally, Eq. (1.3.58) suggests that the metric in the theory of General Relativity is what replaces Newtons gravitational potential since we now find a relationship between the two in the Newton limit. Moreover, Newtons law of motion in a gravitational field is replaced in General Relativity by the geodesic equation (1.3.45) which has a geometric interpretation in terms of the geometry of space-time. Hence, in General Relativity we have that

$$\text{Gravity} = \text{Geometry}\,. \tag{1.3.62}$$

But we still have not understood how to determine $g_{\mu\nu}$. In Newtonian mechanics one determines the gravitational potential from Poissons equation $\vec{\nabla}^2\phi = 4\pi G\rho_m$ where G is Newtons gravitational constant and ρ_m the mass density. What is the analogue of this equation in General Relativity? We will return to this in Section 1.6 after developing further basic geometric concepts that one needs in order to address this question.

1.3.5 Local Inertial System

In Section 1.2 we introduced Einsteins Equivalence Principle (EEP). A consequence of this is the gravitational time-dilation effect, which naturally led us to introduce more general space-time geometries than Minkowski space as a framework to understand gravity. However, this means that the Newtonian concept of Inertial Systems cannot hold in the presence of gravity, since one cannot find a coordinate system for the space-time geometry in which its metric reduces to the Minkowski metric $\eta_{\mu\nu}$.

In General Relativity, what replaces Inertial Systems is the concept of *Local Inertial Systems*. As we shall see, the existence of Local Inertial Systems for a given space-time is both natural from point of view of the EEP, as well as from the point of view that our space-time is described by a four-dimensional space-time geometry.

We begin with the geometric understanding. Let us assume we are given a space-time. Then one can show the following:

Local Inertial System: Consider an event p in a space-time. Then one can always find a coordinate system x^μ for which the metric obeys

$$g_{\mu\nu}|_p = \eta_{\mu\nu} \ , \quad \left.\frac{\partial g_{\mu\nu}}{\partial x^\rho}\right|_p = 0 \, . \tag{1.3.63}$$

A coordinate system with this property is known as a *Local Inertial System* for the event p.

In other words, given any space-time, and any event p in this space-time, we can always find such a Local Inertial System (1.3.63). This makes sense geometrically, since it means that locally any space-time geometry reduces to the Minkowskian line-element in a small neighborhood around p

$$ds^2|_p \simeq -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \, , \tag{1.3.64}$$

when using a LIS for the event p. Another way to say this is that any curved space-time geometry is anchored in the geometry of Minkowski space, since for sufficiently small neighborhoods around p one can use the Minkowskian line-element (if one is in a Local Inertial System). This is analogous to the statement that spatial geometry is anchored in the geometry of Euclidean space, in that for a sufficiently small neighborhood around a given point one can use Pythagoras' theorem. Thus, the existence of Local Inertial Systems is basically part of what defines a space-time geometry.

Regarding the EEP, Local Inertial Systems are equally natural. EEP, as stated in Section 1.2.2, says that in small enough regions of space-time the laws of physics reduce to those of Special Relativity. We can now use the concept of Local Inertial Systems to give a more precise formulation of EEP:

Einsteins Equivalence Principle: Given a Local Inertial System x^μ for an event p, thus with the metric obeying (1.3.63), the laws of physics in a sufficiently small region around p should reduce to those of Minkowski space in an Inertial System.

As an immediate example of this, we note that for a Local Inertial System x^μ for an event p we have that the Christoffel Symbol is zero at p: $\Gamma^\mu_{\nu\rho}|_p = 0$. Hence the geodesic equation (1.3.45) in such coordinates reduces to $d^2x^\mu/d\tau^2 = 0$ sufficiently near p, which we recognize as the equation for unaccelerated motion in an Inertial System within the context of Special Relativity. This is in perfect agreement with our conclusion above in Section 1.2.2 that freely falling motion is unaccelerated.

Finally, it is interesting to re-examine the thought-experiment with the two rockets and the tower of Section 1.2.3 in view of this more precise formulation of the EEP that we now have found. In our more precise version of the thought-experiment we consider two events for which we apply the EEP: Event E1) Rocket 1/bottom of tower emits a light signal. Event E2) Rocket 2/top of tower receives a light signal. Since the two rockets are not subject to gravity, they are in Minkowski space for which we can use the metric $g_{\mu\nu} = \eta_{\mu\nu}$ everywhere. However, the tower is not in Minkowski space since Earth sources a gravitational field. Indeed, for the tower we have found Eq. (1.3.58) where we can use $\phi = -GM/r$ with M being the mass of Earth. This means that g_{00} is not the same at the two events and hence we are working in two different coordinate systems at the two events E1 and E2 when applying the EEP, and one could worry whether this means the formula (1.2.11) for the redshift would be invalidated. However, the consequence of this difference is rather small. If the tower is 20 meters high, the redshift (1.2.11) is of the order $a_g L/c^2 \sim 10^{-6}$ and the effect of using two different Local Inertial Systems gives a difference in the two time coordinates used in the two Local Inertial Systems also of order $a_g L/c^2 \sim 10^{-6}$ which means that the correction to the formula (1.2.11) is of order $(a_g L/c^2)^2 \sim 10^{-12}$. Thus, we were right in neglecting this correction for computing the leading order redshift.

1.4 Tensors and the Principle of General Covariance

The geometry of space-time is given by its line-element (1.3.12). In a specific coordinate system x^μ the line-element is given by the metric $g_{\mu\nu}$. However, one is free to choose other coordinate systems, provided the metric transforms according to (1.3.23). With this arbitrariness in the choice of coordinates, how can one formulate laws of physics? Something that appears static in one coordinate system could appear as accelerating in another. How does one formulate physical laws that takes this into account?

In this section we will address these questions by formulating the principle of general covariance. We then develop the necessary mathematical tools called tensors and the covariant derivative of tensors to implement this principle. Finally, we consider specific applications, such as a general definition of what is meant by acceleration in the theory of General Relativity.

1.4.1 Principle of general covariance

How do we formulate laws of physics in General Relativity? Space-time geometry is defined by the line-element (1.3.12) which is invariant under coordinate transformations of the metric (1.3.23). This enables us to define what we mean by the geometry of space-time in a coordinate-independent manner. Similarly, we would like to formulate laws of physics that also is not specific to what coordinates that we use. This is formulated by the *principle of general covariance*:

> **The principle of general covariance**: The general laws of nature are to be expressed by equations that are covariant with respect to coordinate transformations, meaning that the equations can be formulated in a covariant form such that they preserve that form under any coordinate transformation.

We have seen a precursor to this principle when discussing the formulation of Maxwells equations in Special Relativity in Section 1.1.6. In that case we could formulate Maxwells equations in a form that is the same for all Inertial Systems. In General Relativity this idea is generalized to all possible coordinate systems, no matter if they are accelerated in complicated ways with respect to each other.

EEP means that the laws of physics reduce to those of Special Relativity in a local Inertial System. As we shall see below, combining this with the principle of general covariance will allow us to generalize laws of physics to a general space-time without need of further input. This works in particular for Maxwells equations, as we discuss in Section 1.4.5. To do this, we need first to develop a mathematical object called tensors.

1.4.2 Tensors

We now introduce the concept of tensors. The idea behind tensors is that one should be able to formulate any physical quantity in General Relativity in a way that does not depend on the specific coordinate system one uses to describe the space-time. Thus, given a physical quantity in a particular coordinate system - this could for instance be an electromagnetic field configuration - one should be able to translate this quantity to any other coordinate system in a consistent manner.

Scalar fields

Consider a given space-time geometry. The simplest type of tensor on such a geometry is the *scalar field*. A scalar field Φ assigns a number to each event of the space-time. Thus, for a given coordinate system x^μ it is a function $\Phi(x)$ of x^μ. In more detail, this means that the scalar field Φ in the coordinate system x^μ is a function of four variables $\Phi(x) = \Phi(x^0, x^1, x^2, x^3)$.

Make now a coordinate transformation $x^\mu \to \tilde{x}^\mu(x)$. Such a transformation means that the point that before was labelled x^μ now is labelled $\tilde{x}^\mu(x)$. Hence:

Transformation of scalar fields: A scalar field $\Phi(x)$ transforms as

$$\tilde{\Phi}(\tilde{x}) = \Phi(x)\,, \tag{1.4.1}$$

under a coordinate transformation $x^\mu \to \tilde{x}^\mu(x)$.

Note that $\tilde{x}^\mu = \tilde{x}^\mu(x)$ in (1.4.1). The transformation rule (1.4.1) ensures that one assigns the same number to a given event irrespective of what coordinate system one uses. Writing this out in more detail, it states $\tilde{\Phi}(\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3) = \Phi(x^0, x^1, x^2, x^3)$.

Vectors and vector fields

We now turn to vectors. Our previous knowledge of vectors are from Euclidean space. In Euclidean space a vector is not thought of as associated to a particular point in that space. Instead it can be thought of as the difference between two given points. Also, we think of the vector as given by a specific list of numbers, *i.e.* one number for each coordinate axis.

In a curved space-time we have to discard all of these ways to think about vectors. In a space-time a vector is something we define in a particular point (event) and one cannot just move that vector to a different point without having a prescription of how to do that (see Section 1.7). A vector is not just a list of numbers because, as we shall see, the components of a vector depend on what coordinate system we use.

We define a vector as a tangent vector to a curve. This gives a definition of vectors that is independent of coordinate systems, since a curve can be defined independently of coordinate systems.

The precise definition of what we mean by a vector in a general space-time is as follows. Consider a space-time with coordinate system x^μ. For a given event p, consider a curve

$x^\mu(\lambda)$ going through p for $\lambda = 0$. Then the derivative

$$V^\mu = \left.\frac{dx^\mu}{d\lambda}\right|_{\lambda=0} , \tag{1.4.2}$$

defines a vector V^μ at the event p, as illustrated in Figure 12. The numbers V^μ, $\mu = 0, 1, 2, 3$, are called the components of the vector for the particular coordinate system x^μ.

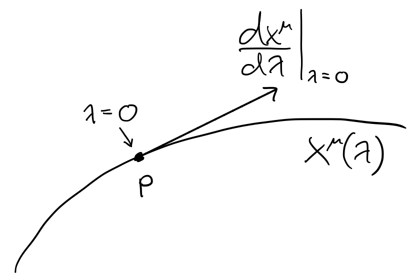

Figure 12: Illustration of the vector (1.4.2) at p obtained as a tangent vector of a curve that goes through p.

After a coordinate transformation $x^\mu \to \tilde{x}^\mu(x)$ the same curve is parametrized by $\tilde{x}^\mu(x(\lambda))$ in the new coordinates. Hence the vector at p in the new coordinates is

$$\tilde{V}^\alpha = \left.\frac{d}{d\lambda}\tilde{x}^\alpha(x(\lambda))\right|_{\lambda=0} = \left.\frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\right|_p \left.\frac{dx^\mu}{d\lambda}\right|_{\lambda=0} = \left.\frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\right|_p V^\mu , \tag{1.4.3}$$

using the chain-rule of differentiation. From this we can read off how vectors transform under coordinate transformations.

The vector V^μ at p in the coordinate system x^μ is the *same* as the vector $\tilde{V}^\mu$ at p in the coordinate system $\tilde{x}^\mu$. Why? Because they are both defined as the tangent vector to the same curve at the same point (event) p. The only difference is that we expressed this tangent vector in two different coordinate systems. Thus, while the components V^μ and $\tilde{V}^\mu$ in general can be quite different, given the two different coordinate systems, they nevertheless parametrize the same vector at p.

A *vector field* is defined such that for each event in the space-time geometry we assign a vector to that event. Given coordinates x^μ a vector field is written as $V^\mu(x)$ meaning that for each event x^μ we have a vector $V^\mu(x)$ at that event. Consider now a coordinate transformation. From the above considerations we see that we have the following transformation rule for vector fields:

Transformation of vector fields: A vector field $V^\mu(x)$ transforms as

$$\tilde{V}^\alpha(\tilde{x}) = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} V^\mu(x)\,, \tag{1.4.4}$$

under the coordinate transformation $x^\mu \to \tilde{x}^\mu(x)$.

The above transformation property defines what we mean by a vector field on a general space-time since it gives a prescription on how to transform a vector field from one coordinate system to another.

One-forms and one-form fields

Suppose we are given a general space-time in a coordinate system x^μ. A *one-form* at an event p is a linear map from vectors at p to numbers at p. Since vectors have four components, also the one-form needs to have four components. Write now a given one-form at p as A_μ. Then the linear map at p that this one-form defines is

$$A_\mu V^\mu\,, \tag{1.4.5}$$

since this maps any vector V^μ at p to a number, and since the map is linear in V^μ. In order for this linear map to give the same number for a given vector in all coordinate systems we need that under a coordinate transformation $x^\mu \to \tilde{x}^\mu(x)$ the one-form A_μ transforms as

$$\tilde{A}_\alpha = \left.\frac{\partial x^\mu}{\partial \tilde{x}^\alpha}\right|_p A_\mu\,, \tag{1.4.6}$$

which means one-forms transforms oppositely to vectors. Using (1.4.3) this gives

$$\tilde{A}_\alpha \tilde{V}^\alpha = A_\mu V^\mu\,, \tag{1.4.7}$$

which indeed shows that the linear map gives the same number in both coordinate systems. One-forms are also called *dual vectors*.

A *one-form field* is defined such that for each event in the space-time geometry we assign a one-form to that event. Given a one-form field $A_\mu(x)$ in a coordinate system x^μ it transforms as

Transformation of one-form fields: A one-form field $A_\mu(x)$ transforms as

$$\tilde{A}_\alpha(\tilde{x}) = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} A_\mu(x)\,, \tag{1.4.8}$$

under the coordinate transformation $x^\mu \to \tilde{x}^\mu(x)$.

We see that $A_\mu(x)V^\mu(x)$ transforms as a scalar field

$$\tilde{A}_\alpha(\tilde{x})\tilde{V}^\alpha(\tilde{x}) = A_\mu(x)V^\mu(x)\,. \tag{1.4.9}$$

This is in accordance with the fact that in any particular event the one-form field can be seen as a coordinate-independent linear map from vectors to numbers for that event.

Tensors

The scalar, vector and one-form fields are the simplest types of tensors that one can have on a space-time geometry. In general a tensor can have many indices

$$T^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m}(x)\,. \tag{1.4.10}$$

Such a tensor has the transformation rule:

Transformation of tensors: A tensor $T^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m}(x)$ transforms as

$$\tilde{T}^{\alpha_1\cdots\alpha_n}{}_{\beta_1\cdots\beta_m}(\tilde{x}) = \frac{\partial\tilde{x}^{\alpha_1}}{\partial x^{\mu_1}}\cdots\frac{\partial\tilde{x}^{\alpha_n}}{\partial x^{\mu_n}}\frac{\partial x^{\nu_1}}{\partial\tilde{x}^{\beta_1}}\cdots\frac{\partial x^{\nu_m}}{\partial\tilde{x}^{\beta_m}}T^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m}(x)\,, \tag{1.4.11}$$

under the coordinate transformation $x^\mu \to \tilde{x}^\mu(x)$.

Note that in general a tensor does not need to have all the upper indices on the left of the lower indices, *e.g.* one can consider tensors with index structures like $T_\mu{}^\nu{}_\rho(x)$ or $T_\mu{}^\nu(x)$ for instance. The transformation rule is always that the upper indices transform like that of a vector and the lower indices transform like that of a one-form.

A very important feature of tensors is that one can use them to write down covariant equations. Suppose the equation

$$T^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m}(x) = 0\,, \tag{1.4.12}$$

is satisfied everywhere in the space-time for the tensor in the particular coordinate system x^μ. Make now an arbitrary coordinate transformation $x^\mu \to \tilde{x}^\mu(x)$. Then it follows from (1.4.11) that the equation

$$\tilde{T}^{\alpha_1\cdots\alpha_n}{}_{\beta_1\cdots\beta_m}(\tilde{x}) = 0\,, \tag{1.4.13}$$

also is satisfied everywhere in the space-time. Thus, we see that the equation (1.4.12) is a covariant equation since it keeps the same form in all coordinate systems.

It follows from (1.4.11) that the sum of two tensors of the same type, *i.e.* the same index structure, is a tensor of that type. Moreover, the product of any two tensors is a tensor, regardless of what type they are. For instance, given a vector field $V^\mu(x)$ the product $L^{\mu\nu}(x) = V^\mu(x)V^\nu(x)$ is a tensor (with two upper indices). If we are also given a one-form field we can make a tensor of the form $M^\mu{}_\nu(x) = V^\mu(x)A_\nu(x)$.

In this connection we define in general a *contraction* of two tensors to be the product of two tensors with sums over one or more of the indices. Each contraction should be over a lower index and an upper index. This ensures that the resulting object is a tensor. In general any contraction between two given tensors is again a tensor. For instance for a tensor $T_{\mu\nu}(x)$ and a vector field $V^\mu(x)$ the contraction $T_{\mu\nu}V^\nu$ is a one-form field. Or, as already noted above, the contraction of a one-form field with a vector field is a scalar field.

We have already encountered two special tensors in defining what we mean by a space-time: the metric $g_{\mu\nu}(x)$ and the inverse metric $g^{\mu\nu}(x)$. Their transformations (1.3.23) and (1.3.28) means that they transform as tensors.

Using the metric tensor $g_{\mu\nu}(x)$ and the inverse metric tensor $g^{\mu\nu}(x)$ we can raise and lower indices on tensors. For instance, if we are given a vector field $V^\mu(x)$ we can make a one-form field by the contraction

$$V_\mu(x) = g_{\mu\nu}(x)V^\nu(x)\,. \tag{1.4.14}$$

One can check that this transforms as a one-form (1.4.8). Similarly one can make a one-form field into a vector field.

We can also use the metric to define the *norm* of a vector as

$$V^2 = g_{\mu\nu}V^\mu V^\nu\,. \tag{1.4.15}$$

Note that V^2 is not restricted to be positive. Indeed, for a given event x^μ the vector $V^\mu(x)$ can be divided in three categories:

$$V^\mu(x) \;\text{ is }\; \begin{cases} \text{space-like if } \;V^2 > 0\,, \\ \text{time-like if } \;V^2 < 0\,, \\ \text{null if } \;V^2 = 0\,. \end{cases} \tag{1.4.16}$$

One can make the same categorization for one-forms $A_\mu(x)$ using the norm $A^2 = g^{\mu\nu}A_\mu A_\nu$. Note also that a vector field $V^\mu(x)$ which for instance is space-like in one region of the space-time can be time-like or null in another region of the same space-time.

Another tensor we have encountered is the Kronecker delta (1.3.26). This is a highly special example of a tensor which has the same components in all coordinate systems. Indeed, it follows from (1.3.29) that

$$\delta^\alpha_\beta = \frac{\partial\tilde{x}^\alpha}{\partial x^\mu}\frac{\partial x^\nu}{\partial\tilde{x}^\beta}\delta^\mu_\nu\,, \tag{1.4.17}$$

which is the transformation of a tensor with one upper and one lower index.

1.4.3 Covariant derivative

Having defined the concept of tensors on a general space-time we now consider how to define a derivative of tensors. Having a concept of derivatives is crucial since that enables us to compare quantities at different events of the space-time and to quantify the rate of change of a given tensor. However, we need that the derivative of a tensor also is a tensor. Otherwise any statements we would make about the rate of change of a tensor would depend on what coordinate system we use.

Consider first a scalar field $\Phi(x)$ on a space-time in a coordinate system x^μ. Then

$$\partial_\mu \Phi \,, \tag{1.4.18}$$

is a one-form, where we introduced the following short-hand notation for the partial derivative

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \,, \tag{1.4.19}$$

that we will use repeatedly from now on. That (1.4.18) transforms as a one-form is easily checked:

$$\tilde{\partial}_\alpha \tilde{\Phi} = \frac{\partial \tilde{\Phi}}{\partial \tilde{x}^\alpha} = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial \Phi}{\partial x^\mu} = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \partial_\mu \Phi \,, \tag{1.4.20}$$

where one uses the chain-rule of differentiation for the second equal sign. However, that one can get a tensor by taking the partial derivative of a scalar field is unique to the scalar field. For any other type of tensor, taking the partial derivative of the tensor is *not* a tensor. Thus, we have to make a definition of a derivative of tensors that is a tensor while still being related to the partial derivative in some way. This is what we will do in the following.

Covariant derivative of vector fields

Consider a vector field $V^\mu(x)$ in a space-time with coordinate system x^μ. The quantity $\partial_\mu V^\nu$ is not a tensor. This can be checked by doing an arbitrary coordinate transformation $x^\mu \to \tilde{x}^\mu(x)$. We compute

$$\begin{aligned} \tilde{\partial}_\alpha \tilde{V}^\beta &= \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \partial_\mu \left(\frac{\partial \tilde{x}^\beta}{\partial x^\nu} V^\nu \right) = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial \tilde{x}^\beta}{\partial x^\nu} \partial_\mu V^\nu + \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial^2 \tilde{x}^\beta}{\partial x^\mu \partial x^\nu} V^\nu \\ &= \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial \tilde{x}^\beta}{\partial x^\nu} \left(\partial_\mu V^\nu + \frac{\partial x^\nu}{\partial \tilde{x}^\gamma} \frac{\partial^2 \tilde{x}^\gamma}{\partial x^\mu \partial x^\rho} V^\rho \right) \,. \end{aligned} \tag{1.4.21}$$

This clearly shows that $\partial_\mu V^\nu$ does not transform as a tensor.

We define now the *covariant derivative* of $V^\mu(x)$ as the tensor $D_\mu V^\nu$ that reduces to the partial derivative in a local Inertial System. That it should be a tensor means that it transforms as

$$\tilde{D}_\alpha \tilde{V}^\beta = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha}\frac{\partial \tilde{x}^\beta}{\partial x^\nu} D_\mu V^\nu\,. \tag{1.4.22}$$

for any coordinate transformation $x^\mu \to \tilde{x}^\mu(x)$.

Consider an event p in the space-time. Then from Section 1.3.5 we know that one can transform from the coordinates x^μ to a new coordinate system $\tilde{x}^\mu(x)$ in which one has a local Inertial System at p

$$\tilde{g}_{\alpha\beta}|_p = \eta_{\alpha\beta}\ ,\quad \tilde{\partial}_\alpha \tilde{g}_{\beta\gamma}|_p = 0\,. \tag{1.4.23}$$

According to our definition of the tensor $D_\mu V^\nu$ we require that in the local Inertial System $\tilde{x}^\mu$ at p the covariant derivative of $\tilde{V}^\mu$ should be the partial derivative

$$\tilde{D}_\alpha \tilde{V}^\beta|_p = \tilde{\partial}_\alpha \tilde{V}^\beta|_p\,. \tag{1.4.24}$$

Requiring this can be seen as a realization of EEP in the sense that the covariant derivative reduces to the derivative used in Special Relativity in a local Inertial System.

Combining (1.4.24) with the general result (1.4.21) we find

$$\tilde{D}_\alpha \tilde{V}^\beta|_p = \tilde{\partial}_\alpha \tilde{V}^\beta|_p = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha}\frac{\partial \tilde{x}^\beta}{\partial x^\nu}\left(\partial_\mu V^\nu + \frac{\partial x^\nu}{\partial \tilde{x}^\gamma}\frac{\partial^2 \tilde{x}^\gamma}{\partial x^\mu \partial x^\rho} V^\rho\right)\Bigg|_p\,. \tag{1.4.25}$$

Using that the covariant derivative should transform as a tensor (1.4.22) we get

$$D_\mu V^\nu|_p = \left(\partial_\mu V^\nu + \frac{\partial x^\nu}{\partial \tilde{x}^\gamma}\frac{\partial^2 \tilde{x}^\gamma}{\partial x^\mu \partial x^\rho} V^\rho\right)\Bigg|_p\,. \tag{1.4.26}$$

The next step is to express the RHS of this equation only in terms of quantities computed in the x^μ coordinates. Transforming (1.4.23) to the x^μ coordinates we find

$$\begin{aligned} g_{\mu\nu}|_p &= \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\frac{\partial \tilde{x}^\beta}{\partial x^\nu}\eta_{\alpha\beta}\ ,\quad g^{\mu\nu}|_p = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha}\frac{\partial x^\nu}{\partial \tilde{x}^\beta}\eta^{\alpha\beta}\,, \\ \partial_\rho g_{\mu\nu}|_p &= \frac{\partial^2 \tilde{x}^\alpha}{\partial x^\mu \partial x^\rho}\frac{\partial \tilde{x}^\beta}{\partial x^\nu}\eta_{\alpha\beta} + \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\frac{\partial^2 \tilde{x}^\beta}{\partial x^\nu \partial x^\rho}\eta_{\alpha\beta}\,, \end{aligned} \tag{1.4.27}$$

at the event p. Hence

$$\begin{aligned}
\Gamma^\nu_{\mu\rho}|_p &= \frac{1}{2} g^{\nu\sigma}(\partial_\mu g_{\rho\sigma} + \partial_\rho g_{\mu\sigma} - \partial_\sigma g_{\mu\rho})|_p \\
&= \frac{1}{2}\frac{\partial x^\nu}{\partial \tilde{x}^\alpha}\frac{\partial x^\sigma}{\partial \tilde{x}^\beta}\eta^{\alpha\beta}\eta_{\gamma\delta}\left(\frac{\partial^2 \tilde{x}^\gamma}{\partial x^\mu \partial x^\rho}\frac{\partial \tilde{x}^\delta}{\partial x^\sigma} + \frac{\partial \tilde{x}^\gamma}{\partial x^\rho}\frac{\partial^2 \tilde{x}^\delta}{\partial x^\mu \partial x^\sigma}\right. \\
&\quad \left. + \frac{\partial^2 \tilde{x}^\gamma}{\partial x^\rho \partial x^\mu}\frac{\partial \tilde{x}^\delta}{\partial x^\sigma} + \frac{\partial \tilde{x}^\gamma}{\partial x^\mu}\frac{\partial^2 \tilde{x}^\delta}{\partial x^\rho \partial x^\sigma} - \frac{\partial^2 \tilde{x}^\gamma}{\partial x^\sigma \partial x^\mu}\frac{\partial \tilde{x}^\delta}{\partial x^\rho} - \frac{\partial \tilde{x}^\gamma}{\partial x^\mu}\frac{\partial^2 \tilde{x}^\delta}{\partial x^\sigma \partial x^\rho}\right) \\
&= \frac{\partial x^\nu}{\partial \tilde{x}^\alpha}\frac{\partial x^\sigma}{\partial \tilde{x}^\beta}\eta^{\alpha\beta}\eta_{\gamma\delta}\frac{\partial^2 \tilde{x}^\gamma}{\partial x^\mu \partial x^\rho}\frac{\partial \tilde{x}^\delta}{\partial x^\sigma} = \frac{\partial x^\nu}{\partial \tilde{x}^\alpha}\eta^{\alpha\beta}\eta_{\gamma\beta}\frac{\partial^2 \tilde{x}^\gamma}{\partial x^\mu \partial x^\rho} = \frac{\partial x^\nu}{\partial \tilde{x}^\alpha}\frac{\partial^2 \tilde{x}^\alpha}{\partial x^\mu \partial x^\rho}\,.
\end{aligned} \tag{1.4.28}$$

Thus, we have derived

$$D_\mu V^\nu|_p = \left(\partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho\right)\big|_p\,. \tag{1.4.29}$$

Notice now that nothing on the RHS depends on the coordinate system $\tilde{x}^\mu$. Since the event p is chosen arbitrarily, we conclude:

Covariant derivative of vector fields: Given a general space-time with metric $g_{\mu\nu}(x)$ in a coordinate system x^μ the covariant derivative of a vector field $V^\mu(x)$ is

$$D_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho\,, \tag{1.4.30}$$

where $\Gamma^\nu_{\mu\rho}$ is the Christoffel symbol (1.3.46).

We note that it follows from our construction that $D_\mu V^\nu$ transforms as a tensor since one can always find a local Inertial System at any event in the space-time. Alternatively, one can check explicitly that (1.4.30) obeys the transformation (1.4.22) for any coordinate transformation.

It is important to notice that since $D_\mu V^\nu$ transforms as a tensor but $\partial_\mu V^\nu$ does not, then it follows from (1.4.30) that the Christoffel Symbol is *not* a tensor.

Covariant derivative of tensors

One can now generalize our notion of covariant derivative to all tensors. Thus, for a general tensor the covariant derivative should itself be a tensor and it should reduce to the partial derivative of the tensor in a local Inertial System.

Since the partial derivative (1.4.18) of a scalar field $\Phi(x)$ transforms as a tensor (as checked in (1.4.20)) we conclude:

Covariant derivative of scalar fields: Given a general space-time with metric $g_{\mu\nu}(x)$ in a coordinate system x^μ the covariant derivative of a scalar field $\Phi(x)$ is

$$D_\mu \Phi = \partial_\mu \Phi \,. \tag{1.4.31}$$

One can use similar arguments for one-form fields as we did above for vector fields to find the covariant derivative. This gives:

Covariant derivative of one-form fields: Given a general space-time with metric $g_{\mu\nu}(x)$ in a coordinate system x^μ the covariant derivative of a one-form field $A_\mu(x)$ is

$$D_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\rho_{\mu\nu} A_\rho \,. \tag{1.4.32}$$

Finally, one can generalize this to a more general tensor of the form (1.4.10). This can again be done using the approach that we used to find the covariant derivative of vector fields. One finds:

Covariant derivative of tensors: Given a general space-time with metric $g_{\mu\nu}(x)$ in a coordinate system x^μ the covariant derivative of a tensor $T^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m}(x)$ is

$$\begin{aligned} &D_\rho T^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m} = \partial_\rho T^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m} \\ &+ \Gamma^{\mu_1}_{\rho\sigma} T^{\sigma\mu_2\cdots\mu_n}{}_{\nu_1\cdots\nu_m} + \Gamma^{\mu_2}_{\rho\sigma} T^{\mu_1\sigma\mu_3\cdots\mu_n}{}_{\nu_1\cdots\nu_m} + \cdots + \Gamma^{\mu_n}_{\rho\sigma} T^{\mu_1\mu_2\cdots\mu_{n-1}\sigma}{}_{\nu_1\cdots\nu_m} \\ &- \Gamma^\sigma_{\rho\nu_1} T^{\mu_1\cdots\mu_n}{}_{\sigma\nu_2\cdots\nu_m} - \Gamma^\sigma_{\rho\nu_2} T^{\mu_1\cdots\mu_n}{}_{\nu_1\sigma\nu_3\cdots\nu_m} - \cdots - \Gamma^\sigma_{\rho\nu_m} T^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_{m-1}\sigma} \,. \end{aligned} \tag{1.4.33}$$

Notice that the covariant derivative for a general tensor (1.4.33) is the partial derivative plus terms with the Christoffel symbol for each upper index corresponding to what one has for a vector field (1.4.30) and minus terms with the Christoffel symbol for each lower index corresponding to what one has for a one-form field (1.4.32).

Covariant derivative along a curve

An important concept that one can introduce using the covariant derivative of tensors is the covariant derivative along a curve. Consider a curve $x^\mu(\lambda)$ in a general space-time parametrized by the parameter λ. We define the covariant derivative along the curve as

$$\frac{D}{d\lambda} = \frac{dx^\mu}{d\lambda} D_\mu \,. \tag{1.4.34}$$

This derivative can act on any tensor that one has given along the curve.

In particular, for a given curve $x^\mu(\lambda)$ and vector field $V^\mu(x)$ the covariant derivative of $V^\mu(x)$ along the curve is

$$\frac{D}{d\lambda}V^\mu = \frac{dx^\rho}{d\lambda}D_\rho V^\mu\,. \tag{1.4.35}$$

If one reparametrizes a curve $x^\mu(\lambda)$ with a new parameter $\zeta = \zeta(\lambda)$ so that one can write the curve instead as $x^\mu(\zeta)$ then we have

$$\frac{D}{d\zeta} = \frac{dx^\mu}{d\zeta}D_\mu = \frac{d\lambda}{d\zeta}\frac{dx^\mu}{d\lambda}D_\mu = \frac{d\lambda}{d\zeta}\frac{D}{d\lambda}\,. \tag{1.4.36}$$

Properties of the covariant derivative

The covariant derivative is a linear operator. For instance, for two vector fields $V^\mu(x)$ and $W^\mu(x)$ we have

$$D_\rho(V^\mu + W^\mu) = D_\rho V^\mu + D_\rho W^\mu\,, \tag{1.4.37}$$

and similarly for other types of tensors.

The covariant derivative also has a product rule. For instance, for the tensor $A_{\mu\nu}(x)$ and the vector field $V^\mu(x)$ we have

$$D_\mu(A_{\nu\rho}V^\rho) = (D_\mu A_{\nu\rho})V^\rho + A_{\nu\rho}D_\mu V^\rho\,, \tag{1.4.38}$$

and similarly for other product of tensors. This works regardless of how many indices are contracted, and also if one does not contract any indices.

Another important property of the covariant derivative is:

Covariant derivative of the metric: Given a general space-time with metric $g_{\mu\nu}(x)$ in a coordinate system x^μ the metric and inverse metric are covariantly constant

$$D_\rho g_{\mu\nu} = D_\rho g^{\mu\nu} = 0\,. \tag{1.4.39}$$

It is a straightforward computation to show this explicitly. The reason that the metric is covariantly constant is that by Section 1.3.5 one can always for any event p go to a local Inertial System where the partial derivative of the metric is zero at p. From the definition of the covariant derivative it follows then that $D_\rho g_{\mu\nu} = 0$ since the covariant derivative should be equal to the partial derivative at p.

That the covariant derivative of the metric is zero means that it does not matter whether one raises an index before or after the covariant derivative. For instance, we have $D_\rho(g_{\mu\nu}V^\nu) = g_{\mu\nu}(D_\rho V^\nu)$.

1.4.4 Acceleration in General Relativity

We can now make a covariant definition of the acceleration of a particle. Suppose a particle follows a time-like curve $x^\mu(\tau)$ parametrized by its proper time τ. We define the velocity of the particle as

$$u^\mu = \frac{dx^\mu}{d\tau} \,. \tag{1.4.40}$$

This is a vector for each point on the curve, hence it is a vector field defined on the curve. Therefore, we can take the covariant derivative of this vector field along the curve using the definition (1.4.34). We use this as a covariant definition of the acceleration

$$a^\mu = \frac{D}{d\tau} u^\mu \,. \tag{1.4.41}$$

Since this is defined using the covariant derivative this is again a vector field defined on the curve. Hence, we have found a definition of the acceleration that can be used in all coordinate systems. This is quite striking, if one thinks about it, since something that stands still from point of view of one coordinate system can seem to accelerate from point of view of another coordinate system. Here we resolve this issue by making a covariant definition of whether a particle is accelerating or not, as well as how much and in what direction it is accelerating. With the definition (1.4.41), those statements do not depend on what coordinate system we choose to work in.

Using (1.4.34) and (1.4.30) we can write out the definition (1.4.41) as

$$a^\mu = \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} \,. \tag{1.4.42}$$

This reveals that in a local Inertial System $\tilde{x}^\mu$ in the neighbourhood of an event p on the curve (obeying (1.4.23)) we have

$$a^\mu|_p = \left. \frac{d^2 \tilde{x}^\mu}{d\tau^2} \right|_p \,. \tag{1.4.43}$$

Thus, our definition of the acceleration reduces to the one in Special Relativity in a local Inertial System. Together with demanding that a^μ transforms as a vector field this means that (1.4.41) is the only possible covariant definition of acceleration.

According to (1.4.41), an unaccelerated particle obeys

$$\frac{D}{d\tau} u^\mu = 0 \,. \tag{1.4.44}$$

Using (1.4.42) we see that this is equivalent to

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0 \,. \tag{1.4.45}$$

We recognize this as the geodesic equation (1.3.45). Thus, (1.4.44) is a covariant formulation of the geodesic equation. Hence it is the equation of motion for freely falling particles. We see now that the fact that this corresponds to the covariant acceleration being zero $a^\mu = 0$ is in accordance with Einsteins Equivalence Principle (EEP).

Unlike the original formulation (1.3.45), the covariant formulation of the geodesic equation (1.4.44) satisfies both the criteria for a covariant formulation of a law of physics, namely:

- The equation has the same form in all coordinate systems, *i.e.* it is covariant.
- The equation reduces to the one of Special Relativity in a local Inertial system (as one can deduce from (1.4.43)).

We conclude from the above that the covariant acceleration (1.4.41) makes explicit that a freely falling particle in a gravitational field is unaccelerated, as previously deduced as consequence of EEP. This confirms that acceleration due to gravity is not a well-defined concept since it is a coordinate dependent statement. Another way to think about this is that Newtons gravitational force is proportional to the derivative of the metric in the Newton limit (see Section 1.3.4)

$$\vec{F}_g = -m\vec{\nabla}\phi = \frac{1}{2}m\vec{\nabla}g_{00}\,. \tag{1.4.46}$$

But we can always transform a first derivative of the metric away by going to a local Inertial System. Thus, a covariant formulation of Newtons force of gravity must necessarily be that the gravitational force is zero since that is the only covariant statement one can make. Of course, this is what we are already saying when we state that freely falling motion is unaccelerated. Therefore, from point of view of General Relativity, there is no such thing as the force of gravity. Hence, acceleration can only be caused by non-gravitational forces.

1.4.5 Maxwells equations in a general space-time

We apply now the principle of general covariance to find Maxwells equations for electromagnetism in vacuum for a general space-time. To do this, we should formulate them in a covariant form, *i.e.* a form that is the same in all coordinate systems, and ensure that they reduce to Maxwells equations of Special Relativity in a local Inertial System.

In Special Relativity we reformulated Maxwells equations (1.1.34) and (1.1.36) for electromagnetism in vacuum as equations (1.1.43) and (1.1.39), respectively.

The covariant version of Maxwells equations for electromagnetism in vacuum can be written as the equations

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \tag{1.4.47}$$

$$D_\mu F^{\mu\nu} = -J^\nu \,. \tag{1.4.48}$$

Here $F_{\mu\nu}$ is the electromagnetic field strength, A_μ is the one-form potential, J^μ is the electromagnetic current vector and we require

$$F^{\mu\nu} = g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma} \,. \tag{1.4.49}$$

Note that it follows from (1.4.47) that $F_{\mu\nu}$ is antisymmetric $F_{\nu\mu} = -F_{\mu\nu}$.

We now show that equations (1.4.47)-(1.4.48) are the correct equations for electromagnetism in vacuum in a general space-time. First of all, we see that these equations correctly reduce to Eqs. (1.1.39) and (1.1.43) in a local Inertial System, as required. Hence, if we can argue that Eqs. (1.4.47) and (1.4.48) are covariant then we have shown that they are the correct equations. The covariance of Eq. (1.4.48) follows from demanding that $F_{\mu\nu}$ and J^μ transform correctly as tensors. That ensures that Eq. (1.4.48) looks the same in all coordinate systems. Regarding Eq. (1.4.47) we require that the potential A_μ transforms as a one-form field. Using (1.4.32) one can show that for any one-form field A_μ

$$D_\mu A_\nu - D_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu \,. \tag{1.4.50}$$

Hence it follows from this that since A_μ transforms as a one-form, $F_{\mu\nu}$ transforms correctly as a tensor. This completes the argument.[9]

One can furthermore use the principle of general covariance and local Inertial Systems to generalize the Lorentz force (1.1.44) to a general space-time. In particular, if we have a particle with charge q and rest mass m in a general space-time with metric $g_{\mu\nu}$ and we have a electromagnetic field strength $F_{\mu\nu}$ in the space-time, then the covariant acceleration of the particle is given by

$$a^\mu = -\frac{q}{m} g_{\nu\rho} u^\nu F^{\rho\mu} \,, \tag{1.4.51}$$

assuming the particle is only subject to the electromagnetic force.

1.4.6 Null geodesics

So far we have only discussed the motion of massive particles in General Relativity. While massive particles follow time-like curves, massless particles folllow null curves. In

[9] Note that a tensor $F_{\mu\nu}$ with two lower indices which is antisymmetric $F_{\nu\mu} = -F_{\mu\nu}$ is known as a two-form field. Eq. (1.4.47) thus gives the two-form field $F_{\nu\mu}$ in terms on the one-form field A_μ.

particular, freely falling massless particles follow null geodesics. Using the principle of general covariance we find:

Null geodesic: For a general space-time with metric $g_{\mu\nu}$ in a coordinate system x^μ, a null geodesic is a curve $x^\mu(\lambda)$ that satisfy

$$\frac{D}{d\lambda}\frac{dx^\mu}{d\lambda} = 0\ , \quad g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0\,. \tag{1.4.52}$$

λ is called an *affine parameter* of the null geodesic.

This follows from the principle of general covariance since it is the covariant generalization of the equations (1.1.28) for motion at the speed of light in Special Relativity.

The second condition of (1.4.52) means that a null geodesic is a null curve, *i.e.* that any infinitesimally separated points on the curve are null separated, as one would expect of a particle that travels at the speed of light. The first condition of (1.4.52) can be written more explicitly as

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0\,. \tag{1.4.53}$$

We see that this resembles the time-like geodesic equation (1.3.45) apart that the proper time now is replaced by the affine parameter λ.

If one wants to change parametrization of a null geodesic $x^\mu(\lambda)$ obeying (1.4.52) to another parameter $\zeta = \zeta(\lambda)$ then we see that in order for the curve in the new parametrization $x^\mu(\zeta)$ to obey $\frac{D}{d\zeta}\frac{dx^\mu}{d\zeta} = 0$ we need that the two parameters are related linearly to each other $\zeta = c_1\lambda + c_2$.

A particular convenient choice of affine parameter λ is the one for which we have

$$p^\mu = \frac{dx^\mu}{d\lambda}\,, \tag{1.4.54}$$

where p^μ is the relativistic momentum of the massless particle. That this is possible follows from the fact that $\frac{D}{d\lambda}p^\mu = 0$ for any affine parameter λ which is the covariant generalization of the statement in Special Relativity that p^μ is constant for a massless particle.

1.5 Riemann Curvature Tensor

1.5.1 Curvature of space-time

We have learnt that in General Relativity there is no force of gravity. This follows from Einsteins Equivalence Principle (EEP), as explained in Section 1.2.2. Indeed, consider a freely falling point particle, which means a particle not subject to any non-gravitational forces. Then the point particle follows a time-like geodesic $x^\mu(\tau)$ given by

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0\,. \tag{1.5.1}$$

As explained in Section 1.4.4, a time-like geodesic corresponds to zero covariant acceleration $a^\mu = 0$. Since the acceleration is zero, there is no external force.

That there is no force on the freely falling point particle means that we cannot infer whether it is subject to gravity, or not, if all we can observe is this single point particle. Thus, following a single geodesics, we are not able to observe the effects of gravity. Again, this is simply due to the fact that a geodesic corresponds to zero covariant acceleration.

That we cannot infer the presence of gravity from a single point particle is due to the fact that one cannot measure gravity locally. This is also what we have learned from the existence of Local Inertial Systems (LIS's) combined with Einsteins Equivalence Principle (EEP). For any event in space-time, one can find a LIS (1.3.63), and then EEP tells us that for a sufficiently small region around the event, the laws of physics reduce to those of Special Relativity in an Inertial System. Thus, for any event in a space-time we can ignore the effects of gravity in a small enough region around the event.[10]

So how can we see the effect of gravity? We need to consider a non-local measurement, *i.e.* a measurement where one compares what happens in different events of the space-time. A way to do this is to compare the motion of two nearby freely falling point particles.

To understand this better, let us first consider what happens in Minkowski space. In this case, one can choose an inertial system so that the line-element is $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$. This means that any geodesic $x^\mu(\tau)$ follows a straight line $\frac{d^2x^\mu}{d\tau^2} = 0$. Thus, if two geodesics start out as parallel (*e.g.* with parallel initial velocities) then they would always be parallel. For this reason we call Minkowski space a *flat* space-time. In contrast to this, a *curved* space-time should be one in which initially parallel geodesics do not necessarily remain parallel. In other words, the *curvature* of space-time can bend geodesics so that they are not straight lines anymore. With a single geodesic, one cannot see this. But with two geodesics, one should be able to see a curvature of space-time from the relative motion of the geodesics, which in turn means that one can see the effect of gravity.

With this in mind, we shall now consider the relative motion of two nearby freely falling point particles. Consider again the freely falling point particle following the time-like

[10]We can also tie this to the above-mentioned freely falling particle by picking an event p on the geodesic curve. Then the geodesic equation (1.5.1) would reduce to $\frac{d^2x^\mu}{d\tau^2}|_p = 0$ which is the equation for zero acceleration in Special Relativity.

geodesic $x^\mu(\tau)$ written in Eq. (1.5.1) in a space-time with metric $g_{\mu\nu}$ and in a coordinate system x^μ. Imagine now a second freely falling point particle in the same space-time. Hence it follows a time-like geodesic which we denote as $\hat{x}^\mu(\tilde{\tau})$. We want to compare the motion of this second particle to that of the first. We choose some particular starting position for the two particles, and adjust τ and $\tilde{\tau}$ so that they correspond to $\tau = 0$ and $\tilde{\tau} = 0$. Then we compare their motion after the same amount of time passed for both their respective proper times. *I.e.* we compare $x^\mu(\tau)|_{\tau=0}$ to $\hat{x}^\mu(\tilde{\tau})|_{\tilde{\tau}=0}$, $x^\mu(\tau)|_{\tau=1\,\text{second}}$ to $\hat{x}^\mu(\tilde{\tau})|_{\tilde{\tau}=1\,\text{second}}$, $x^\mu(\tau)|_{\tau=2\,\text{seconds}}$ to $\hat{x}^\mu(\tilde{\tau})|_{\tilde{\tau}=2\,\text{seconds}}$, and so on. In effect, this means we should compare $x^\mu(\tau)$ with $\hat{x}^\mu(\tau)$ for all τ, *i.e.* we equate $\tilde{\tau} = \tau$. We have illustrated this in Figure 13.

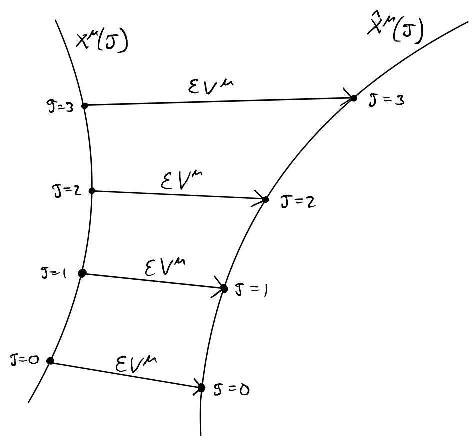

Figure 13: Illustration of the deviation between two geodesics $x^\mu(\tau)$ and $\hat{x}^\mu(\tau)$.

Now we further assume that the freely falling point particles are close to each other during their motion. This means we can write

$$\hat{x}^\mu(\tau) = x^\mu(\tau) + \xi^\mu(\tau)\,, \tag{1.5.2}$$

where $\xi^\mu(\tau)$ is small. A way to think of the smallness of $\xi^\mu(\tau)$ is to write it as

$$\xi^\mu(\tau) = \epsilon V^\mu(\tau)\,, \tag{1.5.3}$$

where ϵ is an infinitesimally small number while $V^\mu(\tau)$ is finite and transforms as a vector field on the curve $x^\mu(\tau)$.[11] See again Figure 13 for an illustration.

To compare the motion of the two particles we should compare the geodesic equation of the first particle (1.5.1) to geodesic equation of the second particle

$$\frac{d^2(x^\mu + \epsilon V^\mu)}{d\tau^2} + \Gamma^\mu_{\nu\rho}(x + \epsilon V)\frac{d(x^\nu + \epsilon V^\nu)}{d\tau}\frac{d(x^\rho + \epsilon V^\rho)}{d\tau} = 0\,, \tag{1.5.4}$$

[11] One can see this from the fact that the infinitesimal line element $dx^\mu = \epsilon V^\mu$ transforms the same way as a vector under coordinate transformations. This is seen explicitly by comparing (1.3.20) to (1.4.4).

in the limit where ϵ is arbitrarily small. Expanding (1.5.4) to first order in ϵ and subtracting (1.5.1) we get

$$\frac{d^2V^\mu}{d\tau^2} + V^\sigma \partial_\sigma \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} + 2\Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dV^\rho}{d\tau} = 0\,. \tag{1.5.5}$$

We would now like to write this as a covariant expression. The covariant derivative of the vector field V^μ along the curve $x^\mu(\tau)$ is

$$\frac{DV^\mu}{d\tau} = \frac{dV^\mu}{d\tau} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}V^\rho\,. \tag{1.5.6}$$

This gives a new vector field $DV^\mu/d\tau$ on the curve $x^\mu(\tau)$. Hence we can again take the covariant derivative along the curve, giving

$$\begin{aligned}
\frac{D^2V^\mu}{d\tau^2} &= \frac{d}{d\tau}\frac{DV^\mu}{d\tau} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{DV^\rho}{d\tau} \\
&= \frac{d^2V^\mu}{d\tau^2} + 2\Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dV^\rho}{d\tau} + \Gamma^\mu_{\nu\rho}\frac{d^2x^\nu}{d\tau^2}V^\rho + \frac{dx^\sigma}{d\tau}\partial_\sigma\Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}V^\rho + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\Gamma^\rho_{\sigma\alpha}\frac{dx^\sigma}{d\tau}V^\alpha \\
&= \frac{d^2V^\mu}{d\tau^2} + 2\Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dV^\rho}{d\tau} + \Gamma^\mu_{\alpha\rho}\frac{d^2x^\alpha}{d\tau^2}V^\rho + \left(\partial_\sigma\Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\alpha\sigma}\Gamma^\alpha_{\nu\rho}\right)V^\rho\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau}\,,
\end{aligned} \tag{1.5.7}$$

where in the last equality we renamed contracted indices in two of the terms.[12] Plugging in the geodesic equation (1.5.1) for $x^\mu(\tau)$ gives

$$\frac{D^2V^\mu}{d\tau^2} = \frac{d^2V^\mu}{d\tau^2} + 2\Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dV^\rho}{d\tau} + \left(-\Gamma^\mu_{\alpha\rho}\Gamma^\alpha_{\nu\sigma} + \partial_\sigma\Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\alpha\sigma}\Gamma^\alpha_{\nu\rho}\right)V^\rho\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau}\,. \tag{1.5.8}$$

Finally, using the geodesic equation (1.5.5) for the second particle on the first two terms gives

$$\frac{D^2V^\mu}{d\tau^2} = -\left(\partial_\rho\Gamma^\mu_{\nu\sigma} + \Gamma^\mu_{\alpha\rho}\Gamma^\alpha_{\nu\sigma} - \partial_\sigma\Gamma^\mu_{\nu\rho} - \Gamma^\mu_{\alpha\sigma}\Gamma^\alpha_{\nu\rho}\right)V^\rho\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau}\,. \tag{1.5.9}$$

As we shall see below, this equation gives the sought-after connection between the relative motion of two geodesics and the space-time geometry in which they move. However, before getting to that, notice first that on the LHS $\frac{D^2V^\mu}{d\tau^2}$ is a tensor, while on the RHS, we have the expression in the parenthesis contracted with the tensor $V^\rho\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau}$. This suggests one can define a tensor from the expression in the parenthesis on the RHS. In fact, we define:

The Riemann curvature tensor: For a general space-time with metric $g_{\mu\nu}$ in a coordinate system x^μ, we define the *Riemann curvature tensor* as

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho\Gamma^\mu_{\nu\sigma} - \partial_\sigma\Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\alpha\rho}\Gamma^\alpha_{\nu\sigma} - \Gamma^\mu_{\alpha\sigma}\Gamma^\alpha_{\nu\rho}\,, \tag{1.5.10}$$

where $\Gamma^\mu_{\nu\rho}$ is the Christoffel symbol (1.3.46).

[12] In particular, $\Gamma^\mu_{\nu\rho}\Gamma^\rho_{\sigma\alpha}V^\alpha\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = \Gamma^\mu_{\nu\alpha}\Gamma^\alpha_{\sigma\rho}V^\rho\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = \Gamma^\mu_{\sigma\alpha}\Gamma^\alpha_{\nu\rho}V^\rho\frac{dx^\sigma}{d\tau}\frac{dx^\nu}{d\tau} = \Gamma^\mu_{\sigma\alpha}\Gamma^\alpha_{\nu\rho}V^\rho\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau}$.

This new tensor is highly significant in General Relativity, as it gives a covariant measurement of whether a space-time is curved or flat, and hence whether one has gravity, or not. Using the definition (1.5.10), Eq. (1.5.9) now gives:[13]

The geodesic deviation equation: For two nearby geodesics, the deviation between them obeys the *geodesic deviation equation:*

$$\frac{D^2V^\mu}{d\tau^2} = -R^\mu{}_{\nu\rho\sigma}V^\rho \frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau}\,. \tag{1.5.11}$$

Here the two geodesics are parametrized as $x^\mu(\tau)$ and $\hat{x}^\mu(\tau) = x^\mu(\tau) + \epsilon V^\mu(\tau)$ with ϵ small. The proper time τ is equated between the two geodesics in the comparison.

We now discuss the geodesic deviation equation (1.5.11) as well as the Riemann curvature tensor (1.5.10).

Regarding Eq. (1.5.11), $\epsilon\frac{D^2V^\mu}{d\tau^2}$ on the LHS can be thought of as the covariant acceleration of the deviation ϵV^μ between the two geodesics. Thus, we see that if the tensor (1.5.10) is zero everywhere in the space-time, then $\epsilon\frac{D^2V^\mu}{d\tau^2} = 0$ for any nearby geodesics. Choosing initially parallel geodesics, one can see from this that they are always parallel. Since this is true for all geodesics, it suggests that the space-time is flat, with all geodesics being straight lines. Conversely, if the tensor (1.5.10) is non-zero, it means that $\epsilon\frac{D^2V^\mu}{d\tau^2}$ is non-zero at least for some nearby geodesics, which in turn suggests that if they were initially parallel they cannot stay parallel, as their deviation is accelerating, suggesting that the space-time is curved. Thus, the tensor (1.5.10) seems indeed to be able to characterize whether a space-time is flat, or is curved, justifying its name.

To understand this better, let's take a closer look at the Riemann curvature tensor (1.5.10). We see from Eq. (1.5.10) that the Riemann curvature tensor is given in terms of the metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$, along with the first derivative $\partial_\rho g_{\mu\nu}$ and the second-derivative $\partial_\rho\partial_\sigma g_{\mu\nu}$ of the metric, in a rather complicated non-linear expression. Therefore $R^\mu{}_{\nu\rho\sigma}$ is determined solely by the metric of the space-time, which means it is a purely geometric quantity, as one would expect from a curvature tensor.

One could ask: why should a curvature include a second derivative of the metric? This is clear from the fact that when using a LIS at an event p, all the first-derivatives of the metric at p are equal to zero. Hence it is necessary to go beyond first derivatives of the metric to include covariant information about the curvature of the geometry. One could

[13]Note that on the last term inside the parenthesis $\Gamma^\mu_{\alpha\nu}\Gamma^\alpha_{\sigma\rho}$ we can exchange ν and σ since the expression outside the parenthesis is symmetric under this exchange.

also ask: does the Riemann curvature tensor contain all the covariant information about the second derivatives? This is indeed the case, since one can show the following:

Zero curvature in a point: For a given space-time, consider an event p in which the Riemann curvature tensor at p is zero, *i.e.* $R^\mu{}_{\nu\rho\sigma}|_p = 0$ (this is a coordinate independent statement since $R^\mu{}_{\nu\rho\sigma}$ is a tensor). Then one can always find a coordinate system x^μ for which the metric obeys

$$g_{\mu\nu}|_p = \eta_{\mu\nu} \ , \quad \partial_\rho g_{\mu\nu}\big|_p = 0 \ , \quad \partial_\rho \partial_\sigma g_{\mu\nu}\big|_p = 0 \, . \tag{1.5.12}$$

This shows that the Riemann curvature tensor (1.5.10) at an event p contains all the coordinate-independent information about the second-derivative of the metric at that event.

With this in hand, we are ready to address a crucial question: Can we use the Riemann curvature tensor to distinguish between when a space-time is flat, *i.e.* Minkowski space, and when it is not?

First of all, we can easily check that for Minkowski space, the Riemann curvature tensor is zero everywhere. This is seen by using an Inertial System with metric $g_{\mu\nu} = \eta_{\mu\nu}$. Then the Christoffel Symbol is zero in that coordinate system, hence $R^\mu{}_{\nu\rho\sigma} = 0$. Since $R^\mu{}_{\nu\rho\sigma}$ is a tensor we conclude that

$$R^\mu{}_{\nu\rho\sigma} = 0 \, , \tag{1.5.13}$$

in any coordinate system for Minkowski space. Thus, the Riemann curvature tensor is zero in a flat space-time geometry, *i.e.* in Minkowski-space, precisely as one would expect.

However, the more important question is the converse statement. If the Riemann curvature tensor is zero everywhere, is the space-time geometry given by Minkowski space?[14] The answer is affirmative:

[14]Note that mathematically this can at most be a local statement. We show below that the Riemann curvature being zero implies a flat space-time. This implies locally that it is Minkowski space. But in principle the global structure of space-time could be different, for instance with one of the spatial directions wrapped on a circle in a particular Inertial System. However, for our purposes this is not a physically viable scenario.

Flat space-time from zero curvature: Consider a space-time in which the Riemann curvature tensor is zero everywhere, *i.e.* at all events. Then one can always find a coordinate system x^μ for which the metric is that of Minkowski space

$$g_{\mu\nu} = \eta_{\mu\nu} \,. \tag{1.5.14}$$

This means the space-time geometry is flat and hence there is no gravity.

One can see this using the result (1.5.12). If $R^\mu{}_{\nu\rho\sigma} = 0$ for any event, and not just at p, then one can find a coordinate system in which (1.5.12) holds while at the same time also the third-derivatives of the metric are zero, since all the first-derivatives of $R^\mu{}_{\nu\rho\sigma}$ are zero at p. And then one can go on to argue that also the fourth-order derivatives of the metric can be set to zero, and so on, until one has a coordinate system around p in which all derivatives of the metric are zero at p, and hence the metric must be constant. Note that in practise this takes some work to show as one needs to exploit the freedom in choosing a coordinate system at each level of the number of derivatives. However, this is the idea of the argument.

Since a vanishing Riemann curvature tensor implies that space-time is flat, this means that all possible information about the curvature of space-time must be contained in the Riemann curvature tensor. The reason for this is that if one imagines constructing an alternative curvature tensor, using the metric and its derivatives, then that tensor would become trivial once we have set $g_{\mu\nu} = \eta_{\mu\nu}$, and hence it would not contain any further information about the curvature of space-time beyond $R^\mu{}_{\nu\rho\sigma}$.

Let us now consider again the geodesic deviation equation (1.5.11). As mentioned above, the LHS of (1.5.11) corresponds to a covariant acceleration of the deviation between two nearby geodesics. We notice that while the covariant accelerations of the two geodesics are always zero, the relative acceleration $\frac{D^2 V^\mu}{d\tau^2}$ can instead be non-zero for a space-time with curvature. Thus, while we cannot associate a gravitational force to an individual freely falling particle, there are actually forces associated with the relative motion of nearby freely falling particles. These forces are known as *tidal forces* in General Relativity, named after the tidal forces in Newtonian gravity that Newton originally found could explain the ocean tides, as due to the variation in the gravitational field of the Moon around the Earth. In Exercise 1.27 one shows that Newtons tidal forces can be derived from the Newton limit of the geodesic deviation equation (1.5.11).

These gravitational tidal forces are what can be used to distinguish whether one has gravity, or not. As we said above, there is no gravitational force on a single point particle. But for two point particles, or more, one has tidal forces, which can be measured. As we see from Eq. (1.5.11), these tidal forces are directly tied to the curvature of space-time, *i.e.* the Riemann curvature tensor (1.5.10). Thus, we conclude that curvature of space-time is tied to gravity. If there is no gravity, there is also no curvature, and vice versa. So, in the theory of General Relativity, a more accurate statement than saying that "gravity equals geometry" is to say:

$$\text{Gravity} = \text{Curvature} \tag{1.5.15}$$

1.5.2 Properties of the Riemann curvature tensor

In this section we consider various properties of the Riemann curvature tensor defined by Eq. (1.5.10).

Commutator of covariant derivatives

By repeated use of the covariant derivative one can show:

Commutator of covariant derivatives: For a general space-time with metric $g_{\mu\nu}$ in a coordinate system x^μ, we have for a given vector field V^μ that the commutator of the covariant derivative is

$$(D_\mu D_\nu - D_\nu D_\mu)V^\rho = R^\rho{}_{\sigma\mu\nu}V^\sigma \,, \tag{1.5.16}$$

where $R^\rho{}_{\sigma\mu\nu}$ is the Riemann curvature tensor (1.5.10).

For Minkowski space we have $R^\rho{}_{\sigma\mu\nu} = 0$. Thus, in this case the formula (1.5.16) gives that $(D_\mu D_\nu - D_\nu D_\mu)V^\rho = 0$. Hence the covariant derivatives commute in a flat space-time. Indeed, this makes sense since if one goes to an Inertial System x^μ of Minkowski space the covariant derivative reduces to the partial derivative $D_\mu = \partial_\mu$ and hence in such a coordinate system the commutativity of the covariant derivatives is equivalent to the commutativity of the partial derivatives $(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)V^\rho = 0$.

For a general space-time the formula (1.5.16) instead shows that covariant derivatives do not commute with each other. Thus, turning on gravity one gets a curved space-time for which covariant derivative do not commute.

Symmetries with indices of Riemann curvature tensor

In the following we use the Riemann curvature tensor (1.5.10) with all four indices down

$$R_{\mu\nu\rho\sigma} = g_{\mu\alpha} R^{\alpha}{}_{\nu\rho\sigma} \,. \tag{1.5.17}$$

It is straightforward to compute the following result

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}(\partial_\nu\partial_\rho g_{\mu\sigma} - \partial_\nu\partial_\sigma g_{\mu\rho} + \partial_\mu\partial_\sigma g_{\nu\rho} - \partial_\mu\partial_\rho g_{\nu\sigma}) + g_{\alpha\beta}(\Gamma^\alpha_{\mu\sigma}\Gamma^\beta_{\nu\rho} - \Gamma^\alpha_{\mu\rho}\Gamma^\beta_{\nu\sigma}) \,. \tag{1.5.18}$$

From this one finds the following identities for permuting the first two or the last two indices

$$R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho} = -R_{\nu\mu\rho\sigma} \,, \tag{1.5.19}$$

for exchanging the first two and the last two indices

$$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu} \,, \tag{1.5.20}$$

and for cyclic permutation of the last three indices

$$R_{\mu\nu\rho\sigma} + R_{\mu\sigma\nu\rho} + R_{\mu\rho\sigma\nu} = 0 \,. \tag{1.5.21}$$

Without any symmetries of the indices a tensor with four components would have $4^4 = 256$ independent components. One can show that the symmetries (1.5.19), (1.5.20) and (1.5.21) this reduces the number of independent components to 20 for the Riemann curvature tensor.

Bianchi identity for Riemann curvature tensor

Consider the tensor

$$D_\alpha R_{\mu\nu\rho\sigma} + D_\nu R_{\alpha\mu\rho\sigma} + D_\mu R_{\nu\alpha\rho\sigma} \,. \tag{1.5.22}$$

Given an event p, use now the formula (1.5.18) for $R_{\mu\nu\rho\sigma}$ in a local Inertial System for p. Since the Christoffel symbol is zero the only non-zero terms in (1.5.22) are those with three partial derivatives acting on the metric. These terms are computed simply by replacing the covariant derivatives in (1.5.22) with partial derivatives and keeping only the terms in (1.5.18) with two partial derivatives acting on the metric. Writing this up, one gets twelve terms that on closer inspection are seen to cancel out with each other. Hence, one finds that the tensor (1.5.22) is zero at the event p in a local Inertial System for p. If a tensor is zero at p in a local Inertial System then it is zero at p in all coordinate systems, as one can see from the general transformation rule (1.4.11). Thus, since this works for any given event p, we conclude:

Bianchi identity for Riemann curvature tensor: The Riemann curvature tensor obeys the identity

$$D_\alpha R_{\mu\nu\rho\sigma} + D_\nu R_{\alpha\mu\rho\sigma} + D_\mu R_{\nu\alpha\rho\sigma} = 0\,. \tag{1.5.23}$$

Ricci tensor and scalar curvature

From the Riemann curvature tensor we define the *Ricci tensor* by

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu} = g^{\rho\sigma} R_{\rho\mu\sigma\nu}\,. \tag{1.5.24}$$

Using the identity (1.5.20) we see that the Ricci tensor is symmetric

$$R_{\mu\nu} = R_{\nu\mu}\,. \tag{1.5.25}$$

We define furthermore the Ricci scalar as the trace of the Ricci tensor

$$R = g^{\mu\nu} R_{\mu\nu} = R^\mu{}_\mu\,. \tag{1.5.26}$$

The Ricci scalar is a scalar field. Using the Bianchi identity (1.5.23) as well as (1.5.19) we find

$$\begin{aligned} 0 &= g^{\alpha\beta} g^{\nu\rho} (D_\mu R_{\alpha\nu\beta\rho} + D_\nu R_{\mu\alpha\beta\rho} + D_\alpha R_{\nu\mu\beta\rho}) \\ &= g^{\alpha\beta} g^{\nu\rho} (D_\mu R_{\alpha\nu\beta\rho} - D_\nu R_{\alpha\mu\beta\rho} - D_\alpha R_{\nu\mu\rho\beta}) \\ &= D_\mu R - 2D^\nu R_{\nu\mu}\,. \end{aligned} \tag{1.5.27}$$

Thus, we have the identity $D^\nu R_{\nu\mu} = \frac{1}{2} D_\mu R$. We can write this as

$$D^\mu (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) = 0\,. \tag{1.5.28}$$

This identity is clearly a direct consequence of the Bianchi identity for the Riemann curvature tensor (1.5.23). It will be highly important below in Section 1.6 for the derivation of Einsteins equations.

1.6 Einsteins Equations

So far we have seen that the geometry of space-time is deeply linked to what we know as the force of gravity in Newtonian physics. We have been considering how the motion of particles and the laws of physics are affected by replacing the Minkowski space with a general space-time. In particular, the geodesic equation (1.3.45) and the equation for null geodesics (1.4.52) address how matter and energy are influenced by gravity in the theory of General Relativity. But we have not yet addressed a major question: how is the space-time geometry influenced by the presence of matter and energy?

Another way to pose this question is: what is the analogue in the theory of General Relativity of the Poisson equation for Newtonian gravity

$$\vec{\nabla}^2 \phi = 4\pi G \rho_m \,, \tag{1.6.1}$$

where ϕ is the Newtonian gravitational potential, ρ_m is the mass density and G is Newtons gravitational constant[15]

$$G = 6.674 \cdot 10^{-11} \frac{\mathrm{m}^3}{\mathrm{kg}\,\mathrm{s}^2} \,. \tag{1.6.2}$$

The analogue of ϕ in General Relativity is the metric $g_{\mu\nu}$. Hence the LHS of (1.6.1) should be generalized to something involving two derivatives of the metric. As we shall see in Section 1.6.2, this will be related to the curvature tensors we found in Section 1.5. Instead on the RHS we have to find the generalization of ρ_m. As we shall see in Section 1.6.1, this is a tensor with two indices called the energy-momentum tensor.

1.6.1 Energy-momentum tensor

In this section we consider the energy-momentum tensor which is a physical quantity that characterizes a continuous configuration of matter and/or energy in General Relativity, thus generalizing the mass density ρ_m in Newtonian physics. We begin by considering the energy-momentum tensor in Special Relativity and then generalize it to General Relativity and general space-times in the end.

Energy-momentum tensor in Special Relativity

Section 1.1 considers massive and massless particles in Special Relativity. However, when discussing Einsteins equations we need to understand the physics of a continuous distribution of matter and energy. We begin by considering this in the case of Special Relativity.

In Newtonian physics gravity is sourced by ρ_m which is the mass density for continuous matter. However, it does not make sense to consider this quantity by itself in Special Relativity. In Special Relativity, mass, momentum and energy are related by the formula (1.1.31) that states $E^2 - \vec{p}^2 = m^2$ where m is the rest mass of the particle. Thus, energy and momentum mix together in the same way as time and space mix together in Special Relativity. Moreover, mass and energy are equivalent in Special Relativity, *i.e.* one has in general $E = \gamma m$ where γ is given by (1.1.20). Thus, one is lead to consider the full

[15] For applications in which we have set speed of light $c = 1$ we record that $G/c^2 = 7.42 \cdot 10^{-28}\mathrm{m/kg}$.

relativistic momentum vector $p^\mu = mu^\mu$ in Special Relativity. In fact, this can be used to characterize massless particles as well. Hence, for a continuous distribution of matter and energy, a proposal could be to consider the density of the relativistic momentum instead of just the mass density.

However, the density of relativistic momentum is not a covariant quantity in Special Relativity. The problem is that this density is measured for a fixed time x^0 and when multiplying with a small volume $dx^1dx^2dx^3$ one gets the total relativistic momentum in the small volume at that time. Thus, since time mixes with space under Lorentz transformations, this means that such a density is not a covariant quantity by itself. The resolution is to instead define a tensor with two indices:

Energy-momentum tensor: Consider Minkowski space in an Inertial System x^μ. The energy-momentum tensor $T^{\mu\nu}(x)$ in Special Relativity is defined by

$$T^{\mu\nu}(x) \prod_{\rho=0,\rho\neq\nu}^{3} dx^\rho = \begin{cases} \text{The total relativistic momentum } p^\mu \\ \text{passing through the 3-dim. volume } \prod_{\rho=0,\rho\neq\nu}^{3} dx^\rho \\ \text{located at a 3-dim. surface of constant } x^\nu \end{cases} \tag{1.6.3}$$

for $\mu, \nu = 0, 1, 2, 3$.

We note that since Minkowski space is four-dimensional a surface of constant x^ν is a three-dimensional surface, possibly including the time-direction.

Consider first the definition (1.6.3) for $\nu = 0$. In this case the definition means that $T^{\mu 0}(x)dx^1dx^2dx^3$ is the total relativistic momentum in the volume $dx^1dx^2dx^3$ given by a cube with sides $[x^i, x^i + dx^i]$ for $i = 1, 2, 3$. The volume is located at a surface of fixed time x^0 since the infinitesimal variations do not include time. Thus, $T^{\mu 0}(x)$ is the density of relativistic momentum at the event x^μ. In particular T^{00} is the energy density and T^{i0} is the x^i-component of the momentum density at the event x^μ.

Consider then the definition (1.6.3) for $\mu = 0$ and $\nu = 3$. In this case the definition means that $T^{03}(x)dx^0 dA$ is the total energy p^0 passing through the area $dA = dx^1dx^2$ during the time from x^0 to $x^0 + dx^0$. dA is the area of the rectangle with sides $[x^1, x^1 + dx^1]$ and $[x^2, x^2 + dx^2]$ which is perpendicular to the x^3 direction. This means that T^{03} is the energy flux (meaning the energy per unit time and area) through a surface perpendicular to x^3. More generally T^{0j} is the energy flux through a surface perpendicular to x^j.

For the components T^{ij} of the energy-momentum tensor one gets from the definition (1.6.3) that these components describe internal forces in the continuous distribution of matter and/or energy.[16] In general one can show that $T^{ij} = T^{ji}$. If we consider a fluid or a gas, and we assume that velocities are much smaller than the speed of light, then one has $T^{ij} = p\delta_{ij}$ where p is the pressure of the fluid or gas.

Thus, in summary:

Energy-momentum tensor: The components of the energy-momentum tensor have the following physical interpretations:

$$\begin{aligned} &T^{00} : \text{energy density} \\ &T^{i0} : \text{density of } x^i\text{-component of the momentum} \\ &T^{0j} : \text{energy flux through surface perpendicular to } x^j \\ &T^{ij} : \text{internal forces per unit area (such as the pressure)} \end{aligned} \tag{1.6.4}$$

where $i, j = 1, 2, 3$.

Since in Special Relativity mass equals energy we have

$$\begin{aligned} T^{0i} &= (\text{energy density}) \cdot (\text{velocity of energy flow})^i \\ &= (\text{mass density}) \cdot (\text{velocity of mass flow})^i = T^{i0} \,. \end{aligned} \tag{1.6.5}$$

Hence we conclude that the energy-momentum tensor is symmetric

$$T^{\mu\nu} = T^{\nu\mu} \,. \tag{1.6.6}$$

Examples

Below we give a few examples of the energy-momentum tensor for certain types of matter and energy. In addition, note that in Exercise 1.21 we consider also the energy-momentum tensor for an electromagnetic field.

A *perfect fluid* is a continuous distribution of matter and/or energy which is locally isotropic and for which we can neglect viscosity and heat conduction (at least to a good

[16]For those who are interested (this is not part of pensum). One finds that $T^{ij} = -\sigma^{ij}$ where σ^{ij} is Cauchy's stress tensor. This can be argued as follows. According to the definition (1.6.3) $T^{i3}(x)dx^0 dA$ is the total momentum p^i passing through the area $dA = dx^1 dx^2$ during the time from x^0 to $x^0 + dx^0$. Using that momentum per unit time is the force, one deduces that T^{i3} is the x^i-component of the force exerted by the surface element $dA = dx^1 dx^2$. This is equal to minus the x^i-component of the force on the surface element $dA = dx^1 dx^2$, thus corresponding to $-\sigma^{i3}$. Since Cauchy's stress tensor is symmetric one finds $T^{ij} = T^{ji}$.

approximation). We introduce the following quantities:

$$\begin{aligned}
&\rho(x) : \text{Energy density in local rest-frame}\,,\\
&p(x) : \text{pressure in local rest-frame}\,,\\
&u^\mu(x) : \text{local velocity of fluid}\,.
\end{aligned} \tag{1.6.7}$$

With this, one can write the energy-momentum tensor as

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p\eta^{\mu\nu}\,. \tag{1.6.8}$$

If we go to a local rest-frame at a given event x^μ where $u^\mu = (1, 0, 0, 0)$ we see that

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}\,. \tag{1.6.9}$$

Hence we see that ρ is the energy density T^{00} in the local rest-frame and p is the pressure $T^{11} = T^{22} = T^{33}$. Moreover, the energy-momentum tensor (1.6.9) in the local rest-frame is rotationally symmetric which means that it is locally isotropic.

The perfect fluid energy-momentum tensor can be used to described a variety of different types of matter and/or energy. The different types of matter/energy are characterized by the equation of state

$$p = p(\rho)\,. \tag{1.6.10}$$

For instance, for continuous matter with $p = 0$ we can neglect the force of pressure and hence we call it *dust*. Instead a relativistic gas such as a gas of photons is described by $p = \frac{1}{3}\rho$. Thus, while it is called a perfect fluid it can be applied to examples in which it is a gas or a distribution of matter. We will return to these examples in Chapter 4 when we apply the theory of General Relativity to Cosmology.

For *Newtonian matter* velocities are much smaller than the speed of light. This means that the spatial components p^i of the relativistic momentum p^μ of a particle is much smaller than the energy p^0. Using the definition (1.6.3) and the symmetry of the energy-momentum tensor (1.6.6) one sees that to a good approximation

$$T^{\mu\nu} = \begin{pmatrix} \rho_m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\,, \tag{1.6.11}$$

where $\rho_m(x)$ is the mass density. It is the mass density rather than the energy density since in Newtonian physics we can only consider massive particles. One can think of this energy-momentum tensor as coming from a perfect fluid with small velocities $u^i \ll u^0$ and neglible pressure $p \ll \rho$. This should be a good approximation to all Newtonian matter. But how can this for instance be a good description of water which is known to have a non-zero pressure? Or the Earth? The pressure of water on Earth is found as

$$p = p_0 + d\,\rho_m\,g \tag{1.6.12}$$

where p is the pressure, p_0 is the pressure at the surface of the water, g is the gravitational acceleration at the surface of the Earth, d is the depth below the water surface and ρ_m is the density of water. We have

$$p_0 = 1 \text{ atm} = 1.01325 \cdot 10^5 \, \frac{\text{kg}}{\text{m}\cdot\text{s}^2} \ , \quad \rho_m = 1 \, \frac{\text{g}}{\text{cm}^3} = 1000 \, \frac{\text{kg}}{\text{m}^3} \ , \quad g = 9.8 \frac{\text{m}}{\text{s}^2} \, . \tag{1.6.13}$$

To compare the pressure of water p to the density of water ρ_m in units with $c = 1$ we should divide the pressure with c^2. Thus, we should compute the quantity

$$\frac{p}{c^2 \rho_m} = \frac{p_0}{c^2 \rho_m} + \frac{d\,g}{c^2} \, . \tag{1.6.14}$$

At the water surface this gives

$$\frac{p}{c^2 \rho_m} = \frac{p_0}{c^2 \rho_m} \sim \frac{10^5}{(3 \cdot 10^8)^2 \cdot 10^3} \sim 10^{-15} \, . \tag{1.6.15}$$

At a depth of 10 kilometers (reached in the deepest parts of the oceans) this gives

$$\frac{p}{c^2 \rho_m} \sim \frac{10^4 \cdot 10}{(3 \cdot 10^8)^2} \sim 10^{-12} \, . \tag{1.6.16}$$

Thus, we see that for water on Earth the highest pressure we can encounter is about $p/\rho_m \sim 10^{-12}$ in units with $c = 1$. Similarly, even if the pressure inside Earth can reach high values one still has that the ratio p/ρ_m is very small and thus the pressure can be neglected to a good approximation.

Conservation of $T^{\mu\nu}$ in Special Relativity

A general property of the energy-momentum tensor $T^{\mu\nu}$ is that it is conserved

$$\partial_\mu T^{\mu\nu} = 0 \, . \tag{1.6.17}$$

This expresses both the conservation of energy and of momentum locally in the continuous distribution of matter and/or energy. As a consequence the total relativistic momentum of an isolated system is conserved. The total momentum is

$$P^\mu = \int_V d^3\vec{x}\, T^{\mu 0}\,, \tag{1.6.18}$$

where V is the volume of the system. Let ∂V be the boundary of V. Then one finds

$$\frac{dP^\mu}{dt} = \int_V d^3\vec{x}\, \partial_0 T^{\mu 0} = -\int_V d^3\vec{x} \sum_{i=1}^{3} \partial_i T^{\mu i} = -\int_{\partial V} dA\, n_i T^{\mu i}\,, \tag{1.6.19}$$

where we used the symmetry of $T^{\mu\nu}$ and Gauss theorem. Here n_i is the unit normal vector to ∂V and dA is an area-element on ∂V. Since the system is assumed to be isolated it means that there is no energy or momentum passing through the boundary ∂V. Hence $T^{\mu\nu} = 0$ on the boundary ∂V. Thus, we conclude that the total relativistic momentum of the system is conserved

$$\frac{dP^\mu}{dt} = 0\,. \tag{1.6.20}$$

$T^{\mu\nu}$ in a general space-time

One can readily promote the energy-momentum tensor $T^{\mu\nu}$ to a general space-time using the connection to Special Relativity via local Inertial Systems. *E.g.* a perfect fluid in a space-time with metric $g_{\mu\nu}$ in a coordinate system x^μ has the energy-momentum tensor

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p g^{\mu\nu}\,. \tag{1.6.21}$$

The local conservation of energy and momentum (1.6.17) is promoted to the fully covariant equation

$$D_\mu T^{\mu\nu} = 0\,. \tag{1.6.22}$$

This follows by applying the principle of general covariance of Section 1.4.1 along with using a local Inertial System 1.3.5. The energy-momentum tensor is required to transform as a tensor under general coordinate transformations $x^\mu \to \tilde{x}^\mu(x)$. It is also required to be symmetric

$$T^{\mu\nu} = T^{\nu\mu}\,. \tag{1.6.23}$$

Below we use the energy-momentum tensor with indices down $T_{\mu\nu} = g_{\mu\rho} g_{\nu\sigma} T^{\rho\sigma}$.

1.6.2 Derivation of Einsteins equations

The energy-momentum tensor $T^{\mu\nu}$ introduced above in Section 1.6.1 is the natural candidate for what should generalize the mass density ρ_m as a source of gravity in the theory of General Relativity. Indeed, the mass density is not a covariant quantity in Special and General Relativity and thus we were led to $T^{\mu\nu}$ as the covariant generalization. But what about the LHS of Poissons equation (1.6.1)? As already remarked, since ϕ is related to the metric in the Newton limit (see Section 1.3.4) it should be something involving second order derivatives of the metric. The natural candidate for that is something involving the curvature tensors introduced in Section 1.5.

Since $T_{\mu\nu}$ is symmetric and has two indices one should equate it to a combination of curvature tensors that has two indices and is symmetric. The Ricci tensor $R_{\mu\nu}$ is an obvious candidate, but also $g_{\mu\nu}R$ is symmetric and involves second order derivatives of the metric. No other candidates for tensors that are symmetric with two indices and have at most two derivatives of the metric can be found. Hence, we conclude

$$R_{\mu\nu} + \alpha g_{\mu\nu} R = \beta T_{\mu\nu}\,, \tag{1.6.24}$$

where α and β are undetermined constants. We now consider how to fix these constants.

First, we notice that $T_{\mu\nu}$ is conserved (1.6.22). This can be written as $D^\mu T_{\mu\nu} = 0$. Using this with (1.6.24) it implies

$$D^\mu (R_{\mu\nu} + \alpha g_{\mu\nu} R) = 0\,. \tag{1.6.25}$$

However, we have the mathematical identity $D^\mu(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) = 0$ of Eq. (1.5.28) that follows from the Bianchi identity (1.5.23). Subtracting this identity from (1.6.25) implies

$$\Big(\alpha + \frac{1}{2}\Big) D_\mu R = 0\,. \tag{1.6.26}$$

Since $D_\mu R$ generically is non-zero for a general space-time, the only consistent choice of α is $\alpha = -\frac{1}{2}$. Hence,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \beta T_{\mu\nu}\,. \tag{1.6.27}$$

Taking the trace on both sides (*i.e.* contracting with $g^{\mu\nu}$) gives

$$-R = \beta g^{\mu\nu} T_{\mu\nu}\,. \tag{1.6.28}$$

Inserting this into (1.6.27) gives

$$R_{\mu\nu} = \beta \Big(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} T_{\rho\sigma} \Big)\,. \tag{1.6.29}$$

To fix β we take the Newton limit described in Section 1.3.4 of a particular component of Equation (1.6.29). The Newton limit involves a weak gravitational field (1.3.48)-(1.3.49), that the metric is independent of time (1.3.50) and small velocities (1.3.51). In this limit we are considering Newtonian matter, hence the energy-momentum tensor is given by (1.6.11). Consider now the 00-component of (1.6.29). We find

$$R_{00} = \beta\Big(\rho_m - \frac{1}{2}\rho_m\Big) = \frac{1}{2}\beta\rho_m\,, \tag{1.6.30}$$

to leading order in the Newton limit (1.3.48)-(1.3.51).

We compute

$$R_{00} = R^\mu{}_{0\mu0} = R^i{}_{0i0}\,, \tag{1.6.31}$$

using (1.5.24) and that (1.5.19) implies $R_{0000} = 0$. Using that the gravitational field is weak and that the metric is independent of time gives

$$R_{00} = R^i{}_{0i0} = \partial_i \Gamma^i_{00}\,. \tag{1.6.32}$$

From (1.3.53) we get

$$\Gamma^i_{00} = -\frac{1}{2}\partial_i h_{00}\,. \tag{1.6.33}$$

Hence we have derived

$$R_{00} = -\frac{1}{2}\partial^i\partial_i h_{00} = -\frac{1}{2}\vec{\nabla}^2 h_{00}\,, \tag{1.6.34}$$

to leading order in the Newton limit (1.3.48)-(1.3.51). Using the result (1.3.57) from the Newton limit of the geodesic equation we find

$$R_{00} = \vec{\nabla}^2\phi\,. \tag{1.6.35}$$

Combining (1.6.30) and (1.6.35) we get

$$\vec{\nabla}^2\phi = \frac{1}{2}\beta\rho_m\,. \tag{1.6.36}$$

Comparing this to the Poisson equation (1.6.1) we see that $\beta = 8\pi G$. Thus, from (1.6.27) we conclude:

Einsteins equations: A general space-time with metric $g_{\mu\nu}$ and with energy-momentum tensor $T^{\mu\nu}$ should obey the equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}\,. \tag{1.6.37}$$

These equations are known as *Einsteins equations*.

On the LHS of Einsteins equations are curvature terms for the given space-time. Thus, Einsteins equations states that the curvature of space-time is sourced by the matter and energy present in that space-time. Note that they can also be formulated as

$$R_{\mu\nu} = 8\pi G\Big(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}T_{\rho\sigma}\Big)\,. \tag{1.6.38}$$

as seen from (1.6.29). We see from this that in the special case where we are considering a region without matter and energy, and thus $T_{\mu\nu}(x) = 0$, Einsteins equations reduce to

$$R_{\mu\nu} = 0 \tag{1.6.39}$$

called the *vacuum Einstein equations*.

Together with the geodesic equations (1.3.45) and (1.4.52), Einsteins equations (1.6.37) define the theory of General Relativity. While the geodesic equations determine the motion of matter and energy in a general space-time geometry, Einsteins equations determine how the space-time geometry is curved due to presence of matter and energy.

One can see from the definition of the Riemann curvature tensor (1.5.10) that it is non-linear in the metric $g_{\mu\nu}$. Concretely, while some of the terms on the LHS of Einsteins equations (1.6.37) are linear in the metric, others are quadratic. This means that Einsteins equations are non-linear. This non-linearity of General Relativity is one of the crucial differences from Newtonian gravity in which one can find the gravitational field of two point masses by superposing their individual fields. That the equations are non-linear can be interpreted as a self-interaction of the gravitational field, *i.e.* that it couples to itself. One way to think of this is that the universality of gravity means that *everything* couples to gravity, even gravity itself, since a gravitational field can carry energy.

1.7 Parallel Transport and Curvature

Section 1.7 is not part of the pensum of the course.

In this section we consider the parallel transport of vectors along a curve. This is defined using the covariant derivative along a curve. This will enable us to compare vectors at different events in the space-time. We shall see that in general the parallel transport of a vector from one event to another can depend on the curve you choose to transport the vector along if the space-time geometry has a non-zero curvature.

1.7.1 Parallel transport

Consider a vector V^μ at the event p and a vector W^μ at the event q in a general space-time with metric $g_{\mu\nu}$ and coordinate system x^μ. The question we address in this section is:

how can we compare these two vectors?

In the case of Special Relativity, one can just compare their components V^μ and W^μ directly without worrying that they have been given at different events (assuming we work in an Inertial System of Minkowski space with metric $\eta_{\mu\nu}$). However, this is not the case in General Relativity. For instance, one can well imagine having two coordinate systems, where in one coordinate system the components V^μ and W^μ are equal, and in the other they are different.

The answer to the above question is that we can use the covariant derivative of a vector along a curve (1.4.35) to compare vectors at different events in the space-time. To do this we consider a curve $x^\mu(\lambda)$ that goes from p to q as λ goes from 0 to Λ. We then extend the vector V^μ to be a vector field on the curve by demanding that the covariant derivative along the curve is zero

$$\frac{D}{d\lambda}V^\mu = \frac{dV^\mu}{d\lambda} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\lambda}V^\rho = 0\,. \tag{1.7.1}$$

From this we get the *parallel transported vector*

$$V^\mu|_{\lambda=\Lambda}\,. \tag{1.7.2}$$

This is a vector at the event q that we can compare to the vector W^μ at q. See Figure 14 for an illustration.

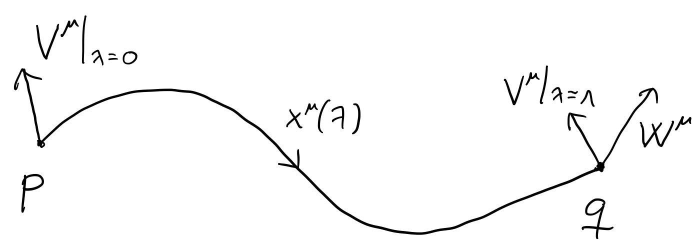

Figure 14: Illustration of the parallel transport of the vector $V^\mu|_{\lambda=0}$ at p to $V^\mu|_{\lambda=\Lambda}$ at q.

In the special case of Minkowski space, we see that in an Inertial System x^μ with metric $\eta_{\mu\nu}$ the Christoffel symbol is zero $\Gamma^\mu_{\nu\rho} = 0$. Thus, for Minkowski space parallel transport of a vector V^μ along a curve $x^\mu(\lambda)$ means that $\frac{dV^\mu}{d\lambda} = 0$. From this we see that the components V^μ are constant along the curve $x^\mu(\lambda)$. Therefore, in Minkowski space parallel transport from p to q simply means that the vector V^μ is the same at p and q. Note in particular that the parallel transport does not depend on what curve one uses to transport the vector along.

1.7.2 Curvature from path-dependence of parallel transport

In Minkowski space the parallel transport of a vector from one event to another does not depend on the path, *i.e.* it does not depend on what curve one transports the vector along. As we shall see in the following, this is *not* the case for a general space-time. The difference lies in the fact that while Minkowski space is flat, a general space-time can be curved. In this section we use the path-dependence of parallel transport to define a new tensor that can quantify the curvature of general space-times.

We are given a general space-time with metric $g_{\mu\nu}$ in a coordinate system x^μ. We consider a vector V^μ defined at the event x_0^μ in the space-time. We then compare the parallel transport of V^μ along two different infinitesimal paths.

Path 1 consists in parallel transporting V^μ from x_0^μ to $x_0^\mu + a^\mu$ and then to $x_0^\mu + a^\mu + b^\mu$ where a^μ and b^μ are infinitesimal. We denote the resulting vector at $x_0^\mu + a^\mu + b^\mu$ as V_1^μ. Path 2 consists in transporting V^μ first to $x_0^\mu + b^\mu$ and then to $x_0^\mu + a^\mu + b^\mu$. The resulting vector at $x_0^\mu + a^\mu + b^\mu$ from following Path 2 is denoted as V_2^μ. See Figure 15 for an illustration.

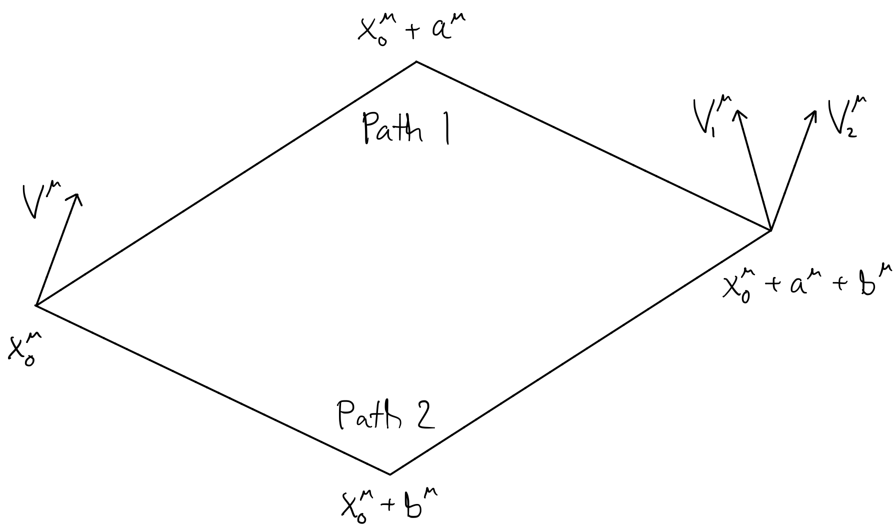

Figure 15: Illustration of the parallel transport of V^μ at x_0^μ along two different infinitesimal paths to the event $x_0^\mu + a^\mu + b^\mu$ with two different parallel transported vectors V_1^μ and V_2^μ depending on the path.

Using (1.7.1) one finds that transporting a vector V^μ from x^μ to $x^\mu + dx^\mu$ gives a new vector $V^\mu + dV^\mu$ with

$$dV^\mu = -\Gamma^\mu_{\nu\rho} V^\nu dx^\rho \,. \tag{1.7.3}$$

This we shall use repeatedly below.

Consider first Path 1. From (1.7.3) we find

$$V^\mu(x_0 + a) = V^\mu(x_0) - \Gamma^\mu_{\nu\rho}(x_0)V^\nu(x_0)a^\rho \,. \tag{1.7.4}$$

Transporting this vector to $x_0^\mu + a^\mu + b^\mu$ we find

$$\begin{aligned} V_1^\mu &= V^\mu(x_0 + a) - \Gamma^\mu_{\nu\rho}(x_0 + a)V^\nu(x_0 + a)b^\rho \\ &= V^\mu(x_0) - \Gamma^\mu_{\nu\rho}(x_0)V^\nu(x_0)a^\rho \\ &\quad - \Big(\Gamma^\mu_{\nu\rho}(x_0) + \partial_\sigma\Gamma^\mu_{\nu\rho}(x_0)a^\sigma\Big)\Big(V^\nu(x_0) - \Gamma^\nu_{\alpha\beta}(x_0)V^\alpha(x_0)a^\beta\Big)b^\rho \\ &\quad + \text{higher order terms}\,, \end{aligned} \tag{1.7.5}$$

where we included terms up to second order in the infinitesimal quantities a^μ and b^μ. Since now all the quantities are evaluated at x_0 we drop the reference to x_0 and find

$$V_1^\mu = V^\mu - \Gamma^\mu_{\nu\rho}V^\nu(a^\rho + b^\rho) + V^\nu a^\rho b^\sigma(-\partial_\rho\Gamma^\mu_{\nu\sigma} + \Gamma^\mu_{\alpha\sigma}\Gamma^\alpha_{\nu\rho}) + \text{higher order terms}\,, \tag{1.7.6}$$

where we included terms up to second order in the infinitesimal quantities a^μ and b^μ.

For Path 2, the only difference is that we should interchange a^μ and b^μ. Hence

$$V_2^\mu = V^\mu - \Gamma^\mu_{\nu\rho}V^\nu(a^\rho + b^\rho) + V^\nu a^\rho b^\sigma(-\partial_\sigma\Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\alpha\rho}\Gamma^\alpha_{\nu\sigma}) + \text{higher order terms}\,. \tag{1.7.7}$$

Consider now the difference

$$\Delta V^\mu = V_2^\mu - V_1^\mu\,, \tag{1.7.8}$$

between the two vectors at $x_0^\mu + a^\mu + b^\mu$. Using now the definition of the Riemann curvature tensor (1.5.10) we have

$$\Delta V^\mu = R^\mu{}_{\nu\rho\sigma}V^\nu a^\rho b^\sigma + \text{higher order terms}\,, \tag{1.7.9}$$

The result (1.7.9) means that for a space-time with a non-zero Riemann curvature tensor $R^\mu{}_{\nu\rho\sigma}$, the parallel transport of a vector will in general depend on the path that one chooses to transport the vector along. The physical reason for the difference between V_1^μ and V_2^μ is that they were subject to different gravitational fields along the two paths. Instead for a flat space-time like Minkowski space there is no path-dependence since $R^\mu{}_{\nu\rho\sigma} = 0$, in accordance with the physical interpretation that flat space-times do not have gravity.

1.8 Exercises for Chapter 1

Exercise 1.1. Invariance of the line-element in Special Relativity.
In Special Relativity the line-element in an Inertial System x^μ is given by $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$

(see Eq. (1.1.16)). Here $\eta_{\mu\nu}$ is the Minkowski metric given by (1.1.10). Show that the line element is invariant under the translation (1.1.5), the rotation (1.1.6) and the boost (1.1.7) coordinate transformations, *i.e.* show that under these three coordinate transformations we have

$$\eta_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu = \eta_{\mu\nu} dx^\mu dx^\nu \,. \tag{1.8.1}$$

[Hint: Do this by inserting $\tilde{x}^\mu$ on the LHS of (1.8.1) as given from the transformations (1.1.5), (1.1.6) and (1.1.7) and show that it gives the RHS.]

Exercise 1.2. Newtons second law in Special Relativity.
Please read Section 1.1.5 before starting on this exercise. In this exercise we consider dynamics in the theory of Special Relativity and in particular the formulation of Newtons second law of mechanics. In the following we write $t = x^0$.

- The relativistic velocity u^μ is defined by Eq. (1.1.19). Show that this is related to the ordinary 3-dimensional velocity vector $\vec{v}$ as in Eq. (1.1.20).

- Consider a particle of rest mass m assumed in this exercise to be conserved. Newtons second law for the particle in Special Relativity can be formulated as

$$\vec{F} = \frac{d\vec{p}}{dt} \,, \tag{1.8.2}$$

  where $\vec{F}$ is the external force and the momentum of the particle $\vec{p}$ is given by

$$\vec{p} = \gamma m \vec{v} \,. \tag{1.8.3}$$

  Show that this implies the relativistic version of Newtons second law as formulated by Eqs. (1.1.32) and (1.1.33) where we defined the relativistic force as

$$\mathcal{F}^0 = \gamma \vec{F} \cdot \vec{v} \,, \quad \mathcal{F}^i = \gamma F^i \,, \tag{1.8.4}$$

  using also the relation (1.1.31).

Exercise 1.3. Maxwells equations in Special Relativity.
Please read Section 1.1.6 before starting this exercise. In this exercise we consider Maxwells equations for electromagnetism in vacuum which can be written as Eqs. (1.1.34) and (1.1.36) where $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field, $\vec{J}$ is the current density and ρ_e is the charge density.

- Show that Eqs. (1.1.34) and (1.1.36) can be written as Eqs. (1.1.43) and (1.1.39), respectively, if we define the field strength $F_{\mu\nu}$ by Eqs. (1.1.37) and (1.1.38) and $F^{\mu\nu}$ by (1.1.42) along with the relativistic current density J^μ defined so that (J^1, J^2, J^3) is the current density $\vec{J}$ and $J^0 = \rho_e$.

- Show that the Lorentz force $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$ can be written in terms of the relativistic force $\mathcal{F}^\mu$ defined by Eq. (1.8.4) and the electromagnetic field strength $F^{\mu\nu}$ as in Eq. (1.1.44). Use this with Newtons second law (1.1.33) to show that the relativistic acceleration a^μ of a charged particle in a electromagnetic field in vacuum is given by Eq. (1.1.45).

Exercise 1.4. Charged particle in an electric field.
Consider a particle with charge q and rest mass m in an Inertial System x^μ. We write $t = x^0$. Assume the particle is subject to a constant electric field $\vec{E} = (\mathcal{E}, 0, 0)$, that the magnetic field is zero $\vec{B} = (0, 0, 0)$ and that the particle is at rest at $t = \tau = 0$ with position $\vec{x} = 0$ in the Inertial System x^μ. Define the quantity

$$\omega = \mathcal{E}q/m \,, \tag{1.8.5}$$

for use below. In the following we explore the motion of the particle. The exercise should make it clear that one can treat accelerated motion in Special Relativity.

- Find the relativistic velocity $u^\mu(\tau)$ of the particle using Eq. (1.1.45). Express the answer in terms of ω.

- Find the coordinate time t as function of proper time τ. This is the relation between the time seen from a static observer in the Inertial System x^μ and an observer travelling with the particle.

- Find the ordinary 3-dimensional velocity $\vec{v}$ (see Eq. (1.1.20)) both as function of proper time τ and of coordinate time t, *i.e.* the functions $\vec{v}(\tau)$ and $\vec{v}(t)$.

- Find the position of the particle $\vec{x}(t)$ as a function of the coordinate time t.

Exercise 1.5. GPS satellites.
A GPS satellite orbits the Earth at 20 000 km above the surface. The radius of the Earth is 6 400 km. It takes about 12 hours to complete an orbit.

- Based on the above information, compute the number of microseconds that the clock on the GPS satellite runs slower in 24 hours compared to a clock on Earth due to the time dilation of Special Relativity, *i.e.* that time runs slower for a clock in relative velocity compared to the observer.

- For time dilation in a weak gravitational field we imagine two clocks being in fixed positions relative to each other, with clock 1 being in gravitational potential ϕ_1 and clock 2 in gravitational potential ϕ_2. An observer at clock 1 now measures a time interval T_1. The observer then sees that clock 2 instead has measured the time T_2. The relation between the two time intervals is

$$\frac{T_2}{T_1} = 1 + \frac{\phi_2 - \phi_1}{c^2} \,. \tag{1.8.6}$$

Use the above information to compute how many microseconds faster a clock in the same distance from Earth as a GPS satellite runs compared to a clock on Earth in 24 hours.

- Argue that you can add up the two effects and find the final result for how many microseconds a clock onboard the GPS satellite runs faster than a clock on Earth in 24 hours, due to General Relativity.

Comment: The imprecision of the clock, as calculated above, would potentially make the position measurement inaccurate with time. Using the above, one finds easily that the inaccuracy in the clock onboard the satellite would correspond to an inaccuracy of a few centimeters for the position on Earth that corresponds to the position of the satellite, after one orbit. This effect is accumulative, so after a month its around a meter, and so on. For a GPS receiver on Earth, one measures radiosignals from four GPS satellites (note that a smartphone instead calibrates its position using the cell towers and wifi hotspots). This goes into a complicated computation of the position and time of the GPS receiver. If one had a atomic clock build in to the GPS receiver, one could do this with three satellites using triangulation. However, it is obviously impractical, and it would also be highly expensive, to build in an atomic clock in a GPS receiver that you bring for a trip to the forest. So one uses instead radiosignals from four satellites to also calibrate the time coordinate of the GPS receiver with atomic clock precision. To do this correctly, the GPS receiver needs to take into account that time for the satellites run faster than for the GPS receiver. Thus, General Relativity is a prerequisite for making the computation of the GPS receiver.

Exercise 1.6. Spherical coordinates.
In this exercise we consider three-dimensional Euclidean space and Minkowski space in spherical coordinates.

- Consider three-dimensional Euclidean space $\mathbb{R}^3$ with line-element

$$ds^2 = dx^2 + dy^2 + dz^2 \,. \tag{1.8.7}$$

Make the following coordinate transformation to spherical coordinates r, θ, ϕ

$$x = r\sin\theta\cos\phi\ , \quad y = r\sin\theta\sin\phi\ , \quad z = r\cos\theta\,. \tag{1.8.8}$$

What is the line-element in these coordinates?

- One should restrict θ to be in the interval $0 \le \theta \le \pi$. Why is this the case?
- One can define the unit sphere as

$$x^2 + y^2 + z^2 = 1\,. \tag{1.8.9}$$

What is this in spherical coordinates? Use this to argue that the line-element of the unit sphere is

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2\,. \tag{1.8.10}$$

where $0 \le \theta \le \pi$ and $0 \le \phi < 2\pi$. Correspondingly, the line-element for a sphere is radius a is

$$ds^2 = a^2 d\Omega^2 = a^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,. \tag{1.8.11}$$

- Argue that the line-element of Minkowski space in spherical coordinates is

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,. \tag{1.8.12}$$

Exercise 1.7. Geodesic equation in polar coordinates.
In this exercise we derive the geodesic equation for Minkowski space in polar coordinates.

- We start from the Minkowski space line element

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2\,. \tag{1.8.13}$$

Go now to polar coordinates (with t and z kept the same)

$$x = r\cos\phi\ , \quad y = r\sin\phi\,. \tag{1.8.14}$$

Show that the line element in these coordinates is

$$ds^2 = -dt^2 + dr^2 + r^2 d\phi^2 + dz^2\,. \tag{1.8.15}$$

- Compute the Christoffel symbol and show that the non-zero components are

$$\Gamma^r_{\phi\phi} = -r\ , \quad \Gamma^\phi_{\phi r} = \frac{1}{r}\,. \tag{1.8.16}$$

- Show that the geodesic equation in polar coordinates have the components

$$\ddot{t} = 0 \ , \quad \ddot{r} = r\dot{\phi}^2 \ , \quad \ddot{\phi} = -\frac{2}{r}\dot{r}\dot{\phi} \ , \quad \ddot{z} = 0 \, , \tag{1.8.17}$$

where $\dot{x}^\mu = dx^\mu/d\tau$ and $\ddot{x}^\mu = d^2x^\mu/d\tau^2$.

- Show using only (1.8.14) that (1.8.17) is equivalent to

$$\ddot{t} = 0 \ , \quad \ddot{x} = 0 \ , \quad \ddot{y} = 0 \ , \quad \ddot{z} = 0 \, . \tag{1.8.18}$$

Exercise 1.8. Rigid rotation and the centrifugal force.
We consider a coordinate system that follows a rigid rotation with angular velocity ω in the $z = 0$ plane. This corresponds to changing coordinates to

$$\Phi = \phi - \omega t \, , \tag{1.8.19}$$

with t, r and z kept the same.

- Consider Minkowski space in polar coordinates (1.8.15). Show that in these new coordinates the line element is

$$ds^2 = -(1-\omega^2 r^2)dt^2 + 2\omega r^2 d\Phi dt + dr^2 + r^2 d\Phi^2 + dz^2 \, . \tag{1.8.20}$$

- What happens for $r = 1/\omega$?

- Using (1.8.17) and (1.8.19), derive the components of the geodesic equation in rotating polar coordinates

$$\ddot{t} = 0 \ , \quad \ddot{r} = r(\dot{\Phi} + \omega\dot{t})^2 \ , \quad \ddot{\Phi} = -\frac{2}{r}\dot{r}(\dot{\Phi} + \omega\dot{t}) \ , \quad \ddot{z} = 0 \, . \tag{1.8.21}$$

  We note that by taking the Newton limit so that $t = \tau$ and $\dot{t} = 1$ one can show that the terms proportional to ω^2 correspond to the centrifugal force while the terms proportional to ω correspond to the coriolis force. Hence one gets the ficticious forces for a rotating coordinate system in this way. Consider in particular a particle with zero velocity $\dot{r} = \dot{\Phi} = \dot{z} = 0$ at a given moment in time. Argue that in this case we find (at that moment)

$$\ddot{r} = r\omega^2 \ , \quad \ddot{\Phi} = 0 \ , \quad \ddot{z} = 0 \, . \tag{1.8.22}$$

  We see that this corresponds to the centrifugal force on the particle in the radial direction.

Exercise 1.9. Transformation of the metric and its inverse.
Under a general coordinate transformation $x^\mu \to \tilde{x}^\mu(x)$ the metric transform as (1.3.23).

- Show that the line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ is invariant under a general coordinate transformation $x^\mu \to \tilde{x}^\mu(x)$.

- Using the chain rule, show the relations (1.3.29).

- The inverse metric $g^{\mu\nu}$ if defined by the statement that $g^{\mu\nu}g_{\nu\rho} = \delta^\mu_\rho$. Argue that if this should hold in all coordinate systems the inverse metric should transform as (1.3.28).

Exercise 1.10. Zero Christoffel symbol equals zero derivatives of metric.
Consider a general space-time with line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ and a given event p. The Christoffel symbol is defined by Eq. (1.3.46). From the definition of the Christoffel symbol it is obvious that if $\frac{\partial g_{\mu\nu}}{\partial x^\rho}|_p = 0$ then $\Gamma^\rho_{\mu\nu}|_p = 0$. We shall now see that the opposite is true as well.

Define

$$A_{\rho\mu\nu} = 2g_{\rho\sigma}\Gamma^\sigma_{\mu\nu} = \frac{\partial g_{\nu\rho}}{\partial x^\mu} + \frac{\partial g_{\mu\rho}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho}\,. \tag{1.8.23}$$

Compute

$$A_{\mu\rho\nu} + A_{\nu\rho\mu}\,. \tag{1.8.24}$$

Use the result of this computation to argue that if $\Gamma^\rho_{\mu\nu}|_p = 0$ then $\frac{\partial g_{\mu\nu}}{\partial x^\rho}|_p = 0$.

Exercise 1.11. Vector field in the plane.
Consider the two dimensional plane in Cartesian coordinates x, y with line element

$$ds^2 = dx^2 + dy^2\,. \tag{1.8.25}$$

Consider a vector field on the plane $V = (V^x, V^y)$ given by

$$V^x = 1\ , \quad V^y = 0\,. \tag{1.8.26}$$

- Make a coordinate transformation to polar coordinates $x = r\cos\phi$ and $y = r\sin\phi$. What are the components of the vector field V^r and V^ϕ in polar coordinates?

- Explain that in the Cartesian coordinate system we have

$$\partial_\mu V^\nu = 0\,. \tag{1.8.27}$$

  Argue that this means

$$D_\mu V^\nu = 0\,, \tag{1.8.28}$$

in the Cartesian coordinate system. Suppose now one makes a coordinate transformation to a new coordinate system. Does (1.8.28) hold in the new coordinate system as well?

- Compute $\partial_\mu V^\nu$ and $D_\mu V^\nu$ in polar coordinates.

Exercise 1.12. Contraction of the Christoffel Symbol.
Define the determinant of the metric as

$$g = \det(g_{\mu\nu})\,, \tag{1.8.29}$$

meaning that we take the determinant of the four by four matrix (1.3.14). One can show in general that the partial derivative of this determinant is

$$\partial_\mu g = g\, g^{\nu\rho} \partial_\mu g_{\nu\rho}\,. \tag{1.8.30}$$

See Exercise 1.16 for a derivation of this result.

- Show using (1.8.30) that

$$\partial_\mu \sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{\nu\rho}\partial_\mu g_{\nu\rho}\,. \tag{1.8.31}$$

- The Christoffel symbol is defined in (1.3.46). Use Eq. (1.8.31) to show that

$$\Gamma^\rho_{\rho\mu} = \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g} = \partial_\mu \log\sqrt{-g}\,. \tag{1.8.32}$$

- Use Eq. (1.8.32) to show

$$D_\mu V^\mu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}V^\mu)\,, \tag{1.8.33}$$

$$D_\mu D^\mu \Phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\Phi)\,, \tag{1.8.34}$$

  for a vector field V^μ and a scalar field Φ.

- Show that for an antisymmetric tensor $F^{\mu\nu} = -F^{\nu\mu}$ one has

$$D_\mu F^{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}F^{\mu\nu})\,. \tag{1.8.35}$$

Exercise 1.13. Covariant Maxwell equation and electric field from point source.
We consider an electric field from a point source in two different coordinate systems for Minkowski space. The purpose of this exercise is to illustrate that the covariant Maxwells equations (1.4.47) and (1.4.48) are useful also in Minkowski space in coordinate systems that are not Inertial Systems.

- Consider Minkowski space in an Inertial System x^μ. Then Maxwells equations can be written as (1.1.39) and (1.1.43). Consider a point charge q sitting at the origin $x^i = 0$ (q is assumed to be constant). Show that the corresponding electromagnetic field strength
$$F^{0i} = \frac{qx^i}{4\pi r^3}\ , \quad F^{ij} = 0\,, \tag{1.8.36}$$
satisfies the Maxwell equation (1.1.43) away from the point source, where $i, j = 1, 2, 3$ and $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$.

- Transform the electromagnetic field strength (1.8.36) to spherical coordinates for Minkowski space $t = x^0$, r, θ and ϕ, given by
$$(x^1, x^2, x^3) = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\,. \tag{1.8.37}$$
[**Hint:** Use that $F^{\mu\nu}$ is a tensor. Alternative route: Use that (1.4.47) works in all coordinate systems. First find potential A_μ in the Inertial System (with only A_0 non-zero), make the (very simple) transformation of A_μ to spherical coordinates for Minkowski space, and use this to find $F^{\mu\nu}$.]

- Show that the transformed field strength in spherical coordinates for Minkowski space satisfy the covariant Maxwell equation (1.4.48). [Hint: Instead of computing Christoffel symbols you can use the result (1.8.35) of Exercise 1.12.]

Exercise 1.14. Christoffel Symbol is not a tensor.
Argue why the Christoffel Symbol $\Gamma^\mu_{\nu\rho}$ is not a tensor.
[**Hint:** There are two different ways: 1) Consider how it transform under coordinate transformations. 2) Use that one can go to a local inertial system.]

Exercise 1.15. Properties of the covariant derivative.

- Check explicitly that
$$D_\rho g_{\mu\nu} = 0\,. \tag{1.8.38}$$

- Check explicitly the product rule in the following case
$$D_\mu(A_{\nu\rho}V^\rho) = (D_\mu A_{\nu\rho})V^\rho + A_{\nu\rho}D_\mu V^\rho\,. \tag{1.8.39}$$

- Use Eqs. (1.8.38) and (1.8.39) to show
$$D^\mu V_\mu = D_\mu V^\mu\,, \tag{1.8.40}$$
where $D^\mu V_\mu = g^{\mu\nu}(D_\nu V_\mu)$.

Exercise 1.16. Variation of determinant of the metric.

The determinant of the metric is defined by Eq. (1.8.29). In this exercise we show that an infinitesimal variation of g satisfies the relation

$$\delta g = g\, g^{\mu\nu} \delta g_{\mu\nu}\,, \tag{1.8.41}$$

and that the partial derivative of g is

$$\partial_\mu g = g\, g^{\nu\rho} \partial_\mu g_{\nu\rho}\,. \tag{1.8.42}$$

- Consider a matrix A that is the exponential of another matrix B

  $$A = e^B\,. \tag{1.8.43}$$

  For an infinitesimal variation A to $A + \delta A$ we have

  $$\delta A = e^B \delta B\,. \tag{1.8.44}$$

  Use this to show that

  $$A^{-1} \delta A = \delta B\,. \tag{1.8.45}$$

- One can show that the determinant of A is the exponential of the trace of B

  $$\det(A) = e^{\mathrm{Tr}\, B}\,. \tag{1.8.46}$$

  Use this to show that

  $$\delta(\det(A)) = \det A\ \mathrm{Tr}(\delta B)\,. \tag{1.8.47}$$

  Use this with (1.8.45) to show that

  $$\delta(\det(A)) = \det A\ \mathrm{Tr}(A^{-1}\delta A)\,. \tag{1.8.48}$$

  Note: One can show that this formula is valid also for matrices that are not exponentials as in (1.8.43).

- Use (1.8.48) to show that for an infinitesimal variation of the metric from $g_{\mu\nu}$ to $g_{\mu\nu} + \delta g_{\mu\nu}$ the variation of the determinant of the metric satisfies the relation (1.8.41). Using this, argue that the partial derivative of g is given by Eq. (1.8.42).

Exercise 1.17. Vector field in the plane II.

Consider the two dimensional plane. In polar coordinates (r, ϕ) the line element is

$$ds^2 = dr^2 + r^2 d\phi^2 \tag{1.8.49}$$

Consider a vector field on the plane $V = (V^r, V^\phi)$ given in polar coordinates as

$$V^r = 1\,, \quad V^\phi = 0\,. \tag{1.8.50}$$

- Make a drawing of the vector field.
- Make a coordinate transformation to Cartesian coordinates $x = r\cos\phi$ and $y = r\sin\phi$. What are the components of the vector field V^x and V^y in Cartesian coordinates? Does this fit with your drawing?
- Explain that in the polar coordinate system we have

$$\partial_\mu V^\nu = 0\,. \tag{1.8.51}$$

  Does this mean that one has $D_\mu V^\nu = 0$? Preferably answer this question without making further computations.
- Compute $D_\mu V^\nu$ in polar coordinates.

Exercise 1.18. Curvature of a sphere.
The sphere of radius a has line-element

$$ds^2 = a^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{1.8.52}$$

with $0 \le \theta \le \pi$ and $0 \le \phi < 2\pi$.

- Check that the non-zero Christoffel symbol components are

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta\ ,\quad \Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta}\,. \tag{1.8.53}$$

- A general formula for the Ricci tensor (1.5.24) is (see Eq. (2.1.28) in Section 2.1.2)

$$R_{\mu\nu} = \partial_\rho\Gamma^\rho_{\mu\nu} - \partial_\mu\partial_\nu \log\sqrt{|g|} + \Gamma^\sigma_{\mu\nu}\partial_\sigma \log\sqrt{|g|} - \Gamma^\rho_{\sigma\mu}\Gamma^\sigma_{\rho\nu}\,, \tag{1.8.54}$$

  where $|g|$ is the numerical part of the determinant of the metric $g = \det(g_{\mu\nu})$ so in this case $|g| = g$. Use this to compute the components of the Ricci tensor

$$R_{\theta\theta} = 1\ ,\quad R_{\theta\phi} = 0\ ,\quad R_{\phi\phi} = \sin^2\theta\,. \tag{1.8.55}$$

- Using (1.8.55), show that the Ricci scalar (1.5.26) for the sphere of radius a is

$$R = \frac{2}{a^2}\,. \tag{1.8.56}$$

  Can you explain why it makes sense that the curvature goes like $1/a^2$?

Exercise 1.19. Alternative form of Einsteins equations.
Einsteins equations are written in Eq. (1.6.37). Show that one can write them on the form (1.6.38).

Exercise 1.20. Commutator of covariant derivatives.
Consider a vector field V^μ in the coordinate system x^μ on a space-time geometry.

- Show that taking two covariant derivatives of V^μ gives the result

$$\begin{aligned} D_\mu D_\nu V^\rho &= D_\mu(D_\nu V^\rho) = \partial_\mu(D_\nu V^\rho) - \Gamma^\sigma_{\mu\nu} D_\sigma V^\rho + \Gamma^\rho_{\mu\sigma} D_\nu V^\sigma \\ &= \partial_\mu \partial_\nu V^\rho - \Gamma^\sigma_{\mu\nu} D_\sigma V^\rho + \Gamma^\rho_{\mu\sigma}\partial_\nu V^\sigma + \Gamma^\rho_{\nu\sigma}\partial_\mu V^\sigma + (\partial_\mu \Gamma^\rho_{\nu\sigma} + \Gamma^\rho_{\alpha\mu}\Gamma^\alpha_{\nu\sigma}) V^\sigma \,. \end{aligned} \tag{1.8.57}$$

- Show that the commutator of the covariant derivatives acting on V^μ gives

$$(D_\mu D_\nu - D_\nu D_\mu) V^\rho = (\partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\alpha\mu}\Gamma^\alpha_{\nu\sigma} - \Gamma^\rho_{\alpha\nu}\Gamma^\alpha_{\mu\sigma}) V^\sigma \tag{1.8.58}$$

  Check that this commutator can be expressed as (1.5.16) in terms of the Riemann curvature tensor (1.5.10).

Exercise 1.21. Energy-momentum tensor in electromagnetism.
The energy-momentum tensor for electromagnetic fields is

$$T^{\mu\nu}_{\mathrm{EM}} = F^\mu{}_\rho F^{\nu\rho} - \frac{1}{4} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \tag{1.8.59}$$

In the following we are in Minkowski space in an inertial system x^μ hence the metric is $g_{\mu\nu} = \eta_{\mu\nu}$. We consider electromagnetic fields in vacuum, hence the relativistic current density $J^\mu = 0$.

- Show that T^{00}_{EM} is equal to the energy density of electromagnetic field $\rho_{\mathrm{EM}} = \frac{1}{2}(\vec{E}^2 + \vec{B}^2)$. Does this make sense?

- Show that T^{0i}_{EM} is given by the Poynting vector of electromagnetism $\vec{S} = \vec{E} \times \vec{B}$. Does this make sense?

- Using the Maxwell's equations 1.1.39-(1.1.43) with $J^\mu = 0$, show that $T^{\mu\nu}_{\mathrm{EM}}$ is conserved (for $g_{\mu\nu} = \eta_{\mu\nu}$).

- Argue using the result for Minkowski space, that for a general space-time geometry the energy-momentum tensor for electromagnetic fields (1.8.59) is conserved provided the covariant Maxwell's equations 1.4.47-(1.4.48) holds with $J^\mu = 0$.

Exercise 1.22. Riemann Normal coordinates.
Consider a general space-time with line-element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ in the coordinate system x^μ. Let p be any event in the space-time. Write p in the x^μ coordinates as x^μ_p.

We define a new coordinate system $\tilde{x}^\mu(x)$ in a neighborhood of p called *Riemann Normal coordinates*. Consider any point q in the space-time (written as x_q^μ in the x^μ coordinates) that can be reached by a curve $x^\mu(\lambda)$ obeying the geodesic equation (note that the curve is not required to be time-like)

$$\frac{D}{d\lambda}\left(\frac{dx^\mu}{d\lambda}\right) = 0\,, \tag{1.8.60}$$

so that $x^\mu(0) = x_p^\mu$ and $x^\mu(1) = x_q^\mu$. We define the new coordinates $\tilde{x}^\mu(x)$ at q to be equal to the components of the tangent vector of the curve at p

$$\tilde{x}^\mu(x_q) = \frac{dx^\mu}{d\lambda}\bigg|_{\lambda=0}\,. \tag{1.8.61}$$

- Consider the above-mentioned curve $x^\mu(\lambda)$ from x_p^μ to x_q^μ as λ goes from 0 to 1 and obeying (1.8.60). Make now a linear reparametrization of the form

  $$\zeta(\lambda) = \frac{\lambda}{b}\,, \tag{1.8.62}$$

  where b is a constant in the interval $0 < b < 1$. Show that

  $$\frac{dx^\mu}{d\zeta} = b\frac{dx^\mu}{d\lambda}\,. \tag{1.8.63}$$

  Argue that $x^\mu(\zeta)$ fulfils

  $$\frac{D}{d\zeta}\left(\frac{dx^\mu}{d\zeta}\right) = 0\,, \tag{1.8.64}$$

  and that one has

  $$\frac{dx^\mu}{d\zeta}\bigg|_{\zeta=0} = b\frac{dx^\mu}{d\lambda}\bigg|_{\lambda=0}\,. \tag{1.8.65}$$

- Consider the point $x^\mu(\zeta = 1)$. This is the point $x^\mu(\lambda = b)$ in the old parametrization. Show using the above that

  $$\tilde{x}^\mu(x(\lambda = b)) = \tilde{x}^\mu(x(\zeta = 1)) = b\frac{dx^\mu}{d\lambda}\bigg|_{\lambda=0}\,. \tag{1.8.66}$$

  Use this to argue that the curve $\tilde{x}^\mu(\lambda) = \tilde{x}^\mu(x(\lambda))$ is given by

  $$\tilde{x}^\mu(\lambda) = \lambda\frac{dx^\mu}{d\lambda}\bigg|_{\lambda=0}\,, \tag{1.8.67}$$

  for $0 \le \lambda \le 1$.

- Argue from (1.8.67) that

  $$\frac{d^2\tilde{x}^\mu}{d\lambda^2}\bigg|_{\lambda=0} = 0\,. \tag{1.8.68}$$

- Argue that the above shows that Eq. (1.8.68) is the geodesic equation at the point p in the $\tilde{x}^\mu$ coordinates. Use this to argue that the Christoffel symbol in the $\tilde{x}^\mu$ coordinate system is zero at p

$$\tilde{\Gamma}^\mu_{\nu\rho}\Big|_p = 0\,. \tag{1.8.69}$$

Conclude using Exercise 1.10 that as consequence of this we have

$$\tilde{\partial}_\rho \tilde{g}_{\mu\nu}\Big|_p = 0\,. \tag{1.8.70}$$

Exercise 1.23. Local Inertial System.
In this exercise we show that for any non-singular event in space-time one can find a local Inertial System for that event.

- Consider a general space-time with line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. Consider a given point p of the space-time. Show that by a rigid coordinate transformation

$$\tilde{x}^\mu = A^\mu{}_\nu x^\nu\,, \tag{1.8.71}$$

where $A^\mu{}_\nu$ is constant, we can diagonalize the metric at the point p, *i.e.* that $\tilde{g}_{\mu\nu}|_p$ is only non-zero for $\mu = \nu$. Argue that by a further rescaling of the coordinates the diagonal entries can be chosen to be ± 1. For the metric of a space-time, this requires that diagonal entries have one -1 and three 1. Thus, we have shown that for any given point p we can make a rigid coordinate transformation (1.8.71) such that

$$\tilde{g}_{\mu\nu}\Big|_p = \eta_{\mu\nu}\,. \tag{1.8.72}$$

- Consider a general space-time with line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. Consider a given point p of the space-time. Using the results of Exercise 1.22 go to Riemann normal coordinates $\tilde{x}^\mu$ near p. In these coordinates $\tilde{\partial}_\rho \tilde{g}_{\mu\nu}|_p = 0$. Perform now a further rigid coordinate transformation $\hat{x}^\mu = A^\mu{}_\nu \tilde{x}^\mu$ as above such that $\hat{g}_{\mu\nu}|_p = \eta_{\mu\nu}$. Argue that the new coordinate system $\hat{x}^\mu$ obeys

$$\hat{g}_{\mu\nu}\Big|_p = \eta_{\mu\nu}\;,\quad \hat{\partial}_\rho \hat{g}_{\mu\nu}\Big|_p = 0\,. \tag{1.8.73}$$

We have thus found a local Inertial System for the event p.

Exercise 1.24. Bianchi identity for Riemann curvature tensor.
Derive the Bianchi identity (1.5.23) following the prescription written below Eq. (1.5.22). Use this identity to derive (1.5.28).

Exercise 1.25. Norm and dot product of parallel transported vectors.
Read Section 1.7.1. Consider two parallel transported vector fields V^μ and W^μ along a curve $x^\mu(\lambda)$. Show that their dot product $g_{\mu\nu}V^\mu W^\nu$ is constant along the curve. Argue that also the norm (1.4.15) of the vector field V^μ is constant along the curve.

Exercise 1.26. Parallel transport of a tangent vector.
Read Section 1.7.1. Consider a geodesic $x^\mu(\lambda)$ going through the event $x^\mu(0)$. Let $V^\mu(\lambda)$ be a parallel transported vector field

$$\frac{D}{d\lambda}V^\mu = 0\,, \tag{1.8.74}$$

on the geodesic $x^\mu(\lambda)$. Show that if $V^\mu(0)$ at the event $x^\mu(0)$ is parallel to the tangent vector of the curve

$$V^\mu(0) = a\frac{dx^\mu}{d\lambda}\bigg|_{\lambda=0}\,, \tag{1.8.75}$$

where a is a constant, then it is parallel to the tangent vector everywhere on the curve, with a constant ratio between them, *i.e.*

$$V^\mu(\lambda) = a\frac{dx^\mu}{d\lambda}\,, \tag{1.8.76}$$

for all λ.

Exercise 1.27. Newtons tidal forces from geodesic deviation equation.
We consider the geodesic deviation equation (1.5.11) in the Newton limit Eqs. (1.3.48)-(1.3.51).

- Argue that to leading order in the Newton limit the geodesic deviation equation (1.5.11) reduces to

  $$\frac{d^2V^i}{dt^2} = -R^i{}_{0j0}V^j \tag{1.8.77}$$

  where $x^0 = t$, $i = 1, 2, 3$ and we have a hidden sum over $j = 1, 2, 3$.

- Argue that in the Newton limit $\Gamma^i_{00} = \partial_i\Phi$ with $i = 1, 2, 3$.

- Argue that in the Newton limit $R^i{}_{0j0} = \partial_i\partial_j\Phi$.

- Argue that the geodesic deviation equation (1.5.11) in the Newton limit becomes

  $$\frac{d^2V^i}{dt^2} = -\partial_i\partial_j\Phi\, V^j \tag{1.8.78}$$

  with a hidden sum over $j = 1, 2, 3$.

- Consider Newtonian mechanics for two point particles A and B in a gravitational potential Φ. Write their spatial locations as x^i_A and x^i_B. Argue that when the particles are nearby, their separation $x^i_B - x^i_A = \epsilon V^i$, with ϵ small, obeys the same equation as (1.8.78). This is Newtons equation for tidal forces, which he used to show that the gravitational pull of the Moon is responsible for the ocean tides on Earth.

2 Schwarzschild Metric: Derivation and Geodesics

In Chapter 1 we introduced the notion of a general space-time defined by its line-element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. In general, the metric $g_{\mu\nu}$ is determined from Einsteins equations (1.6.37). In this chapter we shall consider the most important example of how this is done in practise, namely in Section 2.1 we derive what is known as the Schwarzschild metric for spherically symmetric matter distributions. Subsequently we briefly discuss the interpretation of the metric, which for example can be used to understand the gravitational field around the Sun.

Two other key ingredients in the theory of General Relativity are the geodesic equations (1.3.45) and (1.4.52) for massive and massless particles in a general space-time. We apply these equations in Section 2.2 to the Schwarzschild metric and through that learn about the planetary orbits in our solar system as well as the deflection of light by massive objects.

2.1 Derivation

In this section we derive the metric for spherically symmetric distributions of matter.

2.1.1 Spherically symmetric line-elements

The gravitational field from a spherically symmetric source is relevant for many important physical situations. The Sun is spherically symmetric to a good approximation and from its gravitational field we can determine the motion of the planets in the solar system. Another important case is the gravitational field from Earth. Since the source is spherically symmetric, the space-time geometry it induces by Einsteins equations (1.6.37) is also spherically symmetric. Hence, we start by considering a general spherically symmetric metric line-element that possibly can be time-dependent.

We know already an example of a spherically symmetric space-time: Minkowski space with line-element

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = -dt^2 + d\vec{x}\cdot d\vec{x} = -dt^2 + dr^2 + r^2 d\Omega^2 \,, \tag{2.1.1}$$

where

$$\begin{gathered} \vec{x} = (x^1, x^2, x^3) = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\,, \\ r^2 = \vec{x}\cdot\vec{x}\ , \quad d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2\,. \end{gathered} \tag{2.1.2}$$

The line-element (2.1.1) for Minkowski space is written both using Cartesian coordinates t and $\vec{x}$, corresponding to an Inertial System, and spherical coordinates (t, r, θ, ϕ) with

$0 \leq \theta \leq \pi$.

What is the most general spherically symmetric line-element? Spherical symmetry means that the line-element is invariant under any rotation of $\vec{x}$. Thus, a term like $x^1 dx^2$ is for instance not invariant under rotations. Instead we can use the ingredients t, dt, $\vec{x}$ and $d\vec{x}$ to build spherically symmetric combinations. In this way we see that a spherically symmetric line-element can only depend on

$$t \ , \quad dt \ , \quad \vec{x}\cdot\vec{x} \ , \quad \vec{x}\cdot d\vec{x} \ , \quad d\vec{x}\cdot d\vec{x} \, . \tag{2.1.3}$$

Using (2.1.2) along with $\vec{x}\cdot d\vec{x} = rdr$ and $d\vec{x}\cdot d\vec{x} = dr^2 + r^2 d\Omega^2$ we see that in terms of spherical coordinates (t,r,θ,ϕ) a spherically symmetric line-element can only depend on

$$t \ , \quad dt \ , \quad r \ , \quad dr \ , \quad d\Omega^2 \, , \tag{2.1.4}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. From this we see that the most general spherically symmetric line-element is

$$ds^2 = -A(t,r)dt^2 + B(t,r)dr^2 + 2C(t,r)dtdr + D(t,r)r^2 d\Omega^2 \, , \tag{2.1.5}$$

involving four functions of r and t. We now simplify this line-element by employing coordinate transformations involving t and r but not θ and ϕ since that would ruin the spherical symmetry. First, we make the transformation[17]

$$t' = t \ , \quad r' = r\sqrt{D(t,r)} \, , \tag{2.1.6}$$

that gives a line-element of the form[18]

$$ds^2 = -a(t',r')dt'^2 + b(t',r')dr'^2 + 2c(t',r')dtdr' + r'^2 d\Omega^2 \, , \tag{2.1.7}$$

with three functions of r' and t'.

Make then the coordinate transformation

$$\tilde{t} = \tilde{t}(t',r') \ , \quad \tilde{r} = r' \, , \tag{2.1.8}$$

[17] We assume here that $r^2 D(t,r)$ is not constant. If it was constant the space-time would be a product of a two-dimensional space-time times a sphere of constant radius. This is not a relevant class of space-times for us since we want space-times that can become asymptotically flat when the radius goes to infinity, see Sec. 3.4.1.

[18] One finds $ds^2 = -Adt'^2 + B\left(\frac{dr'}{\partial_r(r\sqrt{D})} - \frac{\partial_t(r\sqrt{D})}{\partial_r(r\sqrt{D})}dt'\right)^2 + 2Cdt'\left(\frac{dr'}{\partial_r(r\sqrt{D})} - \frac{\partial_t(r\sqrt{D})}{\partial_r(r\sqrt{D})}dt'\right) + r'^2 d\Omega^2$. Writing out all the terms one can see that this is on the general form (2.1.7).

where we require that $d\tilde{t}$ is of the form

$$d\tilde{t} = \eta(t', r')\Big(a(t', r')dt' - c(t', r')dr'\Big) . \tag{2.1.9}$$

This is equivalent to the relations

$$\frac{\partial \tilde{t}}{\partial t'} = \eta a \ , \quad \frac{\partial \tilde{t}}{\partial r'} = -\eta c \, . \tag{2.1.10}$$

A consistent transformation requires

$$\frac{\partial}{\partial r'}\left(\frac{\partial \tilde{t}}{\partial t'}\right) = \frac{\partial^2 \tilde{t}}{\partial r' \partial t'} = \frac{\partial^2 \tilde{t}}{\partial t' \partial r'} = \frac{\partial}{\partial t'}\left(\frac{\partial \tilde{t}}{\partial r'}\right) . \tag{2.1.11}$$

Combining this with (2.1.10) we find

$$\frac{\partial}{\partial r'}(\eta a) = -\frac{\partial}{\partial t'}(\eta c) \, . \tag{2.1.12}$$

This relation determines the function $\eta(t', r')$ in (2.1.9) if one specifies it for all r' at a given value of t'. Compute now from (2.1.9)

$$d\tilde{t}^2 = \eta^2 \left(a^2 dt'^2 + c^2 dr'^2 - 2ac dt' dr'\right) \ , \tag{2.1.13}$$

which gives

$$-\frac{1}{a\eta^2} d\tilde{t}^2 = -a dt'^2 - \frac{c^2}{a} dr'^2 + 2c dt' dr' \, . \tag{2.1.14}$$

Using this with (2.1.7) we get the line-element

$$ds^2 = -\frac{1}{a\eta^2} d\tilde{t}^2 + \left(b + \frac{c^2}{a}\right) d\tilde{r}^2 + \tilde{r}^2 d\Omega^2 \, . \tag{2.1.15}$$

We see that we have eliminated the $d\tilde{t} d\tilde{r}$ cross term in the line-element and hence the above metric is written in terms of only two functions. Renaming the coordinates by removing the tildes and defining two new functions, we have derived:

Spherically symmetric line-element: A general spherically symmetric line-element can always be written on the form

$$ds^2 = -e^{2\alpha(t,r)} dt^2 + e^{2\beta(t,r)} dr^2 + r^2 d\Omega^2 \, , \tag{2.1.16}$$

where $\alpha(t, r)$ and $\beta(t, r)$ are two undetermined functions and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ with $0 \leq \theta \leq \pi$.

2.1.2 Derivation of the Schwarzschild metric

In this section we derive the metric outside a spherically symmetric distribution of matter employing Einsteins equations (1.6.37). We assume that any mass or energy present outside the matter distribution have a neglible influence on the gravitational field. *E.g.* an apple in the gravitational field of the Earth has a completely neglible influence on the gravitational field around the Earth. Thus, we can assume $T_{\mu\nu} = 0$ to a good approximation outside the matter distribution. This means that the metric $g_{\mu\nu}(x)$ obeys the vacuum Einstein equations (1.6.39) outside the spherically symmetric matter distribution.

Since the matter distribution is spherically symmetric, the metric around it should also be spherically symmetric. Hence the line-element can be written on the form (2.1.16). To simplify our task in the following, we impose in addition to the spherical symmetry that the line-element is static, *i.e.* independent of time. For the line-element (2.1.16) this means that the functions α and β should not depend on t. Hence, we use in the following the line-element

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2 \,, \tag{2.1.17}$$

with $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. We comment below on what happens in the time-dependent case.

The task in the following is thus to impose the vacuum Einstein equations $R_{\mu\nu} = 0$ on the line-element (2.1.17). To compute $R_{\mu\nu}$ for the line-element (2.1.17) we first need to compute the Christoffel symbol (1.3.46). To this end, we record that the metric components corresponding to (2.1.17) are

$$g_{tt} = -e^{2\alpha(r)} \;,\quad g_{rr} = e^{2\beta(r)} \;,\quad g_{\theta\theta} = r^2 \;,\quad g_{\phi\phi} = r^2\sin^2\theta \,. \tag{2.1.18}$$

The components of the inverse metric are

$$g^{tt} = -e^{-2\alpha(r)} \;,\quad g^{rr} = e^{-2\beta(r)} \;,\quad g^{\theta\theta} = \frac{1}{r^2} \;,\quad g^{\phi\phi} = \frac{1}{r^2\sin^2\theta} \,. \tag{2.1.19}$$

Consider now the following components of the Christoffel symbol

$$\Gamma^t_{\mu\nu} = \frac{1}{2}g^{tt}(\partial_\mu g_{\nu t} + \partial_\nu g_{\mu t} - \partial_t g_{\mu\nu}) \,. \tag{2.1.20}$$

This can only be non-zero if $(\mu,\nu) = (t,r)$ (or equivalently if $(\mu,\nu) = (r,t)$). We compute

$$\Gamma^t_{tr} = \partial_r\alpha \,. \tag{2.1.21}$$

Consider

$$\Gamma^r_{\mu\nu} = \frac{1}{2}g^{rr}(\partial_\mu g_{\nu r} + \partial_\nu g_{\mu r} - \partial_r g_{\mu\nu}) \,. \tag{2.1.22}$$

The non-zero components are

$$\Gamma^r_{tt} = e^{2\alpha-2\beta}\partial_r\alpha \ , \quad \Gamma^r_{rr} = \partial_r\beta \ , \quad \Gamma^r_{\theta\theta} = -re^{-2\beta} \ , \quad \Gamma^r_{\phi\phi} = -re^{-2\beta}\sin^2\theta \, . \tag{2.1.23}$$

Proceeding this way we also find

$$\Gamma^\theta_{r\theta} = \frac{1}{r} \ , \quad \Gamma^\phi_{r\phi} = \frac{1}{r} \ , \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \ , \quad \Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta} \, . \tag{2.1.24}$$

Using (1.5.24) and (1.5.10) we find the general expression for the Ricci tensor

$$R_{\mu\nu} = \partial_\rho\Gamma^\rho_{\mu\nu} - \partial_\nu\Gamma^\rho_{\rho\mu} + \Gamma^\rho_{\rho\sigma}\Gamma^\sigma_{\mu\nu} - \Gamma^\rho_{\sigma\mu}\Gamma^\sigma_{\rho\nu} \, . \tag{2.1.25}$$

In Exercise 1.12 we have shown

$$\Gamma^\rho_{\rho\mu} = \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g} = \partial_\mu \log\sqrt{-g} \, , \tag{2.1.26}$$

where g is defined as the determinant of the metric

$$g = \det(g_{\mu\nu}) \, , \tag{2.1.27}$$

viewing the metric as a matrix (1.3.14). Using this in (2.1.25) we find

$$R_{\mu\nu} = \partial_\rho\Gamma^\rho_{\mu\nu} - \partial_\mu\partial_\nu \log\sqrt{-g} + \Gamma^\rho_{\mu\nu}\partial_\rho \log\sqrt{-g} - \Gamma^\rho_{\sigma\mu}\Gamma^\sigma_{\rho\nu} \, . \tag{2.1.28}$$

This is a general result for the Ricci tensor. For the concrete case at hand we compute

$$g = \det(g_{\mu\nu}) = -e^{2\alpha}e^{2\beta}r^4\sin^2\theta \, , \tag{2.1.29}$$

hence

$$\sqrt{-g} = e^{\alpha+\beta}r^2\sin\theta \, . \tag{2.1.30}$$

Using (2.1.21), (2.1.23), (2.1.24) and (2.1.30) with (2.1.28), we can compute the rr-component of the Ricci tensor

$$\begin{aligned} R_{rr} &= \partial_r\Gamma^r_{rr} - \partial_r^2 \log\sqrt{-g} + \Gamma^r_{rr}\partial_r\sqrt{-g} - (\Gamma^t_{tr})^2 - (\Gamma^r_{rr})^2 - (\Gamma^\theta_{\theta r})^2 - (\Gamma^\phi_{\phi r})^2 \\ &= -\partial_r^2\alpha + \partial_r\alpha\partial_r\beta - (\partial_r\alpha)^2 + \frac{2}{r}\partial_r\beta \end{aligned} \tag{2.1.31}$$

Similarly we compute

$$\begin{aligned} R_{tt} &= e^{2\alpha-2\beta}\left(\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha\right) \\ R_{\theta\theta} &= 1 + e^{-2\beta}\Big(r(\partial_r\beta - \partial_r\alpha) - 1\Big) \\ R_{\phi\phi} &= \sin^2\theta R_{\theta\theta} \end{aligned} \tag{2.1.32}$$

These are the only non-zero components of $R_{\mu\nu}$.

We now impose $R_{\mu\nu} = 0$ on (2.1.31) and (2.1.32). First, we notice

$$e^{2\beta-2\alpha} R_{tt} + R_{rr} = \frac{2}{r}(\partial_r \alpha + \partial_r \beta) \tag{2.1.33}$$

Thus, since the LHS is zero, we find that $\alpha + \beta$ is constant. What is the natural value of this constant? Since we would like that the metric (2.1.17) asymptotes to the Minkowski space metric in spherical coordinates (1.8.12) when we are far away from the matter distribution, this implies α and β should both goes to zero as $r \to \infty$. Thus, a natural choice of the constant would be that it is zero. We can set the constant to zero by making a redefinition of the time coordinate $t \to e^{\kappa} t$ with κ a constant. Choosing κ appropriately, one gets $\alpha + \beta = 0$. Hence

$$\beta(r) = -\alpha(r) \tag{2.1.34}$$

We have now the remaining equations

$$\begin{aligned} R_{rr} &= -\partial_r^2 \alpha - 2(\partial_r \alpha)^2 - \frac{2}{r}\partial_r \alpha = 0 \\ R_{\theta\theta} &= 1 - e^{2\alpha}(1 + 2r\partial_r \alpha) = 0 \end{aligned} \tag{2.1.35}$$

We compute from this that

$$\partial_r R_{\theta\theta} = 2re^{2\alpha} R_{rr} \tag{2.1.36}$$

Thus, at this point we only need to impose $R_{\theta\theta} = 0$ to satisfy $R_{\mu\nu} = 0$. Compute

$$\partial_r(re^{2\alpha}) = e^{2\alpha} + 2re^{2\alpha}\partial_r \alpha = e^{2\alpha}(1 + 2r\partial_r \alpha) \tag{2.1.37}$$

Hence we can rewrite $R_{\theta\theta} = 0$ as

$$\partial_r(re^{2\alpha}) = 1 \tag{2.1.38}$$

The general solution of this is $re^{2\alpha} = r - r_0$ where r_0 is a constant. Thus, we have

$$e^{2\alpha} = 1 - \frac{r_0}{r} \tag{2.1.39}$$

We have thus derived that any static and spherically symmetric line-element solving $R_{\mu\nu} = 0$ can be written on the form

$$ds^2 = -\left(1 - \frac{r_0}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{r_0}{r}} + r^2 d\Omega^2 \tag{2.1.40}$$

where r_0 is a constant.

We now fix r_0 in (2.1.40) in terms of known quantities. For $r \gg r_0$ we see that the line-element is close to that of Minkowski space (2.1.1). Indeed the correction to the

Minkowski space metric $\eta_{\mu\nu}$ is of order r_0/r. This means that we can use our results on the Newton limit for $r \gg r_0$. The Newton limit is valid provided we consider a weak gravitational field (1.3.48)-(1.3.49), that the metric is independent of time (1.3.50) and that we have small velocities (1.3.51). Thus, it is valid for an observer with $r \gg r_0$ which is moving slow compared to the speed of light. In the Newton limit we found $g_{tt} = -1-2\phi$ where ϕ is the gravitational potential (see Section 1.3.4). Hence the metric (2.1.40) for $r \gg r_0$ corresponds to the Newtonian potential

$$\phi = -\frac{r_0}{2r} \tag{2.1.41}$$

We should compare this to the Newtonian gravitational potential from a static, spherically symmetric distribution of matter by solving Poissons equation (1.6.1). For a total mass M we know from Newtonian physics that

$$\phi = -\frac{GM}{r} \tag{2.1.42}$$

where G is Newtons gravitational constant (1.6.2). Hence, by correspondence with Newtonian physics which is a good approximation for $r \gg r_0$ we identify

$$r_0 = 2MG \tag{2.1.43}$$

We have thus derived:

Schwarzschild metric: Any static and spherically symmetric line-element solving $R_{\mu\nu} = 0$ can be written on the form

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2 d\Omega^2 \tag{2.1.44}$$

with $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. Here M is the total mass as measured for $r \gg r_0 = 2GM$. The radius $r_0 = 2GM$ is called the *Schwarzschild radius* and the metric corresponding to this line-element is known as the *Schwarzschild metric*.

We note that in our formulation of the above we have not specified what sources the metric (2.1.44). Below in Section 2.1.3 we consider Newtonian matter as the source, hence the total mass M can alternatively be computed by integrating over the mass density. This is not the case in Chapter 3 where we use the metric (2.1.44) to describe black holes.

The Schwarzschild radius $r_0 = 2GM$ will be discussed extensively in Chapter 3.

In deriving the Schwarzschild metric (2.1.44) we assumed that the metric (2.1.16) is static $\partial_t\alpha = \partial_t\beta = 0$. However, actually one gets the same result without this assumption, as formulated by *Birkhoffs theorem*:

Birkhoffs theorem: Any spherically symmetric line-element solving $R_{\mu\nu} = 0$ can be written on the form (2.1.44) where M is the total mass as measured for $r \gg r_0 = 2GM$.

See Exercise 2.4 for a way to show this explicitly.

2.1.3 Metric outside planets and stars

If we for a moment consider the Schwarzschild metric (2.1.44) to be valid for all radii $r \geq 0$ then it has two radii where components of the metric goes to zero or infinity: $r = 0$ and $r = r_0 = 2GM$. As we shall see below in Section 3.1, $r = 0$ corresponds to a true curvature singularity of the space-time geometry, while the Schwarzschild radius $r = r_0$ corresponds to an event horizon. However, these two radii are not relevant for astrophysical objects such as planets and stars, as we now shall see.

Consider a star or a planet of total mass M. To a good approximation we can assume spherical symmetry of the matter distribution. Hence, since $R_{\mu\nu} = 0$ outside the matter distribution, the Schwarzschild metric (2.1.44) describes the space-time geometry and thereby the gravitational field around the planet or star.

In detail, let $r = r_\star$ be the surface radius of the planet or star (in the coordinate system of the metric (2.1.44)). Thus, the Scwarzschild metric is valid for $r \geq r_\star$. Instead for $r < r_\star$ it is not valid since one has $T_{\mu\nu} \neq 0$ and $R_{\mu\nu} \neq 0$. Now, for all stars and planets we have $r_\star \gg r_0 = 2GM$ which indeed means that the Schwarzschild radius $r_0 = 2GM$ is not physically relevant. In particular, we record

$$\begin{aligned} &\text{Sun:}\ \ r_\star \simeq 700\,000\ \text{km}\ , \quad M \simeq 2 \cdot 10^{30}\ \text{kg}\ , \quad r_0 = 2GM \simeq 3\ \text{km}\,, \\ &\text{Earth:}\ \ r_\star \simeq 6\,400\ \text{km}\ , \quad M \simeq 6 \cdot 10^{24}\ \text{kg}\ , \quad r_0 = 2GM \simeq 9\ \text{mm}\,. \end{aligned} \tag{2.1.45}$$

2.2 Geodesics

In this section we consider the geodesics of massive and massless particles moving in the background of the Schwarzschild metric (2.1.44).

2.2.1 Time-like geodesics

In the following we use this to study geodesics in the background of the Schwarzschild metric (2.1.44). Define

$$\dot{x}^\mu = \frac{dx^\mu}{d\tau}\ , \quad \ddot{x}^\mu = \frac{d^2x^\mu}{d\tau^2}\,. \tag{2.2.1}$$

Then the geodesic equation (1.3.45) for a freely falling massive particle following a time-like curve $x^\mu(\tau)$ is

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu \dot{x}^\rho = 0 \,. \tag{2.2.2}$$

The condition $g_{\mu\nu}dx^\mu dx^\nu = -d\tau^2$ for time-like curves gives

$$g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -1 \,. \tag{2.2.3}$$

We now consider (2.2.2) for the Schwarzschild metric (2.1.44). Using the components of the Christoffel symbol Eqs. (2.1.21), (2.1.23) and (2.1.24), computed in Section 2.1.2, we find

$$\Gamma^t_{tr} = \frac{1}{2}\partial_r \log\left(1 - \frac{r_0}{r}\right) \ , \ \ \Gamma^\theta_{r\theta} = \Gamma^\phi_{r\phi} = \frac{1}{r} \ , \ \ \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \ , \ \ \Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta} \,. \tag{2.2.4}$$

Note that there are also non-zero components of the form $\Gamma^r_{\mu\nu}$ but those we will not need in the following. Using (2.2.4) in (2.2.2) we find for $\mu = t, \theta, \phi$:

$$\begin{aligned} \ddot{t} + \partial_r \log\left(1 - \frac{r_0}{r}\right)\dot{t}\dot{r} &= 0 \,, \\ \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 &= 0 \,, \\ \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + \frac{2\cos\theta}{\sin\theta}\dot{\theta}\dot{\phi} &= 0 \,. \end{aligned} \tag{2.2.5}$$

We can write these equations as

$$\frac{d}{d\tau}\left[\dot{t}\left(1 - \frac{r_0}{r}\right)\right] = 0 \ , \ \ \frac{d}{d\tau}\left[r^2\dot{\theta}\right] = r^2 \sin\theta\cos\theta\dot{\phi}^2 \ , \ \ \frac{d}{d\tau}\left[r^2\sin^2\theta\dot{\phi}\right] = 0 \,. \tag{2.2.6}$$

Instead of using the geodesic equation (2.2.2) for $\mu = r$ we use the condition (2.2.3) which gives

$$1 = \left(1 - \frac{r_0}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{1 - \frac{r_0}{r}} - r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) \,. \tag{2.2.7}$$

where the 1 on the LHS comes from the RHS in (2.2.3). We now have four equations (2.2.6) and (2.2.7) to determine four functions $t(\tau)$, $r(\tau)$, $\theta(\tau)$ and $\phi(\tau)$. Thus, this is sufficient to determine the time-like geodesics of the Schwarzschild metric (2.1.44).

One can show that all geodesic motion of a single particle takes place in a plane that includes the center of mass of the source at $r = 0$. Since the metric is spherically symmetric we can choose the plane of motion to be

$$\theta = \frac{\pi}{2} \,, \tag{2.2.8}$$

without loss of generality. Indeed, starting with the initial conditions $\theta = \pi/2$ and $\dot{\theta} = 0$ one sees from the second equation of (2.2.6) that $\ddot{\theta} = 0$ which shows explicitly that one stays within the plane $\theta = \pi/2$.

The remaining three equations of (2.2.6) and (2.2.7) are

$$\frac{d}{d\tau}\left[\dot{t}\left(1-\frac{r_0}{r}\right)\right]=0\ ,\quad \frac{d}{d\tau}\left[r^2\dot{\phi}\right]=0\ ,\quad 1=\left(1-\frac{r_0}{r}\right)\dot{t}^2-\frac{\dot{r}^2}{1-\frac{r_0}{r}}-r^2\dot{\phi}^2\,. \tag{2.2.9}$$

We define

$$E=\dot{t}\left(1-\frac{r_0}{r}\right)\ ,\quad L=r^2\dot{\phi}\,. \tag{2.2.10}$$

We see from (2.2.9) that these quantities are conserved

$$\frac{dE}{d\tau}=0\ ,\quad \frac{dL}{d\tau}=0\,. \tag{2.2.11}$$

Here E is the energy per unit rest mass of the freely falling particle, while L is the angular momentum per unit rest mass. Thus, one has conservation of energy and angular momentum, just as in the case of Newtonian mechanics.

The definitions (2.2.10) of E and L are for the specific coordinate system used in the metric (2.1.44). One can define them more generally as

$$E=-g_{\mu\nu}T^\mu\dot{x}^\nu\ ,\quad L=g_{\mu\nu}J^\mu\dot{x}^\nu\,, \tag{2.2.12}$$

where $T^\mu=(1,0,0,0)$ and $J^\mu=(0,0,0,1)$ are constant vector fields in the coordinate system used in (2.1.44). One can then transform the vector fields T^μ and J^μ to other coordinate systems along with the metric to define energy and angular momentum more generally. The reason that the vectors T^μ and J^μ give conserved quantities for geodesics is because they are so-called *Killing vector fields* for the metric which means that they are associated to symmetries of the metric, in this case the symmetries of time-translation and rotations in the plane with $\theta=\pi/2$ (see Exercise 2.5 for more on this).

Inserting E and L in the third equation of (2.2.9) we get

$$1=\frac{E^2-\dot{r}^2}{1-\frac{2GM}{r}}-\frac{L^2}{r^2}\,, \tag{2.2.13}$$

where we also inserted $r_0=2GM$. One can now proceed to solve this ordinary differential equation for $r(\tau)$ and thereby obtain the time-like geodesics of the Schwarzschild metric. A trick to accomplish this is to introduce the new variable

$$X=\frac{GM}{r}\,. \tag{2.2.14}$$

Regard now $X=X(\phi)$ to be a function of the rotation angle ϕ. We have

$$\frac{dX}{d\phi}=\frac{d\tau}{d\phi}\frac{dX}{d\tau}=-\frac{GM\dot{r}}{r^2\dot{\phi}}=-\frac{GM}{L}\dot{r}\,, \tag{2.2.15}$$

Inserting this in (2.2.13) we get

$$1 = \frac{1}{1-2X}\left[E^2 - \frac{L^2}{(GM)^2}\left(\frac{dX}{d\phi}\right)^2\right] - \frac{L^2}{(GM)^2}X^2\,, \tag{2.2.16}$$

We can write this as

$$\frac{L^2}{(GM)^2}\left(\frac{dX}{d\phi}\right)^2 + 1 - 2X + \frac{L^2}{(GM)^2}X^2(1-2X) = E^2\,, \tag{2.2.17}$$

Differentiating with respect to ϕ and dividing with $dX/d\phi$ we get[19]

$$\frac{d^2X}{d\phi^2} + X - 3X^2 = \frac{(GM)^2}{L^2}\,. \tag{2.2.18}$$

One can study the solutions of (2.2.18) to obtain all the geodesics of the Schwarzschild metric (2.1.44). However, in the following we choose to focus only on the geodesics that are close to the Newtonian circular or elliptic orbits.

2.2.2 Precession of elliptic orbits

The study of orbits around massive objects is useful for many applications, ranging from the motion of the planets in our Solar System to stars that move around the black hole in the Milky Way galaxy. These orbits are to a good approximation described by Newtonian physics. However, as we shall see, General Relativity gives an important correction to the orbits that can be measured.

Newtonian approximation

Using Newtonian mechanics and Newtons force of gravity, one can write down the equation of motion for a particle moving around a point source of mass M. This gives[20]

$$\text{Newton:}\;\; \dot{r}^2 + 1 - \frac{2GM}{r} + \frac{L^2}{r^2} = E^2\,, \tag{2.2.19}$$

[19] Strictly speaking this derivation does not hold for $dX/d\phi = 0$. However, the equation of motion (2.2.18) is nevertheless valid also in this case. See footnote for (2.2.21) for a comment on this.

[20] A Newtonian analysis gives that the conserved energy is $H = \frac{1}{2}m\dot{r}^2 + m\frac{L^2}{2r^2} - \frac{GMm}{r}$ for a point particle of mass m. We can write this as $\frac{2H}{m} = \dot{r}^2 + \frac{L^2}{r^2} - \frac{2GM}{r}$ which suggests that we should identify the conserved quantities as $\frac{2H}{m} = E^2 - 1$. We can understand this relation by considering a free particle. A free particle with restmass m has relativistic energy $E_{\rm rel} = \sqrt{m^2+p^2}$ with speed of light set to one. Using $E = E_{\rm rel}/m$ this gives $E^2 = 1 + \frac{p^2}{m^2}$. The kinetic energy is $H = \frac{p^2}{2m}$ which corresponds to the Newtonian energy of a free particle. Hence we get the relation $E^2 = 1 + \frac{2H}{m}$, as promised.

where $\dot{r} = dr/dt$. We can rewrite this as

$$\text{Newton:} \quad \frac{L^2}{(GM)^2}\left(\frac{dX}{d\phi}\right)^2 + 1 - 2X + \frac{L^2}{(GM)^2}X^2 = E^2\,. \tag{2.2.20}$$

Comparing to (2.2.17) we see that the difference is the absence of the term going like X^3 on the LHS. Differentiating with respect to ϕ and dividing with $dX/d\phi$ we get[21]

$$\text{Newton:} \quad \frac{d^2X}{d\phi^2} + X = \frac{(GM)^2}{L^2}\,. \tag{2.2.21}$$

Comparing (2.2.21) to (2.2.18) we see that the extra $3X^2$ term in GR is suppressed when $X \ll 1$ and $X \ll GM/L$ corresponding to $r \gg GM$ and $r \gg L$. Moreover, inserting $X \ll 1$ in (2.2.21) we see that since the LHS is small also the RHS is small and hence $L \gg GM$. Thus, Newtonian gravity is approximately valid for the geodesic motion when

$$r \gg L \gg GM\,. \tag{2.2.22}$$

Considering (2.2.21) we find that

$$X(\phi) = \frac{(GM)^2}{L^2}\,, \tag{2.2.23}$$

is a solution. This corresponds to circular motion with constant radius

$$r = \frac{L^2}{GM}\,. \tag{2.2.24}$$

Notice for the solution we have $r \gg L$ provided we require

$$L \gg GM\,. \tag{2.2.25}$$

An elliptic orbit in Newtonian mechanics corresponds to the solution

$$X(\phi) = \frac{(GM)^2}{L^2}(1 + e\cos\phi)\,, \tag{2.2.26}$$

of Eq. (2.2.21) with $0 < e < 1$. Indeed, using (2.2.14) we find the standard formula

$$r(\phi) = \frac{(1-e^2)a}{1 + e\cos\phi}\,, \tag{2.2.27}$$

[21] This derivation of the equation of motion (2.2.21) does not work when $dX/d\phi = 0$ which is the case for a circular motion. However, a proper derivation still gives the result (2.2.21) also in this case. A way to see this is by starting with the Lagrange function $L = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\phi}^2 - \frac{GM}{r}$, write out the Lagrange equation $\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}$, and then translate this equation into the X and ϕ variables. One can also do this for the full GR equation (2.2.18) by including the extra GR terms in the Lagrange function.

for an ellipse with the semi major axis a given by

$$\frac{L^2}{GM} = (1-e^2)a\,. \tag{2.2.28}$$

Here e is the eccentricity of the ellipse which is in the interval $0 < e < 1$. For $e = 0$ we have a circle. For the Newtonian approximation (2.2.22) to be valid we need (2.2.25).

Inserting the solution (2.2.26) into the integrated Newtonian equation (2.2.20) for X one finds that it is satisfied provided

$$E^2 = 1 + \frac{(GM)^2}{L^2}(e^2-1)\,. \tag{2.2.29}$$

From this one sees that the lowest energy E is for a circle $e = 0$. In the range $0 < e < 1$ one has an ellipse. For $e \geq 1$ one has instead an unbounded and non-periodic motion which means the energy E is too high for the massive particle to be in a bounded motion.

Post-Newtonian correction

Consider the regime in which Newtonian mechanics is valid to a good approximation. For a circular or elliptic orbit this requires (2.2.25). In this regime we can get an accurate description of the orbit in General Relativity by first finding the orbit using Newtonian mechanics, and then adding a small correction due to General Relativity. We write the full General Relativity orbit as

$$X(\phi) = X_0(\phi) + X_1(\phi)\,. \tag{2.2.30}$$

Here $X_0(\phi)$ solves the Newtonian mechanics equation (2.2.21). Using (2.2.26) we write

$$X_0(\phi) = \frac{(GM)^2}{L^2}(1 + e\cos\phi)\,, \tag{2.2.31}$$

Instead $X_1(\phi)$ is the leading correction to the Newtonian orbit due to General Relativity, known as a *post-Newtonian correction*. Inserting (2.2.30) into the General Relativity equation (2.2.18) we get to first order in $X_1(\phi)$

$$\frac{d^2X_1}{d\phi^2} + X_1 = 3X_0^2\,, \tag{2.2.32}$$

Since X_0 is of order $(GM/L)^2$ this means X_1 is of order $(GM/L)^4$ and hence is a small correction in the regime (2.2.25). We impose the boundary condition that $X(\phi)$ starts at a (local) maximum at $\phi = 0$

$$\left.\frac{dX}{d\phi}\right|_{\phi=0} = 0\;, \quad \left.\frac{d^2X}{d\phi^2}\right|_{\phi=0} < 0\,. \tag{2.2.33}$$

This corresponds to a (local) minimum for the radius $r(\phi)$. Notice that the Newtonian solution (2.2.31) obeys this boundary condition.

We now solve (2.2.32) for $X_1(\phi)$. From (2.2.31) we find

$$X_0^2 = \frac{(GM)^4}{L^4}\left(1 + e^2\cos^2\phi + 2e\cos\phi\right)$$
$$= \frac{(GM)^4}{L^4}\left(1 + \frac{e^2}{2} + \frac{e^2}{2}\cos 2\phi + 2e\cos\phi\right) . \qquad (2.2.34)$$

Inserting this in Eq. (2.2.32) gives

$$\frac{d^2X_1}{d\phi^2} + X_1 = \frac{(GM)^4}{L^4}\left(3 + \frac{3e^2}{2} + \frac{3e^2}{2}\cos 2\phi + 6e\cos\phi\right) . \qquad (2.2.35)$$

Since

$$\frac{d^2(\phi\sin\phi)}{d\phi^2} + \phi\sin\phi = 2\cos\phi \ , \quad \frac{d^2(\cos 2\phi)}{d\phi^2} + \cos 2\phi = -3\cos 2\phi \, , \qquad (2.2.36)$$

we find the solution

$$X_1(\phi) = \frac{(GM)^4}{L^4}\left(3 + \frac{3e^2}{2} - \frac{e^2}{2}\cos 2\phi + 3e\,\phi\sin\phi\right) , \qquad (2.2.37)$$

that satisfies the boundary condition (2.2.33). Note that one can add a term proportional to $\cos\phi$ to this since that solves the homogenous equation $\frac{d^2X_1}{d\phi^2} + X_1 = 0$. However, this can be regarded as part of the Newtonian solution (2.2.31) by redefining e.

The term $3e\,\phi\sin\phi$ in (2.2.37) is not periodic. This has the important consequence that the elliptic orbit has a precession, *i.e.* that it makes the ellipse rotate slightly with increasing ϕ so that the orbit is not quite periodic. To see this consider

$$\frac{dX}{d\phi} = \frac{dX_0}{d\phi} + \frac{dX_1}{d\phi} . \qquad (2.2.38)$$

Without the $3e\,\phi\sin\phi$ term $X(\phi)$ would reach the next maximum for $\phi = 2\pi$ (starting from the maximum at $\phi = 0$ and increasing ϕ) since all the other terms in $X(\phi)$ are periodic with period 2π. But because of the $3e\,\phi\sin\phi$ term the maximum gets shifted. Denote the shifted angle of the maximum as $2\pi + \Delta\phi$ where $\Delta\phi$ is due to the post-Newtonian correction. Then we should impose

$$\left.\frac{dX}{d\phi}\right|_{\phi=2\pi+\Delta\phi} = \left.\frac{dX_0}{d\phi}\right|_{\phi=2\pi+\Delta\phi} + \left.\frac{dX_1}{d\phi}\right|_{\phi=2\pi+\Delta\phi} = 0 . \qquad (2.2.39)$$

We compute to leading order in the post-Newtonian correction (thus neglecting higher order corrections)

$$\left.\frac{dX_0}{d\phi}\right|_{\phi=2\pi+\Delta\phi} = -\frac{(GM)^2}{L^2}e\sin(2\pi+\Delta\phi) = -\frac{(GM)^2}{L^2}e\sin(\Delta\phi) \simeq -\frac{(GM)^2}{L^2}e\Delta\phi \, , \quad (2.2.40)$$

$$\left.\frac{dX_1}{d\phi}\right|_{\phi=2\pi+\Delta\phi} \simeq \left.\frac{dX_1}{d\phi}\right|_{\phi=2\pi} = \frac{(GM)^4}{L^4}\,\frac{d}{d\phi}(3e\phi\sin\phi)\Big|_{\phi=2\pi} = \frac{(GM)^4}{L^4}6\pi e\,. \tag{2.2.41}$$

Inserting this in (2.2.39) gives

$$\Delta\phi = \frac{6\pi(GM)^2}{L^2}\,. \tag{2.2.42}$$

Using the formula (2.2.28) for the semi major axis and reinstating the speed of light c, we find

Precession of elliptic orbits: In the regime $r \gg L \gg GM$ in which Newtonian mechanics is valid to a good approximation, the precession of an ellipse due to the leading post-Newtonian correction from General Relativity is given by

$$\Delta\phi = \frac{6\pi GM}{(1-e^2)ac^2}\,. \tag{2.2.43}$$

This angle represents the precession of the ellipse, *i.e.* how much the ellipse is shifted after what would have been a single orbit in the Newtonian approximation.

Precession of Mercury

The effect of the precession of elliptic orbits (2.2.43) means that all planets in the Solar System have a precession due to General Relativity. However, since the effect is proportional to GM/a, it is bigger when the planet is closer to the Sun. The closest planet to the Sun is Mercury. At the time Einstein developed his theory one had observations of Mercurys orbit dating back to 1765. Mercury has an orbital period of about 88 days. According to these observations it has a precession of

$$(\Delta\phi)_{\text{observed}} = 5\,601\;\frac{\text{arcsec}}{100\text{ years}}\,. \tag{2.2.44}$$

Here

$$\text{arcsec} = \frac{1}{3600}\text{degree} = \frac{2\pi}{3600\cdot 360}\,. \tag{2.2.45}$$

Using Newtonian physics one could account for

$$(\Delta\phi)_{\text{Newton}} = 5\,558\;\frac{\text{arcsec}}{100\text{ years}}\,, \tag{2.2.46}$$

from the gravitational attraction between Mercury and the other planets in the Solar System. The discrepancy between the Newtonian result (2.2.46) and the measured precession (2.2.44) was pointed out already in 1859. Using the formula (2.2.43) for the precession of Mercury, Einstein computed in 1916

$$(\Delta\phi)_{\text{GR}} = 43\;\frac{\text{arcsec}}{100\text{ years}}\,. \tag{2.2.47}$$

Amazingly, this adds up perfectly:

$$(\Delta\phi)_{\text{observed}} = (\Delta\phi)_{\text{Newton}} + (\Delta\phi)_{\text{GR}} \,. \tag{2.2.48}$$

Thus, General Relativity could explain a known discrepancy in the orbit of Mercury! At the time that Einstein published his theory in 1915-1916 this was the most important confirmation of it.

2.2.3 Deflection of light

According to Newtonian physics, light moves along straight lines. Instead in General Relativity, the path of light in a general space-time can be bended. A massive body such as the Sun curves the space-time around it, as described by the Schwarzschild metric (2.1.44). Thus, in General Relativity, the path of light is deflected when passing a massive body, due to the gravitational field around the massive body.

Light moves along null geodesics. A null geodesic is a curve $x^\mu(\lambda)$ that obeys (1.4.52). Define

$$\dot{x}^\mu = \frac{dx^\mu}{d\lambda} \ , \quad \ddot{x}^\mu = \frac{d^2x^\mu}{d\lambda^2} \,, \tag{2.2.49}$$

where λ is the affine parameter for the null geodesic $x^\mu(\lambda)$. We can then write the equations for a null geodesics (1.4.52) as

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu\dot{x}^\rho = 0 \,, \tag{2.2.50}$$

and

$$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0 \,. \tag{2.2.51}$$

We notice that Eq. (2.2.50) is the same as (2.2.2) while (2.2.51) has a 0 instead of a -1 on the RHS as compared to (2.2.3). Thus, to obtain the equations for null geodesics of the Schwarzschild metric (2.1.44) we can use the results of Section 2.2.1 if we formally identify $\dot{x}^\mu$ and $\ddot{x}^\mu$ and track where the -1 goes in the derivation so that we can put a 0 instead. Indeed, (2.2.6) is the same for a null geodesic while (2.2.7) has 0 on the LHS. It is also true for null geodesics that they move in a plane that includes $r = 0$. Hence by spherical symmetry we can again choose to be in the plane $\theta = \pi/2$. Define

$$E = \frac{dt}{d\lambda}\left(1 - \frac{r_0}{r}\right) \ , \quad L = r^2\frac{d\phi}{d\lambda} \,. \tag{2.2.52}$$

This parallels the definition (2.2.10). With this we get (2.2.13) with 0 on the LHS. Introduce the same variable X defined by (2.2.14). Then we get (2.2.16) with 0 on the

LHS, hence

$$0 = \frac{1}{1-2X}\left[E^2 - \frac{L^2}{(GM)^2}\left(\frac{dX}{d\phi}\right)^2\right] - \frac{L^2}{(GM)^2}X^2 \,. \tag{2.2.53}$$

We can write this as

$$\left(\frac{dX}{d\phi}\right)^2 + X^2(1-2X) = \frac{(GM)^2}{L^2}E^2 \,, \tag{2.2.54}$$

Differentiating with respect to ϕ and dividing with $dX/d\phi$ we get

$$\frac{d^2X}{d\phi^2} + X = 3X^2 \,. \tag{2.2.55}$$

Using Newtonian mechanics one finds instead

$$\text{Newton:} \;\; \frac{d^2X}{d\phi^2} + X = 0 \,. \tag{2.2.56}$$

which, as we shall see below, means that light moves in straight lines. We notice that the X^2 term is absent in comparison to (2.2.55), similarly to what happens for time-like geodesics. For the Newtonian equation (2.2.56) to be approximately valid, we see that we need $X \ll 1$ and hence

$$r \gg GM \tag{2.2.57}$$

This corresponds to having the null geodesics being far away from the Schwarzschild radius $r_0 = 2GM$.

We now consider the leading post-Newtonian correction to a null geodesic in the regime (2.2.57) for which Newtonian physics is approximately valid. Write the full General Relativity solution as

$$X(\phi) = X_0(\phi) + X_1(\phi) \,, \tag{2.2.58}$$

where $X_0(\phi)$ is the Newtonian contribution that solves the Newtonian mechanics equation (2.2.56) and $X_1(\phi)$ is the leading post-Newtonian correction from General Relativity, obeying to first order in X_1 the equation

$$\frac{d^2X_1}{d\phi^2} + X_1 = 3X_0^2 \,. \tag{2.2.59}$$

We impose the boundary condition (2.2.33) that $\phi = 0$ corresponds to a (local) maximum for $X(\phi)$. The solution to the Newtonian contribution is

$$X_0(\phi) = \frac{GM}{b}\cos\phi \,, \tag{2.2.60}$$

which is seen to solve (2.2.56) and obey (2.2.33). This corresponds to

$$r(\phi) = \frac{b}{\cos\phi} \,. \tag{2.2.61}$$

This parametrizes a straight line. Writing the Cartesian coordinates

$$x(\phi) = r(\phi)\cos\phi\ , \quad y(\phi) = r(\phi)\sin\phi\ , \quad z = 0\,, \tag{2.2.62}$$

we see that the Newtonian solution (2.2.61) corresponds to the line $x(\phi) = b$. This is illustrated in Figure 16. The constant b is the minimal distance from the center of the matter distribution at $r = 0$ and is known as the *impact parameter*. Hence we need $b \gg GM$ for the Newtonian approximation to be valid. While $\phi = 0$ is where one has the minimal distance to the center, one has $r \to \infty$ for $\phi \to \pm\pi/2$. This corresponds to $X_0(\phi) \to 0$ for $\phi \to \pm\pi/2$.

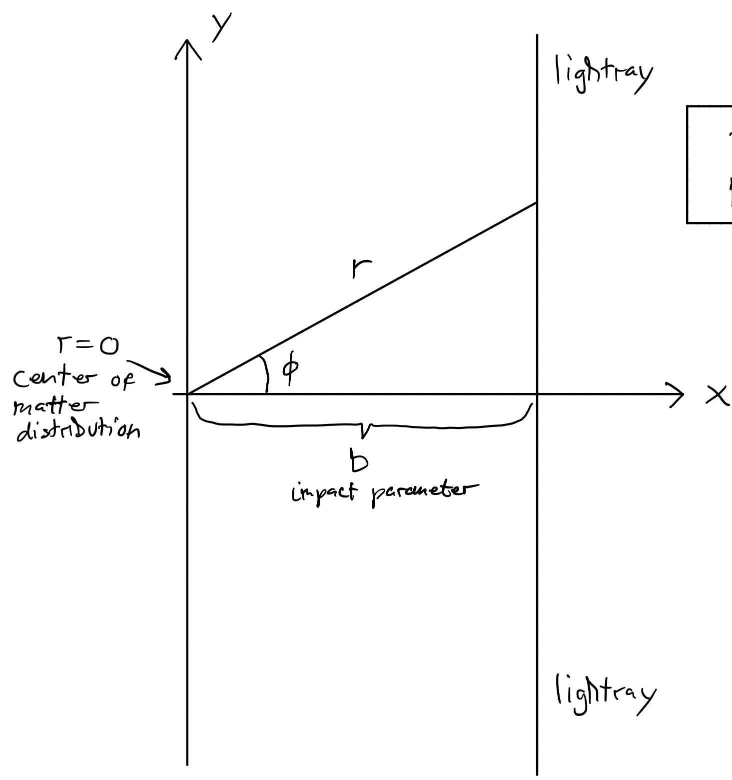

Figure 16: Light follows a straight line in Newtonian mechanics.

The post-Newtonian correction $X_1(\phi)$ obeys Eq. (2.2.59). From (2.2.60) we get

$$\frac{d^2X_1}{d\phi^2} + X_1 = \frac{3(GM)^2}{b^2}\cos^2\phi = \frac{3(GM)^2}{2b^2}(1+\cos 2\phi)\,. \tag{2.2.63}$$

Using the second relation of Eq. (2.2.36) we find

$$X_1(\phi) = \frac{(GM)^2}{2b^2}(3-\cos 2\phi)\,. \tag{2.2.64}$$

To compute the deflection angle $\Delta\phi$ that a lightray would be bend by, we should find the zeroes of $X(\phi)$. Since $X(\phi)$ is an even function, the zeroes are at $\pm(\frac{\pi}{2} + \frac{\Delta\phi}{2})$. See Figure 17 for an illustration of this using again the Cartesian coordinates (2.2.62).

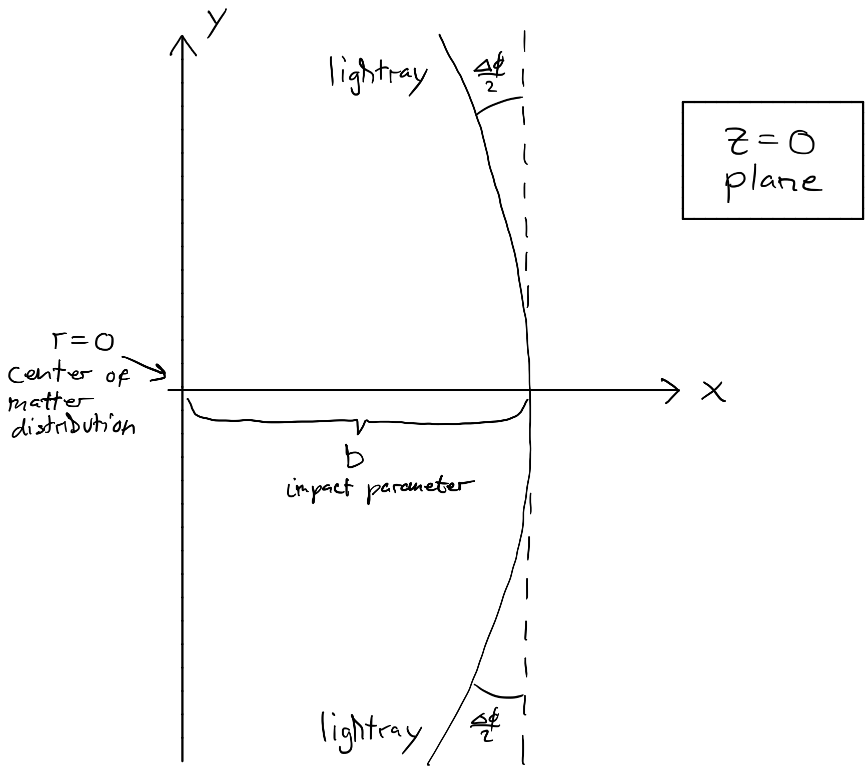

Figure 17: Illustration of the bending of light.

We compute

$$X_0\Big(\frac{\pi}{2} + \frac{\Delta\phi}{2}\Big) = \frac{GM}{b} \cos\Big(\frac{\pi}{2} + \frac{\Delta\phi}{2}\Big) \simeq -\frac{GM}{2b}\Delta\phi\,. \tag{2.2.65}$$

For $X_1(\phi)$ we need only the leading order since it is already a small correction, giving

$$X_1\Big(\frac{\pi}{2} + \frac{\Delta\phi}{2}\Big) \simeq X_1\Big(\frac{\pi}{2}\Big) = \frac{2(GM)^2}{b^2}\,. \tag{2.2.66}$$

Thus, $X(\frac{\pi}{2}+\frac{\Delta\phi}{2}) = 0$ gives $\Delta\phi = 4GM/b$. Reinstating the speed of light, we have derived:

Deflection of light: The deflection angle $\Delta\phi$ for a lightray with impact parameter b due to the gravitational field from a matter distribution with total mass M is

$$\Delta\phi = \frac{4GM}{bc^2}\,. \tag{2.2.67}$$

This is valid when $b \gg GM/c^2$. The matter distribution should be approximately spherically symmetric or have an extension that is much smaller than the impact parameter.

Experimental tests

In 1919 an experiment led by Sir Arthur Eddington and Sir Frank Watson Dyson took place involving two expeditions that sailed out from England to take pictures of the stars seen close to the Sun during the solar eclipse May 29, 1919: one expedition went to the island of Principe on the west coast of Africa, the other to the city of Sobral in northern Brazil. The idea was to take pictures of the stars close to the Sun during the solar eclipse, then take pictures of the same piece of sky in the night when the Sun is no longer there, and then compare the two sets of pictures to see if the stars have moved. Combining the results of the two expeditions, the experiment measured a deflection angle $\Delta\phi$ of 1.60 arcseconds. Using the formula (2.2.67) one finds $\Delta\phi = 1.75$ arcseconds as the prediction of General Relativity, using that M is the mass of the Sun and b is the radius of the Sun (see Exercise 2.1) [4]. This measurement was considered a major breakthrough in showing that Einsteins theory of General Relativity is right. See Figure 18 for an article in New York Times from shortly after the discovery was announced.[22]

Since then many observations of deflections of light have been observed. Typically from the deflection of light from one galaxy due to the gravitational field of another galaxy. This effect is also known as *gravitational lensing*. The most spectacular phenomenon occurs when the positions of the two galaxies relative to Earth is such that light rays from the same galaxy can reach us from several directions at once, forming a ring known as an *Einstein ring*. This is illustrated in Figure 19. The first complete ring was observed in 1998. Figure 20 is a recent observation of an Einstein ring.

[22] For more information you can read the article "The 1919 measurement of the deflection of light" by Clifford M. Will, arXiv:1409.7812, about the history of this discovery and how they measure these things today.

ECLIPSE SHOWED GRAVITY VARIATION

Diversion of Light Rays Accepted as Affecting Newton's Principles.

HAILED AS EPOCHMAKING

British Scientist Calls the Discovery One of the Greatest of Human Achievements.

Special Cable to THE NEW YORK TIMES.

LONDON, Nov. 8.—What Sir Joseph Thomson, President of the Royal Society, declared was "one of the greatest—perhaps the greatest—of achievements in the history of human thought" was discussed at a joint meeting of the Royal Society and the Royal Astronomical Society in London yesterday, when the results of the British observations of the total solar eclipse of May 29 were made known.

There was a large attendance of astronomers and physicists, and it was generally accepted that the observations were decisive in verifying the prediction of Dr. Einstein, Professor of Physics in the University of Prague, that rays of light from stars, passing close to the sun on their way to the earth, would suffer twice the deflection for which the principles enunciated by Sir Isaac Newton accounted. But there was a difference of opinion as to whether science had to face merely a new and unexplained fact or to reckon with a theory that would completely revolutionize the accepted fundamentals of physics.

The discussion was opened by the Astronomer Royal, Sir Frank Dyson, who described the work of the expeditions sent respectively to Sobral, in Northern Brazil, and the Island of Principe, off the west coast of Africa. At each of these places, if the weather were propitious on the day of the eclipse, it would be possible to take during totality a set of photographs of the obscured sun and a number of bright stars which happened to be in its immediate vicinity.

The desired object was to ascertain whether the light from these stars as it passed by the sun came as directly toward the earth as if the sun were not there, or if there was a deflection due to its presence. And if the deflection did occur the stars would appear on the photographic plates at measurable distances from their theoretical positions. Sir Frank explained in detail the apparatus that had been employed, the corrections that had to be made for various disturbing factors, and the methods by which comparison between the theoretical and observed positions had been made. He convinced the meeting that the results were definite and conclusive, that deflection did take place, and that the measurements showed that the extent of deflection was in close accord with the theoretical degree predicted by Dr. Einstein, as opposed to half of that degree, the amount that would follow if the principles of Newton were correct.

Dr. Crommelin, one of the observers at Sobral, who spoke next, said that eight exposures of twenty-eight seconds each were made during the totality of the eclipse. Seven of these plates showed seven stars in each. One showed no stars, owing to the presence of a thin cloud, but gave well-defined images of the inner corona of the sun and of great prominence. Seven exposures of the same star field were made for comparison between July 14 and July 18 in the morning sky, the sun being then 45 degrees or more away from it. The results reduced to the sun's limb were 2.08 seconds and 1.94 seconds respectively. The combined result was 1.98 seconds, with a probable error of about 6 per cent. This was a strong confirmation of Einstein's theory, which gave a shift at the limb of 1.7 seconds. The evidence in favor of the gravitational bending of light was overwhelming, and there was a decidedly stronger case for the Einstein shift than for the Newtonian one.

Though the results were fairly conclusive, Dr. Crommelin said the question of the revision of Newton's law of gravitation was one of such fundamental importance that consideration was already being given to the next total eclipse in September, 1922, visible in the Maldive Islands and Australia.

Two of the consequences of Einstein's theory, he continued, namely, the motion of Mercury's perihelion and the bending of light by gravitation, might now be looked on as established, "at least with great probability." There was, however, a third predicted consequence, which was a shift of the lines in the spectrum toward the red in a strong gravitational field. The effect in the solar spectrum would amount to one-twentieth of the Angstrum unit, the same as that due to a motion of one-half kilometer per second away from the sun. Dr. St. John had looked for this effect without success. If this failure were taken as final it would mean that parts of Einstein's theory would need revision, but the parts already verified would remain.

The effects on practical astronomy, Dr. Crommelin said, of the verification of Einstein's theory were not very great. It was chiefly in the field of philosophic thought that the change would be felt. Space would no longer be looked on as extending indefinitely in all directions. Euclidian straight lines could not exist in Einstein's space. They would all be curved, and if they traveled far enough they would regain their starting point.

Sir Joseph Thomson, summing up the discussion, said:

"These are not isolated results that have been obtained. It is not the discovery of an outlying island, but of a whole continent of new scientific ideas of the greatest importance to some of the most fundamental questions connected with physics. It is the greatest discovery in connection with gravitation since Newton enunciated that principle."

Figure 18: Article in The New York Times, November 9, 1919 (page 6).

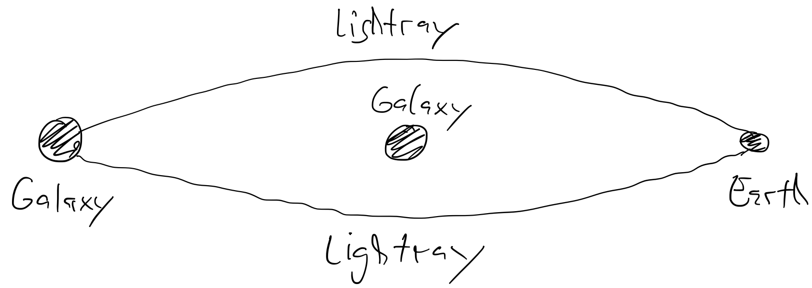

Figure 19: Illustration of bending of light leading to an Einstein ring.

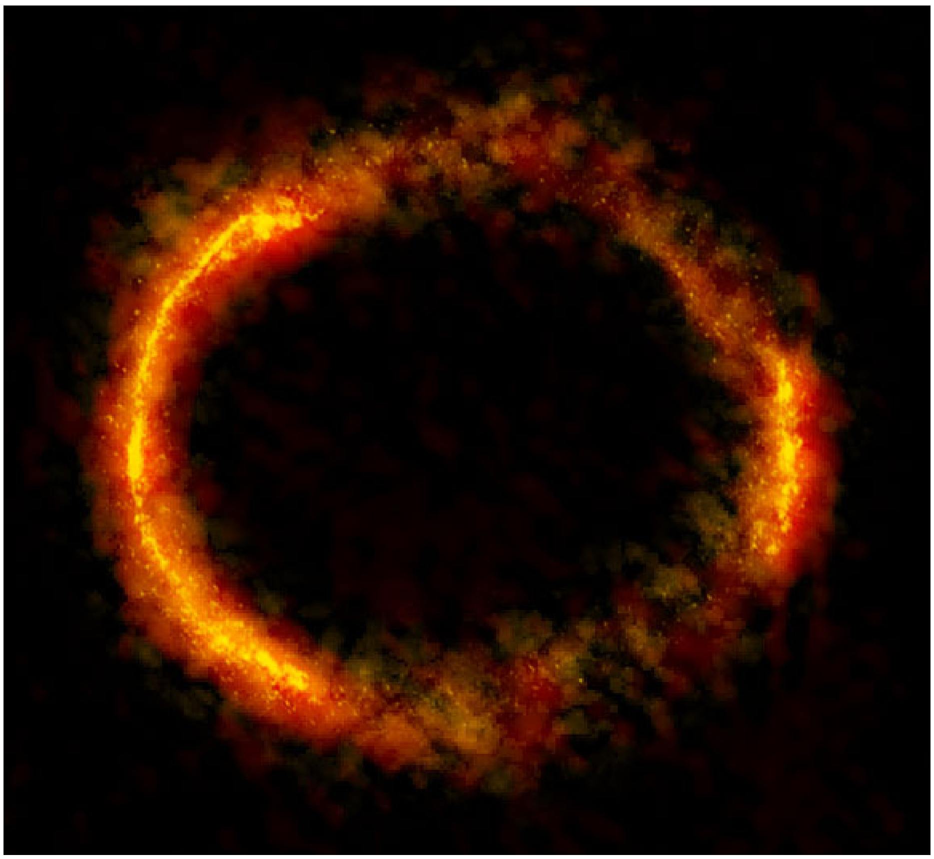

Figure 20: Picture of an observed Einstein ring for the gravitationally lensed galaxy SDP.81 taken by the Atacama Large Millimeter/submillimeter Array (ALMA) in 2015.

2.3 Exercises for Chapter 2

Exercise 2.1. Deflection angle for the Sun.
Compute the deflection angle for light passing close to the Sun. You can use that the radius of the Sun is 696 000 km and the mass of the Sun is $M_\circ = 2.0 \cdot 10^{30}$ kg.

Exercise 2.2. The orbit of star S2 around Sagittarius A*.

The star S2 is observed to have an elliptic orbit around Sagittarius A* (Sgr A*) located in the center of the Milky Way galaxy. Sgr A* is believed to be a supermassive black hole with mass $M = 4.2 \cdot 10^6 M_\circ$ where $M_\circ = 2.0 \cdot 10^{30}$ kg is the mass of our Sun. We assume below that it is a Schwarzschild black hole. The star S2 has a period $T = 15.6$ years, eccentricity $e = 0.88$ and semi-major axis $a = 1.4 \cdot 10^{14}$ meter.

- What is the Schwarzschild radius of Sgr A*?
- The precession of an elliptic orbit due to the leading post-Newtonian correction of General Relativity is
$$\Delta\phi = \frac{6\pi GM}{(1-e^2)ac^2} \,. \tag{2.3.1}$$
What is the precession angle $\Delta\phi$ of the elliptic orbit of S2? What is it in units of arcminutes per orbit? What is it in units of arcseconds per 100 years (*i.e.* the same units as used for Mercurys precession)?
- How close does the star S2 comes to the event horizon of Sgr A*? Is the approximation used above warranted (*i.e.* that the post-newtonian correction is small)?

  [Hint: You can find the minimal radius using Eq. (2.2.27).]

Exercise 2.3. Circular orbits for Schwarzschild metric.
We consider here the circular orbits for freely falling massive particles moving in the background of the Schwarzschild metric (2.1.44).

- Consider the geodesic equation (2.2.17). We can interpret this equation as a particle with unit mass moving in one dimension parametrized by X with velocity $dX/d\phi$ and total energy $\mathcal{E}$, hence obeying
$$\frac{1}{2}\left(\frac{dX}{d\phi}\right)^2 + V(X) = \mathcal{E} \,. \tag{2.3.2}$$
What are $\mathcal{E}$ and $V(X)$ such that we can write (2.2.17) in this form?
- Using your intuition and knowledge from classical mechanics, argue that for circular orbit one needs to find a radius r_c corresponding to an $X_c = GM/r_c$ such that
$$\left.\frac{dV(X)}{dX}\right|_{X=X_c} = 0 \,. \tag{2.3.3}$$
- Argue furthermore that for a stable circular orbit, one needs
$$\left.\frac{d^2V(X)}{dX^2}\right|_{X=X_c} > 0 \,. \tag{2.3.4}$$

Here a stable circular orbit means that a small perturbation of the circular motion remains small.

- Find the radii for which we have (2.3.3). What is the condition on L ensuring that this equation has solutions?

- Find a radius that both satisfies (2.3.3) and (2.3.4). What is the condition on L ensuring that a stable circular orbit is possible?

- How close to the Schwarzschild radius can a stable circular orbit be?

- Compare now all the above answers to what one obtains starting with the Newtonian geodesic equation (2.2.20). What are the differences?

Exercise 2.4. Birkhoffs theorem.
Assuming a spherically symmetric space-time, we have shown in Section 2.1.1 that the line-element can be written in the form (2.1.16). The Christoffel symbol for this line-element has components

$$\begin{gathered}
\Gamma^t_{tt} = \partial_t \alpha \ , \quad \Gamma^t_{tr} = \partial_r \alpha \ , \quad \Gamma^t_{rr} = e^{2\beta - 2\alpha} \partial_t \beta \, , \\
\Gamma^r_{tt} = e^{2\alpha - 2\beta} \partial_r \alpha \ , \quad \Gamma^r_{tr} = \partial_t \beta \ , \quad \Gamma^r_{rr} = \partial_r \beta \ , \quad \Gamma^r_{\theta\theta} = -r e^{-2\beta} \ , \quad \Gamma^r_{\phi\phi} = -r e^{-2\beta} \sin^2\theta \, , \\
\Gamma^\theta_{r\theta} = \frac{1}{r} \ , \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \, , \\
\Gamma^\phi_{r\phi} = \frac{1}{r} \ , \quad \Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta} \, .
\end{gathered} \tag{2.3.5}$$

Below we solve the vacuum Einstein equations $R_{\mu\nu} = 0$ for the spherically symmetric line-element (2.1.16).

- Calculate the three components Γ^t_{tt}, Γ^t_{tr} and Γ^t_{rr} and check that they are the same as in (2.3.5).

- Using the general formula for the Ricci tensor (2.1.28) compute

  $$R_{tr} = \frac{2}{r} \partial_t \beta \, . \tag{2.3.6}$$

  Argue that this means

  $$\beta = \beta(r) \, . \tag{2.3.7}$$

- Compute

  $$R_{\theta\theta} = 1 + e^{-2\beta} (r \partial_r \beta - r \partial_r \alpha - 1) \, . \tag{2.3.8}$$

Show using (2.3.7) that this means $\partial_t \partial_r \alpha = 0$ and hence

$$\alpha(r,t) = f(r) + g(t)\,. \tag{2.3.9}$$

- Argue that one can absorb the function $g(t)$ into a redefinition of the time-coordinate in the line element (2.1.16). Hence we end up with the line element (2.1.17) that was our starting point for the derivation of the Schwarzschild metric in Section 2.1.2.

  Comment: In conclusion, if we combine the above result with the derivation of the Schwarzschild metric in Section 2.1.2 we have shown that if one starts with a general spherically symmetric line-element (2.1.16), with a possible time-dependence included, then the vacuum Einstein equations still lead to the Schwarzschild metric (2.1.44). This proves Birkhoffs theorem mentioned in Section 2.1.2.

Exercise 2.5. Killing vector fields and conservation of energy.
Consider a time-like geodesic $x^\mu(\tau)$ and a vector field $T^\mu(x)$ in a space-time with line-element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. Given this, we consider the scalar field

$$E = -T^\mu g_{\mu\nu}\frac{dx^\nu}{d\tau}\,. \tag{2.3.10}$$

In this exercise we consider under which circumstances E is a conserved quantity. The results of this exercise are generalizing the statement that E defined by (2.2.10) is invariant for the Schwarzschild metric (2.1.44).

- Show by applying $\frac{D}{d\tau}$ on both sides of the equation that

  $$\frac{dE}{d\tau} = -\frac{1}{2}(D_\mu T_\nu + D_\nu T_\mu)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\,. \tag{2.3.11}$$

  From this it follows that if the vector field T^μ obeys the so-called *Killing equation*

  $$D_\mu T_\nu + D_\nu T_\mu = 0\,, \tag{2.3.12}$$

  then E is constant along the geodesic. A vector field T^μ that obeys (2.3.12) is known as a *Killing vector field.*

- Consider a given vector field $T^\mu(x)$. Argue that if E, as defined in Eq. (2.3.10), is a conserved quantity for any geodesic $x^\mu(\tau)$ then T^μ is a Killing vector field.

- Consider a vector field $T^\mu = (1,0,0,0)$. Show that the Killing equation (2.3.12) reduces to $\partial_0 g_{\mu\nu} = 0$. Argue that this means we have rederived the result of Section 2.2.1 that E defined by (2.2.10) is a conserved quantity for a geodesic in the Schwarzschild metric.

Exercise 2.6. Acceleration at the surface of the Earth.

Use the covariant formula for the acceleration (1.4.42) to compute the direction and the magnitude of the acceleration of somebody standing still on the surface of the Earth. You can use that the radius of the Earth is 6 370 km and the mass is $5.97 \cdot 10^{24}$ kg.

3 Black Holes

3.1 Schwarzschild Black Hole

In Section 2.1 we found that the unique line-element with spherical symmetry solving the vacuum Einstein equations is (2.1.44). The metric of the line-element (2.1.44), called the Schwarzschild metric, is valid outside a given spherically symmetric matter distribution with total mass M. We assumed in Chapter 2 that the radius of the matter distribution exceeded the Schwarzschild radius $r_0 = 2GM$. In this section we shall explore what happens if this is not the case, *i.e.* that the matter distribution is inside the Schwarzschild radius r_0.

3.1.1 Event horizon

We assume in the following that the Schwarzschild metric (2.1.44) is valid for all r, *i.e.* that the vacuum Einsteins equations are valid everywhere (apart from at $r = 0$, see Section 3.1.3 below). Considering the Schwarzschild metric (2.1.44), we see that approaching the Schwarzschild radius $r \to r_0 = 2GM$ the time component of the metric goes to zero $g_{tt} \to 0$ while the radial component blows up $g_{rr} \to \infty$. In this section we shall explore the physical meaning of this.

Infalling spaceship

Suppose we are watching from far away (*i.e.* $r \gg r_0$) a spaceship that falls freely towards the center of the Schwarzschild metric. Write the time-like curve of the spaceship as $x^\mu(\tau)$ where τ is its proper time. As far away observers we measure time according to the coordinate time t since the metric for $r \gg r_0$ is approximately the Minkowski metric. Instead the spaceship measures a proper time τ determined by its motion. We can assume $\theta = \pi/2$ without loss of generality. Hence, we can use the results on geodesics of Section 2.2.1. From (2.2.10) we have the conserved quantity

$$E = \frac{dt}{d\tau}\left(1 - \frac{r_0}{r}\right) , \tag{3.1.1}$$

which is the energy per unit rest mass of the spaceship. We can rewrite this as

$$d\tau = \frac{1 - \frac{r_0}{r}}{E} dt \,. \tag{3.1.2}$$

We see from this that as the spaceship approaches the Schwarzschild radius r_0 the proper time τ of the spaceship goes slower and slower compared to the coordinate time t which is the proper time for us as observers,

$$\frac{d\tau}{dt} \to 0 \quad \text{for} \quad r \to r_0 \,. \tag{3.1.3}$$

In fact, for $r = r_0$ we have $d\tau = 0$ which means that time is standing still. However, as we shall see below, the spaceship will actually never reach $r = r_0$.

To consider this in more detail, we assume for simplicity that the spaceship follows a time-like radial geodesic, *i.e.* one with fixed angles θ and ϕ. Thus,

$$\frac{d\theta}{d\tau} = \frac{d\phi}{d\tau} = 0 \,. \tag{3.1.4}$$

Moreover, we impose for convenience here the initial condition on the freely falling motion that for $r/r_0 \to \infty$ the proper time τ should be equal to the coordinate time t since one approaches Minkowski space in this limit. This means that $E = 1$ which in fact corresponds to the statement in Special Relativity that energy is equal to mc^2 (if one reinstates the speed of light c). Thus, since $E = 1$ we have

$$\frac{dt}{d\tau} = \frac{1}{1 - \frac{r_0}{r}} \,. \tag{3.1.5}$$

Using this in (2.2.7) we find

$$\frac{dr}{d\tau} = -\sqrt{\frac{r_0}{r}} \,, \tag{3.1.6}$$

where the minus sign is because r is decreasing with time. In Section 3.1.5 we consider freely falling motion with $E \neq 1$ as well, corresponding to more general initial conditions.

Use now (3.1.5) and (3.1.6) to compute

$$\frac{dr}{dt} = \frac{dr}{d\tau}\frac{d\tau}{dt} = -\left(1 - \frac{r_0}{r}\right)\sqrt{\frac{r_0}{r}} \,. \tag{3.1.7}$$

For $r \simeq r_0$ this gives

$$\frac{dr}{dt} \simeq -1 + \frac{r_0}{r} \simeq -\frac{r - r_0}{r_0} \,. \tag{3.1.8}$$

Hence, once the spaceship is close to the Schwarzschild radius, we have

$$r(t) - r_0 \simeq C \exp\left(-\frac{t}{r_0}\right) \,, \tag{3.1.9}$$

where C is a constant. This shows that it takes an infinite time for $r(t)$ to reach r_0, as seen from point of view of the far away observers which measures proper time using t. Thus, as far away observers we will indeed never see the spaceship reach $r = r_0$.

If we instead consider the spaceship point of view we should use the proper time τ. From (3.1.6) we get $\sqrt{r}dr = -\sqrt{r_0}d\tau$ and hence

$$\frac{2}{3}\left(r^{\frac{3}{2}} - r_0^{\frac{3}{2}}\right) = -\sqrt{r_0}\tau\,, \tag{3.1.10}$$

where $r(0) = r_0$. This shows that from the spaceship point of view, the spaceship reaches the Schwarzschild radius r_0 in a finite time. This is quite a remarkable consequence of the idea that time is an observer-dependent concept in General Relativity. Depending on who is observing, one either reaches the Schwarzschild radius in finite time, or one never reaches it.

The Schwarzschild radius $r = r_0$ in the metric (2.1.44) defines a sphere in the space-time (for fixed t). We call this sphere the *event horizon* because in a certain sense it works similarly to a horizon on Earth: due to the curvature of the surface of the Earth, one cannot see further than the horizon in the distance. However, moving towards it, one is able to travel beyond the point that one saw as the horizon.

We call the part of space-time that is inside and on the event horizon a *black hole*. The reason for the name is that nothing, not even light, can escape from being behind or at the event horizon.[23] This will be shown below in Section 3.1.2. We denote the black hole associated with the Schwarzschild line-element (2.1.44) a *Schwarzschild black hole*, in order to distinguish it from other black holes that we shall discuss below in Section 3.3.

Closing of light cones

Light travels along null curves, and nothing can move faster than the speed of light. Hence it is interesting to examine the consequences of the event horizon at $r = r_0$ for null curves.

Consider for simplicity only radial null curves. Then $d\theta = d\phi = 0$ and we have

$$\left(1 - \frac{r_0}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{r_0}{r}} = 0\,. \tag{3.1.11}$$

Hence, we get

$$\frac{dr}{dt} = \pm\left(1 - \frac{r_0}{r}\right)\,, \tag{3.1.12}$$

where the plus (minus) sign is for outgoing (infalling) radial null curves. Far away from the event horizon $r \gg r_0$ this means we have $dr \simeq \pm dt$. Thus, in the (t, r) coordinates the speed of light is one. But when r is close to r_0, we see that the apparent speed of light in the (t, r) coordinates is diminished, and at $r = r_0$ it is zero. We have illustrated

[23] This is according to the theory of General Relativity. It is conceivable that in a quantum theory of gravity information from behind the event horizon is allowed to escape to the outside.

this in Figure 21 in terms of the light cones in the (t, r) coordinates. As one can see in this figure, the light cones close up near the event horizon at $r = r_0$.

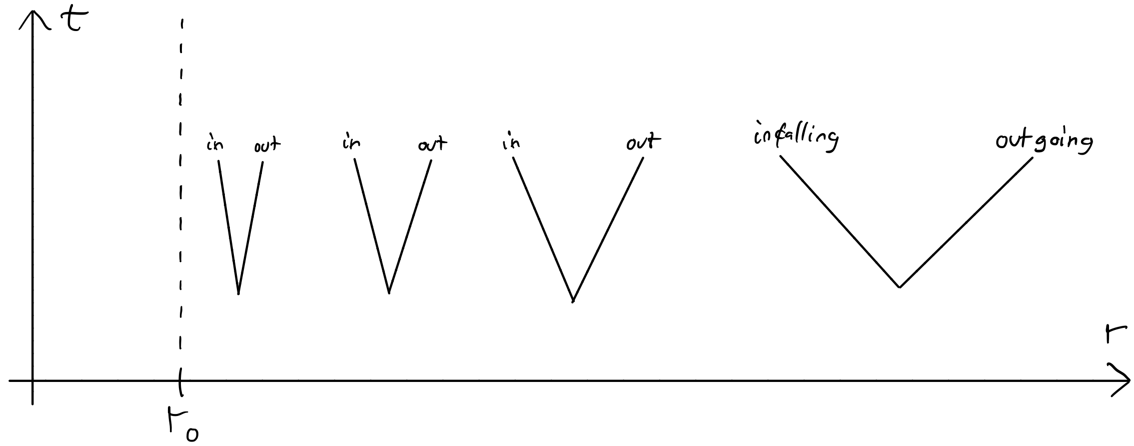

Figure 21: Illustration of lightcones in the t and r coordinates of the Schwarzschild metric (2.1.44).

How should we interpret the closing of the light cones? It suggests that light cannot move radially at $r = r_0$ and move very slowly near $r = r_0$. This means that in the (t, r) coordinates, it is both difficult to reach the event horizon with a light ray, and to escape the event horizon with a light ray. Indeed, for the infalling radial null curves it takes infinite time t to reach the event horizon. This follows by using (3.1.8) and (3.1.9) with exact equal signs replacing the approximate equal signs. This fits well with our conclusion above that ingoing radial time-like geodesics also takes infinite time t to reach the event horizon. However, we saw also that in terms of the proper time τ of the spaceship it actually reaches and passes through the event horizon in a finite time. As we shall see below in Section 3.1.2, a similar phenomena is true for an infalling radial light ray, once we use a different time coordinate than t.

Turning to outgoing radial null curves, we show below in Section 3.1.2 that they are unable to pass through, or leave, the event horizon. This is true regardless of what time coordinate one uses.

Note that one can show similar behavior of general null curves. For a general null curve in the Schwarzschild metric (2.1.44) it is straightforward to derive

$$\frac{dr^2}{dt^2} = \left(1 - \frac{r_0}{r}\right)^2 - \left(1 - \frac{r_0}{r}\right) r^2 \left(\frac{d\theta^2}{dt^2} + \sin^2 \frac{d\phi^2}{dt^2}\right) . \tag{3.1.13}$$

Hence, we get that $dr/dt = 0$ on the event horizon is also true when including the sphere part.

3.1.2 Eddington-Finkelstein coordinates

As we have seen above, strange phenomena appears near the event horizon $r = r_0$ when using the (t, r, θ, ϕ) coordinates. These coordinates are appropriate to use for an observer far away, as t for $r \gg r_0$ would correspond to a proper time for someone with $dr = d\theta = d\phi$. Such an observer would see that massive particles, as well as light, would take infinite time to reach the event horizon. Therefore, to be able to show that ingoing light rays actually can reach and pass through the event horizon, we need to find a new coordinate system. This will also show us that outgoing light rays cannot pass through the event horizon. In other words, we need to find a new set of coordinates in which the light cones do not close up, as in Figure 21.

An issue that is closely connected to this is that the Schwarzschild metric (2.1.44) behaves singularly at the event horizon, since $g_{tt} \to 0$ and $g_{rr} \to \infty$. Does this mean that space-time geometry is singular at $r = r_0$? This would be at odds with the fact that the spaceship that we discussed above passes $r = r_0$ unscathed. We will show in this section that the singular behavior of the Schwarzschild metric (2.1.44) is an artifact of the particular coordinate system used, rather than a singular behavior of the space-time geometry itself. This is again addressed by finding a new set of coordinates.

To find our new coordinates, we first define the so-called *tortoise radial coordinate* $r^\star$ as[24]

$$r^\star(r) = r + r_0 \log \left| \frac{r}{r_0} - 1 \right| . \tag{3.1.14}$$

We compute

$$dr^\star = \frac{dr}{1 - \frac{r_0}{r}} . \tag{3.1.15}$$

Hence in terms of $r^\star$ the radial null curves (3.1.12) obey

$$r^\star = \pm t + \text{constant} . \tag{3.1.16}$$

Define now the u and v coordinates as

$$u = t - r^\star \ , \quad v = t + r^\star . \tag{3.1.17}$$

Then

$$\begin{aligned} &v = \text{constant: infalling radial null curve} , \\ &u = \text{constant: outgoing radial null curve} . \end{aligned} \tag{3.1.18}$$

[24] Here we defined the numerical value as $|x| = x$ for $x \geq 0$ and $|x| = -x$ for $x < 0$. We put the argument in the logarithm with the numerical value since we shall later apply this formula also for $r < r_0$.

We are now ready to define our new set of coordinates for the Schwarzschild metric (2.1.44). The new coordinates are the *Eddington-Finkelstein coordinates* (v, r, θ, ϕ). To write the Schwarzschild metric in these new coordinates, we should make a coordinate transformation from the old (t, r, θ, ϕ) coordinates. To accomplish this, we record

$$dt = dv - dr^\star = dv - \frac{dr}{1 - \frac{r_0}{r}}\,, \tag{3.1.19}$$

since $t = v - r^\star$. Inserting this in the line-element (2.1.44) one finds the line-element

$$ds^2 = -\left(1 - \frac{r_0}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2\,. \tag{3.1.20}$$

This is the Schwarzschild metric in Eddington-Finkelstein coordinates.

We first observe that the metric in Eddington-Finkelstein coordinates (3.1.20) does not break down at $r = r_0$. To see this, set $r = r_0$ to find

$$g_{\mu\nu}|_{r=r_0} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r_0^2 & 0 \\ 0 & 0 & 0 & r_0^2 \sin^2\theta \end{pmatrix}. \tag{3.1.21}$$

Thus, seen as a four by four matrix, the metric is both finite and invertible. Hence the Schwarzschild line-element in Eddington-Finkelstein coordinates (3.1.20) is regular (*i.e.* non-singular) at $r = r_0$, unlike in the coordinates used in (2.1.44). Therefore, we conclude that the apparent breakdown of the metric (2.1.44) at $r = r_0$ is due to a *coordinate singularity* and not a genuine singularity in the space-time geometry.

We can now further examine radial null curves in this new coordinate system. From (3.1.18) we have that $dv = 0$ for infalling radial null curves. Instead outgoing radial null curves have $dt = dr^\star = dr/(1 - \frac{r_0}{r})$ and hence $dv = dt + dr^\star = 2dr^\star = 2dr/(1 - \frac{r_0}{r})$. Thus, in Eddington-Finkelstein coordinates radial null curves are given by

$$\frac{dv}{dr} = \begin{cases} 0 & \text{for infalling}\,, \\ \frac{2}{1 - \frac{r_0}{r}} & \text{for outgoing}\,. \end{cases} \tag{3.1.22}$$

One sees immediately from this that the lightcones no longer close up in Eddington-Finkelstein coordinates. We have illustrated this in Figure 22.

It is clear from (3.1.22) and Figure 22 that infalling radial light rays actually do pass through the event horizon, as we have anticipated above. Furthermore, we see that outgoing radial light rays actually cannot escape the event horizon, nor pass through it. This is in fact true for all outgoing light rays, not just the radially directed ones. Thus,

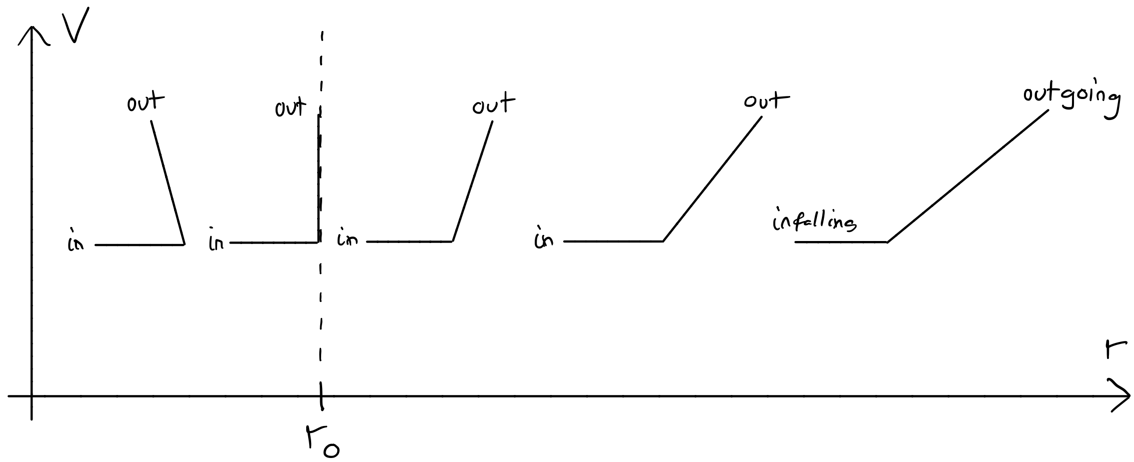

Figure 22: Illustration of lightcones in the Eddington-Finkelstein coordinates v and r.

light cannot escape from the black hole. Since nothing travels faster than light this means that nothing can escape the black hole. Therefore, an outside observer will never be able to know what goes on inside a black hole.[25]

3.1.3 Inside the black hole

We have shown above that the Eddington-Finkelstein coordinates (v, r, θ, ϕ) are well-behaved across the event horizon at $r = r_0$. This leads to the conclusion that the metric (3.1.20) is valid not only for $r \geq r_0$ but also for $r \leq r_0$. Thus, using the metric (3.1.20) we can actually get a peek at what is behind the event horizon of the black hole. Note that we assume in this section that one does not have any matter distribution inside the black hole, except very close to $r = 0$. We consider the validity of this assumption in the end of the section.

The radial null curves in Eddington-Finkelstein coordinates, illustrated by Figure 22, show that inside the black hole, *i.e.* for $r < r_0$, a light ray can only decrease the radial r coordinate. This is in contrast to outside a black hole where r can both decrease and increase. Since a massive particle travels slower than the speed of light, a radial time-like curve must also have decreasing r inside the black hole. Thus, no matter what, one has to travel towards the center of the black hole.

It is interesting to further consider the inside of the black hole in different coordinates. Define for $r < r_0$ the coordinate t as

$$t(v, r) = v - r - r_0 \log\left(1 - \frac{r}{r_0}\right) \tag{3.1.23}$$

[25] Again, here we ignore effects of a quantum theory of gravity.

in accordance with (3.1.14) and (3.1.17). One finds

$$dt = dv - \frac{dr}{1 - \frac{r_0}{r}} \tag{3.1.24}$$

Inserting this in the metric (3.1.20) we get back the Schwarzschild metric (2.1.44). Thus, we have shown that the Schwarzschild metric (2.1.44) is actually also valid for $r < r_0$.[26]

Using now the Schwarzschild metric (2.1.44) we see in these coordinates that the g_{tt} and g_{rr} terms both changes sign when we enter the black hole. Clearly this means that t cannot be a time-coordinate anymore, since a curve with fixed (r, θ, ϕ) would have $ds^2 > 0$ and hence not be time-like. Instead a curve with fixed (t, θ, ϕ) is time-like since $ds^2 < 0$. Thus, r can be used as a time-coordinate inside the black hole. Since all time-like or null curves are decreasing in r, one gets that decreasing r corresponds to moving forward in time, just like increasing t does outside the black hole. From this one deduce that no matter what one does inside a Schwarzschild black hole, one can only move towards the center of the black hole at $r = 0$.

Using (3.1.10) one sees that a massive particle following a geodesic would take a finite proper time to reach $r = 0$. Indeed, starting from $r = r_0$ it would take the proper time

$$\tau = \frac{2}{3} r_0 \,, \tag{3.1.25}$$

to reach the center $r = 0$. Since a geodesic maximizes the proper time, this is an upper bound on how long it takes to travel from the event horizon to the center of the Schwarzschild black hole.[27]

Consider now the center $r = 0$ of the Schwarzschild black hole. We see that the metric behaves in a singular way both in the coordinates of the Schwarzschild metric (2.1.44) as well as in the Eddington-Finkelstein coordinates (3.1.20). This is because there is a singularity in the space-time geometry at $r = 0$, thus all coordinate systems break down at $r = 0$. To show this, we need to construct a scalar field out of the Riemann curvature tensor (1.5.10) that diverges at $r = 0$ since that would show that the geometry is singular irrespective of what coordinate system we describe it in. One finds in the coordinates of the Schwarzschild metric (2.1.44)

$$R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \frac{12 r_0^2}{r^6} \,. \tag{3.1.26}$$

[26]It is important to note that to connect $r > r_0$ and $r < r_0$ one needs strictly speaking to use coordinates without a coordinate singularity at $r = r_0$ such as the Eddington-Finkelstein coordinates. Of course, some computations might very well work fine even if one disregard this.

[27]Note that the proper time (3.1.25) depends on the initial conditions of the freely falling motion from $r = r_0$ to $r = 0$ (see Section 3.1.5). Indeed if one starts with $dr/d\tau = 0$ at $r = r_0$ one would get a longer proper time. Thus, (3.1.25) is only an upper bound given that one has the same initial velocity at $r = r_0$.

This is clearly a scalar field made out of the Riemann curvature tensor, and it diverges for $r \to 0$. Thus, we conclude that the space-time geometry of the Schwarzschild line-element (2.1.44) has a curvature singularity at $r = 0$.

Using the so-called geodesic deviation equation one can show that tidal forces inside the black hole would pull apart any extended object such as an astronaut or a spaceship before one reaches $r = 0$. This effect is increased for smaller black holes. For sufficiently large Schwarzschild black holes one can survive passing the event horizon without being torn apart. So in this case one survives a bit inside the black hole before reaching certain death.

So far we have assumed in this section that one does not have any spherically symmetric matter distribution for $r > 0$. However, even if one did, the above considerations show that such matter would shrink towards $r = 0$. Note that if one is in a spaceship inside the black hole it would not help if there is matter closer to the center that possibly could smoothen out the singularity, since that matter would shrink away before one reaches the singularity at the center (in tiny pieces).

Finally, note that one of the biggest question marks in modern theoretical physics is if the picture of General Relativity of what is inside a black hole is true. There are several new ideas and theories that points towards that General Relativity breaks down already around the event horizon of a black hole. This is in accordance with the fact that according to General Relativity, one cannot communicate anything from inside the black hole to the outside. Thus, if quantum gravity effects kicks in already near the event horizon this would not alter significantly the physics outside the event horizon. But it could fundamentally change our understanding of what happens inside a black hole and what a black hole is made of. We shall briefly come back to these questions below in Section 3.5.3.

3.1.4 Global time coordinate

As we have seen above in Section 3.1.3, the coordinate t of the Schwarzschild metric (2.1.44) is only a time coordinate outside the event horizon since a curve with fixed r, θ and ϕ is only time-like for $r > r_0$. Inside the event horizon it is instead r that can be seen as a time-coordinate. Thus, the Schwarzschild metric (2.1.44) in the (t, r, θ, ϕ) coordinates does not have a global time coordinate, *i.e.* a coordinate that can be used as a time coordinate everywhere in the space-time. Also for the Eddington-Finkelstein coordinates (v, r, θ, ϕ) with metric (3.1.20) there is no global time coordinate. In this section we consider a new coordinate system for the Schwarzschild space-time with a global time coordinate that works both inside and outside the event horizon. This clarifies the fact that the direction of the flow of time is well-defined for the Schwarzschild black hole.

The new coordinates are called *Lemaître coordinates* (T, R, θ, ϕ). They are defined in terms of the Eddington-Finkelstein coordinates (v, r, θ, ϕ) as

$$T(v,r) = v - r + 2\sqrt{r_0 r} - 2r_0 \log\left(1 + \sqrt{\frac{r}{r_0}}\right) \;,\quad R(v,r) = \frac{2r^{\frac{3}{2}}}{3\sqrt{r_0}} + T(v,r)\,. \quad (3.1.27)$$

The angles θ and ϕ are the same as in (3.1.20). Notice that $r \geq 0$ implies

$$T \leq R\,. \quad (3.1.28)$$

From (3.1.27)

$$dT = dv - \frac{dr}{1 + \sqrt{\frac{r_0}{r}}} \;,\quad dR = dT + \sqrt{\frac{r}{r_0}}dr\,. \quad (3.1.29)$$

Using this in (3.1.20), one finds that the Schwarzschild metric in Lemaître coordinates becomes

$$ds^2 = -dT^2 + \frac{r_0}{r}dR^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \quad (3.1.30)$$

where $r = r(T, R)$ seen as a function of the new coordinates T and R is

$$r(T,R) = \left(\frac{3}{2}\sqrt{r_0}(R - T)\right)^{\frac{2}{3}}\,. \quad (3.1.31)$$

This follows from the second equation of (3.1.27). We see that the the metric (3.1.30) has T as time-coordinate both inside, on, and outside the event horizon since all curves with fixed R, θ and ϕ are time-like. Thus, T is a global time-coordinate. Correspondingly, R, θ and ϕ are always spatial coordinates.

In terms of the original coordinates (t, r, θ, ϕ) for the Schwarzschild metric we find

$$dR = dt + \sqrt{\frac{r}{r_0}}\frac{dr}{1 - \frac{r_0}{r}}\,. \quad (3.1.32)$$

This can be obtained by employing

$$dv = dt + \frac{dr}{1 - \frac{r_0}{r}}\,. \quad (3.1.33)$$

We can use (3.1.32) to give an interpretation of the Lemaître coordinates. Consider having R, θ and ϕ constant. Comparing (3.1.32) for $dR = 0$ with (3.1.7) we see that they give the same equation for dr/dt. Hence a motion with R, θ and ϕ constant corresponds to a free fall along a radial time-like geodesic with $E = 1$. Moreover, for such a motion we have $ds^2 = -dT^2$ which means T corresponds to the proper time. Thus, the coordinate system

(T, R, θ, ϕ) follows observers which fall freely along radial geodesics with $E = 1$, *i.e.* the same type of radial motion that we have explored in Section 3.1.1. The time-coordinate T thus measures the proper time for such observers.

The location of the singularity $r = 0$ is at $R = T$. Thus, for a motion with constant R, θ and ϕ, $T = R$ is the moment in time in which the singularity is reached. Therefore, in the case of a radial free fall the condition (3.1.28) expresses the fact that it will end at the singularity. We have illustrated this in Figure 23.

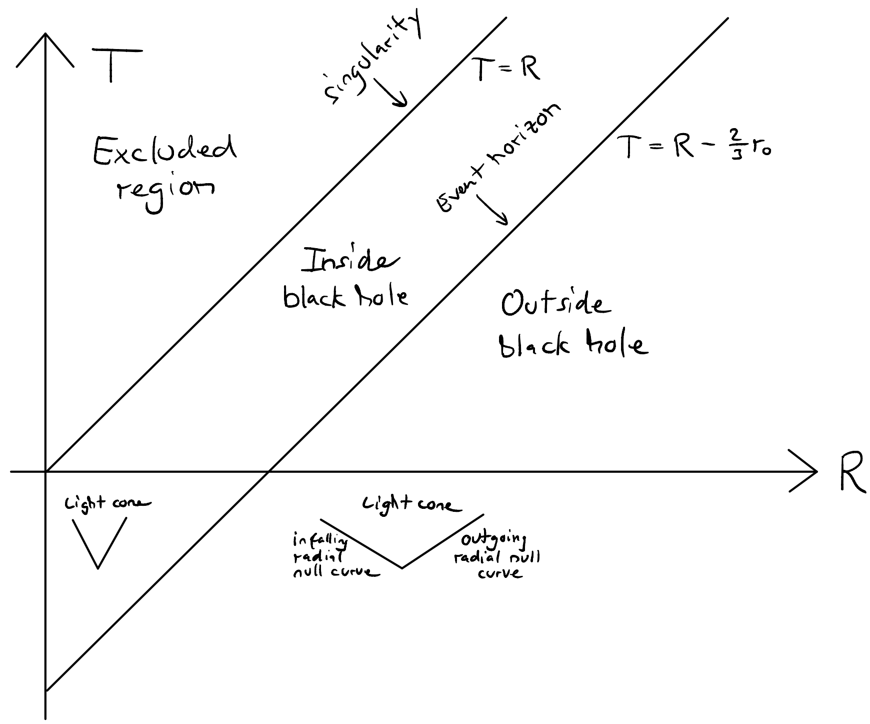

Figure 23: Illustration of the Schwarzschild black hole space-time in Lemaître coordinates. A radial free fall with $E = 1$ corresponds a curve with R held constant.

From (3.1.31) we see that the event horizon is at

$$R - T = \frac{2}{3} r_0 \,. \tag{3.1.34}$$

Thus, we are outside the event horizon for $R - T > \frac{2}{3} r_0$ and inside for $0 \leq R - T < \frac{2}{3} r_0$. We see that for a motion with constant R, θ and ϕ the difference in proper time T for such freely falling observers is $\frac{2}{3} r_0$ which reproduces the result (3.1.25).

Finally, we can consider the lightcones in these coordinates. For a radial null curve we need

$$dT = \pm \sqrt{\frac{r_0}{r}} dR \,, \tag{3.1.35}$$

in order to get $ds^2 = 0$. The plus (minus) sign corresponds to the outgoing (infalling) radial null curve. Thus, we find that $dT/dR > 1$ outside the event horizon and $dT/dR < 1$ inside the event horizon. This fits with the fact that the outgoing radial null curve can increase $R - T$ and hence escape the event horizon when we are outside the event horizon. Instead when we are inside the event horizon $R - T$ can only decrease which means we inevitably end up in the singularity. See also Figure 23 for an illustration.

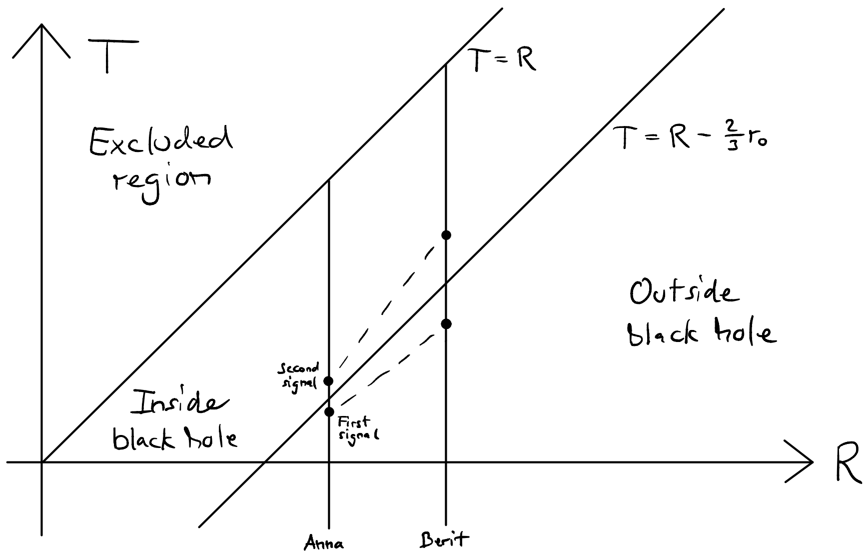

Figure 24: Illustration of two astronauts Anna and Berit in radial free fall towards the black hole.

We can use the Schwarzschild metric in Lemaître coordinates to address the following scenario which is illustrated in Figure 24. Suppose two astronauts are in a radial free fall. The one closest to the event horizon is Anna and the one slightly further away is Berit. Anna keeps sending light signals to Berit with a flashlight. The question is: suppose Anna sends a light signal just after she passed the event horizon. Wouldn't Berit, who is just outside the event horizon, receive the light signal before entering the black hole, and thereby violate the fact that light cannot escape the black hole? The answer is no. If Anna sends a light signal before she enters the event horizon this can reach Berit before Berit enters the black hole since the outgoing part of the lightcone permits this. However, if Anna sends a light signal after she enters the event horizon the lightcone has narrowed such that the outgoing radial null curve can only reach Berit after Berit herself enters the black hole, as illustrated in Figure 24.

3.1.5 Redshift for infalling observer

We consider here how the curved space-time geometry of the Schwarzschild metric affects sending light signals to and from the vicinity of the event horizon. In addition we consider a more general solution to an infalling observer.

Infalling observer with $E = 1$

We consider here again the two observers of Section 3.1.1. An observer *at infinity*, meaning very far away from the black hole, *i.e.* with r so large that we can ignore r_0/r in the metric (2.1.44). The second observer is inside an infalling spaceship, meaning a freely falling spaceship with $r > r_0$ which is close to the event horizon. The scenario we consider is that light signals are sent and received between the two observers. In the following we use for simplicity the same initial conditions as described in Section 3.1.1 with $E = 1$. Below we generalize the results to initial conditions with any E.

Consider the motion of the infalling spaceship as described in (t, r) coordinates by Eqs. (3.1.5)-(3.1.6) in terms of the proper time of the spaceship. Using Eqs. (3.1.15) and (3.1.17) we find that in terms of the (u, v) coordinates this motion is

$$du = \frac{d\tau}{1 - \sqrt{\frac{r_0}{r}}} \ , \quad dv = \frac{d\tau}{1 + \sqrt{\frac{r_0}{r}}} \ . \tag{3.1.36}$$

This describes the motion of a freely falling massive particle when the motion is only in the radial direction, *i.e.* with fixed angles θ and ϕ.

The first scenario we consider is that the infalling spaceship sends light signals towards the observer at infinity (along a radial null curve).

We represent this as two infinitesimally close events, the event A1 that corresponds to the first light signal from the infalling observer and the event A2 that corresponds to a subsequent light signal, with proper time $d\tau$ between them. This is illustrated in Figure 25. The proper time $d\tau$ could be taken to correspond to the beginning and end of one period of an electromagnetic wave, with $d\tau$ being the period (not necessarily in the visible part of the spectrum). The event A1 is at $r = r_\mathrm{A}$ and $t = t_\mathrm{A}$ while the event A2 is at $r = r_\mathrm{A} + dr$ and $t = t_\mathrm{A} + dt$ with dr and dt given by (3.1.5)-(3.1.6) with $d\tau$ being the proper time of the infalling observer between the two events. In (u, v) coordinates this corresponds to

$$\text{A1: } (u, v) = (u_\mathrm{A}, v_\mathrm{A}) \ , \quad \text{A2: } (u, v) = \left(u_\mathrm{A} + \frac{d\tau}{1 - \sqrt{\frac{r_0}{r_\mathrm{A}}}}, v_\mathrm{A} + \frac{d\tau}{1 + \sqrt{\frac{r_0}{r_\mathrm{A}}}} \right) \ , \tag{3.1.37}$$

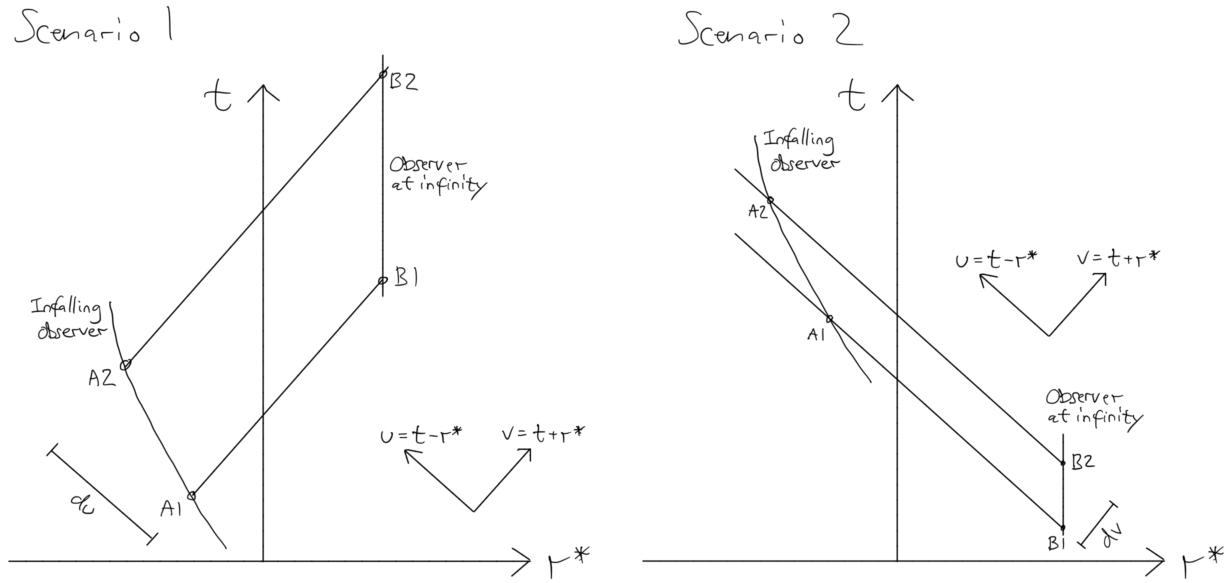

Figure 25: Illustration of scenario 1 with the infalling observer that sends a light signal towards the observer at infinity as well as scenario 2 with the infalling observer receiving a light signal from the observer at infinity.

using Eq. (3.1.36) where $(u_\mathrm{A}, v_\mathrm{A}) = (t_\mathrm{A} - r^\star(r_\mathrm{A}), t_\mathrm{A} + r^\star(r_\mathrm{A}))$.

The light signals are then received by the observer at infinity. The first light signal is received at the event B1 and the second at the event B2, as illustrated in Figure 25. The event B1 is at $r = r_\mathrm{B}$ and $t = t_\mathrm{B}$. Since the event B2 occurs at the same position it is at $r = r_\mathrm{B}$ and $t = t_\mathrm{B} + d\tau_\infty$ where $d\tau_\infty$ is the proper time between the two events. Note that $dt = d\tau_\infty$ since we assume r_B is so large that we can neglect r_0/r in the metric (2.1.44). Thus,

$$\text{B1: } (u, v) = (u_\mathrm{B}, v_\mathrm{B}) \ , \quad \text{B2: } (u, v) = (u_\mathrm{B} + d\tau_\infty, v_\mathrm{B} + d\tau_\infty) \,, \tag{3.1.38}$$

in (u, v) coordinates where $(u_\mathrm{B}, v_\mathrm{B}) = (t_\mathrm{B} - r^\star(r_\mathrm{B}), t_\mathrm{B} + r^\star(r_\mathrm{B}))$.

We impose now that the two light signal are outgoing radial null curves. From (3.1.18) this means that u is constant along the null curves. Thus, for the first light signal we have that $u_A = u_B$ and for the second $u_A + d\tau_\infty = u_B + d\tau/(1 - \sqrt{r_0/r_\mathrm{A}})$. We conclude that[28]

$$d\tau_\infty = \frac{d\tau}{1 - \sqrt{\frac{r_0}{r_{\rm A}}}}\,. \tag{3.1.39}$$

From this we see that $d\tau_\infty > d\tau$. This means that if the two light signals are send with 1 second apart from the infalling observer, they are further apart in time when the reach the observer at infinity. If $r_{\rm A}$ is very close to r_0 one can have that $d\tau_\infty$ could be hours, or years. Thus, the closer the infalling observer is to the event horizon, the more she will appear to be frozen in time. Moreover, interpreting $d\tau$ as a period of an electromagnetic wave, we see that the wave will be redshifted in general, and if $r_{\rm A}$ is very close to r_0 the redshift can be arbitrarily large. Therefore, not only the infalling observer will appear to freeze in time, she will also fade away to the infrared part of the spectrum. Since our instruments naturally have physical limits in how long wavelengths we can observe (the longest radiowaves are measured in meters) eventually we will not be able to see the infalling observer from afar.

The second scenario we consider is that the observer at infinity sends light signals towards the infalling observer. We represent this again at two infinitesimally close events, the event B1 that corresponds to the first light signal sent from the observer at infinity, and the event B2 corresponding to the second light signal, with proper time $d\tau_\infty$ between them. The infalling observer then receives the first light signal at the event A1 and the second at the event A2 with proper time $d\tau$ between the events. This is illustrated in Figure 25. We can use the same parametrization of the four events as above in (3.1.37)-(3.1.38). For this situation we should impose that the two light signal are ingoing radial null curves. From (3.1.18) this means that v is constant along the null curves. Thus, for the first light signal we have that $v_A = v_B$ and for the second $v_A + d\tau_\infty = v_B + d\tau/(1+\sqrt{r_0/r_{\rm A}})$. We conclude that

$$d\tau_\infty = \frac{d\tau}{1 + \sqrt{\frac{r_0}{r_{\rm A}}}}\,. \tag{3.1.40}$$

From this we see that $d\tau > d\tau_\infty$. This means light reaching the infalling observer is redshifted, just like in the other situation. However, in this case the redshift factor at maximum becomes $d\tau \simeq 2d\tau_\infty$ when the infalling observer is close to the event horizon. In fact, only close to $r = 0$ the redshift factor diverges. This makes sense as the light

[28] Note that one cannot use the formula (3.1.5) to relate the proper time between the light signals for the two observers since dt in that formula should be the difference in the coordinate time for the motion of the spaceship (the infalling observer) near the horizon between the reception of the two light signals, and this difference in coordinate time does not have to be equal to the difference in coordinate time that occur when the two signals were sent from the observer at infinity.

signal should be able to travel inside the event horizon, and in fact one should be able to receive light signals from the observer at infinity.

The key to understand this is to remember that while from point of view of the observer at infinity nothing can go out of the black hole. Hence for this observer the event horizon is a very special region in space-time. In contrast to this, from point of view of an infalling observer nothing special happens when reaching the Schwarzschild radius. There is no local physical measurement that one can perform to see that one is passing the Schwarzschild radius. This follows from Einsteins Equivalence Principle as formulated in Section 1.3.5, and the fact that there is no singularity in the space-time at this radius.

Infalling observer with any E

We consider here the most general initial conditions for an infalling observer falling along a radial geodesic. In Section 3.1.1 we imposed $E = 1$ corresponding to a freely falling motion that has zero velocity for $r/r_0 \to \infty$. However, as we explain below, all values of E corresponds to physically reasonable initial conditions. Taking E to be arbitrary in (3.1.1) we have

$$\frac{dt}{d\tau} = \frac{E}{1 - \frac{r_0}{r}} . \tag{3.1.41}$$

Inserting this in Eq. (2.2.7) we find

$$\frac{dr}{d\tau} = -\sqrt{E^2 - 1 + \frac{r_0}{r}} . \tag{3.1.42}$$

For $E > 1$ one has a motion for which the kinetic energy is non-zero for $r/r_0 \to \infty$. This corresponds to a scenario in which the spaceship starts far a way from the black hole with an initial acceleration, giving an initial velocity, and afterwards turn off the engines of the spaceship with the observer so that it is freely falling. Turning to $E < 1$ one can see from (3.1.42) that $dr = 0$ at the radius $r = \frac{r_0}{1-E^2}$. Thus, $E < 1$ is realized by starting a freely falling motion with zero velocity at this radius.

Using Eqs. (3.1.15) and (3.1.17) with Eqs. (3.1.41)-(3.1.42) we get the change in the u and v coordinates

$$du = \frac{E + \sqrt{E^2 - 1 + \frac{r_0}{r}}}{1 - \frac{r_0}{r}} d\tau \ , \quad dv = \frac{E - \sqrt{E^2 - 1 + \frac{r_0}{r}}}{1 - \frac{r_0}{r}} d\tau \,, \tag{3.1.43}$$

given the change in proper time $d\tau$ of the freely falling observer.

We can now again look at the two scenarios, as illustrated in Figure 25. We use again the parametrization (3.1.37)-(3.1.38). In scenario 1, where the infalling observer sends out light (or electromagnetic waves) towards the observer at infinity, we get

$$d\tau_\infty = \frac{E + \sqrt{E^2 - 1 + \frac{r_0}{r_A}}}{1 - \frac{r_0}{r_A}} d\tau \,. \tag{3.1.44}$$

We see again that $d\tau_\infty > d\tau$ and that $d\tau_\infty$ diverges for $r \to r_0$. This means that essentially one has the same physics as for $E = 1$, namely that there is a redshift of the light signals received from the infalling observer, and this redshift becomes arbitrarily large as the infalling observer approaches the event horizon.

In scenario 2, where the infalling observer receives light (or electromagnetic waves) from the observer at infinity, we get

$$d\tau_\infty = \frac{E - \sqrt{E^2 - 1 + \frac{r_0}{r_A}}}{1 - \frac{r_0}{r_A}} d\tau \,. \tag{3.1.45}$$

First we observe that for $r_A = r_0$ we have

$$d\tau_\infty = \frac{1}{2E} d\tau \,. \tag{3.1.46}$$

Thus, there is no divergency at $r = r_0$ for any E. We notice that for $E > \frac{1}{2}$ one has a redshift when the infalling observer reaches the Schwarzschild radius since $d\tau > d\tau_\infty$ at $r = r_0$. Instead for $E < \frac{1}{2}$ there is a blueshift when the infalling observer reaches the Schwarzschild radius. More generally, one can see that for $E \geq 1$ the infalling observer always receives the light with a redshift. Instead for $E < 1$ one has a blueshift above a certain critical radius, and a redshift below the critical radius. In particular, at the radius $r_\mathrm{A} = \frac{r_0}{1-E^2}$ for which the velocity is zero, one finds

$$d\tau_\infty = \frac{1}{\sqrt{1 - \frac{r_0}{r_A}}} d\tau \,. \tag{3.1.47}$$

Thus, the closer one starts to the event horizon with the freely falling motion, the higher a blueshift factor one gets.

Observer at fixed position

Finally we consider the case in which we keep the spaceship with an observer aboard at a fixed position outside the events horizon. More precisely, we consider a motion for which the coordinates (r, θ, ϕ) do not change. To accomplish this, one needs the spaceship to accelerate, and the closer one gets to the event horizon the larger the acceleration has to be. The observer at fixed position has $dr = d\theta = d\phi = 0$ and hence from the metric (2.1.44) we get

$$d\tau^2 = \Big(1 - \frac{r_0}{r}\Big)\, dt^2\,. \tag{3.1.48}$$

Using the u and v coordinates defined by Eqs. (3.1.15) and (3.1.17) this is

$$du = dv = \frac{d\tau}{\sqrt{1-\frac{r_0}{r}}}\,. \tag{3.1.49}$$

For the observer at infinity we have again $du = dv = d\tau_\infty$ as above. Thus, we derive the relation (3.1.47), both in the scenario where the observer at fixed position sends light towards the observer at infinity, and the scenario where the observer at fixed position receives light from the observer at infinity. In the first scenario, we get a redshift which, as in the cases above, becomes arbitrarily large close to the Schwarzschild radius. In the second scenario, one gets a blueshift that becomes arbitrarily large close to the Schwarzschild radius. We see that this ties up nicely with the $E < 1$ case of the freely falling observer at the radius $r = \frac{r_0}{1-E^2}$ for which the velocity is zero.

3.1.6 Maximal extension

Section 3.1.6 is not part of the pensum of the course.

We have seen that the Schwarzschild space-time geometry (2.1.44) that initially was found for $r > r_0$ can be extended to include $0 \le r \le r_0$ by considering what happens in the future to freely falling observers that starts with $r > r_0$. The question we ask and answer in the following is: can one extend the Schwarzschild geometry further? We show below that this indeed is possible if one also considers the possible past of radial free fall, or of radial null curves.

We define the Kruskal-Szekeres coordinates by

$$\mathcal{T} = \frac{1}{2}(e^{\frac{v}{2r_0}} - e^{-\frac{u}{2r_0}})\ ,\ \ \mathcal{R} = \frac{1}{2}(e^{\frac{v}{2r_0}} + e^{-\frac{u}{2r_0}})\,, \tag{3.1.50}$$

These coordinates are defined in terms of the t and r coordinates of (2.1.44) through $u = u(t,r)$ and $v = v(t,r)$ given by (3.1.17) and (3.1.14). In Exercise (3.5) one shows that the Schwarzschild metric (2.1.44) in these coordinates becomes

$$ds^2 = \frac{4r_0^3}{r} e^{-\frac{r}{r_0}} \Big(- d\mathcal{T}^2 + d\mathcal{R}^2 \Big) + r^2 (d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{3.1.51}$$

where the function $r(\mathcal{T},\mathcal{R})$ is defined indirectly by

$$\mathcal{T}^2 - \mathcal{R}^2 = -\frac{r-r_0}{r_0} e^{\frac{r}{r_0}}\,. \tag{3.1.52}$$

We see that the coordinate $\mathcal{T}$ is a global time-coordinate since $g_{\mathcal{T}\mathcal{T}} < 0$ everywhere. Similarly, $\mathcal{R}$ is a global spatial coordinate since we have $g_{\mathcal{R}\mathcal{R}} > 0$ everywhere. Moreover, the metric (3.1.51) is well-behaved everywhere except for $r = 0$. Thus, just like the Eddington-Finkelstein coordinates, the Kruskal-Szekeres coordinates show that the singular behavior near $r = r_0$ in the metric (2.1.44) is because the coordinate system (t, r, θ, ϕ) breaks down, not because there is a genuine singularity in the space-time geometry.

Using (3.1.52) we find

$$\begin{aligned} r = r_0 \ &\Leftrightarrow \mathcal{R} = \pm\mathcal{T}\,, \\ r > r_0 \ &\Leftrightarrow |\mathcal{R}| > |\mathcal{T}|\,, \\ r < r_0 \ &\Leftrightarrow |\mathcal{R}| < |\mathcal{T}|\,, \\ r = 0 \ &\Leftrightarrow \mathcal{T}^2 = 1 + \mathcal{R}^2\,. \end{aligned} \tag{3.1.53}$$

This maps the important regions of the Schwarzschild metric (2.1.44) to the Kruskal-Szekeres coordinates. We have illustrated the space-time given by the metric (3.1.51) in Figure 26.

Considering (3.1.53) we notice that $r = r_0$ can both be achieved for $\mathcal{R} = \mathcal{T}$ and $\mathcal{R} = -\mathcal{T}$. What is the physical meaning of this? To understand this, we consider the motion of outgoing and infalling radial null curves. For outgoing radial null curves u is constant, while for infalling radial null curves v is constant. Hence

$$\begin{aligned} &\text{outgoing radial null curve :} \quad \mathcal{R} = \mathcal{T} + \text{constant}\,, \\ &\text{infalling radial null curve :} \quad \mathcal{R} = -\mathcal{T} + \text{constant}\,. \end{aligned} \tag{3.1.54}$$

Thus, since the event horizon that we have been considering so far is characterized by capturing the outgoing radial null curves, such an event horizon must be with $\mathcal{R} = \mathcal{T}$. In the rest of this section we call the region $\mathcal{R} = \mathcal{T}$ with $\mathcal{T} > 0$ and $\mathcal{R} > 0$ the *future event horizon*.

What is instead the meaning of the region $\mathcal{R} = -\mathcal{T}$ with $\mathcal{T} < 0$ and $\mathcal{R} > 0$? This region we call the *past event horizon* since it appears at earlier times compared to the future event horizon. This past event horizon is characterized by the fact that outgoing radial null curves can leave the region with $r \leq r_0$, while infalling radial null curves cannot enter the region $r \leq r_0$. Thus, we have a region with the opposite physics than that of a black hole. For a black hole, light cannot escape from behind the event horizon, but it is possible to enter from outside. Instead for the region behind the past event horizon, light cannot enter, but it can escape. Thus, one calls this a *white hole*. See also Exercise 3.1 for more on this. We have illustrated the past and future event horizons for $\mathcal{R} > 0$ in Figure 26 as well. Note here also the singularity at $r = 0$ is mapped both to a singularity

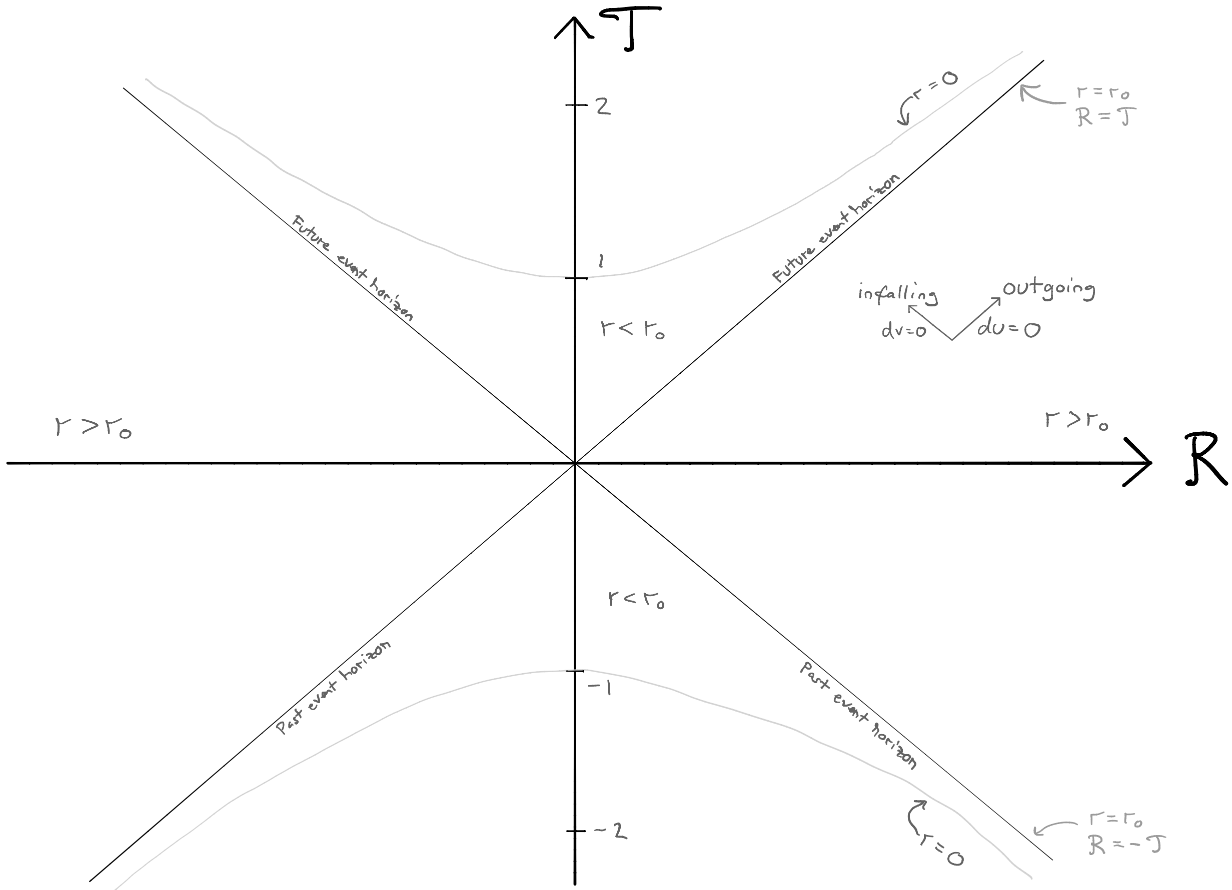

Figure 26: Illustration of the Kruskal-Szekeres space-time with metric (3.1.51) which is the maximal extension of the Scwarzschild metric (2.1.44).

in the future given by $\mathcal{T} = \sqrt{1+\mathcal{R}^2}$ which is hidden behind the future event horizon, and a singularity in the past given by $\mathcal{T} = -\sqrt{1+\mathcal{R}^2}$ which is hidden behind the past event horizon.

Finally, one notices that it is possible to analytically extend the metric (3.1.51) to negative values of $\mathcal{R}$ just like we extended the metric (3.1.20) to include $r \leq r_0$. In this way one obtains the maximally extended space-time that one can obtain from the original Schwarzschild metric (2.1.44), known as the Kruskal-Szekeres space-time. As illustrated in Figure 26, one has a future event horizon at $\mathcal{R} = -\mathcal{T}$ with $\mathcal{T} > 0$ and $\mathcal{R} < 0$, *i.e.* a black hole, and a past event horizon at $\mathcal{R} = \mathcal{T}$ with $\mathcal{T} < 0$ and $\mathcal{R} < 0$, *i.e.* a white hole in the $\mathcal{R} < 0$ region as well. The region for $-\mathcal{R} > |\mathcal{T}|$ corresponds to a region outside a past and future event horizon, analogous to the region $\mathcal{R} > |\mathcal{T}|$. How should we think of this? The point here is that both on the right side $\mathcal{R} > 0$ and the left side $\mathcal{R} < 0$ in Figure 26 we have a universe with a black hole in its future and a white hole in its past. Thus, we have really two distinct universes. For $\mathcal{T} < -1$ these two universes are disconnected from each other, and we have a white hole in each of the universes. For $-1 < \mathcal{T} < 1$ the two universes are connected by a wormhole that starts as a white hole and ends as a black hole after $\mathcal{T} = 0$. Finally, for $\mathcal{T} > 1$ the two universes disconnect again and we have a black hole in each of them. This process is illustrated in Figure 27. As one can see from Figure 26 this wormhole is not traversible, in the sense that one cannot connect the two universes by a time-like curve. Thus, it is not possible to travel from one universe to the other. But two space-ships entering the future event horizons from the two different universes could (briefly) meet and share notes before they are torn apart from the tidal forces near the future singularity at $\mathcal{T} = \sqrt{1 + \mathcal{R}^2}$.

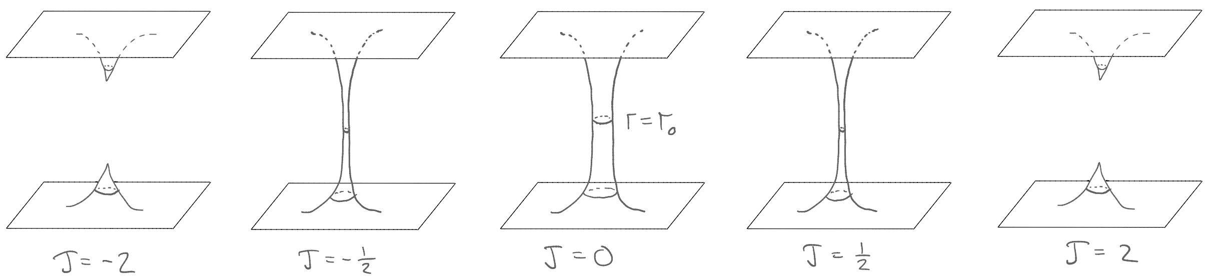

Figure 27: Illustration of the flow of time in the Kruskal-Szekeres space-time with a wormhole that appears and then disappears again.

One should stress that the sitation described by the Kruskal-Szekeres space-time is very likely completely unrealistic. From what we know about black holes, they are created by the gravitational collapse of ordinary matter. Hence, in such a space-time there is only a future event horizon and no past event horizon. Indeed, one should think of white holes as the time-reverse of black holes. Suppose I am making cornflakes with milk in a bowl for breakfast, but then I manage to drop everything on the floor, so that all the cornflakes and milk lie on the floor, together with pieces of the bowl that broke. This is a realistic physical scenario. But the time-reverse is not. It is not likely that a bunch of cornflakes on kitchen floor, together with small lakes of milk and the scattered pieces of a bowl, all fly up and end up as a nice breakfast that is ready for me to eat. In the same sense, a white hole, which is the time-reverse of a black hole, is not a realistic physical scenario. It occurs in the Kruskal-Szekeres space-time only because we derived this space-time assuming that we have a time-reserval symmetry, *i.e.* $t \to -t$ or $\mathcal{T} \to -\mathcal{T}$ of the line-element ds^2. But in nature, such a symmetry is not natural.

3.2 Observations of Black Holes

In the last thirty years, the subject of black holes have changed from being one of conjecture to being a full-blown observational science. I will not discuss all aspects of observations of black holes since this is too rich a subject for a theoretical course. Nevertheless, I will try to give a few of the most important facts.

There are three classes of black holes that all have been observed in astrophysics:

- Stellar mass black holes with masses between 3 and 100 solar masses.
- Intermediate-mass black holes with masses ranging from 100 to 10^5 solar masses.
- Supermassive black holes with masses ranging from 10^5 to 10^{10} solar masses.

Here a solar mass is the mass of the Sun $M_\circ = 2.0 \cdot 10^{30}$ kg.

Stellar mass and intermediate-mass black holes

The burning of a star is fueled by thermonuclear fusion in the core. This gives an outward directed force in the star. That force is balanced by the gravitational force (in Newtonian terms) directed inwards. When the star runs out of fuel it undergoes a gravitational collapse. In this process it can explode (supernova) leaving a remnant that undergoes gravitational collapse, or the whole star can collapse if it is sufficiently heavy.

If the remnant undergoing gravitational collapse is less than $3M_\circ$ then it becomes a white dwarf or a neutron star. If instead the mass of the remnant is more than $3M_\circ$ then it will collapse to a black hole.

Regarding observations of stellar mass black holes, one possibility is to measure the mass of objects in a binary system of a black hole and a star. The method is then to exclude that the compact object orbiting the star can be anything else than a black hole, in the sense that it is too compact to be anything else. The first reliable observation was the Cygnus X-1 with mass about $15M_\circ$ and a companion star denoted HDE 226868 (with mass about 25 to 35 times $M_\circ$). It was originally discovered in 1964 as a radiosource. In 1990 the observational data became convincing enough to conclude that Cygnus X-1 was indeed a black hole in the binary system. This is considered the first observation of a black hole. Since then one has had several other observations of stellar mass black holes in binary systems. In Table 1 we have collected a list of ten observed black holes.

Another type of observations that we will discuss in Chapter 5 is the observations of the collision and merger of stellar mass black holes using gravitational waves. The first

Stellar black hole	Mass of black hole $M/M_\circ$	Rotation parameter $J/(GM^2)$
Cygnus X-1	14.8	> 0.95
LMC X-1	10.9	0.92
M33 X-7	15.7	0.84
GRS 1915+105	10.1	> 0.95
4U 1543-47	9.4	0.80
GRO J1655-40	6.3	0.70
XTE J1550-564	9.1	0.34
H1743-322	8	0.2
LMC X-3	7.6	< 0.3
A0620-00	6.6	0.12

Table 1: Masses and rotation parameters for ten stellar mass black holes. Data taken from [5] where one can read about the methods and uncertainties of the measurements.

observation of gravitational waves, known as GW150914, was made September 14, 2015, by LIGO [6]. It originated from two colliding black holes, approximately $30M_\circ$ each, that collided 1.3 billion years ago. This is the first observation of black holes that relies directly on the physics of the gravitational field as described by General Relativity. We discuss this observation more in Sections 3.5.2 and Chapter 5.

Since then, many additional mergers of black holes have been observed, see Section 5.2.3 for more details and in particular Figure 37 . One of the particularly interesting observations is the event called GW190521 that resulted in the first observed black hole in the intermediate mass-range, since the black hole resulting from the merger has a mass of approximately $142M_\circ$. Subsequent to this event, further gravitational-wave observations of intermediate-mass black holes have been made as well. Whether intermediate-mass black holes can only arise from mergers of stellar black holes, or whether there are other mechanisms for their creation, is at this point unknown.

Supermassive black holes

Astrophysicists suspect that most - if not all - galaxies have a supermassive black hole in their center. The breakthrough in observational evidence for this came with the Hubble telescope launched in 1990. The most convincing evidence is for our own galaxy - the

Milky Way - as well as for a few nearby galaxies.

In the Milky Way there is a black hole in the center of the galaxy with mass $4 \cdot 10^6 M_\circ$. It is called Sagittarius A*, or, in short Sgr A*. The most compelling evidence is the star S2 that is orbiting Sgr A* (see Exercise 2.2). From this orbit one can infer the mass of Sgr A* as well as the maximal possible radius of Sgr A*. The only known type of object that is able to be sufficiently compact to fit inside this radius is a black hole. The star S2 was discovered in 1992 and since one orbit takes about 16 years, one has increasingly good measurements of the orbit of S2 around Sgr A*. One has also observed other stars orbiting Sgr A*, with same conclusions.

In 2019, the Event Horizon Telescope for the first time published an image of a black hole shadow [7], see Figure 28. The image depicts the supermassive black hole M87* which is located in the center of the supergiant elliptical galaxy Messier 87 (M87) of approximate mass $6.5 \cdot 10^9 M_\circ$. One can observe the gravitational bending of light by the black hole on the image. In 2022 the Event Horizon Telescope has published an image of Sgr A* as well [8].

Figure 28: Image of M87* from [7].

It is not clear how supermassive black holes have been formed. One theory is that they were formed before the galaxies that surrounds them, thus being a seed for the galaxies.

3.3 Kerr Black Hole

3.3.1 Astrophysical motivation

The Schwarzschild black hole with metric (2.1.44) is not the only type of black hole that one can find as solution of Einsteins equations (1.6.37). We made three assumptions in deriving the Schwarzschild metric (2.1.44):

- The metric satisfy the vacuum Einstein equations (1.6.39).
- Spherical symmetry of line-element.
- The metric is invariant under time translations.

Note that the last assumption follows from the other two by Birkhoffs theorem.

There exist more general black hole space-times that do not fulfil the first two assumptions. One can find electrically charged black holes that are spherically symmetric (known as *Reissner-Nordström black holes*). These do not obey the vacuum Einstein equations (1.6.39) since the charge induces an electromagnetic field around the black hole. However, for applications in astrophysics it is inconceivable to have an electric charge on a macroscopic scale since that would immediately attract particles of the opposite charge. For the same reason one does not find electrically charged stars or planets.

Thus, for astrophysical applications one can safely assume that black holes are solutions of the vacuum Einstein equations (1.6.39). Note that matter and electromagnetic fields around an astrophysical black hole can be neglected when solving Einsteins equations for the black hole since their mass and energy are insignificant compared to the mass of the black hole. This includes accretion discs, jets and electromagnetic fields.

However, the Kerr metric is an example of a metric for a black hole that solves the vacuum Einstein equations (1.6.39), is invariant under time translations, but that is not spherically symmetric. Thus, it breaks the second of the two above assumptions. The absence of spherical symmetry is tied to the fact that the Kerr metric describes a rotating black hole. This means that it has an angular momentum J that is comparable in size to GM^2 where M is its mass (See Sec. 3.4.1).

In astrophysics one has observational evidence for rotating black holes. In Table 1 we have listed ten observed stellar mass black holes, all having a measurable value of the angular momentum J in units of GM^2. Moreover, the recent observation GW150914 of gravitational waves from the collision and merger of two black holes gives firm evidence for rotating black holes since the final black hole has angular momentum. We shall come back to this in Sections 3.5.2 and Chapter 5.

In conclusion, we have ample motivation to go on with describing black holes with rotation.

3.3.2 Kerr metric

The Kerr metric is

$$\begin{aligned} ds^2 = &-\left(1-\frac{r_0 r}{\Sigma}\right)dt^2 - \frac{2ar_0 r}{\Sigma}\sin^2\theta dt d\phi \\ &+ \frac{(r^2+a^2)^2-\Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2\,, \end{aligned} \tag{3.3.1}$$

where we defined the functions

$$\Sigma(r,\theta) = r^2 + a^2 \cos^2\theta \ , \quad \Delta(r) = r^2 - r_0 r + a^2 \, . \tag{3.3.2}$$

We see that the metric depends on two parameters r_0 and a. The components of the metric depends only on the coordinates r and θ. Hence it is invariant under time translations, $t \to t + \text{constant}$, and rotations around the axis at $\sin\theta = 0$, *i.e.* translations $\phi \to \phi + \text{constant}$. The metric is also invariant under the combined transformation $a \to -a$, $\phi \to -\phi$ hence we can choose $a \geq 0$ without loss of generality. For $a = 0$ the Kerr metric (3.3.1)-(3.3.2) reduces to the Schwarzschild metric (2.1.44).

We see that g_{rr} blows up when Δ goes to zero. The zeroes of $\Delta(r)$ are

$$r_\pm = \frac{r_0}{2} \pm \frac{r_0}{2}\sqrt{1 - \frac{4a^2}{r_0^2}} \, . \tag{3.3.3}$$

Note that for $a > r_0/2$ one does not have any zeroes while for $a = r_0/2$ one has a single zero. See Section 3.4.3 for comments on what happens for $a \geq r_0/2$. We assume here and in the following that $0 \leq a < r_0/2$ which gives $r_- < r_+$. Using (3.3.3) we can write

$$\Delta = (r - r_-)(r - r_+) \, . \tag{3.3.4}$$

r_+ is the highest radius for which g_{rr} blows up. This suggests that we have an event horizon at $r = r_+$. To show that this indeed is the case, consider a light ray with $r \geq r_+$ that moves away from $r = r_+$, *i.e.* with $\frac{dr}{dt} > 0$. Since it follows a null curve we have

$$0 = g_{tt} dt^2 + 2 g_{t\phi} dt d\phi + g_{\phi\phi} d\phi^2 + g_{rr} dr^2 + g_{\theta\theta} d\theta^2 \, . \tag{3.3.5}$$

From this we get

$$\frac{dr^2}{dt^2} = -\frac{g_{tt} + 2 g_{t\phi}\frac{d\phi}{dt} + g_{\phi\phi}\frac{d\phi^2}{dt^2} + g_{\theta\theta}\frac{d\theta^2}{dt^2}}{g_{rr}} \, . \tag{3.3.6}$$

Since $g_{rr} \to \infty$ as $r \to r_+$, while the other components of the metric are finite, one gets that $\frac{dr}{dt} \to 0$ for $r \to r_+$. This shows that from point of view of an observer far away, *i.e.* with t as the proper time, light appears to move slower and slower in the r-direction the closer one gets to $r = r_+$. This is analogous to what happens in the case of the Schwarzschild black hole with the closing of the light cones, as seen in Section 3.1.1. Since also infalling matter moves slower and slower it means that one has an event horizon at $r = r_+$. Thus, we call the object that the Kerr metric describes the *Kerr black hole*. Note that one needs to verify as well that light cannot escape the event horizon. This is shown in Exercise 3.4 by introducing coordinates that are analogous to the Eddington-Finkelstein coordinates for Schwarzschild (3.1.20).

For the Schwarzschild black hole (2.1.44) one has that g_{tt} is zero at the event horizon. Instead for the Kerr black hole we see from (3.3.1)-(3.3.2) that g_{tt} is non-zero for $r = r_+$. The reason for this is that the metric is off-diagonal in the t and ϕ directions. Consider the determinant

$$\det \begin{pmatrix} g_{tt} & g_{t\phi} \\ g_{t\phi} & g_{\phi\phi} \end{pmatrix} = -\Delta \sin^2\theta \,. \tag{3.3.7}$$

We see that this is zero at the event horizon $r = r_+$.

As we shall see below, the reason for the off-diagonal term $g_{t\phi}$ is that the Kerr black hole is rotating. It is rotating around the axis $\sin\theta = 0$ in the direction of the ϕ angle. The effect of this rotation near the horizon is explored in Section 3.3.3. Moreover, in Section 3.4.1 we find that the Kerr black hole has angular momentum.

3.3.3 Ergoregion and frame dragging

Consider the g_{tt} component in the Kerr metric (3.3.1)-(3.3.2). This is zero when $\Sigma = r_0 r$ which is equivalent to $r^2 - r_0 r + a^2\cos^2\theta = 0$. The solutions of this are

$$r = \frac{r_0}{2} \pm \frac{r_0}{2}\sqrt{1 - \frac{4a^2}{r_0^2}\cos^2\theta} \,. \tag{3.3.8}$$

Comparing to r_+ in (3.3.3) we see that the root with the plus gives a radius that is larger than r_+ while the root with the minus gives a radius that is smaller than r_+. Write the root with a plus as

$$r_{\text{ergo}}(\theta) = \frac{r_0}{2} + \frac{r_0}{2}\sqrt{1 - \frac{4a^2}{r_0^2}\cos^2\theta} \,. \tag{3.3.9}$$

The surface

$$r = r_{\text{ergo}}(\theta) \,, \tag{3.3.10}$$

is called the *ergosphere* and it is surrounding the event horizon at $r = r_+$ except at the axis of rotation $\sin\theta = 0$ where they intersect. In between the event horizon and the ergosphere is the *ergoregion*, defined by

$$r_+ < r < r_{\text{ergo}}(\theta) \,. \tag{3.3.11}$$

See Figure 29 for an illustration of the event horizon, ergosphere and ergoregion for the Kerr black hole.

Outside the ergosphere $r > r_{\text{ergo}}(\theta)$ we have $g_{tt} < 0$. This means that keeping the three coordinates (r, θ, ϕ) fixed gives a time-like curve and is thus a possible path for a massive particle (note that it is an accelerated path, not a geodesic). At the ergosphere $r = r_{\text{ergo}}(\theta)$

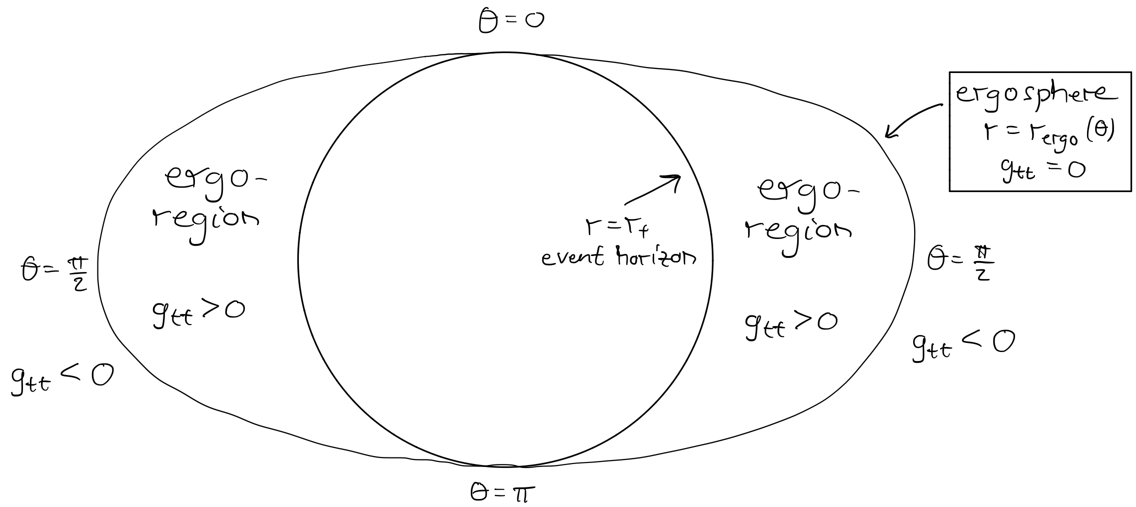

Figure 29: Illustration of the event horizon $r = r_+$, the ergoregion $r_+ < r < r_{\rm ergo}(\theta)$ and the ergosphere $r = r_{\rm ergo}(\theta)$ of the Kerr black hole for fixed t and ϕ. The axis of rotation is at $\sin\theta = 0$.

we have $g_{tt} = 0$. Thus, keeping (r, θ, ϕ) fixed corresponds to a null curve. Moreover, in the ergoregion $r_+ < r < r_{\rm ergo}(\theta)$ we have $g_{tt} > 0$ which means that keeping (r, θ, ϕ) fixed is a space-like curve. Thus, it is impossible to stand still inside the ergoregion. A related consequence of this is that t cannot be used as a time on the ergosphere or in the ergoregion. But what is the physical interpretation of the ergosphere and ergoregion? This is what we now shall explore.

Consider a massive particle in the ergoregion. Since it moves on a time-like curve (also in case it is accelerating) we have

$$-d\tau^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 \,. \tag{3.3.12}$$

This gives

$$-2g_{t\phi}\frac{d\phi}{dt} = \left(\frac{d\tau}{dt}\right)^2 + g_{tt} + g_{\phi\phi}\left(\frac{d\phi}{dt}\right)^2 + g_{rr}\left(\frac{dr}{dt}\right)^2 + g_{\theta\theta}\left(\frac{d\theta}{dt}\right)^2 \,. \tag{3.3.13}$$

In the ergoregion all the terms on the RHS are positive, as one can infer from (3.3.1)-(3.3.2). Since $g_{t\phi} < 0$ we deduce the angular velocity of the particle must be positive

$$\frac{d\phi}{dt} > 0 \,. \tag{3.3.14}$$

Thus, any massive particle moving in the ergoregion, also if its accelerating, must have that the angle ϕ increases with time t (*i.e.* the time a far away observer measures). This is rather striking since we could for instance have a very powerful rocket that accelerates all it can, but no matter what it does, it is not able to decrease ϕ. What we observe here is the effect of *frame-dragging*, namely that the rotation of the Kerr black hole is dragging particles in the ergoregion along with it, making it impossible to go in the direction of decreasing ϕ. It is called frame-dragging since one can view this effect as the local Inertial Systems in the ergoregion (also known as local inertial frames) being dragged along with the rotating event horizon of the Kerr black hole.

We now want to find a lower and upper bound on the angular velocity $d\phi/dt$ of the particle. Since g_{rr} and $g_{\theta\theta}$ are positive in the ergoregion we have

$$-2g_{t\phi}\frac{d\phi}{dt} - g_{\phi\phi}\left(\frac{d\phi}{dt}\right)^2 \geq \left(\frac{d\tau}{dt}\right)^2 + g_{tt}\,. \tag{3.3.15}$$

To find the lower and upper bound of $d\phi/dt$ we should consider particles that travel arbitrarily close to the speed of light, hence with arbitrarily small $d\tau$. Thus,

$$-2g_{t\phi}\frac{d\phi}{dt} - g_{\phi\phi}\left(\frac{d\phi}{dt}\right)^2 > g_{tt}\,. \tag{3.3.16}$$

See Figure 30 for an illustration of this inequality.

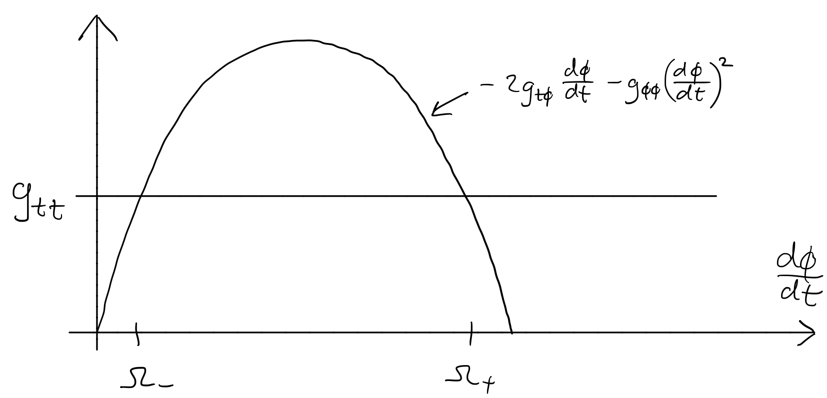

Figure 30: Illustration of the inequality (3.3.16) and how it leads to the inequality (3.3.19) via $\Omega_\pm(r,\theta)$ defined by (3.3.18).

Solving the equation

$$-2g_{t\phi}\frac{d\phi}{dt} - g_{\phi\phi}\left(\frac{d\phi}{dt}\right)^2 = g_{tt}\,, \tag{3.3.17}$$

gives

$$\frac{d\phi}{dt} = \Omega_\pm(r,\theta)\ ,\quad \Omega_\pm(r,\theta) = \frac{-g_{t\phi} \pm \sqrt{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}}{g_{\phi\phi}} = \frac{-g_{t\phi} \pm \sin\theta\sqrt{\Delta}}{g_{\phi\phi}}\,, \tag{3.3.18}$$

where we used (3.3.7). Since $g_{t\phi} < 0$, $g_{tt} > 0$ and $g_{\phi\phi} > 0$ both solutions are positive $\Omega_\pm(r,\theta) > 0$. Hence, we see that the lower and upper bound on a massive particle in the ergoregion is

$$\Omega_-(r,\theta) < \frac{d\phi}{dt} < \Omega_+(r,\theta)\,, \tag{3.3.19}$$

where r and θ are given by the momentary location of the particle. See Figure 30 for an illustration of this. Note in particular that $\Omega_-(r,\theta) > 0$ inside the ergoregion. On the ergosphere we have $g_{tt} = 0$, hence one finds $\Omega_-(r,\theta) = 0$. This is in accordance with the fact that outside the ergosphere the effect of frame-dragging is weakened so much that it is possible to have negative angular velocity $d\phi/dt < 0$.

If we instead consider the limit of approaching the event horizon $r \to r_+$ we find that the lower and upper bounds on the angular velocity both asymptote to the angular velocity

$$\Omega_{\mathrm{H}} = \Omega_\pm(r_+,\theta) = -\frac{g_{t\phi}}{g_{\phi\phi}}\bigg|_{r=r_+} = \frac{a}{r_0 r_+}\,, \tag{3.3.20}$$

where we used $r_+^2 + a^2 - r_0 r_+ = \Delta(r_+) = 0$. Thus, right outside the event horizon the angular velocity of a massive particle is in a small interval around Ω_{H}, and the closer the particle is to the event horizon, the closer the angular velocity $d\phi/dt$ of the particle must be to Ω_{H}. For this reason we interpret Ω_{H} as the *angular velocity of the Kerr black hole*. Indeed, the frame-dragging effect means that all particles close to the event horizon must approximately rotate with this angular velocity. Notice that Ω_{H} is independent of θ. This is not a priori an obvious fact since the Kerr black hole is not spherically symmetric.

For massless particles in the ergoregion one evidently get the lower and upper bounds

$$\Omega_-(r,\theta) \le \frac{d\phi}{dt} \le \Omega_+(r,\theta)\,, \tag{3.3.21}$$

since they have $d\tau = 0$. Thus, also massless particles have an angular rotation close to Ω_{H} near the event horizon.

3.4 Asymptotics and uniqueness of black holes

3.4.1 Asymptotic region

Imagine that one starts with Minkowski space, and then one places a localized object in it (*i.e.* an object of finite extension) then that object would curve space-time around it. But far away from the object one should get Minkowski space to a good approximation, and the farther one is from the object, the closer the metric will asymptote to that of

Minkowski space in a suitable coordinate system. A space-time with this property is called *asymptotically flat*.

A way to formulate this more precisely is as follows. For an asymptotically flat space-time one should be able to find a coordinate system (t, r, θ, ϕ) such that the metric $g_{\mu\nu}$ in these coordinates asymptotes to

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{3.4.1}$$

for $r \to \infty$. The Kerr metric (3.3.1)-(3.3.2) and the Schwarzschild metric (2.1.44) are examples of *asymptotically flat metrics*, *i.e.* metrics for asymptotically flat space-times. This makes sense since when one travels very far away from a black hole the gravitational pull should become very small and hence one should be able to use Minkowski space to a good approximation.

As explained in Section 2.1.2, one can read off the total mass of the Schwarzschild black hole by considering the leading correction to g_{tt} for $r \to \infty$. One can generalize this to include angular momentum as well. Imagine that we have a localized object with total mass M and angular momentum J around a particular rotation axis. The metric is asymptotically flat so it asymptotes to (3.4.1). Define now the angles such that the rotation axis for which we have an angular momentum coincides with $\sin\theta = 0$ which means that the associated rotation angle is ϕ. Then one can show that the leading corrections to g_{tt} and $g_{t\phi}$ are given by

$$g_{tt} \simeq -1 + \frac{2GM}{r}\ , \quad g_{t\phi} \simeq -2GJ\frac{\sin^2\theta}{r}\,, \tag{3.4.2}$$

for $r \to \infty$ up to corrections of order $1/r^2$.[29]

Consider now the Kerr metric (3.3.1)-(3.3.2). We compute

$$g_{tt} = -1 + \frac{r_0 r}{r^2 + a^2\cos^2\theta} \simeq -1 + \frac{r_0}{r}\ , \quad g_{t\phi} = -\frac{a r_0 r \sin^2\theta}{r^2 + a^2\cos\theta} \simeq -\frac{a r_0 \sin^2\theta}{r}\,. \tag{3.4.3}$$

Comparing this to (3.4.2) we find that for the Kerr black hole

$$M = \frac{r_0}{2G}\ , \quad J = \frac{a r_0}{2G}\,, \tag{3.4.4}$$

where M is the mass and J is the angular momentum associated with the rotation around the $\sin\theta = 0$ axis. Note that from (3.4.4) we have

$$\frac{J}{GM^2} = \frac{2a}{r_0}\,, \tag{3.4.5}$$

hence the dimensionless ratio a/r_0 is directly tied to $J/(GM^2)$.

[29]See Section 5.3 for a derivation of this (note that Section 5.3 is not part of this course).

3.4.2 Black hole uniqueness

The Schwarzschild metric (2.1.44) and the Kerr metric (3.3.1)-(3.3.2) both have the following properties

(1) The metric is asymptotically flat.

(2) There is an event horizon.

(3) Outside the event horizon the metric is a solution to the vacuum Einstein equations (1.6.39).

(4) The metric is stationary.

We have already discussed the properties (1), (2) and (3). Regarding (4), the definition of a *stationary* metric is that it is invariant under time translations. This is fulfilled provided that we can find coordinates (t, r, θ, ϕ) such that the metric is asymptotically flat for $r \to \infty$ and such that all the components of the metric are independent of the t coordinate. If in addition to being stationary, the metric is also invariant under the time-reversal coordinate transformation $t \to -t$ we say that it is *static*.

Clearly both the Schwarzschild metric (2.1.44) and the Kerr metric (3.3.1)-(3.3.2) are stationary. Instead, only the Schwarzschild metric (2.1.44) is static since the $g_{t\phi}$ term in the Kerr metric (3.3.1)-(3.3.2) changes sign under $t \to -t$.

Given that the Schwarzschild metric (2.1.44) and the Kerr metric (3.3.1)-(3.3.2) obey all four properties (1), (2), (3) and (4), one could ask: do there exist any other metrics that have these four properties?

Amazingly, the answer is no. This is the content of the so-called *uniqueness theorems* for black holes. From these uniqueness theorems we know that *any* metric that has the four properties (1), (2), (3) and (4) is either the Schwarzschild metric (2.1.44) or the Kerr metric (3.3.1)-(3.3.2), possibly in a different coordinate system. This means that since the Kerr metric is completely specified by its mass M and angular momentum J, we know that the black hole space-time is unique given M and J. Obviously, for the special case $J = 0$ we have the Schwarzschild black hole.

This is quite striking if one compares to matter distributions, *e.g.* planets or stars. For a planet or a star there are an infinite number of possible matter configurations for each value of M and J. Hence, one would need many more parameters to characterize them with some precision. Instead for black holes, one needs just two parameters.

This, in turn, has another striking consequence: seemingly we can lose information if something falls into the black hole. *E.g.* for a star collapsing to a black hole we lose the information about the precise composition of the star. Another way to say this is that it seems that we can lower the entropy of an isolated system containing a black hole if we let matter fall into the black hole. If so, it would contradict the second law of thermodynamics. We will come back to this point in Section 3.5.3 when discussing black hole thermodynamics.

3.4.3 Cosmic censorship hypothesis

Consider the metric (3.3.1)-(3.3.2) for the Kerr black hole. We now consider what happens for various values of a.

As already explained, we can assume $a \geq 0$ without loss of generality. We have shown above that one has an event horizon at r_+ given by (3.3.3) when $0 \leq a < r_0/2$, *i.e.* for $0 \leq J < GM^2$ using (3.4.5). Note in particular that $a = 0$ corresponds to the Schwarzschild black hole (2.1.44).

What happens for larger values of a? For $a = r_0/2$ one finds $J = GM^2$. This gives $r_+ = r_- = r_0/2$. This is known as the *extremal Kerr black hole*. Also in this case one has an event horizon.

Finally, what happens for $a > r_0/2$, *i.e.* $J > GM^2$? In this case $\Delta(r) > 0$ for all $r \geq 0$. Thus, we do not have an event horizon. Moreover, the space-time geometry is singular at $r = 0$, and this singularity is not covered by an event horizon in this case. Hence, for such a space-time the physics of the surrounding space-time can be affected by the singular part of the space-time. This is unlike the case $J \leq GM^2$ where the event horizon shields the curvature singularity from affecting the space-time outside the event horizon. In this case, where the singular part of the space-time is not shielded by an event horizon, one says that the space-time has a *naked singularity*.

It is generally accepted that one should not be able to create naked singularities from matter undergoing gravitational collapse. However, this has not been proven. The hypothesis that this is not possible is known as the *cosmic censorship hypothesis* (it was first stated by Sir Roger Penrose).

3.5 Black Hole Mechanics

In this section we discuss the laws of black hole mechanics, most importantly the first and second law. These are general laws that enable one to grasp aspects of a black hole seen as a whole, without needing to know the details about the space-time. For instance, from the second law of black hole mechanics one gets restrictions on how much energy a gravitational wave can have if it arises from a merger of two black holes. Related to the first and second laws are also the Penrose process that explains that one can extract energy from a Kerr black hole. Finally, we briefly comment on how the laws of black hole mechanics can be turned into laws of thermodynamics for black holes.

3.5.1 First law of black hole mechanics

The first law of black hole mechanics expresses a general relation for the change in mass of a black hole for a process that changes the black hole slightly, *e.g.* if one throws in an astronaut into a black hole. We begin by considering this in the case of the Schwarzschild black hole, defining two new physical quantities that one needs to be able to formulate the first law of black hole mechanics. These are the surface gravity and the area of the event horizon. After considering the first law for the Schwarzschild black hole we turn to the Kerr black hole where the first law of black hole mechanics involve the angular velocity Ω_{H} and the angular momentum J as well.

First law for the Schwarzschild black hole

Consider a Schwarzschild black hole with metric (2.1.44). Suppose we want to keep a massive particle at a certain fixed position outside the event horizon. Thus, we consider a particle with fixed r, θ and ϕ in the coordinates of (2.1.44). Since the particle feels a gravitational pull from the black hole it requires an acceleration to keep the particle there. We now compute this acceleration. Since the particle is not moving in the coordinate system of (2.1.44) the relativistic velocity is

$$u^t = \frac{1}{\sqrt{-g_{tt}}} = \left(1 - \frac{r_0}{r}\right)^{-\frac{1}{2}} \ , \quad u^r = u^\theta = u^\phi = 0 \,. \tag{3.5.1}$$

The normalization of u^t comes from the requirement $g_{\mu\nu}u^\mu u^\nu = -1$. From (1.4.41)-(1.4.42) we see that the acceleration is

$$a^\mu = \frac{D}{d\tau}u^\mu = \frac{du^\mu}{d\tau} + \Gamma^\mu_{\nu\rho}u^\nu u^\rho = \Gamma^\mu_{tt}(u^t)^2 = \frac{1}{1-\frac{r_0}{r}}\Gamma^\mu_{tt} \,. \tag{3.5.2}$$

We compute from the metric (2.1.44)

$$\Gamma^\mu_{tt} = -\frac{1}{2}g^{\mu\sigma}\partial_\sigma g_{tt} = -\delta^\mu_r \frac{1}{2}g^{rr}\partial_r g_{tt} = \delta^\mu_r \left(1 - \frac{r_0}{r}\right)\frac{r_0}{2r^2} \,. \tag{3.5.3}$$

Hence, the acceleration is

$$a^r = \frac{r_0}{2r^2} \ , \quad a^t = a^\theta = a^\phi = 0 \,. \tag{3.5.4}$$

This expresses the expected result that to keep the particle at a fixed position outside the event horizon of a Schwarzschild black hole, one needs an acceleration that is directed away from the black hole, *i.e.* in the direction of increasing r. However, a^r is not coordinate invariant, so to get a coordinate invariant measure of the magnitude of the acceleration required to keep the particle at a fixed position we compute the norm of a^μ

$$\sqrt{g_{\mu\nu} a^\mu a^\nu} = \frac{1}{\sqrt{1-\frac{r_0}{r}}} \frac{r_0}{2r^2} \,, \tag{3.5.5}$$

which is a coordinate invariant way of defining the magnitude of the acceleration. This is the magnitude of the acceleration as seen from point of view of an observer that is fixed in the same position as the particle, since one uses the proper time τ to measure the acceleration. We see that this quantity goes to infinity as one approaches the event horizon at $r = r_0$. This expresses that the closer the particle is placed to the event horizon, the more it needs to be accelerated.

If we instead want to know how the acceleration of the particle near the black hole is measured from point of view of an observer far away from the black hole, *i.e.* for $r \gg r_0$, we should measure the acceleration using the time t since that is the proper time for such an observer. In other words, we consider the path of the particle as function of t rather than τ. Hence, the acceleration is

$$a^\mu_{(t)} = \frac{D}{dt} u^\mu = \frac{d\tau}{dt} \frac{D}{d\tau} u^\mu = \frac{d\tau}{dt} a^\mu = \sqrt{1 - \frac{r_0}{r}} a^\mu \,, \tag{3.5.6}$$

where we used (1.4.36) and $d\tau = \sqrt{-g_{tt}} dt$. The magnitude of this acceleration is

$$\sqrt{g_{\mu\nu} a^\mu_{(t)} a^\nu_{(t)}} = \frac{r_0}{2r^2} \,. \tag{3.5.7}$$

Suppose now we put the particle just outside the event horizon. What is the magnitude of the acceleration that one needs to keep the particle there, as seen from point of view of a far away observer? This quantity is called the *surface gravity* of the Schwarzschild black hole, denoted as κ. Evaluating (3.5.7) at $r = r_0$ we find

$$\kappa = \frac{1}{2r_0} = \frac{1}{4GM} \,. \tag{3.5.8}$$

This represents the gravitational pull of the black hole close to the event horizon, as seen from point of view of a far away observer.

We now turn to measuring the area of the event horizon. This will turn out to be a very important quantity for black holes in general. Consider the Schwarzschild metric (2.1.44). For fixed time t and radius r we find

$$ds^2|_{r,t \text{ fixed}} = r^2 d\Omega^2 \,, \tag{3.5.9}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric of the unit sphere. Taking the limit $r \to r_0$ we get

$$\lim_{r\to r_0} ds^2|_{r,t \text{ fixed}} = r_0^2 d\Omega^2 \,. \tag{3.5.10}$$

This is the metric for a sphere of radius r_0. Thus, we can compute the area of this sphere

$$\mathcal{A} = 4\pi r_0^2 = 16\pi G^2 M^2 \,. \tag{3.5.11}$$

This quantity is called the *area of the event horizon*.

Note that while the Schwarzschild metric (2.1.44) has a coordinate singularity at $r = r_0$, which is why we took the limit $r \to r_0$ above, one can also compute the area of the event horizon directly from the setting $r = r_0$ in the metric (3.1.20) in Eddington-Finkelstein coordinates, with the same result.

Consider a small perturbation of the Schwarzschild black hole (2.1.44) in which the mass M changes from M to δM. This could be caused from the black hole absorbing matter, for instance. Given this perturbation, we compute using Eqs. (3.5.8) and (3.5.11)

$$\kappa\,\delta\mathcal{A} = \frac{1}{4GM}\delta(16\pi G^2 M^2) = 8\pi G\delta M \,. \tag{3.5.12}$$

Thus, we have derived:

First law of black hole mechanics for Schwarzschild black hole: Consider a small perturbation of the Schwarzschild black hole such that when it settles down again to a new stationary state it is again described as a Schwarzschild black hole. Then the changes in mass M and the area $\mathcal{A}$ of the event horizon obey

$$\delta M = \frac{\kappa}{8\pi G}\delta\mathcal{A} \,, \tag{3.5.13}$$

where κ is the surface gravity of the Schwarzschild black hole. This is known as *the first law of black hole mechanics* in the special case of the Schwarzschild black hole.

For astrophysical applications, the requirement that the new stationary state is described by the Schwarzschild black hole consists in demanding that one does not perturb it in a way that gives rise to a rotation, since then the new stationary state is instead described by the Kerr black hole. Thus, in astrophysical terms, we get a general first law of black hole mechanics if we can formulate it for the Kerr black hole. This is what we consider now.

First law for the Kerr black hole

One can also define the surface gravity κ for the Kerr black hole (3.3.1)-(3.3.2). However, the physical interpretation does not work as well as in the case of the Schwarzschild black hole. If we want a particle to be at fixed radial position just above the event horizon $r = r_+$ then from Section 3.3.3 we know that it will have to rotate with the event horizon with angular velocity $\Omega_{\rm H}$. Thus, we are interested in a particle with r and θ fixed, $\phi - \Omega_{\rm H} t$ fixed, and with r being infinitesimally close to r_+. Change now to the so-called *co-moving coordinates*

$$\tilde{t} = t \ , \quad \tilde{\phi} = \phi - \Omega_{\rm H} t \, . \tag{3.5.14}$$

In these coordinates the particle has fixed $\tilde{\phi}$. In line with (3.5.6) we consider the acceleration

$$a^\mu_{(\tilde{t})} = \frac{D}{d\tilde{t}} u^\mu = \frac{d\tau}{d\tilde{t}} a^\mu = \sqrt{-g_{\tilde{t}\tilde{t}}}\, a^\mu \, , \tag{3.5.15}$$

where a^μ is the covariant acceleration of the particle (1.4.41). After a considerable amount of computations, one finds that the magnitude of this acceleration for $r \to r_+$ is finite and equal to

$$\kappa = \sqrt{g_{\mu\nu} a^\mu_{(\tilde{t})} a^\nu_{(\tilde{t})}}\Big|_{r=r_+} = \frac{r_+ - r_-}{2 r_0 r_+} \, , \tag{3.5.16}$$

which is how we define the surface gravity of the Kerr black hole. Note in particular that this is independent of θ, which is not an obvious fact as the Kerr black hole is not spherically symmetric. The issue with the physical interpretation of this is that the time $\tilde{t}$ in the co-moving coordinates is measured by an observer at infinity that should be travelling arbitrarily faster than the speed of light, since the observers velocity for a given radius r is $r\Omega_{\rm H}$. This is related to the fact that t is not a good time-coordinate in the ergoregion where we are placing the above-mentioned particle. If one wants to make sense of all this, one has to instead introduce the notion of Killing horizons, where the surface gravity comes out naturally as a quantity that one can measure for any event horizon.

For the area of the event horizon of the Kerr black hole, we can follow the same procedure as for the Schwarzschild black hole without any issues. Thus, consider the metric (3.3.1)-(3.3.2). Keeping fixed t and r we find

$$ds^2|_{r,t \text{ fixed}} = \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma} \sin^2\theta d\phi^2 + \Sigma d\theta^2 \, . \tag{3.5.17}$$

Taking the limit $r \to r_+$ we get

$$\lim_{r\to r_+} ds^2|_{r,t \text{ fixed}} = \frac{(r_+^2 + a^2)^2}{\Sigma(r_+,\theta)} \sin^2\theta d\phi^2 + \Sigma(r_+,\theta) d\theta^2 \, . \tag{3.5.18}$$

To compute the area of a two-dimensional surface with a given two-dimensional metric, one should integrate over the squareroot of the determinant of the metric. In this case, the metric of the two-dimensional surface is the one written on the RHS of Eq. (3.5.18). The determinant of this metric is easy to compute since it is a diagonal metric, hence we get

$$\frac{(r_+^2+a^2)^2}{\Sigma(r_+,\theta)}\sin^2\theta\,\Sigma(r_+,\theta) = (r_+^2+a^2)^2\sin^2\theta\,. \tag{3.5.19}$$

Thus, the squareroot of the determinant is

$$(r_+^2+a^2)\sin\theta\,. \tag{3.5.20}$$

To find the area, we should integrate this over the whole surface. We get

$$\mathcal{A} = \int_0^\pi d\theta\int_0^{2\pi} d\phi (r_+^2+a^2)\sin\theta = 2\pi(r_+^2+a^2)\int_0^\pi d\theta\sin\theta = 4\pi(r_+^2+a^2)\,. \tag{3.5.21}$$

Using $r_+^2+a^2-r_0r_+ = \Delta(r_+) = 0$ we can write this as

$$\mathcal{A} = 4\pi r_0 r_+ = 8\pi G^2M^2\left(1+\sqrt{1-\frac{J^2}{G^2M^4}}\right)\,. \tag{3.5.22}$$

This is the area of the event horizon of the Kerr black hole.

We have defined the five parameters M, J, $\Omega_{\rm H}$, κ and $\mathcal{A}$ for the Kerr black hole in Eqs. (3.4.4), (3.3.20), (3.5.16) and (3.5.22). These can be seen as functions of the two parameters r_0 and a where $r_\pm$ is given by (3.3.3). One can now compute

$$\frac{\partial M}{\partial r_0} = \frac{\kappa}{8\pi G}\frac{\partial\mathcal{A}}{\partial r_0} + \Omega_{\rm H}\frac{\partial J}{\partial r_0}\,,\quad \frac{\partial M}{\partial a} = \frac{\kappa}{8\pi G}\frac{\partial\mathcal{A}}{\partial a} + \Omega_{\rm H}\frac{\partial J}{\partial a}\,. \tag{3.5.23}$$

This shows that for an arbitrary small change in the parameters of the Kerr black hole we have

$$\delta M = \frac{\partial M}{\partial r_0}\delta r_0 + \frac{\partial M}{\partial a}\delta a = \frac{\kappa}{8\pi G}\delta\mathcal{A} + \Omega_{\rm H}\delta J\,. \tag{3.5.24}$$

Thus, we have derived:

First law of black hole mechanics: Consider a small perturbation of a black hole in a stationary state. When it again settles down to a new stationary black hole the changes in the quantities M, J, $\Omega_{\rm H}$, κ and $\mathcal{A}$ obey

$$\delta M = \frac{\kappa}{8\pi G}\delta\mathcal{A} + \Omega_{\rm H}\delta J\,. \tag{3.5.25}$$

This is known as *the first law of black hole mechanics*. We assume here that one starts with a Kerr or a Schwarzschild black hole and that the perturbation does not involve charge of a magnitude comparable to M and J.

That one starts with either the Kerr or a Schwarzschild black hole and that the perturbation does not involve charge, are highly well-founded assumptions in astrophysics, as already discussed above. However, we note that one can also generalize the first law of black hole mechanics to include the possibility of an electric and/or magnetic charge that potentially can be useful for black holes of subatomic size.

3.5.2 Second law of black hole mechanics and the Penrose process

Increase of area of event horizon in absorption process

Consider the setting of the first law of black hole mechanics for the Schwarzschild black hole (3.5.13). A perturbation that should change the mass of the Schwarzschild black hole can only involve the black hole absorbing matter or radiation with a positive energy. Therefore one has always $\delta M > 0$. Using the first law (3.5.13) we see that this means the change in area is positive as well $\delta \mathcal{A} > 0$.

Let us consider instead the Kerr black hole. In this case, the presence of the ergoregion makes the analysis of a perturbation of the Kerr black hole considerably more interesting. Outside the ergoregion, one can use the coordinate t for the flow of time. This corresponds to the vector field T^μ given by

$$T^t = 1 \ , \quad T^\phi = T^r = T^\theta = 0 \, . \tag{3.5.26}$$

The energy of a particle with restmass m associated to the flow of time T^μ outside the ergoregion is measured as

$$E = -m g_{\mu\nu} T^\mu \frac{dx^\nu}{d\tau} \, , \tag{3.5.27}$$

in accordance with (2.2.12). Note that for geodesics this is a conserved quantity for the Kerr black hole since the metric does not depend on t (see Exercise 3.3). If the particle moves on a geodesic and it at some point is located in the asymptotic region far away from the black hole $r \gg r_+$, then we need to require that E is positive since in the asymptotic region $E = m dt/d\tau$ and hence E is positive if the particle moves forward in time t (so that $dt/d\tau > 0$).

However, inside the ergoregion t is not a good time coordinate. This can be seen from the fact that $T^2 = g_{tt} > 0$ in the ergoregion. How can we remedy this? We can go to the co-moving coordinates (3.5.14). In these coordinates we have

$$\tilde{g}_{\tilde{t}\tilde{t}} = g_{tt} + 2\Omega_{\rm H} g_{t\phi} + \Omega_{\rm H}^2 g_{\phi\phi} \, . \tag{3.5.28}$$

One can check that $\tilde{g}_{\tilde{t}\tilde{t}} = 0$ on the event horizon $r = r_+$ and that $\tilde{g}_{\tilde{t}\tilde{t}} < 0$ sufficiently close to the event horizon. The coordinate $\tilde{t}$ is thus a good time-coordinate sufficiently close to the event horizon (but not necessarily in the whole ergoregion). The flow of time associated with $\tilde{t}$ is the vector field $\tilde{\chi}^\mu$ that in the co-moving coordinates is given by $\tilde{\chi}^{\tilde{t}} = 1$ and $\tilde{\chi}^{\tilde{\phi}} = \tilde{\chi}^r = \tilde{\chi}^\theta = 0$. Transforming that vector field back to the original coordinates (t, r, θ, ϕ) for the Kerr metric (3.3.1)-(3.3.2), we find that the vector field χ^μ is given by

$$\chi^t = 1 \, , \quad \chi^\phi = \Omega_{\rm H} \, , \quad \chi^r = \chi^\theta = 0 \, . \tag{3.5.29}$$

Now we find that the energy of a particle with restmass m associated to the flow of time χ^μ near the event horizon is measured as

$$\mathcal{E} = -m g_{\mu\nu} \chi^\mu \frac{dx^\nu}{d\tau} \, . \tag{3.5.30}$$

The energy $\mathcal{E}$ of a particle should be positive if it is sufficiently close to the horizon since then the sign of $\mathcal{E}$ is the same as $d\tilde{t}/d\tau$ which is required to be positive as the particle moves forward in time $\tilde{t}$.[30] In line with (2.2.12) one can define the angular momentum of the particle as

$$L = m g_{\phi\nu} \frac{dx^\nu}{d\tau} \, . \tag{3.5.31}$$

For geodesics this is also a conserved quantity for the Kerr black hole as the metric does not depend on ϕ (see Exercise 3.3). Using this, we find

$$\mathcal{E} = E - \Omega_{\rm H} L \, . \tag{3.5.32}$$

Consider now a particle that is being absorbed by a Kerr black hole with mass M and angular momentum J. Just before the particle reaches the event horizon it should have $\mathcal{E} > 0$. Thus, one gets that $E > \Omega_{\rm H} L$ for the particle. From the conservation of energy and angular momentum one gets that $\delta M = E$ and $\delta J = L$. Thus, one finds

$$\delta M > \Omega_{\rm H} \delta J \, , \tag{3.5.33}$$

for the absorption of the particle. Combining this with the first law of black hole mechanics (3.5.25) we get

$$\delta \mathcal{A} > 0 \, , \tag{3.5.34}$$

where we used that $\kappa > 0$. Thus, we see that also in absorption processes for the Kerr black hole the area of the event horizon always increases.

[30] One can see this in the co-moving coordinates since for $r \to r_+$ we have $\tilde{g}_{\tilde{t}\tilde{\phi}} \to 0$ and hence $\mathcal{E} \simeq -m \tilde{g}_{\tilde{t}\tilde{t}} \frac{d\tilde{t}}{d\tau}$.

The Penrose process

In the above argument we see that having positive energy of the particle that is absorbed by the Kerr black hole means $\mathcal{E} > 0$. But $\mathcal{E}$ is not the energy that an asymptotic observer would measure. Instead, that is $E = \mathcal{E} + \Omega_{\mathrm{H}} L$. This difference gives rise to the possibility that one can extract energy out of the Kerr black hole in a process known as *the Penrose process* (named after Roger Penrose), that we now briefly describe.

In the Penrose process, we start with Particle A that travels toward a Kerr black hole from far away. Particle A has the energy E_A. We have $E_A > 0$ since it starts out in the asymptotic region. Suppose now that Particle A enters the ergoregion of the Kerr black hole and splits up into two particles near the event horizon: Particle B and Particle C. Particle B is absorbed by the black hole, while Particle C exits the ergoregion and flies away from the black hole. We assume the particles move along geodesics, hence their energies are conserved. Measuring the energy of particle C when it is far away from the black hole we must have $E_C > 0$. However, Particle B is created close to the event horizon where χ^μ gives the flow of time with time coordinate $\tilde{t}$. Thus, while particle B necessarily has $\mathcal{E}_B = E_B - \Omega_{\mathrm{H}} L_B > 0$, it does not need to have $E_B > 0$. Indeed, if L_B is sufficiently negative, this can happen, corresponding to Particle B having an angular momentum with the opposite sign of the Kerr black hole. Seen from far away, the black hole is now absorbing a particle with the energy $E_B < 0$ which means that the change in mass of the black hole is negative $\delta M = E_B < 0$. Hence, in the Penrose process the mass of the black hole is decreasing. Another way to see this is that the energy E_C of particle C is greater than the energy E_A of particle A. Thus, by energy conservation the black hole must have lost some of its mass. We conclude from this that the Penrose process makes it possible to extract energy out of a Kerr black hole, basically by letting it absorb a particle with negative energy. To understand this process better let us consider the angular momentum. The angular momentum L_B of Particle B has to be sufficiently negative for E_B to be negative, and we have that $\delta J = L_B$ in the absorption process. Thus, the Penrose process always involves a decrease of the angular momentum of the Kerr black hole. In conclusion, what is happening is that we are extracting some of the rotational energy of the Kerr black hole.

The Penrose process is very important for astrophysics. In astrophysics, there are highly spectacular phenomena associated with jets of particles that are accelerated away from rotating black hole along its rotation axis. It is believed that what drives this process is the extraction of the rotational energy of the rotating black hole similarly to what we described in the Penrose process, *i.e.* that the rotating black hole is absorbing negative energy and angular momentum [9, 10].

General formulation of the second law

We have shown above that for the absorption of particles assumed to be much smaller than the black hole itself, the area of the event horizon increases. More generally one can show *the second law of black hole mechanics*:

> **Second law of black hole mechanics:** The area $\mathcal{A}$ of the event horizon of a black hole can not decrease as a function of time (as observed asymptotically) under any process that can be described in the framework of General Relativity.

This is also known as *Hawkings area theorem* after Stephen Hawking that proved it mathematically in the early 1970's.[31]

An important consequence of the second law of black hole mechanics is that it introduces irreversability for physical processes involving black holes. Suppose we consider two Schwarzschild black holes with masses M_1 and M_2. Thus, their event horizons have areas $\mathcal{A}_i = 16\pi G^2 M_i^2$, $i = 1, 2$. Imagine now a process in which these two Schwarzschild black holes merge into a new Schwarzschild black hole. The resulting black hole should have mass $M_1 + M_2$, at least approximately (we assume only very few energy loss in the process). The area of its event horizon is therefore

$$\mathcal{A}_{12} = 16\pi G^2 (M_1 + M_2)^2 \,. \tag{3.5.35}$$

We see now that the resulting area is bigger than the combined area before the merger

$$\mathcal{A}_{12} = \mathcal{A}_1 + \mathcal{A}_2 + 32\pi G^2 M_1 M_2 > \mathcal{A}_1 + \mathcal{A}_2 \,. \tag{3.5.36}$$

Thus, this is a process that is allowed by the second law. But the reverse is not: it is not allowed that a black hole splits up into two black holes. This is clear from the second law of black hole mechanics since that would mean the total area would be decreasing. Hence, the merger of two black holes is an irreversible process.

[31] Note that in addition to the first and second laws of black hole mechanics one has also a zeroth and a third law. The zeroth law of black hole mechanics states that the surface gravity κ and the angular velocity Ω_{H} are constant on the event horizon, as one can infer from Eqs. (3.3.20) and (3.5.16). The third law of black hole mechanics says that the extremal Kerr black hole has zero surface gravity $\kappa = 0$. One can see this from (3.5.16) since the extremal Kerr black hole has $r_+ = r_-$.

Consider the GW150914 event mentioned above in Section 3.2 (see also Chapter 5). This actually gives us a concrete experiment in which we can test the second law of black hole mechanics. The origin of the GW150914 event was a merger of two black holes with masses

$$M_1 = 35.6 M_\circ \ , \quad M_2 = 30.6 M_\circ \, , \tag{3.5.37}$$

with neglible angular momenta. The merger resulted in a final black hole with

$$M = 63.1 M_\circ \ , \quad \frac{J}{GM^2} = 0.69 \, . \tag{3.5.38}$$

The total energy radiated as a gravitational wave is $3.1 M_\circ$. Using (3.5.11) we find that the total area before the merger is

$$\frac{\mathcal{A}_{\text{before}}}{16\pi G^2 M_\circ^2} = 35.6^2 + 30.6^2 = 2204 \, . \tag{3.5.39}$$

After the merger the area is

$$\frac{\mathcal{A}_{\text{after}}}{16\pi G^2 M_\circ^2} = \frac{63.1^2}{2} \left(1 + \sqrt{1 - 0.69^2} \right) = 3432 \, . \tag{3.5.40}$$

Hence we have $\mathcal{A}_{\text{after}} > \mathcal{A}_{\text{before}}$ in accordance with the second law. Thus, the second law of black hole mechanics holds up in this concrete experiment.

3.5.3 Black hole thermodynamics

The laws of black hole mechanics bear a striking resemblance to the laws of thermodynamics. As we now shall discuss, this is not a coincidence. Stephen Hawking discovered in 1975 that black holes radiates particles with a black body spectrum of temperature

$$T_{\text{BH}} = \frac{\hbar}{k_{\text{B}}} \frac{\kappa}{2\pi} \, , \tag{3.5.41}$$

where κ is the surface gravity of the black hole, $\hbar$ is the reduced Planck constant and k_{B} is the Boltzmann constant.[32] This is known as *Hawking radiation*. The argument for (3.5.41) uses Quantum Field Theory in the curved space-time background of the black hole. Thus, it is a quantum effect that goes beyond what can be described by the theory of General Relativity.

Hawking and Bekenstein argued around the same time that black holes obey the laws of thermodynamics. Based on (3.5.41) we see that this means that a black hole has the entropy

$$S_{\text{BH}} = \frac{k_{\text{B}}}{\hbar} \frac{\mathcal{A}}{4G} \, , \tag{3.5.42}$$

[32]To reinstate the speed of light one should have a c^3 in the numerator.

where $\mathcal{A}$ is the area of the event horizon, since then the first law of black hole mechanics (3.5.25) becomes the first law of thermodynamics

$$\delta M = T_{\mathrm{BH}} \delta S_{\mathrm{BH}} + \Omega_{\mathrm{H}} \delta J \,. \tag{3.5.43}$$

Moreover, the second law of black hole mechanics gives that S_{BH} is a non-decreasing function of time for processes that are described by General Relativity.

Bekenstein went further and defined a generalized entropy for an isolated system consisting of a black hole and the surrounding matter, radiation, electromagnetic fields and what ever else might be in its vicinity. This entropy has a contribution from the black hole as well as from everything surrounding it

$$S_{\mathrm{total}} = S_{\mathrm{BH}} + S_{\mathrm{rest}} \,, \tag{3.5.44}$$

where S_{rest} is the entropy of everything outside the event horizon of the black hole in the isolated system. S_{total} is thus meant to represent the total entropy of everything in the isolated system, including inside the event horizon. In line with the second law of thermodynamics, Bekenstein conjectured that S_{total} never decreases in any physical process.

For one thing, this can potentially resolve the issue raised in connection with the black hole uniqueness, namely that the entropy S_{rest} can decrease when it is created in a gravitational collapse from a remnant of a star. Since S_{BH} is macroscopically large, it is conceivable that the decrease in S_{rest} is offset by an increase in S_{BH} giving that S_{total} does not decrease. However, this requires that the non-trivial restriction that the entropy of the remnant, which is proportional to its volume, is less than the entropy of the resulting black hole, which is proportional to the area of the black hole.

Another problem that Bekensteins proposal has the potential to resolve is that the Hawking radiation of an isolated Schwarzschild black hole will make the Schwarzschild black hole lose mass since the radiation carries energy. If a Schwarzschild black hole loses mass it also decreases its area, and hence its entropy S_{BH}. However, the Hawking radiation also has an entropy which is part of S_{rest}. Thus, a logical resolution of the problem is that that the increase in entropy S_{rest} due to the Hawking radiation offset the decrease in S_{BH}. However, Hawking argued as part of his famous information paradox that the entropy of Hawking radiation is rather low since it only carries information about the temperature of the radiation and hence the resolution of this problem is part of the information paradox.

The introduction of an entropy for black holes (3.5.42) raises crucial questions that we are still working on answering:

- Can we find S_{BH} by counting the quantum states of a black hole?
- Why is S_{BH} proportional to an area rather than a volume? What does this mean for the quantum states of a black hole?
- Hawkings information paradox: in some cases a black hole can evaporate after loosing all its mass via Hawking radiation. But the Hawking radiation seemingly carries only the information about the temperature (3.5.41). Does this mean that one can lose the information that was contained in the black hole? If so, Quantum Mechanics would be in trouble.

All three questions go beyond General Relativity. Answering them requires a theory that in one way or the other can unify Quantum Mechanics and Quantum Field Theory with the theory of General Relativity.

3.6 Exercises for Chapter 3

Exercise 3.1. White holes.
We consider the Schwarzschild line-element (2.1.44).

- Express the Schwarzschild line-element (2.1.44) in the so-called *outgoing Eddington-Finkelstein coordinates* (u, r, θ, ϕ) where u is defined by (3.1.17).
- Find du/dr for infalling and outgoing radial null curves. Use this result to draw a r-u diagram illustrating the lightcones formed by the infalling and outgoing radial null curves for outgoing Eddington-Finkelstein coordinates u and r. This should be the analogue to the r-v diagram for the lightcones in Figure 22 for Eddington-Finkelstein coordinates v and r.
- Argue that r necessarily will increase as time moves forward for $r < r_0$.
- Argue that a particle inside the Schwarzschild radius r_0 necessarily either ends up at $r = r_0$ or that it goes outside the Schwarzschild radius $r > r_0$. Argue that a particle with $r > r_0$ can never arrive inside the Schwarzschild radius $r < r_0$.

Comment: The interpretation of the above results is that the Schwarzschild line-element (2.1.44) not only can be used to describe a black hole with an event horizon at $r = r_0$, for which one can never escape. It can also describe a *white hole* which has the opposite behavior of a black hole, hence the name. The surface at $r = r_0$ in the white hole is called the past event horizon (as opposed to future event horizon for black holes) and the region $r \leq r_0$ is the white hole. Particles in a white hole will necessarily leave it, and it is never possible to enter a white hole from outside. Unlike black holes, white holes are not thought to exist since there does not seem to be any physical scenario under which they could be formed. See also Section 3.1.6 for further comments on this.

Exercise 3.2. Photon sphere.

Consider light rays (photons) moving on the Schwarzschild black hole space-time (2.1.44). For motion in the plane $\theta = \pi/2$ we can use the equation of motion (2.2.55) where $X = GM/r$.

- Using Eq. (2.2.55) find the radius r for which a photon would travel on a circle. The sphere with this radius of known as the *photon sphere* of the Schwarzschild black hole.

- Using the ansatz
$$X(\phi) = \frac{1}{3} + \delta X(\phi)\,, \tag{3.6.1}$$
where δX is considered small, derive photon orbits near the photon sphere obey
$$\frac{d^2\delta X}{d\phi^2} = \delta X\,. \tag{3.6.2}$$

- Argue that the most general solution to (3.6.2) is
$$\delta X(\phi) = A\cosh\phi + B\sinh\phi\,, \tag{3.6.3}$$
and that this translates to
$$r(\phi) = \frac{3}{2}r_0 - \frac{9r_0}{2}(A\cosh\phi + B\sinh\phi)\,, \tag{3.6.4}$$
to leading order for photon orbits near the photon sphere.

- Give a physical interpretation of the solution
$$r(\phi) = \frac{3}{2}r_0 + L\cosh\phi\,. \tag{3.6.5}$$
with a positive $L \ll r_0$.

- Supposing that $L = \frac{r_0}{1000}$, roughly how many times would a photon revolve around the photon sphere, before it escapes the black hole again?

Exercise 3.3. Symmetries of Kerr metric.

In this exercise we consider symmetries of the Kerr metric (3.3.1)-(3.3.2).

- Define the vector fields T^μ and J^μ by

$$T^\mu = (T^t, T^r, T^\theta, T^\phi) = (1, 0, 0, 0) \ , \quad J^\mu = (J^t, J^r, J^\theta, J^\phi) = (0, 0, 0, 1) \,. \tag{3.6.6}$$

  Show that T^μ and J^μ are Killing vector fields as defined in Exercise 2.5. Show that T^μ being a Killing vector in the coordinates of the Kerr metric (3.3.1)-(3.3.2) is equivalent to

$$\partial_t g_{\mu\nu} = 0 \,, \tag{3.6.7}$$

  and show that J^μ being a Killing vector in the coordinates of the Kerr metric (3.3.1)-(3.3.2) is equivalent to

$$\partial_\phi g_{\mu\nu} = 0 \,. \tag{3.6.8}$$

- Define for any time-like curve $x^\mu(\tau)$ the scalar fields

$$E = -m g_{\mu\nu} T^\mu \frac{dx^\nu}{d\tau} \ , \quad L = m g_{\mu\nu} J^\mu \frac{dx^\nu}{d\tau} \,. \tag{3.6.9}$$

  Argue using the results of Exercise 2.5 that E and L are conserved on any time-like geodesic $x^\mu(\tau)$.

Exercise 3.4. Ingoing Kerr coordinates for the Kerr metric.

Consider the Kerr metric (3.3.1)-(3.3.2) in (t, r, θ, ϕ) coordinates. Define a new radial coordinate $\rho(r)$ by

$$\frac{d\rho}{dr} = \frac{r^2 + a^2}{\Delta} \,. \tag{3.6.10}$$

This is analogous to the tortoise radial coordinate (3.1.14) for Schwarzschild. One can find $\rho(r)$ explicitly by integrating the above, but this will not be important below. Using this, we define the two new coordinates $v(t, r)$ and $\chi(\phi, r)$ by

$$v = t + \rho(r) \ , \quad d\chi = d\phi + \frac{a}{\Delta} dr \,, \tag{3.6.11}$$

With this we define a new coordinate system (v, r, θ, χ) for the Kerr metric called *ingoing Kerr coordinates*.

- Show that the Kerr metric (3.3.1) in the new coordinates (v, r, θ, χ) takes the form

$$\begin{aligned} ds^2 = &- \left(1 - \frac{r_0 r}{\Sigma}\right) dv^2 + 2 dv dr - \frac{2 a r_0 r}{\Sigma} \sin^2\theta dv d\chi - 2a \sin^2\theta d\chi dr \\ &+ \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma} \sin^2\theta d\chi^2 + \Sigma d\theta^2 \,, \end{aligned} \tag{3.6.12}$$

where Σ and Δ are defined in (3.3.2). [NOTE: If this becomes a long calculation you can also just select one or two particular component to check, *e.g.* g_{vr}, $g_{\chi r}$ or g_{rr}.]

- In Section 3.3.2 we have shown that $dr/dt \to 0$ for $r \to r_+$ which means that in the (t, r, θ, ϕ) coordinates the light cones close up at the event horizon $r = r_+$. Explain why dr/dv does not need to go to zero for $r \to r_+$ in the ingoing Kerr coordinates, and hence that one avoids the closing of the light cones.

- Consider radial null curves in the ingoing Kerr coordinates. Show that

$$\begin{aligned}
&\text{Infalling radial null curves:} \quad dv = 0\,, \\
&\text{Outgoing radial null curves:} \quad \frac{dr}{dv} = \frac{1}{2}\left(1 - \frac{r_0 r}{\Sigma}\right)\,.
\end{aligned} \tag{3.6.13}$$

  Use this to argue that light cannot escape from behind the event horizon at $r = r_+$.

Exercise 3.5. Schwarzschild black hole in Kruskal-Szekeres coordinates.
We consider the line-element (2.1.44) for the Schwarzschild black hole. This line-element is in coordinates $x^\mu = (t, r, \theta, \phi)$. In this exercise we consider the coordinate transformation to the so-called Kruskal-Szekeres coordinates $\tilde{x}^\mu = (\mathcal{T}, \mathcal{R}, \theta, \phi)$ given by (3.1.50) where u and v are defined by (3.1.17) and (3.1.14).

- Show that in coordinates (u, v, θ, ϕ) the line-element (2.1.44) is

$$ds^2 = -\left(1 - \frac{r_0}{r}\right) du dv + r^2 (d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{3.6.14}$$

  where u and v are defined by (3.1.17) and (3.1.14).

- Show that the line-element (2.1.44) in Kruskal-Szekeres coordinates $\tilde{x}^\mu = (\mathcal{T}, \mathcal{R}, \theta, \phi)$ is given by Eq. (3.1.51) where $r(\mathcal{T}, \mathcal{R})$ is given indirectly from Eq. (3.1.52). [Hint: You can use (3.6.14) as an intermediate step.]

- Using (3.1.52) and by considering outgoing radial null curves (*i.e.* with constant u) in the line-element (3.1.51), show that the event horizon of the Schwarzschild black hole is located at $\mathcal{T} = \mathcal{R}$. Where is the region outside the event horizon? And where is the inside of the black hole?

- Argue that $\mathcal{T}$ in the line-element (3.1.51) can be used as a time-coordinate both inside and outside the Schwarzschild black hole.

- Using (3.1.52) and by considering infalling radial null curves (*i.e.* with constant v) in the line-element (3.1.51), show that there is a past event horizon of a white hole (see Exercise 3.1) located at $\mathcal{T} = -\mathcal{R}$.

Comment: The Kruskal-Szekeres space-time (3.1.51) gives the maximal possible extension of the Schwarzschild line-element (2.1.44). This not only includes inside the black hole, but also a past region with a white hole, and if one allows $\mathcal{R}$ to be negative as well, another asymptotic region outside a black hole and white hole for negative $\mathcal{R}$. Since it includes a white hole (see Exercise 3.1), the whole extended Kruskal-Szekeres space-time cannot arise from gravitational collapse, and for this reason the parts with negative $\mathcal{R}$ and the past event horizon $\mathcal{T} = -\mathcal{R}$ are not relevant for applications to black holes in astrophysics. In any case, one can still use it for studying the inside of the Schwarzschild black hole. And it is interesting to study it as a theoretical laboratory to learn more about General Relativity. See Section 3.1.6 for more on the Kruskal-Szekeres space-time.

Exercise 3.6. Merger of black holes and the second law of black hole mechanics. Consider two Schwarzschild black holes with masses M_1 and M_2 that spiral around each other until they merge together into a new Kerr black hole with mass M and angular momentum $J = \alpha G M^2$ where $0 < \alpha \leq 1$. The event releases a gravitational wave with energy E.

- Using the second law of black hole mechanics, show that

$$\frac{1+\sqrt{1-\alpha^2}}{2} \geq \left(\frac{M_1}{M}\right)^2 + \left(\frac{M_2}{M}\right)^2 . \tag{3.6.15}$$

- If the masses of the two merging black holes are equal $M_1 = M_2$ and the resulting Kerr black hole has $\alpha = \sqrt{3}/2$, what is maximal possible energy (measured in units of M) of the gravitational radiation that the event can emit according to the second law of black hole mechanics?

- If $M_1 = 3M_2$ and the resulting energy of the emitted gravitational waves is $E = \frac{1}{5}M$ what is the maximal possible value of $\alpha = J/(GM^2)$ (*i.e.* the maximal possible value of J measured in units of GM^2) for the resulting black hole according to the second law of black hole mechanics?

- In the GW170814 event (see also Section 5.2.3) two black holes with approximately no angular momenta and masses $M_1 = 31 M_\circ$ and $M_2 = 25 M_\circ$ inspiraled and merged 1.8 billion years ago into a Kerr black hole with mass $M = 53 M_\circ$ and angular momentum (in units of GM^2) $\alpha = 0.7$ while emitting the energy $3M_\circ$ as gravitational waves. Is the GW170814 event in accordance with the second law of black hole mechanics?

Exercise 3.7. First law of black hole mechanics.

For the Kerr black hole we have the mass M and angular momentum J given by (3.4.4), the angular velocity Ω_{H} by (3.3.20), the surface gravity κ by (3.5.16) and the area of the event horizon $\mathcal{A}$ by (3.5.22).

- Show the two relations in Eq. (3.5.23).
- Show using Eq. (3.5.23) the first law of black hole mechanics (3.5.25).

4 Cosmology

4.1 Friedmann-Lemaître-Robertson-Walker Metric

4.1.1 The Cosmological Principle

Cosmology is the study of the evolution and dynamics of the universe as a whole. This means one is describing the universe on the very largest scales. The *Cosmological Principle* asserts that the universe for a given moment in time looks the same everywhere, *i.e.* that no region or direction of the universe is special. This principle concerns the largest scales in the universe, thus one should imagine averaging over all the matter and energy present at smaller scales (*e.g.* stars, planets, dust, galaxies, clusters of galaxies, etc.) to find the average distribution of matter at the largest scales. According to the Cosmological Principle this averaged distribution of matter looks the same everywhere.

The observable universe today is about 100 billion light years in diameter. The cosmological principle should apply to scales above at least 100 million light years.[33] While one has some evidence for the Cosmological Principle from observing the visible universe, the best evidence for the Cosmological Principle is not from looking at todays universe, but instead from the cosmic microwave background (CMB) which tells us about the state of the universe at a very early moment. It is found that the CMB has a constant temperature of 2.725 Kelvin with variations less than 1/1000 Kelvin.

Since the space-time geometry of the universe is sourced by the matter and energy in it, it follows that at the largest scales the space-time geometry should look the same everywhere for any given time. We shall consider the consequence of this in the following.

4.1.2 Derivation of the Friedmann-Lemaître-Robertson-Walker metric

We now consider the space-time geometry of the universe as a whole, on the largest scales. We begin by assuming the existence of a coordinate system

$$(t, x^1, x^2, x^3)\,, \tag{4.1.1}$$

in which t is a *cosmic standard time* that can parametrize the evolution of the universe. For a given t it follows from the Cosmological Principle that the space-time geometry should be

[33] 1 million lightyears corresponds to 0.3066 Mpc (Megaparsecs). Hence 100 billion light years is about 30 000 Mpc and 100 million light years is about 30 Mpc.

- **Homogeneous.** The geometry should look the same everywhere for the given t, *i.e.* one has invariance under spatial translations.

- **Isotropic.** That the geometry is isotropic around a given point x^i means that it should look the same in all directions from that point x^i, *i.e.* it should be spherically symmetric around this point. Combining this with homogeneity, it means the universe should be spherically symmetric around any point in the universe, for the given t.

The averaged distribution of matter and energy is at rest for constant x^i in this coordinate system. If this was not true, it would mean that the velocity $u^i = dx^i/d\tau$ would be non-zero, which would mean that the direction of the velocity would be special and this would clearly be at odds with having isotropy around all points of the universe.

As a consequence of the averaged matter and energy being at rest for constant x^i it follows that any path with the position x^i fixed is a time-like geodesic. Consider a massive test particle at constant x^i. One has $d\tau^2 = -g_{tt}dt^2$ where τ is the proper time of the particle. Clearly g_{tt} cannot depend on x^i because of the homogenuity of the universe and the t dependence can be absorbed in the definition of t without affecting the fact that the averaged matter and energy are at rest. Hence, one can choose $g_{tt} = -1$. This is natural since then the cosmic standard time t is equal to the proper time τ of the matter and energy at rest.

One can furthermore argue that $g_{ti} = 0$. A non-zero g_{ti} for a particular time t and point x^i would imply a preferred direction at that point which is in contradiction with having isotropy in that point. Thus, the line-element in the coordinates (4.1.1) can be written

$$ds^2 = -dt^2 + g_{ij}dx^i dx^j \,. \tag{4.1.2}$$

It follows from this line-element that a particle at a fixed position x^i is a geodesic. This corresponds to the relativistic velocity

$$u^\mu = \frac{dx^\mu}{d\tau} = (1,0,0,0)\,. \tag{4.1.3}$$

One can see that this fulfil the geodesic equation (1.3.45) provided $\Gamma^\mu_{tt} = 0$. This follows from $2\partial_t g_{t\mu} = \partial_\mu g_{tt}$ that is easily seen to be satisfied for the line-element (4.1.2).

For a fixed time $t = t_0$ the metric $g_{ij}|_{t=t_0}$ should describe a homogeneous and isotropic space. Consider a three-dimensional space with line-element

$$d\sigma^2 = \gamma_{ij}dx^i dx^j \,. \tag{4.1.4}$$

Demanding this to be homogeneous and isotropic means that it should be *maximally symmetric*. For such a space the high degree of symmetry means that the Riemann curvature tensor should take the form

$$ {}^{(\gamma)}R_{ijkl} = \frac{k}{a^2}(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}) \,, \tag{4.1.5} $$

with $a > 0$ a constant of dimensional length and with

$$ k \in \{-1, 0, 1\} \,. \tag{4.1.6} $$

Here ${}^{(\gamma)}R_{ijkl}$ is our notation for the Riemann curvature tensor of the metric γ_{ij} where $i, j, k, l = 1, 2, 3$. From (4.1.5) we compute the Ricci tensor

$$ {}^{(\gamma)}R_{ij} = \frac{2k}{a^2}\gamma_{ij} \,. \tag{4.1.7} $$

Since this geometry is isotropic around any point we can choose at random a point, and then the geometry will be spherically symmetric around that point. Hence, we can use that we have derived in Section 2.1.1 that the line-element for a spherically symmetric space-time always can be written in the form (2.1.16). Taking away the time-direction this means that we can put the line-element (4.1.4) on the form

$$ d\sigma^2 = e^{2\beta(\tilde{r})}d\tilde{r}^2 + \tilde{r}^2(d\theta^2 + \sin^2\theta d\phi^2) \,. \tag{4.1.8} $$

We can now find the Ricci tensor for the line-element (4.1.8) from the Ricci tensor (2.1.31)-(2.1.32) calculated for the line-element (2.1.17) in Section 2.1.2 by setting $\alpha = 0$. This gives

$$ {}^{(\gamma)}R_{\tilde{r}\tilde{r}} = \frac{2}{\tilde{r}}\partial_{\tilde{r}}\beta \;, \quad {}^{(\gamma)}R_{\theta\theta} = 1 + e^{-2\beta}(\tilde{r}\partial_{\tilde{r}}\beta - 1) \;, \quad {}^{(\gamma)}R_{\phi\phi} = \sin^2\theta \, {}^{(\gamma)}R_{\theta\theta} \,. \tag{4.1.9} $$

For this to be consistent with (4.1.7) we need

$$ {}^{(\gamma)}R_{\tilde{r}\tilde{r}} = \frac{2k}{a^2}\gamma_{\tilde{r}\tilde{r}} = e^{2\beta}\frac{2k}{a^2} \;, \quad {}^{(\gamma)}R_{\theta\theta} = \frac{2k}{a^2}\gamma_{\theta\theta} = \frac{2k\tilde{r}^2}{a^2} \,. \tag{4.1.10} $$

This is equivalent to the equations

$$ \partial_{\tilde{r}}\beta = \frac{k}{a^2}\tilde{r}e^{2\beta} \;, \quad 1 + e^{-2\beta}(\tilde{r}\partial_{\tilde{r}}\beta - 1) = \frac{2k\tilde{r}^2}{a^2} \,. \tag{4.1.11} $$

Eliminating $\partial_{\tilde{r}}\beta$ by combining these two equations we find

$$ e^{-2\beta} = 1 - \frac{k\tilde{r}^2}{a^2} \,, \tag{4.1.12} $$

which is seen to satisfy both equations (4.1.11). With this, we get the line-element

$$d\sigma^2 = \frac{d\tilde{r}^2}{1 - \frac{k\tilde{r}^2}{a^2}} + \tilde{r}^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{4.1.13}$$

Introducing the dimensionless radial coordinate

$$r = \frac{\tilde{r}}{a}\,, \tag{4.1.14}$$

we can write the line-element (4.1.13) as

$$d\sigma^2 = a^2\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]\,. \tag{4.1.15}$$

This metric can describe any three-dimensional maximally symmetric space. The space is characterized by the length $a > 0$ and $k \in \{-1, 0, 1\}$. With this, we have shown that the metric $g_{ij}|_{t=t_0}$ can be put in the form (4.1.15). Since k cannot be time-dependent this means that any time dependence should be put in the length $a = a(t)$ in (4.1.15). Thus, we have derived:

Friedmann-Lemaître-Robertson-Walker (FLRW) metric: The space-time geometry at the largest scales is described by the *Friedmann-Lemaître-Robertson-Walker (FLRW) metric*

$$ds^2 = -dt^2 + a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]\,. \tag{4.1.16}$$

This metric is determined by $k \in \{-1, 0, 1\}$ and the function $a(t)$ known as the *scale factor*. Since averaged matter at the largest scales are at rest in the metric (4.1.16) one says that the coordinates of (4.1.16) are *co-moving*, *i.e.* moving along with the matter.

4.1.3 Geometric interpretation

Consider the three-dimensional maximally symmetric space (4.1.15) corresponding to the geometry of the FLRW metric (4.1.16) for fixed time t. From (4.1.5) we see that if $k = 0$ the Riemann curvature tensor is zero. Instead for $k \neq 0$ we see from (4.1.7) that $k = -1$ corresponds to negative curvature and $k = 1$ to positive curvature.

Introduce the new radial coordinate χ by demanding

$$d\chi = \frac{dr}{\sqrt{1-kr^2}}\,, \tag{4.1.17}$$

and that $r = 0$ corresponds to $\chi = 0$. We have

$$r(\chi) = \begin{cases} \sin\chi & \text{for } k = 1\,, \\ \chi & \text{for } k = 0\,, \\ \sinh\chi & \text{for } k = -1\,. \end{cases} \tag{4.1.18}$$

Using this one finds that the three possible values of k corresponds to the following geometries

- For $k = 1$ the metric (4.1.15) is

$$d\sigma^2 = a^2\Big[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\Big]\,. \tag{4.1.19}$$

  This is the metric of a three-sphere S^3 with radius a.[34] Thus, this is a periodic space with finite extension and volume. A universe with $k = 1$ is called a *closed universe*.

- For $k = 0$ the metric (4.1.15) is

$$d\sigma^2 = a^2\Big[d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\phi^2)\Big]\,. \tag{4.1.20}$$

  This is the metric for three-dimensional Euclidean space in spherical coordinates. Depending on the choice of topology this can be of finite or infinite extension. A universe with $k = 0$ is called a *flat universe*.

- For $k = -1$ the metric (4.1.15) is

$$d\sigma^2 = a^2\Big[d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\Big]\,. \tag{4.1.21}$$

  This is the metric of a three-dimensional hyperboloid (also known as a hypersphere) which is a maximally symmetric space with negative curvature. Depending on the global topology the space can either be of finite or infinite extension. A universe with $k = -1$ is called an *open universe*.

4.2 Hubble's Law

Depending on whether the time-derivative of the scale factor

$$\dot{a}(t) = \frac{da}{dt}\,, \tag{4.2.1}$$

[34] A three-sphere is a three-dimensional sphere. The three-sphere of radius a is the three-dimensional surface in four-dimensional Euclidean space $\mathbb{R}^4$ that solves the equation $a^2 = \sum_{n=1}^{4}(y^n)^2$ where $\mathbb{R}^4$ is parametrized by the Cartesian coordinates (y^1, y^2, y^3, y^4). The embedding of the three-sphere that gives the metric (4.1.19) is $y^1 = a\cos\chi$, $y^2 = a\sin\chi\cos\theta$, $y^3 = a\sin\chi\sin\theta\cos\phi$ and $y^4 = a\sin\chi\sin\theta\sin\phi$.

is positive, negative or zero at a given time t, the FLRW metric (4.1.16) describes an expanding universe, a contracting universe, or a universe at rest. Before Hubble made his observations in 1929 the consensus was that the universe is static, *i.e.* always at rest.

4.2.1 Derivation of Hubble's law

Consider the situation where a lightray is sent from a galaxy different from ours and received by us (in our galaxy). We consider this in terms of the FLRW metric (4.1.16) using the radius χ defined by (4.1.17)-(4.1.18). We choose the coordinates so that our galaxy is at $\chi = 0$ and the galaxy from which the light is emitted is at radius $\chi = \chi_0$. Since averaged matter is at rest in the metric (4.1.16) the position of our galaxy $\chi = 0$ and the other galaxy $\chi = \chi_0$ do not change with time.[35] We consider a lightray with period T_i sent out from $\chi = \chi_0$ at time $t = t_i$ and received at $\chi = 0$ at time $t = t_f$ now with period T_f. See Figure 31 for an illustration.

Figure 31: Illustration of a lightray sent from another galaxy to our galaxy.

Since the lightray is travelling along a radial null curve we have

$$0 = ds^2 = -dt^2 + a(t)^2 d\chi^2 \,. \tag{4.2.2}$$

Hence

$$\frac{dt}{a(t)} = -d\chi \,. \tag{4.2.3}$$

Integrating, we get

$$\int_{t_i}^{t_f} \frac{dt}{a(t)} = -\int_{\chi_0}^{0} d\chi = \chi_0 \,. \tag{4.2.4}$$

From time t_i to $t_i + T_i$ the wave of the lightray at the source has gone through one period. Correspondingly, this is received at t_f and $t_f + T_f$. Hence, since χ_0 is time-independent

[35] We assume here that the two galaxies are both approximately at rest relative to the averaged matter of the universe. To leading order, this is a good approximation.

we get

$$\int_{t_i}^{t_f} \frac{dt}{a(t)} = \chi_0 = \int_{t_i+T_i}^{t_f+T_f} \frac{dt}{a(t)} \,. \tag{4.2.5}$$

This gives

$$\int_{t_i}^{t_f} \frac{dt}{a(t)} = \int_{t_i}^{t_f} \frac{dt}{a(t)} + \int_{t_f}^{t_f+T_f} \frac{dt}{a(t)} - \int_{t_i}^{t_i+T_i} \frac{dt}{a(t)} \,. \tag{4.2.6}$$

Hence,

$$\int_{t_f}^{t_f+T_f} \frac{dt}{a(t)} = \int_{t_i}^{t_i+T_i} \frac{dt}{a(t)} \,. \tag{4.2.7}$$

Since a typical lightray has a period of order 10^{-14} seconds and the scale factor $a(t)$ concerns the largest scales in the universe we have that $a(t)$ is constant over the time scale of one period. Hence,

$$\frac{T_f}{a(t_f)} = \frac{T_i}{a(t_i)} \,. \tag{4.2.8}$$

A lightray with period T has frequency $f = 1/T$ and wavelength $\lambda = 1/f = T$. Let λ_i be the emitted wavelength (at $\chi = \chi_0$) and λ_f be the received wavelength (at $\chi = 0$). Then we define the *redshift parameter* z by

$$z = \frac{\lambda_f - \lambda_i}{\lambda_i} \,. \tag{4.2.9}$$

We see that if $z > 0$ then $\lambda_f > \lambda_i$, corresponding to a redshift. Instead if $z < 0$ then $\lambda_f < \lambda_i$, correponding to a blueshift of the light. From (4.2.8) we get

$$z = \frac{a(t_f) - a(t_i)}{a(t_i)} \,. \tag{4.2.10}$$

Thus, if the universe expands ($a(t_f) > a(t_i)$) the light is redshifted, while if it contracts ($a(t_f) < a(t_i)$) it is blueshifted. All this is in accordance with the intuition that in an expanding (contracting) universe the wavelength of a lightray will expand (contract).

Consider now a lightray sent out from a nearby galaxy. Then we can approximate

$$a(t_i) \simeq a(t_f) + \dot{a}(t_f)(t_i - t_f) \,, \tag{4.2.11}$$

where $\dot{a} = da/dt$. Inserting this in (4.2.10) we get

$$z \simeq \frac{\dot{a}(t_f)}{a(t_f)}(t_f - t_i) \,. \tag{4.2.12}$$

The distance to the galaxy where the lightray is emitted is approximately

$$L \sim t_f - t_i \,, \tag{4.2.13}$$

since it is nearby. Using this with (4.2.12) we have derived:

Hubble's law: For nearby galaxies the redshift parameter z is proportional to the distance L to the galaxy to a good approximation

$$z \simeq H_0 L \,, \tag{4.2.14}$$

where

$$H_0 = \frac{\dot{a}(t_f)}{a(t_f)} \,, \tag{4.2.15}$$

is called the *Hubble constant*. This linear relation (4.2.14) between z and L is known as *Hubble's law*. The difference in arrival time t_f for the measurements taken is many orders of magnitude smaller than the cosmological time-scales over which $a(t)$ varies (*e.g.* hundred years as compared to a billion years). Thus, in this sense one can regard H_0 as a constant, *i.e.* as measuring the current value of $\dot{a}(t)/a(t)$.

4.2.2 Measurement of the Hubble constant

In 1929 Hubble measured that light from nearby galaxies are redshifted and he found that the redshift was proportional to the distance. From this he conjectured Hubble's law (4.2.14). This was the first experimental evidence of the expansion of the universe. It is only in the last 20 years that a precise measurement of the Hubble constant H_0 has been possible. Current measurements give (Planck 2015) [36]

$$H_0 = 67.7\,\mathrm{km\,s^{-1}\,Mpc^{-1}} \,, \tag{4.2.16}$$

where Mpc stands for mega parsec which is equal to

$$1 \text{ Mpc} = 3.262 \text{ million light years} = 3.09 \cdot 10^{22}\,\mathrm{m} \,. \tag{4.2.17}$$

This gives the current rate of expansion of the universe. One can also express it as a time

$$\frac{1}{H_0} = 14.4 \text{ billion years} \,, \tag{4.2.18}$$

and as a length

$$\frac{c}{H_0} = 14.4 \text{ billion light years} = 4420 \text{ Mpc} = 1.36 \cdot 10^{26}\,\mathrm{m} \,. \tag{4.2.19}$$

[36] Here and below when writing *Planck 2015* we are referring to the measurements of cosmic microwave background by the Planck satellite from 2009 to 2013 that after an extensive data analysis was released by the Planck collaboration in 2015. An updated analysis was released in 2018 with minor modifications.

These are the cosmological length and time scales. They provide an order-of-magnitude estimate of how big and old the universe can be, and what we mean when we talk about large scales in cosmology.

However, recent measurement from supernovae observations gives instead a quite different value of the Hubble constant

$$H_0 = (74.0 \pm 1.4)\,\mathrm{km\,s^{-1}\,Mpc^{-1}}\,, \tag{4.2.20}$$

in contrast with the Planck 2015 measurement which is $(67.74 \pm 0.46)\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. The difference between these two measurements of the Hubble constant is quite a big puzzle in modern cosmology and is known as the *Hubble tension*. For completeness we note that the Planck 2018 result is $(67.4 \pm 0.5)\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ which is even further away from the supernovae data.

4.3 The Friedmann Equations

Up to now we established two things:

- We found the FLRW metric (4.1.16) that provides a general description of the space-time geometry at the largest scales.
- We found using Hubble's law that the universe at this moment in time is expanding.

To understand the past and future of the universe we need to use Einstein's equations (1.6.37). In order to do this, we first need to understand the energy-momentum tensor of the universe at the largest scales.

4.3.1 Energy-momentum tensor

We model the energy-momentum tensor of the universe at the largest scales as that of a perfect fluid (1.6.21). Homogeneity and isotropy gives that ρ and p only depends on the cosmic standard time t. In the co-moving coordinates used in the FLRW metric (4.1.16) we have that averaged matter is at rest $u^\mu = (1,0,0,0)$. Thus, in the co-moving coordinates (1.6.21) gives

$$T_{00} = \rho\ ,\quad T_{0i} = 0\ ,\quad T_{ij} = p g_{ij}\,. \tag{4.3.1}$$

The different types of cosmological perfect fluids are characterized by their equation of state

$$p = p(\rho)\,. \tag{4.3.2}$$

As we shall see, we can restrict ourselves to fluids with the equation of state being of the form

$$p = w\rho\,, \tag{4.3.3}$$

where w is a constant. The three types of fluids we consider are:

- **Matter:** The equation of state is

  $$p = 0\,. \tag{4.3.4}$$

  This includes all matter for which the pressure is neglible compared to the energy density ρ, as for instance stars, planets, dark matter, water, etc. See also discussion of why $p \ll \rho$ for Newtonian matter in Section 1.6.1.

- **Radiation:** The equation of state is

  $$p = \frac{1}{3}\rho\,. \tag{4.3.5}$$

  This is the equation of state for a gas of photons. One can derive this from demanding $T^\mu{}_\mu = 0$, which is true for the energy-momentum tensor of an electromagnetic field.

- **Dark energy:** The equation of state is

  $$p = -\rho\,. \tag{4.3.6}$$

  In our derivation of Einsteins equations (1.6.37) we could have added a term $\Lambda g_{\mu\nu}$ to the LHS

  $$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}\,. \tag{4.3.7}$$

  Provided Λ is constant this obeys $D^\mu(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}) = 0$ thus generalizing the Bianchi identity (1.5.28). Λ is known as the cosmological constant. However, one can equivalently regard the term $\Lambda g_{\mu\nu}$ as belonging to the RHS of Einsteins equations (1.6.37). This corresponds to an cosmological perfect fluid with

  $$\rho = -p = \frac{\Lambda}{8\pi G}\,, \tag{4.3.8}$$

  since we see from (1.6.21) that this gives an energy momentum tensor $T_{\mu\nu} = -\frac{\Lambda}{8\pi G}g_{\mu\nu}$. In 1998 it was measured that $\Lambda > 0$ and hence it gives a positive energy density $\rho > 0$. For this reason we call it *dark energy*.

4.3.2 Deriving the Friedmann equations

We consider now Einsteins equations (1.6.37) for the FLRW metric (4.1.16) with the energy-momentum tensor of the form (4.3.1). The components of the Christoffel symbol are

$$\begin{gathered}\Gamma^t_{ij} = \frac{\dot{a}}{a} g_{ij} \ , \quad \Gamma^i_{tj} = \delta^i_j \frac{\dot{a}}{a} \ , \quad \Gamma^r_{rr} = \frac{kr}{1-kr^2} \ , \quad \Gamma^r_{\theta\theta} = -r(1-kr^2) \, , \\ \Gamma^r_{\phi\phi} = -r(1-kr^2)\sin^2\theta \ , \quad \Gamma^\theta_{r\theta} = \Gamma^\phi_{r\phi} = \frac{1}{r} \ , \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \ , \quad \Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta} \, ,\end{gathered} \tag{4.3.9}$$

with $i,j = r,\theta,\phi$. Using this we compute the non-zero components of the Ricci tensor

$$R_{tt} = -3\frac{\ddot{a}}{a} \ , \quad R_{ij} = \left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{2k}{a^2} \right) g_{ij} \, , \tag{4.3.10}$$

where $\ddot{a} = d^2a/dt^2$. This gives the Ricci scalar

$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) \, . \tag{4.3.11}$$

Thus the components of the LHS of Einsteins equations (1.6.37) are

$$R_{tt} - \frac{1}{2} g_{tt} R = 3\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) \ , \quad R_{ij} - \frac{1}{2} g_{ij} R = -\left(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) g_{ij} \, . \tag{4.3.12}$$

Using this with (4.3.1) we can write Einsteins equations as:

The Friedmann equations: The scalar factor $a(t)$ obeys

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \, , \tag{4.3.13}$$

called the *Friedmann equation*. This is obtained from the tt-component of Einsteins equations (1.6.37). If one knows the energy-density as a function of time Eq. (4.3.13) determines the evolution of the scale factor $a(t)$. In addition the scalar factor obeys

$$\frac{\ddot{a}}{a} = -4\pi G \left(p + \frac{1}{3}\rho \right) \, , \tag{4.3.14}$$

obtained from the remaining non-zero components of Einsteins equations. This is known as the *second Friedmann equation*.

It is also interesting to consider the conservation of energy and momentum (1.6.22). We have

$$0 = D_\mu T^\mu{}_t = \partial_\mu T^\mu{}_t + \Gamma^\mu_{\mu\rho} T^\rho{}_t - \Gamma^\rho_{\mu t} T^\mu{}_\rho = -\dot{\rho} - 3\frac{\dot{a}}{a}\rho - 3\frac{\dot{a}}{a}p \, , \tag{4.3.15}$$

where $\dot{\rho} = d\rho/dt$. This gives:

Energy-momentum conservation: The conservation of energy and momentum is equivalent to the equation

$$\dot{\rho} = -\frac{3\dot{a}}{a}(\rho + p)\,. \tag{4.3.16}$$

It is important to note that this equation holds for each type of cosmological fluid separately, with ρ being the energy density and p the pressure of the cosmological fluid, so not just for the total density and pressure. This is due to the fact that in our approximation the different fluid components do not interact with each other, and hence they separately have to satisfy the conservation of energy and momentum (1.6.22).

One can check that (4.3.16) is consistent with (4.3.13)-(4.3.14) when ρ and p are the total energy density and pressure, respectively. Indeed, if one differentiates (4.3.13) with respect to t one finds

$$\frac{\dot{a}\ddot{a}}{a^2} = \frac{\dot{a}^3}{a^3} + \frac{k\dot{a}}{a^3} + \frac{4\pi G}{3}\dot{\rho}\,. \tag{4.3.17}$$

Substituting (4.3.14) on the LHS and (4.3.13) on the RHS one finds indeed (4.3.16).

4.4 Evolution of the Scale Factor

We have now all the necessary ingredients to find the time-dependence of the scale factor $a(t)$.

4.4.1 Friedmann equations revisited

We introduce the *Hubble parameter*

$$H(t) = \frac{\dot{a}}{a}\,. \tag{4.4.1}$$

This is equal to the Hubble constant H_0 when t is our present time. This is given in Eqs. (4.2.16), (4.2.18) and (4.2.19). We introduce also the so-called *critical density*

$$\rho_{\text{crit}}(t) = \frac{3H^2}{8\pi G}\,, \tag{4.4.2}$$

as well as the *density parameter*

$$\Omega(t) = \frac{\rho(t)}{\rho_{\text{crit}}(t)} = \frac{8\pi G}{3H^2}\rho\,. \tag{4.4.3}$$

With this the Friedmann equation (4.3.13) can be written as

$$\Omega - 1 = \frac{k}{H^2 a^2} \,. \tag{4.4.4}$$

We see that the sign of k is correlated with the sign of $\Omega - 1$, hence

$$\begin{aligned} \Omega < 1 \Leftrightarrow k = -1 \quad & \text{(open universe)}\,, \\ \Omega = 1 \Leftrightarrow \ k = 0 \quad & \text{(flat universe)}\,, \\ \Omega > 1 \Leftrightarrow \ k = 1 \quad & \text{(closed universe)}\,. \end{aligned} \tag{4.4.5}$$

Current measurements (Planck 2015) gives

$$\Omega = 1.00 \,, \tag{4.4.6}$$

for the value of Ω today. Hence it seems the universe is either flat or very close to be flat.[37] For this reason we shall set $k = 0$ in the following. Assuming a flat universe the Friedmann equation (4.4.4) is simply

$$\Omega = 1 \,. \tag{4.4.7}$$

4.4.2 Composite model

We consider now a flat universe composed of four different species of cosmological fluids:[38]

- Radiation with energy-density ρ_{R}. $(w = 1/3)$. This is electromagnetic radiation.
- Baryonic matter with energy-density ρ_{B} $(w = 0)$. This consist of all the matter made out of baryons in the universe. All matter that we can describe using the standard-model of particle physics is made out of baryons.
- Dark matter with energy-density ρ_{DM} $(w = 0)$. This is matter that we can only infer through its gravitational interaction with the baryonic matter and radiation. It is not known what dark matter is made of.
- Dark energy with energy-density ρ_{Λ} $(w = -1)$. One can describe this macroscopically using General Relativity. However, the microscopic origin of dark energy -

[37]As discussed in Exercise 4.2 Ω tends to get closer to 1 as one goes back in time. Hence since Ω is close to 1 now, it must have been much closer to 1 in the past.

[38]Here we do not include gravitational radiation.

also known as the cosmological constant - is an unsolved problem in theoretical physics.[39]

Thus, the total energy-density is

$$\rho = \rho_{\mathrm{R}} + \rho_{\mathrm{B}} + \rho_{\mathrm{DM}} + \rho_{\Lambda}\,. \tag{4.4.8}$$

Correspondingly, we have

$$1 = \Omega = \Omega_{\mathrm{R}} + \Omega_{\mathrm{B}} + \Omega_{\mathrm{DM}} + \Omega_{\Lambda}\,. \tag{4.4.9}$$

At present time $t = t_0$, *i.e.* in the universe right now, the density parameters have been measured to be (Planck 2015)

$$\Omega_{\mathrm{R}}^{(0)} < 0.001\ ,\quad \Omega_{\mathrm{B}}^{(0)} = 0.05\ ,\quad \Omega_{\mathrm{DM}}^{(0)} = 0.26\ ,\quad \Omega_{\Lambda}^{(0)} = 0.69\,, \tag{4.4.10}$$

which indeed gives (4.4.9). The total density is measured to be

$$\rho^{(0)} = 8.6 \cdot 10^{-27}\,\mathrm{kg/m^3}\,. \tag{4.4.11}$$

Thus, in our current state the universe is dominated by dark energy.

Writing the equation of states for each species as (4.3.3) we can employ (4.3.16) to get

$$\frac{\dot\rho}{\rho} = -3(1+w)\frac{\dot a}{a}\,. \tag{4.4.12}$$

Therefore, employing the ansatz

$$\frac{\rho(t)}{\rho(t_0)} = \left(\frac{a(t)}{a(t_0)}\right)^{-n}\,, \tag{4.4.13}$$

we find

$$n = 3(1+w)\,. \tag{4.4.14}$$

We record

$$n_{\mathrm{R}} = 4\ ,\quad n_{\mathrm{B}} = n_{\mathrm{DM}} = 3\ ,\quad n_{\Lambda} = 0\,, \tag{4.4.15}$$

which means that

$$\rho_{\mathrm{R}} \propto a^{-4}\ ,\quad \rho_{\mathrm{B}} \propto a^{-3}\ ,\quad \rho_{\mathrm{DM}} \propto a^{-3}\ ,\quad \rho_{\Lambda} \propto a^{0}\,. \tag{4.4.16}$$

[39] A natural microscopic origin of a cosmological constant is the vacuum energy in Quantum Field Theory. However, using the vacuum energy of the standard-model of particle physics seems to give a prediction of the cosmological constant which is 10^{120} times as big as the one we measure in cosmology. This is fortunate, because if it was as predicted from the standard-model of particle physics, the universe would be too hot for the creation of life.

With all the species included the Friedmann equation is

$$H^2 = \frac{8\pi G}{3} \sum_i \rho_i \,, \tag{4.4.17}$$

where $i = \mathrm{R}, \mathrm{B}, \mathrm{DM}, \Lambda$. Since $\rho_i > 0$ for each species we get

$$H > 0 \,, \tag{4.4.18}$$

suggesting that the universe is always expanding, never contracting. From the second Friedmann equation (4.3.14) we get

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \sum_i (1 + 3w_i)\rho_i \,. \tag{4.4.19}$$

Combining $\dot{H} = \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}$ with (4.4.17) and (4.4.19) we get

$$\dot{H} = -4\pi G \sum_i (1 + w_i)\rho_i \,. \tag{4.4.20}$$

Since $w_i \geq -1$ we see that

$$\dot{H} \leq 0 \,. \tag{4.4.21}$$

This means that the Hubble parameter is never increasing with time.

4.4.3 Future of the universe

From (4.4.18) we conclude that the universe will keep expanding. Considering (4.4.16) we see that the fraction of dark energy will keep growing. Hence the dark energy will dominate more and more in the future.

A universe dominated completely by dark energy has

$$\dot{H} = 0 \,, \tag{4.4.22}$$

as seen from (4.4.20). Hence it has a constant Hubble parameter $H = H_\Lambda$. We see from (4.4.20) and (4.4.21) that

$$H_\Lambda < H_0 \,. \tag{4.4.23}$$

From (4.4.22) we get

$$a(t) = a_\Lambda \exp(H_\Lambda t) \,, \tag{4.4.24}$$

where a_Λ is a constant. Thus, the universe will expand exponentially.

4.4.4 Past of the universe

From (4.4.18) we get that the universe is contracting when going back in time. Using (4.4.16) we see that since the matter currently comprise about 31% of the universe and the radiation less than 0.1%, then we enter a matter-dominated phase when we go back in time. A convenient way to parametrize this is in terms of the redshift parameter z. We can write (4.2.10) as

$$\frac{a(t_0)}{a(t)} = 1 + z\,, \tag{4.4.25}$$

where $t = t_0$ is the current value of the cosmic standard time. From (4.4.10), (4.4.11) and (4.4.16)

$$\rho_{\rm R} = \Omega_{\rm R}^{(0)}\rho^{(0)}(1+z)^4\ ,\quad \rho_{\rm M} = \Omega_{\rm M}^{(0)}\rho^{(0)}(1+z)^3\ ,\quad \rho_\Lambda = \Omega_\Lambda^{(0)}\rho^{(0)}\,, \tag{4.4.26}$$

where $\rho_{\rm M}$ is the density of matter, including both the baryonic and the dark matter. In the above we have defined $\Omega_{\rm M}^{(0)}$ as the current value of the density parameter of matter

$$\Omega_{\rm M}^{(0)} = \Omega_{\rm B}^{(0)} + \Omega_{\rm DM}^{(0)} = 0.31 \tag{4.4.27}$$

which is the sum of the baryonic matter and dark matter today. The ratio of matter is

$$\Omega_{\rm M} = \frac{\rho_{\rm M}}{\rho_\Lambda + \rho_{\rm M} + \rho_{\rm R}}\,. \tag{4.4.28}$$

When $z = 3$ we find $\Omega_{\rm M} > 0.95$. Thus, $z = 3$ clearly corresponds to a matter dominated phase.

For a universe dominated by an energy density

$$\rho \propto a^{-n}\ ,\ n > 0\,, \tag{4.4.29}$$

we see from (4.4.17) that this gives $H^2 \propto a^{-n}$ and hence

$$\dot{a} \propto a^{1-\frac{n}{2}}\,. \tag{4.4.30}$$

The most general solution to this is

$$a(t) = C(t - t_\star)^{\frac{2}{n}}\,, \tag{4.4.31}$$

where C and $t_\star$ are constants. Hence for a matter-dominated universe we have

$$a(t) = C(t - t_\star)^{\frac{2}{3}}\,, \tag{4.4.32}$$

using (4.4.15). We see that such a universe has

$$a(t_\star) = 0\,, \tag{4.4.33}$$

which means that the universe had a beginning!

One puts the matter-dominated phase of the universe to be for redshift parameter

$$0.4 < z < 3600\,, \tag{4.4.34}$$

and the dark energy dominated phase as

$$z < 0.4\,. \tag{4.4.35}$$

What happens for $z > 3600$? In this case the redshift is so large that we enter a radiation dominated phase where the majority of the energy is from radiation. The ratio of radiation is

$$\Omega_{\mathrm{R}} = \frac{\rho_{\mathrm{R}}}{\rho_{\Lambda} + \rho_{\mathrm{M}} + \rho_{\mathrm{R}}}\,. \tag{4.4.36}$$

Setting $\Omega_{\mathrm{R}}^{(0)} = 0.001$ gives $\Omega_{\mathrm{R}} > 0.9$ for $z = 3600$. In the radiation dominated phase we see from (4.4.15) and (4.4.31) that

$$a(t) = \tilde{C}\sqrt{t - t_1}\,, \tag{4.4.37}$$

which again points to a beginning of our universe.

According to current measurements our universe is 13.8 billion years old (Planck 2015). The beginning is known as *Big Bang*. Shortly after it began it entered an inflationary epoch where it grew exponentially. This ended around 10^{-32} seconds after Big Bang. After that the universe entered a radiation-dominated era lasting until 47 000 years after Big Bang (corresponding to $z = 3600$). Then the universe entered the matter dominated era lasting until about 10 billion years after Big Bang ($z = 0.4$). Now we have entered the dark energy dominated era that could last forever, for all we know.

4.4.5 Temperature of the universe

The cosmic microwave background (CMB) is electromagnetic black body radiation with a current temperature of $T_{(0)} = 2.725$ K. One can think of it as a gas of photons. Going back in time, the universe is contracting. This makes wavelengths proportionally shorter, as shown by (4.2.9) and (4.2.10). Since the temperature grows proportionally with the average energy of a photon in the gas, which is inversely proportional to the average wavelength, this means

$$T = T_{(0)}(1 + z)\,, \tag{4.4.38}$$

using (4.2.9). Thus, as we rewind the universe the CMB gets hotter. The origin of CMB is at $z = 1100$ (380 000 years after Big Bang). For $z < 1100$ the temperature become low enough for photons to decouple from matter, thus giving rise to the CMB. Before this point ($z > 1100$) the universe was more than 4000 K and it was a hot dense plasma of photons, electrons and protons. Recombination happened slightly prior to the photon decoupling. This is the temperature below which neutral atoms can form.

4.5 Exercises for Chapter 4

Exercise 4.1. Age of the universe.

According to the 2015 data from the Planck satellite the Hubble constant is measured to be (4.2.16) corresponding to the time (4.2.18). Moreover, the ratios of matter (including dark matter) and dark energy in the universe today are

$$\Omega_M^{(0)} = 0.31\ , \quad \Omega_\Lambda^{(0)} = 0.69\,. \tag{4.5.1}$$

- Assuming we live in a flat universe and that the universe only consists of matter and dark energy, *i.e.* that we can neglect the contribution from the radiation, write down the Friedmann equation (4.3.13) in terms of $a(t)$, $\dot{a}(t)$ and constants.

- Let a_0 be the current value of the scale factor. Define the function

$$y(t) = \frac{a(t)}{a_0}\,. \tag{4.5.2}$$

  Show that one can write the Friedmann equation (4.3.13) as

$$\dot{y}^2 \frac{1}{H_0^2} = \frac{\Omega_M^{(0)}}{y} + \Omega_\Lambda^{(0)} y^2\,. \tag{4.5.3}$$

- Show that the age of the universe (with above assumptions) is given by the integral

$$T = \frac{1}{H_0} \int_0^1 \frac{dy}{\sqrt{\frac{\Omega_M^{(0)}}{y} + \Omega_\Lambda^{(0)} y^2}}\,. \tag{4.5.4}$$

- Using the definite integral

$$\int_0^1 \frac{dy}{\sqrt{\frac{b}{y} + (1-b)y^2}} = \frac{2\log(1+\sqrt{1-b}) - \log b}{3\sqrt{1-b}}\,, \tag{4.5.5}$$

  find the age of the universe T.

Exercise 4.2. The flatness problem.

The Planck satellite has measured the ratios (4.5.1) for the energy densities of matter and dark energy. The radiation $\Omega_R^{(0)} < 0.001$. This gives a total density parameter

$$\Omega^{(0)} = \Omega_M^{(0)} + \Omega_\Lambda^{(0)} + \Omega_R^{(0)} = 1.00\,. \tag{4.5.6}$$

Including uncertainties the Planck measurement gives

$$\Omega^{(0)} = 1.002 \pm 0.005\,, \tag{4.5.7}$$

for the current value of the density parameter. In the following we consider what happens if Ω is not necessarily exactly equal to one. In accordance with the Planck measurements we assume

$$|\Omega^{(0)} - 1| < 0.01\,, \tag{4.5.8}$$

for the current value of the density parameter.

- Let ρ_0 be the current value of the total energy density. Show that

$$\frac{8\pi G}{3}\rho_0 = \Omega^{(0)} H_0^2\,. \tag{4.5.9}$$

- Let a_0 be the current value of the scale factor. Show that one can write the Friedmann equation (4.3.13) as

$$\frac{\dot{a}^2}{\dot{a}_0^2} = \Omega_M^{(0)}\frac{a_0}{a} + \Omega_\Lambda^{(0)}\frac{a^2}{a_0^2} + \Omega_R^{(0)}\frac{a_0^2}{a^2} - \frac{k}{a_0^2 H_0^2}\,, \tag{4.5.10}$$

  where $\dot{a}_0$ is the value of $\dot{a}(t)$ today.

- Use (4.5.10) to argue that provided we have (4.5.8) then

$$\frac{|k|}{a_0^2 H_0^2} < 0.01\,. \tag{4.5.11}$$

- The beginning of the matter-dominated era occured when the redshift factor z was around 3600. This is the point in which the energy from radiation was about half of all the energy in the universe (and after this point in time it became less than half). Use this information to compute an approximate value for $\Omega_R^{(0)}$.

- Given (4.5.8), how close should Ω be to 1 when $z = 3600$? [Hint: First show using the Friedmann equation Eq. (4.4.4) that $\Omega - 1 = \frac{\dot{a}_0^2}{\dot{a}^2}(\Omega^{(0)} - 1)$ for $k \neq 0$. Then find the ratio $\dot{a}_0^2/\dot{a}^2$ using (4.5.10) and (4.5.11) and insert the result together with (4.5.8).]

- The inflationary epoch ended around 10^{-32} seconds after Big Bang with a redshift factor of the order $z \sim 10^{21}$. How close should Ω be to 1 at this point?

- **Comment:** There are no mechanisms in General Relativity that can explain why Ω should be exactly one. On the other hand, it seems that Ω would have had to be extremely close to one right after the inflationary epoch. This is known as the *flatness problem*: Why should the early universe be so close to flat? Logically, there are only two possibilities: 1) Either there is some argument for why Ω is exactly equal to one, perhaps in a quantum theory of gravity, or 2) There exists a mechanism that can make $\Omega - 1$ very small just after the universe started. This is one of the reasons the theory of inflation was introduced, since inflation actually gives a mechanism of the second type that can make $\Omega - 1$ very small during the inflationary epoch.

Exercise 4.3. Einsteins static universe.

Before Hubble released his measurements in 1929 showing the universe is expanding, Einstein considered a cosmological solution to Einsteins equations in which the universe is static. We take this to mean that the FLRW metric (4.1.16) has a constant scale factor, *i.e.* that $a(t)$ does not depend on the time t. In the following we use the two Friedmann equations (4.3.13) and (4.3.14).

- Assume the static universe has matter with energy density ρ_M and that the energy density of radiation is so small that we can neglect it. Find the necessary dark energy density ρ_Λ to sustain a static universe.

- Should a static universe be flat, open or closed?

- What is the relation between the (constant) scale factor and the matter energy density ρ_M for a static universe?

- **Comment:** Einstein proposed his static universe in [11]. As part of this, he introduced the possibility of a cosmological constant, in order to explain the negative pressure needed for a static solution. However, Einsteins static universe is inherently unstable, as shown by Eddington in [12]. If one adds a little matter to the static universe, this will cause a contraction of the universe, that will not stop. Similarly, if one subtract some matter, it will cause the universe to expand without end.

5 Linearized Gravity and Gravitational Waves

5.1 Linearized Gravity

One of the distinguishing features of General Relativity compared to Newtonian gravity is that the gravitational field can interact with itself due to the non-linearity of Einsteins equations (1.6.37). However, even in the weak field limit where Einsteins equations become approximately linear, General Relativity has new features compared to Newtonian gravity. This is what we shall explore in the following.

5.1.1 Weak field limit of gravity

The metric $g_{\mu\nu}(x)$ is the gravitational field. When there is no gravity, $g_{\mu\nu}(x)$ is the metric for Minkowski space that in an Inertial System is $g_{\mu\nu}(x) = \eta_{\mu\nu}$. When the gravitational field is weak the metric $g_{\mu\nu}(x)$ is approximately that of Minkowski space. We can write this as

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)\,, \tag{5.1.1}$$

where all the components of $h_{\mu\nu}(x)$ are small

$$|h_{\mu\nu}(x)| \ll 1\,. \tag{5.1.2}$$

This is the weak field limit of gravity.[40] The weak field limit should not be confused with the Newton limit of Section 1.3.4. In that case we assume in addition small velocities (1.3.51) and no time-dependence (1.3.50). We are not making these assumptions above.

We now expand General Relativity to first order in $h_{\mu\nu}(x)$. This will give us what we call *linearized gravity*. To first order in $h_{\mu\nu}(x)$ the inverse metric is

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}\,, \tag{5.1.3}$$

where we raised the indices of $h_{\mu\nu}$ with the inverse Minkowski metric,

$$h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}\,. \tag{5.1.4}$$

In the following we will raise and lower indices using $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$. One can easily check that (5.1.3) is the correct inverse metric to first order in $h_{\mu\nu}$,

$$\begin{aligned}(\eta^{\mu\nu} - h^{\mu\nu})(\eta_{\nu\rho} + h_{\nu\rho}) &= \eta^{\mu\nu}\eta_{\nu\rho} + \eta^{\mu\nu}h_{\nu\rho} - h^{\mu\nu}\eta_{\nu\rho} - h^{\mu\nu}h_{\nu\rho} \\ &= \delta^\mu_\rho + \eta^{\mu\nu}h_{\nu\rho} - \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}\eta_{\nu\rho} + \mathcal{O}(h^2) \\ &= \delta^\mu_\rho + \mathcal{O}(h^2)\,,\end{aligned} \tag{5.1.5}$$

[40] For use below, one should strictly speaking also assume that $|\partial_\rho h_{\mu\nu}|$ is either of the same order as $|h_{\mu\nu}|$ or smaller.

where $\mathcal{O}(h^2)$ denotes any term of second or higher order in $h_{\mu\nu}$. From (5.1.1) we see that the partial derivative of $g_{\mu\nu}$ is

$$\partial_\rho g_{\mu\nu} = \partial_\rho h_{\mu\nu} \,. \tag{5.1.6}$$

Hence to first order in $h_{\mu\nu}$ the Christoffel symbol is

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}\eta^{\rho\sigma}(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}) = \frac{1}{2}(\partial_\mu h_\nu{}^\rho + \partial_\nu h_\mu{}^\rho - \partial^\rho h_{\mu\nu}) \,, \tag{5.1.7}$$

where we raised the indices with $\eta^{\mu\nu}$, *i.e.* $h_\mu{}^\nu = \eta^{\nu\rho} h_{\mu\rho}$. Using this we can find the weak field limit of the geodesic equation (1.3.45)

$$\frac{d^2 x^\rho}{d\tau^2} = -\left(\partial_\mu h_\nu{}^\rho - \frac{1}{2}\partial^\rho h_{\mu\nu}\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \,, \tag{5.1.8}$$

known as the *linearized geodesic equation*.

We now consider the weak field limit of the Ricci tensor. From the general formula (2.1.25) we see that the terms with the products of two Christoffel symbols can be neglected since they are of order $\mathcal{O}(h^2)$. Thus, we get

$$R_{\mu\nu} = \frac{1}{2}\Big(\partial_\mu\partial_\rho h_\nu{}^\rho + \partial_\nu\partial_\rho h_\mu{}^\rho - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h_\rho{}^\rho\Big) \,, \tag{5.1.9}$$

to first order in $h_{\mu\nu}$ where we introduced the notation

$$\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = \partial_\mu\partial^\mu \,, \tag{5.1.10}$$

called the *d'Alembert operator*, or the *box operator*. Note also that $h_\rho{}^\rho = \eta^{\mu\nu} h_{\mu\nu}$. Using Eq. (5.1.9) one can find the Ricci scalar R and hence the left-hand side $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ of Einsteins equations (1.6.37) to first order in $h_{\mu\nu}$. Doing this, one gets the equations for linearized gravity. However, as we shall see below one can simplify these equations considerably by using coordinate transformations.

5.1.2 Gauge transformations

Consider a coordinate transformation $x^\mu \to \tilde{x}^\mu(x)$

$$\tilde{x}^\mu(x) = x^\mu - \epsilon^\mu(x) \,, \tag{5.1.11}$$

where we require

$$|\partial_\mu \epsilon^\nu| \ll 1 \,. \tag{5.1.12}$$

To first order in $\partial_\mu \epsilon^\nu$ we have

$$\frac{\partial \tilde{x}^\mu}{\partial x^\nu} = \delta^\mu_\nu - \partial_\nu \epsilon^\mu \,, \tag{5.1.13}$$

and hence

$$\frac{\partial x^\mu}{\partial \tilde{x}^\nu} = \delta^\mu_\nu + \partial_\nu \epsilon^\mu \,. \tag{5.1.14}$$

We assume in the following that all the components of $|\partial_\mu \epsilon^\nu|$ are of the same order or less than the largest component of $|h_{\mu\nu}|$. Using the transformation of the metric (1.3.23) we find to first order in $h_{\mu\nu}$

$$\tilde{g}_{\mu\nu}(\tilde{x}) = (\delta^\rho_\mu + \partial_\mu \epsilon^\rho)(\delta^\sigma_\nu + \partial_\nu \epsilon^\sigma)(\eta_{\rho\sigma} + h_{\rho\sigma}) = \eta_{\mu\nu} + h_{\mu\nu}(x) + \partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x) \,, \tag{5.1.15}$$

where $\epsilon_\mu(x) = \eta_{\mu\nu}\epsilon^\nu(x)$. Since the components of $|\partial_\mu \epsilon^\nu|$ is of same order (or less) as $h_{\mu\nu}$ and since $\eta_{\mu\nu}$ does not depend on x^μ we find to first order in $h_{\mu\nu}$

$$\tilde{g}_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) + \partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x) \,. \tag{5.1.16}$$

Thus, the transformed metric is also in the weak field limit (5.1.1)-(5.1.2). For this reason we define $\tilde{h}_{\mu\nu}(x)$ by

$$\tilde{g}_{\mu\nu}(x) = \eta_{\mu\nu} + \tilde{h}_{\mu\nu}(x) \,, \tag{5.1.17}$$

with

$$\tilde{h}_{\mu\nu}(x) = h_{\mu\nu}(x) + \partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x) \,. \tag{5.1.18}$$

In this sense one can view the coordinate transformation (5.1.11) as a transformation of $h_{\mu\nu}(x)$. This class of coordinate transformation for linearized gravity are known as *gauge transformations* for $h_{\mu\nu}$. The reason for this name is that such a transformation does not change the weak field limit of the Ricci tensor,

$$\begin{aligned}
\tilde{R}_{\mu\nu} &= \frac{1}{2}\Big(\partial_\mu \partial_\rho \tilde{h}_\nu{}^\rho + \partial_\nu \partial_\rho \tilde{h}_\mu{}^\rho - \Box \tilde{h}_{\mu\nu} - \partial_\mu \partial_\nu \tilde{h}_\rho{}^\rho\Big) \\
&= R_{\mu\nu} + \frac{1}{2}\Big(\partial_\mu \partial_\rho (\partial_\nu \epsilon^\rho + \partial^\rho \epsilon_\nu) + \partial_\nu \partial_\rho (\partial_\mu \epsilon^\rho + \partial^\rho \epsilon_\mu) \\
&\quad - \Box(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) - 2\partial_\mu \partial_\nu \partial_\rho \epsilon^\rho\Big) \\
&= R_{\mu\nu} \,.
\end{aligned} \tag{5.1.19}$$

This in turns means that the weak field limit of the LHS of Einsteins equations (1.6.37) is invariant under gauge transformations (5.1.18) of $h_{\mu\nu}$. This is analogous to the gauge transformation $A_\mu \to A_\mu + \partial_\mu \chi$ of the potential A_μ in Electromagnetism under which the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ remains the same.

Lorenz gauge

The gauge transformation (5.1.18) of $h_{\mu\nu}$ can be used to simplify Einsteins equations (1.6.37). A particularly nice gauge choice is the *Lorenz gauge* in which $h_{\mu\nu}$ obeys[41]

$$\partial^\mu\Big(h_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}h_\rho{}^\rho\Big)=0\,. \tag{5.1.20}$$

Given $h_{\mu\nu}$ in Lorenz gauge, one can make a gauge transformation (5.1.18) with

$$\Box\epsilon^\mu=0\,, \tag{5.1.21}$$

and still be in the Lorenz gauge. We check this explicitly:

$$\begin{aligned}\partial^\mu\Big(\tilde{h}_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}\tilde{h}_\rho{}^\rho\Big)&=\partial^\mu\Big(h_{\mu\nu}+\partial_\mu\epsilon_\nu+\partial_\nu\epsilon_\mu-\frac{1}{2}\eta_{\mu\nu}(h_\rho{}^\rho+2\partial_\rho\epsilon^\rho)\Big)\\&=\Box\epsilon_\nu+\partial^\mu\partial_\nu\epsilon_\mu-\partial^\mu(\eta_{\mu\nu}\partial_\rho\epsilon^\rho)=0\,.\end{aligned} \tag{5.1.22}$$

This shows that $\tilde{h}_{\mu\nu}$ is in Lorenz gauge. Thus, even if one is in Lorenz gauge (5.1.20) there is still a remnant gauge symmetry left, given by the gauge transformations (5.1.18) obeying (5.1.21). This will be useful when considering gravitational waves.

Consider now $h_{\mu\nu}$ in Lorenz gauge (5.1.20). One has $\partial_\mu\partial_\rho h_\nu{}^\rho=\partial_\nu\partial_\rho h_\mu{}^\rho=\frac{1}{2}\partial_\mu\partial_\nu h_\rho{}^\rho$. Inserting this in (5.1.9) we see that the weak field limit of the Ricci tensor in Lorenz gauge is

$$R_{\mu\nu}=-\frac{1}{2}\Box h_{\mu\nu}\,. \tag{5.1.23}$$

This gives the weak field limit of the LHS of Einsteins equations in the form (1.6.38). To get the RHS one should use that $g_{\mu\nu}g^{\rho\sigma}T_{\rho\sigma}$ to leading order in the weak field limit is $\eta_{\mu\nu}\eta^{\rho\sigma}T_{\rho\sigma}$. Thus, we have derived:

Linearized Einstein equations: In the weak field limit of gravity (5.1.1)-(5.1.2) Einsteins equations can be written in the form

$$\Box h_{\mu\nu}=-16\pi G\left(T_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}T_{\rho\sigma}\right)\,, \tag{5.1.24}$$

where we impose the Lorenz gauge (5.1.20) on $h_{\mu\nu}(x)$. These equations are known as the *linearized Einstein equations*.

[41] One can show that it is always possible to go to Lorenz gauge. Consider a given $h_{\mu\nu}$ which is not in Lorenz gauge. The gauge transformations (5.1.18) has four arbitrary functions $\epsilon_\mu(x)$. Thus, one can impose the four equations $\Box\epsilon_\nu=-\partial^\mu(h_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}h_\rho{}^\rho)$ and find solutions for the four functions $\epsilon_\mu(x)$. Using these four functions in the gauge transformations (5.1.18) one finds $\partial^\mu(\tilde{h}_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}\tilde{h}_\rho{}^\rho)=0$ which means that $\tilde{h}_{\mu\nu}$ is in Lorenz gauge.

5.2 Gravitational Waves

5.2.1 Gravitational wave solution

Consider the linearized Einstein equations (5.1.24) in Lorenz gauge (5.1.20). In vacuum $T_{\mu\nu} = 0$ they become

$$\Box h_{\mu\nu} = 0\,. \tag{5.2.1}$$

We recognize this as the relativistic wave equation for each component of $h_{\mu\nu}$. Consider the following ansatz

$$h_{\mu\nu}(x) = A_{\mu\nu} \exp\left(ik_\rho x^\rho\right)\,, \tag{5.2.2}$$

where $A_{\mu\nu}$ and k_ρ are constants. Since $h_{\mu\nu} = h_{\nu\mu}$ we should require $A_{\mu\nu} = A_{\nu\mu}$. For convenience this ansatz is for now allowed to be complex valued. Later we shall restrict to the real part of the expression. The ansatz (5.2.2) satisfies the Lorenz gauge condition (5.1.20) provided

$$k^\mu A_{\mu\nu} = \frac{1}{2} k_\nu \eta^{\rho\sigma} A_{\rho\sigma}\,. \tag{5.2.3}$$

Inserting (5.2.2) in (5.2.1) we get that it is a solution provided

$$k_\mu k^\mu = 0\,. \tag{5.2.4}$$

With these conditions on k^μ and $A_{\mu\nu}$ imposed, (5.2.2) is a solution of the linearized vacuum Einstein equations (5.2.1).

Monochromatic plane wave at speed of light

The solution (5.2.2)-(5.2.4) of (5.2.1) corrresponds to a monochromatic plane wave propagating at the speed of light. Write

$$k^\mu = (\omega, k^1, k^2, k^3)\,. \tag{5.2.5}$$

Then from standard theory of waves we know that ω is the angular frequency, related to the frequency f as

$$\omega = 2\pi f\,. \tag{5.2.6}$$

Writing $k = \sqrt{k_i k^i}$ we have that k is the wave number related to the wavelength λ as

$$k = \frac{2\pi}{\lambda}\,. \tag{5.2.7}$$

Note that our interpreation of ω and k is for an observer at rest in the coordinate system x^μ. From (5.2.4) we get the condition

$$\omega = k\,. \tag{5.2.8}$$

Using this we find that the speed of the wave is

$$v = \lambda f = \frac{2\pi}{k}\frac{\omega}{2\pi} = 1\,, \tag{5.2.9}$$

which indeed means that it is propagating at the speed of light. The wave is propagating in the direction of the unit vector

$$\frac{k^i}{k}\,. \tag{5.2.10}$$

The wave is monochromatic since it only has one frequency. However, since Eq. (1.6.39) is linear one can superpose waves of different frequencies.

Polarizations

We count now the number of physically distinct solutions that the wave solution (5.2.2)-(5.2.4) corresponds to. This is what is known as the possible *polarizations* of the wave.

Consider $A_{\mu\nu}$. This has 16 components since it has two indices. Since we should impose $A_{\mu\nu} = A_{\nu\mu}$ this brings the number of independent components of $A_{\mu\nu}$ from sixteen to ten. Imposing the Lorenz gauge condition (5.2.3) puts four conditions on $A_{\mu\nu}$ thus bringing the number of independent components to six.

However, there is still the remnant gauge symmetry (5.1.21) in the Lorenz gauge. This remnant gauge symmetry means that the 6 remaining linearly independent components are not all physically distinct. Consider the gauge transformation (5.1.18) with

$$\epsilon^\mu(x) = -ib^\mu \exp\left(ik_\rho x^\rho\right)\,, \tag{5.2.11}$$

with b^μ constant. Clearly, this obeys (5.1.21). Under this transformation we get

$$\tilde{h}_{\mu\nu}(x) = \tilde{A}_{\mu\nu} \exp\left(ik_\rho x^\rho\right)\,, \tag{5.2.12}$$

with

$$\tilde{A}_{\mu\nu} = A_{\mu\nu} + k_\mu b_\nu + k_\nu b_\mu\,. \tag{5.2.13}$$

Since b^μ has four components we can use this to eliminate four of the six remaining linearly independent components of $\tilde{A}_{\mu\nu}$. In particular, one can choose b^μ such that

$$\eta^{\mu\nu}\tilde{A}_{\mu\nu} = 0\,,\quad \tilde{A}_{\mu 0} = 0\,. \tag{5.2.14}$$

One can argue that this amounts to four conditions as follows. Since we impose $\eta^{\mu\nu}\tilde{A}_{\mu\nu} = 0$ then the Lorenz gauge condition for $\tilde{A}_{\mu\nu}$ is

$$k^\mu \tilde{A}_{\mu\nu} = 0\,. \tag{5.2.15}$$

Thus, in particular $k^\mu \tilde{A}_{\mu 0} = 0$ which means that $\tilde{A}_{\mu 0} = 0$ only corresponds to three new independent conditions on $\tilde{A}_{\mu\nu}$. Including $\eta^{\mu\nu}\tilde{A}_{\mu\nu} = 0$ we count a total of four conditions on $\tilde{A}_{\mu\nu}$.

We have now used all the gauge freedom. What remains are only two physically distinct components of $\tilde{A}_{\mu\nu}$. Thus, we can conclude that the wave (5.2.2) has two polarizations.

Gravitational wave solution

We have found the wave solution (5.2.2) with the conditions (5.2.3) and (5.2.4). In addition to this we can impose (5.2.14). Thus, in summary, we have the following requirements on k^μ and $A_{\mu\nu}$

$$k_\mu k^\mu = 0\ , \quad A_{\mu\nu} = A_{\nu\mu}\ , \quad k^\mu A_{\mu\nu} = 0\ , \quad A_{\mu 0} = 0\ , \quad \eta^{\mu\nu} A_{\mu\nu} = 0\,. \tag{5.2.16}$$

From (5.1.1)-(5.1.2) we see that this corresponds to a small perturbation $h_{\mu\nu}$ of the Minkowski metric with

$$h_{\mu\nu} = \mathrm{Re}\Big[A_{\mu\nu} \exp\left(ik_\rho x^\rho\right) \Big]\,, \tag{5.2.17}$$

where we take the real value of the solution (5.2.2). Being in the weak field limit requires $|A_{\mu\nu}| \ll 1$. Since this is a wave solution for the metric in the weak field limit we call this a *gravitational wave*.

Consider a gravitational wave propagating along the x^3 axis. Then

$$k^\mu = (\omega, 0, 0, \omega)\,. \tag{5.2.18}$$

Imposing the conditions (5.2.16) on $A_{\mu\nu}$ we find

$$A_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{12} & -A_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\,, \tag{5.2.19}$$

which makes it explicit that there are only two linearly independent components of $A_{\mu\nu}$. Write now $A_{11} = B_1 e^{i\psi_1}$ and $A_{12} = B_2 e^{i\psi_2}$ where B_1, B_2, ψ_1 and ψ_2 are real. Then

$$h_{11} = -h_{22} = B_1 \cos\Big(\omega(x^3 - t) + \psi_1\Big)\ , \quad h_{12} = h_{21} = B_2 \cos\Big(\omega(x^3 - t) + \psi_2\Big)\,, \tag{5.2.20}$$

with all other components of $h_{\mu\nu}$ being zero and $t = x^0$. With this we can write the gravitational wave metric in terms of the line-element

$$ds^2 = -dt^2 + (1 + h_{11})(dx^1)^2 + 2h_{12}dx^1dx^2 + (1 - h_{11})(dx^2)^2 + (dx^3)^2\,, \tag{5.2.21}$$

where h_{11} and h_{12} are given by (5.2.20).

5.2.2 Relative motion of test particles

Consider the gravitational wave solution (5.2.20)-(5.2.21). In these coordinates one finds that the curve

$$x^0 = \tau \ , \quad x^i = \text{constant} \, , \tag{5.2.22}$$

is a solution to the linearized geodesic equation (5.1.8). This is easily seen using $\frac{dx^0}{d\tau} = 1$, $\frac{dx^i}{d\tau} = 0$, $\frac{d^2 x^\mu}{d\tau^2} = 0$ and $h_{0i} = h_{00} = 0$. Thus, any test particle lying still in the coordinate system of (5.2.20)-(5.2.21) is on a time-like geodesic. This shows that we cannot measure a gravitational wave by looking at the motion of a single test particle. Instead, as we now shall see, two or more test particles that are put in different locations will move relative to each other. This effect is what one uses to detect gravitational waves.

Consider two test particles A and B at fixed positions

$$\begin{aligned} &\text{Particle A:} \ \ \vec{x}_A = (x_A^1, x_A^2, x_A^3) \, , \\ &\text{Particle B:} \ \ \vec{x}_B = (x_A^1 + L_0 \cos\theta, x_A^2 + L_0 \sin\theta, x_A^3) \, . \end{aligned} \tag{5.2.23}$$

See Figure 32 for an illustration. For simplicity we assume that the two test particles are in the same plane with $x^3 = x_A^3$, thus a plane which is perpendicular to the direction of the propagation of the gravitational wave (see Exercise 5.1 for the general case). Without the gravitational wave, the distance between the two particles is L_0.

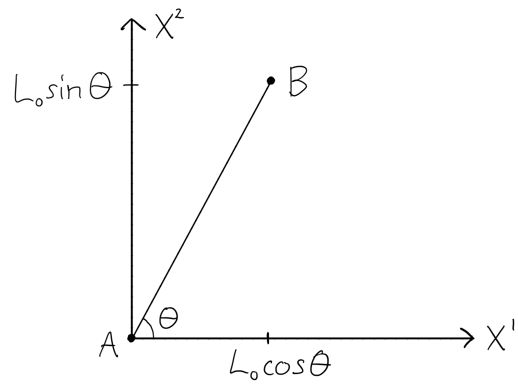

Figure 32: Illustration of the positions of the two test particles A and B in (5.2.23). For simplicity we put particle A in the origin of the (x^1, x^2)-plane.

We consider now the proper distance between the two test particles in the presence of the gravitational wave (5.2.20)-(5.2.21). Since x^3 is fixed h_{ij} only depends on the time t. We will measure this proper distance for a fixed time t. Note that, as we discuss below, in practise one does not measure a length instantenously. One can now parametrize the path between the two test particles as

$$(x_A^1 + r\cos\theta, x_A^2 + r\sin\theta, x_A^3) \, , \tag{5.2.24}$$

where r goes from 0 to L_0. Note that this path is a line since h_{ij} for the purpose of this computation can be regarded as constant. Using $dx^1 = \cos\theta\, dr$ and $dx^2 = \sin\theta\, dr$ we see using (5.2.21) that the line-element along this line is

$$\begin{aligned} ds^2 &= (1+h_{11})\cos^2\theta\, dr^2 + 2h_{12}\cos\theta\sin\theta\, dr^2 + (1-h_{11})\sin^2\theta\, dr^2 \\ &= \Big[1 + h_{11}\cos(2\theta) + h_{12}\sin(2\theta)\Big] dr^2 \,. \end{aligned} \tag{5.2.25}$$

Since h_{ij} is small, we have that the infinitesimal proper distance is

$$ds = \left[1 + \frac{1}{2}h_{11}\cos(2\theta) + \frac{1}{2}h_{12}\sin(2\theta)\right] dr \,. \tag{5.2.26}$$

Integrating r from 0 to L_0 gives the proper distance $L(t)$ between the two test particles. We have thus derived:

Relative motion of test particles: Consider the gravitational wave (5.2.20)-(5.2.21). For two test particles that both lie in a plane perpendicular to the propagation of the gravitational wave, as parametrized in (5.2.23), the proper distance between them is

$$L(t) = \left[1 + \frac{1}{2}h_{11}(t)\cos(2\theta) + \frac{1}{2}h_{12}(t)\sin(2\theta)\right] L_0 \,, \tag{5.2.27}$$

to first order in $h_{\mu\nu}$ with

$$h_{11}(t) = B_1\cos\Big(\omega t + \chi_1\Big) \ , \quad h_{12}(t) = B_2\cos\Big(\omega t + \chi_2\Big) \,, \tag{5.2.28}$$

and where L_0 is the distance in the absence of the gravitational wave.

Note that in (5.2.28) we have redefined the phases, which we can do since we hold x^3 fixed for the two particles.

The proper distance $L(t)$ in (5.2.27) is time-dependent due to the time-dependence of the gravitational wave solution (5.2.20)-(5.2.21). Thus, even if the two test particles are not moving individually in the coordinate system of (5.2.20)-(5.2.21), the distance between them varies with time. Hence they are moving relative to each other. It is this relative motion of test particles that one measures in a gravitational wave detector. This is in accordance with the geodesic deviation equation (1.5.11). While both particles are following geodesics, they move relative to each other, since the gravitational wave solution (5.2.20)-(5.2.21) has a non-zero Riemann curvature tensor.

Note that a detector cannot measure a distance instantaneously. One uses light to measure the distances, which means that the changes in length should happen sufficiently slowly in order for light to be able to move back and forth. Or, equivalently, the size of the detector should be much smaller than the wavelength of the gravitational wave that we want it to detect. Hence, we require

$$L_0 \ll \lambda = \frac{2\pi}{\omega} \,. \tag{5.2.29}$$

We see also from (5.2.27) that there are two contributions to $L(t)$: one from each polarization of the gravitational wave $h_{11}(t)$ and $h_{12}(t)$. These two polarization affect the relative motion of the two test particles differently due to the fact that $\sin(2(\theta+\pi/4)) = \cos(2\theta)$. This means that the effect of h_{12} is rotated 45 degrees compared to the effect of h_{11}. Thus, they are two linear independent effects.

We now consider this in more detail. We consider a ring of test particles in the (x^1, x^2)-plane. Consider first the case with $B_2 = 0$ and for simplicity $\chi_1 = 0$ (for $B_2 = 0$ one can redefine t to get this). Then we have

$$L(t) = L_0 \left[1 + \frac{1}{2} B_1 \cos(\omega t) \cos(2\theta) \right] \,. \tag{5.2.30}$$

Thus, for two test particles that lie along the x^1 axis with $\theta = 0$ we get $\cos(2\theta) = 1$. Instead two test particles that line along the x^2 axis with $\theta = \pi/2$ has $\cos(2\theta) = -1$. This means that when the particles along the x^1 axis move towards each other, the particles along the x^2 move away from each other, and vice versa. We have illustrated this in Figure 33.

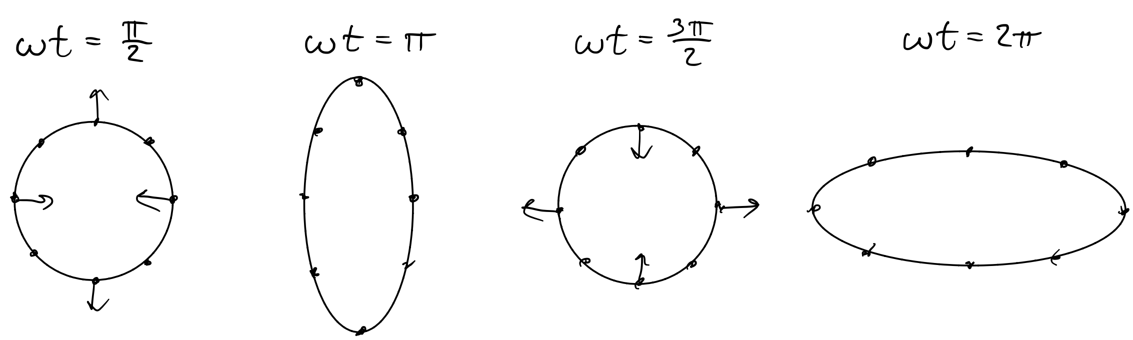

Figure 33: Illustration of how a ring of test particles is affected by a gravitational wave polarized such that $h_{12} = 0$.

Consider instead the other possible polarization with $B_1 = 0$ and for simplicity $\chi_2 = 0$. Then we have

$$L(t) = L_0 \left[1 + \frac{1}{2} B_2 \cos(\omega t) \sin(2\theta) \right] \,. \tag{5.2.31}$$

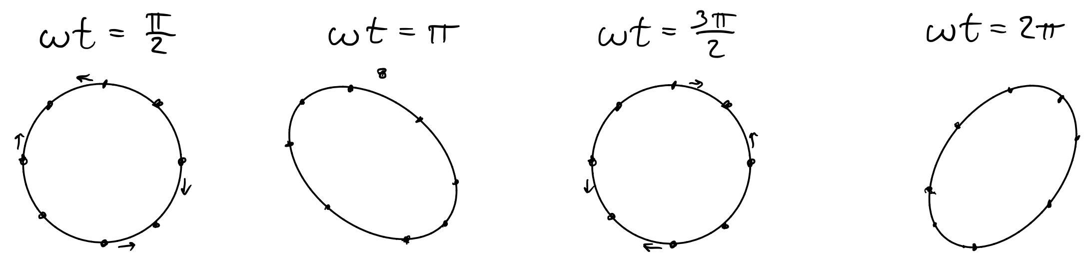

Figure 34: Illustration of how a ring of test particles is affected by a gravitational wave polarized such that $h_{11} = 0$.

The motion is now rotated with 45 degrees, and we have illustrated this in Figure 34.

The general solution $L(t)$ in (5.2.27) is thus a linear combination of the two above polarizations.

5.2.3 Detection of gravitational waves

The first ever observation of gravitational waves was done by Advanced LIGO on September 14, 2015 [6]. The name for this event is GW150914. The two detectors of Advanced LIGO are placed in Livingston, State of Louisiana and in Hanford, Washington State, USA. See Figure 35 for a picture. They both have two perpendicular 4 km long arms, see illustration in Figure 36. The detectors can then use interferometry to measure whether the two arms have equal length or not.

When a gravitational wave passes one of the detectors, one arm will be slightly shorter than the other, as one can see from the relative motion of test particles as explained in Section 5.2.2. Advanced LIGO can measure changes in distances of order 10^{-18} meter (about 1/1000 the size of a proton).

The LIGO detectors are illustrated in Figure 36. What happens is that the laser sends out a light signal. Half of it is reflected in the beamsplitter and goes to the 4 km long arm in the perpendicular direction. The other half is not reflected and goes in the 4 km long parallel arm. In both arms the light is reflected multiple times between the mirrors until they both go towards the detector. The detector can then check possible interference between the lightsignals from the two arms. If they are perfectly in phase, the arms have the same length. But if they are not in phase, the arms have different lengths.

The GW150914 event detected a gravitational wave with a wavelength λ and a frequency f of order

$$\lambda \sim 2000 \text{ km}\,, \quad f \sim 150 \text{ Hz}\,. \tag{5.2.32}$$

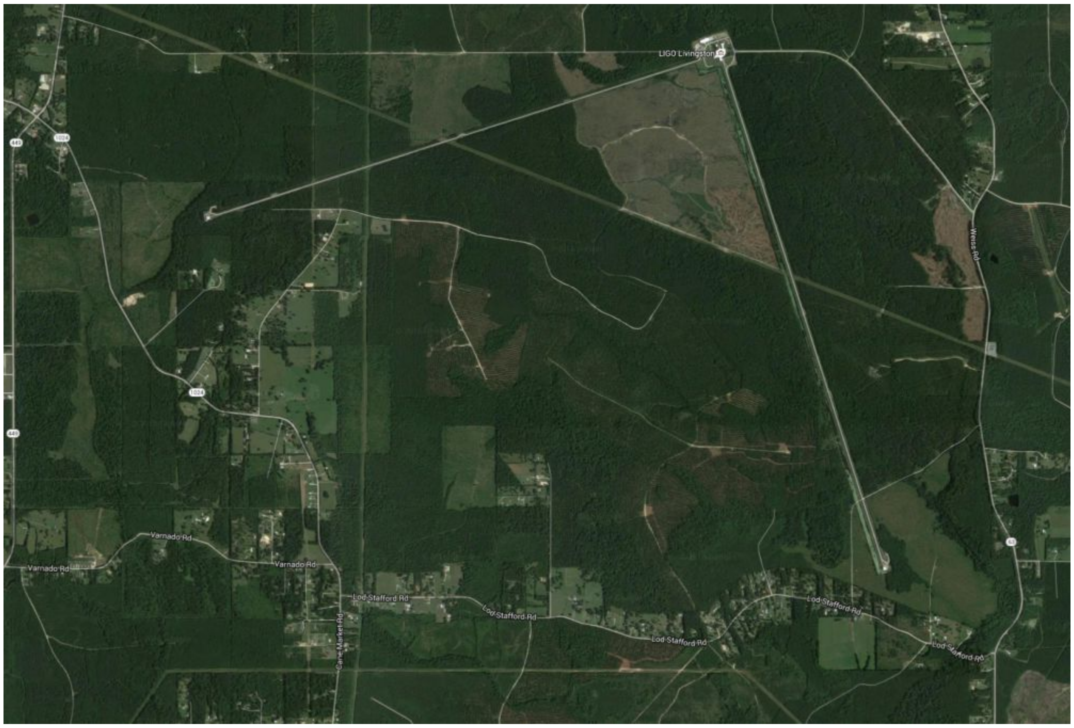

Figure 35: Picture of the LIGO detector in Livingston, State of Louisiana, USA.

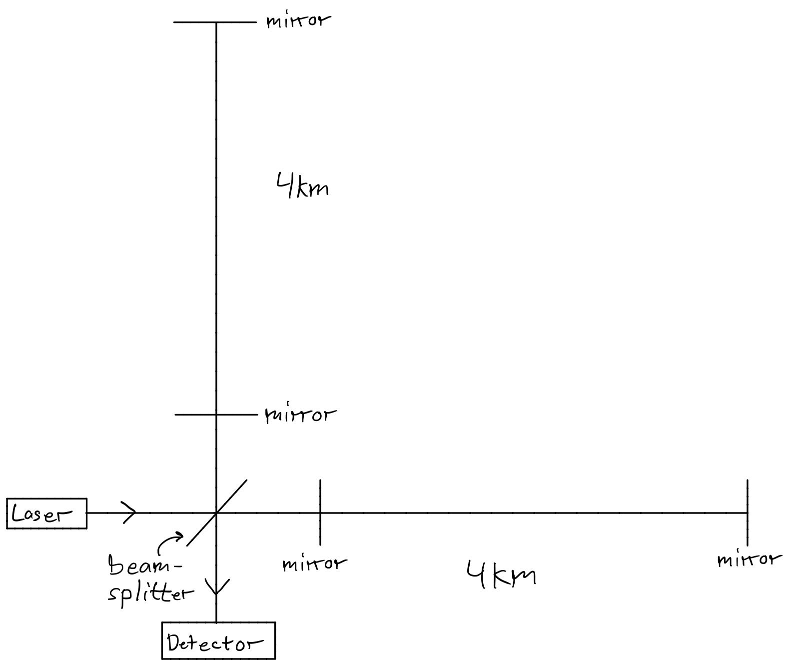

Figure 36: Illustration of how the LIGO detector works.

The wave took 0.1 seconds to pass. The origin of this gravitational wave is a collision and merger of two black holes that happened 1.3 billion years ago.

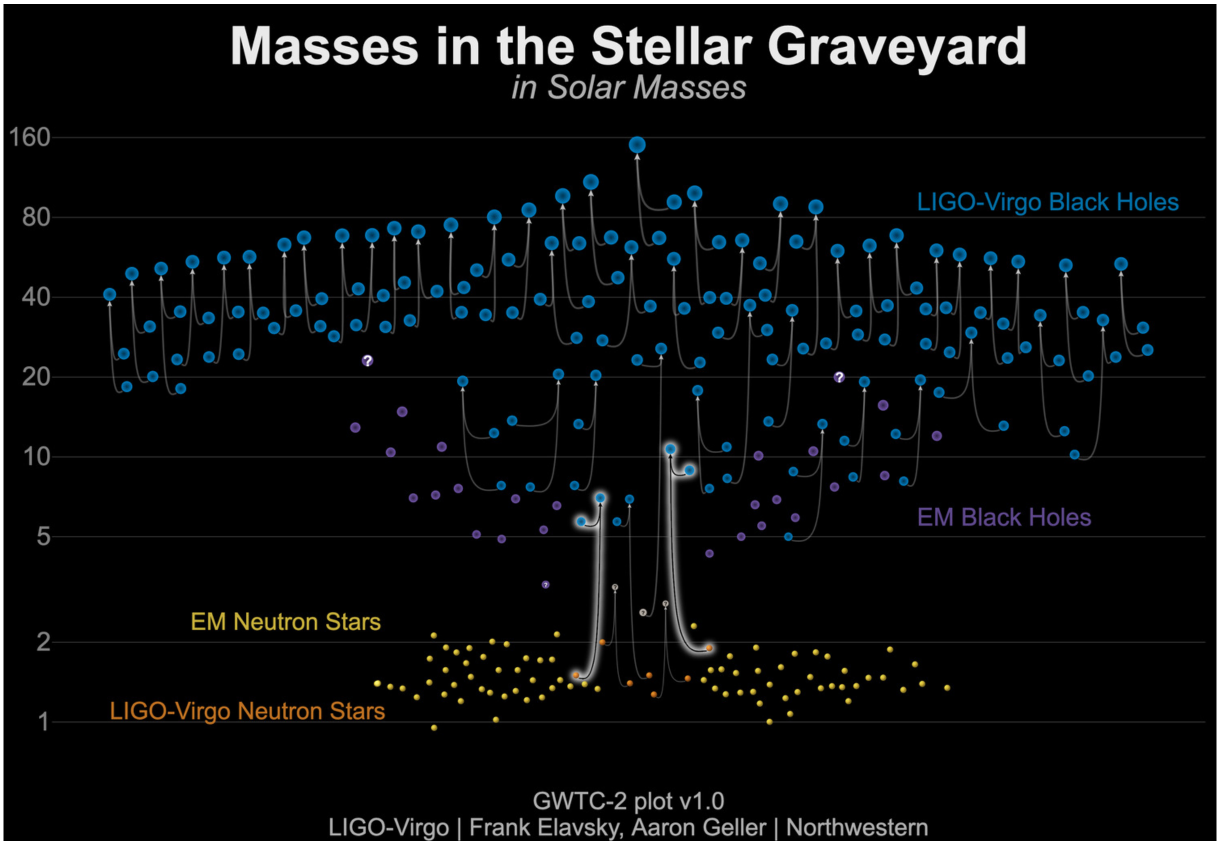

Figure 37: An illustration of all the observations of LIGO and VIRGO, as known public on June 29, 2021. The two new observations of neutron-star-black-hole mergers are highlighted. Credits: LIGO-Virgo / Frank Elavsky, Aaron Geller / Northwestern University.

Up to August 2021 there have been three observational runs at LIGO and VIRGO. See Figure 37 for an illustration of all the detections of binary mergers in the first three runs.

The first observing run O1 were from September 12, 2015 to January 19, 2016. In O1 gravitational waves from three binary black hole mergers were detected.

The second observing run O2 were from November 30, 2015 to August 25, 2017. In this run several new binary black hole mergers were detected. Of these the event GW170814 is interesting since it was the first time that Advanced VIRGO, a gravitational wave detector with 3 km long arms placed near Pisa in Italy, has participated in the detection of a gravitational wave. This is important since having three detectors in different locations on Earth means that one can pinpoint a direction from which the signal is coming.

The event GW170817 in O2 was the first detection of a collision of two neutron stars. This gave a further important breakthrough in the observation of gravitational waves, as it was the first time that one observed gravitational waves and electromagnetic waves from the same event. In addition to LIGO and VIRGO, the event was detected by 70 observatories on seven continents, and in space, observing various frequency ranges in the electromagnetic spectrum, including short gamma-ray bursts.

Since then a third observing run O3 has given new detections. O3 was from April 1, 2019 and lasted about a year. Included among the discoveries were two events GW200105 and GW200115 that for the first time correspond to mergers of binary systems with one black hole and one neutron star. In addition 48 binary black hole mergers and two binary neutron star mergers have been detected.

5.3 Stationary Matter

In this section we solve perturbatively at large distances the linearized Einstein equations (5.1.24) for the case of stationary matter. *This section is not part of the pensum of the course.*

We begin by rewriting the linearized Einstein equations (5.1.24). Define the new field $\Phi_{\mu\nu}(x)$ by

$$\Phi_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h_\rho{}^\rho \,. \tag{5.3.1}$$

In terms of $\Phi_{\mu\nu}(x)$ we see that the Lorenz gauge (5.1.20) on $h_{\mu\nu}$ is equivalent to imposing

$$\partial^\mu \Phi_{\mu\nu} = 0 \,. \tag{5.3.2}$$

Using (5.3.1) one can get $\Phi_{\mu\nu}$ from $h_{\mu\nu}$. Similarly, one can get $h_{\mu\nu}$ from $\Phi_{\mu\nu}$ using

$$h_{\mu\nu} = \Phi_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Phi_\rho{}^\rho \,. \tag{5.3.3}$$

That this is equivalent to (5.3.1) is easily seen by noticing that it follows from (5.3.1) that $\Phi_\rho{}^\rho = -h_\rho{}^\rho$. It is straightforward to show that the linearized Einstein equations (5.1.24) takes the form

$$\Box \Phi_{\mu\nu} = -16\pi G T_{\mu\nu} \,, \tag{5.3.4}$$

for $\Phi_{\mu\nu}$ in the Lorenz gauge (5.3.2).

General solution

A case for which one can readily solve the linearized Einstein equations (5.3.4) is that of a stationary matter distribution. In this case we have

$$\partial_0 T_{\mu\nu} = 0 \ , \quad \partial_0 \Phi_{\mu\nu} = 0 \,, \tag{5.3.5}$$

where $x^0 = t$. Then the linearized Einstein equations (5.3.4) become

$$\vec{\nabla}^2 \Phi_{\mu\nu} = -16\pi G T_{\mu\nu} \,, \tag{5.3.6}$$

where $\vec{\nabla}^2$ is the Laplacian

$$\vec{\nabla}^2 = \partial_i \partial^i = \partial_x^2 + \partial_y^2 + \partial_z^2 \,, \tag{5.3.7}$$

where we have introduced the coordinates $x^1 = x$, $x^2 = y$ and $x^3 = z$. Write now

$$\vec{r} = (x, y, z) \,. \tag{5.3.8}$$

One can show that

$$\vec{\nabla}^2 \frac{1}{|\vec{r} - \vec{r}'|} = -4\pi \delta^3(\vec{r} - \vec{r}') = -4\pi \delta(x - x')\delta(y - y')\delta(z - z') \,, \tag{5.3.9}$$

with $\vec{r}' = (x', y', z')$. Using this, one can solve (5.3.6) as

$$\Phi_{\mu\nu}(\vec{r}) = 4G \int_V d^3\vec{r}' \frac{T_{\mu\nu}(\vec{r}')}{|\vec{r} - \vec{r}'|} \,, \tag{5.3.10}$$

where V is a volume that includes all the matter (we assume V is finite, *i.e.* that the matter distribution is localized). See Figure 38 for an illustration. Thus, (5.3.10) gives the first-order correction to the Minkowski metric for any matter distribution that has sufficiently low density of energy and momentum such that it only gives rise to a weak gravitational field (5.1.1)-(5.1.2).

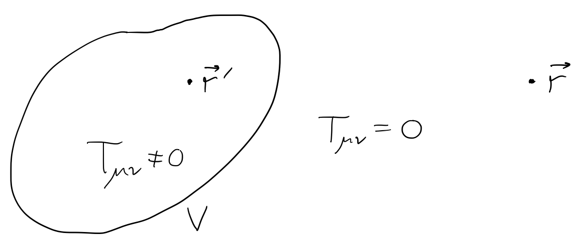

Figure 38: Illustration of the setup for the general solution (5.3.10) for $\Phi_{\mu\nu}(\vec{r})$ in terms of an integral of $T_{\mu\nu}$ over the volume V.

Multipole expansion

We are considering a localized matter distribution. When $r = |\vec{r}|$ is sufficiently large we are far away from the matter at V we can use the expansion

$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{r} + \frac{\vec{r} \cdot \vec{r}'}{r^3} + \mathcal{O}(r^{-3}) \,, \tag{5.3.11}$$

in powers of $1/r$. Inserting this in (5.3.10) we find

$$\Phi_{\mu\nu}(\vec{r}) = \frac{4G}{r}\int_V d^3\vec{r}'\, T_{\mu\nu}(\vec{r}') + \frac{4G\vec{r}}{r^3}\cdot\int_V d^3\vec{r}'\, T_{\mu\nu}(\vec{r}')\vec{r}' + \mathcal{O}(r^{-3})\,. \tag{5.3.12}$$

This is called a multipole expansion. The first and second term of the expansion in Eq. (5.3.12) are called the *monopole* term and *dipole* term, respectively. One can readily continue this expansion to higher order.

Newton limit

For small velocities $v^i \ll 1$ we have that T^{00} is of order one in an expansion in v^i, T^{0i} goes like v^i and T^{ij} goes like v^iv^j. Hence the Newton limit corresponds to keeping only T^{00}. This is what we shall consider in the following.

Assume we are in the rest frame for the center of mass. Then

$$\int_V d^3\vec{r}'\, T_{00}(\vec{r}')\vec{r}' = 0\ ,\quad \int_V d^3\vec{r}'\, T_{0i}(\vec{r}') = 0\,. \tag{5.3.13}$$

For the $\mu = \nu = 0$ component this means

$$\Phi_{00} = \frac{4GM}{r} + \mathcal{O}(r^{-3})\,, \tag{5.3.14}$$

where M is the total mass

$$M = \int_V d^3\vec{r}\, T_{00}(\vec{r})\,, \tag{5.3.15}$$

from the monopole term in (5.3.12). The other components of the monopole term are zero to this order in the Newton limit. This is in precise correspondence with what we found previously in the Newton limit. Indeed using (5.3.3) we get

$$h_{00} = \Phi_{00} - \frac{1}{2}\eta_{00}(-\Phi_{00}) = \frac{1}{2}\Phi_{00}\,. \tag{5.3.16}$$

Thus,

$$h_{00} = \frac{2GM}{r} + \mathcal{O}(r^{-3})\,. \tag{5.3.17}$$

This is the formula that we use in Section 3.4.1 to measure the mass of the Kerr black hole, as one can see by comparing to Eq. (3.4.2).

First post-Newtonian correction

The first post-Newtonian correction term is sourced by T^{0i} since that goes like v^i in the Newton limit. From (5.3.12) we see that T^{0i} indeed sources the dipole term. This term picks up the angular momentum of the mass distribution. In general one defines the angular momentum in the ij-plane as

$$J_{ij} = \int_V d^3\vec{r}\left(x^i T^{0j} - x^j T^{0i}\right). \tag{5.3.18}$$

In particular, the angular momentum in the 12-plane (the xy-plane) is J_{12}, corresponding to rotation around the x^3 axis (z axis).

Energy-momentum conservation (1.6.17) gives

$$\partial_i T^{0i} = 0\,. \tag{5.3.19}$$

Using this and partial integration we find

$$\begin{aligned}\int_V d\vec{r}\, x T^{0x} &= \int_V d^3\vec{r}\left(\partial_x(\frac{1}{2}x^2T^{0x}) - \frac{1}{2}x^2\partial_x T^{0x}\right) = \int_V d^3\vec{r}\,\frac{1}{2}x^2\left(\partial_y T^{0y} + \partial_z T^{0z}\right) \\ &= \int_V d^3\vec{r}\left(\partial_y\left(\frac{1}{2}x^2T^{0y}\right) + \partial_z\left(\frac{1}{2}x^2T^{0z}\right)\right) = 0\,.\end{aligned} \tag{5.3.20}$$

Here the total derivative terms are zero since $T_{\mu\nu} = 0$ is zero outside V. Moreover,

$$\begin{aligned}\int_V d\vec{r}\, x T^{0y} &= \int_V d^3\vec{r}\Big(\partial_y(xyT^{0y}) - xy\partial_y T^{0y}\Big) = \int_V d^3\vec{r}\, xy\Big(\partial_x T^{0x} + \partial_z T^{0z}\Big) \\ &= \int_V d^3\vec{r}\Big(\partial_x(xyT^{0x}) - yT^{0x} + \partial_z(xyT^{0z})\Big) = -\int_V d\vec{r}\, yT^{0x}\,.\end{aligned} \tag{5.3.21}$$

From (5.3.20) and (5.3.21) we infer

$$\int_V d\vec{r}\, x^i T^{0j} = -\int_V d\vec{r}\, x^j T^{0i}\,. \tag{5.3.22}$$

Hence from (5.3.18) we get

$$\int_V d\vec{r}\, x^i T^{0j} = \frac{1}{2}J_{ij}\,. \tag{5.3.23}$$

Thus, the dipole term in (5.3.12) gives

$$\Phi_{0i} = \frac{2G}{r^3}J_{ij}x^j\,. \tag{5.3.24}$$

From (5.3.3) we find

$$h_{0i} = \frac{2G}{r^3}J_{ij}x^j\,. \tag{5.3.25}$$

With a rotation one can make $J_{12} = -J_{21}$ the only non-zero components of J_{ij}. Changing to spherical coordinates

$$(x, y, z) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\,, \tag{5.3.26}$$

we find

$$
\begin{aligned}
h_{0\phi} &= \frac{\partial x}{\partial \phi} h_{0x} + \frac{\partial y}{\partial \phi} h_{0y} + \frac{\partial z}{\partial \phi} h_{0z} = -y h_{0x} + x h_{0y} = \frac{2G}{r^3}(-y J_{12} y + x J_{21} x) \\
&= \frac{2G}{r^3} J_{12}(-x^2 - y^2) = -2G J_{12} \frac{\sin^2\theta}{r} .
\end{aligned}
\tag{5.3.27}
$$

Thus, the dipole term corresponds to the first post-Newtonian correction and it is non-zero if the matter distribution has an angular momentum. We see that (5.3.27) is the formula we use in Section 3.4.1 to measure the angular momentum of the Kerr black hole, as one can see by comparing to Eq. (3.4.2).

5.4 Exercises for Chapter 5

Exercise 5.1. General formula for relative motion of test particles.
Consider a gravitational wave (5.2.20)-(5.2.21) propagating along the x^3 direction. Consider two test particles A and B at the positions

$$
\begin{aligned}
&\text{Particle A:} \;\; \vec{x}_A = (x_A^1, x_A^2, x_A^3) \, , \\
&\text{Particle B:} \;\; \vec{x}_B = (x_A^1 + L_\parallel \cos\theta, x_A^2 + L_\parallel \sin\theta, x_A^3 + L_\perp) \, .
\end{aligned}
\tag{5.4.1}
$$

- Derive that the proper distance $L(t)$ between the test particles is

$$
L(t) = \sqrt{L_\parallel^2 + L_\perp^2} + \frac{L_\parallel^2}{\sqrt{L_\parallel^2 + L_\perp^2}} \left(\frac{1}{2} \cos(2\theta) h_{11}(t) + \frac{1}{2} \sin(2\theta) h_{12}(t) \right) , \tag{5.4.2}
$$

  to first order in $h_{\mu\nu}$.

- For the gravitational wave event GW170817 (the first measurement of two colliding neutron stars) the two LIGO detectors in Hanford and Livingston, USA, measured a signal, but the VIRGO detector in Pisa, Italy, did not, even if the signal was strong enough for the VIRGO detector to be able to measure it. Explain using Eq. (5.4.2) how this fact can be interpreted?

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