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ABJM and the Associated Holographic Duality

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Abstract

The thesis presents a wide scope of topics all falling under the subject of AdS/CFT. Specifically, the focus will be on the AdS_4/CFT_3 case dubbed ABJM, where a review will be done mainly regarding the general aspect of the AdS/CFT correspondence, ABJM and tools to motivate it such as Superstrings, Supergravity and M-theory. Next, the spin-chain and Penrose limit framework is established as it will be important for calculations and interpretations. In the last part, the focus will lie on Spin Matrix theory, and how to use it to study decoupling limits in the context of Partition functions on $\mathbb{R} \times S^2$ and BPS-backgrounds for $AdS_4 \times \mathbb{C}P^3$. We find new backgrounds that exhibit a peculiar case compared to what is already known.

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Sometimes in life, things come in Packages, which has been the case for me and my brother, (not by blood), B-boy Cazo a.k.a Casper Juul Lorentzen. He already knows what his friendship has meant to me through our time at university, breakdance, this thesis and everything in between, and I will always cherish that. Then, this project would for obvious reasons never have been realized if it wasn´t for our supervisors Niels A. Obers and Troels Harmark. Their trust and belief in us, for giving us this big topic with immense potential, and truly, without making an understatement, their guidance and sharing of knowledge, comparable to Dante and Virgil, in this vast landscape of academia and theoretical physics has been a blessing to a pair of boys in a confusing world. For the sometimes (and sometimes not) long days spent at the office, I was always surrounded and comforted by what has undoubtedly become good friends where discussions and laughs on all possible ideas and topics were exchanged whether it was love, physics, life or any other random thoughts. My thanks go to Pablo, Kostas, Carl and Max-Emil for all the struggles and good times that was faced during our time at the office and elsewhere too. Also to one of my best friends Emil, thank you for always beeing there through thick and thin, in good and bad times. During the period of writing this thesis, I was lucky enough to have met what might be my first love. Even though academia is time-consuming, I managed to find time to spend with a truly special, smart, kind and beautiful girl who will always have a special place in my heart. Thank you Lisa.

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Contents

Chapter 1

Introduction

In the field of physics, one is often led to believe in radical and at first sight, crazy ideas. Going through the ages, certain figures come to mind. Aristotle, Copernicus, Newton, Maxwell, Bohr, Einstein, etc. all which have made their own contribution to the field, and revolutionized how we think about fundamental properties of nature. The foundation for all progress done today has its roots in these same crazy ideas. In more recent times 60 years or so, people have aimed higher than ever in the quest to find the pinnacle, a theory of everything. This task at first seems daunting and incomprehensible, and still to this day is an unsolved paradigm haunting physics. Nevertheless, bravery still resides within the human heart, and the boundaries are being pushed day by day to uncover the apogee of understanding nature.

A first attempt in a modern sense, was done throughout the 60´s and 70´s where through the work of Weinberg, Salam, Georgi, Glashow, Gross, Wilczek and many others great names, the electroweak unification saw the first daylight together with the strong force. It was attempted to unify all the fundamental forces leading to the Standard Model[132, 68, 67, 60, 61, 63], but there will always be a member not willing to cooperate. Gravity did not seem to get involved in the party, so the search was still on. Instead, supergravity became a realization[44], where quantization of General relativity took place with the help of supersymmetry. While this was happening, string theory came about as a relatively new idea that circulated around academic masses. But it was not until the mid 90´s that the equivalent of the renaissance took place in theoretical physics amounting to the second superstring revolution. This sparked new life in supergravity again since they became two sides of the same coin. Many proposals and crazy ideas seemed to have gained popularity, some with more experimental evidence to back up than others.

Turning to the main problem of this thesis, the most crazy idea was yet to be proposed. In the early 90´s, t´Hooft came up with idea of a Holographic principle while later on Susskind provided the bridge to connect with string theory[140, 85]. This seemingly crazy line of thought, was used to argue that our 3-dimensional universe is a hologram described by a distant 2-dimensional boundary. This certainly goes beyond even the wildest sci-fi. But as it stands, this led to what has since bloomed into a big field in theoretical physics. During the late 90´s, a remarkable connection was made [106]. Through the holographic principle, Maldacena conjectured that gravity and gauge theories were dual to each other in consecutive dimensions. By viewing the world as a hologram, it was possible to interpret a gravity theory in the bulk of space-time to be equivalent to a gauge theory on the boundary of the volume of such space-time. This came to be known as the AdS_{d+1}/CFT_d correspondence. This specifically connects a certain space-time, known as Anti-de Sitter space, to Conformal field theories.

Since Maldacena´s proposal, a countless amount of work has been initiated in the name of

AdS/CFT. But before it truly took off, Gubser, Klebanov, Polyakov and Witten (GKPW)[144, 69] refined the idea which formalized the essence of the duality for later use. In the first 10 years, it was mainly $\mathcal{N} = 4$ Super Yang-Mills (SYM) who was the star of the show corresponding to AdS_5/CFT_4. Various surprising results were unveiled, such as the connection to spin-chains[111, 24, 99], establishing a connection to condensed matter and material physics. Even further, Penrose limits, partition functions, Hagedorn temperatures and magnons are aspects considered worth noting[84, 139, 29, 80, 4]. While SYM became an established milestone in the duality, an example in a dimension lower saw the first light 17 years ago. It came to be known as ABJM after its creators (Aharony, Bergman, Jafferis, Maldacena)[5]. The same program was then used to understand, this time for the AdS_4/CFT_3 case, if it had the same properties as $\mathcal{N} = 4$ SYM. ABJM proved to be the tedious little brother. Anyhow, major progress has been made and it still stands as a hot topic today, ergo why this thesis was written in the first place.

Due to the lack of supersymmetry in ABJM, that is present in SYM, one finds a rougher journey in obtaining direct correspondence between the field/operator map. This could mean that the spectrum for operators on the gauge side is different than for what is found on the string side [120]. Interpolating between weak and strong coupling does in general become a more difficult task, but still possible nevertheless. But as for SYM, the same field of interest has been applied and elaborated on [28, 15, 81, 47, 1, 59, 66, 93]. Another interesting aspect of AdS/CFT which has gained popularity over the years, is considering non-relativistic strings[83, 121]. Using the framework of Spin Matrix Theory [79], it becomes possible to study decoupled BPS sectors for different backgrounds given by the Geometry prescribed by pp-waves of $AdS_5 \times S^5$ and $U(1)$ Galilean backgrounds as well.

The main idea of the thesis is to explore the general landscape of AdS/CFT and then specify to ABJM. With the structure established, it should be possible to analyze previous work done for $\mathcal{N} = 4$ SYM [83, 79, 80, 78] and establish to some extent how this can be modified for ABJM instead. The structure of the thesis goes as follows. In **chapter 2,3,4**, the foundation will be established to know what language we speak in gauge/gravity duality. We briefly review the build-up of AdS/CFT, which includes superstrings, type IIA and B SUGRA, T-duality and M-theory. Then we motivate the conjecture through holography and give a dictionary for translation between gravity and fields. Lastly, a deep dive is taken into the specific structure ABJM has as a theory, to be able to use this for later chapters.

In **chapter 5,6**, both spin-chains and Penrose limits will be the main stars of the show, since they become valuable tools for analyzing and interpretation in the upcoming computation. **chapter 7** introduces Spin Matrix theory and its connection to SYM, where we in the end twist the story towards ABJM and find some new connections for the structure of the theory. **chapter 8** is a more technical and heavy computation-filled part, where the calculations from [81] are reproduced. One finds via a Sigma model limit and Penrose limit Landau-Lifshitz spin chains models in the $SU(2) \times SU(2)$ sector. An extension of this sector has been calculated where spin has been added. **chapter 8** is devoted to the work of [80, 78], where the corresponding partition function for ABJM is derived in agreement with known results [47, 32]. The last chapters, **chapter 10,11** are the main results found in this thesis, which extends the previously known decoupled backgrounds in SMT [82], to $AdS_4 \times \mathbb{C}P^3$, where previously known result show up. We comment on how the geometry alters the computations and why the same truncation from the maximal BPS-sector in the Super Yang-Mills case is not present in ABJM.

Chapter 2

From the Gauge/Gravity Cradle to a Stack of $M2$-Branes: The Road to ABJM

2.1 Pre AdS/CFT: Strings and Supergravity

Before arriving at the main entrance, we have to make a detour first to a more familiar landscape, namely strings and supergravity, and then get a feel for they relate to each other. This will be foundational for the whole buisness of the thesis and in general AdS/CFT

2.1.1 Superstrings, GSO projections and SUGRA

We start from a well known place, namely the familiar superstring. As it is known, going from the 26- dimensional bosonic string, cutting down to 11 dimensions is possible by adding fermions to theory via supersymmetry

$$S = -\frac{T}{2}\int d^2\xi(\partial^\alpha X^\mu \partial^\alpha X_\mu - i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu) \tag{2.1.1}$$

Here ψ is as usual the two-component Dirac spinor for for fermion, and ρ is the two-dimensional representation of the Dirac matrices. By the addition of fermions, one needs periodicity conditions on the left and right moving sectors from the EOM´s [21, 141]. This leads to what is called the Ramond (R) and Neveu-Schwarz (NS) conditions. Counting gives four types of periodic conditions namely $NS - NS, R - R, NS - R, R - NS$. Consider now the spectrum of the closed string, which also motivates the SUGRA-theory. In the process a problem will be encountered when level matching is established. Depending on the choice of sector, part of the spectrum seems to be projected out. To ensure consistency, including modular invariance and the removal of unphysical states (tachyons), the GSO projection is applied on either NS or R states. In the R sector, it enforces a chirality condition via the Γ_{11} operator, analogous to γ_5 in four dimensions. The choice of chirality for the left- and right-moving sectors determines the type of theory through the parameter $s = \pm 1$: if both sectors have the same chirality ($s = 1$), one obtains Type IIB (chiral) supergravity; if they have opposite chirality ($s = -1$), the result is Type IIA (non-chiral) supergravity. We briefly go through the actions for both superstrings and motivate the content through irreducible representations. Then we proceed to connect the two sides via T-duality and in the end describe M-theory and branes shortly

2.1.2 Supergravity theories and different species

Since the late 70´s, supersymmetry has been used to construct various supergravity (SUGRA) theories. Originally, the 11-dimensional SUGRA was constructed, which was shown to be supersymmetric which sparked a vast amount of work in the field[44, 50, 118, 92, 70, 117]. SUGRA has had its era of ups and downs, especially in the wake of the second superstring revolution due to Witten [145]. We briefly review type IIA/B SUGRA, since it will be an essential underlying frame for the entire theory and thesis

Type IIB Supergravity

Let us start with the most classical type of SUGRA namely type IIB. Looking at the mass spectrum results in the decomposition of the irreducible representations. The mass spectrum transforms as a vector in $SO(D-2)$ as the Wigner little group. In 10 dimensions this is $SO(8)$. These irreducible representations are given by $\{\mathbf{1}, \mathbf{8}_v, \mathbf{8}_c, \mathbf{8}_s, \mathbf{28}, \mathbf{35}_v, \mathbf{35}_s, \mathbf{56}_c, \mathbf{56}_s\}$. The table below translates into field content for the respective irreps. For $SO(N)$, decomposing tensor products into direct sums is given as $N \otimes N = (\frac{1}{2}N(N+1)-1) \oplus (\frac{1}{2}N(N-1) \oplus 1$. IIB superstrings have the following structure

$$(\mathbf{8}_{\mathrm{v}} \oplus \mathbf{8}_{\mathrm{c}}) \otimes (\mathbf{8}_{\mathrm{v}} \oplus \mathbf{8}_{\mathrm{c}}) \tag{2.1.2}$$

The first term that will be common in both cases is the tensor product for the (NS, NS) sector

$$\mathbf{8}_{\mathrm{v}} \otimes \mathbf{8}_{\mathrm{v}} = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_v \tag{2.1.3}$$

Specifying for the remainder of bosonic DOF from the $R-R$ sector for type IIB gives

$$\mathbf{8}_{\mathrm{c}} \otimes \mathbf{8}_{\mathrm{c}} = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_{\mathrm{c}} \tag{2.1.4}$$

Lastly are the fermionic fields. They are found in either (or both) $NS-R$ or $R-NS$, and for IIB it is

$$\mathbf{8}_{\mathrm{v}} \otimes \mathbf{8}_{\mathrm{c}} = \mathbf{8}_{\mathrm{v}} \oplus \mathbf{56}_{\mathrm{c}} \tag{2.1.5}$$

Field content of type IIB supergravity		
Field	SO(8) representation	content
g_{mn}	$\mathbf{35}$	metric (graviton)
$C_{(0)} + \exp(-i\phi)$	$\mathbf{1}^2$	Axion and Dilaton
$B_{(2)}, C_{(2)}$	$\mathbf{28}^2$	two-form
$C_{(4)}$	$\mathbf{35}_+$	self-dual four-form
$\lambda^M_{I\alpha}, I = 1, 2$	$\mathbf{56}'^2$	Majorana–Weyl gravitinos
$\lambda_{I\alpha}, I = 1, 2$	$\mathbf{8}'^2$	Majorana–Weyl dilatinos

Table 2.1: Summary of the bosonic and fermionic fields in type IIB supergravity and their $SO(8)$ representations

The low-energy action for type IIB superstrings can now be obtained in the string frame, using the direct sums decomposed from the various sectors, and using table (1.1), one builds the SUGRA action from the components[6]

$$\begin{aligned}\mathcal{S}_{IIB} = \frac{1}{2\kappa_{10}^2}\Bigg[\int d^{10}X\sqrt{-g}\Big(e^{-2\phi}(R + 4\partial_M\phi\partial^M\phi - \frac{1}{2}|H_{(3)}|^2 - \frac{1}{2}|F_{(1)}|^2 - \\ \frac{1}{2}|\tilde{F}_{(3)}|^2 - \frac{1}{4}|\tilde{F}_{(5)}|^2) - \frac{1}{2}\int C_{(4)} \wedge H_{(3)} \wedge F_{(3)}\Bigg]\end{aligned} \tag{2.1.6}$$

The factor in front is the ten-dimensional gravitational constant $2\kappa_{10}^2 = (2\pi)^7\alpha'^4$. To get the Newton constant one simply modifies with the coupling constant. The final part is to introduce the fields composed in one-forms

$$\begin{aligned} &F_{(p)} = dC_{(p-1)}, \quad H_{(3)} = dB_{(2)}, \quad \tilde{F}_{(3)} = F_{(3)} - C_{(0)}H_{(3)}, \\ &F_{(1)} = dC_{(0)}, \quad \tilde{F}_{(5)} = F_{(5)} - \frac{1}{2}C_{(2)} \wedge H_{(3)} + \frac{1}{2}B_{(2)} \wedge F_{(3)} \end{aligned} \tag{2.1.7}$$

Here the d is the exterior derivative, and further one must impose a self-duality constraint for $*\tilde{F}_{(5)} = \tilde{F}_{(5)}$. This is added as a constraint on the solutions and not imposed on the action, which leads to wrong EOM´s [127]. It is also worth mentioning the EOM´s which will provide the metric that a stack of Dp-branes will provide by curving space-time. The general ansatz is

$$ds^2 = F(r)\Big[-dt^2 + \sum_{i=1}^{p} dz_i^2\Big] + G(r)\Big[dr^2 + r^2 d\Omega_{D-p-2}^2\Big], \quad C_{tz_1\ldots z_p} = C(r), \quad e^{\Phi} = I(r) \tag{2.1.8}$$

It is a relatively straightforward exercise to determine the unknown functions by inserting them in the EOM´s which produces

$$F(r) = H(r)^{\frac{p-7}{8}}, \quad G(r) = H(r)^{\frac{p+1}{8}}, \quad C(r) = 1 - H(r)^{-1}, \quad I(r) = H(r)^{\frac{3-p}{4}} \tag{2.1.9}$$

One obtains the characteristic harmonic function on the transverse space $H(r) = 1 + \frac{\alpha}{r^{7-p}}$ and $\alpha = (4\pi)^{\frac{5-p}{2}}\Gamma(\frac{7-p}{2})l_s^{7-p}g_sN$. In the context of AdS/CFT, we consider the case $D = 10$ and $p = 3$. This will describe a stack of $D3$-branes curving a 10-dimensional spacetime with resulting solutions being

$$ds^2 = H(r)^{-1/2}\Big[-dt^2 + \sum_{i=1}^{3} dz_i^2\Big] + H(r)^{1/2}\Big[dr^2 + r^2 d\Omega_5^2\Big], \quad C_{tz_1\ldots z_p} = 1 - H(r)^{-1}, \quad \Phi = 0 \tag{2.1.10}$$

In this context, the harmonic function takes the value

$$H(r) = 1 + \frac{\alpha}{r^4}, \quad \alpha = 4\pi l_s^4 g_s N \tag{2.1.11}$$

This is a starting point for the original Ads_5/CFT_4 where Maldacena saw the connection between open and closed strings[106]. For further elaboration see Appendix E[1]

Type IIA Supergravity

Similarly, one can construct an action for type IIA supergravity given the product representations as done for type IIB. For type IIA the general structure is

$$(\mathbf{8}_{\mathrm{v}} \oplus \mathbf{8}_{\mathrm{c}}) \otimes (\mathbf{8}_{\mathrm{v}} \oplus \mathbf{8}_{\mathrm{s}}) \tag{2.1.12}$$

As before, we can find the remaining bosonic DOF and fermion fields by considering the remaining sector

$$R-R : \mathbf{8}_{\mathrm{c}} \oplus \mathbf{8}_{\mathrm{s}} = \mathbf{8}_{\mathrm{v}} \oplus \mathbf{56}_{\mathrm{v}}, \quad NS-R : \mathbf{8}_{\mathrm{v}} \otimes \mathbf{8}_{\mathrm{s}} = \mathbf{8}_{\mathrm{c}} \oplus \mathbf{56}_{\mathrm{s}} \tag{2.1.13}$$

Then as before, interpreting the SUGRA action via the irreps. an explicit expression can be written

$$\begin{aligned} \mathcal{S}_{IIA} = \frac{1}{2\kappa_{10}^2}\Big[\int d^{10}X\sqrt{-g}\Big(&e^{-2\phi}(R + 4\partial_M\phi\partial^M\phi - \frac{1}{2}|H_{(3)}|^2) \\ &-\frac{1}{2}|F_{(2)}|^2 - \frac{1}{2}|\tilde{F}_{(4)}|^2 - \frac{1}{2}\int B \wedge F_{(4)} \wedge F_{(4)}\Big] \end{aligned} \tag{2.1.14}$$

[1] An interesting sidenote: Type IIB SUGRA has a global $SL(2,\mathbb{R})$ (Möbius) symmetry (not manifestly) and is holographically related to the Montonen-Olive duality of N = 4 SYM[113]

One defines $\tilde{F}_{(4)} = dA_{(3)} - A_{(1)} \wedge F_{(3)}$. We will see below that the action can be obtained by dimensional reduction of an eleven-dimensional supergravity that is unique in the sense that it is the only (local) supersymmetric theory in eleven dimensions containing only massless particles of spin ≤ 2. In particular, it contains two bosonic fields, the metric G_{MN} and a three-form potential $A_{(3)} = A_{MNR}dx^M \wedge dx^N \wedge dx^R$. This will arise in the context of M-theory[2]

2.1.3 The Bridge between A and B: T-duality

To make a full circle, string theory manages to connect both types of SUGRA theories[141, 21]. The idea is to consider the Minkowski background in $\mathbb{R}^{1,8} \times S^1$ without any Kalb-Ramond, Dilaton or other field strengths turned on. The circle comes from assuming $X^9 \equiv X^9 + 2\pi R$. Using this on the embedding coordinates in target space, a winding number appears through $X^9(\tau, \sigma + 2\pi) = X^9(\tau, \sigma) - 2\pi m R, m \in \mathbb{Z}$. As a consequence, the mass spectrum acquires an extra term, depending on the sector of the form $(\frac{k}{R} \pm \frac{mR}{l_s^2})^2$. Hence, T-duality can be stated as the symmetry where momentum and winding modes can be interchanged $k \leftrightarrow m$. But the caveat is M^2 should stay invariant. Looking at $(\frac{k}{R} \pm \frac{mR}{l_s^2})^2$ this becomes possible if we also exchange $R \leftrightarrow \frac{l_s^2}{R}$. It can also be shown that the chirality operators change signs under T-duality, meaning that $s \to -s$. The conclusion is that type IIA string theory on $\mathbb{R}^{1,8} \times S^1$ with radius R, is dual to type IIB string theory on $\mathbb{R}^{1,8} \times S^1$ with radius $\frac{l_s^2}{R}$. Physics on either type of strings can be mapped to another since they are T-dual. It can be inferred that the target space is not a fundamental property in string theory surprisingly, following this compactification of one-dimension. This means that there isn´t a unique correct space-time since one can translate between them which is a powerful implication. T-duality is just one of the few dualities used in the realm of string theories. Famously S and U-dualities exist as well, but for the purpose we just review the connection between type II A and B strings. To finish the story of string theory and start the story of AdS/CFT, we consider the connecting piece; M-theory

2.1.4 M-Theory, $M2$-Branes, BLG and Near-Horizon Geometry

Whenever people talk about M-theory, some mythical beast seems to be thought of. The reason seems to be that nobody understands its structure properly. A good way of representing how M-theory relates to known physics is through the following diagram displaying the connection to 11 d SUGRA and type IIA backgrounds Thus M- theory, depending on a low energy limit or

$$\begin{array}{ccc} \text{M-theory} & \underset{S^1}{\longrightarrow} & \text{type IIA string theory} \\ \downarrow & & \downarrow \\ \text{11D supergravity} & \underset{S^1}{\longrightarrow} & \text{10D type IIA supergravity} \end{array}$$

compactification, can relate to either of them, or doing both, to a 10 d type IIA supergravity. Originally M-theory was found to spit out a relation between the radius and the string coupling constant after a Kaluza-Klein reduction also;

$$R_s/l_p = g_s^{2/3}, \quad l_p^3 = g_s l_s^3. \tag{2.1.15}$$

Here R_s is the radius associated to an extra compact dimension with finite coupling in units of an eleven-dimensional Planck length l_p. The relation is obtained by comparing coefficients from

[2]Foreshadowing BPS bounds, for type IIA theories, one can establish $M \geq \frac{c_0}{\lambda}|W|$, where is the central charge of the supersymmetry algebra, $\lambda = e^{\Phi/2}$ is the ten-dimensional string coupling and c_0 some constant. One can find soliton solution (black hole) of type-IIA supergravity with the required properties, mimicking the Kaluza-Klein mechanism for the spectrum of BPS-states $M = \frac{c}{\lambda}|n|, n \in \mathbb{Z}$[94]

the action of low energy 11 D string theory and 10 D SUGRA.
In the context of this thesis, the center of attention will be on $M2$-branes. There are various ways to argue for the existence of them. Looking at the EOM´s of 11 D SUGRA magnetic and electric branes are found, that exactly correspond to either $M2$ or $M5$-branes[107]. This can also be seen from the SUGRA action containing a 3-form antisymmetric field $A^{(3)}_{\mu\nu k}$ which couples to the $M2$ [3]. Prior to ABJM, what led up to the conjecture was a proposal by Baggert, Lambert and Gustavsson (BLG) [15, 14, 71], that proposed $M2$-branes and Chern-simons theories were dual. The connection was established as follows. Consider first as motivation pure CS field theory in $2+1$ dimensions The lagrangian contains 3-form field strengths[4] which seemingly implies that the Lagrangian can be integrated over a 3-manifold. It preserves topological invariance without needing a coupling to a metric. The moment such a theory couples to scalar or fermionic matter fields, the topological invariance is broken, since the metric is needed to define the matter kinetic terms and couplings. But, preserving conformal invariance remains a viable option. Theories of this kind would prove to be crucial candidates when thinking of world volume field theories on multiple membranes in M-theory. With this observation, the BLG program proceeded by considering supersymmetries that $M2$-branes preserved. By introducing terms that are non-linear in the scalar fields (mimicking interactions), a peculiar triple-product or 3-bracket was constructed from cubic terms arising[5]. As a consequence, after going through supersymmetry transformations, a Chern-Simons term appears in the full Lagrangian, which is the first sign of an interacting Lagrangian in quantum field theory not dependent on SYM-terms. Furthermore, the desired structure of bi-fundamental and anti bi-fundamental fields has its origin in the gauge transformations which fixes two separate terms one for each representation each at level k. The cutting point happens at the level of supersymmetry where $\mathcal{N}=8$ fails at describing an arbitrary number of $M2$´s. This establishes the $\mathcal{N}=6$ supersymmetry which we discuss in the next section. To establish a bridge from gravity to gauge fields, we first want obtain a near-horizon limit that will serve an important purpose. It is through the brane construction of ABJM that it was found to be dual to a stack of $M2$´s transverse to a $\mathbb{C}^4/\mathbb{Z}_k$ orbifold geometry. In terms of an eleven-dimensional metric, this is best described by [95, 6]

$$\begin{aligned} ds^2 &= H(r)^{-2/3}(-dt^2+dx_1^2+dx_2^2)+H(r)^{1/3}(dr^2+r^2d\Omega_7^2) \\ H(r) &= 1+\frac{L^6}{r^6}, \quad L^6=32\pi^2 N l_p^6, \quad F_{(4)}=-dt\wedge dx_1\wedge dx_2\wedge dH(r)^{-1} \end{aligned} \tag{2.1.16}$$

In the near-horizon limit where $r \ll L$, the harmonic function can be approximated to

$$\left(1+\frac{L^6}{r^6}\right)^{-2/3} \simeq \frac{r^4}{L^4}, \quad \text{and} \quad \left(1+\frac{L^6}{r^6}\right)^{1/3} \simeq \frac{L^2}{r^2},$$

Which simplifies the metric to

$$ds^2=\frac{r^4}{L^4}(-dt^2+dx_1^2+dx_2^2)+\frac{L^2}{r^2}(dr^2+r^2d\Omega_7^2). \tag{2.1.17}$$

Recalling that the Anti-de Sitter metric in Poincaré coordinates can be written

$$ds^2_{AdS_4}=\frac{L^2}{r^2}dr^2+\frac{r^2}{L^2}dx^\mu dx^\nu \eta_{\mu\nu}$$

[3] Another way is considering a $D2$-brane [16]. Relating the coupling in SYM to type IIA string via $g^2_{\mathrm{YM}}=\frac{g_s}{\sqrt{\alpha'}}$, and writing the lagrangian for a non-dynamical field $B_{\mu\nu}$, the EOM´s and Equation 2.1.15 generate the lagrangian for a single $M2$-brane

[4] $\mathcal{L}_{\mathrm{cs}}=\frac{k}{4\pi}(A\wedge dA-\frac{2i}{3}A\wedge A\wedge A)$

[5] The structure is the same as usual commutators constituting a lie algebra, this time only with three elements $[T^a,T^b,T^c]=f^{abc}_d T^d$ (this also constitutes a lie algebra)

and using the transformation $z = L^3/2r^2 \implies dz^2 = L^6/r^6 dr^2$, the $M2$-brane metric in the near-horizon limit can be found as

$$
\begin{aligned}
ds^2 &= \frac{L^2}{4z^2} dx^\mu dx^\nu \eta_{\mu\nu} + \frac{L^2}{4z^2} dz^2 + L^2 d\Omega_7^2 \\
&= L^2 \left(\frac{1}{4} ds^2_{Ads_4} + ds^2_{S^7} \right). \qquad (2.1.18)
\end{aligned}
$$

So it can be seen that a stack of $M2$-branes in a near horizon limit probes a geometry just like $D3$-branes does. This will constitute the gravity side when ABJM is established later on.

2.2 Aspects of AdS/CFT and Motivation

Arriving at the front step, we must confront the conceptual landscape before actually getting dirty with any calculations. Thus, we introduce The Holographic Principle as the leading guide to the whole arena

2.2.1 The holographic principle and motivation for AdS/CFT

If we want to trace the origin of holography and for that case AdS/CFT, one must go back to t´Hooft and Susskind in their seminal papers[85, 140]. Given some quantum theory, and remembering thermodynamics with its third law, it is possible to relate the entropy $\mathcal{S}$ of a system to the total number of DOF; Following t´Hooft, we first consider the Hilbert space dimension $\mathcal{N}$. Assuming a discrete model of boolean variables, for instance, n spins taking only two values, this is regarded as sufficient. Entropy and boolean DOF are related by $e^{\mathcal{S}} = \mathcal{N} = 2^n$[85]. Bekenstein and Hawking famously derived the bound that showed that entropy of a black hole[26], is proportional to its area $\mathcal{S}_{BH} = \frac{A}{4G}$. A is the surface area of the black hole and G is the Newton constant. Further, it is postulated that the generic entropy of a system cannot violate $\mathcal{S} < \mathcal{S}_{BH}$. Combining the bound for entropy and DOF we can get the relation $\mathcal{N} \leq e^{\frac{A}{4G}}$. Going to Susskind [140], the world, which by default is taken to be 3-dimensional, is a lattice of binary quantum DOF. For concreteness assume that the lattice spacing is the Planck length l_P with the same kind of discretized model. Thus, the number of quantum states in a volume V is $\mathcal{N}(V) = 2^n, n = \frac{V}{l_p^d}$, so n just counts the lattice sites in V. The logarithm of $N(V)$ is the maximum possible entropy, so one obtains in the system $\log N(V) = \frac{V}{l_P^3} \log 2$. Combining with the previous bound we get

$$
\mathcal{S} \leq \log(\mathcal{N}(V)) = \frac{V}{l_P^d} \log 2 \qquad (2.2.1)
$$

Rather than being bounded by the area A, the largest possible entropy scales as the volume V it seems. Above the Planck threshold, it holds that $A \leq V$ amounting to a larger entropy bound. Suppose that the entropy estimate holds. Then by our first bound, the dimensionality of the Hilbert space describing the region is $N \sim e^V$. But, supposing that the region collapses to a black hole, the total entropy has decreased to $e^{\frac{A}{4G}}$ together with the number of states. This seems to mean, that the second law has been violated. Thus the original bound Bekenstein proposed must be valid. Concluding with the radical idea from t´Hooft, one can describe all phenomena within V by a set of DOF which resides on the surface with area A bounding V. The DOF should not exceed that of a two-dimensional lattice with approximately one boolean DOF per Planck area. This leads to the interpretation that the world can be seen as a two-dimensional lattice of spins. To sum up the holographic principle we can state:

Susskind–'t Hooft Holographic Principle: *A region of spacetime with boundary of area A is fully described by at most $A/4G$ degrees of freedom*

Taking the holographic principle at face value, we are led to conclude that if we have a theory of quantum gravity on some manifold $\mathcal{M}$, the theory will be entirely determined by some other theory living on the boundary $\partial\mathcal{M}$, and the theories are said to be dual. This would rather seem to be a good place to start if one would construct a theory concerning dualities between gravity in bulk space-time and some more mystical landscape at the boundary. This would be the case for AdS/CFT at least. If one tries to count the number of DOF per Planck area, it becomes tedious since a CFT has an infinite amount. The solution is then to make a cutoff for the DOF on the field theory side and compare to the bulk. Writing the metric in an approriate form with singularities [6], will amount to the UV[7]. It can be shown that the entropy scales the same for $\mathcal{N} = 4$ SYM in both the field and gravity theories, saturating the holographic bound[8][3]

Another argument given by Polchinski [128] bypassing string theory has its ties to a no-go theorem by Witten and Weinberg. As with so many no-go theorems, there always seems to be a slick way out of things by going beyond the underlying hidden assumption. A great example is the S-matrix. Going beyond bosonic symmetries, supersymmetry was realized, hence the Coleman-Mandula lemma was beaten. Back to Witten and Weinberg, a No-go theorem was established for a system of gravitons being bound states of gauge bosons:

(Witten & Weinberg): *All theories with a Lorentz-covariant energy-momentum tensor, such as all known renormalizable quantum field theories, composite as well as elementary massless particles with $j > 1$ are forbidden*

So can it be beaten one might ask. The answer is yes, but the idea is to realize that if one has a gauge boson propagating through spacetime in dimension d, then the graviton will have to move in one dimension higher, to circumvent the no-go theorem. As holography constrains entropy to be most that of a black hole proportional to its area in Planck units, one gets that quantum gravity in any volume is naturally formulated in terms of DOF on its surface, one per Planck unit as well. Thus QG lives in a dimension higher than the gauge theory. One might ask what this extra dimension is. There seems to be an interpretation in terms of Wilson Renormalization and RG equations. Having motivated AdS/CFT as a consequence of the holographic principle, establishing the general correspondence between fields and operators will be the next task at hand.

2.2.2 The Correspondence

As the name suggests, AdS/CFT makes a connection between gauge theories and gravity as prescribed by holography. But to translate between the two, objects of use need to be identified. Since CFT´s do not have asymptotic states or an S-matrix, the natural objects to consider are operators. Operators as it turns out are dual to closed strings or particles in the bulk. Further, the mass of particles in the bulk is related to the conformal dimension of the gauge theory on the boundary. We proceed to show this and further state how correlation functions of field theories relate to partition functions of strings with diagrammatic interpretations as well.

[6] Singularities at $r = 1$ or $z = 0$ correspond to metrics $ds^2 = R^2\Big[-(\frac{1+r^2}{1-r^2})^2 dt^2 + \frac{4}{(1-r^2)^2}(dr^2 + r^2 d\Omega^2)\Big]$ and $ds^2 = R^2 \frac{-dt^2 + d\vec{x}^2 + dz^2}{z^2}$

[7] What is meant is setting $r = 1 - \delta$ and taking the limit $\delta \to 0$ makes the field theory flow to the UV

[8] Using δ as the UV-cutoff one finds that the entropy scales like $S \sim N^2\delta^{-3}$

Boundary asymptotics and Breitenlohner-Freedman bound

To relate mass and conformal dimensions, start by considering a scalar field dual to a primary operator in the CFT for an action toy model

$$S[\phi] = -\frac{1}{2}\int dzd^dx\sqrt{-g}(g^{mn}\partial_m\phi\partial_n\phi + m^2\phi^2) \tag{2.2.2}$$

The idea behind the action is splitting the integration up into a "radial" coordinate which in the Ads metric contains the z-coordinate, we associate this to some kind of energy scale $r = \frac{L^2}{z} \sim \mu$. Now to continue we find the Euler Lagrange equations given as $\partial_m \frac{\delta\mathcal{L}}{\delta\partial_m\phi} = \frac{\delta\mathcal{L}}{\delta\phi}$. Using that $\mathcal{L} = \sqrt{-g}(g^{mn}\partial_m\phi\partial_n\phi + m^2\phi)$ and taking the appropriate derivative while also reminding ourselves that the Laplace-Beltrami operator can be written as

$$\Box_g\phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) \tag{2.2.3}$$

The EOM is just the Klein-Gordon equation $(\Box_g - m^2)\phi = 0$. It may be convenient to express box at the boundary of the metric

$$\Box_g|_{AdS} = \frac{1}{L^2}(z^2\partial_z^2 - (d-1)z\partial_z + z^2\eta_{\mu\nu}\partial^\mu\partial^\nu) \tag{2.2.4}$$

Doing a plane wave-ansatz for the scalar field $\phi(z,x) = e^{ip^\mu x_\mu}\phi_p(z)$, and inserting this into the KG equation one finds

$$z^2\partial_z^2\phi_p - (d-1)z\partial_z\phi_p - (m^2l^2 + p^2z^2)\phi_p = 0 \tag{2.2.5}$$

The main interest will be to look for asymptotic solutions as $z \to 0$. Surely in that case the term p^2z^2 becomes very small and can be neglected, leaving us with

$$z^2\partial_z^2\phi_p - (d-1)z\partial_z\phi_p - m^2l^2\phi_p = 0 \tag{2.2.6}$$

Assuming a power-law ansatz as a solution of the form $\phi_p(z) = z^\Delta$ leaves us with.

$$z^2\partial_z^2 z^\Delta - (d-1)z\partial_z z^\Delta - m^2l^2z^\Delta = \Delta(\Delta-1)z^\Delta - (d-1)\Delta z^\Delta - m^2l^2z^\Delta = 0 \iff \Delta(\Delta-d) = m^2l^2 \tag{2.2.7}$$

Solving the standard quadratic equation in Δ one finds two roots admitting the solutions

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2l^2}, \quad \Delta_+ > \Delta_- \tag{2.2.8}$$

The general solution can be written as a linear combination in the asymptotic limit $\phi \sim \phi_0 z^{\Delta_-} + \phi_0 z^{\Delta_+}$. Using the ansatz in the Poincaré AdS action, it may be written as

$$S[\phi] = -\frac{l^{d-1}}{2}\int dzd^dx\frac{1}{z^{d+1}}(z^2\partial_z\phi\partial_z\phi + m^2l^2\phi^2) \tag{2.2.9}$$

Splitting up the integral into some uv-cutoff and letting the second integral run, the integral becomes $\int_0^\infty dzI(z) = \int_0^\epsilon dzI(z) + \int_\epsilon^\infty dzI(z)$. This computation boils down to finding a dimensional bound such that the integral is normalizable

$$S[\phi] = -\frac{l^{d-1}}{2}\int_0^\epsilon dzd^dx\frac{1}{z^{d+1}}(z^2(\partial_z z^\Delta)^2 + m^2l^2z^{2\Delta}) = \tag{2.2.10}$$

$$-\frac{l^{d-1}}{2(2\Delta-d)}(\Delta^2 + m^2l^2)\int d^dx[z^{2\Delta-d}]_0^\epsilon \tag{2.2.11}$$

So in order not to get any logarithmic divergence in the integral, we impose that the bound $\Delta > \frac{d}{2}$, such that the integral is normalizable.

Consider performing integration by parts in the action

$$S[\phi] = -\frac{l^{d-1}}{2}\int dz d^d x \frac{1}{z^{d+1}}(-z^2\phi\partial_z^2\phi + (d-1)z\phi\partial_z\phi + m^2l^2\phi^2) \tag{2.2.12}$$

The ansatz we use here goes to next order, such that $\phi(x,z) \sim C_1 e^{\Delta} + C_2 e^{\Delta+2}$. Inserting this into the action, expanding and doing some gymnastics, one may find the reduced expression

$$S[\phi] = -\frac{l^{d-1}}{2}C_1C_2\int d^d x\int_0^{\epsilon} dz(2d - 4(\Delta+1) - m^2l^2)z^{2\Delta-d+1} = \tag{2.2.13}$$

$$-\frac{l^{d-1}}{2}\frac{2d-4(\Delta+1)-m^2l^2}{2\Delta-d+1}C_1C_2\int d^d x[z^{2\Delta-d+2}]_0^{\epsilon} \tag{2.2.14}$$

For the integral to be normalizable, the bound $\Delta > \frac{d-2}{2}$ must be satisfied. Requiring that the square root in the solution for Δ must be positive, exactly gives the Breitenlohner-Freedman bound

$$m^2l^2 > -\frac{d^2}{4} \tag{2.2.15}$$

in the case of the positive solution of Δ_+. Note further that $\Delta_+ \geq \Delta_-$ as well as $\Delta_- = d - \Delta_+$, which implies that under boundary conformal rescaling $x \to x' = \lambda x$ (same for z), the boundary field $\phi_{(0)}(x)$ transforms as

$$\phi'_{(0)}(\lambda x) = \lim_{z'\to 0}(z')^{-\Delta_-}\phi'(z',x') = \lambda^{-\Delta_-}\lim_{z'\to 0} z^{-\Delta_-}\phi(z,x) = \lambda^{d-\Delta_+}\phi_{(0)}(x) \tag{2.2.16}$$

where we have used the fact that the bulk field is invariant under the AdS isometry. One can infer that $\phi_{(0)}(x)$ transforms as a source for a primary operator with dimension Δ_+, leading us to identify the boundary field $\phi_{(0)}(x)$ as a source for a dual field theory operator $\mathcal{O}_{\Delta_+}$ and similarly $\phi_+(x)$ is the VEV of $\mathcal{O}_{\Delta_+}$. Turning to the other case, for $(d-2)/2 \leq \Delta < d/2$, it turns out that we have to identify the conformal dimension of the field with Δ, so on the overlap $d^2/4 \leq m^2l^2 \leq d^2/4 + 1$, the identification of VEV and sources of the field theory operator can be interchanged, thus modifying the boundary conditions of the problem. We make a summary for fields and how they change in the case of $\mathcal{N} = 6$ Chern-Simons and $\mathcal{N} = 4$ SYM

Mass-Dimension Relations		
Field	$\mathcal{N} = 6$ **CS:** m–Δ **Relation**	$\mathcal{N} = 4$ **SYM:** m–Δ **Relation**
Scalars, massive spin-2 fields	$m^2L^2 = \Delta(\Delta-3)$	$m^2L^2 = \Delta(\Delta-4)$
Massless spin-2 fields	$\Delta = 3$	$\Delta = 4$
p-form fields	$m^2L^2 = (\Delta-p)(\Delta+p-3)$	$m^2L^2 = (\Delta-p)(\Delta+p-4)$
Spin-$\frac{1}{2}$, spin-$\frac{3}{2}$	$\lvert m\rvert L = \Delta - \frac{3}{2}$	$\lvert m\rvert L = \Delta - 2$
Rank-s symmetric traceless tensor	$(\Delta+s-2)(\Delta-s-1)$	$(\Delta-2)^2 - s^2$

Table 2.2: Mass-dimension relations in $\mathcal{N} = 6$ Chern–Simons and $\mathcal{N} = 4$ SYM.

From Quantum Fields to Conformal Fields: GKPW

Even though Maldacena put forward the conjecture[106], it was not until a little while after, that the realization of a "precise" statement for the link between the conformal field operators and the bulk gravity action could be stated. In the notorious paper of Witten[144], we get the missing link for the partition function that binds the two sides. The generating functional $W_{[\phi_0]}$ for connected Green's functions has the scalar $\phi_{(0)}$ in the bulk source the operator $\mathcal{O}_\Delta$ of composite fields at the boundary. Then the partition function in Euclidean signature for the CFT takes the form

$$Z_{CFT}[\phi_0] = e^{W_{[\phi_0]}} = \left\langle \exp\left(\int d^d x \phi_{(0)}(x) \mathcal{O}_\Delta \right) \right\rangle_{\text{CFT}} \tag{2.2.17}$$

Moving to the bulk, gravity dominates so the expected partition function is the one for string theory. But as it turns out, the partition function for strings is unknown. For the weak form of the correspondence, a saddle point approximation is used to get the superstring partition function Z_{string} which is given by the low energy string namely type IIB supergravity. The action is $\mathcal{S}_{\text{Sugra}}[\phi]$, which relates to connected Green's function $W_{[\phi_0]}$ as

$$W_{[\phi_0]} = \mathcal{S}_{\text{Sugra}}[\phi]\Big|_{\lim_{z\to 0}(z^{\Delta-d}\phi(z,x))=\phi_{(0)}(x)} \tag{2.2.18}$$

The story might be more involved than what is suggested. The On-Shell action for Supergravity is flooded with IR divergences, and this needs to be taken care of as one does in QFT. To this mean, holographic renormalization is used to cancel out whatever infinities might arise. Another interesting pictorial aspect of AdS/CFT is that Feynmann diagrams have been substituted with Witten diagrams, which we touch upon in the next section. But in the end, the starting point for the holographic calculation of n-point correlation functions of composite gauge invariant operators is done by taking derivatives to the sources of the connected greens function.

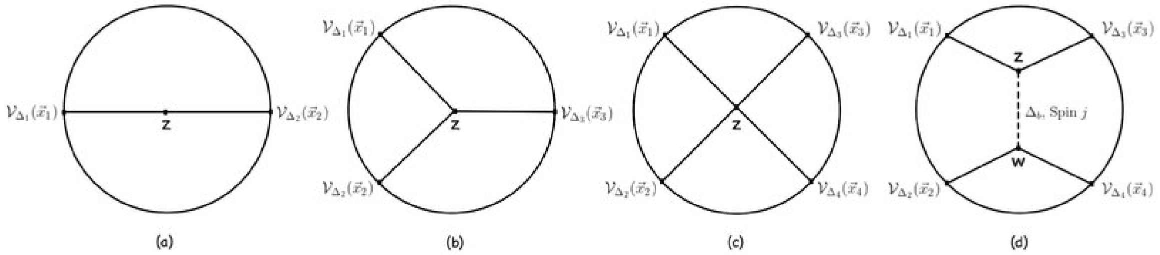
$$\left\langle \mathcal{O}_1(x_1)...\mathcal{O}_n(x_n) \right\rangle_{\text{CFT}} = \frac{\delta \mathcal{S}_{\text{sugra}}[\phi_{\Delta_i}]|_{\lim_{z\to 0}(z^{\Delta-d}\phi_{\Delta_i}(z,x))=\phi_{(0)\Delta_i}(x)}}{\delta\phi_{(0)\Delta_1}(x_1)...\delta\phi_{(0)\Delta_n}(x_n)}\Bigg|_{\phi_{(0)\Delta_i}=0} \tag{2.2.19}$$

With this under the belt, the natural progression is to establish how to organize correlators with Witten diagrams.

Witten Diagrams: Feynmann Diagrams Imprisoned

As analogously for QFT, to compute correlators establishing a sense of propagation in the system is crucial. The theory at hand is a bit different than the usual Feynmann rules, since processes depend on the whereabouts in the space that is being considered. Whereas only different particles had different propagators depending on their nature, one must also take into account the fact that we have a conformal boundary where operators exist. The previous argument establishes that the whole picture is governed by both bulk and boundary propagators. This is the root for computations of holographic amplitudes in AdS_{d+1} space, that are dual to correlation functions in the conformal field theory living on the boundary of AdS. Witten diagrams are represented by a circle denoting the conformal boundary of AdS_{d+1} and the interior denoting the bulk. Since the supergravity approximation is being used, we do not go beyond tree-level. The points on the boundary are labeled by chiral primaries dual to the bulk field $\phi(z)$, sourced by $\phi_0(z)$. The solid line propagators that emanate from a source on the boundary to a point in the bulk are called bulk-boundary propagators, and the dotted lines in the bulk (z and w) are called bulk-bulk propagators. The vertices in the bulk are governed by the interaction terms in the SUGRA action. As an example, take the vertices in (b) and (d) that arise from cubic coupling terms.

The starting point is the euclidean AdS_{d+1} metric $ds^2 = \frac{L^2}{z^2}(dz^2 + \delta_{\mu\nu}dx^\mu dx^\nu)$. We want to follow geodesic from boundary to bulk, and bulk-bulk also. To this mean defining a chordal



distance ξ due to the isometry of the AdS space makes computations more convenient. We define the length $d(z,x;w,y)$ for a geodesic connecting the points (z,x_μ) and (w,y_μ) as

$$d(z,x;w,y) = \int_{(z,x)}^{(w,y)} ds = \log\left(\frac{1+\sqrt{1-\xi^2}}{\xi}\right) \tag{2.2.20}$$

From this one can extract the chordal distance as

$$\xi = \frac{2zw}{z^2+w^2+(x-y)^2} \tag{2.2.21}$$

The general method to compute these propagators[57] is to first find the terms containing the bulk field ϕ in the action S_{sugra}, derive the EOM, and then compute the boundary propagator G the EOM with appropriate source J. We see how this is the case for scalars, and comment for gauge fields with $S=1$ and tensor fields with $S=2$ in D, not to exceed the purpose of the thesis.

2.2.3 Outlook

So far, we have established the underlying principles of what structures AdS/CFT builds upon, plus what motivated the search for a duality. Over the years, countless applications have been found in various fields to extend its use, and to maybe even gain observable quantities that can give a clue about a phenomenological side to the story. Nevertheless, the search is still ongoing, and the field is still evolving. From the cradle when $\mathcal{N}=4$ SYM was subjected as patient zero to the framework, other dualities have found their way to the table. We proceed to investigate certain aspects of the AdS/CFT framework in the context of a newer duality dubbed ABJM.

2.3 ABJM: $AdS_4 \times \mathbb{CP}^3$ and $\mathcal{N}=6$ Chern-Simons

The road starts at a place in pure mathematics, where due to Chern and Simons [40], it became possible to get a whole framework for multiple areas in physics. As always in physics, we are interested in an action or Lagrangian. To that mean, consider a topological field theory in $2+1$ dimensions, described by a gauge group $\mathcal{G}$ and level k. Written compactly in differential forms, the celebrated Chern-Simons theory has either the following action or Lagrangian

$$\mathcal{S}_{cs} = \frac{k}{4\pi}\int_{\mathcal{M}} \text{Tr}(A\wedge dA + \frac{2}{3}A\wedge A\wedge A), \quad \mathcal{L}_{cs} = \kappa\epsilon^{\mu\nu\rho}\text{Tr}(A_\mu\partial_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_{\rho)} \tag{2.3.1}$$

The domain of integration is over a topological 3-manifold $\mathcal{M}$, a 1-form associated with the gauge field A which also transforms in the adjoint representation of $\mathcal{G}$. A rather interesting feature compared to usual QFT´s, is the absence of a metric, hence why the theory i called topological. One is also led to question if the action is gauge invariant because it involves the gauge field A_μ itself, rather than just the (manifestly gauge invariant) field strength $F_{\mu\nu}$. Luckily this is the case[51]. An important feature is to consider a gauge transformation that will have a great consequence for the rest of the thesis. The gauge field and action transforms as [9]

$$\begin{aligned} &A_\mu \to A_\mu^g = g^{-1}A_\mu g + g^{-1}\partial_\mu g, \qquad \mathcal{L}_{cs} \to \mathcal{L}_{cs} - w(g) \\ &w(g) = \frac{k}{24\pi^2}\epsilon^{\mu\nu\rho}\mathrm{Tr}(g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\rho g), \qquad \mathcal{S}_{cs} \to \mathcal{S}_{cs} - 8\pi^2 kN \end{aligned} \tag{2.3.2}$$

Under the gauge transformation, one picks up $w(g)$ which is commonly known as the winding number[10]. In terms of QFT´s, for the path integral to remain gauge invariant, it must be required that the level takes discrete values $k \in \mathbb{Z}$[11]. This will have a big influence on supersymmetry for the theory which will become apparent shortly.

2.3.1 $\mathcal{N} = 2$ Chern-Simons action from superspace: A First Glance towards ABJM

To go from the pure CS theory, we need to add fermionic degrees of freedom, which will come through the vector and chiral multiplets.

$$\mathbf{V} : \{A_\mu, \chi, \sigma, D\}, \quad \Phi : \{\phi, \psi, F\} \tag{2.3.3}$$

For the vector multiplet, A_μ is the gauge field, χ is the two Majorana spinors combined into one complex spinor and σ is a real scalar and D is a real auxiliary scalar. For the chiral superfield we have that ϕ is a complex scalar, ψ is the two Majorana spinors combined into a complex spinor and F is a complex auxiliary scalar. To proceed, one is in dire need of superspace formalism Appendix G. The first attempt to construct such theories came from[119, 12, 133, 90]. They constructed it by considering $(d, \mathcal{N}) = (3, 2)$ vector superfields can be obtained by dimensional reduction from $(d, \mathcal{N}) = (4, 1)$ vector superfields. Thus, $\mathcal{N} = 2$ Chern-Simons matter theory can be obtained by dimensional reduction of SYM, besides the kinetic part of the vector multiplet which is replaced by the supersymmetric version in the pure Chern-Simons lagrangian. One can ponder what superspace action would do this, and in a very non-trivial way it can be chosen to be [58]"

$$\mathcal{S}_{CSM}^{\mathcal{N}=2} = \int d^3x \int d^4\theta \{\frac{k}{2\pi}\int_0^1 dt \mathrm{Tr}[V\bar{D}^\alpha(e^{-tV}D_\alpha e^{tV})] + \sum_{i=1}^{N_f} \bar{\Phi}^i e^V \Phi^i\} \tag{2.3.4}$$

The index i is a global $U(N_f)$ flavor symmetry acting on Φ. The trace is in the fundamental representation for either $U(N)$ or $SU(N)$. Thus the generators T^a obeys $\mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$. Φ^i is a vector which is acted on by the representation R_i of the group. This reduces in components after integrating out the superfields, where for the kinetic part we get

$$\begin{aligned} \mathcal{S}_{CS}^{\mathcal{N}=2} &= \int d^3x \int d^4\theta \{\frac{k}{2\pi}\int_0^1 dt \mathrm{Tr}[V\bar{D}^\alpha(e^{-tV}D_\alpha e^{tV})] \\ &= \frac{k}{4\pi}\int \mathrm{Tr}(A^aT^a \wedge dA^aT^a + \frac{2}{3}A^aT^a \wedge A^bT^b \wedge A^cT^c) + \bar{\chi}^a\chi^b\delta^{ab} + D^a\sigma^b\delta^{ab} \\ &= \frac{k}{4\pi}\int \mathrm{Tr}(A \wedge dA + \frac{2}{3}A^3 + \bar{\chi}\chi + 2D\sigma) \end{aligned} \tag{2.3.5}$$

Here $a \in \{1, 2, ...dim(\mathcal{G})\}$ is the index related to the fundamental generators, such that as an example $A_\mu = A_\mu^a T^a$, and the same for the rest of the components in the multiplets. Focusing

[9]In the Lagrangian, one finds an additional term containing a total derivative $-k\epsilon^{\mu\nu\rho}\partial_\mu \mathrm{Tr}(\partial_\nu g g^{-1}A_\rho)$, which vanishes with suitable boundary conditions

[10]This is purely an artifact of the theory being non-abelian

[11]It is conventionally written with a factor $\frac{k}{4\pi}$

on the chiral field, we note that the covariant derivative is defined by

$$D_\mu\{\phi^i,\psi^i\} = \partial_\mu\{\phi^i,\psi^i\} + \frac{i}{2}A_\mu^a T_{R_i}^a\{\phi^i,\psi^i\} \tag{2.3.6}$$

Using the covariant derivative, it is possible to spill out the superspace integral and get

$$\begin{aligned}\int d^4\theta \sum_{i=1}^{N_f}\bar{\Phi}^i e^V\Phi^i &= \sum_{i=1}^{N_f}(D_\mu\bar{\phi}^i D^\mu\phi^i - i\bar{\psi}^i\gamma^\mu D_\mu\psi^i - \frac{1}{4}\bar{\phi}^i\sigma^a\sigma^b T_{R_i}^a T_{R_i}^b\Phi^i + \frac{1}{2}\bar{\phi}^i D^a T_{R_i}^a\Phi^i \\ &- \frac{1}{2}\bar{\psi}^i\sigma^a T_{R_i}^a\psi^i + \frac{i}{\sqrt{2}}\bar{\phi}^i\chi^a T_{R_i}^a\psi^i - \frac{i}{\sqrt{2}}\bar{\psi}^i T_{R_i}^a\bar{\chi}^a\phi^i + \bar{F}^i F^i)\end{aligned} \tag{2.3.7}$$

$T_{R_i}^a, a \in \{1,2,...dim(\mathcal{G})\}$ are the generators of gauge group $\mathcal{G}$ in the R_i representation and furthermore $A_\mu^i = A_\mu^a T_{R_i}^a$ and so forth. By solving the EOM´s it is found that

$$\begin{aligned} &D^\alpha : \sigma^a = -\frac{2\pi}{k}\bar{\phi}^i T_{R_i}^a\phi^i, \quad F,\bar{F} : F = 0, \bar{F} = 0 \\ &\chi^a : \bar{\chi}^a = \frac{4\pi i}{\sqrt{2}k}\bar{\phi}^i T_{R_i}^a\psi^i, \quad \bar{\chi}^a : \chi^a = -\frac{4\pi i}{\sqrt{2}k}\bar{\psi}^i T_{R_i}^a\phi^i\end{aligned} \tag{2.3.8}$$

For the grand finale, after all the parts have been manipulated, we can combine and integrate out the fields. This will in the end yield[5]

$$\begin{aligned}\mathcal{S}_{CSM}^{\mathcal{N}=2} = \mathcal{S}_{CS} + \int d^3x(&D_\mu\bar{\phi}^i D^\mu\phi^i - i\bar{\psi}^i\gamma^\mu D_\mu\psi^i + \frac{\pi^2}{k^2}(\bar{\phi}^i T_{R_i}^a\phi^i)(\bar{\phi}^j T_{R_j}^b\phi^j)(\bar{\phi}^k T_{R_k}^a T_{R_k}^b\phi^k) \\ &+ \frac{\pi}{k}(\bar{\phi}^i T_{R_i}^a\phi^i)(\bar{\psi}^j T_{R_j}^a\psi^j) + \frac{2\pi}{k}(\bar{\psi}^i T_{R_i}^a\phi^i)(\bar{\phi}^j T_{R_j}^a\psi^j))\end{aligned} \tag{2.3.9}$$

This concludes the case for $\mathcal{N} = 2$ CS in superspace. But to be compatible with ABJM, the theory has to be enhanced to $\mathcal{N} = 3$. The trick is to replace the kinetic part of $\mathcal{N} = 4$ SYM with the trilinear gauge field and derivatives, which in return breaks supersymmetry to $\mathcal{N} = 3$. It has been argued, compared to the $\mathcal{N} = 2$, that the enhancement requires adding an auxiliary chiral multiplet ϕ in the adjoint representation to the vector. Additionally one must assume that the chiral multiplets come in pairs in the conjugate representation of the gauge group forming a hpyermultiplet

$$\begin{aligned}&\mathbf{V} : \{A_\mu, \chi, \sigma, D\}, \quad \phi :_{\{}\, q, \lambda, S\} \\ &\Phi^i : \{\phi^i, \psi^i, F^i\}, \quad \tilde{\Phi}^i : \{\tilde{\phi}^i, \tilde{\psi}^i, \tilde{F}^i\}\end{aligned} \tag{2.3.10}$$

The relevant action can be found in [59], where extra terms have been added corresponding to the hypermultiplet contributions and a $\mathcal{N} = 4$ superpotential $W_{\mathcal{N}=4} = \tilde{\Phi}_i\phi\Phi_i$. The breaking of supersymmetry comes from the Chern-simons superpotential $W_{\mathcal{N}=3} = -\frac{k}{8\pi}\mathrm{Tr}(\phi^2)$. By integrating out the superfields, the superpotentials can be reduced to a single term with dependence on the Chern-Simons level

$$W = \frac{4\pi}{k}(\tilde{\Phi}_i T_{R_i}^a\Phi_i)(\tilde{\Phi}_j T_{R_j}^b\Phi_j) \tag{2.3.11}$$

Using the same prescription as was done for the $\mathcal{N} = 2$ and going through rather lengthy computations, an expression for the full action can be obtained [114]. This will set us up for how to proceed when considering the construction of the ABJM action. For further details, one can consult [89, 88, 90]

2.3.2 The Conjecture

$\mathcal{N}$=6 superconformal Chern–Simons matter theory in 2+1 dimensions with gauge group U(N) × U(N) and Chern–Simons levels $(k, -k)$, referred to as ABJM theory is dynamically equivalent to

M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$ with N units of R-R four-form flux $F_{(4)}$ through AdS_4. The 't Hooft coupling is given by $\lambda = \frac{N}{k}$ and is related to the AdS_4 radius L and the eleven-dimensional Planck length l_p by

$$\frac{L^3}{l_p^3} = 4\pi\sqrt{2kN} = 4\pi k\sqrt{2\lambda}, \quad g_s \sim \left(\frac{N}{k^5}\right)^{1/4} = \frac{\lambda^{5/4}}{N}, \quad \frac{R^2}{\alpha'} = 4\pi\sqrt{2\lambda} \tag{2.3.12}$$

2.3.3 General aspects of ABJM

As it was teased in M-theory, what pushed the forthcoming of ABJM, was the previous work of BLG an others [15, 14, 16, 13, 71, 5]. Looking at the conjecture, we lay out the general structure and aspects of the theory to reach as many corners of both gravity and gauge theory as possible. Starting as general as possible, both theories are controlled by two parameters, k and N (where N is the number of $M2$-branes), which take integer values. These parameters determine all other quantities like coupling constants and the effective string tension. Here $1/k$ acts like a coupling constant. Considering the 't Hooft limit $N \to \infty$, where the t´Hooft coupling is defined as $\lambda = \frac{N}{k}$ and is kept fixed, gives leading contributions of planar Feynmann diagrams. In the language of λ, this constitutes a continuous coupling. ABJM theory is weakly coupled for $\lambda \ll 1$ giving the emerging pertubative regime of $\mathcal{N} = 6$ Chern-Simons. As the conjecture states, ABJM is dual to M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$ with N units of four-form flux $F_{(4)}$. In the limit of large k one has roughly speaking that $S^7/\mathbb{Z}_k \simeq \mathbb{C}P^3 \times S^1$[12] making ABJM dual to type IIA string theory on $AdS_4 \times \mathbb{C}P^3$. This duality is valid for $\lambda \gg 1$ and the type IIA string description holds when $k^5 \gg N$. With these general remarks, we delve deep and investigate both scenarios which will constitute the $\mathcal{N} = 6$ CS and $AdS_4 \times \mathbb{C}P^3$ gravity.

The Superconformal Group OSp(4|6) and the algebra

Facing ABJM, one might find this duality to contain more peculiarities than SYM due to the lack of supercharges. We start by Considering both global bosonic and fermionic symmetries which can be encapsulated by the Lie superalgebra OSp(4|6)[147, 34]. The bosonic sub-algebra is SO(6) × Sp(4) ≅ SU(4) × SO(3, 2). SU(4) × SO(3, 2). This is nothing but the $3D$ conformal algebra. It has 10 components, 6 of which belong to the Poincaré algebra which contains the Lorentz group so(2, 1) ≅ sl(2, $\mathbb{R}$) with generators $M_{\mu\nu}$. Additionally, there is also the space-time generators P_μ. The remainder is the dilatation operator D and special conformal transformations K_μ. It is a standard exercise to derive the commutation relations, but they are just stated for convenience in terms of spinor indices[34]

$$\begin{aligned} &[K^{\alpha\beta}, P_{\delta\gamma}] = 4\delta^{[\alpha}_{[\gamma}M^{\beta]}_{\delta]} + 4\delta^{\alpha}_{[\gamma}\delta^{\beta}_{\delta]}D \quad [D, P_{\alpha\beta}] = P_{\alpha\beta} \quad [D, K_{\alpha\beta}] = -K_{\alpha\beta} \quad [D, M_{\alpha\beta}] = 0 \\ &[M^{\beta}_{\alpha}, M^{\gamma}_{\delta}] = \delta^{[\beta}_{[\gamma}M^{\delta]}_{\alpha]}, \quad [M_{\alpha\beta}, P_{\gamma\delta}] = \delta^{\beta}_{[\delta}P_{\gamma]\alpha} - \delta^{\beta}_{\alpha}P_{\gamma\delta} \quad [M_{\alpha\beta}, K_{\gamma\delta}] = \delta^{\beta}_{[\delta}K_{\gamma]\alpha} - \delta^{\beta}_{\alpha}K_{\gamma\delta} \end{aligned} \tag{2.3.13}$$

Moving on, we have the $SU(4)$ part of the algebra which contains the R-symmetry generators. They are denoted as R^I_J, where $I, J = \{1, 2, 3, 4\}$ and $R^I_I = 0$. Thus the commutators of the R-symmetry generators can be written as

$$[R_{IJ}, R_{KL}] = i(\delta_{I[L}R_{K]J} - \delta_{J[L}R_{K]I}) \tag{2.3.14}$$

Lastly, we have $2\mathcal{N}$[13] supercharges or fermionic generators $Q_{\alpha I}$ and $S_{\alpha I}$ whose conformal dimension is $\{1/2, -1/2\}$. This can be seen from the commutator with the dilatation operator

$$[D, Q_{\alpha I}] = \frac{1}{2}Q_{\alpha I}, \quad [D, S_{\alpha I}] = -\frac{1}{2}S_{\alpha I} \tag{2.3.15}$$

[12] This is accomplished by hopf fibrating along a $U(1)$ fiber bundle

[13] For $\mathcal{N} = 6$ we have 24 supercharges indicating that supersymmetry has been partially reduced or broken from the maximal case of 32

Since they have fermionic nature, they obey anti-commutation relations and generate all the bosonic operators as well

$$\begin{aligned} \{Q_{\alpha I}, Q_{\beta J}\} = 2\delta_{IJ} P_{\alpha\beta} \quad \{S_{\alpha I}, S_{\beta J}\} = -2\delta_{IJ} K_{\alpha\beta} \\ \{Q_{\alpha I}, S_{\beta J}\} = 2i(M_{\alpha\beta} - \epsilon_{\alpha\beta} D) - 2\epsilon_{\alpha\beta} R_{IJ} \end{aligned} \tag{2.3.16}$$

Lastly, the only remaining commutators are the Poincare generators and the supercharges. They will give us

$$\begin{aligned} [M_{\alpha}^{\beta}, Q_{\gamma I}] = \delta_{\gamma}^{\beta}, Q_{\alpha I} - \frac{1}{2}\delta_{\alpha}^{\beta}, Q_{\gamma I} \quad [K^{\alpha\beta}, Q_{\gamma I}] = -i\delta_{\gamma}^{[\alpha} S_{I}^{\beta]} \\ [M_{\alpha}^{\beta}, S_{I}^{\gamma}] = -\delta_{\alpha}^{\gamma} S_{\beta}^{I} + \frac{1}{2}\delta_{\alpha}^{\beta} S_{I}^{\gamma} \quad [P_{\alpha\beta}, S_{I}^{\gamma}] = -i\delta_{[\alpha}^{\gamma} S_{\beta]I} \end{aligned} \tag{2.3.17}$$

A nice way of categorizing the operators of the superconformal algebra, is by grouping them in terms of the dimension of the corresponding operators. This can be put in a form of a diagram

$$\begin{array}{cc} -1 & K_{\alpha\beta} \\ -\frac{1}{2} & S_{\alpha}^{I} \\ 0 & M_{\alpha\beta} \quad \Delta \quad R_{IJ} \\ \frac{1}{2} & Q_{\alpha}^{I} \\ 1 & P_{\alpha\beta} \end{array} \tag{2.3.18}$$

The left column represents how each operator raises or lowers the conformal dimension. Using the oscillator picture, one can find relations between operators through the hermitian conjugates

$$\begin{aligned} (P_{\alpha\beta})^{\dagger} = K_{\alpha\beta}, \quad (K_{\alpha\beta})^{\dagger} = P_{\alpha\beta}, \quad (M_{\alpha\beta})^{\dagger} = M_{\alpha\beta}, \quad D^{\dagger} = D \\ (Q_{\alpha}^{I})^{\dagger} = -iS_{\alpha}^{I}, \quad (S_{\alpha}^{I})^{\dagger} = -iQ_{\alpha}^{I} m \quad (R_{IJ})^{\dagger} = R_{IJ} \end{aligned} \tag{2.3.19}$$

This means that if a superconformal primary state is considered, then $\{K_{\alpha\beta}, S_{\alpha}^{I}\}$ will annihilate the state and $\{P_{\alpha\beta}, Q_{\alpha}^{I}\}$ will raise the conformal dimensions of the primary to make *descendants*. Finding unitary irreducible representations (irrep.), one faces the problem of $OSp(\mathcal{N}|4)$ being non-compact, hence the irreps are infinite-dimensional, but this obstacle can be bypassed. Our final goal is to somewhat classify a general notion BPS-operators in $OSp(6|4)$[14]. This means that there exist unitary multiplets that will obey certain inequalities specific to the superconformal algebra[47, 42]. Each multiplet can be described by the Cartan generators given by the vector $\{\Delta, s, r, q, p\}$. In total one finds three types of multiplets which are defined below

$$\begin{aligned} &\Delta > s + r + \frac{1}{2}(q+p) + 1 \quad - \quad \text{Long Multiplet} \\ &\Delta = s + r + \frac{1}{2}(q+p) + 1 \quad - \quad \text{Semishort Multiplet (A-Type)} \\ &\Delta = r + \frac{1}{2}(q+p) \quad - \quad \text{Short Multiplet (B-Type)} \end{aligned} \tag{2.3.20}$$

We see that depending on the type of multiplet and the choice of spin and $SU(4)$ Dynkin labels, different BPS operators can be obtained. We list the possible choices and their BPS fractions in the table below[34]

[14] A finite dimensional subspace of states V_B called the $(OSp(\mathcal{N}|4), B)$ module (Harish-Chandra module), consisting of B-finite vectors with respect to a maximal compact subgroup$B \subset OSp(\mathcal{N}|4)$, is enough to recover the entire original state space V

Type	Δ	Spin	Multiplet	$SU(4)_R$	BPS
Long	$> \Delta_B + s + 1$	s	Long	$[r, q, p]$	0
A	$\Delta_B + s + 1$	s	(A,1)	$[r, q, p]$	1/12
			(A,2)	$[0, q, p]$	1/6
			(A,+)	$[0, q, 0]$	1/4
			(A,-)	$[0, 0, p]$	1/4
			(a,const.)	$[0, 0, 0]$	1/3
B	Δ_B	0	(B,1)	$[r, q, p]$	1/6
			(B,2)	$[0, q, p]$	1/3
			(B,+)	$[0, q, 0]$	1/2
			(B,-)	$[0, 0, p]$	1/2
			Trivial	$[0, 0, 0]$	1

Table: Multiplets of $\mathfrak{osp}(6|4)$ and the quantum numbers of their superconformal primary, where $\Delta_B = r + \frac{1}{2}(p + q)$.

Field and matter content

To follow up where we left for the $\mathcal{N} = 3$ case, we briefly state how things change. From the brane-construction of ABJM, it is prescribed that CS theories arise naturally in the low-energy limit[15]. Requiring parity invariance, one gets that the action must be of a product gauge group[5, 16]. In terms of CS theories, this corresponds to different levels for each gauge group $(k, -k)$ of the moduli space[16] $U(N)_k \times U(N)_{-k}$. Looking at the matter content, it can be recognized that it has the structure of a quiver gauge theory with two nodes corresponding to the product gauge groups, and arrows indicating bifundamental fields. The field contents consist of gauge

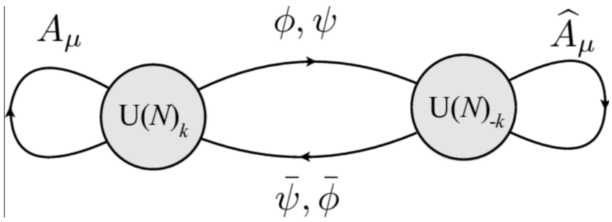


Figure 2.1: Quiver Gauge diagram for ABJM theory containing Gauge fields $\{A_\mu, \hat{A}_\mu\}$ on the nodes and matter fields along the arrows between nodes $\{Y^a, Y_a^\dagger\}$

fields A_μ and $\hat{A}_\mu$ transforming in the adjoint representation of the groups $U(N)_k$ and $U(N)_{-k}$ respectively. Then we have four complex scalars accompanied by the same amount of Weyl fermions given by $Y^A, \psi_A, A \in \{1, 2, 3, 4\}$. Firstly, one identifies the $N \times \bar{N}$ chiral superfields containing the complex scalars A_1, A_2. Same goes for $\bar{N} \times N$ with B_1, B_2. They can be grouped into supermultiplets of the $SU(4)$ R-symmetry which transform in the $\mathbf{4}$ and $\bar{\mathbf{4}}$ of $SU(4)$[17]

$$Y^a = (A_1, A_2, B_1^\dagger, B_2^\dagger) \quad Y_a^\dagger = (A_1^\dagger, A_2^\dagger, B_1, B_2) \tag{2.3.21}$$

The fermions are superpartners of the scalars so they transform in the fundamental and anti-fundamental of $SU(4)$ as well. The difference lies in the fact that scalars have conformal dimensions $\Delta = 1/2$ and transform in the trivial representation of $SO(3)$[18]. On the other hand,

[15]To be more precise, the brane construction of Type IIB string theory can have at most $\mathcal{N} = 3$ supersymmetry, but when generalized to $U(N) \times U(N)$ gauge group with CS terms at levels $(k, -k)$ and matter in the bi-fundamental rep. the theory flows in the IR to $\mathcal{N} = 6$ CS

[16]The discussion is more involved but can be extended to $SU(N)_k \times SU(N)_{-k}$ as well

[17]Representations and dimensions of scalars and operators can be found in Appendix I

[18]The $SO(3)$ isometry is manifesting on the geometry side through $S^2 \subset AdS_4$

fermions have conformal dimension $\Delta = 1$ and transform in spin 1/2 representation of $SO(3)$. As known from Klebanov-Witten theory[96] with the same structure, a superpotential must be added to maintain the $\mathcal{N} = 6$ symmetry.

$$W = \frac{2\pi}{k}\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\mathrm{Tr}(A_a B_{\dot{a}} A_b B_{\dot{b}}) \tag{2.3.22}$$

which exhibits $SU(2) \times SU(2)$ symmetry separately acting on the A´s and B´s. Lastly, we have in addition a covariant derivative D_μ transforming in the spin 1 representation of $SO(3)$ and in the trivial representation of $SU(4)$. with conformal dimension $\Delta = 1$. We write the three components as D, D_0 and D_+ according to the Cartan generator S of SO(3) with eigenvalues $(1, 0, 1)$ This should sum up the most important notions of the matter and field content.

The Action and additional components

Obtaining the action for ABJM was consequently done in the aftermath of its creation. Using the BLG construction in van Raamsonk´s product gauge formalism [142, 28] it was shown that breaking the global $SO(8)$ symmetry to $SU(4)_R \times U(1)_R$ precisely achieved what was conjectured to be the case for ABJM. The action can be separated into three main components accounting for the full expression. Writing it as $S = S_{CS} + S_{mat} + S_{pot}$, we will have a chern-simons contribution, matter part and potential part, being the main sources of ABJM. Spilling them out separately gives

$$S_{CS} = -iK\int d^3x d^4\theta \int_0^1 dt \mathrm{Tr}\big[\mathcal{V}\bar{D}^\alpha(e^{t\mathcal{V}}D_\alpha e^{-t\mathcal{V}}) - \hat{\mathcal{V}}\bar{D}^\alpha(e^{t\hat{\mathcal{V}}}D_\alpha e^{-t\hat{\mathcal{V}}})\big] \tag{2.3.23}$$

$$S_{mat} = -\int d^3x d^4\theta \mathrm{Tr}\Big[-\bar{\mathcal{Z}}_A e^{-\mathcal{V}}\mathcal{Z}^A e^{\hat{\mathcal{V}}} - \bar{\mathcal{W}}^A e^{-\hat{\mathcal{V}}}\mathcal{W}_A e^{\mathcal{V}}\Big] \tag{2.3.24}$$

$$S_{pot} = L\int d^3x d^2\theta \mathrm{W}(\mathcal{Z}) + L\int d^3x d^2\bar{\theta}\bar{\mathrm{W}}(\bar{\mathcal{Z}}) \tag{2.3.25}$$

It is worth mentioning the promoting features going from BLG to ABJM. First of all, the CS action is unchanged, but the matter and potential action acquires additional terms. In BLG, only a single chiral superfield is present whereas in the product gauge group languages, an extra anti-chiral superfield $\mathcal{W}$ is present. The difficulty in the transition was giving up $SU(4)$ invariance in the superpotential, by splitting the BLG scalars into complex combinations of bifundamental fields

$$Z^1 = X^1 + iX^5, \quad Z^2 = X^2 + iX^6, \quad W_1 = X^{3\dagger} + iX^{7\dagger}, \quad W^1 = X^{4\dagger} + iX^{8\dagger} \tag{2.3.26}$$

In the matter sector, one gets contribution from both $\mathcal{Z}$ and $\mathcal{W}$ as the promoting feature then. For the superpotentials, there seems only to be a $SU(2) \times SU(2) \times U(1)$ global symmetry. This can be enhanced however by relating the normalization factors as $K = \frac{1}{L}$ to $SU(4)_R$. It can be shown by combining the scalars as in 2.3.21, the R-symmetry gains the symmetry enhancement. Since the isomorphism $SU(4)_r \simeq SO(6)_R$ is viable, the 2 + 1 dimensional theory has a $\mathcal{N} = 6$ supersymmetry[19]. This will then be the building blocks for the full action. It is quite long and hairy to get to the final expression writing out all the components of the superfields and so forth. Nevertheless, in its full glory, it can compactly be written as

$$\begin{aligned} S = \frac{k}{4\pi}\int d^3x\Big[&\epsilon^{\mu\nu\lambda}\mathrm{Tr}(A_\mu\partial_\nu A_\lambda + \frac{2i}{3}A_\mu A_\nu A_\lambda - \hat{A}_\mu\partial_\nu\hat{A}_\lambda - \frac{2i}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\lambda \\ &- \mathrm{Tr}(D_\mu Y^\dagger)D^\mu Y - i\mathrm{Tr}(\psi^\dagger\gamma^\mu D_\mu\psi) - V_{\text{ferm}} - V_{\text{bos}}\Big] \end{aligned} \tag{2.3.27}$$

[19]Interestingly enough, at levels $k = 1$ supersymmetry is revived to $\mathcal{N} = 8$ for $M2$-branes in $\mathbb{R}^8$, while for $k = 2$ the space becomes $\mathbb{R}^8/\mathbb{Z}_2$

The fermionic and bosonic potentials are expressions containing combinations of scalar and fermionic fields coupled to each other. They as well have the manifest $SU(4)$ invariance. Writing out the sextic bosonic and quartic mixed potentials one obtains

$$\begin{aligned} V_{\text{bos}} = -\frac{1}{12}\text{Tr}(&Y^A Y_A^\dagger Y^B Y_B^\dagger Y^C Y_C^\dagger + Y_A^\dagger Y^A Y_B^\dagger Y^B Y_C^\dagger Y^C \\ &+ 4Y^A Y_B^\dagger Y^C Y_A^\dagger Y^B Y_C^\dagger - 6Y^A Y_B^\dagger Y^B Y_A^\dagger Y^C Y_C^\dagger) \end{aligned} \tag{2.3.28}$$

$$\begin{aligned} V_{\text{ferm}} = -\frac{i}{2}\text{Tr}(&Y_A^\dagger Y^A \psi^{\dagger B}\psi_B + Y^A Y_A^\dagger \psi_B \psi^{\dagger B} + 2Y^A Y_B^\dagger \psi_A \psi^{\dagger B} - 2Y_A^\dagger Y^B \psi^{\dagger A}\psi_B \\ &- \epsilon^{ABCD} Y_A^\dagger \psi_B Y_C^\dagger \psi_D + \epsilon^{ABCD} Y^A \psi^{\dagger B} Y^C \psi^{\dagger D}) \end{aligned} \tag{2.3.29}$$

It has been mentioned many times by now, but the construction of the $U(N) \times U(N)$ CS action accomplishes describing N coincident $M2$-branes at the $\mathbb{Z}_k$ orbifold of $\mathbb{C}^4$ which further solidifies the conjecture. The whole story can be accompanied by the brane configuration, which is explained in [16, 5, 2]. The next step is to summarize the emergence of $\mathbb{C}P^3$ geometry accompanied by type IIA strings in the $k^5 \gg N$ limit

Geometry of $AdS_4 \times S^7/\mathbb{Z}_k$

Going from the gauge theory side, the focus will be on gravity now. As it was seen through M-theory and BLG, a stack of $M2$-branes on a $\mathbb{Z}_k$ orbifold of $\mathbb{C}^4$, give in a near horizon limit

$$ds^2 = R^2(\frac{1}{4}ds^2_{AdS_4} + ds^2_{S^7/\mathbb{Z}_k}), \quad R^2 = \sqrt{\frac{32\pi^2 N}{k}} = 4\pi\sqrt{2\lambda} \tag{2.3.30}$$

Let us first remark how the orbifolding or quotient acts. If the S^7 is defined as four complex coordinates z_i which satisfy the condition $\sum_{i=1}^4 |X_i|^2 = 1$, then Orbifolding is implemented[20] as R-symmetry rotations for the scalars

$$z_i \to e^{2\pi i/k} z_i \tag{2.3.31}$$

Using the quotient $\mathbb{Z}_k$[21] on S^7 it can be interpreted as a S^1 hopf fibration over $\mathbb{CP}^3$. The circle has a constant radius, where the orbifolding then decreases it. This means that the $ds^2_{S^7}$ metric can be written as [5, 62]

$$ds^2_{S^7} = (d\gamma' + \omega)^2 + ds^2_{CP^3}, \quad ds^2_{CP^3} = \frac{\sum_i dX_i d\bar{X}_i}{\rho^2} + \frac{\sum_i |X_i d\bar{X}_i|^2}{\rho^2}, \quad \rho^2 \equiv \sum_{i=1}^4 |X_i|^2 \tag{2.3.32}$$

ω is related, if considered as a one-form dω, to the Kahler form on $\mathbb{C}P^3$ and γ' is periodic with period 2π. The action of $\mathbb{Z}_k$ changes the periodicity of S^1 from $2\pi L \to \frac{2\pi}{k}L$. The regime of supergravity is valid in the $N \to \infty$ so the radius of S^1 in M-theory is $\frac{L}{k}$. In the $Nk \gg 1$ limit, the $\mathbb{C}P^3$ radius becomes large. On the other hand, the radius of $d\gamma$ becomes very small when k increases, so the regime becomes $k^5 \gg N$[22] and the theory essentially reduces to a

[20] For a more elegant geometric implementation, one defines $z_i = \mu_i e^{i\phi_i}$ where each angle generates an isometry in terms of angular momenta $J_i = -i\partial_{\phi_i}$, which can be related to $\{R_1, R_2, R_3\}$ with additional features [81]

[21] Another way orbifolding can be implemented is using 2.3.31 on spinors which transform as $\psi \to e^{2\pi i(s_1+s_2+s_3+s_4)/k}\psi$, with $s_i = \pm 1/2$. Demanding the chirality sum to be even $\sum_{i=1}^4 s_i \pmod{k}$, only six spinors are left invariant and break $\mathcal{N} = 8$ to $\mathcal{N} = 6$ (for $k = 1, 2$, $\mathcal{N} = 8$ still holds)

[22] The radius of γ in Planck units is of the order $\frac{R}{kl_p} \propto (Nk)^{\frac{1}{6}}/k$, so M-theory is valid when $k^5 \ll N$

ten-dimensional background describing Type IIA string theory (supergravity) on $AdS^4 \times \mathbb{C}P^3$ with the string frame metric, dilaton and field strengths

$$ds^2 = R^2(\frac{1}{4}ds^2_{AdS_4} + ds^2_{CP^3}), \quad e^{2\phi} = \frac{1}{N^2}\left(\frac{N}{k}\right)^{5/2}, \quad F_4 = \frac{3}{8}R^3\hat{\epsilon}_4, \quad F_{(2)} = kd\gamma = kJ \quad (2.3.33)$$

The compactification to type IIA super string theory has N units of $F_{(4)}$ flux on AdS_4 and k units of $F_{(2)}$ on $\mathbb{C}P^1 \subset \mathbb{C}P^3$. Furthermore, $\hat{\epsilon}_4$ is the unit radius of AdS_4. The radius of curvature for the string can also be expressed in terms of the t´Hooft coupling $R^3 = 2^{5/2}\pi k\sqrt{\lambda}$. This turns out to have the same scaling for λ as for $\mathcal{N} = 4$ SYM. With the general picture established, it would be convenient to express the metric for both AdS_4 and $\mathbb{CP}^3$ in terms of angular coordinates (this will prove handy later).

AdS_4 **metric and parametrization**:

Having established the general notion of the metric, it becomes straightforward to parametrize the coordinates. Start by writing the components as

$$\begin{aligned} &Z_0 = R\cosh\rho\cos t, \quad Z_1 = R\cosh\rho\sin t \\ &Z_2 = R\sinh\rho\cos\theta, \quad Z_3 = R\sinh\rho\sin\theta\cos\phi \\ &Z_4 = R\sinh\rho\sin\theta\sin\phi \end{aligned} \quad (2.3.34)$$

Inserting each component and summing over it in $ds^2 = \sum_{i=0}^{4} |dz_i|^2$, will add the needed contributions to obtain the metric

$$ds^2_{AdS_4} = -\cosh\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_2^2, \quad d\Omega_2^2 = \frac{1}{4}(d\theta^2 + \sin\theta d\phi^2) \quad (2.3.35)$$

Instead of S^3, the geometry has a S^2 or rather a S^1 hopf fibration over $\mathbf{CP}^1$. The AdS part of the metric is in this context not changing, whereas the expression for $\mathbb{C}P^3$ space can vary depending on the physical system considered. But this makes us able to write the near-horizon limit as

$$\begin{aligned} ds^2 &= R^2(\frac{1}{4}ds^2_{AdS_4} + ds^2_{S^7/\mathbb{Z}_k}) \\ &= \frac{R^2}{4}(-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_2^2) + R^2 ds^2_{S^7/\mathbb{Z}_k} \end{aligned} \quad (2.3.36)$$

$\mathbb{C}P^3$ **metric and parametrization in Homogenous coordinates**:

There is a freedom of choice when it comes to picking the $\mathbb{CP}^3$. Describing it can be done by a coset space using the hopf fibration $S^7 \xrightarrow{U(1)} \mathbb{C}P^3$ such that $\mathbb{C}P^3 = \frac{S^7}{U(1)}$. Similarly this can be described by another coset $\mathbb{C}P^3 = \frac{SU(4)}{U(3)}$. For the first parametrization of the metric, it can be found using[120, 45]

$$\begin{aligned} X_1 = \cos\xi\cos\frac{\theta_1}{2}e^{i\alpha_1} \quad X_2 = \cos\xi\sin\frac{\theta_1}{2}e^{i\alpha_2} \\ X_3 = \sin\xi\cos\frac{\theta_2}{2}e^{i\alpha_3} \quad X_4 = \sin\xi\sin\frac{\theta_2}{2}e^{i\alpha_4} \end{aligned} \quad (2.3.37)$$

Using the formula for generating the $\mathbb{C}P^3$ metric and inserting the elements will amount to

$$\begin{aligned} ds^2_{\mathbb{C}P^3} = d\xi^2 &+ \frac{1}{4}\cos^2\xi(d\theta_1^2 + \sin^2\theta_1 d\alpha_1^2 + \cos^2\theta_1 d\alpha_2^2) \\ &+ \frac{1}{4}\sin^2\xi(d\theta_2^2 + \sin^2\theta_2 d\alpha_3^2 + \cos^2\theta_2 d\alpha_4^2) \end{aligned} \quad (2.3.38)$$

This expression has its benefits for upcoming interpretation and can be used when considering BPS-bounds for string backgrounds.

$\mathbb{C}P^3$ **metric and parametrization in Inhomogenous coordinates**: Lastly, using inhomogenous coordinates, the amount of components needed to describe the metric is reduced by considering ratios X_i/X_4. The coordinates are

$$\begin{aligned} X_1/X_4 &= \tan\xi\cos\alpha e^{i\alpha_1} \\ X_2/X_4 &= \tan\xi\sin\alpha\sin(\theta/2)e^{i\alpha_2} \\ X_3/X_4 &= \tan\xi\cos\alpha\cos(\theta/2)e^{i\alpha_3} \end{aligned} \tag{2.3.39}$$

For now, the full metric does not need to be explicitly written, but it will be used when considering transformations of angles in computations later on.

Isometries Having assembled all the parts, one can deduce that $AdS_4 \times \mathbb{C}P^3$ admits five different Killing vectors for this background

$$\Delta = -i\partial_t, \quad S = -i\partial_\phi, \quad J_1 = -i\partial_{\phi_1}, \quad J_2 = -i\partial_{\phi_2}, \quad J_3 = -i\partial_\chi \tag{2.3.40}$$

The last charge will only be relevant to the last mentioned parametrization of $\mathbb{C}P^3$, so we put χ as the angle associated too the charge. But these are the generators that make up the Cartan subalgebra of the isometries. The condition for S^7 and $\mathbb{C}P^3$ are both met by $SO(3,2)$ and $SU(4)$ which as we already know was the R-symmetry and 3-dimensional conformal symmetry. The charge Δ gives the space-time energy of a string state[23].

2.3.4 Subsectors and Decoupling Limits for $\mathcal{N} = 6$ Chern-Simons

Consider the letters in $\mathcal{N} = 6$ Chern-Simons. The procedure is the same, but the weights and Dynkin labels change. In the literature, different weights can be used, but they will amount to the same physics, even though there might be a difference in the values of coefficients on the chemical potentials. Using data from [97], tables can be arranged as follow

	D_-	D_0	D_+
SO(3)	-1	0	1
SU(4)	(0,0,0)	(0,0,0)	(0,0,0)

Table 2.3: Weight of derivative operators in SO(3) and SU(4) representation

	Y_1 $\psi^\dagger_{1\pm}$	Y_2,$\psi^\dagger_{2\pm}$	Y_3,$\psi^\dagger_{3\pm}$	Y_4,$\psi^\dagger_{4\pm}$
SU(4)	(1,0,0)	(-1,1,0)	(0,1,-1)	(0,0,-1)

Table 2.4: Weight of Scalars in bi-fundamental representation

	$Y^\dagger_1$,$\psi_{1\pm}$	$Y^\dagger_2$,$\psi_{2\pm}$	$Y^\dagger_3$,$\psi_{3\pm}$	$\psi_{4\pm}$,$Y^\dagger_4$
SU(4)	(-1,0,0)	(1,-1,0)	(0,1,-1)	(0,0,1)

Table 2.5: Weight of Scalars in Anti Bi-Fundamental representation

[23]Or the dimension of the corresponding operator on the gauge side

	Y^A, $Y_A^\dagger$	$\psi_{A\pm}$, $\psi^{\dagger A\pm}$
$SO(3)$	0	$\pm 1/2$

Table 2.6: $SO(3)$ weights of scalars and fermions.

Interestingly enough, depending on the use of $SO(6)$ or $SU(4)$, the weights will take a completely different look. But using the following relation $[r, q, p] \rightarrow [q+r, r-q, q-p]$[24], which in general is related to the Gelfand-Tsetlin Map[47], changing between weights in the two representations becomes possible and equivalent due to the isomorphism between $SO(6) \cong SU(4)$. The supergroup $OSp(4|2)$ will be the main frame, that is the maximal subgroup of the full $OSp(6|4)$ considered. From the geometry side of ABJM, a thorough analysis was done

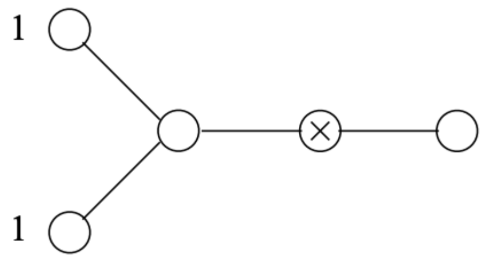


Figure 2.2: A specific choice for the $OSp(6|4)$ Dynkin diagram, related to the $SU(4)$ spin chain in [110]

relating angular momentum generators to R-symmetry generators of $SU(4)$ through orbifolding of S^7. The inequality for operators takes the form[81]

$$\Delta_0 \geq m_1 R_1 + m_2 R_2 + m_3 R_3 + m_4 S \tag{2.3.41}$$

The Cartan generators manifest themselves as R_i for $SU(4)$ while S is the Cartan generator for $SO(3)$. This can alternatively be written in the language of the angular momenta

$$\Delta_0 \geq n_1 J_1 + n_2 J_2 + n_3 J_3 + n_4 J_4 + n_5 S \tag{2.3.42}$$

But this comes with the restriction $\sum_{i=1}^{4} J_i = 0$. Using the same prescription as for SYM, it becomes a matter of systematically obtaining the subsectors. The problem can be solved in two ways. Either constructing a matrix with Dynkin labels as columns and weights as rows and then diagonalizing the matrix gives you the BPS-vector $(m_1R_1, m_2R_2, m_3R_3, m_4S)$ that can be used to determine the letter content and spin group. Otherwise, reverse engineering by putting restrictions on coefficients and generators will also work. An example is determining all sectors with derivatives, which immediately gives $S = \pm 1$. We summarize the result in the table below, where BPS-vector, Spin group and letter content are presented

One can look at the type of operators appearing for different cases in the subsectors[25]. An example is the $SU(2) \times SU(2)$ sector which has been studied extensively. In later sections, we review calculations that will show features chosen for the $SU(2) \times SU(2)$ sector involving finding

[24]The $SO(6)$ representation has HWS $[1/2, 1/2, 1/2] \rightarrow [1, 0, 0]$

[25]A graphical way of representing the full BPS-landscape is using Dynkin diagrams, where blank dots are bosonic roots, and those with a cross are fermionic. One can separate the root system so it admits the subsectors obtained in this section. For supergroups, it is not unique how one constructs such diagrams, as it was considered in [110]. Depending on the choice of root- system, this changes the diagram and corresponding Bethe equations as well

BPS-vector $(m_1R_1, m_2R_2, m_3R_3, m_4S)$	G_s	**Letter Content**
$(1/2, m_2R_2, 1/2, 0)$	Vacuum	$Y_1, Y_4^\dagger$
$(1/2, 0, 1/2, 0)$	$SU(2)\times SU(2)$	$Y_1, Y_4^\dagger, Y_2, Y_3^\dagger$
$(1/2, 1, 1/2, 0)$	$SU(2)$	$Y_1, Y_4^\dagger, Y_2$
$(1/2, 0, 1/2, 1/2)$	$SU(1,1)$	$Y_1, Y_4^\dagger, \psi_{4+}$
$(1/2, 1, 1/2, 1)$	$SU(2\vert 1)$	$Y_1, Y_4^\dagger, Y_2, \psi_{4+}$
$(1/2, 1/2, 1/2, 1/2)$	$OSp(4\vert 2)$	$Y_1, Y_4^\dagger, Y_2, Y_3^\dagger, \psi_{4+}, \psi_+^{\dagger 1}, \psi_{3+}, \psi_+^{\dagger 2}, D_+$
$(1/2, 0, 1, 1/2)$	$OSp(2\vert 2)$	$Y_1, Y_4^\dagger, \psi_{4+}, \psi_+^{\dagger 1}, D_+$
$(m_1R_1, m_2R_2, m_3R_3, m_4S)$?	$SU(3\vert 2)$	$Y_1, Y_4^\dagger, Y_2, Y_3^\dagger, \psi_{4+}, \psi_+^{\dagger 1}$

Table 2.7: Subsectors of the full $OSp(6|4)$ group, with letter content, spin group and bps vector

a spectrum for operators on the string side. On the other hand, more extensive use will be done with upcoming sectors when considering non-relativistic string backgrounds arising from Spin Matrix theory in the last section. In the spirit of [81], we end the section by writing the operator structure for specific and clean BPS-sectors[26]

$$\text{SU(3) BPS-Sector} : \Delta \geq R_1 + R_2 + R_3 \rightarrow \quad \mathcal{O} = \mathrm{Tr}(Y^{A_1}B_2 \dots Z^{A_L}B_2)\chi_{A_1A_2\dots A_L},\ A_j = 1,2,3$$

$$\text{OSp(2|2) BPS-Sector} : \Delta \geq R_1 + R_2 + S \rightarrow \quad \mathcal{O} = \mathrm{Tr}(Y^{A_1}Y^\dagger_{B_1}Y^{A_2}Y^\dagger_{B_2}\dots Y^{A_L}Y^\dagger_{B_L})\chi^{B_1\dots B_L}_{A_1\dots A_L}$$

$$\text{OSp(4|2) BPS-Sector} : \Delta \geq R_1 + R_2 + R_3 + S \rightarrow \quad \mathcal{O} = \mathrm{Tr}(Y^{A_1}Y^\dagger_{B_1}Y^{A_2}Y^\dagger_{B_2}\dots Y^{A_L}Y^\dagger_{B_L})\chi^{B_1\dots B_L}_{A_1\dots A_L} \tag{2.3.43}$$

It looks strange why the two last BPS-sectors have the same looking operators, but we can distinguish them by writing up their respective modules[27]

$$\begin{aligned} &\text{OSp(2|2) BPS-Sector} : \mathcal{V} = \{D_+^n A_{1,2}, D_+^n\psi_{A_{1,2}}\}, \quad \bar{\mathcal{V}} = \{D_+^n B_{1,2}, D_+^n\psi_{B_{1,2}}\} \\ &\text{OSp(4|2) BPS-Sector} : \mathcal{V} = \{D_+^n(A_{1,2}, B_1^\dagger), D_+^n\psi_{(A_{1,2},B_1^\dagger)}\}, \quad \bar{\mathcal{V}} = \{D_+^n B_2, D_+^n\psi_{B_2}\} \end{aligned} \tag{2.3.44}$$

The barred modules live on the even sites, while unbarred on the odd. All superpartners to the scalars have $S = 1/2$ as well. This was a taste of how the structure of operators looks when restricting to certain charges in the BPS language. Later chapters will utilize this to a great extent.

[26]In the table above, it seems impossible to determine the BPS-vector due to solving the set of linear equations $a = 1/2, -a + b = 1/2, -b + c = 1/2$, which ambiguously gives both $b = 1$ and $b = 0$. Either conventions are not followed properly, or weights are off. Maybe having used $SO(6)$ would have given more aesthetic BPS-vectors

[27]This is explained in section 3.1 about spin-chains

Chapter 3

Peculiarities of AdS/CFT: The Emergence of Spin Chains and pp-waves

3.1 The Planar Limit: Here Comes The Spin Chain and Integrability

A surprising result that came in the early 2000´s presented a peculiar side of AdS/CFT. Taking the planar limit when $N \to \infty$[1], it was found that one can get a spin chain description by interpreting the gauge invariant operators consisting of scalars and other letters as spin particles. We set out to show how one might construct these types of operators which may differ depending on the particular duality (for a detailed and brilliant reviews see [125, 126, 138]).

3.1.1 From Spin-Chains and operators to $\mathcal{N} = 4$ SYM

The first link between Spin chains and operators in $\mathcal{N} = 4$ SYM was found by Minahan and Zarembo[111]. Considering operators of the form

$$\mathcal{O}[\psi] = \psi^{i_1,\ldots,i_L}\mathrm{Tr}(\phi_{i_1}\ldots\phi_{i_1}) \tag{3.1.1}$$

The holographic dictionary states that ϕ_{i_j} are scalars in the bulk of AdS. Many operators, most notably chiral primary and BMN operators, belong to this class. Knowing the letter content, the six scalars are defined as three complex ones namely $\{Z, W, X\}$ along with their conjugates[2]. Knowing that correlators between such operators give conformal n-point functions. As it is known, the dilatation operator (D) makes the two-point function depend on Δ, which can receive corrections from higher loop orders in D. In pertubation theory, the dilatation operator can be expanded in powers of the coupling constant[22, 24]

$$D = \sum_{k=0}^{\infty} \left(\frac{g_{ym}^2}{16\pi^2}\right) D_{2k} \tag{3.1.2}$$

The sum denotes for k-loop order. As it was found, the one-loop renormalization can be calculated by considering the two-point correlation function between in and out states with the bosonic part of $\mathcal{N} = 4$ SYM as the interaction part[111, 112]. Renormalized operators in general

[1] The t´Hooft coupling for SYM is defined through the coupling constant in the theory $\lambda = g_{\mathrm{ym}}^2 N$ where in the planar limit λ is kept fixed

[2] One defines the scalars as $Z = \phi_1 + i\phi_2, W = \phi_3 + i\phi_4, X = \phi_5 + i\phi_6$, where conjugates are with signs flipped

are linear combinations of bare operators considered above. Choosing a particular operator basis $\mathcal{O}^A_{\mathrm{Ren}} = Z^A_B \mathcal{O}^B$, one requires finiteness of the correlation function and gets Z. All renormalization factors depend on the UV cutoff Λ and the t´Hooft coupling in the large-N limit. The renormalization factor determines the matrix of anomalous dimensions by extracting logarithmically divergent pieces from it, giving the expression

$$\Gamma = \frac{dZ}{d\ln\Lambda} Z^{-1} \tag{3.1.3}$$

To calculate this, one shows that contractions between flavor indices of the fields that arise from the quartic terms, lead to defining permutation and trace operators. Once this has been found, it is possible to renormalize all the relevant Feynmann diagrams via dim-reg in $d = 4 - 2\epsilon$. The computation is usually done in momentum space[102] using a Fourier transform, which after evaluating all relevant integrals arising from interaction pieces in the SYM action, will give the anomalous dimension

$$\Gamma = \frac{\lambda}{8\pi^2} \sum_{l=1}^{L} (1 - P_{l,l+1} + \frac{1}{2} K_{l,l+1}) \tag{3.1.4}$$

In all generality, $P_{l,l+1}$ and $K_{l,l+1}$ could contribute to operator mixing, but this is avoided at one loop level. Nevertheless, they are defined as follows $P_{l,l+1}$ is the exchange operator, and as its name implies it exchanges the flavor indices of the l and the $l+1$ sites inside the trace. Its action on the δ-functions appearing when contracting indices is

$$P_{l,l+1} \delta^{J_1}_{I_1} ... \delta^{J_l}_{I_l} \delta^{J_l+1}_{I_l+1} ... \delta^{J_L}_{I_L} = \delta^{J_1}_{I_1} ... \delta^{J_l+1}_{I_l} \delta^{J_l}_{I_l+1} ... \delta^{J_L}_{I_L} \tag{3.1.5}$$

$K_{l,l+1}$ is the trace operator that contracts the flavor indices of neighboring fields. Its action on the δ-functions is

$$K_{l,l+1} \delta^{J_1}_{I_1} ... \delta^{J_l}_{I_l} \delta^{J_l+1}_{I_l+1} ... \delta^{J_L}_{I_L} = \delta^{J_1}_{I_1} ... \delta_{I_l I_{l+1}} \delta^{J_l J_{l+1}} ... \delta^{J_L}_{I_L} \tag{3.1.6}$$

Facing the dilation operator, it commutes with the Lorentz and R-symmetry generators. Since this is true for all λ, all loop orders of D_{2k} are commuting as well. Hence, the Lorentz and R-charges are preserved by the mixing[112]. It appears then that mixing only occurs between operators with the same R-charges, Lorentz charges, and bare dimensions. From this fact, closed subsectors can be found, where the range of operators is restricted to the weights chosen, described by the Dynkin labels. Take for instance the $SU(2)$ sector. It contains Z and W scalars that have weights $(1,0,0;1,0,0)$ and $(1,0,0;0,1,0)$. But here comes the big twist. Looking at D_2 for this sector, there is no contribution from $K_{l,l+1}$ since there are no conjugate fields for Z and W. Thus, the $SU(2)$ sector has the reduced D_2

$$\Gamma_{SU(2)} = \frac{\lambda}{8\pi^2} \sum_{l=1}^{L} (1 - P_{l,l+1}) \tag{3.1.7}$$

Amazingly this can exactly be interpreted as the $XXX_{1/2}$ Heisenberg spin chain Appendix C. We make the identification between spin up and down states with the scalar operators. Looking at vacuum defined as $|0\rangle = |\uparrow^L\rangle = \mathrm{Tr}Z^L$ one is led to identify the scalar Z with a spin up state. Then excitations would be cases where operators $\mathrm{Tr}(ZZW...Z)$, such that W is a spin down state. This idea extends beyond this particular case and can be analyzed while conjugate fields are present as well as fermions etc. This has led to a whole business in AdS/CFT concerning how operators and spin-chains interplay. For concreteness, various operators can be analyzed, but we just mention some classical examples. Consider a chiral primary and the Konishi operator defined as

$$\mathcal{Q}_{nm} = \mathrm{Tr}(\phi_n \phi_m) - \frac{1}{6} \delta_{nm} \mathrm{Tr}(\phi_k \phi_k) \quad \text{and} \quad \mathcal{K} = \mathrm{Tr}(\phi_k \phi_k) \tag{3.1.8}$$

Using an alternate form of D_2 involving normal-ordering of traces of products between fields[22], it can be found that the chiral primary is protected to quantum corrections $D_2\mathcal{Q}_{nm} = 0$ while the Konishi scalars take contributions $D_2\mathcal{K} = 12N\mathcal{K}$. Using the pre-factor of the coupling constant then amount to

$$\Delta_{\mathcal{Q}} = 2 \quad \text{and} \quad \Delta_{\mathcal{K}} = 2 + \frac{3\lambda}{4\pi^2} \tag{3.1.9}$$

This process can be carried on for more complex operators containing either more fields or rather being multi-trace operators. There is also a whole class of operators called BMN, which plays an important role as we will see in the context of Penrose limits and pp-waves. For more details, see C where the $SU(2)$ Heisenberg spin chain and Yang Baxter equations are discussed asserting integrability. An important thing to note is that there is a non-planar case to consider as well which goes as $1/N^2$ compared to the $1/N$ in the planar limit. In the $N \to \infty$, these contributions can be neglected. But this is valid only when $L << N$. If $L \sim N$, then the number of non-planar diagrams exceeds those for planar cases and this regime is broken. The field of non-planarity is also considered extensively in the literature [22, 24, 25] Finally, it is worth noting that Sigma models also appear in the context of spin chains. It was shown [99] defining coherent states, that an action can be obtained in direct correspondence to the action obtained by spinning strings on a submanifold of $AdS_5 \times S^5$. Sigma models will also appear later on related to spin-chains but in the case of $\mathcal{N} = 6$ CS instead.

3.1.2 From operators in $\mathcal{N} = 6$ Chern-Simons to the $SU(4)$ Spin-Chain

When considering operators in $\mathcal{N} = 6$ CS, one finds a significant difference in the composition of operators compared to $\mathcal{N} = 4$ SYM. As a consequence of the gauge theory and representation $U(N) \times U(\bar{N})$, we have alternation for the matter fields on the odd and even sites in the trace operators. This means that the general class of gauge invariant operators are constructed from fields that transform in an alternating fashion in the representations $(\mathbf{N}, \bar{\mathbf{N}})$ and$(\bar{\mathbf{N}}, \mathbf{N})$

$$\mathcal{O} = \mathrm{Tr}(Y^{A_1} Y^\dagger_{B_1} Y^{A_2} Y^\dagger_{B_2} ... Y^{A_L} Y^\dagger_{B_L}) \chi^{B_1...B_L}_{A_1...A_L} \tag{3.1.10}$$

The operator is of even length, and we will see that they correspond to (in the planar limit), spin chains. The matter allowed on even and odd sites are the $4_B + 8_F$ fields $(Y^A, \psi_{A\alpha})$ for sites $2l_i + 1$, and on the even sites, we can have any of the $4_B + 8_F$ fields $(Y^\dagger_A, \psi^{A\dagger}_\alpha)$ for sites $2l_i$. Finally, both derivatives and field strength insertions can be acted onto the fields, but this will not introduce extra sites. We can express this in terms of modules and BPS-operators. Considering the Dynkin labels $[\Delta, S, J_1, J_2, J_3]$, states that are 1/2-BPS[3] are of interest. The highest-weight state can be written for Y^1 as $[J, 0, 2J, 0, 0]$ and for the anti-fundamental $Y^\dagger_4$ as $[J, 0, 0, 0, 2J]$. Choosing $J = 1/2$ exactly matches the weights in [97] giving $\{[1/2, 0, 0, 0, 1], [1/2, 0, 1, 0, 0]\}$. This makes it possible to create modules, by acting with lowering operators, providing the $4_B + 8_F$ matter fields on the odd and even sites

$$\mathcal{V} = \{D^n Y^A, D^n \psi_{A\alpha}\}, \quad \bar{\mathcal{V}} = \{D^n Y^\dagger_A, D^n \psi^{A\dagger}_\alpha\} \tag{3.1.11}$$

The scalars and gauge fields obey transformations corresponding to the product gauge group of $(U, \hat{U}) \in U(N) \times \hat{U}(N)$

$$\begin{aligned} U(N) &: Y^A \to U Y^A \hat{U}^\dagger, \quad A_\mu \to U A_\mu U^\dagger - iU\partial_\mu U^\dagger \\ \hat{U}(N) &: Y^\dagger_A \to \hat{U} Y^\dagger_A U^\dagger, \quad \hat{A}_\mu \to \hat{U} \hat{A}_\mu \hat{U}^\dagger - i\hat{U}\partial_\mu \hat{U}^\dagger \end{aligned} \tag{3.1.12}$$

The bare dimension of the operators is L and one considers it a chiral primary if χ is symmetric in all A_i, B_i indices and all traces are zero. What will be interesting is when this is not the case,

[3] The condition $\Delta = J$ exactly defines a *BPS-operator* as we will see in decoupling limits and subsectors

where operators pick up quantum loop contributions from the anomalous dimension. Going to the dimension of operators, one can In $SU(4)$ language express the dimension of the operators in terms of the generic $SU(4)$ Dynkin labels $[r, q, p]$, where we state the most important for ABJM [4]

$$\begin{aligned} &\text{Dim}[0,0,0] = \mathbf{1} && \text{Trivial representation} \\ &\text{Dim}[1,0,0] = \mathbf{4} && \text{Fundamental representation} \\ &\text{Dim}[0,0,1] = \bar{\mathbf{4}} && \text{Anti-fundamental representation} \end{aligned} \tag{3.1.13}$$

Considering the vacuum, it can be seen that operators of length L can be described as $(\text{Dim}(1,0,0) \otimes \text{Dim}(0,0,1))^{\otimes L} = (\mathbf{4} \otimes \bar{\mathbf{4}})^{\otimes L}$. The usual choice and convention for ground state operators is $|0\rangle = \text{Tr}(Y^1 Y_4^\dagger)^L$, which is considered a 1/3-bps state since $\{Q_\pm^{12}, Q_\pm^{13}\}$ annihilates $(Y^1 Y_4^\dagger)^L$ and is thus a chiral primary. It also has the weights of the considered Dynkin Labels, [97]. Then building more complicated operators is just a matter of changing respectively on odd and even sites scalars or fermions that coincide in the same module and introducing derivatives or field strengths. As it stands, it might not come as a surprise that this description exactly fits spin-

$$\mathbf{tr}\left(Y^1 Y_4^\dagger Y^1 Y_4^\dagger Y^1 Y_4^\dagger Y^1 Y_4^\dagger\right) \quad \leftrightarrow$$

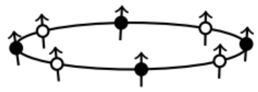


Figure 3.1: The single trace operator of $(Y^1 Y_4^\dagger)^L$ can be interpreted as having spin-up states on all sites, which is the lowest energy-configuration $E = \Delta - J = 0$

$$\mathbf{tr}\left(Y^1 Y_4^\dagger Y^2 Y_4^\dagger Y^1 Y_3^\dagger Y^2 Y_4^\dagger\right) \quad \leftrightarrow$$

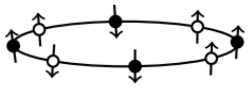


Figure 3.2: Adding impurities such as $\{Y^2, Y_3^\dagger\}$ alters the Energy with $\delta E = 1/2$ for each impurity, such that the vacuum is broken, seen by spin down states

chains in the planar limit[5]. After their original work for SYM in $d = 4$ [111], it was found in the wake of ABJM, spin-chains and integrability should hold for the $d = 3$ CS case as well[110]. Using the vacuum identification found for operators, it so happens that the full ABJM symmetry group OSp(6|4) is broken down to $SU(2|2) \times U(1)$ for the spin chain model[6].

Switching gears, we want to establish integrability and deduce what the anomalous dimension for the system is. The same procedure lies ahead as the case for SYM. But this time, due to the CS action, things turn out to be different. First of all, considering contributions to the anomalous dimensions from Feynmann diagrams, it appears that the lowest loop order contributing to the mixing matrix is at two-loop compared to one-loop in SYM[7]. This peculiar feature stems from a difference in the quartic interaction in the SYM action whereas the bosonic potential has sextic interaction terms (Y^6) in CS. Going through the Feynmann diagrams in dim-reg again, the anomalous dimension is found to be[54, 110]

$$\Gamma = \frac{\lambda^2}{2} \sum_{i=1}^{2L} (2 - 2P_{l,l+2} + P_{l,l+2} K_{l,l+1} + K_{l,l+1} P_{l,l+2}) \tag{3.1.14}$$

Integrability for the spin chain is established with the R-matrix[55]. Defining the spectral parameter u and using the mapping $R_{ab}(u) : V_a \otimes V_b \to V_a \otimes V_b$, gives rise to defining the matrix

[4] see Appendix I for details on $SU(4)$ group

[5] Just to remind, the planar limit or t´Hooft limit is $k, N \to \infty, \lambda = \frac{N}{k}$ Fixed

[6] For further discussion on the sub-groups of $SU(2|2) \times U(1)$ and different kind of exications, see [97, 58]

[7] The four-loop has also been considered as the next order [103, 104]

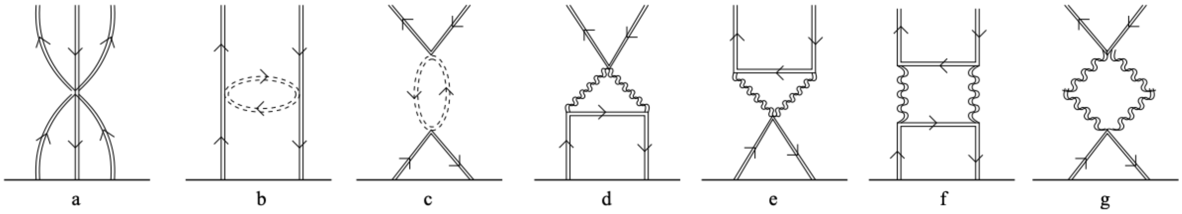


Figure 3.3: Feynmann diagrams in the planar limit contributing to mixing at two-loop for the $SU(4)$ spin chain. Only diagram (a),(b) and (d) affects the anomalous dimension[110]

as $R_{ab}(u) = u - P_{ab}$, such that it satisfies the usual YB equation (section C.2) with additional spectral parameter v.

$$R_{ab}(u-v)R_{ac}(u)R_{bc}(v) = R_{bc}(v)R_{ac}(u)R_{ab}(u-v) \tag{3.1.15}$$

The twist to the story is that for SYM, the vector space considered did not have a $U(N)_k \times U(N)_{-k}$ gauge theory associated to it. Taking into account that the fundamental and anti-fundamental representations occur such that indices can mix as $\{ab, a\bar{b}, \bar{a}\bar{b}\}$, additional R-matrices are defined as $R_{a\bar{b}}(u) = u + K_{a\bar{b}}$ and $R_{\bar{a}\bar{b}}(u) = u - P_{\bar{a}\bar{b}}$. This will add two extra YB equations to the system, plus modified versions as well[110]. Defining appropriate monodromy matrices, and solving the system of equations for the odd and even sited Hamiltonians, given in terms of Bethe-like solutions, amounts to the same structure of the energy as it was found in $\mathcal{N} = 4$ SYM

$$E = \lambda^2 \left(\sum_{j=1}^{M_u} \frac{1}{u_j^2 + \frac{1}{4}} + \sum_{j=1}^{M_v} \frac{1}{v_j^2 + \frac{1}{4}} \right) \tag{3.1.16}$$

The conclusion is that this represents a $SU(4)$ spin chain where, alternating on the odd and even sites, letters transform in the fundamental and anti-fundamental respectively without mixing, as anticipated from the structure of operators. Considering BPS-sectors, one can just as well

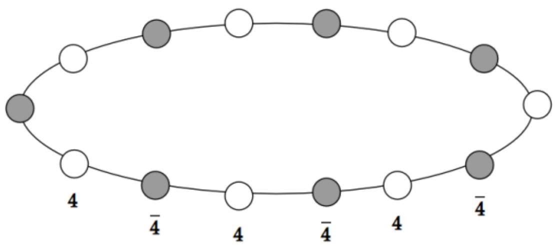


Figure 3.4: $SU(4)$ spin chain with bi and anti Bi-fundamental fields on odd and even sites

find such regimes, where the most notorious and well-studied example is the $SU(2) \times SU(2)$ sector. Operators for spin-chains are considered of this form (following notation [58])

$$\mathrm{Tr}(A_{i_1}B_1A_{i_2}B_1A_{i_3}B_1...) \tag{3.1.17}$$

These are chiral, but in general not primaries due to the existence of a superpotential in the theory. Due to the sextic interactions between scalar potential one gets contributions to Γ

$$\frac{16\pi^2}{k^2}\mathrm{Tr}\big[(A_1B_1A_2 - A_2B_1A_1)(A_1B_1A_2 - A_2B_1A_1)^\dagger\big] \tag{3.1.18}$$

The anomalous dimension reduces to the familiar one that was found for the $XXX_{1/2}$ Heisenberg spin chain as in SYM, but with a twist. Consider sending $B_1 \to B_2$. This translation does not affect the amplitude across the potential[8], hence it can be argued that there is a double structure of what was found in SYM. Instead of a single $SU(2)$ spin chain, one finds two decoupled $XXX_{1/2}$ Heisenberg spin chains in the fundamental and anti-fundamental representation respectively, living up to the $SU(2) \times SU(2)$ name. As a little teaser, this can be translated into a BPS bound related to the anomalous dimension as

$$\Delta - J = \lambda^2 \sum_{l=1}^{2J} (1 - P_{l,l+2}) = \lambda^2 \sum_{l=1}^{J} (1 - P_{2l-1,2l+1} + 1 - P_{2l,2l+2}) \tag{3.1.19}$$

One can also study an infinite $SU(2|2)$ chain as was done in [58] and. In conclusion, depending on the specific duality at hand, outcomes from the spin-chain picture can vary significantly, making it an interesting task to establish integrable models for AdS/CFT. A whole side to the story which will not be touched upon, is magnons arising as spin chain excitations. In the context of AdS/CFT this has been considered in various aspects of both SYM and ABJM [84, 98, 23, 81, 58, 65, 134, 101, 105, 131]. We will see how the magnon dispersion emerges in the next two sections for both dualities. As a concluding remark, it seems that the cases for $d = 3, 4$ which are the established cases for SYM and ABJM are well understood, while $d = 2$ is under construction. Beyond this, for $AdS_{6,7}$ and other cases, integrability seems not to be an emerging feature. This should nevertheless not discourage the study of these theories.

3.2 Penrose Limits and PP-waves

From the revolutionary work Penrose did in his famous work [124], it was quickly adapted in the AdS/CFT field for extensive use. In this section, we look at a particular type of metric that arises in certain decoupled theories when considering both SYM and ABJM. They are known as pp-wave backgrounds. We will investigate and define the notion of a Penrose limit, which essentially is a zoom on null geodesics in the space-time. The introductory part on pp-waves can be found here [37, 35, 36]. Lastly, the spectrum of operators will be seen through the eyes of BMN[29] in the case of SYM and ABJM[120]

3.2.1 Linearized gravity and pp-waves

Usually one talks about plane wave solutions when one considers the linearized Einstein equations. This is accompanied by adding a small pertubation to the flat Minkowski background $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. Finding solutions to what is reduced to a wave equation gives transversally polarised gravitational waves. Taking a specific solution in the (t, z)-direction for instance, will reduce the metric, employing light cone coordinates, it can be written as

$$ds^2 = 2dUdV + g_{ij}(U)dy^i dy^j \tag{3.2.1}$$

This is a plane wave metric written in what is called Rosen coordinates. But more conventionally, there lies a translation between what is called Brinkman coordinates and Rosen coordinates. The metric can take on a more familiar look

$$ds^2 = 2dudv + A_{ab}(u)x^a x^b du^2 + d\vec{x}^2 \tag{3.2.2}$$

In Brinkmann coordinates, a plane wave metric is characterized by a single symmetric matrix-valued function $A_{ab}(u)$. There is very little redundancy in the description of plane waves in

[8]exchanging A_1 and A_2 across B_1 and B_2 and vice versa

Brinkmann coordinates. Only a few residual coordinate transformations are left, which leaves the form of the metric invariant. Lastly, the metric is specified almost uniquely by $A_{ab}(u)$. In particular a plane wave metric is flat if and only if $A_{ab}(u) = 0$. Why this is useful in the first place, is due to Penrose as mentioned. That any space-time has emerging pp-wave, making it a tractable feature to use when looking for null geodesics e.g. The physical origin will be explored in the next section

3.2.2 Penrose limit and the emergence of pp-waves

Given a metric $g_{\mu\nu}$ or a line element $ds^2 = g_{\mu\nu}x^\mu x^\nu$, one can consider the Penrose Limit for a choice of null geodesic γ, that amounts to a plane wave metric. One first writes the coordinates adapted to γ as

$$ds^2 = 2dUdV + a(U,V,Y^k)dV^2 + 2b_i(U,V,Y^k)dVdY^i + g_{ij}(U,V,Y^k)dY^i dY^j \tag{3.2.3}$$

One then performs a set of coordinate transformation with rescaling rescaling $(U,V,Y^k) = (u, \lambda^2\tilde{v}, \lambda y_k)$. Then the Penrose limit of the metric is defined as

$$d\bar{s}^2 = \lim_{\lambda\to 0}\lambda^{-2}ds^2_{\gamma,\lambda} = 2dud\tilde{v} + g_{ij}(U)dy^i dy^j \tag{3.2.4}$$

We recognize this as a plane wave metric in Rosen coordinates[36]. As a rule of thumb, one can almost algorithmically get the Penrose limit by writing the metric into adapted coordinates, set a, b to 0, and restricting g_{ij} to the null-geodesic. Then readily it is a matter of translating between Rosen and Brinkmann coordinates to obtain the wanted metric. Having established this fact, we might want to take a step back and reflect upon the physical meaning of these Limits for space-time. Penrose put it best into words himself.

We envisage a succession of observers travelling in a space-time M whose world lines approach the null geodesic γ, more and more closely; *so we picture these observers as travelling with greater and greater speeds,approaching that of light. As their speeds increase they must correspondingly recalibrate their clocks to run faster and faster (assuming that all space-time measurements are referred to clock measurements in the standard way), so that in the limit the clocks measure the affine parameter x^0 along γ.(Without clock recalibration a degenerate space-time metric would result) In the limit the observers measure the space-time to have the plane wave structure W_γ* (3.2.5)

In other words, the Penrose limit can be understood as a boost accompanied by a commensurate uniform rescaling of the coordinates in such a way that the affine parameter along the null geodesic remains invariant. To implement this procedure in practice, we consider a Lorentzian space-time with a metric $g_{\mu\nu}$, choose some null geodesic γ, and locally write the metric in an adapted coordinates $x^\mu \to (U,V,Y^k)$. If we use the Penrose prescription one starts by making a boost

$$(U,V,Y^k) \to (\lambda^{-1}U, \lambda V, Y^k) \tag{3.2.6}$$

Trying to take the infinite boost limit $\lambda \to 0$ without recalibrating one's coordinates(clocks and measuring rods) seems to result in a singular metric. To avoid singularities arising, we rescale again such that

$$(U,V,Y^k) \to (\lambda U, \lambda V, \lambda Y^k) \tag{3.2.7}$$

What is found is an asymmetric scaling that overall goes as

$$(U,V,Y^k) \to (U, \lambda^2 V, \lambda Y^k) \tag{3.2.8}$$

This leaves our affine parameter $U = u$ invariant. As before, it leads back to the coordinates $(U, V, Y^k) \to (u, \lambda^2\tilde{v}, \lambda y_k)$. The metric now contains the rescaling parameter $ds^2_\gamma \to ds^2_{\gamma,\lambda}$, boiling the problem down counting and book-keeping. Making the last overall rescaling

$$ds^2_{\gamma,\lambda} \to \lambda^{-2} ds^2_{\gamma,\lambda} \tag{3.2.9}$$

gives the metric in powers of λ

$$ds^2 = 2dUdV + \lambda^2 a(U, V, Y^k)dV^2 + 2\lambda b_i(U, V, Y^k)dVdY^i + g_{ij}(U, V, Y^k)dY^i dY^j \tag{3.2.10}$$

The combined infinite boost and large volume limit $\lambda \to 0$ results in a well-defined and non-degenerate metric that provides Equation 3.2.4 again

3.2.3 pp waves as limits of AdS space-times

To narrow in on the the problem related to space-times in AdS we consider different pp-waves emerging, from the classic $AdS_5 \times S^5$ and variations together with $AdS_4 \times \mathbb{CP}^3$. Further, we investigate BMN theory and see how strings and operators arise for $\mathcal{N} = 4$ SYM and $\mathcal{N} = 6$ CS

The starting point: pp waves in $AdS_5 \times S^5$ and BNM

Consider the trajectory of a particle that is moving very fast along S^5 focusing on the geometry that it sees. Suppose it sits at the center of AdS, and rotates in $S^3 \subset S^5$ along an angular direction ψ. Following Bernstein, Maldacena and Nastase (BMN)[29], this was realized starting with the $AdS_5 \times S^5$ metric

$$ds^2 = R^2\big[-dt^2\cosh^2\rho + d\rho^2 + \sinh^2\rho d\Omega_3^2 + d\psi^2\cos^2\theta + d\theta^2 + \sin^2\theta d\Omega_3'^2\big] \tag{3.2.11}$$

One looks at a particle moving along the ψ direction while being placed at $\rho = 0$ and $\theta = 0$. The geometry near this trajectory will soon resemble a null geodesic. To do this, introduce light cone coordinates $\tilde{x}^\pm = \frac{t\pm\psi}{2}$ and then perform a rescaling of the coordinates w

$$x^+ = \tilde{x}^+, \quad x^- = R^2\tilde{x}^-, \quad \rho = \frac{r}{R^2}, \quad \theta = \frac{y}{R^2} \tag{3.2.12}$$

Taking the $R \to \infty$ limit is exactly what the Penrose limit is equivalent to. Expanding around the parameters in the rescaled variables amounts to

$$ds^2 = R^2\big[-dt^2(1-\rho^2) + dr^2 + r^2 d\Omega_3^2 + d\psi^2(1-\theta^2) + dy^2 + y^2 d\Omega_3'^2\big] \tag{3.2.13}$$

By the new coordinate $\tilde{x}^\pm$ we see that $-dt^2 + d\psi^2 = -4d\tilde{x}^+ d\tilde{x}^-$. The squared terms will contribute via the light cone coordinates which can easily be seen by

$$dt = dx^+ - \frac{dx^-}{R^2}, \quad d\psi = dx^+ + \frac{dx^-}{R^2} \tag{3.2.14}$$

Inserting this and keeping terms to the order of $\mathcal{O}(\frac{1}{R^2})$ exactly gives

$$ds^2 = R^2\big[-4d\tilde{x}^+ d\tilde{x}^- - (y^2 + r^2)(dx^+)^2 + d\vec{r}^2 + d\vec{y}^2\big] \tag{3.2.15}$$

This is following the geodesic around $\rho = 0$ and $\theta = 0$ where y and r parametrize points on R^4. Equivalently, one can also introduce a mass parameter via transformations $x^- \to x^-/\mu$ and $x^+ \to \mu x^+$ and parameterize to a coordinate $\vec{z}$ for points in R^8, such that the metric takes the form (as well as the fluxes)

$$ds^2 - 4d\tilde{x}^+ d\tilde{x}^- - \mu^2\vec{z}^2(dx^+)^2 + d\vec{z}^2, \quad F_{+1234} = F_{+5678} = c\mu \tag{3.2.16}$$

Generally throughout the next sections, we will be interested in BPS-bounds, motivating us to understand, how the energy and angular momentum along ψ scale in the rescaling limit. Energy is associated to time through $E = i\partial_t$ and angular momenta $J = -i\partial_\psi$. The beauty lies in the interpretation of the CFT where the dual of the geometry becomes the energy and R-charge of a state of the field theory on $\mathbb{R} \times S^3$. Alternatively, we can say that $E = \Delta$ is the conformal dimension of an operator on R^4. This will prove very useful when considering decoupling limits. It is easily seen, if the light cone momenta and Hamiltonian are defined as

$$\mathcal{H} = 2p^- = i\partial_{x^+} = i(\partial_t + \partial_\psi) = \Delta - J, \quad 2p^+ = \frac{1}{R^2} i\partial_{x^-} = \frac{1}{R^2} i(\partial_t + \partial_\psi) = \frac{\Delta + J}{R^2}, \quad (3.2.17)$$

From BPS condition $|\Delta| \geq J$, the momenta are positive. Amazingly, considering a pp-wave background, the quantization of strings yields a σ-model that can be quantized, and provides a light-cone Hamiltonian by expanding all fields in Fourier modes accompanied by a level matching condition such that the total momentum of the string vanishes[9]

$$2p^- = \mathcal{H}_{lc} = \sum_{n=-\infty}^{\infty} N_n \sqrt{\mu^2 + \frac{n^2}{(\alpha' p^+)^2}}, \quad P = \sum_{n=-\infty}^{\infty} n N_n = 0 \quad (3.2.18)$$

n is the Fourier mode label, while N_n denotes the total occupation number of modes. A limit to consider now is $1 \ll \mu\alpha' p^+$, where it was found that the contributions of $(\Delta - J)_n$ for the curved background are the frequencies of each oscillator. To build a bridge between gauge and gravity, it is realized that the eight transverse directions in the metric are just eight gauge bosons in the action. Similarly, the coupling to the RR-background provides the mass of the eight transverse fermions. This can be utilized by employing the Hamiltonian to find the spectrum of operators with finite $\Delta - J$. To excite the particles, a vacuum must be considered first corresponding to $\Delta - J = 0$ given by a scalar operator $|0\rangle = \mathrm{Tr}[Z^J]$ which is a chiral primary. Vacuum in light-cone gauge corresponds to

$$\frac{1}{\sqrt{J} N^{J/2}} \mathrm{Tr}[Z^J] \leftrightarrow |0, p_+\rangle_{lc} \quad (3.2.19)$$

Where then normalization comes from combinatorics of field contractions. Now consider $\Delta - J = 1$ with eight bosonic and fermionic oscillators $a_0^i, b_0^j, i, j \in \{1, ..., 8\}$. Applying the zero momentum bosonic and fermionic operators gives for instance results in

$$a_0^i |0, p_+\rangle_{lc} = \frac{1}{\sqrt{J} N^{J/2+1/2}} \mathrm{Tr}[\phi_i Z^J], \quad a_0^{\dagger i} b_0^{\dagger j} |0, p_+\rangle_{lc} = \frac{1}{\sqrt{J} N^{J/2+1/2}} \sum_{l=1}^{J} \mathrm{Tr}[\phi_i Z^l \psi^b_{J=1/2} Z^{J-l}] \quad (3.2.20)$$

With this procedure at hand, it enables one to build towers of operators for different values of finite $\Delta - J$ in $\mathcal{N} = 4$ SYM. Lastly, note that the two-point correlator, for insertions of fields, contributes to the one-loop anomalous dimension

$$(\Delta - J)_n = \mu + \frac{2\pi g N n^2}{\mu J^2} \quad (3.2.21)$$

Due to to insertion of impurities, operator mixing occurs giving corrections at one-loop level under the dilatation operator. But Equation 3.2.21 exactly matches the $1 \ll \mu\alpha' p^+$ limit. It can be seen that the spectrum is uniform for strongly and weakly coupled theories in λ. Due to this accomplishment, BMN became a celebrated framework. For this reason, ABJM went through the same machinery. But as we will see, the spectrum might behave very differently for Penrose limits on $AdS_4 \times \mathbb{C}P^3$

[9]When only the $n = 0$ modes are excited then one reproduces the spectrum of massless supergravity modes propagating on the plane wave geometry

pp waves in $AdS_4 \times \mathbb{CP}^3$

Through a Penrose limit, it was first after ABJM was established that the subsequent analysis came. We elaborate on it and consider the differences to BMN on a $AdS_4 \times \mathbb{C}P^3$ geometry. Interestingly enough, the geometry admitted has 24 supercharges [120, 86] which breaks the full 32 which distinguishes type IIA and B backgrounds in the Penrose limits[10]. Starting at the type IIA considered in Equation 2.3.38, a null geodesic can be considered where $\rho = 0, \theta_1 = \theta_2 = 0, \xi = \pi/4$ and further for convenience a new angular coordinate can be defined as $\tilde{\psi} = \psi + \frac{\phi_1 - \phi_2}{2}$. This can be implemented as a Penrose limit with rescaled coordinates

$$\frac{t + \tilde{\psi}}{2} = x^+, \quad \tilde{R}^2 \frac{t - \tilde{\psi}}{2} = x^-, \quad \rho = \frac{r}{\tilde{R}}, \quad \theta_i = \frac{\sqrt{2} y_i}{\tilde{R}}, \quad \xi = \frac{\pi}{4} + \frac{y_3}{2\tilde{R}} \tag{3.2.22}$$

Expanding the metric in $1/\tilde{R}$, and taking the $\tilde{R} \to \infty$ limit, the type IIA metric is

$$\begin{aligned} ds^2 &= -dt^2(1 - r^2/\tilde{R}^2) + dr^2/\tilde{R}^2 + r^2 d\Omega_2^2/\tilde{R}^2 + \frac{dy_3^2}{\tilde{R}^2} \\ &+ \frac{1}{4}\left(1 - \frac{y_3^2}{\tilde{R}^2}\right)\left(d\psi + \frac{1}{2}\left(1 - \frac{y_1}{\tilde{R}}\right)d\phi_1 - \frac{1}{2}\left(1 - \frac{y_2}{\tilde{R}}\right)d\phi_2\right)^2 \\ &+ \frac{1}{2}\left(1 - \frac{y_3}{\tilde{R}}\right)\left(\frac{2(dy_1^2 + y_1^2 d\phi_1^2)}{\tilde{R}^2}\right) + \frac{1}{2}\left(1 + \frac{y_3}{\tilde{R}}\right)\left(\frac{2(dy_2^2 + y_2^2 d\phi_2^2)}{\tilde{R}^2}\right) \end{aligned} \tag{3.2.23}$$

In the limit, this will reduce to a light-cone metric

$$\begin{aligned} ds_{IIA}^2 = -4dx^+dx^- - (r^2 + y_3^2)(dx^+)^2 + dx^+(-y_1^2 d\phi_1 + y_2^2 d\phi_2) \\ + dr^2 + r^2 d\Omega_2^2 (dy_1^2 + y_1^2 d\phi_1) + (dy_2^2 + y_2^2 d\phi_2) + dy_3^2 \end{aligned} \tag{3.2.24}$$

The RR fluxes change accordingly. To get the recognizable pp-wave, defining new angles $\tilde{\phi}_1 = \phi_1 - \frac{x^+}{2}$ and $\tilde{\phi}_2 = \phi_2 + \frac{x^+}{2}$ will produce a metric reminiscent of the Brinkman coordinates. We can also express this in cartesian coordinates $(x_1, ..., x_8)$

$$\begin{aligned} ds_{IIA}^2 &= -4dx^+dx^- - (r^2 + y_3^2 + \frac{y_1^2 + y_2^2}{4})(dx^+)^2 \\ &+ dr^2 + r^2 d\Omega_2^2 (dy_1^2 + y_1^2 d\tilde{\phi}_1) + (dy_2^2 + y_2^2 d\tilde{\phi}_2) + dy_3^2 \\ &= -4dx^+dx^- - \left(\sum_{i=1}^{4} x_i^2 + \frac{1}{4}\sum_{i=5}^{8} x_i^2\right)(dx^+)^2 + \sum_{i=1}^{8}(dx_i^2) \end{aligned} \tag{3.2.25}$$

The procedure follows the same framework as BMN now. The obvious detail with two representations for scalars changes the light cone momenta as follows

$$\begin{aligned} 2p_b^- &= \sum_{n=-\infty}^{\infty} N_n^{(1)} \sqrt{1 + \frac{n^2}{(p^+)^2}} + \sum_{n=-\infty}^{\infty} N_n^{(2)} \sqrt{\frac{1}{4} + \frac{n^2}{(p^+)^2}} \\ &= \sum_{n=-\infty}^{\infty} N_n^{(1)} \sqrt{1 + \frac{2\pi^2 n^2}{J^2}\lambda} + \sum_{n=-\infty}^{\infty} N_n^{(2)} \sqrt{\frac{1}{4} + \frac{2\pi^2 n^2}{J^2}\lambda} \end{aligned} \tag{3.2.26}$$

with level matching condition

$$\sum_{n=-\infty}^{\infty} n(N_n^{(1)} + N_n^{(2)}) = 0 \tag{3.2.27}$$

[10]Due to toroidal compactification, 8 of the supercharges are inevitably broken on the spatial isometry direction

Stating the light cone momenta in terms of gauge theory quantities is crucial as we will see in a moment. The bosons and fermions arise in the same fashion as excitations of the eight transverse directions, but the impurity operators this time will be $\mathrm{Tr}(A_1B_1)^J$ which is the chiral primary with $\Delta - J = 0$. An example of the type of operators are

$$\mathcal{O}_n = \frac{1}{2\sqrt{J}} \sum_{l=0}^{J} \mathrm{Tr}[((A_1B_1)^l A_1B_1(A_1B_1)^{J-l}A_1B_1]e^{2\pi i l n/J} \tag{3.2.28}$$

For the anomalous dimension, the story does get peculiar. Computing the two-point correlator, it was obtained that the leading order correction to the anomalous dimension is $\delta_n^{CS} = 4\pi^2 \frac{N^2 n^2}{k^2 J^2}$. Curiously, this does not seem to match what was found in the type IIA spectrum. This gives a non-trivial dependence on the t'Hooft coupling as was noted in [64]

$$f(\lambda) = \begin{cases} 4\lambda^2, & \lambda \to \infty \\ 2\lambda, & \lambda \to 0 \end{cases} \tag{3.2.29}$$

What the cause is, still poses as a mystery. Some argue that it is due to the violation of BMN scaling, others might say reduction to 24 supersymmetry generators is the cause etc. But this poses a crucial difference compared to what what found in BMN. Nevertheless, it has still proven fruitful to apply the formalism of BMN to ABJM through type IIA pp-waves. Later on, we compute how the spectrum on the string side can be extracted concretely

Chapter 4

Spin Matrix theory (SMT): A Quantum Model of the AdS/CFT Correspondence

4.1 Spin Matrix Theory: Quantum Mechanical Theory for AdS/CFT

So far we have been discussing the gauge/gravity duality in the Maldacena case, where one takes two opposite limits and gets that massive stings curving $d+1$ dimensional AdS space which is conjectured to be dual to a d-dimensioal CFT at two different limits of the t´Hooft coupling, usually taken in the planar limit $N \to \infty$. The link between this duality is interpreted in terms of integrable spin chains, thus the integrability is the interpolation from going from one side of the duality to the other. But what if we want the case when $N < \infty$, then we need to start revising our strategy. To interpolate between strong and weak t´Hooft coupling at finite N, must consider new possibilities then. And the motivations for doing this are plenty [79]. Thus the idea is to consider non-relativistic limits of the AdS/CFT correspondence in the grand canonical ensemble, such that it corresponds to the approach of critical temperatures $T = 0$. Here we will let $\vec{\Omega}$ denote the respective chemical potentials conjugate to the global symmetry charges. In the original work for SMT, one considers $\mathcal{N} = 4$ SYM, And the following limit is considered

$$(T, \vec{\Omega}) \to (0, \vec{\Omega}^{(c)}), \quad \lambda = 0, \quad \text{with} \quad \frac{\lambda}{T}, \frac{\vec{\Omega} - \vec{\Omega}^{(c)}}{T} \quad \text{kept fixed} \tag{4.1.1}$$

This will give a simple spin chain with nearest-neighbour interaction, based on the fact that we build our Hilbert space out of harmonic oscillators giving a non-relativistic quantum mechanical theory. The idea is to go over the case for $\mathcal{N} = 4$ SYM and see if the same procedure can be applied to $\mathcal{N} = 6$ Chern-simons theory. For details of the construction, see [79].

4.1.1 Spin Matrix Theory for $\mathcal{N} = 4$ SYM

The framework developed has a gateway to $\mathcal{N} = 4$ SYM if considered at near-zero temperatures critical points in the grand canonical ensemble. In this ensemble, a partition function can be constructed with chemical potentials present, given by the bosonic subalgebra for the field theory. It was previously established what generators are present given by the $SU(4)$ R-symmetry and $SO(4,2)$ conformal group. The partition function is $Z(\beta, \vec{\Omega}) = \mathrm{Tr}(e^{-\beta D + \beta \vec{\Omega} \cdot \vec{J}})$, with $T = \frac{1}{\beta}$ and the dot product for chemical potentials are given by a weight $\vec{\Omega} = (\omega_1, \omega_2, \Omega_1, \Omega_2, \Omega_3)$ times the generators $\vec{J} = (S_1, S_2, J_1, J_2, J_3)$. This is obviously $\vec{\Omega} \cdot \vec{J} = \omega_1 S_1 + \omega_2 S_2 + \Omega_1 R_1 + \Omega_2 R_2 + \Omega_3 R_3$. The theory is considered on $\mathbb{R} \times S^3$ due to isometries of the geometry and groups as well. Like in the case of renormalizing the conformal two-point function, it becomes the dilatation operator that is expanded in powers of the t´Hooft coupling λ such that $D = D_0 + \delta D$. Working only up to one-loop gives $\delta D = \lambda D_2 + \mathcal{O}(\lambda^{3/2})$. To obtain one loop corrections, an explicit form of D_2 is needed, which luckily has been studied intensively for $PSU(2,2|4)$[22, 24]. Acting with D_2 hits two letters at a time in the singleton representation $\mathcal{A}$ of $PSU(2,2|4)$. The product of two singletons are irreps. $\mathcal{V}_j$, labelled uniquely by the quadratic Casimir. This gives

$$\mathcal{A} \otimes \mathcal{A} = \sum_{j=0}^{\infty} \mathcal{V}_j \tag{4.1.2}$$

. Having asserted this, D_2 has the following structure

$$D_2 = -\frac{1}{8\pi^2 N} \sum_{j=0}^{\infty} h(j)(p_j)^{AB}_{CD} : \mathrm{Tr}[W_A, \partial_{W_c}][W_B, \partial_{W_D}] : \tag{4.1.3}$$

Here $h(j) = \sum_{k=1}^{j} \frac{1}{k}, h(0) = 0$ are the harmonic numbers, P_j is the projection operator from the product of singletons to the irreps. and $W_A, A \in \mathcal{A}$ represents all letters in SYM while maintaining normal ordering. Amazingly, one can identify components between SMT and SYM. Raising operators becomes letters $a_s^\dagger \leftrightarrow W_s$, D_2 can be interchanged with H_{int} if $\mathcal{J} \leftrightarrow j$ and $V_{\mathcal{J}} \leftrightarrow \mathcal{V}_j$ such that $C_j = \frac{1}{8\pi^2} h(j), j = 0, 1, 2$. The twist to the naive story is that it only holds in a non-relativistic limit when $\lambda = 0$. This restricts us to subsectors of the space of operators simplifying matters. Using this, SMT has found many applications in non-relativistic string theory, which we demonstrate later on. The essential features of ner BPS-limits will be considered now

4.1.2 Near BPS-limit for subsectors and zero-temperature critical points

From the definition of the partition function, it becomes apparent that confinement/deconfinement might happen considering certain bounds, such that the system undergoes a phase transition. The zero-temperature critical points is defined as a continuation of a submanifolds of phase transitions to zero temperature meaning $(T, \vec{\Omega}) \to (0, \vec{\Omega}^{(c)})$. The critical points exactly correspond to choices of weight-vectors for Cartan generators when obtaining subsectors for a theory. By specific choices, the BPS-bound considered is given by $D \geq \vec{\Omega}^{(c)} \cdot \vec{J}$ for all operators while there should still be some that saturate the bound. This will be the most crucial feature. The discussion of spin matrix theory is long and can be extended to find SMT theories for subsectors by appropriate use of the translation between D_2, H_{int} and the respective representations. Consider now raising the temperature. One encounters a singularity in $Z(\beta, \vec{\Omega})$ at $T_H(g)$ then. This is known as the Hagedorn temperature. This had already been analyzed for both ends of coupling regimes in earlier works[78, 77, 80]. The resemblance between this and AdS/CFT is evident in the planar limit interpretation of spin chains. This motivates the investigation of how to extend the formalism to fit for $\mathcal{N} = 6$ Chern-Simons theories as well. We will see the construction of the Free letter partition function later, but first, the first attempt at connecting pieces from SMT to the structure in ABJM is done.

4.1.3 Spin Matrix Theory for $\mathcal{N} = 6$ Chern-Simons

We proceed as for SYM and list the obvious differences. Constructing the grand canonical partition function will be identical, up to differences in the bosonic subalgebra. The same R-symmetry is present, but the conformal theory goes down a dimension $SO(3,2)$. The weights and generators are shortened to $\vec{\Omega} = (\omega, \Omega_1, \Omega_2, \Omega_3, \Omega_4)$ times the generators $\vec{J} = (S_1, J_1, J_2, J_3, J_4)$. This is obviously $\vec{\Omega} \cdot \vec{J} = \omega S + \Omega_1 J_1 + \Omega_2 J_2 + \Omega_3 J_3 + \Omega_4 J_4$. One can reformulate this in terms of R-generators, but due to orbifolding, we have a translation between J_i and R_i by a linear set of equations. This theory now is considered on $\mathbb{R} \times S^2$, due to one less DOF of S. The major differences occur when we consider loop-order of D, since the first contribution D_2 comes at two-loop. Another major difference occurs when translating the singleton representations of the theory. The global symmetry is now $OSp(6|4)$ which tied to $\mathcal{N} = 6$ CS has both a fundamental and anti-fundamental representation $(\mathbf{N}, \bar{\mathbf{N}})$ and $(\bar{\mathbf{N}}, \mathbf{N})$. This will in all give three different types of ways, D_2 can act on two letters (or modules) at a time. Following Zwiebel[147], the tensor product of a conjugate pair of modules $\mathcal{V}_\phi$ and $\mathcal{V}_{\bar{\phi}}$ has one highest-weight state for each nonnegative integer spin j. Similarly, a like pair of modules has one highest-weight state with spin $(j - 1/2)$ for each nonnegative integer j. This gives the combinations

$$\mathcal{V}_\phi \otimes \mathcal{V}_{\bar{\phi}} = \sum_{j=0}^{\infty} \mathcal{V}_j, \quad \mathcal{V}_\phi \otimes \mathcal{V}_\phi = \sum_{j=0}^{\infty} \mathcal{V}_{j-1/2}, \quad \mathcal{V}_{\bar{\phi}} \otimes \mathcal{V}_{\bar{\phi}} = \sum_{j=0}^{\infty} \mathcal{V}_{j-1/2} \tag{4.1.4}$$

The structure of the tensor products both accounts for nearest and next to nearest neighbour interactions since the vector space is composed as $(\mathcal{V} \otimes \bar{\mathcal{V}})^L$. The irreducible representation is again labeled uniquely by the quadratic Casimir which in $OSp(6|4)$ takes the form

$$J^2 = \frac{1}{8}([Q_{ij,\alpha}, S^{ij,\alpha}] - 2R^i_j R^j_i + 2M^\alpha_\beta M^\beta_\alpha + 4D^2 - \{P_{\alpha\beta}, K^{\alpha\beta}\}) \tag{4.1.5}$$

Acting with J^2 on highest weight states (HWS), this reduces to an expression in terms of Dynkin labels

$$\begin{aligned} J^2 = \frac{1}{2}\Bigg(D(D+3) + s(s+2) + 3J^1_1 + 2J^2_2 + J^3_3 + \frac{1}{2}\sum_{i=1}^{4}(J^i_i)^2\Bigg) = \frac{1}{2}\Bigg(D(D+3) \\ + s(s+2) + \frac{1}{4}q_1(q_1+2) + \frac{1}{4}q_2(q_2+2) + \frac{1}{8}(2p+q_1+q_2)^2 - (2p+q_1+q_2)\Bigg) \end{aligned} \tag{4.1.6}$$

D is the dimension and s is the Lorentz spin. The first expression uses eigenvalues of all diagonal entries of the traceless matrix of R-symmetry generators, while the second uses the standard $SU(4)$ Dynkin labels

$$q_1 = J^2_2 - J^1_1, \quad q_2 = J^3_3 - J^2_2, \quad q_3 = J^4_4 - J^3_3 \tag{4.1.7}$$

which satisfies the relation $j(j+1) = J^2$. With this in mind, we can write the full $OSp(6|4)$ two-loop dilatation operator

$$\begin{aligned} D_2 = \sum_{i=0}^{2L}\Bigg(2\log 2 + \sum_{j=0}^{\infty} h(j)\mathcal{P}^{(j)}_{i,i+1} + \sum_{j_1,j_2,j_3=0}^{\infty} (-1)^{j_1+j_3}\frac{1}{2}h(j_2 - 1/2) \\ \times \left(\mathcal{P}^{(j_1)}_{i,i+1}\mathcal{P}^{(j_2-1/2)}_{i,i+2}\mathcal{P}^{(j_3)}_{i,i+1} + \mathcal{P}^{(j_1)}_{i+1,i+2}\mathcal{P}^{(j_2-1/2)}_{i,i+2}\mathcal{P}^{(j_3)}_{i+1,i+2}\right)\Bigg) \end{aligned} \tag{4.1.8}$$

We see to some extent the same structure as in SYM, but this time, it becomes altered by the $SU(4)$ spin-chain picture. $h(j)$ are still the harmonic numbers, and the projectors $\mathcal{P}$ come from one of the tensor products combinations. But, the same type of one-to-one mapping is not as trivial[1]. The terms in D_2 can be summarized as nearest and next to nearest types of interactions, thus the coefficient C_j has to be split up into two pieces

$$C_j = C_0 + C^{\text{Near}}_j + C^{\text{Next Near}}_j = (2\log 2 + h(j)) + (-1)^{j_1+j_3}\frac{1}{2}h(j_2 - 1/2) \tag{4.1.9}$$

[1] The author of [147] notes that it would be nice to have an expression for these projectors in components, which are given for $OSp(4|2)$ in appendix D

This seems to be the only natural way of describing the coefficients of the interactions in terms of SMT language. But the same problem is still facing us, as it was originally noted. $\mathcal{N}=6$ cannot be a SMT since it is a QFT. Following the same line of reasoning, it becomes only sensible at critical points and BPS-limits, to consider the validity of SMT. No more work has been done as of yet, but it seems like a possibility using the derived subsectors, to accordingly construct an $SU(2)\times SU(2)$ (and the remaining sectors as well) spin matrix theory by right identification and further elaborations.

4.2 Subsector in the string dual of N=6 superconformal Chern-Simons theory

In this section, we go through explicit calculations [64] to show that in a large wavelength limit, a specific spin chain namely the Landau-Lifshitz model emerges from a type IIA string theory on $AdS_4\times\mathbb{C}P^3$. Both a Sigma-model limit and a Penrose limit of the $SU(2)\times SU(2)$ sector are considered. We infer from the letters, that this corresponds to the BPS-bound $\Delta\geq J_1+J_2$. It will be apparent that the same picture emerges in the two cases. We extend what is known to the same bound with the addition of spin

4.2.1 Sigma-Model limit

The starting point is considering the geometry of M-theory background $AdS_4\times S^7$ as we know it. In getting the $SU(2)\times SU(2)$ sector one can split $\mathbb{C}^4=\mathbb{C}^2\times\mathbb{C}^2$. Each SU(2) correspond to certain parts of our SU(4) multiplets, the first being for $A_{1,2}$ which is associated to $z_{1,2}$, while $B_{1,2}$ is associated to $\bar{z}_{3,4}$. This motivates S^7 to be split into two S^3´s giving the metric

$$ds^2_{s^7}=d\theta^2+\cos^2(\theta)d\Omega_3^2+\sin^2(\theta)d\Omega_3'^2 \tag{4.2.1}$$

After various manipulations of angle and definitions, one is led to consider an 11-dimensional metric of the form

$$ds^2_{11}=-\frac{\hat{R}^2}{4}dt^2+\frac{\hat{R}^2}{2}(d\Omega_3^2+d\Omega_3'^2) \tag{4.2.2}$$

Using the definitions of the S^3´s from [64], the final form of the metric becomes

$$ds^2_{11}=-\frac{\hat{R}^2}{4}dt^2+\hat{R}^2(d\gamma+A)^2+\hat{R}^2[\frac{1}{8}d\Omega_2^2+\frac{1}{8}d\Omega_2'^2+(d\delta+\omega)^2] \tag{4.2.3}$$

There is a presence from one-forms $A=\frac{1}{4}(\sin\theta_1 d\phi_1-\sin\theta_2 d\phi_2)$ and $\omega=\frac{1}{4}(\sin\theta_1 d\phi_1+\sin\theta_2 d\phi_2)$. To obtain a type IIA background, the prescription from ABJM states that the radius of dγ becomes very small. The term effectively drops out giving the ten-dimensional type IIA background

$$ds^2=-\frac{\hat{R}^2}{4}dt^2+\hat{R}^2[\frac{1}{8}d\Omega_2^2+\frac{1}{8}d\Omega_2'^2+(d\delta+\omega)^2] \tag{4.2.4}$$

Getting back to the sigma-model limit, the zoom we have to make to get a narrow window of momenta, such that we will see individual magnon states in the spin chain, is given by the coordinate transformations $\bar{t}=\frac{1}{J^2}t$ and $\chi=\delta-\frac{1}{2}t$. Defining $J\equiv J_1+J_2$ and letting $J\to\infty$, this precisely give us the correct $SU(2)\times SU(2)$ BPS-bound $\Delta-J$ regime. Using the coordinate transformations one obtains a type IIA metric

$$\begin{aligned}ds^2&=R^2\left[(J^2d\bar{t}+d\chi+\omega)(d\chi+\omega)+\frac{1}{8}d\Omega_2^2+\frac{1}{8}d\Omega_2'^2\right]\\&=R^2\left[J^2d\bar{t}d\chi+J^2\omega d\bar{t}+\omega^2+2\omega d\chi+d\chi^2+\frac{1}{8}d\Omega_2^2+\frac{1}{8}d\Omega_2'^2\right]\end{aligned} \tag{4.2.5}$$

Consider now the standard bosonic sigma-model Lagrangian.

$$\mathcal{L} = -\frac{1}{2}h^{\alpha\beta}G_{\mu\nu}\partial_\alpha x^\mu \partial_\beta x^\nu \tag{4.2.6}$$

Apparently a good gauge choice is the one where $\bar{t} = \kappa\tau$, $p_\chi = \text{const}$, $h_{\alpha\beta} = \eta_{\alpha\beta}$ with $2\pi l_s^2 p_\chi = \partial\mathcal{L}/\partial\partial_\tau\chi$. Via the metric and gauge, one can write out the respective coordinates and retrieve the full Lagrangian. The gauge choice makes $h_{\alpha\beta}$ diagonal giving us two cases to consider[2]

$$\begin{aligned}\frac{2}{R^2}\mathcal{L} &= \kappa J^2\partial_\tau\chi + \kappa J^2\omega_\tau + \partial_\tau\chi^2 + \omega_\tau^2 + 2\omega_\tau\partial_\tau\chi - \chi'^2 - \omega_\sigma^2 - 2\omega_\sigma\chi' \\ &+ \sum_{i=1}^{2}(\partial_\tau\theta_i)^2 - \theta_i'^2 + \cos^2\theta_i[(\partial_\tau\phi_i)^2 - \phi_i'^2] = (\kappa J^2 + \partial_\tau\chi + \omega_\tau)(\partial_\tau\chi + \omega_\tau) \\ &- (\chi' + \omega_\sigma)^2 + \frac{1}{8}\sum_{i=1}^{2}(\partial_\tau\theta_i)^2 - \theta_i'^2 + \cos^2\theta_i[(\partial_\tau\phi_i)^2 - \phi_i'^2]\end{aligned} \tag{4.2.7}$$

We write $\omega = \omega_\tau d\tau + \omega_\sigma d\sigma$ where letters with primes translate to derivatives with respect to σ. Proceeding, one considers the Virasoro constraints. This is done by taking the energy-momentum tensor and imposing conformal symmetry by traceless condition $T^\alpha_\alpha = 0$ (off-diagonal elements contribute as well). In the case of the given metric, this becomes

$$\begin{aligned}&T_{\alpha\beta} = G_{\mu\nu}\partial_\alpha x^\mu\partial_\beta x^\nu - \frac{1}{2}\eta_{\alpha\beta}\eta^{\delta\gamma}G_{\mu\nu}\partial_\delta x^\mu\partial_\gamma x^\nu = G_{\mu\nu}\partial_\alpha x^\mu\partial_\beta x^\nu - \eta_{\alpha\beta}\mathcal{L} = \\ &\kappa J^2 d\tau d\chi + J^2\omega_\tau d\tau^2 + \kappa J^2\omega_\tau d\tau d\sigma + d\chi^2 + 2\omega_\tau d\tau d\chi \\ &+ 2\omega_\sigma d\sigma d\chi + \omega_\tau^2 d\tau^2 + \omega_\sigma^2 d\sigma^2 + 2\omega_\tau\omega_\sigma d\tau d\sigma + \frac{1}{8}\sum_{i=1}^{2} d\theta_i^2 + \cos^2\theta_i d\phi_i\end{aligned} \tag{4.2.8}$$

Looking at the contributions one gets the corresponding Virasoro constraints.

$$\begin{aligned}T_{\tau\tau} &= G_{\mu\nu}\partial_\tau x^\mu\partial_\tau x^\nu - \mathcal{L} = (\partial_\tau\chi)^2 + (\kappa J^2 + 2\omega_\tau)\partial_\tau\chi + \kappa J^2\omega_\tau + \omega_\tau^2 + \frac{1}{8}\sum_{i=1}^{2}\partial_\tau\theta_i^2 + \cos^2\theta_i\partial_\tau\phi_i - \mathcal{L} \\ T_{\sigma\sigma} &= G_{\mu\nu}\partial_\tau x^\mu\partial_\tau x^\nu + \mathcal{L} = \chi'^2 + 2\omega_\sigma\chi + \omega_\sigma^2 + \frac{1}{8}\sum_{i=1}^{2}\theta_i'^2 + \cos^2\theta_i\phi_i'^2 + \mathcal{L}\end{aligned} \tag{4.2.9}$$

Adding the two diagonal elements, using the traceless condition will give us the first constraint

$$T_{\sigma\sigma}+T_{\tau\tau} = (\kappa J^2+\partial_\tau\chi+\omega_\tau)(\partial_\tau\chi+\omega_\tau)+(\chi'+\omega_\sigma)^2+\frac{1}{8}\sum_{i=1}^{2}(\partial_\tau\theta_i)^2+\theta_i'^2+\cos^2\theta_i[(\partial_\tau\phi_i)^2+\phi_i'^2] = 0 \tag{4.2.10}$$

For the off-diagonal element, the second constraint immediately follows

$$\begin{aligned}T_{\sigma\tau} &= \kappa J^2\omega_\sigma + 2\omega_\sigma\omega_\tau + \kappa J^2\chi'^2 + 2\omega_\tau\chi' + 2\omega_\sigma\partial_\tau\chi + \partial_\tau\chi\chi' + \frac{1}{8}\sum_{i=1}^{2}[\partial_\tau\theta_i\theta_i' + \cos^2\theta_i\partial_\tau\phi_i\phi_i'] \\ &= (\kappa J^2 + \partial_\tau\chi + \omega_\tau)(\chi' + \omega_\sigma) + \frac{1}{8}\sum_{i=1}^{2}[\partial_\tau\theta_i\theta_i' + \cos^2\theta_i\partial_\tau\phi_i\phi_i'] = 0\end{aligned} \tag{4.2.11}$$

[2]We also multiply by $\frac{2}{R^2}$ to remove as much redundancy from LHS

These will prove handy. But first, our attention is directed to the momentum constraint. Explicit computation shows

$$2\pi l_s^2 p_\chi = \partial\mathcal{L}/\partial\partial_\tau\chi = \frac{R^2}{2}(\kappa J^2 + 2\partial_\tau\chi + 2\omega_\tau) \quad \rightarrow \quad p_\chi = \frac{R^2}{2\pi l_s^2}(\frac{1}{2}\kappa J^2 + \partial_\tau\chi + \omega_\tau) \tag{4.2.12}$$

Using an argument of time scales relevant to the energy scale of $\tilde{H}$ one can consider in the $J \to \infty$ velocities of $\tilde{t}$ to be finite. Thus $\partial_\tau\chi = \kappa\partial_{\tilde{t}}\chi \to 0$. Using also that $2J = \int_0^{2\pi} p_\chi$

$$2J = \frac{R^2}{l_s^2}\frac{1}{2}\kappa J^2 \rightarrow \kappa = \frac{4l_s^2}{JR^2} \tag{4.2.13}$$

This goes to zero in the $J \to \infty$ as considered. Applying the limit we can reduce both the Lagrangian and the Virasoro constraints such that terms with κ and J with the same power survive

$$\begin{aligned} \frac{2}{R^2}\mathcal{L} &= \frac{16l_s^4}{R^4}(\dot{\chi} + \omega_\tau) - (\chi' + \omega_\sigma)^2 - \frac{1}{8}\sum_{i=1}^{2}[\theta_i'^2 + \cos^2\theta_i\phi_i'^2] \\ \chi' + \omega_\sigma = 0 \quad & \frac{16l_s^4}{R^4}(\dot{\chi} + \omega_\tau) + \frac{1}{8}\sum_{i=1}^{2}[\theta_i'^2 + \cos^2\theta_i\phi_i'^2] = 0 \end{aligned} \tag{4.2.14}$$

The dot denotes the derivative with respect to $\tilde{t}$. Further χ is constrained by the angular variables, such that we can gauge away the non-dynamical angle and get a gauge fixed Lagrangian

$$\frac{2}{R^2}\mathcal{L} = \frac{16l_s^4}{R^4}\omega_{\tilde{t}} - \frac{1}{8}\sum_{i=1}^{2}[\theta_i'^2 + \cos^2\theta_i\phi_i'^2] \tag{4.2.15}$$

In the end, we can obtain an action for the Sigma-model in the $J \to \infty$ limit as, using also the relation $\frac{l_s^4}{R^4} = \frac{1}{64\pi^2\lambda}$

$$I = \frac{J}{4\pi}\sum_{i=1}^{2}\int d\tilde{t}\int_0^{2\pi} d\sigma\Big[\sin\theta_i\dot{\phi} - \pi^2\lambda(\theta_i'^2 + \cos^2\theta_i\phi_i'^2)\Big] \tag{4.2.16}$$

supplemented by a momentum constraint

$$\sum_{i=1}^{2}\int_0^{2\pi} d\sigma \sin\theta_i\phi_i' = 0 \tag{4.2.17}$$

this amazingly can be interpreted in the $SU(2) \times SU(2)$ sigma-model limit as two Landau Lifshitz models added together for each $SU(2)$. The LL spin chain is just a long wavelength limit $J \to \infty$ which reduces to the standard $XXX_{1/2}$ Heisenberg spin chain. What might be even more amazing, is that by a Penrose limit of $AdS_4 \times \mathbb{C}P^3$, we might find the same interpretation

4.2.2 $\mathbf{SU}(2) \times \mathbf{SU}(2)$ Penrose limit

Instead of attacking the problem at hand with a sigma-model limit we consider a Penrose limit where through certain coordinate transformations and rescaling, a pp-wave background emerges. Start by considering the $\mathbb{C}P^3 \times AdS_4$ metric

$$ds^2 = \frac{R^2}{4}(-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\hat{\Omega}_2^2) + R^2 ds_{\mathbb{C}P^3}^2 \tag{4.2.18}$$

Writing the $\mathbb{C}P^3$ in the following way will be useful

$$ds^2_{\mathbb{C}P^3} = d\theta^2 + \frac{\cos^2\theta}{4} d\Omega_2^2 + \frac{\sin^2\theta}{4} d\Omega_2^{'2} + 4\cos^2\theta \sin^2\theta (d\delta + \omega)^2 \tag{4.2.19}$$

The one-form ω is the same as in the previous calculation for the sigma model. In addition, the same transformations apply for coordinates t and χ. We group terms that go as dt'^2 together and write $\cosh^2\rho = 1 - \sinh^2\rho$. Then the metric reads.

$$\begin{aligned} ds^2 = &\frac{R^2}{4} dt'^2 (1 - 4\cos^2\theta\sin^2\theta + \sinh^2\rho) + \frac{R^2}{4}(d\rho^2 + \sinh^2\rho d\hat{\Omega}_2^2) \\ &R^2[d\theta^2 + \frac{\cos^2\theta}{4} d\Omega_2^2 + \frac{\sin^2\theta}{4} d\Omega_2'^2 + 4\cos^2\theta\sin^2\theta(dt' + d\chi + \omega)(d\chi + \omega)] \end{aligned} \tag{4.2.20}$$

As noted before we still have the same conserved quantities in terms of killing vectors and isometries, namely $\Delta - J = i\partial_{t'}$ and $2J = -i\partial_\chi$. The quantity $\Delta - J$ is of course the energy we want to measure for the $SU(2) \times SU(2)$ sector. From this peculiar space-time, one might zoom in to a null geodesic that corresponds to the $SU(2) \times SU(2)$ limit at hand. This will be done by considering a Penrose limit of the metric. Introducing rescaled coordinates

$$v = R^2\chi, \quad u_4 = R(\theta - \frac{\pi}{4}), \quad r = \frac{R}{2}\rho, \quad x_a = R\phi_a, \quad y_a = R\theta_a, \quad a = 1, 2 \tag{4.2.21}$$

The Penrose limit is realized when the $R \to \infty$ limit is performed which exactly corresponds to zooming in on the null geodesic. The metric will reduce down to a type IIA pp-wave background when the terms have been expanded to first (second) order around the respective coordinates. This reduces the metric

$$\begin{aligned} ds^2 = &\frac{R^2}{4} dt'^2 (4u_4 + 4r^2)\frac{1}{R^2} + \frac{R^2}{4}(4dr^2 + 4r^2 d\hat{\Omega}_2^2)\frac{1}{R^2} \\ &R^2[du_4^2 + dy_1^2 + dx_1^2 + dy_2^2 + dx_2^2 + 2dt'(y_1 dx_1 + y_2 dx_2) + dt' dv]\frac{1}{R^2} \end{aligned} \tag{4.2.22}$$

The full calculation is a little tedious, but essentially it boils down to keeping terms that have a $\frac{1}{R^2}$ dependence such that they don´t vanish in the $R \to \infty$ limit. If we define $r^2 = \sum_{i=1}^3 u_i^2$ and $dr^2 + r^2 d\hat{\Omega}_2^2 = \sum_{i=1}^3 du_i^2$, the metric can compactly be written as

$$ds^2 = dt' dv + \sum_{i=1}^4 (du_i^2 - u_i^2 dt'^2) + \frac{1}{8}\sum_{a=1}^2 (dx_a^2 + dy_a^2 + 2dt' y_a dx_a) \tag{4.2.23}$$

To accommodate the pp-wave on type IIA background, there is a RR-field strength unique to the type of SUGRA theory as we know, which in these coordinates read

$$F_{(2)} = dt' du_4, \quad F_{(4)} = dt' du_1 du_2 du_3 \tag{4.2.24}$$

Compared to the type IIB background that contain 32 supercharges, as we noted the type IIA background only has 24. Proceeding, one picks the gauge

$$t' = c\tau, \quad h_{\alpha,\beta} = \eta_{\alpha,\beta} \tag{4.2.25}$$

Using the Lagrangian in Equation 4.2.6 we find

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^4 \left[(\partial_\tau u_i)^2 - u'^2 + c^2 u_i^2\right] + \frac{c}{8}\sum_{i=1}^2 y_a \partial_\tau x_a + \frac{1}{16}\sum_{i=1}^4 \left[(\partial_\tau x_a)^2 + (\partial_\tau y_a)^2 - x_a'^2 - y_a'^2\right] \tag{4.2.26}$$

The final goal is to obtain the spectrum, so we both need to determine the mode expansions and the light cone Hamiltonian[3] The relevant components at hand are u, x, y. Using the standard Legendre transform and integrating over σ, gives

$$\mathcal{H}_{lc} = \frac{1}{2\pi l_s^2}\int_0^{2\pi} d\sigma \bigg\{ \sum_{i=1}^{4}\left[(\partial_\tau u_i)^2 + u'^2 + c^2 u_i^2\right] + \frac{1}{16}\sum_{i=1}^{4}\left[(\partial_\tau x_a)^2 + (\partial_\tau y_a)^2 + x_a'^2 + y_a'^2\right]\bigg\} \quad (4.2.28)$$

The spectrum can be retrieved by the mode expansion from the fields. It will be convenient to define $z_a(\tau,\sigma) = x_a(\tau,\sigma) + iy_a(\tau,\sigma)$. Finding the EOM for u_i, z is the last piece of the puzzle before assembling. For u_i this is just a plane wave equation $(\partial_\tau^2 - \partial_\sigma^2 - c)u_i = 0$. Doing a standard plane wave ansatz in terms of raising and lowering operators given by

$$u_i(\tau,\sigma) = \frac{i}{\sqrt{2}}\sum_{n\in\mathbb{N}} \frac{1}{\sqrt{\Omega_n}} \hat{a}_n^i e^{-i(\Omega_n\tau - n\sigma)} - (\hat{a}_n^i)^\dagger e^{i(\Omega_n\tau - n\sigma)} \quad (4.2.29)$$

one can by the ansatz find from $(\Omega_n^2 - n^2 - c^2)u_i = 0$ the dispersion $\Omega_n = \sqrt{n^2+c^2}$. This can similarly be done for $z_a(\tau,\sigma)$, this time the EOM just changes along with the dispersion. One finds

$$z_i(\tau,\sigma) = 2\sqrt{2}e^{i\frac{c\tau}{2}}\sum_{n\in\mathbb{N}} \frac{1}{\sqrt{\omega_n}} a_n^a e^{-i(\omega_n\tau - n\sigma)} - (\tilde{a}_n^a)^\dagger e^{i(\omega_n\tau - n\sigma)} \quad (4.2.30)$$

where $\omega_n = \sqrt{\frac{c^2}{4} + n^2}$. The mode expansions contain three types of raising and lowering operators in total. This will establish the following commutation relations between the fields and conjugates and the operators themselves.

$$\begin{gathered}[x_a(\tau,\sigma), p_{x_b}(\tau,\sigma')] = i\delta_{ab}\delta(\sigma-\sigma'), \quad [y_a(\tau,\sigma), p_{y_b}(\tau,\sigma')] = i\delta_{ab}\delta(\sigma-\sigma') \\ [u_i(\tau,\sigma), p_j(\tau,\sigma')] = i\delta_{ij}\delta(\sigma-\sigma')\end{gathered} \quad (4.2.31)$$

$$[\hat{a}_m^i, (\hat{a}_n^j)^\dagger] = i\delta_{mn}\delta_{ij}, \quad [\tilde{a}_m^a, (\tilde{a}_n^b)^\dagger] = i\delta_{mn}\delta_{ab}, \quad [a_m^a, (a_n^b)^\dagger] = i\delta_{mn}\delta_{ab} \quad (4.2.32)$$

Using these relations, one can expand the fields in the Hamiltonian, and expand in terms of dispersions and number operators

$$c\mathcal{H}_{lc} = \sum_{i=1}^{4}\sum_{n\in\mathbb{N}} \sqrt{n^2+c^2}\hat{N}_n^i + \sum_{a=1}^{2}\sum_{n\in\mathbb{N}}\left(\sqrt{\frac{c^2}{4}+n^2} - \frac{c}{2}\right)M_n^a + \sum_{a=1}^{2}\sum_{n\in\mathbb{N}}\left(\sqrt{\frac{c^2}{4}+n^2} + \frac{c}{2}\right)N_n^a \quad (4.2.33)$$

The number operators are defined as $\hat{N}_n^i = (\hat{a}_n^i)^\dagger\hat{a}_n^i$, $M_n^a = (a^a)_n^\dagger a_n^a$,$N_n^a = (\tilde{a}^a)_n^\dagger\tilde{a}_n^a$ and supplemented with the level matching condition

$$\sum_{n\in\mathbb{N}} n\left[\sum_{i=1}^{4}\hat{N}_n^i + \sum_{a=1}^{2}(M_n^a + N_n^a)\right] = 0 \quad (4.2.34)$$

As we did for κ, the constant c can be fixed via the constraint on the term that has been omitted $\frac{c}{2}\partial_\tau v$ coming from $dt'dv$ in the full lagrangian. Using $2\pi l_s^2 p_v = \partial\mathcal{L}/\partial\partial_\tau v$ gives $c = \frac{4l_s^2 J}{R} = \frac{J}{\pi\sqrt{2\lambda}}$, where momentum conservation $\int_0^{2\pi} d\sigma p_\chi = 2J$ was used. Using these pieces, the spectrum reads

$$c\mathcal{H}_{lc} = \sum_{i=1}^{4}\sum_{n\in\mathbb{N}}\sqrt{1+\frac{2\pi^2\lambda}{J^2}n^2}\hat{N}_n^i + \sum_{a=1}^{2}\sum_{n\in\mathbb{N}}\left[\left(\sqrt{\frac{1}{4}+\frac{2\pi^2\lambda}{J^2}n^2} - \frac{1}{2}\right)M_n^a + \left(\sqrt{\frac{1}{4}+\frac{2\pi^2\lambda}{J^2}n^2} + \frac{1}{2}\right)N_n^a\right] \quad (4.2.35)$$

[3] The Hamiltonian can be found via the Legendre transform

$$H_{lc} = \frac{\partial\mathcal{L}}{\partial(\partial_\tau x^\mu)}\partial_\tau x^\mu - \mathcal{L} \quad (4.2.27)$$

We can connect the spectrum, to the spin-chain description we found previously by writing $\Delta = \sqrt{\frac{1}{4} + \frac{\lambda}{2}p^2}, p^2 = 2\pi n/J$, where p is the momentum of the magnons arising. One finds that this is consistent with the sigma-model approach. The spectrum also corresponds to the result presented for BMN operators on a slightly different Penrose limit in [120]. But the dichotomy still stands with the gauge theory side as we saw. To end things off, a way to bridge the two sides, is by looking at a large charge limit $J \to \infty$ where the action

$$I = \frac{J}{16\pi^2\sqrt{2\lambda}l_s^2} \sum_{i=1}^{2} \int dt' d\sigma \Big[y_i \partial_{t'} x_i - \frac{\pi^2 \lambda}{J^2}(x_i'^2 + y_i'^2)\Big] \tag{4.2.36}$$

The same structure will also appear in a different context of non-relativistic strings as we will see later on

4.2.3 New sector: $OSp(2|2)$ Sigma-Model limit

Picking up where we left the $SU(2) \times SU(2)$ sector, it should not be difficult to extend the BPS bound such that we have a spin DOF included. One can use the same geometry constructed as in the previous case; we only consider extra effects from $S^2 \subset AdS_4$. To zoom in, we take $\theta = \pi/2$ and $\xi = \pi/4$. This should correspond to a null geodesic in our sector. With this in mind, we can write up the type IIA background for our 10-Dimensional metric

$$ds^2/R^2 = -\frac{1}{4}(-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Phi^2) + [\frac{1}{8}d\Omega_2^2 + \frac{1}{8}d\Omega_2'^2 + (d\delta + \omega)^2] \tag{4.2.37}$$

Considering the same coordinates as before, we only need to adjust for our spin-parameter. This is just the polar angle on the S^2 that transforms as $\phi = \Phi - t$. The Hamiltonian and $-i\partial_\chi$ are the same, but we include an additional parameter for the new polar angle

$$\tilde{H} = J^2(\Delta - J - S), \quad 2J = -i\partial_\chi, \quad S = -i\partial_\phi \tag{4.2.38}$$

We still want to preserve the $1/J^2$ scaling to be able to see the magnon states in the $J \to \infty$ limit. We see that the extra zoom is happening around $\Phi = t$ which is a null geodesic in the metric. This probably corresponds on the field theory side to a chiral primary where $\Delta = S$, such that we get a connection to $\tilde{H}$. Employing the coordinate we write the type IIA metric as

$$ds^2 = R^2 \left[\frac{1}{4}(d\rho^2 + \sinh^2 \rho d\phi(2J^2 d\tilde{t} + d\phi)) + (J^2 d\tilde{t} + d\chi + \omega)(d\chi + \omega) + \frac{1}{8}d\Omega_2^2 + \frac{1}{8}d\Omega_2'^2\right] \tag{4.2.39}$$

If we consider the same Sigma-model Lagrangian and gauge as for the other BPS sector, with the addition of $p_\phi = \text{const}$ with $2\pi l_s^2 p_\phi = \partial\mathcal{L}/\partial\partial_\tau\phi$ Then one can find the Lagrangian to be

$$\begin{aligned}\frac{2}{R^2}\mathcal{L} = (\kappa J^2 + \partial_\tau \chi + \omega_\tau)(\partial_\tau \chi + \omega_\tau) - (\chi' + \omega_\sigma)^2 + \frac{1}{8}\sum_{i=1}^{2}(\partial_\tau \theta_i)^2 - \theta_i^{'2} + \cos^2\theta_i[(\partial_\tau \phi_i)^2 - \phi_i^{'2}] \\ + (\partial_\tau \rho)^2 - \rho_i^{'2} + \sinh^2 \rho[(\partial_\tau \phi)^2 - (\phi')^2 + 2\kappa J^2 \partial_\tau \phi]\end{aligned} \tag{4.2.40}$$

The Virasoro constraints can be directly copied and used again, this time we only need to add the AdS-part to the diagonal and off-diagonal pieces

$$\begin{aligned}(\kappa J^2 + \partial_\tau \chi + \omega_\tau)(\partial_\tau \chi + \omega_\tau) + (\chi' + \omega_\sigma)^2 + \frac{1}{8}\sum_{i=1}^{2}(\partial_\tau \theta_i)^2 + (\theta_i')^2 + \cos^2 \theta_i[(\partial_\tau \phi_i)^2 + \phi_i'^2] \\ + (\partial_\tau \rho)^2 + (\rho_i')^2 + \sinh^2 \rho[(\partial_\tau \phi)^2 + (\phi')^2 + 2\kappa J^2 \partial_\tau \phi] = 0\end{aligned} \tag{4.2.41}$$

$$(\kappa J^2+\partial_\tau\chi+\omega_\tau)(\chi'+\omega_\sigma)+\frac{1}{8}\sum_{i=1}^{2}[\partial_\tau\theta_i\theta_i'+\cos^2\theta_i\partial_\tau\phi_i\phi_i']+(\partial_\tau\rho)\rho_i'+\sinh^2\rho[(\partial_\tau\phi)\phi'+2\kappa J^2\phi']=0 \tag{4.2.42}$$

Using the standard relation for κ found with p_χ, we can reduce the Lagrangian and constraints to the following

$$\begin{aligned}\frac{2}{R^2}\mathcal{L}=\frac{16l_s^4}{R^4}(\dot{\chi}+\omega_\tau)-(\chi'+\omega_\sigma)^2-\frac{1}{8}\sum_{i=1}^{2}[\theta_i^{'2}+\cos^2\theta_i\phi_i^{'2}]-(\rho')^2-\sinh^2\rho(\phi')^2+2\kappa^2J^2\sinh^2\rho\dot{\phi}\\ \frac{16l_s^4}{R^4}(\dot{\chi}+\omega_\tau)+\frac{1}{8}\sum_{i=1}^{2}[\theta_i^{'2}+\cos^2\theta_i\phi_i^{'2}]+(\rho')^2+\sinh^2\rho(\phi')^2+2\kappa^2J^2\sinh^2\rho\dot{\phi}=0\\ \chi'+\omega_\sigma+\frac{8l_s^2}{R^2}\sinh^2\rho\phi'=0\end{aligned} \tag{4.2.43}$$

The gauge fixed Lagrangian, using the constraints, is now available to us

$$\frac{2}{R^2}\mathcal{L}=\omega_{\tilde{t}}-\pi^2\lambda\sum_{i=1}^{2}[\theta_i^{'2}+\cos^2\theta_i\phi_i^{'2}]-(\rho')^2-\sinh^2\rho(1+\pi^2\lambda\sinh^2\rho)(\phi')^2+2\kappa^2J^2\sinh^2\rho\dot{\phi} \tag{4.2.44}$$

The construction of the action is the same as for the sigma-model in the $J\to\infty$

$$I=\frac{J}{4\pi}\int d\tilde{t}\int_0^{2\pi}d\sigma\Big[\sum_{i=1}^{2}[\sin\theta_i\dot{\phi}-\pi^2\lambda(\theta_i'^2+\cos^2\theta_i\phi_i'^2)]-(\rho')^2+\frac{1}{2}\sinh^2\rho\dot{\phi}-\sinh^2\rho(1+\pi^2\lambda\sinh^2\rho)(\phi')^2\Big] \tag{4.2.45}$$

The Action, as it can be seen, contains the previous Landau-Lifshitz model as expected, but contains Wess-Zumino like terms given by hyperbolic coordinates from AdS. This Structure is reminiscent of the coherent state representation of an integrable spin chain Hamiltonian with symmetry group $SL(2,\mathbb{R})$, which was also found in a sigma model limit on $AdS_5\times S^5$[27]. As for the $SU(2)\times SU(2)$ sector, this action will emerge once again in the spin matrix context of non-relativistic backgrounds

4.3 Partition functions, Decoupling, and Hagedorn Temperature in AdS/CFT

On a thermodynamic level, systems in AdS/CFT can be regarded in finite temperature regimes where confined theories undergo a phase transition to deconfinment states. Over the last two and a half decades, intensive studies of this phenomenon have been performed with great advancement. Usually, this is analyzed via free theory partition functions, since strong coupling regimes break pertubative methods, making it harder to extract exact results. Through a logarithmic partition function, Hagedorn temperatures occur at which point, in the standard setting[73], hadronic matter turns into a quark-gluon plasma. This motivates us to study the same effect in the context of the bulk scalars and fields of AdS/CFT. This is well established for $\mathcal{N}=4$ SYM where numerous authors have contributed to the thermodynamic aspect [4, 139, 136, 146]. The focus at hand will lie in applying the framework used in [80, 78, 74, 77, 76, 75]. The spirit is to establish this for ABJM, since work on this was last done more than 15 years ago in this specific context.

4.3.1 Subsectors and Decoupling Limits for $\mathcal{N}=6$ Chern-Simons

Following the discussion from [80, 79], and defining the t´Hooft coupling to be $\lambda=\frac{N}{k}$, it can be established as was done for SYM, how decoupling limits emerge for $\mathcal{N}=6$ CS. Firstly; the type of operators that are present on S^2 are gauge singlets. This gives all possible linear combinations of the multi-trace operators for the alternating fundamental and anti-fundamental structure of the operators

$$\prod_{i=1}^{k} \mathrm{Tr}(A_i^{(i)} B_i^{(i)} ... A_{L_{k-1}}^{(i)} B_{L_k}^{(i)}) \tag{4.3.1}$$

The quantum numbers associated are found in 2.3.40. If we wish to construct the partition function in the grand canonical ensemble for $U(N) \times U(\bar{N})$ (or $SU(N) \times SU(\bar{N})$ we can write

$$Z_{\lambda,N}(\beta, \omega, \Omega_1, \Omega_2, \Omega_3) = \mathrm{Tr}_M[\exp(-\beta D + \beta\omega S + \beta \sum_{i=1}^{3} \Omega_i J_i)] \tag{4.3.2}$$

The factors in front of generators will be chemical potentials analogous to the construction in statistical mechanics. To get to the point of decoupled theories, we set the chemical potentials to be equal to the same parameter such that $(\omega, \Omega_1, \Omega_2, \Omega_3) = \Omega(n_1, n_2, n_3, n_4)$, here n_i are real numbers and $\Omega \in [0, 1]$ (this can be done in the language of J_i as well). For $\Omega \to 1$ we approach critical values of the set of chemical potentials. Employing this and defining a charge $J = n_1 S + n_2 J_1 + n_3 J_2 + n_4 J_3$, we write

$$Z_{\lambda,N}(\beta, \omega_1, \omega_1, \Omega_1, \Omega_2, \Omega_3) = \mathrm{Tr}_M[e^{-\beta\Delta + \beta\Omega J}] = \mathrm{Tr}_M[e^{-\beta(\Delta - J) - \beta(1-\Omega)J}] \tag{4.3.3}$$

It becomes clear that in the limit, a decoupled theory appears, where the contribution purely comes from $\Delta - J$. In general, the dilatation operator can be expanded in powers for small λ such that $\Delta = \Delta_0 + \lambda\Delta_2 + \lambda^{\frac{3}{2}}\Delta_3 + ...$ The coupling only enters through the dilatation, thus for each term we take into account, e another loop-order is considered. We will look at two cases

$$\begin{aligned} Z_{\lambda=0,N}(\beta, \omega_1, \omega_1, \Omega_1, \Omega_2, \Omega_3) &= \mathrm{Tr}_M[e^{-\beta(D_0 - J) - \beta(1-\Omega)J}] \\ Z_{\lambda,N}(\beta, \omega_1, \omega_1, \Omega_1, \Omega_2, \Omega_3) &= \mathrm{Tr}_M[e^{-\beta(D_0 - J) - \beta\lambda D_2 - \beta(1-\Omega)J + \beta\mathcal{O}(\lambda^{\frac{3}{2}})}] \end{aligned} \tag{4.3.4}$$

In the case of no interactions, we restrict to choices such that $\Delta_0 \geq J$ from the choices of (n_1, n_2, n_3, n_4). Letting $\beta \to \infty$ all states with our chosen condition will decouple from the partition function. Avoiding total decoupling, it is demanded that our choice of the integers or half-integers must satisfy, that some states obey $\Delta_0 = J$. To get a non-trivial partition function, we keep $\beta(1 - \Omega)$ fixed in the $\beta \to \infty$ limit. This enables us to write $Z_N(\tilde{\beta}) = \mathrm{Tr}_M[e^{-\tilde{\beta} D_0}], \tilde{\beta} = \beta(1-\Omega)$. Considering $\mathcal{N} = 6$ Chern-simons, D_2 corresponds to the two-loop[147, 109] dilatation operator, while still demanding the same from the $\lambda = 0$ case. To get a non-trivial interaction the only term needed is $\beta\lambda$. Hence, in the $\beta \to \infty$ limit we find

$$\beta \to \infty, \quad \tilde{\beta} = \beta(1 - \Omega)\text{fixed}, \quad \tilde{\lambda} = \frac{\lambda}{1 - \Omega}\text{fixed}, \quad N \quad \text{fixed} \tag{4.3.5}$$

This will give us $Z_N(\tilde{\beta}) = \mathrm{Tr}_M[e^{-\tilde{\beta} D_0 + \tilde{\lambda} D_2}]$, bringing us close to the zero temperature, $\Omega = 1$ and zero coupling. The following remarks may be good to sum up now. Higher loop terms for $n \geq 3$ in the dilatation operator will be negligible in the considered limit. Further, no assumption on N has been made. This indicates that it works out in finite cases, and $Z_N(\tilde{\beta})$ will in the decoupled theory depend on $\tilde{\lambda}, N, \tilde{\beta}$. Lastly from our choices of n_i this will mean that $(T, \Omega) = (0, 1)$ is a critical point or rather $(T, \omega, \Omega_1, \Omega_2, \Omega_3) = (0, n_1, n_2, n_3, n_4,)$

4.3.2 Partition function for $\mathcal{N} = 6$ Chern-Simons and Hagedorn temperatures in free theories

As it stands, Thermal partition functions have been considered in the literature for ABJM. We review and state known results at tree level and two-loops with no chemical potentials present.

Other authors have via exotic techniques been able to obtain compact expressions for the ABJM partition function [91, 129, 38]. What we face, is the case where Z turns on chemical potentials, making it possible to derive the letter partition function such that it agrees with the result in [47].

Tree level and two-loop with no chemical potential for $\mathcal{N} = 6$ Chern-Simons

The tree-level Hagedorn was first found in [120]. Considering the weak coupling limit $k \to \infty$ on $S^1 \times S^2$, one proceeds by writing the free action of two copies of Chern-Simons terms. The partition function can be expressed as a matrix model in a plethystic exponential

$$Z = \int [dU][dV] \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} z_n(x)(\mathrm{tr}U^n \mathrm{tr}V^{-n} + \mathrm{tr}U^{-n}\mathrm{tr}V^n)\right] \tag{4.3.6}$$

The single particle partition function $z_n(x) = 4z_B(x^n) + (-)^{n+1}4z_F(x^n)$ is given by bosonic and fermionic single particle versions

$$z_F(x^n) = \frac{2x}{(1-x)^2}, \quad z_B(x^n) = \frac{x^{1/2}(1+x)}{(1-x)^2} \tag{4.3.7}$$

The temperture enters as $x = e^{-\beta}$,. In the large-N limit it is stated that the Hagedorn singularity can be found by $z_1(x) = 1$. This gives

$$\frac{4(2x + x^{1/2}(1+x))}{(1-x)^2} = 1 \quad \leftrightarrow \quad x = \frac{1}{17+12\sqrt{2}} \quad \leftrightarrow \quad T_H = \frac{1}{\log\left(17+12\sqrt{2}\right)} \approx 0.283648 \tag{4.3.8}$$

This is the tree-level Hagedorn temperature T_H. Extending on this, one must consider loop-corrections in the dilatation operator at two-loops. It was shown in [123] how to obtain an expression for $< D_2(x) >$, which was found to be

$$< D_2(x) >= 8\sqrt{x}\frac{(1+\sqrt{x})^2}{(1-\sqrt{x})^6}\left[\sqrt{x} + x + (1 - 6\sqrt{x} + x)\log\left(1-\sqrt{x}\right)\right] \tag{4.3.9}$$

With this result, and taking the limit $\lambda \to 0$ and $N \to \infty$, the partition function $\mathcal{Z}$ of ABJM on S^2 is given by a simple expression

$$\frac{\delta T_H}{T_H} = \frac{\lambda^2}{\sqrt{2}} < D_2(x_h) >= 2\lambda^2(\sqrt{2}-1) \tag{4.3.10}$$

Hagedorn has also been considered on the string side, where pertubation theory and the use of Quantum spectral curve (QSC) has proven useful[4], in the context of ABJM, where corrections to higher loops were computed [52].

Constructing partition function with added chemical potential for $\mathcal{N} = 6$ Chern-Simons

Going beyond what has already been done, consider a partition function in the grand canonical ensemble with chemical potentials turned on. Following [47] we look at how characters give the emergence of the single particle free field partition function. The starting point is Superconformal Characters with $SO(2N)$ R-symmetry for ABJM. It is specified to work in $N = 3$ such that characters for unitary irreducible representations of the conformal group are in three dimensions

[4] See [83, 82] for application to Hagedorn for SYM on $AdS_5 \times S^5$

$SO(3,2)$ and have $SO(6)$ manifest R-symmetry. For arbitrary N this can compactly be written as

$$\chi^{(N,B,\pm)}_{(\frac{1}{2};0;\frac{1}{2},\ldots\frac{1}{2},\pm\frac{1}{2})}(s,x,y) = \mathcal{D}_{\text{rac}}(s,x)\chi^{(N)}_{(\frac{1}{2};0;\frac{1}{2},\ldots\frac{1}{2},\pm\frac{1}{2})}(y) + \mathcal{D}_{\text{Di}}(s,x)\chi^{(N)}_{(\frac{1}{2};0;\frac{1}{2},\ldots\frac{1}{2},\mp\frac{1}{2})}(y) \tag{4.3.11}$$

This is composed of characters for the free field representations of $SO(3,2)$, and the Di and Rac, singleton representations. The terms can be expressed as

$$\mathcal{D}_{\text{Rac}}(s,x) = \frac{s+s^3}{(1-s^2x^2)(1-s^2x^{-2})}, \quad \mathcal{D}_{\text{Di}}(s,x) = \frac{s^2(x+x^{-1})}{(1-s^2x^2)(1-s^2x^{-2})} \tag{4.3.12}$$

The expression for the characters of the 3d conformal representation turns out to be 1/2- BPS. For ABJM, we restrict to $N=3$. The field content is already known to us, but it is worth noting how scalars and fermions transform in the $\pm$ states of $(3,B,\pm)$, which can be seen in the table.

Field	Δ	$SO(3,2)$ rep.	$SO(6)$ rep.	$U(n)\times U(n)$ rep.
ϕ_1	$\frac{1}{2}$	Rac	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$(\mathbf{N},\bar{\mathbf{N}})$
ψ_1	1	Di	$(\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	$(\mathbf{N},\bar{\mathbf{N}})$
ϕ_2	$\frac{1}{2}$	Rac	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$(\bar{\mathbf{N}},\mathbf{N})$
ψ_2	1	Di	$(\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	$(\bar{\mathbf{N}},\mathbf{N})$

The character we consider is a trace of what will be interpreted as fugacities to the power of Cartan generators. For free field theory, the single particle partition function is then given by

$$z_{\text{Free}}(s,x,y,u,v) = \text{Tr}(s^{2D}x^{2J_3}y_1^{H_1}y_2^{H_2}y_3^{H_3}u_1^{L_1}...u_n^{L_n}v_1^{M_1}...v_n^{M_n}) \tag{4.3.13}$$

In the context of the letter partition function, contributions from u,v will not be considered. The trace can be expressed in terms of a function given by the characters for $(3,B,\pm)$

$$z_{\text{Free}}(s,x,y) = f_+(s,x,y) + f_-(s,x,y) \tag{4.3.14}$$

where

$$\begin{aligned} f_\pm(s,x,y) &= \chi^{(N,B,\pm)}_{(\frac{1}{2};0;\frac{1}{2},\ldots\frac{1}{2},\pm\frac{1}{2})}(s,x,y) \\ &= (y_1y_2y_3)^{\mp1/2}\Big(\sum_{i=1}^{3} y_i^{\pm1} + (y_1y_2y_3)^{\pm1}\Big)\mathcal{D}_{\text{Rac}}(s,x) \\ &+ (y_1y_2y_3)^{\pm1/2}\Big(\sum_{i=1}^{3} y_i^{\mp1} + (y_1y_2y_3)^{\mp1}\Big)\mathcal{D}_{\text{Di}}(s,x) \end{aligned} \tag{4.3.15}$$

The expression for the partition function can be factorized nicely which will motivate what terms to include in the letter partition function later on. In the latter, we will make the substitution to new letters, such that we are in accordance with [80], giving $(s,x,y)\to(x,\rho,y)$

4.3.3 Letter Partition Function for $\mathcal{N}=4$ SYM on $\mathbb{R}\times S^3$

To motivate the following calculation, it might be insightful to first state what is known. Consider the partition function for N = 4 SYM on $\mathbb{R}\times S^3$ in the presence of non-zero chemical potential for R-charges of the $SU(4)$ R-symmetry and the Cartan generators of $SO(4)$ group on S^3. A method to attack the problem is by spherical harmonic expansion where we expand each field and associate its spherical harmonic to it. There are 3 parts of this calculation before we can add everything up. Partition function for scalars, vectors and fermions has to be considered. It is worth noting that one does not do this for Cartan generators of $SO(4)$ but splits it up into $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. The detailed derivation can be found in Appendix B. Using the letter representations that dictate the spherical harmonics on S^3, will contribute to the overall letter partition function which in all its glory is

$$\begin{aligned} z(x,\omega_j,y_i) = & \prod_{k=1}^{2}((1-xe^{\beta\omega_k})(1-xe^{-\beta\omega_k}))^{-1}\{(x-x^3)\sum_{l=1}^{3}(y_l+y_l^{-1}) \\ + \quad & 2x^2\Big[1+2\cosh(\beta\omega_1)\cosh(\beta\omega_2)-x(\cosh(\beta\omega_1)+\cosh(\beta\omega_1))+x^2\Big] \\ + \quad & 2Y_1x^{\frac{3}{2}}\left[\cosh[\beta(\frac{\omega_1-\omega_2}{2}]-x\cosh[\beta(\frac{\omega_1+\omega_2}{2})]\right] \\ + \quad & 2Y_2x^{\frac{3}{2}}\left[\cosh[\beta(\frac{\omega_1+\omega_2}{2}]-x\cosh[\beta(\frac{\omega_1-\omega_2}{2})]\right]\Bigg\} \end{aligned} \tag{4.3.16}$$

4.3.4 Letter Partition Function for $\mathcal{N}=6$ Chern-Simons on $\mathbb{R}\times S^2$

We proceed to compute the letter partition function and derive the same expression as 4.3.14. Considering the Symmetries, we include the $SO(3)$ group on S^2 with an effective spin S. Otherwise, one obtains the same as for SYM. The exponential parameters involving temperature, spin and R-symmetry can be defined as

$$x=e^{-\beta}, \quad \rho=e^{\beta\omega}, \quad y_i=e^{-\beta\Omega_i}, i=1,2,3 \tag{4.3.17}$$

We proceed to motivate and argue for the structure of R-symmetry generators and then use it when computing the different contributions from scalars and fermions

R-symmetry Chemical potential and structure

To motivate why given a non-abelian group, one introduces chemical potentials to the maximal abelian subgroup[146, 100]. Consider a system with a Hamiltonian $\mathcal{H}$ and internal symmetry group G. Assuming G is a semi-simple and compact lie group makes it possible to write the elements as exponentials. Defining a unitary operator $U(g)$ gives a way of defining the partition function $z(\beta,g)=\mathrm{Tr}[U(g)e^{-\beta\mathcal{H}}]$. Using group theoretical arguments, one can choose an element h of the maximal abelian subgroup which in return can be written as the exponential of a sum of generators of a Cartan subalgebra $h=e^{i\gamma_p Q_p}, p\in\{1,...,\mathrm{rank}(\mathcal{G})\}$ where Q are the generators. Doing analytic continuation $\gamma_p\to -i\beta\mu_p$, a more familiar Grand Canonical partition function appears

$$z(\beta,\mu_p)=\mathrm{Tr}\exp[-\beta(\mathcal{H}-Q_p\mu_p)] \tag{4.3.18}$$

with chemical potentials μ_p associated with a maximal set of commuting conserved charges Q_p. Hence, given a non-Abelian symmetry group, introducing chemical potentials corresponds to a Cartan subalgebra of the group. Having motivated the need for R-symmetry chemical potentials, the structure is needed to implement it.

As for $\mathcal{N}=4$ SYM, the structure of the Generators of R-symmetry can vary in how they couple to fields[78]. Most notable is the SYM example, where scalars transform in the **6** and $\bar{\mathbf{6}}$ of $SU(4)$ and fermions transform in **4**. The consequence is presented explicitly in the partition function[5]. But this problem does not occur in $\mathcal{N}=6$ Chern-Simons, since both fermions and scalars transform in the same (or dual) representation. Thus, the structure is unchanged. The

[5] For characters in the **6** representation one finds $W_{[0,1,0]}=\sum_{i=1}^{3}(y_i+y_i^{-1})$ and for $(\mathbf{4},\bar{\mathbf{4}})$ one finds $W_{[1,0,1]}=\prod_{i=1}^{3}(y_i^{1/2}+y_i^{-1/2})$ [78]

effective potentials defined in [146], can explicitly be realized through $S^7/\mathbb{Z}_k$. The orbifolding imposes a quantization condition on the R-symmetry charges (expressed in terms of angular momenta) such that the quantization condition holds[64]

$$\sum_{i=1}^{4} J_i \in k\mathbb{Z} \tag{4.3.19}$$

Introducing three charges generated by the angles ϕ_i that further generate J_i, given by $R_j = -i\partial_{\eta_j}$, this is immediately recognized as the three Cartan generators for the $SU(4)$ subgroup of $SO(8)$ which is know to be dual to the R-symmetry of ABJM. This allows us to write geometrical angles as effective chemical potentials with the extra constraint that $\sum_{i=1}^{4} J_i = 0$ as linear combinations

$$J_1 = (R_1 + R_2 + R_3), \quad J_2 = (R_1 - R_2 - R_3), \quad J_3 = (-R_1 + R_2 - R_3), \quad J_4 = (-R_1 - R_2 + R_3) \tag{4.3.20}$$

Conversely, one can also express R_i in terms of J_i. This leads to expressions for the fermionic partition function to have the R-symmetry structure of

$$\prod_{i=1}^{3} (y_i^{1/2} + y_i^{-1/2}) \tag{4.3.21}$$

This will not follow automatically in the letter partition function, but via factorization, it can be shown that the added contribution will have this form.

Partition function for scalars and fermions

To compute the Letter Partition Function, spherical harmonic expansion is used for each field. For SYM this was done on S^3, where the authors split $SO(4) = SU(2)_L \times SU(2)_R$. Since we operate on S^2, there is no trivial decomposition as before. But we do note that the spin index will change from integers working with scalars, to half-integers for fermions. The usual spherical harmonics should be considered Y_{jm} with appropriately replaced quantum numbers. The representations are denoted as $[r, p, q]_s$ for $SU(4)$ and $SO(3)$. Since only one ρ is present, the latter case simplifies to one sum where $m \in \{-j, j\}$. As argued in [80], the R-charge chemical potential in $[0, 0, 1]$ or rather $\bar{\mathbf{4}}$ is given by

$$Y_1 = (y_1 y_2 y_3)^{1/2} + y_1^{1/2}(y_2 y_3)^{-1/2} + y_2^{1/2}(y_1 y_3)^{-1/2} + y_3^{1/2}(y_1 y_2)^{-1/2} \tag{4.3.22}$$

Using this, we compute the scalar partition function

$$\eta_s(x, \rho, y_i) = Y_1 \sum_{j=0}^{\infty} \sum_{m=-j}^{j} x^{2(j+1/2)} \rho^{2m} \tag{4.3.23}$$

The factor of 2 follows the convention of [47], to match the results. Evaluating the sum one finds

$$\eta_s(x, \rho, y_i) = Y_1 \frac{x + x^3}{(1 - x^2\rho^2)(1 - x^2\rho^{-2})} \tag{4.3.24}$$

Turning to the fermions, the sum essentially stays the same besides the half-integer contribution giving us

$$\eta_f(x, \rho, y_i) = Y_1 \sum_{j \in \mathbb{Z}+1/2}^{\infty} \sum_{m=-j}^{j} x^{2(j+1/2)} \rho^{2m} \tag{4.3.25}$$

The numerator changes to have an explicit ρ dependence in the following fashion

$$\eta_f(x,\rho,y_i) = Y_1 \frac{x^2(\rho+\rho^{-1})}{(1-x^2\rho^2)(1-x^2\rho^{-2})} \tag{4.3.26}$$

This takes care of the contributions from $[0,0,1]_{1/2}$. Luckily the other case is an exact replication, besides substituting the expression for $Y_1 \to Y_2$, which is defined as for SYM. Putting this together the full letter partition function seems to agree with what we aimed for

$$z(x,\rho,y_i) = \left(\frac{x+x^3}{(1-x^2\rho^2)(1-x^2\rho^{-2})} + \frac{x^2(\rho+\rho^{-1})}{(1-x^2\rho^2)(1-x^2\rho^{-2})}\right) \times (Y_1+Y_2) \tag{4.3.27}$$

Using the fugacities defined above, we can put the partition function in the form of the cartan Generators explicitly.

$$z(x,\rho,y_i) = \frac{\cosh(\beta)+\cosh(\beta\omega)}{\cosh(2\beta)-\cosh(2\beta\omega)} \times (Y_1+Y_2) = \prod_{i=1}^{3} \frac{\cosh\left(\frac{\beta(1+\omega)}{2}\right)\cosh\left(\frac{\beta(1-\omega)}{2}\right)}{\sinh(\beta[1+\omega])\sinh(\beta[1-\omega])} \cosh\left(\frac{\beta\Omega_i}{2}\right) \tag{4.3.28}$$

The partition function presents itself in a nice formula with ordinary functions, explicitly depending on the chemical potentials. One interesting aspect to consider, compared to SYM on $\mathbb{R}\times S^3$, is that no monopole operators appeared in the theory. By inclusion of the baryonic $U(1)_b$, one gets a charge generated by the gauge transformations at the boundary, which gives rise to quantized field strengths. For ABJM this means there exist a non-zero magnetic flux on S^2 surrounding the insertion point of the operator. This is exactly the monopole operators appearing. In [93], it was shown through quantization of the field strengths and the use of Gauss constraint, that an additional term in $f_\pm$ proportional to $x^{|n_i-\tilde{n}_j|}$ appears. But this is accompanied by monopole spherical harmonics, which seems to be a different beast than the usual one when writing and evaluating the sums. This could be interesting for future investigation.

Hagedorn temperature: No Chemical Potentials:

For good sport we look at the easy case which is when the chemical potentials are turned off, which corresponds to $\omega = \Omega_i = 0$, and is just the essential same calculation as was done in [120]. This gives a polynomial where T_H can be extracted

$$\frac{8(x+2x^2+x^3)}{(1-x^2)^2} = 1 \to x = 5 \pm 2\sqrt{6} \tag{4.3.29}$$

Choosing the minimal solution, the Hagedorn temperature is calculated to be $T_H = \frac{-1}{\log\left(5-2\sqrt{6}\right)} \approx 0.436218$. It is worth noting though that the same expression has been derived in an altered form. In [31, 32], the partition functions were found to contain an extra t-parameter which can be interpreted as R-symmetry chemical potentials. The authors set the parameters to $t \to 1/x$ and $y \to -1$ where $y = \rho$. This is the same case as we just considered, but with the identifications, the partition function boils down to

$$z(x) = \frac{x^{1/2}+x^{3/2}}{1-x^2} = \frac{x^{1/2}}{1+x} \tag{4.3.30}$$

One can similarly find the Hagedorn temperature to be $T_H = \frac{-3}{2\log(-1)} \approx 0.477464$.

As it has been advertised, there are various cases to consider where we turn on different chemical potentials and make numerical solutions of dependence on either $\{\omega, \Omega_i\}$ or both simultaneously. Further one can investigate how the partition functions might look for the decoupled sectors, all inspired by [80, 78].

4.4 Spin Matrix String Backgrounds

Using SMT, a novel way of evaluating near BPS-bounds in the AdS/CFT correspondence has been developed. It has already been established for SYM with super lie algebra $PSU(2|2,4)$ that in certain BPS-bounds, one can establish connections to spin chains as an example. This relates to non-relativistic string theory with a non-relativistic target space described by a $U(1)$ Galilean geometries. One direct way to observe it is to look at magnon dispersions [79] which exhibit non-relativistic features in the SMT-limit. The starting point is the Torsional Newton Cartan (TNC) string which was used to derive SMT strings. We briefly review the non-relativistic approach and establish a manifold that will be useful when going to BPS-bounds that describe an emerging $U(1)$ geometry. Calculations for $\mathcal{N}=4$ which has been established are done in A. The extension for geometries dual to $\mathcal{N}=6$ Chern-Simons theory is done here. We look at the spin subgroups of OSp(4|6) in certain limits which we will use to parametrize and isolate specific isometries of $\mathrm{AdS}_4 \times \mathbb{CP}^3$.

4.4.1 Brief review of TNC strings and BPS-bounds in SMT limit

The starting point is to consider a relativistic string that couples to a non-relativistic TNC geometry. To this mean, it will be convenient to consider a $(d+1)$-dimensional Lorentzian geometry with null isometry ∂_u as

$$ds^2 = 2\tau_\mu dx^\mu(du - m_\mu dx^\mu) + h_{\mu\nu}dx^\mu dx^\nu \tag{4.4.1}$$

By our null reduction along u, we see an emerging Torsional Newton-Cartan (TNC) geometry characterized by a clock one-form τ_μ, a symmetric tensor $h_{\mu\nu}$ of rank $d-1$ and the $U(1)$ connection m_μ. Without going into great detail, one can write up gauge transformations for the TNC data which makes the decomposition non-unique. For further elaboration see [83]. The important thing to note is that this will make the Galilean boost and $U(1)$ transformations visible. The trick that was considered was to make the constant momentum P_u off-shell by exchanging a single winding mode in a direction η dual to u. But the question is now how to ensure that when we pick a specific BPS-bound that u will be null on the background geometry. In the SMT-limit [83] it was established that introducing the following new coordinates one could ensure that we face no trouble. Consider the BPS-bound

$$g_s = 0, \quad N = \text{fixed}, \quad \frac{E-Q}{g_s} = \text{fixed} \tag{4.4.2}$$

Depending on the specific duality one can define Q in various ways. ABJM and SYM align in terms of Cartan generators considered in $Q = S + J$. From the time coordinate one can extract that $E = i\partial_t$, $S = i\partial_{\bar{\gamma}}$ and $J = i\partial_\gamma$. Making a coordinate change will give a non-relativistic string on the world sheet. Introduce x_0 and u such that

$$i\partial_{x_0} = E - Q = E - S - J, \quad -i\partial_u = \frac{1}{2}(E - S + J) \tag{4.4.3}$$

Then one can rescale x_0 such that the conserved charge scales as g_s when the limit $g_s \to 0$. Introducing $x_0 = \frac{\tilde{x_0}}{4\pi g_s N}$ one obtains in the SMT limit

$$c \to \infty, \quad x_0 = c^2 \tilde{x_0}, \quad c = \frac{1}{4\pi g_s N}, \quad \text{N and } \tilde{x_0} \text{ fixed} \tag{4.4.4}$$

Using these coordinates, when taking a certain BPS-bound, transforms our global coordinates of the geometry to something depending on x_0 and u. Using the coordinates 2.3.34 and the isometries 2.3.40, it is convenient to define, with the global AdS_4 time t, transformations to $\{x^0, u, w\}$[6] such that 4.4.3 is satisfied

$$\begin{pmatrix} t \\ \bar{\gamma} \\ \gamma \end{pmatrix} = \begin{pmatrix} 1 & -1/2 & 0 \\ 1 & -1/2 & 1 \\ 1 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} x^0 \\ u \\ w \end{pmatrix} \quad \rightarrow \quad \begin{aligned} &-i\partial_{x_0} = E - Q = E - S - J, \\ &-i\partial_u = \frac{1}{2}(E - S + J), \quad -i\partial_w = S \end{aligned} \tag{4.4.5}$$

Lastly, a Polyakov action also exists for the SMT-string [83]. In the context we operate in, it will be convenient to write it in a flat worldsheet gauge. Following the original work, the flat worldsheet gauge-fixed SMT string action was found to be

$$S_{\text{flat,gf}} = -\frac{J}{2\pi} \int d^2\sigma (m_\mu x^\mu + \frac{1}{2} h_{\mu\nu} dx^\mu dx^\nu) \tag{4.4.6}$$

This will be enough to proceed and analyze various BPS-sectors to find reduced geometries on non-relativistic backgrounds

4.4.2 SMT-limits of $\mathcal{N} = 6$ Chern-Simons theory and ABJM

As for the $\mathcal{N} = 4$ SYM, we set out to do the same computations, this time in the context of ABJM theory, where one considers a $\mathrm{AdS_4 \times S^7/\mathbb{Z}_k}$ or rather $\mathrm{AdS_4} \times \mathbb{CP}^3$ geometry for large k. Thus the starting point is to establish a general metric depending on the Cartan generators. The same formalism will be used in this correspondence where we instead now have a S^7 which is associated to the angular momentum $J_j = -i\partial_{\alpha_j}$ and also $S^2 \subset AdS_4$ which associates to spin $S = -i\partial_\phi$. In contrast to SYM, we only have one spin degree of freedom from the AdS part, but a further addition of angular momentum DOF. We proceed by first analyzing the "simplest case" namely the $SU(2) \times SU(2)$ sector.

4.4.3 The $SU(2) \times SU(2)$ background and penrose-limit

We start by considering a doubling of the cousin from SYM, namely the $SU(2) \times SU(2)$ case. ABJM has the same BPS-bound as for SYM $Q = J_1 + J_2$, which is purely constituted by the S^7 part of the metric. From previous calculations and review, our starting point is writing S^7 as two S^3´s and then further to two S^2[64]. This reduced geometry is a type IIA background obtained via M-theory relations considered in 4.2.1

$$\begin{aligned} ds^2/R^2 = -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\hat{\Omega}_2^2 + ds^2_{\mathbb{CP}^3} = -\cosh^2\rho dt^2 + d\rho^2 \\ \sinh^2\rho d\hat{\Omega}_2^2 + d\theta^2 + \frac{1}{4}(\cos^2\theta d\Omega_2^2 + \sin^2\theta d\Omega_2'^2) + 4\cos^2\theta\sin^2\theta(d\delta + \omega)^2 \end{aligned} \tag{4.4.7}$$

The 2 S^2´s and the 1-form ω are parametrized by

$$\omega = \frac{1}{4}\sum_{i=1}^{2} \sin\theta_i d\phi_i, \quad d\Omega_2^2 = d\theta_1^2 + \cos^2\theta_1 d\phi_1^2, \quad d\Omega_2'^2 = d\theta_2^2 + \cos^2\theta_2 d\phi_2^2$$

Having established the metric one may introduce and transform the global AdS time and the fibration over δ via the coordinates x^0 and u as follows

$$\begin{pmatrix} t \\ \delta \end{pmatrix} = \begin{pmatrix} 1 & -1/2 \\ 1 & 1/2 \end{pmatrix} \begin{pmatrix} x^0 \\ u \end{pmatrix} \tag{4.4.8}$$

[6]Originally there was need of gauge fixing, for the w direction since it was a mix of both S^5, S^3. This meant that $-i\partial_w = c_1 S + c_2 J$, but it was found that $c_1 = 1$ and $c_2 = 0$ which we take as default setting throughout the computatios

Reading off from the second column, we get the same type of condition as for $\mathcal{N}=4$ SYM such that ∂_u is null when the following inequality is satisfied

$$4(\partial_u)^2/R^2 = -\cosh^2\rho + 4\cos^2\theta\sin^2\theta \leq 0 \tag{4.4.9}$$

This is exactly met when $\rho = 0$ and $\theta = \pi/4$. Using this and transforming to the new coordinates yield

$$\begin{aligned} ds^2/R^2 &= -(dx^0 - \frac{1}{2}du)^2 + (dx^0 + \frac{1}{2}du + \omega)^2 + \frac{1}{8}(d\Omega_2^2 + d\Omega_2'^2) \\ &= du(2dx^0 + \omega) + \omega^2 + 2dx^0\omega + \frac{1}{8}(d\Omega_2^2 + d\Omega_2'^2) \\ &= 2\tau(du - m) + h_{ij}dx^i dx^j \end{aligned} \tag{4.4.10}$$

It is easy to identify the TNC-variables

$$\tau = dx^0 + \frac{1}{2}\omega, \quad m = -\omega, \quad h_{ij}dx^i dx^j = \frac{1}{8}(d\Omega_2^2 + d\Omega_2'^2) \tag{4.4.11}$$

For comparison, the same corresponding action [82, 75] has been considered. The obvious difference is the addition of a S^2, which stems ABJM-theory. From the construction in 2.3, both the geometry and product gauge group seem to indicate this. That would suggest that the additional sphere represents the same kind of $SU(4)$ spin chain picture that was found in [110], or rather the R-symmetry multiplets of scalars (and fermions) grouped in 2.3.21 in the fundamental and anti-fundamental representation. If we proceed to consider the flat gauge fixed action for the non-relativistic SMT string using the TNC-variables one obtains

$$\begin{aligned} S_{\text{flat,gf}} &= -\frac{J}{2\pi}\int d^2\sigma(m_\mu x^\mu + \frac{1}{2}h_{\mu\nu}dx^\mu dx^\nu) \\ &= \frac{J}{8\pi}\sum_{i=1}^{2}\int d^2\sigma(\sin\theta_i\dot{\phi}_i - \frac{1}{2}[\theta_i'^2 + \cos^2\theta_i\phi'^2]) \end{aligned} \tag{4.4.12}$$

The structure is in agreement with (4.4.1) which was the Landau-Lifshitz model on odd and even sites. The two regimes both for the sigma model limit and SMT-limit for the BPS-bounds $SU(2)\times SU(2)$ seem to overlap.

4.4.4 The OSp(2|2) Background

The subsector defined for the OSp(2|2) background is given by the BPS bound $Q = J_1 + J_2 + S$, that is, we extend the $SU(2)\times SU(2)$ sector by introducing spin. Our starting point will be the metric used in the previous section. The main difference will be the contribution from $d\hat{\Omega}_2^2 = d\theta^2 + \sin^2\theta d\phi^2$. Thus, 4.4.7 can be expanded to

$$\begin{aligned} ds^2/R^2 = &-\cosh^2\rho dt^2 + d\rho^2\sinh^2\rho(d\theta^2 + \sin^2\theta d\phi^2) \\ &+ d\xi^2 + \frac{1}{4}(\cos^2\xi d\Omega_2^2 + \sin^2\xi d\Omega_2'^2) + 4\cos^2\xi\sin^2\xi(d\delta + \omega)^2 \end{aligned} \tag{4.4.13}$$

The isometries that need to be accounted for includes the azimuthal part of S^2 as well. This makes us extend the linear transformations to include the w-parameter as in SYM, this time only controlling how the w-direction is aligned along $S^2 \subset AdS_4$. The matrix translating between global isometries and coordinates on the submanifold with TNC variables is

$$\begin{pmatrix} t \\ \phi \\ \delta \end{pmatrix} = \begin{pmatrix} 1 & -1/2 & 0 \\ 1 & -1/2 & 1 \\ 1 & 1/2 & 0 \end{pmatrix}\begin{pmatrix} x^0 \\ u \\ w \end{pmatrix}, \tag{4.4.14}$$

Reading off from the second column, we get ∂_u is null under the following condition

$$4(\partial_u)^2/R^2 = -\cosh^2\rho + \sinh^2\rho\sin^2\theta + 4\cos^2\xi\sin^2\xi \leq 0 \tag{4.4.15}$$

This condition will exactly be met when $\theta = \pi/2$ and $\xi = \pi/4$. Setting the angles to the found constants, the metric reduces to

$$\begin{aligned} ds^2/R^2 = &-\cosh^2\rho(dx_0 - \frac{1}{2}du)^2 + d\rho^2 + \sinh^2\rho(dx_0 - \frac{1}{2}du + dw)^2 \\ &+ \frac{1}{8}(d\Omega_2^2 + d\Omega_2'^2) + (dx_0 + \frac{1}{2}du + \omega)^2 \end{aligned} \tag{4.4.16}$$

After grouping terms such that it will align with the metric, one finds

$$\begin{aligned} ds^2 =& du(2dx_0 - \sinh^2\rho dw + \omega) + 2\sinh^2\rho dw dx_0 + 2dx_0\omega + \omega^2 \\ &+ d\rho^2 + \frac{1}{8}(d\Omega_2^2 + d\Omega_2'^2) + \sinh^2\rho dw^2 \end{aligned} \tag{4.4.17}$$

In this form, it can easily be seen what the structure of the TNC-variables are for τ and m especially. With some rewriting one finds that the geometry has gained some terms due to the null isometry from $-i\partial_\phi$

$$\begin{aligned} \tau &= dx^0 - \frac{1}{2}(\sinh^2\rho dw - \omega) \\ m &= -(\sinh^2\rho dw + \omega) \\ h &= d\rho^2 + \frac{1}{8}(d\Omega_2^2 + d\Omega_2'^2) + \cosh^2\rho\sinh^2\rho dw^2. \end{aligned} \tag{4.4.18}$$

Notice how the contributions exactly match the combination of structure from the $SU(1,1)$ in SYM and $SU(2)\times SU(2)$ backgrounds from CS. To finish up, we state the flat gauge fixed action on this background to be

$$\begin{aligned} S_{\text{flat,gf}} = -\frac{J}{2\pi}\int d^2\sigma\Bigg[&\dot{w}\sinh^2\rho - \frac{1}{2}\left((\rho')^2 + (w')^2\sinh^2\rho\cosh^2\rho\right) \\ &+ \sum_{i=1}^{2}\left[\sin\theta_i\dot{\phi}_i - \frac{1}{16}(\theta_i'^2 + \cos^2\theta_i\phi'^2)\right]\Bigg]. \end{aligned} \tag{4.4.19}$$

The result is exactly contributions that combines the two sectors found in [81, 27][7].

4.4.5 The SU(2|3) Background

The last example before going to the all background case, we consider the maximal R-symmetry admitted by the bps-sectors, namely $Q = J_1 + J_2 + J_3$. This sector admits SU(3) symmetry and is purely connected to the S^7, thus most of the AdS_4 geometry will be redundant. In 2.3, it was discussed that there are two distinct ways of writing the metric of $\mathbb{C}P^3$. For the particular choice of Q, we tried using the previous metric, that a type IIA background cannot

[7]The action can take another form by using the set of coordiantes defined in [120] given by

$$\begin{aligned} S_{\text{flat,gf}} = -\frac{J}{2\pi}\int d^2\sigma\Big[&\dot{w}\sinh^2\rho - \frac{1}{2}\left((\rho')^2 + (w')^2\sinh^2\rho\cosh^2\rho\right) \\ &+ \frac{1}{2}\dot{\phi}_1\cos\theta_1 - \frac{1}{2}\dot{\phi}_2\cos\theta_2 + \frac{1}{16}\sum_{i=1}^{2}\left((\theta_i')^2 + (\phi_i')^2\sin^2\theta_i\right)\Big]. \end{aligned} \tag{4.4.20}$$

be obtained. With three angular momenta and four complex coordinates to parametrize S^7, one falls short of obtaining a null isometry. This leads to the shift in metric, such that 2.3.39 will be favourable to use. as in the $SU(2|3)$ case of SYM, the metric can be seen decomposed into $\mathbb{CP}^3 \sim \mathbb{CP}^2 + (d\chi + A)^2$, such that we can obtain the Fubini-Study for $\mathbb{CP}^2$ with a non-dynamic angle $d\chi$ which can be factored with Kahler potential. Using the metric, we employ a transformation from angular coordinates on $\mathbb{C}P^3$, namely the α_j´s into three new coordinates

$$\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix} = \begin{pmatrix}1 & -1/2 & 0\\ 1 & 1/2 & -1/2\\ 1 & 1/2 & 1/2\end{pmatrix}\begin{pmatrix}\chi\\ \psi\\ \phi\end{pmatrix} \quad \rightarrow \quad \begin{aligned} z1/z4 &= \tan\xi\cos\alpha e^{i\chi-\psi/2}\\ z2/z4 &= \tan\xi\sin\alpha\sin\frac{\theta}{2}e^{i\chi}e^{i(\psi-\phi)/2}\\ z3/z4 &= \tan\xi\cos\alpha\cos\frac{\theta}{2}e^{i\chi}e^{i(\psi+\phi)/2}\end{aligned} \tag{4.4.21}$$

Following [45], this expression becomes comparable to the $AdS_5 \times S^5$ geometry which is identical up to numerical factors. The resulting metric is then

$$\begin{aligned} ds^2_{\mathbb{CP}^3} &= 4d\xi^2 + 4\sin^2\xi\cos^2\xi(d\chi + \frac{1}{2}(\sin^2\alpha(d\psi + \cos\theta d\phi) - d\psi))^2\\ &\quad + 4\sin^2\xi\Big[d\alpha^2 + \frac{1}{4}\sin^2\alpha(d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\alpha(d\psi + \cos\theta d\phi)^2)\Big], \end{aligned} \tag{4.4.22}$$

Note that the factor of 4 is a consequence of rescaling the overall radius in the near-horizon geometry as for the previous metric. In this case, we take advantage of it for the null conditions. The metric can be put in a nicer form following [82] by defining the quantities corresponding to the Kahler potential

$$B = \sin^2\alpha(d\psi + \cos\theta d\phi) - d\psi, \quad A = \cos\theta d\phi \tag{4.4.23}$$

Looking at the second line of the metric, we can recognize this exactly as the Fubini-Study metric over $\mathbb{CP}^2$ defined as

$$d\Sigma_2^2 = d\alpha^2 + \sin^2\alpha d\Sigma_1^2 + \frac{1}{4}\sin^2\alpha\cos^2\alpha(d\psi + A)^2, \quad d\Sigma_1^2 = \frac{1}{4}(d\theta^2 + \sin^2\theta d\phi^2) \tag{4.4.24}$$

Putting all this together we obtain for the full $\mathrm{AdS}_4 \times \mathbb{CP}^3$ metric on the $SU(3|2)$ background

$$ds^2/R^2 = -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_2^2 + d\xi^2 + 4\sin^2\xi\cos^2\xi(d\chi + \frac{1}{2}B)^2 + 4\sin^2\xi d\Sigma_2^2. \tag{4.4.25}$$

Using the same type of transformation as for the $SU(2) \times SU(2)$ background

$$\begin{pmatrix}t\\ \chi\end{pmatrix} = \begin{pmatrix}1 & -1/2\\ 1 & 1/2\end{pmatrix}\begin{pmatrix}x^0\\ u\end{pmatrix} \tag{4.4.26}$$

we see that the null condition becomes

$$4(\partial_u)^2/R^2 = -\cosh^2\rho + 4\sin^2\xi\cos^2\xi \leq 0, \tag{4.4.27}$$

This holds when $\rho = 0$ and $\xi = \pi/4$. Our Metric reduces as a consequence to the following

$$ds^2/R^2 = -dt^2 + (d\chi + \frac{1}{2}B)^2 + 2d\Sigma_2^2. \tag{4.4.28}$$

Inserting this into the metric will reduce in terms of new coordinates

$$ds^2/R^2 = 2dx^0du + \frac{1}{2}Bdu + Bdx^0 + \frac{1}{4}B^2 + 2d\Sigma_2^2, \tag{4.4.29}$$

which can readily be written in the TNC form

$$ds^2/R^2 = 2\tau(du - m) + h \tag{4.4.30}$$

with the identifications

$$\tau = dx^0 + \frac{1}{4}B, \quad m = -\frac{1}{2}B, \quad h = 2d\Sigma_2^2, \tag{4.4.31}$$

The last step is to find the flat gauge fixed action, given by

$$\begin{aligned} S_{\text{flat,gf}} &= -\frac{J}{2\pi}\int d^2\sigma(m_\mu x^\mu + \frac{1}{2}h_{\mu\nu}dx^\mu dx^\nu) \\ &= \frac{J}{4\pi}\int d^2\sigma\Big(\sin^2\alpha\cos\theta\dot{\phi} - \cos^2\alpha\dot{\psi} - 2\Big[(\alpha')^2 \\ &\frac{1}{4}\sin^2\alpha(\theta'^2 + \sin^2\theta\phi'^2) + \frac{1}{4}\cos^2\alpha\sin^2\alpha(\psi' + \cos\theta\phi')^2\Big]\Big) \end{aligned} \tag{4.4.32}$$

This action is equivalent in structure to the case of $SU(2|3)$ case considered in the A. But this time, the term that would usually go as $-\cos(2\alpha)\dot{\psi}$ is replaced by $-\cos^2\alpha\dot{\psi}$ since the Kahler potentials do not have the exact structure to make it identical[8]

4.4.6 All backgrounds From the OSp$(2|4)$ background

For the grand finale, we obtain the U(1) Galilean geometry for $S + J_1 + J_2 + J_3 = Q \leq E$. This BPS bound leads to the OSp(4|2) spin matrix theory which can be used to obtain the other bounds in spin matrix theory previously considered, by considering different manipulations for angles that give the different backgrounds. The full geometry will be parametrized via a hopf coordinate for the $S^2 \subset \text{AdS}_4$ and the $\mathbb{CP}^3$ using a S^1-fibration over $\mathbb{CP}^2$ for the full in the Fubini-Study coordinates. This will result in the isometries $-i\partial_\phi = S$ and $-i\partial_\chi = J_1 + J_2 + J_3$. The metric will be written in terms of Fubini-Study potentials

$$ds^2/R^2 = -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho(d\bar{\theta}^2 + \sin^2\bar{\theta}d\phi^2) + d\xi^2 + 4\sin^2\xi\cos^2\xi(d\chi + \frac{1}{2}B)^2 + 4\sin^2\xi d\Sigma_2^2 \tag{4.4.33}$$

Following the transformations we have used before, we get

$$\begin{pmatrix} t \\ \phi \\ \chi \end{pmatrix} = \begin{pmatrix} 1 & -1/2 & 0 \\ 1 & -1/2 & 1 \\ 1 & 1/2 & 0 \end{pmatrix}\begin{pmatrix} x^0 \\ u \\ w \end{pmatrix}, \tag{4.4.34}$$

Compared to the PSU$(1,2|3)$ case we do not have that $-i\partial_\phi$ and $-i\partial_\chi$ are of constant length in ABJM, so the null condition we are met with is

$$4(\partial_u)^2/R^2 = -\cosh^2\rho + \sinh^2\rho\sin^2\bar{\theta} + 4\sin^2\xi\cos^2\xi \leq 0 \tag{4.4.35}$$

This is satisfied when $\xi = \pi/4$ and $\bar{\theta} = \pi/2$ Using the transformation reduces our expression for the metric

$$\begin{aligned} ds^2/R^2 &= 2dx^0du + d\rho^2 + \sinh^2\rho(dw^2 + 2dw(dx^0 - \frac{1}{2}du)) \\ &\frac{1}{4}B^2 + B(dx^0 + \frac{1}{2}du) + 2d\Sigma_2^2 = \\ &2\tau(du - m) + h \end{aligned} \tag{4.4.36}$$

[8] In the case of SYM the Kahler potentials are defined as $B = \sin^2\xi(d\psi + A) - \frac{1}{2}d\psi$ and $A = \frac{1}{2}\cos\theta d\phi$. Here one finds $\sin^2\xi d\psi - \frac{1}{2}d\psi = -\frac{1}{2}\cos(2\xi)$ while in our case this will only amount to $\sin^2\alpha - 1 = -\cos^2\alpha$

Where the TNC variables are given by

$$
\begin{aligned}
\tau &= dx^0 + \frac{1}{4}B + \frac{1}{2}\sinh^2\rho dw \\
m &= -\Big(\frac{1}{2}B + \sinh^2\rho dw\Big) \\
h &= d\rho^2 + \sinh^2\rho\cosh^2\rho dw + 2d\Sigma_2^2,
\end{aligned}
$$

On this background the flat gauge-fixed SMT string action gives

$$
\begin{aligned}
S_{\text{flat,gf}} = \frac{J}{4\pi}\int d^2\sigma\Big(&2\sinh^2\rho\dot{w} + \sin^2\alpha\cos\theta\dot{\phi} - \cos^2\alpha\dot{\psi} - (\rho')^2 - \sinh^2\rho\cosh^2\rho(w')^2 \\
&- 2\Big[(\alpha')^2 + \frac{1}{4}\sin^2\alpha((\theta')^2 + \sin^2\theta(\phi')^2) + \frac{1}{4}\sin^2\alpha\cos^2\alpha(\psi' + \cos\theta\phi')^2\Big]\Big)
\end{aligned}
\tag{4.4.37}
$$

It is interesting to consider how this compares (up to numerical factors) to the case for the $PSU(1,2|3)$ background in SYM. All the terms that appear in the action are present in the full background of SYM as well, but the geometry is still not null on the whole space due to 4.4.35. Further, due to the larger isometry, there are additional terms from the S^3 present in SYM that is inherently not part of the $AdS_4 \times \mathbb{C}P^3$ geometry. It is worth reflecting upon some of the results we have arrived at. First, the obvious deviation from the case considered in $\mathcal{N}=4$ SYM, is the fact that there is not a direct way to go from the maximal BPS-bound and then truncate to all other subsectors. As we saw, the change of coordinates was crucial in obtaining the remaining subsectors where the charge contained $\{J_1, J_2, J_3\}$, such that a $U(1)$ Galilean background could be obtained. Seen from the perspective of the supercoset construction[137, 7], it is found in comparison to the Supercoset of SYM, that it only preserves 24 fermionic DOF (As seen for type IIA backgrounds as well). It would seem that the $\mathrm{OSP}(6|4)/\mathrm{SO}(1,3)\times U(3)$ is not a supercoset manifold[135]. In this context, looking at the coset structure $SU(4)/U(3)$, reducing the S^7 into two S^3 consequently reduced the coset to the subset $SU(2)\times SU(2)\times U(1)$ or in terms of BPS language $\Delta \geq J_1 + J_2$. The emergent $U(1)$, is nothing but the hopf fibration along a fiber bundle for S^1. This would shrink through M-theory such that the type IIA background emerged. Turning on J_3, the culprit becomes the orbifolding, removing one of the four $U(1)$ Cartan isometries. Hence, J_3 would move along the $U(1)$ fiber which seems to decouple the S^2 interpretation. This forced the switch, which did not make it possible to make a consistent truncation to the rest of the BPS landscape. These interpretations need to be revised since the physical picture is somewhat elusive to the naked eye and should be taken with a grain of salt.

4.5 Penrose Limits of Spin Matrix String Backgrounds

As was done for $\mathcal{N}=4$ SYM, we have been (almost) able to succeed in finding the $U(1)$ Galilean backgrounds corresponding to the Spin Matrix limits. We do face some complicated and non-linear theories when going to the flat gauge-fixed action. But as was considered, one can take a large charge limit for J and zoom in on excitations around specific angles. One expects that we might obtain free theories that will resemble Penrose limits where we zoom in on the geometry around the null geodesic. Starting from $\mathrm{AdS}_4 \times \mathbb{C}P^3$ and zooming in on the null geodesic along AdS_4 and $\mathbb{C}P^3$, the resulting geometry should become the 10-dimensional maximally symmetric PP-wave. But it is important to remember that the expression will depend on the coordinates we start with for our null geodesic, since it depends on the coordinate pair (x^0, u). When one zooms in around ∂_u on a submanifold where the vector is null we can expect a background of the form

$$
ds^2/R^2 = dx^0(du0 + x_i dy_i) + dx_i dx^i + dy_i dy^i + dx_a dx^a - x_a x^a (dx^0)^2 \tag{4.5.1}
$$

where $i = 1, ..., n$ and $a = 1, ..., 8 - 2n$ The pp-wave can be written in the form of the above metric and has the 2n flat directions (x^i, y^i). There will also be a quadratic potential contained in the $8-2n$ remaining transverse directions x^a. In the rescaling $x^0 = \frac{\tilde{x^0}}{c^2}$ one gets that the slope diverges in the SMT-limit, thus we get dynamics that is suppressed and only the flat directions contribute. In terms of the TNC-data we expect the following structure to appear then

$$\tilde{\tau} = d\tilde{x^0}, \quad m = -\sum_{i=1}^{n} x^i dy^i, \quad h = \sum_{i=1}^{n}((dx^i)^2 + (dy^i)^2) \tag{4.5.2}$$

This is what originally has been dubbed the flat fluxed (FF) backgrounds, since they contain a mass flux term $x_i dy^i$ that supports the dynamics in the flat directions of both components. The goal is now to show that the SMT and large charge limit compared with the Penrose limit should correspond to the same FF U(1) Galilean Geometry. This is done for the cases considered in the previous section

4.5.1 The $SU(2) \times SU(2)$ flat background

Considering the simplest of cases, namely the $SU(2) \times SU(2)$ sector, it was found in a previous section, that the TNC-data can be written as

$$\tilde{\tau} = d\tilde{x}^0, \quad m = -\omega, \quad h_{ij} dx^i dx^j = \frac{1}{8}(d\Omega_2^2 + d\Omega_2'^2) \tag{4.5.3}$$

For the large charge limit of $J \to \infty$, we define the following coordinate transformations

$$\theta_i = x_i/\sqrt{J} - \pi/2, \quad \phi_i = y_i/\sqrt{J} \tag{4.5.4}$$

Using this for m and h we get

$$\begin{aligned} m_0 &= \lim_{J\to\infty} Jm = \frac{1}{4}\sum_{i=1}^{2} x_i dy_i \\ h_0 &= \lim_{J\to\infty} Jm = \frac{1}{8}\sum_{i=1}^{2} dx_i^2 + dy_i^2 \end{aligned} \tag{4.5.5}$$

This corresponds to zooming in on the excitations around $\theta = \pi/2$ and $\phi = 0$

$$S_{\text{flat,gf}} = \frac{J}{8\pi}\sum_{i=1}^{2}\oint d\sigma\Big[x_i\dot{y}_i - \frac{1}{2}[(x_i')^2 + (y_i')^2]\Big] \tag{4.5.6}$$

This is the same action up to factors as in [81]. Now we take the Penrose limit and show that we retrieve the same geometry from a SMT limit of the corresponding pp-wave. To get the pp-wave, the metric can be written in the adapted coordinates we used previously and then zoom in on a null geodesic on the submanifold of $\mathcal{M}$ corresponding to $\rho = 0$ and $\xi = \pi/4$ by defining

$$R = R'/\epsilon, \quad u = U\epsilon^2, \quad \phi_a = y_a\epsilon, \quad \theta_a = x_a\epsilon, \quad \rho = r\epsilon, \quad \xi = \pi/4 + z\epsilon \tag{4.5.7}$$

Writing out the metric first and then taking the limit $\epsilon \to 0$ we get

$$\begin{aligned} ds^2/R^2 = &-\cosh^2\rho(dx^0 - \frac{1}{2}du)^2 + d\rho^2 + \sinh^2\rho(d\bar{\theta}_2^2 + \sin^2\bar{\theta}(dx^0 - \frac{1}{2}du + dw)^2) \\ &+ d\xi^2 + \frac{1}{4}(\cos^2\xi d\Omega_2^2 + \sin^2\xi d\Omega_2'^2) + 4\cos^2\xi\sin^2\xi(dx^0 + \frac{1}{2}du + \omega)^2 \end{aligned} \tag{4.5.8}$$

Expanding in ϵ, and collecting terms that scale like ϵ^2, since this is the global factor defined on the radius of the geometry, one finds

$$\begin{aligned} ds^2/R^2 &= 2dx^0 dU + dr^2 + r^2 d\hat{\Omega}_2^2 + 4dz^2 + \frac{1}{8}dx_1^2 + \frac{1}{8}dx_2^2 \\ &\quad + \frac{1}{2}dx^0(x_1 dy_1 + x_2 dy_2) + \frac{1}{8}dy_1^2 + \frac{1}{8}dy_2^2 - z^2(dx^0)^2 \\ &= 2dx^0(dU - m_0) + h_0 - z^2(dx^0)^2 + 4dz^2 + dr^2 + r^2 d\hat{\Omega}_2^2. \end{aligned} \tag{4.5.9}$$

It can be seen that freezing the Z-direction, one can obtain a similar expression with an infinitely steep potential as noted in [82].

4.5.2 The OSp(2|2) flat background

Turning to the case where we keep the same amount of generators over $\mathbb{C}P^3$, but additionally we add a spin DOF from the $\mathbb{CP}^1 \subset AdS_4$. The relevant quantities to look at are the TNC-data

$$\tau = d\tilde{x}^0, \quad m = -(\sinh^2\rho dw + \omega), \quad h = d\rho^2 + \frac{1}{8}(d\Omega_2^2 + d\Omega_2'^2) + \cosh^2\rho\sinh^2\rho dw^2. \tag{4.5.10}$$

Defining in the large charge limit for $J \to \infty$ the following new coordinates, we find

$$r = \sqrt{J}\rho, \quad x_i = \sqrt{J}(\theta_i - \pi/2), \quad y_a = \sqrt{J}\phi_a \tag{4.5.11}$$

After taking the limit, we are left with a much simpler

$$\begin{aligned} m_0 &= \lim_{J\to\infty} Jm = \frac{1}{4}\sum_{i=1}^{2} x_i dy_i + r^2 dw \\ h_0 &= \lim_{J\to\infty} Jh = \frac{1}{8}\sum_{i=1}^{2}(dx_i^2 + dy_i^2) + dr^2 + r^2 dw^2 \end{aligned} \tag{4.5.12}$$

The situation is not much different than before, only this time we have mixed both the sphere and the AdS part. This will give us the following background action with the addition of zooming in on the region $\rho = 0$

$$S_{\text{flat,gf}} = \frac{J}{8\pi}\sum_{i=1}^{2}\oint d\sigma\left[x_i\dot{y}_i + r^2\dot{w} - \frac{1}{2}\Big([(x_i')^2 + (y_i')^2] + r^2(w')^2 + (r')^2\Big)\right] \tag{4.5.13}$$

The corresponding Penrose limit can be obtained from the $AdS_4\times\mathbb{CP}^3$ coordinates by introducing

$$R = R'/\epsilon, \quad u = U\epsilon^2, \quad \phi_a = y_a\epsilon, \quad \theta_i = x_i\epsilon - \pi/2, \quad \rho = r\epsilon, \quad \xi = \pi/4 + q\epsilon \tag{4.5.14}$$

In the $\epsilon \to 0$ limit, we obtain the metric

$$\begin{aligned} ds^2/R^2 &= 2dx^0(dU + \frac{1}{2}r^2\sin^2\bar{\theta}dw + \frac{1}{4}\sum_{i=1}^{2}x_i dy_i) + \frac{1}{8}\sum_{i=1}^{2}(dx_i^2 + dy_i^2) \\ &\quad + dr^2 + r^2(d\bar{\theta}^2 + \sin^2\bar{\theta}dw^2) + (q^2 + r^2\sin^2\bar{\theta})(dx^0)^2 \end{aligned} \tag{4.5.15}$$

We see that when $q^2 = -r^2$ and $\bar{\theta} = \pi/2$, one exactly retrieve the $U(1)$ Galilean data we derived from the large J limit above.

4.5.3 The $SU(3|2)$ flat background

Instead of adding a spin degree of freedom this time, we extend to the maximal amount of generators for S^7 giving us the $SU(3)$ sector from the BPS-bound $Q = J_1 + J_2 + J_3 \leq E$. If we consider the large charge limit again of the TNC-data

$$\tilde{\tau} = d\tilde{x}^0, \quad m = -\frac{1}{2}B, \quad h = 2d\Sigma_2^2,$$

One can write coordinates in the $J \to \infty$ limit as

$$\alpha = \frac{\pi}{4} + \frac{q}{\sqrt{j}}, \quad \theta = \frac{x}{\sqrt{j}} + \frac{\pi}{2}, \quad \phi = y/\sqrt{J}, \quad \psi = p/\sqrt{J} \tag{4.5.16}$$

and from this obtain

$$\begin{aligned} m_0 &= \lim_{J\to\infty} Jm = \frac{1}{4}xdy - \frac{1}{2}qdp \\ h_0 &= \lim_{J\to\infty} Jh = \frac{1}{2}dq^2 + \frac{1}{4}(dx^2 + dy^2) + \frac{1}{8}dp^2 \end{aligned} \tag{4.5.17}$$

The action subsequently takes the form

$$S_{\text{flat,gf}} = \frac{J}{4\pi} \oint d\sigma \Big[\frac{1}{2}x\dot{y} - q\dot{p} - \frac{1}{2}((x')^2 + (y')^2) + (q')^2 + \frac{1}{4}(p')^2\Big] \tag{4.5.18}$$

Similarly, the coordinates that are chosen for the specific Penrose limit will be

$$\begin{aligned} &R = R'/\epsilon, \quad u = U\epsilon^2, \quad \phi_a = y_a\epsilon, \quad \theta_i = x_i\epsilon + \pi/2 \\ &\psi = p\epsilon, \quad \rho = r\epsilon, \quad \alpha = \pi/4 + q\epsilon, \quad \xi = \pi/4 + z\epsilon \end{aligned} \tag{4.5.19}$$

After painstakingly expanding and bookkeeping powers of ϵ^2, the terms that are left after using the (x^0, u) is

$$ds^2 = 2dx^0(dU - m_0) + h_0 + 2dq^2 + \frac{1}{2}dz^2 + dr^2 + r^2 d\Omega_2^2 - (r^2 + 4z^2)(dx^0)^2 \tag{4.5.20}$$

The same kind of phenomena occurs in the three-charge case as for SYM, where the relativistic string experiences a quadratic potential r^2 in the now three transverse directions $dr^2 + r^2 d\Omega_2^2$. In the SMT limit with $x^0 = \tilde{x^0}/c^2, c \to \infty$, the potential becomes infinitely steep as well. Hence, the geometry is restricted to a $U(1)$ Galilean geometry described by the coordinates in the limit $J \to \infty$. The only notable difference is the extra coordinate dz^2 as an artifact of the geometry and the fact that we are a factor 2 off for some peculiar reason (this will be resolved at some point).

4.5.4 The OSp$(4|2)$ flat background

The last background which corresponds to the OSp(4|2) sector, has to be considered as the maximal one, From the type IIA condition in 4.3.19. Using the TNC-data for the BPS bound $\Delta - J_1 - J_2 - J_3 - S$.

$$\tilde{\tau} = d\tilde{x^0}, \quad m = -(\frac{1}{2}B + \sinh^2\rho dw), \quad h = d\rho^2 + \sinh^2\rho\cosh^2\rho dw + 2d\Sigma_2^2, \tag{4.5.21}$$

Then defining the coordinates that will go into play when taking the large charge limit

$$\alpha = \frac{\pi}{4} + \frac{q}{\sqrt{j}}, \quad \theta = \frac{x}{\sqrt{j}} + \frac{\pi}{2}, \quad \phi = y/\sqrt{J}, \quad \psi = p/\sqrt{J}, \quad \rho = r/\sqrt{J} \tag{4.5.22}$$

Inserting into m and h, we find in the $J \to \infty$ limit

$$\begin{aligned} m_0 &= \lim_{J\to\infty} Jm = \frac{1}{4}xdy - \frac{1}{2}qdp + r^2 dw \\ h_0 &= \lim_{J\to\infty} Jh = dr^2 + r^2 dw^2 + 2dq^2 + \frac{1}{4}(dx^2+dy^2) + \frac{1}{8}dp^2 \end{aligned} \tag{4.5.23}$$

In the large charge limit, we find the maximal extension of the action which we hint towards having a interpretation in action-angle variables and symplectic potential in phase space is given by

$$S_{\text{flat,gf}} = \frac{J}{4\pi}\oint d\sigma \Big[\frac{1}{2}x\dot{y} - q\dot{p} + r^2\dot{w} - (r')^2 - r^2(w')^2 - (q')^2 - \frac{1}{2}\Big((x')^2 + (y')^2\Big) - \frac{1}{4}(p')^2\Big] \tag{4.5.24}$$

Going to the Penrose limit we define as previously all the coordinates in the same fashion

$$\begin{aligned} R = R'/\epsilon, \quad u = U\epsilon^2, \quad \phi_a = y_a\epsilon, \quad \theta_i = x_i\epsilon + \pi/2 \\ \psi = p\epsilon, \quad \rho = r\epsilon, \quad \alpha = \pi/4 + q\epsilon, \quad \xi = \pi/4 + z\epsilon \end{aligned} \tag{4.5.25}$$

Taking $\epsilon \to 0$, we finally obtain the grand piece

$$\begin{aligned} ds^2/R^2 &= 2dx^0(dU + r^2\sin^2\bar{\theta}dw - \frac{1}{4}xdy + \frac{1}{2}qdp) + dr^2 + \frac{1}{2}dz^2 \\ &\quad + r^2(d\bar{\theta}^2 + \sin^2\bar{\theta}dw^2) + \frac{1}{4}(dx^2+dy^2) + 4dq^2 + \frac{1}{8}dp^2 + (r^2\sin^2\bar{\theta} - z^2)(dx^0)^2 \\ &= 2dx^0(dU - m_0) + h_0 + 2dq^2 + dr^2 + \frac{1}{2}dz^2 + (r^2\sin^2\bar{\theta} - z^2)(dx^0)^2 \end{aligned} \tag{4.5.26}$$

In the "maximal" case we are restricted to submanifolds where by setting $\bar{\theta} = \pi/2$ and $z = \frac{1}{2}r$, this exactly reproduces the result obtained in the large charge limit. Compared to SYM, this inherently does not exhibit the pure background as an immediate effect after taking the Penrose limit. One can speculate that due to 4.4.35, making $-i\partial_u$ not null on the entirety of $AdS_4\times\mathbb{C}P^3$, exactly gives these extra terms which have to be removed in a by choosing certain planes in the geometry. And the factor of $2dq^2$ is still following, so for consistency, this has to be removed, but this is hopefully a technical little detail

Phase space, Symplectic Potentials and Action Angle variables:

As it was established in the inspirational work [82] laying the ground for these results, the flat gauge fixed action can be interpreted as phase space coordinates, which has dynamics described by a symplectic form and Hamiltonian

$$\omega = -\frac{J}{2\pi}dm, \quad H = \frac{J}{4\pi}\oint d\sigma^1 h_{\mu\nu}X'^{\mu}X'^{\nu} \tag{4.5.27}$$

The actions found in the large charge limit, also resemble that of action-angle variables. This is natural considering the phase space language exhibited by our action linear in velocities of the target space embedding fields X^μ. This begs the question whether or not more complicated Sigma models can be quantized, and what kind of integrable models and phase space structures are hiding behind these theories.
To end the last chapter, we can make a form of sanity check. The Coordinates obtained in the large charge limit seem to appear as for SYM, besides terms that couple the extra spin DOF. Due to restrictions on operators in the BPS $\Delta \geq J + S$, the maximal space is still restricted to a submanifold such that ∂_u is not null on the entire $AdS_4\times\mathbb{C}P^3$, which can be seen from 4.4.35, and makes additional terms vanish. Other than numerical constants also, the structure that is found might be aligned with SYM. For further investigation, it might be useful to extend from the work of [66], and see if there are possible directions to follow.

Chapter 5

Conclusion and Outlook

And thus the end has arrived. The culmination of a whole year of academic work is reaching its end. So what have we achieved and done? and have we not? First and foremost, since the start of this project, we have traversed the vast landscape of the AdS/CFT correspondence. It has been a great pleasure to catch up with what has happened in theoretical modern physics over the last 30 years or so. The thesis has been angled toward the unknown superstar of the gauge/gravity duality, namely ABJM.
The thesis was intended to be told as much as a fluent story as possible. The intention was to start the story in a familiar place, where everyone is comfortable. Thus superstrings, type II A and B SUGRA, T-duality, M-theory and supersymmetry were introduced in rather layman terms, seen from the perspective of an ordinary physicist. This supposedly lays the ground for partial ingredients needed to motivate and arrive at the actual AdS/CFT correspondence that Maldacena famously conjectured. But to get a sense of the physical idea, t´Hooft and Susskind had to, via holography, take us on a journey for our eyes to be opened. Through the conjecture, it was also possible to connect quantities from gravity and gauge theory, most notably the conformal dimension Δ and the mass of particles m. To end the general story, GKPW was needed to formalize the precise statement, such that nobody was left confused.
Following right up, Chern-Simons theory was constructed in superspace, since this is the main act of the ABJM action. With the help of this, and BLG theory involving $M2$-branes as well, the ABJM conjecture was stated and unpacked as reasonable as possible, but with some aspects still missing, such as the brane construction[1]. Nevertheless, the full algebra, action, field and matter content, geometry and subsectors have been displayed as pedagogical as possible.
With this established, we saw how peculiarities arise in AdS/CFT, when it becomes possible to realize spin-chains in the planar limit. This builds a whole bridge between operators interpreted as spin particles as was seen. Furthermore, Penrose limits were considered as they presented themselves as useful ways to obtain pp-wave backgrounds. Constructing the quantized light-cone Hamiltonian spectrum for strings became possible, which was compared to the gauge theory side where corrections of spectra to loop orders can be compared. The case of SYM was *straight forward* and caused no harm, while ABJM presented that at strong and weak t´Hooft coupling, the spectrum scaled differently.
Moving on to the main part of the show, non-relativistic quantum mechanics became the main framework for AdS/CFT (in our case), which was realized by Spin Matrix theory. It was attempted to link the spin-chain gas to notions considered for ABJM, and while some obvious remarks were made, much work is needed to be established for a satisfactory comprehension of the structure.
The first real new contribution came when considering the computations done in [81], where a new subsector $\Delta \geq J_1 + J_2 + S$ was considered in a sigma model limit. This was low-hanging

[1] For the interested reader see [16, 5, 2]

fruit which was nothing but a simple extension of the previous analysis.
Switching gears and feeling inspired by [80, 78], we set out to find the Hagedorn dependence on chemical potentials for ABJM. While there are known results for the partition function, we could derive it through spherical harmonics, and obtain a similar expression to [47, 32] as a first step toward the desired results. The tree level Hagedorn was computed with no chemical potentials turned on, but numerical work is still in progress to identify the behavior with explicit dependence on (ω, Ω_i)
Lastly, the crown jewel of the project was applying the same framework as in [82], this time only for $AdS_4 \times \mathbb{C}P^3$. The peculiar results which are found, do not amount to the same satisfaction as for the case of SYM for obvious reasons. The culprit seems to be M-theory giving a 10 dimensional type IIA background with the $\mathbb{C}P^3$ metric. But even if this seems like the case, we obtain results that align with structures known from the literature up to numerical factors.

Even though it took some time to pick up the pace, we are at an exciting stage where several directions can be taken to extend this work. First, for future work, a Penrose limit could also be explored of the $OSp(2|2)$ case considered in 2.3, revealing if this sector might have a different dispersion. Furthermore, one could generate a new giant magnon as well. If this is carried out, the spin element can be implemented in various finite-size corrections to both the string, magnons etc. [11, 10, 8, 9, 81].
Second, to advance on the question of Hagedorn, a proper analysis will have to be made of the tree-level structure with all chemical potentials considered, and further look into decoupling limits. Additionally, it seems like a natural progression to consider how the one-loop contribution for T_H changes in the case of chemical potentials corresponding to the bosonic generators of the geometry. Using what was done in [123], this should be possible, but the extension is as of yet still not clear. Using the decoupled sectors, it would also be possible to find T_H for each case and extend to different coupling regimes in λ one could hope. Lastly, some subsectors might have a nice spin-chain description coupled to an external magnetic field e.g. or other interesting configurations. Generally, following and extending results from [78, 80, 77, 75] would be a nice addition to the literature.
On a different note, the whole SMT program has been busy over the last few years [19, 20, 17] providing near-BPS limits of $\mathcal{N} = 4$ SYM which enables the probing of finite N effects like D-branes and black hole physics. The authors of [18] seemed to be hopeful for the extension to ABJM, which also seems to be the motivation for this thesis in the first place and a starting point for a new era in SMT. With the result obtained for the $SU(2) \times SU(2)$ Landau-Lifshitz model etc. one could be hopeful for exciting new work in the near future.

Appendix A

SMT Limits of $\mathcal{N} = 4$ SYM

We go through some of the computations done in [82] for Spin Matrix backgrounds in SYM, to illustrate how the original computations were carried out. The starting point from here is to consider a parametrization of $AdS_5 \times S^5$, in the following way

$$\begin{aligned} z_0 &= R\cosh\rho e^{it}, & w_1 &= R\sin(\beta_1/2)\sin(\beta_2/2)e^{i\alpha_1} \\ z_1 &= R\sinh\rho\sin(\bar{\beta}/2)e^{i\bar{\alpha}_1}, & w_2 &= R\sin(\beta_1/2)\cos(\beta_2/2)e^{i\alpha_2} \\ z_2 &= R\sinh\rho\sin(\bar{\beta}/2)e^{i\bar{\alpha}_2}, & w_3 &= R\cos(\beta_1/2)e^{i\alpha_3} \end{aligned} \tag{A.0.1}$$

The geometry exhibits both features from S^5 which is associated to the angular momentum $J_j = -i\partial_{\alpha_j}$ and also $S^3 \subset AdS_5$ which associates to spin $S_i = -i\partial_{\bar{\alpha}_i}$. By combination of angles appropriately we can define γ and $\bar{\gamma}$ from α_j and $\bar{\alpha}_i$. In addition if we consider the global time coordinate, we can define new coordinates as per the discussion of null isometries of the non-relativistic strings

$$\begin{bmatrix} t \\ \bar{\gamma} \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 0 \\ 1 & -1/2 & c_1 \\ 1 & 1/2 & c_2 \end{bmatrix} \begin{bmatrix} x^0 \\ u \\ w \end{bmatrix} \tag{A.0.2}$$

This matrix equation precisely leads to the relations established in 4.5. The only addition is introducing the parameter w which is aligned along S^3 and is controlled by c_1 and c_2. It turns out that the parameters can be gauge fixed to $c_1 = 1$ and $c_2 = 0$ such that $s = -i\partial_w$. From here on out we should be able to study specific subsectors employing what we have established so far. This leads to reviewing some calculations for specific cases of PSU$(2|2,4)$.

The $SU(2)$ Background

The maybe simplest example is to consider the BPS-bound $E \geq Q = J_1 + J_2$. Since we only concern ourselves with Angular momentum generators we can focus solely on the S^5 part. One can decompose it in terms of a Fubini-Study metric and a fibration over one of the directions on the sphere. Thus we can write

$$d\Omega_5^2 = d\alpha^2 + \sin^2\alpha d\beta^2 + \cos^2\alpha[d\Sigma_1^2 + (d\gamma + A)^2], \quad A = \frac{1}{2}\cos\theta d\phi, \quad d\Sigma_1^2 = \frac{1}{4}(d\theta^2 + \sin^2\theta d\phi^2) \tag{A.0.3}$$

Thus we will focus on t and γ. If we write them as linear combination of the coordinates x_0 and u we get

$$t = x^0 - \frac{1}{2}u, \quad \gamma = x^0 + \frac{1}{2}u \tag{A.0.4}$$

If we want to get conditions for when we exactly have a manifold being null, we impose conditions on the full metric (i will write explicitly what) such that $g_{uu} = 0$ when $\rho = \alpha = 0$ Thus if we insert our transformations in the metric and try to reformulate it in terms of TNC variables

$$\begin{aligned} ds^2/R^2 = &-\cosh^2\rho(dx^0 - \frac{1}{2}du)^2 + d\rho^2 + \sinh^2 d\Omega_3^2 + d\alpha^2 + \sin^2\alpha d\beta^2 \\ &+ \cos^2\alpha[d\Sigma_1^2 + (dx^0 + \frac{1}{2}du)^2 + A^2 + 2A(dx^0 + \frac{1}{2}du)] \end{aligned} \tag{A.0.5}$$

We describe in detail how one might identify The TNC variable in this case which will be the same procedure used in the other cases too. If we group terms firstly that have a factor of du attached we can group terms and get $du(dx^0 + A)$, thus meaning we can identify $\tau = dx^0 + \frac{1}{4}\cos\theta d\phi$ then we look at the rest of the terms left. Since the structure of the TNC variable is of the form $2\tau(du - m)$ we look at terms that fit with τ when factorized. Terms that are left are $A^2 + 2Adx^0$, so we need to satisfy the equation $2\tau(du - m) = dx^0 du + A^2 + 2Adx^0 + Adu$. The choice can easily be seen to be $m = -\frac{1}{2}\cos\theta d\phi$. Lastly we have the term $h_{\mu\nu}dx^\mu dx^\nu$. We look for squared elements in the range of μ, ν meaning that our transformed coordinates are out of question. It can easily be seen that the Fubini-study metric precisely has the structure needed meaning $h_{\mu\nu}dx^\mu dx^\nu = \frac{1}{4}(d\theta^2 + \sin^2\theta d\phi^2)$. We can also group the $\cosh^2\rho$ term and $\cos^2\alpha$ and using standard trig-identities to get $-(\sinh^2\rho + \sin^2\alpha)(dx^0 + \frac{1}{2}du)^2$. Thus assembling it all we get the metric in terms of the TNC variables

$$ds^2/R^2 = 2\tau(du - m) + h_{\mu\nu}dx^\mu dx^\nu \tag{A.0.6}$$

A more elaborate continuation is given in "cite", but we just want to show how one can get this type of metric to begin with. The only thing that could be missing is to take the SMT limit now and obtain $\tau = d\tilde{x}^0$ when combining the BPS-bound with the coordinate transformation. Further one can gauge fix the World-sheet by fixing the zweibeins and and taking a gauge choice on η. This will reduce the sigma-model lagrangian to a Landau-Lifshitz model describing spin chains.

The $SU(2|3)$ Background

From the previous example, we extend the BPS-bounds and consider now the maximal choice of S^5 where $E \geq Q = J_1 + J_2 + J_3$. This subsectors is a $SU(2|3)$ theory with the largest possible compact spin group of $\mathcal{N} = 4$ SYM. Thus we are zooming in on all the commuting generators of S^5 seen from the bulk perspective. There will be an emergence of $\mathbb{CP}^2$ as the compact spatial section parametrized by a Fubini study-metric giving us the U(1)-Galilean background. The strategy is to perform a Hopf-fibration such that the S^5 is described as a circle fibration (parametrized by χ) over the $\mathbb{CP}^2$ space. The way we want to define the fibration coordinate is through $Q = J_1 + J_2 + J_3 = -i\partial_\chi$. The reasoning leads back to this vector being of constant length on particular submanifolds on the geometry. When defining u, this will ensure that ∂_u will be null on specific submanifolds as well. To this mean we perform a set of linear transformations of the α_i´s by the following matrix

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 & -1/2 \\ 1 & -1/2 & 1/2 \\ 1 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} \chi \\ \psi \\ \phi \end{pmatrix} \tag{A.0.7}$$

Since only considering angles on S^5, AdS$_5$ can be disregarded, leading to the background

$$
\begin{aligned}
ds^2/R^2 &= d\xi^2 + \sin^2(\xi)(d\theta + \sin^2(\theta/2)d\alpha_1^2 + \cos^2(\theta/2)d\alpha_2^2) + \cos^2(\xi)d\alpha_3^2 \\
&= d\xi^2 + \sin^2(\xi)(d\theta + \sin^2(\theta/2)(d\chi + \frac{1}{2}d\psi - \frac{1}{2}d\phi)^2 \\
&+ \cos^2(\theta/2)(d\chi - \frac{1}{2}d\psi + \frac{1}{2}d\phi)^2) + \cos^2(\xi)(d\chi + \frac{1}{2}d\psi)^2
\end{aligned}
\tag{A.0.8}
$$

The angles range from $\xi \in (0, \pi/2)$ and $\theta \in (0, \pi)$. Expanding and gathering terms in the fashion such that we have the circle fibration over χ, we can rewrite the full metric in the following form

$$
ds^2/R^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\bar{\Omega}_3^2 + (d\chi + B)^2 + d\Sigma_2^2 \tag{A.0.9}
$$

The metric can be expressed in terms of a Fubini study metric an potentials defined as

$$
\begin{aligned}
&B = \sin^2 \xi (d\psi + A), \quad A = \frac{1}{2}\cos\theta d\phi \\
&d\Sigma_2^2 = d\xi^2 + \sin^2 \xi d\Sigma_1^2 + \cos^2 \xi \sin^2 \xi (d\psi + A), \quad d\Sigma_1^2 = \frac{1}{4}(d\theta^2 + \sin^2 \theta d\phi^2)
\end{aligned}
\tag{A.0.10}
$$

To obtain a $U(1)$ Galilean background, one relates the coordinates considered to a pair of new ones related to a submanifold where you will have null geodesics along the isometry of the considered subsector. Introducing x^0 and u we get

$$
\begin{pmatrix} t \\ \chi \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 1 & -1/2 \end{pmatrix} \begin{pmatrix} x^0 \\ u \end{pmatrix} \tag{A.0.11}
$$

Since u is of constant length across $\mathbb{CP}^2 \subset S^5$ the following condition has to be met

$$
4(\partial_u)^2/R^2 = -\cosh^2 \rho + 1 \leq 0 \tag{A.0.12}
$$

u will be null if and only if $\rho = 0$. This six-dimensional manifold is now described by coordinates $\{x^0, u, \theta, \phi, \xi, \psi\}$, where the last angle is part of the $\mathbb{CP}^2$. One obtains a metric that can be written using the condition on ρ

$$
\begin{aligned}
ds^2/R^2 &= -(dx^0 - \frac{1}{2}du)^2 + (dx^0 + \frac{1}{2}du + B)^2 + d\Sigma_2^2 \\
&= du(2dx^0 + B) + B^2 + 2Bdx^0 + d\Sigma_2^2 \\
&= 2\tau(du - m) + h_{ij}dx^i dx^j
\end{aligned}
\tag{A.0.13}
$$

It is easy to read off what the three different TNC-variables are

$$
\tau = dx^0 + \frac{1}{2}B, \quad m = -B, \quad h = d\Sigma_2^2 \tag{A.0.14}
$$

As a check to see if one is on the right path, one can look for a structure of subsector which in this case would be $SU(2) \subset SU(2|3)$. This seems plausible since it has been engineered via a hopf fibration of $\mathbb{CP}^1$ inside the S^3, corresponding to the previous BPS-bound we considered. Setting $\xi = \pi/2$ and fixing ψ, realizes the same potentials and study-metrics. This is a general trend that can be derived starting from the maximal PSU$(1,2|3)$ background and reducing on it

The $SU(1,1)$ Background

The last real subsector we will look at before the full beast, is a background mixed between the spin and angular momentum mixing both the S^5 and AdS_5. This will correspond to a $SU(1,1)$ background with the particular choice $Q = S_1 + J_1$. Taking the embedding coordinates, one finds the induced metric to be

$$\begin{aligned} ds^2/R^2 = &- \cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho \left(\frac{d\bar{\beta}^2}{4} + \sin^2(\bar{\beta}/2) d\bar{\alpha}_1^2 + \cos^2(\bar{\beta}/2) d\bar{\alpha}_2^2\right) \\ &+ \frac{d\beta_1^2}{4} + \sin^2(\beta_1/2)\left(\frac{d\beta_2^2}{4} + \sin^2(\beta_2/2) d\alpha_1^2 + \cos^2(\beta_2/2) d\alpha_2^2\right) \\ &+ \cos^2(\beta_1/2) d\alpha_3^2 \end{aligned} \tag{A.0.15}$$

From the definitions $S_1 = -i\partial_{\bar{\alpha}_1}$ and $J_1 = -i\partial_{\alpha_1}$. Given this, performing the transformation x^0, u, w to the choices that correspond to our choice in the near BPS-limit gives

$$\begin{pmatrix} t \\ \bar{\alpha}_1 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} 1 & -1/2 & 0 \\ 1 & -1/2 & c_1 \\ 1 & 1/2 & c_2 \end{pmatrix} \begin{pmatrix} x^0 \\ u \\ w \end{pmatrix} \tag{A.0.16}$$

Reading off from the second column and gathering the pre-factors in front of our original coordinates, gives the null condition on u

$$4(\partial_u)^2/R^2 = -\cosh^2 \rho + \sinh^2 \rho \sin^2(\bar{\beta}/2) + \sin^2(\beta_1/2)\sin^2(\beta_2/2) \leq 0 \tag{A.0.17}$$

The condition for u being null is exactly satisfied when $\bar{\beta} = \beta_1 = \beta_2 = \pi$. Inserting this into the metric and using the conditions one finds

$$\begin{aligned} ds^2/R^2 &= -\cosh^2 \rho (dx^0 - \frac{1}{2}du)^2 + d\rho^2 + \sinh^2 \rho (dx^0 - \frac{1}{2}du + c_1 dw)^2 \\ &+ (dx^0 + \frac{1}{2}du + c_2 dw)^2 \\ &= 2dx^0 du + d\rho^2 + \sinh^2 \rho (c_1^2 dw^2 + 2c_1 dw(dx^0 - \frac{1}{2}du)) + c_2^2 dw^2 + 2c_2 dw(dx^0 + \frac{1}{2}du) \\ &= du(2dx^0 - (c_1 \sinh^2 \rho - c_2)dw) + (c_1^2 \sinh^2 \rho + c_2^2)dw^2 + 2dx^0 dw(c_1 \sinh^2 \rho + c_2) \end{aligned} \tag{A.0.18}$$

It can easily be read off what the TNC-data on the submanifold is

$$\begin{aligned} \tau &= dx^0 - \frac{1}{2}(c_1 \sinh^2 \rho - c_2)dw \\ m/R^2 &= -(c_1 \sinh^2 \rho + c_2)dw \\ h/R^2 &= d\rho^2 + c_1^2 \sinh^2 \rho \cosh^2 \rho dw^2 \end{aligned} \tag{A.0.19}$$

Apparently, the spatial slices of the geometry parametrized by ρ, w are non-compact compared to the $SU(3|2)$. Also, it can be shown that the constants can be fixed such that they have the values $c_1 = 1$ and $c_2 = 0$. If we insert this into the gauge-fixed action on this background we arrive at the result

$$\begin{aligned} S_{\text{flat,gf}} &= -\frac{J}{2\pi} \int d^2\sigma (m_\mu x^\mu + \frac{1}{2} h_{\mu\nu} dx^\mu dx^\nu) \\ &= -\frac{J}{2\pi} \int d^2\sigma \Big[\sinh^2 \rho \dot{w} - \frac{1}{2}\left((\rho')^2 + \sinh^2 \rho \cosh^2 \rho (w')^2\right) \Big]. \end{aligned} \tag{A.0.20}$$

To compare, one can make the correct coordinate choices and obtain/reproduce the action obtained from coherent states in the $sl(2)$ spin chain[27] and spinning strings on $\mathrm{AdS}_5 \times \mathrm{S}^5$. The discussion can be extended to the maximal $PSU(1,2|3)$ case as well, but this case just contains all the cases we have reviewed so far. The interested reader is referred to [82] for the details, where by our calculations, it should be realtively easy to understand the details and even go through the computation by one self.

Appendix B

Letter partition function for $\mathcal{N}=4$ SYM on $\mathbb{R}\times S^3$

Consider the partition function for N = 4 SYM on $\mathbb{R}\times S^3$ in the presence of non-zero chemical potential for R-charges of the $SU(4)$ R-symmetry and the Cartan generators of $SO(4)$ group on S^3. We go through the computations to arrive at 4.3.16 for SYM. A method to attack the problem is by spherical harmonic expansion where we expand each field and associate its spherical harmonic to it. There are 3 parts of this calculation before we can add everything together, one partition function for scalars, vectors and fermions. It is worth noting that the Cartan generators for $SO(4)$ is split into $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$.

Partition function for scalars

Scalar partition function in terms of its spherical harmonics are given by $S_{j,m,\bar{m}}(\alpha)$, where α is the coordinates on S^3 and m and $\bar{m}$ are the eigenvalues of $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$. Here $m,\bar{m}=-\frac{j}{2},-\frac{j}{2}+1,...,\frac{j}{2}$. This gives the Partition function

$$\eta_s(x,\rho,\bar{\rho},y_i)=\sum_{i=1}^{3}\sum_{j=0}^{\infty}\sum_{m=-\frac{j}{2}}^{\frac{j}{2}}\sum_{\bar{m}=-\frac{j}{2}}^{\frac{j}{2}}x^{j+1}\rho^m\bar{\rho}^{\bar{m}}(y_i+y_i^{-1}) \tag{B.0.1}$$

For convenience, introducing the two functions makes lif easier

$$\rho=e^{\beta(\omega_1-\omega_2)},\quad \bar{\rho}=e^{\beta(\omega_1+\omega_2)} \tag{B.0.2}$$

To evaluate this, the two first sums that are finite can be handled which are nothing but geometric series

$$\sum_{\bar{m}=-\frac{j}{2}}^{\frac{j}{2}}\bar{\rho}^{\bar{m}}=-\bar{\rho}^{-\frac{j}{2}}\frac{1-\bar{\rho}^{j+1}}{1-\bar{\rho}}\quad\text{and}\quad\sum_{m=-\frac{j}{2}}^{\frac{j}{2}}\rho^{m}=-\rho^{-\frac{j}{2}}\frac{1-\rho^{j+1}}{1-\rho} \tag{B.0.3}$$

Inserting this into B.0.1 we get

$$\eta_s(x,\rho,\bar{\rho},y_i)=\frac{x}{(1-\bar{\rho})(1-\rho)}\sum_{i=1}^{3}\sum_{j=0}^{\infty}x^{j+1}\rho^{-\frac{j}{2}}\bar{\rho}^{-\frac{j}{2}}(1-\bar{\rho}^{j+1})(1-\rho^{j+1})(y_i+y_i^{-1}) \tag{B.0.4}$$

Multiplying everything and splitting it into 4 terms, we recognize each of them as a geometric series for the index j, which can be performed

$$\eta_s(x,\rho,\bar{\rho},y_i) = \sum_{i=1}^{3} \frac{x}{(1-\bar{\rho})(1-\rho)} \times \left(\frac{1}{1-xe^{-\beta\omega_1}} - \frac{\rho}{1-xe^{-\beta\omega_2}} - \frac{\bar{\rho}}{1-xe^{\beta\omega_2}} + \frac{\bar{\rho}\rho}{1-xe^{\beta\omega_1}} \right)(y_i + y_i^{-1}) \tag{B.0.5}$$

This is a tedious expression to evaluate, so instead Mathematica was used to evaluate the sums, which in the end result in

$$\eta_s(x,\rho,\bar{\rho},y_i) = \frac{e^{2\beta\omega_1}(x-x^3)}{(e^{\beta\omega_1}-x)(-e^{\beta\frac{\omega_1+\omega_2}{2}} + xe^{\beta\frac{\omega_1-\omega_2}{2}})(e^{\beta\frac{\omega_1-\omega_2}{2}} - xe^{\beta\frac{\omega_1+\omega_2}{2}})(-1+xe^{\beta\omega_1})} \tag{B.0.6}$$

We want to have the structure of the denominator in all of the expressions to be the same if possible, so by factorizing everything out and writing it as

$$\prod_{k=1}^{2}((1-xe^{\beta\omega_k})(1-xe^{-\beta\omega_k}))^{-1} = \frac{1}{(1-xe^{\beta\omega_1})(1-xe^{-\beta\omega_1})(1-xe^{\beta\omega_2})(1-xe^{-\beta\omega_2})} \tag{B.0.7}$$

To obtain the desired denominator, a $e^{2\beta\omega_1}$ term is picked up that will cancel in the numerator

$$\eta_s(x,\rho,\bar{\rho},y_i) = \frac{x-x^3}{(1-xe^{\beta\omega_1})(1-xe^{-\beta\omega_1})(1-xe^{\beta\omega_2})(1-xe^{-\beta\omega_2})} \sum_{i=1}^{3}(y_i + y_i^{-1}) \tag{B.0.8}$$

This is precisely the reduced scalar partition function that was needed.

Partition function for Vectors

Apparently it seems that vectors are neutral under R-charges. Also the spherical harmonics corresponding to the gauge boson in the representation $[0,0,0]_{(1,0)}$ are denoted with $V^L_{j,m,\bar{m}}(\alpha)$ where $\bar{m} = -\frac{j-1}{2}, ..., \frac{j-1}{2}$ and $m = -\frac{j+1}{2}, ..., \frac{j+1}{2}$ Using this we write the partition function

$$\eta_{VR}(x,\rho,\bar{\rho},y_i) = \sum_{j=1}^{\infty} \sum_{m=-\frac{j+1}{2}}^{\frac{j+1}{2}} \sum_{\bar{m}=-\frac{j-1}{2}}^{\frac{j-1}{2}} x^{j+1}\rho^m \bar{\rho}^{\bar{m}} \tag{B.0.9}$$

To use the same trick for the geometric series over the j-sum, we must re-index the sum which on the contrary will affect the m-sums. Set $i = j-1$, then we can set the j-sum to go from $i = 0$. But this will change $\bar{m} = -\frac{j-1}{2}, ..., \frac{j-1}{2} \to -(\frac{i}{2}+1), ..., \frac{i}{2}+1$ and $m = -\frac{j+1}{2}, ..., \frac{j+1}{2} \to -\frac{i}{2}, ..., \frac{i}{2}$ and also $x^{j+1} \to x^2 \cdot x^i$ giving us

$$\eta_{VR}(x,\rho,\bar{\rho},y_i) = x^2 \sum_{i=0}^{\infty} \sum_{m=-\frac{i}{2}}^{\frac{i}{2}} \sum_{\bar{m}=-(\frac{i}{2}+1)}^{\frac{i}{2}+1} x^i \rho^m \bar{\rho}^{\bar{m}} \tag{B.0.10}$$

Again the two inner sums can be performed as finite geometric series and get

$$\sum_{m=-\frac{i}{2}}^{\frac{i}{2}} \rho^m = \rho^{-\frac{i}{2}} \frac{1-\rho^{i+1}}{1-\rho} \quad \text{and} \quad \sum_{\bar{m}=-(\frac{i}{2}+1)}^{\frac{i}{2}+1} \bar{\rho}^{\bar{m}} = \bar{\rho}^{-(\frac{i}{2}+1)} \frac{1-\bar{\rho}^{i-1}}{1-\bar{\rho}} \tag{B.0.11}$$

Thus if we put this into the computer and let it do the geometric series over the index i, we get after factorizing the denominator and multiplying out the numerator

$$\frac{x^2[1+2\cosh(\beta\omega_1)\cosh(\beta\omega_2)-2x(\cosh(\beta\omega_1)+\cosh(\beta\omega_2))+x^2]}{(1-xe^{\beta\omega_1})(1-xe^{-\beta\omega_1})(1-xe^{\beta\omega_2})(1-xe^{-\beta\omega_2})} \tag{B.0.12}$$

Before going any further, we can check the other sum for the vector which is the su$(2)_L$ sector. We see that it is given by the following sum after re-indexing as before

$$\eta_{VL}(x,\rho,\bar\rho,y_i)=x^2\sum_{i=0}^{\infty}\sum_{m=-\frac{i+2}{2}}^{\frac{i+2}{2}}\sum_{\bar m=-(\frac{i}{2})}^{\frac{i}{2}} x^i\rho^m\bar\rho^{\bar m} \tag{B.0.13}$$

Thus only the index on the m and $\bar m$ sum get switched, resulting in the same expression, giving us only an additional factor of 2

$$\frac{x^2[1+2\cosh(\beta\omega_1)\cosh(\beta\omega_2)-2x(\cosh(\beta\omega_1)+\cosh(\beta\omega_2))+x^2]}{(1-xe^{\beta\omega_1})(1-xe^{-\beta\omega_1})(1-xe^{\beta\omega_2})(1-xe^{-\beta\omega_2})} \tag{B.0.14}$$

This provides the second piece of the puzzle.

Partition function for Fermions

Now for the fermion. The peculiar feature compared to the rest is that they appear in 2 representations, namely both in $[0,0,1]_{(\frac{1}{2},0)}$ and $[0,0,1]_{(0,\frac{1}{2})}$. We can introduce the spherical harmonics as usual. One can for the $[0,0,1]_{(\frac{1}{2},0)}$ representation write for the eigenvalues for S_L and S_R given by $\bar m=-\frac{j-1}{2},...,\frac{j-1}{2}$ and $m=-\frac{j}{2},...,\frac{j}{2}$. One have to take into account that the R-charge chemical potentials for fermions in this representation is given by

$$Y_1=(y_1y_2y_3)^{\frac{1}{2}}+y_1^{\frac{1}{2}}(y_2y_3)^{-\frac{1}{2}}+y_2^{\frac{1}{2}}(y_1y_3)^{-\frac{1}{2}}+y_3^{\frac{1}{2}}(y_1y_2)^{-\frac{1}{2}} \tag{B.0.15}$$

This will give the sum for fermions

$$\eta_{F1}(x,\rho,\bar\rho,y_i)=Y_1\sum_{j=1}^{\infty}\sum_{m=-\frac{j}{2}}^{\frac{j}{2}}\sum_{\bar m=-(\frac{j-1}{2})}^{\frac{j-1}{2}} x^i\rho^m\bar\rho^{\bar m} \tag{B.0.16}$$

Doing the re-indexing as for the vector part, we find setting $i=j-1$, we get $\bar m=-\frac{j-1}{2},...,\frac{j-1}{2}\to-(\frac{i}{2}),...,\frac{i}{2}$ and $m=-\frac{j}{2},...,\frac{j}{2}\to-\frac{i+1}{2},...,\frac{i+1}{2}$ and also $x^{j+1}\to x^{\frac{3}{2}}x^i$ giving us

$$\eta_{F1}(x,\rho,\bar\rho,y_i)=Y_1x^{\frac{3}{2}}\sum_{i=0}^{\infty}\sum_{m=-\frac{i+1}{2}}^{\frac{i+1}{2}}\sum_{\bar m=-(\frac{i}{2})}^{\frac{i}{2}} x^i\rho^m\bar\rho^{\bar m} \tag{B.0.17}$$

Following the previous steps, we solve for the two innermost sums and evaluate the i-sum for the whole thing and find

$$\frac{x^{\frac{3}{2}}(e^{\frac{1}{2}\beta(\omega_1-\omega_2)}+e^{-\frac{1}{2}\beta(\omega_1-\omega_2)}-xe^{\frac{1}{2}\beta(\omega_1+\omega_2)}-xe^{\frac{1}{2}\beta(\omega_1+\omega_2)}}{(1-xe^{\beta\omega_1})(1-xe^{-\beta\omega_1})(1-xe^{\beta\omega_2})(1-xe^{-\beta\omega_2})}=\frac{2x^{\frac{3}{2}}(\cosh[\beta(\frac{\omega_1-\omega_2}{2}]-x\cosh[\beta(\frac{\omega_1+\omega_2}{2})])}{(1-xe^{\beta\omega_1})(1-xe^{-\beta\omega_1})(1-xe^{\beta\omega_2})(1-xe^{-\beta\omega_2})} \tag{B.0.18}$$

This assembles by combining with the $SU(4)$ terms to

$$\eta_{F1}(x,\omega_j,y_i)=Y_1\frac{2x^{\frac{3}{2}}(\cosh[\beta(\frac{\omega_1-\omega_2}{2}]-x\cosh[\beta(\frac{\omega_1+\omega_2}{2})])}{(1-xe^{\beta\omega_1})(1-xe^{-\beta\omega_1})(1-xe^{\beta\omega_2})(1-xe^{-\beta\omega_2})} \tag{B.0.19}$$

This is the exact expression we were looking for. But what about the other sum for the other representation. The only difference is the sign in the exponents between the two chemical potentials swap, making the calculation essentially the same. Thus we can finish the calculation to gain the full letter partition function. Defining Y_2

$$Y_2 = (y_1y_2y_3)^{-\frac{1}{2}} + y_1^{-\frac{1}{2}}(y_2y_3)^{\frac{1}{2}} + y_2^{-\frac{1}{2}}(y_1y_3)^{\frac{1}{2}} + y_3^{-\frac{1}{2}}(y_1y_2)^{\frac{1}{2}} \tag{B.0.20}$$

makes it possible to gather the full expression given by the sum of all the pieces to obtain the full letter partition function

$$\begin{aligned}
z(x,\omega_j,y_i) = \prod_{k=1}^{2}((1-xe^{\beta\omega_k})(1-xe^{-\beta\omega_k}))^{-1}\{(x-x^3)\sum_{l=1}^{3}(y_l+y_l^{-1})\\
+ \quad 2x^2\Big[1+2\cosh(\beta\omega_1)\cosh(\beta\omega_2)-x(\cosh(\beta\omega_1)+\cosh(\beta\omega_1))+x^2\Big]\\
+ \quad 2Y_1x^{\frac{3}{2}}\left[\cosh[\beta(\frac{\omega_1-\omega_2}{2}]-x\cosh[\beta(\frac{\omega_1+\omega_2}{2})]\right]\\
+ \quad 2Y_2x^{\frac{3}{2}}\left[\cosh[\beta(\frac{\omega_1+\omega_2}{2}]-x\cosh[\beta(\frac{\omega_1-\omega_2}{2})]\right]\Bigg\}
\end{aligned} \tag{B.0.21}$$

Appendix C

The SU(2) Heisenberg $XXX_{1/2}$ spin chain, Integrability and Yang-baxter

We review some basics on spin-chains, integrability and Yang-Baxter to connect to the discussion in section 3.1 to give the reader a more complete understanding of the whole landscape

C.1 $XXX_{1/2}$ spin chain

A classic problem appearing in fields like condensed matter physics has to do with spin chains which is of interest in AdS/CFT as well. Thus, it is worth reviewing the original work of Bethe [30]. The story starts at the Hamiltonian for the Heisenberg ferromagnetic $XXX_{1/2}$ spin-chain. This has a ground state with all L sites containing spin-up $|\uparrow\uparrow ... \uparrow\rangle$. We sketch out how to get the S-matrix, Bethe equation and momentum constraints in terms of rapidity comes forth. This will be done for the one-magnon, two-magnon and M-magnon case[112] **One-magnon state**:

Consider a spin-chain where single excitations occur. This gives an eigenvector of the form

$$|p\rangle = \frac{1}{\sqrt{L}}\sum_{l=1}^{L} e^{ipl} S_l^+ |0\rangle \tag{C.1.1}$$

The ground state is defined as $|0\rangle = |\uparrow\uparrow\uparrow ... \uparrow\rangle$, where the Hamiltonian is needed such that the energy spectrum can be derived. The Hamiltonian for this system was found to be

$$\mathcal{H} = \frac{\lambda}{8\pi^2}\sum_{l=1}^{L}(1 - P_{l,l+1}) = \frac{\lambda}{8\pi^2}\sum_{l=1}^{L}(\frac{1}{2} - \frac{1}{2}\vec{S}_l \cdot \vec{S}_{l+1}) \tag{C.1.2}$$

One Defines a permutation operator that switches the position of neighboring sites. This translates into spin operators as well, in a classical QM sense.

$$\begin{aligned}\Gamma |p\rangle &= \frac{\lambda}{8\pi^2}(2\,|\uparrow ... \downarrow\uparrow ... \uparrow\rangle - |\uparrow ... \downarrow\uparrow ... \uparrow\rangle - |\uparrow ... \uparrow\downarrow ... \uparrow\rangle \\ &= \frac{\lambda}{8\pi^2}(2 - e^{ip} - e^{-ip})\,|p\rangle = \frac{\lambda}{8\pi^2}(2(1-\cos(p))\,|p\rangle = \frac{\lambda}{2\pi^2}\sin^2\frac{p}{2}\,|p\rangle\end{aligned} \tag{C.1.3}$$

Lastly, one imposes periodic boundary conditions such that $e^{ip(l+L)} = e^{ipl}$ providing quantization condition $\frac{2\pi n}{L}, n \in \{0, 1, ..., L-1\}$. $|p\rangle$ is a single magnon state. Via a quantization argument, the only allowed state is $p = 0$. So there are no operators that are not chiral primaries when only having a single Y excitation.

Two Magnon state:

Next, we investigate the double magnon state. Some subtleties arise, namely that we can only have scattering if the two magnons are in close proximity. If the two Y-excitations are far from each other, we might as well say that the magnons are independent. But in the case where they are in proximity on sites $(l, l+1)$, new features present themselves. A priori, one considers a superposition of all two-body states parameterized by two momenta p_1, p_2 at sites l_1, l_2

$$|p_1, p_2\rangle = \sum_{l_1<l_2} \psi(l_1, l_2) S^-_{l_1} S^-_{l_2} |0\rangle = \left[\sum_{l_1<l_2} e^{i(p_1 l_1 + p_2 l_2)} + s(p_1, p_2) \sum_{l_1>l_2} e^{i(p_1 l_1 + p_2 l_2)} \right] S^-_{l_1} S^-_{l_2} |0\rangle \tag{C.1.4}$$

There is freedom in adding a phase factor $e^{i\phi}$, which usually associates to the scattering matrix. Following [30], one can consider two situations where either magnons are situated right next to each other, or where they are split apart by more than one site. This gives two equations that will determine both the energy and the S-Matrix

$$\begin{aligned} E(p_1, p_2)\psi(l_1, l_2) &= 4\psi(l_1, l_2) - \psi(l_1+1, l_2) - \psi(l_1-1, l_2) - \psi(l_1, l_2 1) - \psi(l_1, l_2-1) \\ E(p_1, p_2)\psi(l_1, l_2) &= 2\psi(l_1, l_2) - \psi(l_1-1, l_2) - \psi(l_1, l_2+1) \end{aligned} \tag{C.1.5}$$

Inserting the ansatz into the first equation, the energy is found to be

$$E(p_1, p_2)\psi(l_1, l_2) = (4 - e^{-ip_1} - e^{ip_2} - e^{ip_1} - e^{ip_2})\psi(l_1, l_2) = 4\sum_{i=1}^{2} \sin^2 \frac{p_i}{2} \psi(l_1, l_2) \tag{C.1.6}$$

For the S-matrix, the equation can be derived through the second constraint

$$\begin{aligned} &e^{ip_2}(2 - e^{ip_2} - e^{-ip_1}) + e^{ip_1}(2 - e^{ip_1} - e^{-ip_2}) s(p_1, p_2)) \\ &= (4 - e^{-ip_1} - e^{ip_2} - e^{ip_1} - e^{ip_2})(e^{ip_2} + e^{ip_1} s(p_1, p_2)) \leftrightarrow \\ &s(p_1, p_2) = \frac{e^{i(p_1+p_2)} - 2e^{ip_1} + 1}{e^{i(p_1+p_2)} - 2e^{ip_2} + 1} = s(p_2, p_1)^{-1} \end{aligned} \tag{C.1.7}$$

Alternatively, an expression for the S-Matrix depending explicitly on trigonometric functions is also possible to derive

$$s(p_1, p_2) = \frac{\cot \frac{p_2}{2} - \cot \frac{p_2}{2} - 2i}{\cot \frac{p_1}{2} - \cot \frac{p_2}{2} + 2i} \tag{C.1.8}$$

Typically this also associates a phase to the S-matrix $e^{i\phi}$, which implies that for real momenta we have $s(p_1, p_2)s(p_2, p_1) = 1$. Imposing periodic boundary conditions by putting the two magnons back on a cyclic spin chain of length L. To quantize p_1, transport the magnon once around the circle leaving the state invariant. But, transporting will bring the first magnon past the second one, so it also picks up a phase $e^{i\phi}$. This gives $e^{i\phi}e^{ip_1 L} = 1$. Using the momentum constraint, one finds $e^{ip_1 L} = s(p_1, p_2)$ and similarly $e^{ip_2 L} = s(p_2, p_1)$ giving the allowed quantized values $p_1 = \frac{2\pi l}{L-1}$ (same for p_2), making the eigenvalues take the form $E = 4\sum_{i=1}^{2} \sin^2 \frac{\pi l}{L-1}$. Introducing the rapidity as $u = 2\cot \frac{p}{2}$ in the Bethe equation will provide an expression for the S-matrix as

$$s(p_2, p_1) = \frac{u_1 - u_2 - i}{u_1 - u_2 + i} \tag{C.1.9}$$

Using this we can finally obtain the Bethe equation

$$\left[\frac{u_1 + i/2}{u_1 - i/2}\right]^L = \frac{u_1 - u_2 + i}{u_1 - u_2 - i}, \quad \left[\frac{u_2 + i/2}{u_2 - i/2}\right]^L = \frac{u_2 - u_2 + i}{u_2 - u_1 - i} \tag{C.1.10}$$

M-body problem:

Ending with the M-body problem, which at first glance seems daunting, amazingly turns out to have all the relevant data of the scattering and spectrum which is fully encoded in the dispersion relation and the two-body scattering. Considering the M-body problem, the general magnon state as a superposition of all M-excitation states given momenta p_i can be written as

$$|p_i\rangle = \sum_{1\leq x_1\leq ...\leq x_m\leq L} \psi(x_1, ..., x_m) \prod_n^M a^\dagger_{x_n} |0\rangle \tag{C.1.11}$$

Using the Bethe ansatz, we can generalize the whole system to have a wave-function

$$\psi(x_1, ..., x_m) = \sum_{\{\tau\}} A(\tau) \prod_{i=1}^M e^{ip_{\tau_i} x_i} \tag{C.1.12}$$

$\{\tau\}$ represents all possible permutations of the excitations in the M-body state. $A(\tau)$ is the amplitude that will relate to the S-matrix "cite". In the end one obtains periodic boundary conditions for the momenta such that the total phase factor picked up by a magnon when going around the chain is equal to the product of scattering with all of the other magnons

$$e^{ip_k L} = \prod_{j\neq k}^M s(p_k, p_j) \tag{C.1.13}$$

Using the same trick for rapidity, and defining $u_k = \frac{1}{2}\cot\frac{p_k}{2}$, will in exchange give the full M-body Bethe equation and additionally the trace condition

$$\left(\frac{u_k + i/2}{u_k - i/2}\right)^L = \prod_{j\neq k}^M \frac{u_k - u_j + i}{u_k - u_j - i}, \quad \prod_{j=1}^M \frac{u_j + i/2}{u_j - i/2} = 1 \tag{C.1.14}$$

The energy just becomes the sum of the two-body scattering we considered, giving us

$$\sum_k^M E(p_k) = \sum_k^M 4\sin^2\frac{p_k}{2} = \sum_k^M \frac{4}{1+\cot^2\frac{p_k}{2}} = \sum_k^M \frac{1}{u_k^2 + \frac{1}{4}} \tag{C.1.15}$$

Thus we have encoded a very big problem into information containing the data of 2-body scattering. This is why Bethe equations and ansatz are so celebrated since they provide a gateway to integrability at a relatively low price. One can consider other possible sectors where other types of interactions might enter. We touch upon this when the link to conformal field theory is done. But we explore classical integrability and yang-Baxter equations first

C.2 Integrability and the Yang-Baxter Equation

Integrability stems from the classical notion of Hamiltons equations[53, 130], supplemented by Liouvilles Theorem. Considers a function $F(q_i, p_i)$ which are defined by conjugate variables. Then Finding the EOM´s gives

$$\dot{F}(q_i, p_i) = \sum_{i=1}^n \left(\frac{\partial F}{\partial q_i}\dot{q}_i - \frac{\partial F}{\partial p_i}\dot{p}_i\right) \tag{C.2.1}$$

Using Hamiltons equations $\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}$ and $\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$, makes it possible to reformulate it in terms of Poisson brackets, which are the classical analogue to commutators

$$\dot{F}(q_i, p_i) = \sum_{i=1}^n \left(\frac{\partial F}{\partial q_i}\frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial \mathcal{H}}{\partial q_i}\right) = \{F, \mathcal{H}\} \tag{C.2.2}$$

Then Liouvilles theorem states that *the equations of motion of a Liouville integrable system can be solved by quadratures.* This is very powerful since it provides a way to analyze and solve various systems. But other approaches are possible as well. Promoting the functions to $N \times N$ matrices, and Poisson brackets to commutators, one can define a set of Lax pairs H, L such that $\dot{L} = [H, L]$. The Lax pairs can either be free of or depend upon an additional complex variable, called spectral parameter, λ. This proves as a stronger form of integrability. It is possible to relate the lax pairs, to something called the r-matrix. From [130] it can be seen that through the Jacobi identity, assuming r_{ij} does not depend on the dynamical variables and satisfying $r_{ij} = -r_{ji}$, then one can obtain the classical Yang-Baxter equation.

$$[r_{12}(\lambda - \mu), r_{13}(\lambda)] + [r_{12}(\lambda - \mu), r_{23}(\mu)] + [r_{13}(\lambda), r_{23}(\mu)] = 0 \tag{C.2.3}$$

Having motivated this, we move on to the quantum promotion. Defining the quantum R-matrix can be thought of as a function again depending on spectral parameters $R(\lambda, \mu)$ and taking a linear mapping from a tensor product of two Hilbert space $\mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$. Taking the R-matrix again, a quantum Yang-Baxter (qYBE) equation can be defined which just takes $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$ instead. In all its glory the qYBE is

$$R_{12}(\lambda - \mu) R_{13}(\lambda) R_{23}(\mu) = R_{23}(\mu) R_{13}(\lambda) R_{12}(\lambda - \mu) \tag{C.2.4}$$

As in the classical part, we will also have a Lax operator. It can be interpreted as the transport between two consecutive sites $|v_{i+1}\rangle = \mathcal{L}_i |v_i\rangle$. In a similar fashion to the RRR-equation, one can

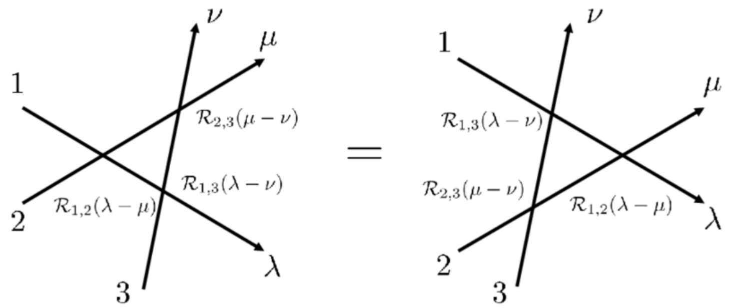


Figure C.1: Graphical representation of the Yang-baxter equation as scattering in $1 + 1$ dimensional QFT which factorize into two-body scattering. The order of scattering is associative, meaning that the S-matrix is not concerned with. Thus physical observables are independent of the order of pairwise scattering

do the same and construct a RLL-equation. Lastly, the lax operators can define a new object called the monodromy matrix. If transport needs to happen over far apart sites, consecutive uses of the lax operator can be used. This gives the monodromy matrix

$$\mathcal{T}_\alpha(\lambda) = \prod_{i=1}^{N} \mathcal{L}_{\alpha, N+1-i}(\lambda) \tag{C.2.5}$$

This will also satisfy a Yang-Baxter equation of the form RTT. One could ask why bother, and just proceed with the coordinate Bethe ansatz. It turns out that there are various types of Bethe ansatz, where in this case, the algebraic version is considered. Through ABA, it is allowed to find both the eigenvalues and eigenvectors of the transfer matrix. But more than that, to find not only the spectrum of the Hamiltonian but the spectrum of all the conserved charges of our model in a straightforward manner. Maybe it lacks the physical intuition that one can get through the S-matrix and magnon states, but using Monodromy-matrices, R-matrices and lax operators, we can prove integrability for relevant systems.

Appendix D

Massive Scalar Field: Propagators for $S = 0$ et al.

D.1

Taking the simplest example for scalars, one might try to solve the source-free Klein-Gordon equation starting from the same action as was considered in the computation for the boundary asymptotics. The usual EOM is $(\Box_g - m^2)\phi_\Delta(x,z) = 0$ subject to boundary conditions $\phi(x,z) = \phi_{(0),\Delta} z^{d-\Delta}$ for $z \to 0$

$$\phi(x,z)_\Delta = \int_{\partial \mathrm{AdS}} d^d y K_\Delta(z,x;y)\phi_{(0),\Delta}(y) \tag{D.1.1}$$

Seemingly the scalar only depends on the boundary value y^μ now. In the same fashion, bulk-to-bulk processes can be considered by a KG equation with a source $(\Box_g - m^2)\phi_\Delta(x,z) = J(x,z)$

$$\phi(x,z)_\Delta = \int_{\partial \mathrm{AdS}} dw d^d y \sqrt{g} G_\Delta(z,x;w,y) J(x,z) \tag{D.1.2}$$

where the coordinates (z,x) denote a point with bulk coordinate z and boundary coordinates x^μ, while (w,y) denotes a point with bulk coordinate w and boundary coordinates y^μ. This means that the bulk-to-bulk propagator must satisfy boundary conditions

$$(\Box_g - m^2)G_\Delta(z,x;w,y) = \frac{\delta(z-w)\delta^d(x-y)}{\sqrt{g}} \tag{D.1.3}$$

It turns out that this boundary value problem in the end is a hypergeometric equation. Using the chordal distance defined we can express the bulk-to-bulk propagator as a hypergeometric with ξ as[56, 144]

$$G_\Delta(\xi) = \frac{C_\Delta}{2^\Delta(2\Delta - d)}\xi^\Delta F_1\left(\frac{\Delta}{2}, \frac{\Delta+1}{2}, \Delta - \frac{d}{2} + 1; \xi^2\right), \quad C_\Delta = \frac{\Gamma(\Delta)}{\pi^{d/2}\Gamma(\Delta - \frac{d}{2})} \tag{D.1.4}$$

From bulk-to-boundary propagator one obtains the bulk-to-bulk propagator by taking the limit $w \to 0$

$$K_\Delta(z, x-y) = \lim_{w\to 0} \frac{2\Delta - d}{w^\Delta} G_\Delta(z,x;w,y) \tag{D.1.5}$$

An illuminating way to obtain the exact expression is by use of Greens second identity

$$\int_M dz d^d x \sqrt{g}[\phi(\Box_g - m^2)\psi - \psi(\Box_g - m^2)\phi] = \int_{\partial\mathcal{M}} d^d x \sqrt{\gamma}[\phi\partial_n\psi - \psi\partial_n\phi] \tag{D.1.6}$$

Here γ is the determinant of the induced metric on the boundary, and ∂_n is the derivative normal to the boundary which would correspond to ∂_z. One could also just explicitly take the limit given. But doing either one will finally give

$$K_\Delta(z,x;y) = C_\Delta\left(\frac{z}{z^2+(x-y)^2}\right)^\Delta \tag{D.1.7}$$

We notice that for the interior points, there seems to be no obstruction making the propagator regular for $z \to \infty$. In the other limit $z \to 0$ it is found

$$\lim_{z\to 0}(z^{\Delta-d}K_\Delta(z,x;y)) = \delta^d(x-y) \tag{D.1.8}$$

This provides near the boundary, a correspondence to a source with a delta-like distribution. This provides a framework for doing n-point correlators. A standard exercise is to derive the conformal two-point function which remarkably doesn´t need holographic renormalization, which makes it a prime example [116]. Furthermore, the case at hand limits a theory only to scalars, but this is possible to extend for gauge and tensor fields. The procedure is the same as for the scalars, but the structure becomes more involved. The main differences are briefly mentioned For the Gauge field, one considers the appropriate part of the action that contains derivatives

and masses coupled to A_μ

$$S_{\text{vector}} = \int d^{d+1}z\sqrt{g}\Big(\frac{1}{2}(\nabla_\mu A_\nu)^2 - \frac{1}{2}(\nabla^\mu A_\mu)^2 + \frac{1}{2}m^2A^\mu A_\mu - A_\mu J^\mu\Big) \tag{D.1.9}$$

As mentioned above, working in the restricted space of covariantly-conserved currents $\nabla_\mu J_\nu = 0$, acting on the EOM with ∇_μ from the left, leads to the gauge $\nabla_\mu A_\nu = 0$. Another thing to note is the mass relation. This gives us $m^2 = (\Delta-1)(\Delta+1-d)$. Lastly, the action is modified with a vector function that reflects the gauge freedom but vanishes when the above equation is multiplied by the covariantly conserved current and integrated over. For further details check [43, 46]

For the tensor field, the mass relation stays the same as for scalars, But we change the action instead. Usually gravitons are the most considered particle with $S = 2$ in the tensor field representation, so one consider an Einstein-Hilbert action with negative cosmological constant

$$S_{Ads_{d+1}} = \int d^{d+1}z\sqrt{g}\left(\frac{1}{2\kappa_{d+1}^2}(R-2\Lambda) + \mathcal{L}_M\right) \tag{D.1.10}$$

where $\mathcal{L}_M$ is the matter lagrangian and κ_{d+1} is the $d-$ dimensional gravitational constant. Besides this the most notable difference is that the solution to this equation is obtained by decomposing $G^\Delta_{\mu\nu:\mu'\nu'}$ onto a basis of five irreducible $SO(d,1)$ tensors $T_{\mu\nu:\mu'\nu'}$. For further details check [39, 115]

Appendix E

The Vanilla Example: $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM

Starting with the Birth of AdS/CFT from its cradle[106], we state the conjecture that Maldacena famously did almost 28 years ago. The perspective for open and closed string and stack of D3 branes is reviewed as well as the action and field content with operators and subsectors

E.1 The Conjeture

$\mathcal{N}$=4 Super Yang-Mills (SYM) theory with gauge group SU(N) and Yang–Mills coupling constant $g_{\rm ym}$ is dynamically equivalent to type IIB superstring theory with string length $l_s = \sqrt{\alpha'}$ and coupling constant g_s on $AdS_5 \times S^5$ with radius of curvature L and N units of $F_{(5)}$ flux on S^5. The parameters of each side of the correspondence are mapped to each other via

$$2g_{YM}^2 N = 2\lambda = \frac{L^4}{\alpha'^2} \quad \text{and} \quad g_{YM}^2 = 2\pi g_s \tag{E.1.1}$$

E.2 The heuristic derivation for Open String and closed strings

It shouldn´t be surprising that open and closed strings are the constituents of duality at hand. The correspondence already prescribes the dual theories between bulk a boundary. In this case, it just specifies different physical scenarios for a stack of N D3 branes in $\mathbb{R}^{9,1}$. Going back to Susskind and t´Hooft, one can assert the realization of the holographic principle as the information of the five-dimensional theory obtained from Kaluza–Klein reduction of type IIB string theory on S^5, is mapped to a four-dimensional theory which lives on the conformal boundary of the five-dimensional spacetime. Unfortunately, only weak/strong duality can be established between the gauge and gravity side. We consider the two sides separately

The open String:

Open strings have endpoints on the D3-branes and are excitations of them also. Considering small energies $E << l_s^{-1}$, we get only massless string modes excitations. In the regime of small coupling constant $g_s << 1$, one finds that the DBI (Dirac-Born-Infeld) action describes the dynamics of the string. We outline for a single D-brane and generalize for multiple straight after. We motivate the particles via the mass spectrum here. Taking the gauge field A_μ, this corresponds to open string excitations parallel to the D brane. This can be seen starting from the physical state conditions that require $a = 1$ and $n = 0$. with this one can write the spectrum for physical states as

$$M^2 |\phi\rangle = -p_a p^a |\phi\rangle = \frac{1}{l_s^2}(N_{\parallel} + N_{\perp} - 1) \tag{E.2.1}$$

First, take the case where $N_{\parallel} = 1$ and $N_{\perp} = 0$. Using the physical state conditions one can for a general linear superposition $A_\mu(k)\alpha_{-1}^a |0; k\rangle$, make a Fourier transform to position space, and get in Lorentz gauge $\partial_a A^a = 0$, that the corresponding gauge transformation is nothing but a U(1) gauge boson with associated field strength $F_{ab} = \partial_a A^b - \partial_b A^a$. For the other case when $N_{\parallel} = 0$ and $N_{\perp} = 1$ the general linear superposition of states can be written as

$$\sum_{I=p+1}^{25} \Phi_I(k)\alpha_{-1}^I |0; k\rangle \tag{E.2.2}$$

Going to position space again via the Fourier transform, one can solve the EOM, and find the usual Klein-Gordon equation in p+1 dimension $\partial_a\partial^a\Phi_I = 0$. Thus scalars arise as transverse excitations on the brane, while the gauge bosons and field arise parallel to the brane. To extend this, we must consider what happens in the case of N coincident D3-branes. Now that the fock space is larger than before one must introduce Chan-Paton factors, which are non-dynamical degrees of freedom assigned to the endpoints of the string and labels strings that go between brane i and j. One can show that λ_{ij} has the Lie algebra $U(N)$, resulting in $U(N)$ gauge theory with effective coupling $g_s N$.

The Closed String

On the other hand, one can also regard closed D-branes as solitonic solutions to SUGRA theories. To make a description that makes sense, we require that the scale of space-time must be large compared to curvature at which the branes exhibit a source of the gravitational field. This should imply that we are at low energy also. We can relate the proportionality between the size of the space-time and the coupling of N D-branes as $\frac{L^4}{\alpha'^2} \propto g_s N$. This leads us to conclude that we work in a regime where $g_s N >> 1$.

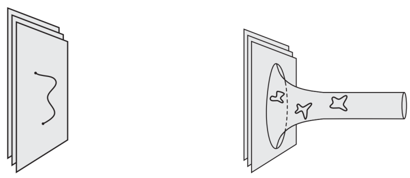


Figure E.1: Open vs. closed strings. Open strings act as transverse and perpendicular fluctuations to produc gauge fields, where closed strings act as gravitational sources in the flat and throat region

To be more concrete, one can also derive the metric in a near horizon geometry that we outline. In the strongly couple limit, N D3 branes arise as BPS solutions to type IIB supergravity. The whole story is quite lengthy, but making an ansatz to the EOM from the SUGRA action, helps one establish the following metric

$$\begin{aligned} ds^2 = H(r)^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + H(r)^{1/2}(dr^2 + r^2 d\Omega_{S^5}), \quad H(r) = 1 + \frac{L^4}{r^4}, L^4 = 4\pi g_s N\alpha'^2 \\ e^{2\phi} = g_s^2, \quad C_{(4)} = (1 - H(r)^{-1})dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 + F_5 \text{ Self duaity terms} \end{aligned} \tag{E.2.3}$$

$dC_{(4)}$ is the four-form gauge field, which is related to the self-dual five-form R-R field via $F_{(5)} = dC_{(4)}$. The D3-branes act as sources for the self-dual five-form F(5), which has a flux on the five-sphere. Taking the near horizon limit $r << L$, the harmonic function reduces to $H(r) \approx \frac{L^4}{r^4}$. This give a reduced metric of the form

$$ds^2_{NH} = (\frac{r^2}{L^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dr^2) + L^2 d\Omega^2_{S^5} \tag{E.2.4}$$

But this is nothing but $AdS_5 \times S^5$ in Poincare patch coordinates. In the other limit $r \gg L$ the metric becomes flat space due to $H(r) = 1$. Hence, two strings seem to propagate in the space-time. One in ten-dimensional Minkowski space (the asymptotically flat region) far away from the horizon and closed strings propagating in the near-horizon geometry $AdS_5 \times S^5$. The regions decouple in what is known as the Maldacena limit where $\alpha' \to 0$ and $u = \frac{r}{\alpha'}$ is fixed. To obtain all details in its full glory, interested readers may be guided to [3]

E.3 The Action

To derive the $\mathcal{N} = 4$ SYM action, the usual freshman route is going through the DBI-action, and chopping up your directions into transverse and perpendicular coordinates. This produces both the gauge field and the scalars in the bosonic sector. To get the whole beast there seems to be two ways of obtaining it. Either one starts from $\mathcal{N} = 1$ superspace for $\mathcal{N} = 4$ SYM and express the action in terms of chiral superfields Φ, as well as a gauge superfield $\mathcal{V}$ with associated field strength $\mathcal{W}$ [6]

$$S_{\mathcal{N}=4} = \int d^4x \Big[\int d^4\theta \Phi^{i\dagger} e^{\mathcal{V}} \Phi^i e^{-\mathcal{V}} + \frac{1}{8\pi}\mathrm{Im}\Big(\tau \int d^2\theta \mathcal{W}_\alpha \mathcal{W}^\alpha\Big) + \Big(ig_{\mathrm{ym}}\frac{\sqrt{2}}{3!}\int d^2\theta \epsilon_{ijk}\Phi^i[\Phi^j, \Phi^k] + \text{h.c}\Big)\Big] \tag{E.3.1}$$

Or one considers dimensional reduction of the $\mathcal{N} = 1$ SYM in ten dimension

$$S_{10D} = \int d^{10}x \mathrm{Tr}\Big(-\frac{1}{2}F_{mn}F^{mn} + i\bar{\Psi}\Gamma^m D_m \Psi\Big) \tag{E.3.2}$$

where Γ_m are 32×32 Dirac matrices in ten dimensions. The field strength tensor has gotten additional structure in terms of a coupling constant glued to the commutators giving a manifest non-abelian gauge theory. This can be compared to the derivation of the bosonic part from DBI. We define it as $F_{mn} = \partial_m A_n - \partial_n A_m + ig[A_m, A_n]$. Ψ represents the Majorana–Weyl fermion and the covariant derivative D_m on Ψ reads $D_m\Psi = \partial_m \Psi_m + ig[A_m, \Psi]$. To obtain the final action, one must in Kaluza-Klein style do dimensional reduction on the six-dimensional torus T^6. The idea is the same as for the DBI-action. One splits the space into two ranges for $\mu \in \{0, 1, 2, 3\}$ and $\phi_{i+3}, i \in \{1, ..., 6\}$, which decompose the gauge field as $A_m = (A_\mu(x^\nu), \phi^i(x^\nu))$. One can go through the compactification procedure, but effectively in its full glory, it is found that the action is give by

$$\begin{aligned} S_{\mathrm{SYM}} = \frac{1}{4\pi g_s}\int d^4\xi \mathrm{Tr}\Big\{ &-\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu} + \frac{\theta_I}{8\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} - \sum_a i\bar{\lambda}^a\bar{\sigma}^m u D_\mu \lambda - \sum_i D_\mu X^i D^\mu X^i \\ &+ \sum_{a,b,i} gC_i^{ab}\lambda_a[X^i, \lambda_b] + \sum_{a,b,i} g\bar{C}_{i,ab}\bar{\lambda}^a[X^i, \bar{\lambda}^b] + \frac{g}{2}\sum_{i,j}[X^i, X^j]^2 \Big\} \end{aligned} \tag{E.3.3}$$

The constants C_i^{ab} and $\bar{C}_{iab}$ are Clebsch-Gordan coefficients and related to the Clifford Dirac matrices for $SO(6)_R \simeq SU(4)_R$, g is the gauge coupling and θ_I is the instanton angle. Further, λ arises as the gaugino fields from the $\mathcal{N} = 1$ superspace expansion one gets from super-symmetry considerations

E.4 Global Symmetries with Field and Matter Content

The three main components of the superconformal algebra in SYM, is the supersymmetry, conformal symmetry and R-symmetry as for ABJM. Together they are a part of the bigger lie supergroup PSU(2, 2|4)[24, 112]. Considering the bosonic subalgebra of the supergroup $SU(2,2) \times SU(4) \simeq SO(4,2) \times SO(6)$. One can through the similarity explicitly see the appearance of both the R-symmetry manifesting as the $SO(6)$ and the conformal group manifesting as $SO(4,2)$. Ten of the generators belong to Poincaré group of $SO(3,1)$ where 4 of the generators are space-time translations, and the last 6 are Lorentz transformations. The remaining generators are devoted to dilatations and special conformal transformations. To go further for completeness, one also finds 32 supercharges $(Q_{\alpha a}, \tilde{Q}^a_{\dot{\alpha}}, S^a_\alpha, \tilde{S}_{\dot{\alpha}a})$, and also R-symmetry generators R_{IJ}. The summary is that SYM seems to be nicer given these features compared to ABJM. The full algebra can be found in [112]

For the field content and matter, there is one gauge field A_μ in the singlet **1** representation of $SU(4)$. One also finds the Weyl fermions $\lambda^a_\alpha, a \in \{1,2,3,4\}$ transforming in the fundamental **4** representation and scalars ϕ^i transforming in the antisymmetric **6** representation Facing the operators, we require that they must be gauge invariant. This provides single trace operator taking the form as $\mathcal{O}(x) = \mathrm{Tr}(\chi_1(x)\chi_2(x)...\chi_L(x))$. A specific class of operators only contains scalars defined as $\mathcal{O}(x) = \mathrm{Str}(\phi^{\{i_1}\phi^{i_2}...\phi^{i_k\}}$, where Str stands for symmetrized trace for the gauge algebra, which for the scalars $\phi^i = \phi^{ia}T_a$ in the adjoint representation is given by the sum over all permutations

$$\mathrm{Str}(T_{a_1}, ..., T_{a_n}) = \sum_{\text{all perm.}\sigma} \mathrm{Tr}(T_{\sigma(a_1)}...T_{\sigma(a_n)}) \tag{E.4.1}$$

This ensures that operators are totally symmetric. We want to construct the simplest operators now in terms of the scalars ϕ^i that combine into three complex scalars defined as

$$Z = \frac{1}{\sqrt{2}}(\phi^1 + i\phi^2), \quad W = \frac{1}{\sqrt{2}}(\phi^3 + i\phi^4), \quad X = \frac{1}{\sqrt{2}}(\phi^5 + i\phi^6) \tag{E.4.2}$$

These will be the building blocks, which is not as restrained as seen in ABJM. The dimension of operators are given by the bosonic subgroup. It has rank 6 which is thesame amount of Cartan generators or charges $(\Delta, S_1, S_2, J_1, J_2, J_3)$. Here Δ is the conformal dimension, S_1, S_2 are the two charges of the $SO(1,3)$ Lorentz group which we call spin, and J_1, J_2, J_3 are the R-symmetry generators. The scalars in $SU(4)$ transformed as $[0,1,0]$ so the dimension was $\mathrm{Dim}(0,L,0) = \frac{1}{12}(L+1)(L+2)^2(L+3)$ Taking $L = 1$ we get the notorious **6** representation of $SU(4)$.

E.5 From DBI and open string to $\mathcal{N} = 4$ SYM

The starting point will be a single D-brane or rather $N = 1$, and then we extend appropriately as we go along. Taking a single D-brane on $\mathbb{R}^{9,1}$ we split the directions of X^D into namely $D = \mu + I$, where $\mu \in \{0,1,2,3\}$ and $I \in \{4, ..., 9\}$. The dynamics are known to be describe by the DBI-action

$$S_{\mathrm{DBI}} = -\frac{1}{(2\pi)^3 g_s l_s} \int d^4\xi \sqrt{-\det(\gamma_{ab} + 2\pi\alpha' F_{ab})} \tag{E.5.1}$$

We have the usual induced metric $\gamma_{ab} = \eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$ and F is the field strength 2-form given as $F = dA$ of the U(1) gauge field. From the splitting of directions, we must choose our coordinates appropriately. First specifying the embedding and choose $X^a(\xi) = \xi^a$ with

$a \in \{0,1,2,3\}$. The transverse directions, which we center at the origin for convenience purposes, can be described by six scalars that are fluctuations in the position of the brane on the world volume

$$X^{i+3}(\xi) = 2\pi\alpha'\phi_i(\xi), \quad i \in \{4, ..., 9\}$$

Applying this to the induced metric we get the Minkowski metric with some fluctuations that we interpret as the scalars in the theory

$$\gamma_{ab} = \eta_{ab} + (2\pi\alpha')^2\partial_a\phi^i\partial_b\phi_i$$

Using these relations, we find that the determinant can be written as

$$\det\left[\eta_{ab} + 2\pi\alpha' F_{ab} + (2\pi\alpha')^2\partial_a\phi^i\partial_b\phi_i\right]$$

In the low energy limit where $\alpha' \to 0$ we can expand around the parameter. For convenience we can look at the determinant and write it as $\eta_{ab} + \epsilon\Lambda_{ab} = \eta_{ac}(\delta^c_b + \epsilon\Lambda^c_b)$. Using this, we can use the homomorphism property of determinants to split it up

$$\det\left[\eta_{ac}(\delta^c_b + \epsilon\Lambda^c_b)\right] = -\det(\delta^c_b + \epsilon\Lambda^c_b)$$

This suggest that we should use the identity $\det(\Gamma) = \exp(\mathrm{Tr}[\log(\Gamma)]$ where Γ is an $n \times n$ matrix. Using that $\Gamma = \mathbb{I} + \epsilon\Lambda$, we can compute

$$\begin{aligned}\det(\mathbb{I}+\epsilon\Lambda) &= \exp(\mathrm{Tr}[\log(\mathbb{I}+\epsilon\Lambda)]) = \exp\left(\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\epsilon\mathrm{Tr}[\Lambda^n]\right)\\ &= 1 + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\epsilon\mathrm{Tr}[\Lambda^n] - \frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\epsilon\mathrm{Tr}[\Lambda^n]\right)\left(\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\epsilon\mathrm{Tr}[\Lambda^m]\right)\\ &= 1 + \epsilon\mathrm{Tr}[\Lambda]\end{aligned} \tag{E.5.2}$$

Finally, we have to consider the square root in the action, so we use the expansion $\sqrt{1+x} = 1 + \frac{1}{2}x + \mathcal{O}(x^2)$. With this in mind, we can write the determinant in the desired form

$$\sqrt{-\det} = 1 + (2\pi\alpha')^2(\frac{1}{2}F_{ab}F^{ab} + \frac{1}{2}\partial_a\phi^i\partial_b\phi_i) \tag{E.5.3}$$

Dropping the contribution from 1, which just integrates to the world volume, we find the action

$$S_{\mathrm{DBI}} = \frac{1}{4\pi g_s}\int d^4\xi(\frac{1}{4}F_{ab}F^{ab} + \partial_a\phi^i\partial_b\phi_i) + \text{Fermions} \tag{E.5.4}$$

This exactly corresponds to the action of $\mathcal{N} = 4$ SYM, with gauge group U(1) given that we make the identification between yang-mills coupling and string coupling $g^2_{\mathrm{ym}} = 4\pi g_s$ If we

extend this to N D3-branes, we get a $U(N)$ gauge theory, which can be represented in the adjoint representation given the generators of the group $\phi^i = \phi^{ia}T_a$ and $A_\mu = A^{ia}_\mu T_a$. For U(N) there are $N^2 - 1$ generator satisfying the commutator relation $[T^a, T^b] = f^{abc}T_c$. The main difference is that one needs to consider covariant derivatives instead of partial only, but the same identifications can be made nevertheless. At the same time, one gets a type IIB SUGRA action which is a free theory in the bulk of $\mathbb{R}^{9,1}$ in the $\alpha' \to 0$ limit. This is why the limit is also known as the decoupling limit, since both SUGRA and interactions both do not contribute to the overall action. This is somewhat the original setup for the conjecture Maldacena proposed. The story is a bit more involved, but the essence stays the same[1]

E.6 Subsectors

Moving on, we list the respective weights of letters in the theory for both the R-symmetry $SU(4)$ and S^3 $SO(4)$, that we use to construct the subsectors. Following "cite" we can organize data into table through the representations

	F_+	F_0	F_-	$\tilde{F}_+$	$\tilde{F}_0$	$\tilde{F}_-$
SO(4)	(1,-1)	(0,0)	(-1,1)	(1,1)	(0,0)	(-1,-1)
SU(4)	(0,0,0)	(0,0,0)	(0,0,0)	(0,0,0)	(0,0,0)	(0,0,0)

Table E.1: Weight of Gauge Field Strength components

	Z	X	W	$\bar{Z}$	$\bar{X}$	$\bar{W}$
SO(4)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
SU(4)	(1,0,0)	(0,1,0)	(0,0,1)	(-1,0,0)	(0,-1,0)	(0,0,-1)

Table E.2: Complex Scalars in $\mathcal{N} = 4$ SYM

	$\chi_1,\chi_3,\chi_5,\chi_7$	$\chi_2,\chi_4,\chi_6,\chi_8$	$\bar{\chi}_1,\bar{\chi}_3,\bar{\chi}_5,\bar{\chi}_7$	$\bar{\chi}_2,\bar{\chi}_4,\bar{\chi}_6,\bar{\chi}_8$
SO(4)	$(\frac{1}{2},-\frac{1}{2})$	$(-\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$(-\frac{1}{2},-\frac{1}{2})$

Table E.3: $SO(4)$ weights of Fermions in $\mathcal{N} = 4$ SYM

	$\chi_1,\chi_2,\bar{\chi}_1,\bar{\chi}_2$	$\chi_3,\chi_4,\bar{\chi}_3,\bar{\chi}_4$	$\chi_5,\chi_6,\bar{\chi}_5,\bar{\chi}_6$	$\chi_7,\chi_8,\bar{\chi}_7,\bar{\chi}_8$
SU(4)	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$	$(-\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	$(-\frac{1}{2},-\frac{1}{2},\frac{1}{2})$

Table E.4: $SO(4)$ weights for $\chi_1, ..., \chi_8$ Fermions in $\mathcal{N} = 4$ SYM

	d_1	d_2	$\bar{d}_1$	$\bar{d}_2$
SO(4)	(1,0)	(0,1)	(-1,0)	(0,-1)
SU(4)	(0,0,0)	(0,0,0)	(0,0,0)	(0,0,0)

Table E.5: Derivative Operators of $\mathcal{N} = 4$ SYM

The way one reads the tables is for the $SU(4)$ one has a vector with entries (J_1, J_2, J_3) and for $SO(4)$ one has (S_1, S_2). From the decoupling prescription, we see that the choices of

[1]Maldacena considers a stack of $N+1$ $D3$-branes, where one moves away in the X^9 direction at a distance r. This theory is described by $U(N) \times U(1)$ gauge theory. For the stack to combine again, one must keep a Higgs expectation value fixed, $< X^9 >= \frac{r}{2\pi\alpha'}$, resulting in the Maldacena limit where $\alpha' = 0$ and $\mathcal{U} = \frac{r}{\alpha'}$ is fixed. This implies that the mass of the stretched strings remains fixed. Apparently, the $U(1) \subset U(N)$ corresponds to singleton fields living on the boundary in the gravity theory that cannot propagate into the bulk and thus decouple, leaving us with $SU(N)$ four-dimensional $\mathcal{N} = 4$ SYM, which is valid for any N

coefficients will pick out the letters for the specific subsector by matching it with$\Delta_0 = J$. We illustrate by example. Take the vector $(0,0,1,1,0)$. This corresponds to $J = J_1 + J_2$ which is the classic $SU(2)$ sector. Once we have this, we go through all the letters and see which are the ones that satisfy $\Delta_0 = J_1 + J_2$. We only take use of the $SU(4)$, since there is no dependence on spin in this case. Looking at the gauge fields, we see that all the components equate to 0 while $\Delta_{0_F} = 2$. Thus no gauge fields are found. It is important to note that the conformal dimension is different for different letters due to the different representations they occupy. For the scalars one finds that Z, X satisfy our condition since $\Delta_{0_{(x,z)}} = 1$ and $J_1 + J_2 = 1$ for both. In the end, after going through all of this, these remain as the only letters contained in this subsector. This is the general fashion of how to determine the complete landscape. One could do a more thorough analysis by considering inequalities of the coefficients and thereby determine different numbers of fermions present in each case. The details can be found in [80]

Appendix F

Gauge/Gravity Duality

As both gravity and Gauge theory is at the heart of the AdS/CFT correspondence, both sides of the story is touched upon in a general fashion

F.1 AdS and gravity theories in the bulk

Facing gravity first, it is no surprise that the starting point is Einstein´s field equations

$$R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{F.1.1}$$

We are interested in a maximally symmetric spacetime with Ricci scalar $R = 2\Lambda\frac{d+1}{d-1}$. This corresponds to scenarios where $\Lambda < 0$ such that the geometry is described by negative curvature. One can consider a $d+1$ dimensional Anti-de Sitter space (AdS_{d+1}) which can be embedded into $\mathbb{R}^{d,2}$. This is a $(d+2)$ dimensional Minkowski space. The metric signature is $\eta = \mathrm{diag}(-,+,+,...,+,-)$ and is given by

$$ds^2 = (dx^0)^2 + (dx^1)^2 + (dx^2)^2 + ... + (dx^{d+1})^2 = \eta_{MN}dx^M dx^N \tag{F.1.2}$$

AdS_{d+1} can also be written in coordinates as a hypersurface

$$\eta_{MN}x^M x^N = -(x^0)^2 + \sum_{i=1}^{d}(x^i)^2 - (x^{d+1})^2 = -L^2 \tag{F.1.3}$$

where L is the radius of curvature of AdS_{d+1}. The embedding is invariant under the Lorentz group for $\mathbb{R}^{d,2}$, $SO(d,2)$, which has dimension $\frac{1}{2}(d+1)(d+2)$. This is the number of Killing vectors associated to AdS_{d+1}, leading us to conclude that the space is maximally symmetric. $SO(d,2)$ is the conformal group of d-dimensional Minkowski space, pointing in the right direction regarding symmetries of the duality. One can parametrize the coordinates in multiple ways. Let us introduce the coordinates $t \in \mathbb{R}, \vec{x} = (x_1, ..., x^{d-1}) \in \mathbb{R}^{d-1}$ and $r \in \mathbb{R}_+$. The parameterization in these coordinates is given by

$$\begin{aligned} X^0 &= \frac{L^2}{2r}(1 + \frac{r^2}{L^4}(\vec{x}^2 - t^2 + L^2)), \quad X^i = \frac{rx^i}{L}, \quad i \in \{1, ..., d-1\} \\ X^D &= \frac{2r}{L^2}(1 + \frac{r^2}{L^4}(\vec{x}^2 - t^2 + L^2)), \quad X^{D+1} = \frac{rt}{L} \end{aligned} \tag{F.1.4}$$

Due to the restriction $r > 0$, we cover only one-half of the AdS_{d+1} spacetime. These local coordinates are referred to as Poincaré patch coordinates. In the Poincaré patch, the metric of the space reads

$$ds^2 = \frac{L^2}{r^2}dr^2 + \frac{r^2}{L^2}(d\vec{x}^2 - dt^2) = \frac{L^2}{r^2}dr^2 + \frac{r^2}{L^2}(\eta_{\mu\nu}dx^\mu dx^\nu) \tag{F.1.5}$$

where we have recognized the metric of d-dimensional Minkowski space. Using this metric, one finds the Ricci scalar to be $R = \frac{-d(d+1)}{L^2}$, implying that L^2 is indeed the radius of curvature. Another useful form of the Poincaré metric is obtained by inverting the radial coordinate, $z = L^2/r$, yielding the metric in Poincaré z-coordinates,

$$ds^2 = \frac{L^2}{z^2}(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu) \tag{F.1.6}$$

Note that the boundary in these coordinates is located at $z = 0$

Another possibility is to introduce global coordinates τ, ρ, θ_i, and describe the space-time via hyperbolic functions

$$X^0 = L\cosh\rho\cos\tau, \quad X^{D+1} = L\cosh\rho\cos\tau, \quad X^i = L\Omega_i\sinh\rho \tag{F.1.7}$$

Here Ω_i with $i = 1, ..., d$ are angular coordinates satisfying $\sum_i \Omega_i^2 = 1$. In other words Ω_i parametrize a $d-1$ dimensional sphere. These coordinates are referred to as global coordinates of AdS_{d+1} since all points of the hypersurface are taken into account exactly once. The induced metric can be found to be

$$ds^2 = L^2(-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_{d-1}^2) \tag{F.1.8}$$

Since the metric above does not depend on τ, we infer the existence of a timelike killing vector ∂_τ, and since this killing vector is defined globally on the manifold, τ acts as a sensible global time coordinate. Near the center $\rho = 0$ the metric assumes the form $ds^2 = -L^2(d\tau^2 + d\rho^2 + \rho^2 d\Omega_{S^{d-1}}^2)$ implying that the space-time has topology, since τ is periodic, of $S^1 \times \mathbb{R}^d$, where S^1 is the periodic time; in particular, since ∂_τ is everywhere timelike, keeping ρ and θ_i fixed while varying τ will produce closed time-like curves. This is, however, not an intrinsic property of this space-time, merely a consequence of our embedding: $\mathbb{R}^{d,2}$ has two timelike directions, so the appearance of closed timelike curves is not so surprising after all.

F.2 Conformal field theories

To start we are interested in understanding how big this conformal algebra is and what bits and pieces it contains. It would be seen that it is an extension of the Poincaré algebra with extra features. What constitutes a conformal transformation is that it´s angle preserving, this we can represent as

$$g_{\alpha\beta}(x) \to \tilde{g}_{\alpha\beta}(x) = e^{2\sigma(x)}g_{\alpha\beta}(x), \quad g_{\mu\nu}(x) \to \Omega^2(x)g_{\mu\nu}(x) \tag{F.2.1}$$

If we want to know the infinitesimal transformations in flat space, we should take the lie derivative of the Minkowski metric and solve the killing equation to get the isometries

$$\mathcal{L}_\epsilon\eta_{\mu\nu} = \eta_{\mu\nu} + \partial_\mu\epsilon\nu + \partial_\nu\epsilon\mu \quad \text{and} \quad e^{2\sigma(x)}\eta_{\mu\nu}(x) \approx (1 + 2\sigma(x))\eta_{\mu\nu} \tag{F.2.2}$$

This leads to the equation after taking the trace of the metric to relate σ to ϵ

$$\partial^\mu\partial_\mu\epsilon_\nu = \frac{1}{d}(2-d)\partial_\nu\partial_\lambda\epsilon^\lambda \tag{F.2.3}$$

There are 2 distinct cases to take into account, $d < 2$ and $d > 2$. The case for $d < 2$ is known for an infinitesimal conformal transformation given by

$$\epsilon^\mu(x) = a^\mu + \sigma^\mu_\nu x^\nu + \lambda x^\mu + b^\mu x^2 - 2x^\mu b\cdot x \tag{F.2.4}$$

We can link each term to an operator which are generators of the conformal group. The first term we recognize as $a^\mu \to p_\mu$ which is nothing but translation. Secondly we have Lorentz transformations $\sigma^\mu_\nu x^\nu - J_{\mu\nu}$, and thirdly we get a new feature namely dilatations D which has the property of scaling coordinates as follows

$$x^\mu \to \lambda x^\mu \tag{F.2.5}$$

The last feature is special conformal transformations $(b^\mu x^2 - 2x^\mu b\cdot x) - K_\mu$. These objects are the building blocks for conformal field theory. This will be the starting point for the algebra. Since the Poincare algebra is a subalgebra of the conformal group, we have all the same commutators, but with our new features, we get non-vanishing commutation relation

$$[D, P_\mu] = -iP_\mu, \quad [J_{\mu\nu}, K_\rho] = -i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu), \quad [D, K_\mu] = iK_\mu, \quad [P_\mu, K_\nu] = 2i(J_{\mu\nu} - \eta_{\mu\nu}D) \tag{F.2.6}$$

The conformal algebra is isomorphic to ISO(d,2) with signature $\{-,+,...,+,-\}$. One can construct elements of the Lorentz matrix consisting of the other generators to manifest the isomorphism such that

$$J_{\mu d} = \frac{K_\mu - P_\mu}{2}, \quad J_{\mu(d+1)} = \frac{K_\mu - P_\mu}{2} \quad J_{d(d+1)} = D \tag{F.2.7}$$

Going from here, deriving the correlator between two or more conformal fields is possible. First we start by expecting states of the form $\phi(x) = e^{ix^\mu P_\mu}\phi(0)$. One can find that the commutator now a field at $x = 0$ with the dilitation operator obeys $[D, \phi(0)] = -i\Delta\phi(0)$. This implies

$$[D, \phi(x)] = [D, e^{ix^\mu P_\mu}\phi(0)] = ([D, e^{ix^\mu P_\mu}] + e^{ix^\mu P_\mu}D)\phi(0) + e^{ix^\mu P_\mu}\phi(0)D \tag{F.2.8}$$

Expanding the exponential and get

$$[D, e^{ix^\mu P_\mu}] = \sum_{i=0}^{\infty} \frac{i^n}{n!} x^{\mu_1}...x^{\mu_n}[D, P_{\mu_1}...P_{\mu_n}] \tag{F.2.9}$$

Where we define $[D, P_{\mu_1}...P_{\mu_n}] = [D, P^n]$. Since $[D, P_\mu] = -iP_\mu$ this leads to proving the following via induction $[D, P^n] = inP^n$. Consider for $n+1$

$$[D, P^{n+1}] = [D, P^n]P + P^n[D, P] = inP^n + iP^n = i(n+1)P^n \tag{F.2.10}$$

Using all these identities, proceeding from where we are left with

$$\begin{aligned}
[D, \phi(x)] &= \sum_{i=0}^{\infty} \frac{i^{n+1}n}{n!}(x^\mu P_\mu)^n \phi(0) + e^{ix^\mu P_\mu}[D, \phi(0)] \\
&= i^2 x^\mu P_\mu \sum_{i=1}^{\infty} \frac{i^{n-1}}{(n-1)!}(x^\mu P_\mu)^{n-1}\phi(0) - i\Delta\phi(x) \\
&= i(x^\mu \partial_\mu - \Delta)\phi(x)
\end{aligned} \tag{F.2.11}$$

We are prepared to look at the two-point function for scalar operators and see what we might expect. From rotational and translational invariance we get

$$< \phi_1(x)\phi_2(y) >= f(|x-y|) \tag{F.2.12}$$

To find the undetermined function, a general ward identity is useful concerning dilatation namely that

$$0 = \sum_{i=1}^{n} (x_i \frac{\partial_i}{\partial x_i^\mu} - \Delta_i) < \phi(x)...\phi_i(x_i)...\phi_n(x_n) > \tag{F.2.13}$$

For the two-point case, it easily comes out by considering the commutator between the dilatation operator and the fields at hand

$$\begin{aligned} 0 &= \langle 0|[D, \phi_1(x)\phi_2(y)]\,|0\rangle = \langle 0|\,\phi_1(x)[D, \phi_2(y)] - [D, \phi_1(x)]\phi_2(y)\,|0\rangle \\ &= (x^\mu \partial_\mu^{(x)} - \Delta_1 + y^\mu \partial_\mu^{(y)} - \Delta_2) < \phi_1(x)\phi_2(y) > \end{aligned} \tag{F.2.14}$$

The superscripts refer to the variable that the derivatives act on. The solution is the usual blueprint of a conformal field theory

$$f(|x-y|) = \frac{C}{|x-y|^{2\Delta}} \tag{F.2.15}$$

The exponent is in actuality $\Delta_1 + \Delta_2$ but by special conformal transformations, one can fix them to be $\Delta_1 = \Delta_2$. This procedure can also be done for 3-point functions, but the story does change for the 4-point where one must consider the cross ratios as your restrictions in the undetermined functions since they are invariants under the symmetry group.

In general one can take a primary field $\phi(x)$ of scaling dimension Δ and get the general conformal transformation

$$\phi(x) \to \phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/d} \phi(x) \tag{F.2.16}$$

Where one must introduce the Jacobian for the coordinate transformation and also the space-time dimension d. This also implies from the coordinate rescaling that $\phi'(\lambda x) = \lambda^{-\Delta}\phi(x)$. Lastly, we just mention primary operators and how one can lower and raise the conformal dimension from the commutations. If we consider the following

$$[D, K_\mu \phi(0)] = K_\mu[D, \phi(0)] - [D, K_\mu]\phi(0) = -i(\Delta - 1)K_\mu \phi(0) \tag{F.2.17}$$

From this one can deduce that by applying an arbitrary number of K_μ operators on an operator, this process must eventually terminate giving us $[K_\mu, \phi(0)] = 0$, meaning that $\phi(0)$ is a primary operator. From primary operators, it is then possible to construct what is called descendants, which can be obtained by applying consecutive momentum operators on such primaries $\prod_{i=1}^n P_{\mu_i}\phi(0)$ giving a conformal weight of $\Delta + n$. This can be stated in commutator language as $[D, P_\mu \phi(0)] = -i(\Delta + 1)P_\mu \phi(0)$. To summarize thus, for an operator to be primary, one must meet these conditions

$$[D, \phi(0)] = -i\Delta\phi(0), \quad [J_{\mu\nu}, \phi(0)] = \mathcal{J}_{\mu\nu}\phi(0), \quad [K_\mu, \phi(0)] = 0 \tag{F.2.18}$$

Appendix G

Supersymmetry and Superspace formalism

Having encountered the Action in ABJM, one is in great need of both supersymmetry and superspace fields. Hence, we give a brief review of the general idea of the formalisms

G.1 Supersymmetry and BPS bounds

As it is known, one can extend the Poincaré algebra to contain further symmetries such as dilatations and SCt´s, which is nothing but the conformal algebras and thus the basis for conformal field theories. One might wonder if there is an even larger group of symmetries present. Due to the Coleman-Mandula no-go theorem[41], it was assumed that conformal symmetry was the largest extension the S-matrix could fulfill. But as it was shown [72], if the notion of supercharges is introduced Q^a, a new symmetry is introduced, which converts between fermionic and bosonic states. Taking the general case consider, in weyl notation, the supercharges $\{Q^a_\alpha, Q_{b\beta}\}, a, b = 1, ..., \mathcal{N}$ with the following commutation relations and algebra [6].

$$\begin{gathered}
[Q_\alpha, J^{\mu\nu}] = (\sigma^{\mu\nu})_\alpha^\beta Q_\beta, \quad [\bar{Q}_{\dot{\alpha}}, J^{\mu\nu}] = \epsilon_{\dot{\alpha}\dot{\beta}}(\bar{\sigma}^{\mu\nu})^{\dot{\beta}}_{\dot{\gamma}}\bar{Q}^{\dot{\gamma}} \\
[Q_\alpha, P^\mu] = 0, \quad [\bar{Q}_{\dot{\alpha}}, P^\mu] = 0 \\
\{Q^a_\alpha, \bar{Q}_{b\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P^\mu \delta^a_b, \quad \{Q^a_\alpha, Q^b_\beta\} = \epsilon_{\alpha\beta} Z^{ab}, \quad \{\bar{Q}_{a\dot{\alpha}}, \bar{Q}_{b\dot{\beta}}\} = \epsilon_{\dot{\alpha}\dot{\beta}} \bar{Z}^{ab}
\end{gathered} \tag{G.1.1}$$

$Z^{ab}, \bar{Z}^{ab}$ are called the central charges and commute with all the other generators of the supersymmetry algebra, or rather generate the center of the algebra. Respecting anticommutator symmetry, Z_{ab} obey the antisymmetric property $Z_{ab} = Z_{ba}$ as well as $Z^{ab} = (\bar{Z}^\dagger)_{ab}$ from the fact that $\bar{Q}_{a\dot{\alpha}} = (Q^a_\alpha)^*$. An important feature is that the algebra is invariant under the global $U(\mathcal{N})$ phase rotation of the supercharges

$$Q^a_\alpha \to Q^{a'}_\alpha = R^a_b Q^b_\alpha, \quad \bar{Q}_{a\dot{\alpha}} \to \bar{Q}'_{a\dot{\alpha}} = \bar{Q}_{b\dot{\alpha}}(R^\dagger)^b_a \tag{G.1.2}$$

This non-abelian symmetry is also known as *R-symmetry* R^a_b are $\mathcal{N} \times \mathcal{N}$ matrices. The charges transform in their respective representations as well,Q^a_α in the fundamental of $U(\mathcal{N})$ and $Q^{a'}_\alpha$ in the conjugate $\bar{\mathcal{N}}$, which is indicated by the upper and lower index of a. Further details are found in [6]

Turning the wheel, we will be interested in the massive representations to get a sense of field multiplets. For the central charges, it is convenient to pick a basis where Z_{ab} are diagonal and have eigenvalues q_i. It can be arranged in an antisymmetric matrix Z^{ab} giving us for $\mathcal{N} = 2$

$$Z^{ab} = \begin{pmatrix} 0 & q_1 \\ -q_1 & 0 \end{pmatrix} \tag{G.1.3}$$

The same construction follows for $\mathcal{N} > 2$. Here we just build block diagonals consisting of Lego blocks for the matrix above[6] Constructing a set of raising a lowering operators, the only ones that are non-zero, will take the additional term of the eigenvalues for the non-vanishing central charges. Using a linear combination

$$\tilde{Q}^j_{\alpha\pm} = Q^{2j-1}_\alpha \pm (Q^{2j}_\alpha)^\dagger, \quad j \in \{1, ..., \frac{\mathcal{N}}{2}\} \tag{G.1.4}$$

and establishing the anti-commutator, it can be found that all non-zero terms can be written as

$$\{\tilde{Q}^i_{\alpha_+}, (\tilde{Q}^j_{\beta_+})^\dagger\} = \delta^j_i \delta^\beta_\alpha (2m + q_i), \quad \{\tilde{Q}^i_{\alpha_-}, (\tilde{Q}^j_{\beta_-})^\dagger\} = \delta^j_i \delta^\beta_\alpha (2m - q_i) \tag{G.1.5}$$

For unitary particle representations, we must insist that both the right-hand sides must stay positive leading to $|q_j| \leq 2m$ for all j. The famous Bogomolnyi–Prasad–Sommerfield (BPS) bound is obtained, when equality holds for $|q_j| = 2m$. Imagine that k of the q_j are fulfilling the BPS bound, then we see that $2\mathcal{N} - 2k$ of these operators satisfy the equality such that we now have $2^{2\mathcal{N}-2k}$ states. This is referred to as $1/2^k$ BPS multiplets. The space of multiplets then become $\{1/2, 1/4, 1/8, 1/16\}$ BPS etc. To summarize for different values of k one finds BPS-multiplets which has following shortening conditions [48]

$$\begin{aligned} &k = 0 \rightarrow \quad 2^{2\mathcal{N}}\text{States} \quad \text{Long Multiplet} \\ &0 < k < \frac{\mathcal{N}}{2} \rightarrow \quad 2^{2(\mathcal{N}-k)}\text{States} \quad \text{Short Multiplet} \\ &k = \frac{\mathcal{N}}{2} \rightarrow \quad 2^{\mathcal{N}}\text{States} \quad \text{Ultra Short Multiplet} \end{aligned} \tag{G.1.6}$$

The relevance of BPS-solutions has proven immense in the fields of SUGRA and string theory, specifically related to black holes and branes making it a tool worth using to study phenomena that exhibit properties characterized by supersymmetry. For further details, there is plenty of literature to consult for the interested reader [108, 49, 143, 33, 6]

G.2 Superspace Formalism

In ordinary Quantum Field Theory, fields are usually functions of x^μ, the coordinates of Minkowski space. But, as we just extended the conformal group to be a larger symmetry by introducing supercharges, one can in the same spirit extend Minkowski space to superfields living in superspace. The idea is to include anti-commuting fields in the mixture. The set of variables will be a triplet $\{x^\mu, \theta_\alpha, \bar{\theta}^{\dot{\alpha}}\}$, which is nothing but space-time coordinates in a Minkowskian sense with the addition of Grassmann spinors. Superspace becomes not a regular manifold of the kind that we know, but instead, it is an example of a supermanifold, with both commuting and anti-commuting dimensions. To get a sense of how to represent the supercharges and covariant derivatives, we first start by looking at the properties of the Grasmann spinors. Indices are raised and lowered as $\theta^\alpha = \epsilon^{\alpha\beta}\theta_\beta$ and $\theta_\alpha = \epsilon_{\alpha\beta}\theta^\beta$ with $\epsilon^{12} = -\epsilon_{12} = 1$. For products, conventions dictate that $\theta^\alpha\theta_\alpha = \theta^2$,$\theta^\alpha\bar{\theta}_\alpha = \theta\bar{\theta}$ etc. There is also surpressed indices $\theta^\alpha\gamma^\mu_{\alpha\beta}\bar{\theta}^\beta$, where we use Dirac matrices $(\gamma^\mu)^\beta_\alpha = (i\sigma^2, \sigma^1, \sigma^3)$. This amounts to $\theta_\alpha\theta_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\theta^2$ and $\theta^\alpha\theta^\beta = \frac{1}{2}\epsilon^{\alpha\beta}\theta^2$. Having defined the properties, one can generate operators for both supercharges and covariant derivatives[1], by considering infinitesimal transformations and using the BHC formula. This provides the basis for writing

$$\begin{aligned} \bar{D}_\alpha &= -\frac{\partial}{\partial\bar{\theta}^\alpha} - \theta^\beta\gamma^\mu_{\beta\alpha}\partial_\mu, \quad D_\alpha = \frac{\partial}{\partial\bar{\theta}^\alpha} + \bar{\theta}^\beta\gamma^\mu_{\alpha\beta}\partial_\mu \\ \bar{Q}_\alpha &= -\frac{\partial}{\partial\bar{\theta}^\alpha} + \theta^\beta\gamma^\mu_{\beta\alpha}\partial_\mu, \quad Q_\alpha = \frac{\partial}{\partial\bar{\theta}^\alpha} - \bar{\theta}^\beta\gamma^\mu_{\alpha\beta}\partial_\mu \end{aligned} \tag{G.2.1}$$

[1]Partial derivative in the superfields are defined as $\partial_\alpha\partial^\alpha = \partial^2$

For the purpose of section (2.3.1), we only comment on Vector and Chiral superfields and leave the details to be studied in [87, 6]. Since grassmann numbers obey anti-commutation $\theta_i^2 = 0$ or rather $\theta_\alpha \theta_\beta = -\theta_\beta \theta_\alpha$, Grassmann variables at most contains quadratic orders in θ. Thus a generic superfield $Y(x, \theta, \bar{\theta})$ can be taylor expanded in the spinor components which truncates at the quadratic $\theta^2 \bar{\theta}^2$. In terms of ABJM and gauge poduct group for chiral vector superfields in $\mathcal{N} = 3$ Chern-Simons theory, one gets fields components

$$\begin{aligned}
V(x, \theta, \bar{\theta}) &= -\theta \gamma^\mu \bar{\theta} A_\mu - \theta \bar{\theta} \sigma(x) + i\theta^2 \bar{\theta} \bar{\chi}(x) - i\bar{\theta}^2 \theta \chi(x) + \frac{1}{2} \bar{\theta}^2 \theta^2 D(x) \\
\Phi(x, \theta, \bar{\theta}) &= \phi(x) + \sqrt{2} \theta \psi(x) + \theta^2 F(x) + i\theta \gamma^\mu \bar{\theta} \partial_\mu \phi(x) - \frac{i}{\sqrt{2}} \theta^2 \partial_\mu \psi(x) \gamma^\mu \bar{\theta} - \frac{1}{4} \theta^2 \bar{\theta}^2 \partial^2 \phi(x) \\
\bar{\Phi}(x, \theta, \bar{\theta}) &= \bar{\phi}(x) + \sqrt{2} \bar{\theta} \bar{\psi(x)} + \bar{\theta}^2 \bar{F}(x) - i\theta \gamma^\mu \bar{\theta} \partial_\mu \bar{\phi}(x) + \frac{i}{\sqrt{2}} \bar{\theta}^2 \gamma^\mu \theta \partial_\mu \bar{\psi}(x) - \frac{1}{4} \bar{\theta}^2 \theta^2 \partial^2 \bar{\phi}(x)
\end{aligned} \tag{G.2.2}$$

Appendix H

Kaluza-Klein compactification on a circle

Usually when one talks about super-gravity, we are dealing with situations where the dimension of the theory exceeds the one we should make phenomenological models for in particle physics. Thus one could consider what could happen if we could shrink the redundant dimensions to be so small, that they would not have an effect in the observable theory. One way is to do Kaluza-klein (KK) compactification on certain geometries (Calabi-Yau manifolds, a Torus or spheres/circles). Thus, we wish to compactify one space dimension on a circle S^1_R of radius R[122]. Thus we make one of our x^μ coordinates into a y-coordinate on a circle and let the remaining ones be called $x^{\bar{\mu}}$. Thus our wave operator can be written as

$$\Box_D = \Box_{D-1} + \frac{\partial^2}{\partial y^2} \tag{H.0.1}$$

We want to investigate how fields transform in the limit where we let the radius $R \to 0$ which will be called dimensional reduction. Starting with a scalar field $\phi(x^\mu)$ obeying periodic boundary conditions on S^1_R, which can then be expanded into its Fourier decomposition

$$\phi(x^{\bar{\mu}}, y) = \sum_{n \in \mathrm{Z}} \phi_n(x^{\bar{\mu}}) e^{\frac{2\pi i n y}{R}} \tag{H.0.2}$$

If we look at a standard kinetic term, of the Klein-Gordon form in d dimensions, we can use dimensional reduction to get

$$\begin{aligned} \int d^d x \phi(-\Box_D + m^2)\phi &= \sum_{n \in \mathrm{Z}} 2\pi R \int d^d x \phi_n(-\Box_{D-1} + \frac{\partial^2}{\partial y^2} + m^2)\phi_n e^{\frac{2\pi i n y}{R}} \\ &= \sum_{n \in \mathrm{Z}} 2\pi R \int d^d x \phi_n(-\Box_{D-1} + \frac{4\pi^2 n^2}{R^2} + m^2)\phi_n \end{aligned} \tag{H.0.3}$$

One defines the mass of the n'th mode as $m_n = \frac{n^2}{R^2}$. In the limit $R \to 0$ the only mode that contributes is for $n = 0$ since all others modes acquire an infinitely heavy mass and thus decouples. Due to the infinity of every increasing mass, this became known as the Kaluza-Klein tower of states. When $L << 1$, the non-zero modes will be immensely heavy and can be safely neglected. These heavy masses truncate the Kaluza-Klein spectrum and are known as the Kaluza-Klein reduction ansatz

Appendix I

Representation of $SU(4)$

As $SU(4)$ seems to be an important player in this whole story, it is worth giving a little background on all the confusing numbers flying around all the time. This is not intended as a group theoretic background to $SU(N)$ groups, this can be found in [6]. Given the group $SU(4)$, the rank is in $SU(N)$ is given by $N-1$, so it has rank 3. This exactly represents the three Dynkin labels $[r, q, p]$. To find the dimension of an arbitrary representation in $SU(N)$, a deep result known as the Weyl character formula can be stated

$$\dim[\lambda_1, ...\lambda_{N-1}] = \prod_{1 \leq i \leq j \leq N} \frac{\lambda_i - \lambda_j + j - i}{j - i} \tag{I.0.1}$$

This is in spirit a very general formula, since it encapsulates all the possible representations, but for practical purposes, we see how it is used for $SU(4)$[1]. Going through all the possible combinations of indices, it can be found that the dimension of any given representation in $SU(4)$ is

$$\mathrm{Dim}[r, q, p] = \frac{1}{12}(r+1)(q+1)(p+1)(r+q+2)(q+p+2)(r+q+p+3) \tag{I.0.2}$$

With this formula, it is nice to summarize the most fundamental cases, which is extensively used in the thesis in the table below

Dynkin Label	**Representation**
$[0, 0, 0]$	1
$[1, 0, 0]$	4
$[0, 0, 1]$	$\bar{4}$
$[0, 1, 0]$	6
$[2, 0, 0]$	10
$[0, 0, 2]$	$10'$
$[1, 0, 1]$	15
$[0, 2, 0]$	20
$[1, 1, 0]$	$20'$

Table I.1: Representation of $SU(4)$ and the dimension of the different representations

[1] One can find the dimension of representations via Young Tableaux as well, but this is a bit more daunting than this plug and play game

Bibliography

[1] Michael C Abbott and Inês Aniceto. "Giant magnons inAdS4×CP3: embeddings, charges and a Hamiltonian". In: *Journal of High Energy Physics* 2009.04 (Apr. 2009), pp. 136–136. ISSN: 1029-8479. DOI: 10.1088/1126-6708/2009/04/136. URL: http://dx.doi.org/10.1088/1126-6708/2009/04/136.

[2] Ofer Aharony, Oren Bergman, and Daniel Louis Jafferis. "Fractional M2-branes". In: *Journal of High Energy Physics* 2008.11 (Nov. 2008), pp. 043–043. ISSN: 1029-8479. DOI: 10.1088/1126-6708/2008/11/043. URL: http://dx.doi.org/10.1088/1126-6708/2008/11/043.

[3] Ofer Aharony et al. "Large N field theories, string theory and gravity". In: *Physics Reports* 323.3–4 (Jan. 2000), pp. 183–386. ISSN: 0370-1573. DOI: 10.1016/s0370-1573(99)00083-6. URL: http://dx.doi.org/10.1016/S0370-1573(99)00083-6.

[4] Ofer Aharony et al. "The deconfinement and Hagedorn phase transitions in weakly coupled large N gauge theories". In: *Comptes Rendus. Physique* 5.9–10 (Nov. 2004), pp. 945–954. ISSN: 1878-1535. DOI: 10.1016/j.crhy.2004.09.012. URL: http://dx.doi.org/10.1016/j.crhy.2004.09.012.

[5] Ofer Aharony et al. " = 6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals". In: *Journal of High Energy Physics* 2008.10 (Oct. 2008), pp. 091–091. ISSN: 1029-8479. DOI: 10.1088/1126-6708/2008/10/091. URL: http://dx.doi.org/10.1088/1126-6708/2008/10/091.

[6] Martin Ammon and Johanna Erdmenger. *Gauge/Gravity Duality: Foundations and Applications*. 1st. USA: Cambridge University Press, 2015. ISBN: 1107010349.

[7] Gleb Arutyunov and Sergey Frolov. "Superstrings on AdS4× 3as a coset sigma-model". In: *Journal of High Energy Physics* 2008.09 (Sept. 2008), pp. 129–129. ISSN: 1029-8479. DOI: 10.1088/1126-6708/2008/09/129. URL: http://dx.doi.org/10.1088/1126-6708/2008/09/129.

[8] Davide Astolfi et al. "Finite-size corrections for quantum strings on $\mathrm{AdS}_4 \times \mathbb{C}P^3$". In: *Journal of High Energy Physics* 2011.5 (May 2011). ISSN: 1029-8479. DOI: 10.1007/jhep05(2011)128. URL: http://dx.doi.org/10.1007/JHEP05(2011)128.

[9] Davide Astolfi et al. "Finite-size corrections in the SU(2) x SU(2) sector of type IIA string theory on AdS(4) x CP**3". In: *Nucl. Phys. B* 810 (2009), pp. 150–173. DOI: 10.1016/j.nuclphysb.2008.10.020. arXiv: 0807.1527 [hep-th].

[10] Davide Astolfi et al. "Finite-size corrections to the rotating string and the winding state". In: *JHEP* 08 (2008), p. 099. DOI: 10.1088/1126-6708/2008/08/099. arXiv: 0804.3301 [hep-th].

[11] Davide Astolfi et al. "Full Lagrangian and Hamiltonian for quantum strings on AdS (4) x CP**3 in a near plane wave limit". In: *JHEP* 04 (2010), p. 079. DOI: 10.1007/JHEP04(2010)079. arXiv: 0912.2257 [hep-th].

[12] L. V. Avdeev, G. V. Grigorev, and D. I. Kazakov. “Renormalizations in Abelian Chern-Simons field theories with matter”. In: *Nucl. Phys. B* 382 (1992), pp. 561–580. DOI: 10.1016/0550-3213(92)90659-Y.

[13] Jonathan Bagger and George Bruhn. “Three-Algebras in N = 5, 6 Superconformal Chern-Simons Theories: Representations and Relations”. In: *Phys. Rev. D* 83 (2011), p. 025003. DOI: 10.1103/PhysRevD.83.025003. arXiv: 1006.0040 [hep-th].

[14] Jonathan Bagger and Neil Lambert. “Comments on multiple M2-branes”. In: *Journal of High Energy Physics* 2008.02 (Feb. 2008), pp. 105–105. ISSN: 1029-8479. DOI: 10.1088/1126-6708/2008/02/105. URL: http://dx.doi.org/10.1088/1126-6708/2008/02/105.

[15] Jonathan Bagger and Neil Lambert. “Modeling multiple M2-branes”. In: *Physical Review D* 75.4 (Feb. 2007). ISSN: 1550-2368. DOI: 10.1103/physrevd.75.045020. URL: http://dx.doi.org/10.1103/PhysRevD.75.045020.

[16] Jonathan Bagger et al. “Multiple membranes in M-theory”. In: *Physics Reports* 527.1 (June 2013), pp. 1–100. ISSN: 0370-1573. DOI: 10.1016/j.physrep.2013.01.006. URL: http://dx.doi.org/10.1016/j.physrep.2013.01.006.

[17] Stefano Baiguera, Troels Harmark, and Yang Lei. “Spin Matrix Theory in near $\frac{1}{8}$-BPS corners of $\mathcal{N} = 4$ super-Yang-Mills”. In: *JHEP* 02 (2022), p. 191. DOI: 10.1007/JHEP02(2022)191. arXiv: 2111.10149 [hep-th].

[18] Stefano Baiguera, Troels Harmark, and Yang Lei. “The Panorama of Spin Matrix theory”. In: *JHEP* 04 (2023), p. 075. DOI: 10.1007/JHEP04(2023)075. arXiv: 2211.16519 [hep-th].

[19] Stefano Baiguera, Troels Harmark, and Nico Wintergerst. “Nonrelativistic near-BPS corners of $\mathcal{N} = 4$ super-Yang-Mills with $SU(1,1)$ symmetry”. In: *JHEP* 02 (2021), p. 188. DOI: 10.1007/JHEP02(2021)188. arXiv: 2009.03799 [hep-th].

[20] Stefano Baiguera et al. “Symmetry structure of the interactions in near-BPS corners of $\mathcal{N} = 4$ super-Yang-Mills”. In: *JHEP* 04 (2021), p. 029. DOI: 10.1007/JHEP04(2021)029. arXiv: 2012.08532 [hep-th].

[21] James Bedford. *An Introduction to String Theory.* 2012. arXiv: 1107.3967 [hep-th]. URL: https://arxiv.org/abs/1107.3967.

[22] N. Beisert, C. Kristjansen, and M. Staudacher. “The dilatation operator of conformal super-Yang–Mills theory”. In: *Nuclear Physics B* 664.1–2 (Aug. 2003), pp. 131–184. ISSN: 0550-3213. DOI: 10.1016/s0550-3213(03)00406-1. URL: http://dx.doi.org/10.1016/S0550-3213(03)00406-1.

[23] Niklas Beisert. “The Analytic Bethe Ansatz for a Chain with Centrally Extended su(2—2) Symmetry”. In: *J. Stat. Mech.* 0701 (2007), P01017. DOI: 10.1088/1742-5468/2007/01/P01017. arXiv: nlin/0610017.

[24] Niklas Beisert. “The complete one-loop dilatation operator of super-Yang–Mills theory”. In: *Nuclear Physics B* 676.1–2 (Jan. 2004), pp. 3–42. ISSN: 0550-3213. DOI: 10.1016/j.nuclphysb.2003.10.019. URL: http://dx.doi.org/10.1016/j.nuclphysb.2003.10.019.

[25] Niklas Beisert and Matthias Staudacher. “The N=4 SYM integrable super spin chain”. In: *Nucl. Phys. B* 670 (2003), pp. 439–463. DOI: 10.1016/j.nuclphysb.2003.08.015. arXiv: hep-th/0307042.

[26] J. D. Bekenstein. “Black holes and the second law”. In: *Lett. Nuovo Cim.* 4 (1972), pp. 737–740. DOI: 10.1007/BF02757029.

[27] S. Bellucci et al. "SL(2) spin chain and spinning strings on". In: *Nuclear Physics B* 707.3 (Feb. 2005), pp. 303–320. ISSN: 0550-3213. DOI: 10.1016/j.nuclphysb.2004.11.020. URL: http://dx.doi.org/10.1016/j.nuclphysb.2004.11.020.

[28] Marcus Benna et al. "Superconformal Chern-Simons theories and AdS4/CFT3correspondence". In: *Journal of High Energy Physics* 2008.09 (Sept. 2008), pp. 072–072. ISSN: 1029-8479. DOI: 10.1088/1126-6708/2008/09/072. URL: http://dx.doi.org/10.1088/1126-6708/2008/09/072.

[29] David Berenstein, Juan Maldacena, and Horatiu Nastase. "Strings in flat space and pp waves from Script N = 4 Super Yang Mills". In: *Journal of High Energy Physics* 2002.04 (Apr. 2002), pp. 013–013. ISSN: 1029-8479. DOI: 10.1088/1126-6708/2002/04/013. URL: http://dx.doi.org/10.1088/1126-6708/2002/04/013.

[30] H. Bethe. "On the theory of metals. 1. Eigenvalues and eigenfunctions for the linear atomic chain". In: *Z. Phys.* 71 (1931), pp. 205–226. DOI: 10.1007/BF01341708.

[31] Jyotirmoy Bhattacharya and Shiraz Minwalla. "Superconformal indices for = 6 Chern Simons theories". In: *Journal of High Energy Physics* 2009.01 (Jan. 2009), pp. 014–014. ISSN: 1029-8479. DOI: 10.1088/1126-6708/2009/01/014. URL: http://dx.doi.org/10.1088/1126-6708/2009/01/014.

[32] Jyotirmoy Bhattacharya et al. "Indices for Superconformal Field Theories in 3,5 and 6 Dimensions". In: *JHEP* 02 (2008), p. 064. DOI: 10.1088/1126-6708/2008/02/064. arXiv: 0801.1435 [hep-th].

[33] Adel Bilal. *Introduction to Supersymmetry.* 2001. arXiv: hep-th/0101055 [hep-th]. URL: https://arxiv.org/abs/hep-th/0101055.

[34] Damon John Binder. "On $N = 6$ Superconformal Field Theories". PhD thesis. Princeton U., Princeton U., 2021.

[35] M Blau et al. "Penrose limits and spacetime singularities". In: *Classical and Quantum Gravity* 21.7 (Mar. 2004), pp. L43–L49. ISSN: 1361-6382. DOI: 10.1088/0264-9381/21/7/l02. URL: http://dx.doi.org/10.1088/0264-9381/21/7/L02.

[36] Matthias Blau and Martin O'Loughlin. "Multiple M2-Branes and Plane Waves". In: *JHEP* 09 (2008), p. 112. DOI: 10.1088/1126-6708/2008/09/112. arXiv: 0806.3253 [hep-th].

[37] Matthias Blau et al. "Penrose limits and maximal supersymmetry". In: *Classical and Quantum Gravity* 19.10 (Apr. 2002), pp. L87–L95. ISSN: 0264-9381. DOI: 10.1088/0264-9381/19/10/101. URL: http://dx.doi.org/10.1088/0264-9381/19/10/101.

[38] Nikolay Bobev, Junho Hong, and Valentin Reys. "Large N partition functions of the ABJM theory". In: *JHEP* 02 (2023), p. 020. DOI: 10.1007/JHEP02(2023)020. arXiv: 2210.09318 [hep-th].

[39] I. L. Buchbinder et al. "Equations of motion for massive spin-2 field coupled to gravity". In: *Nucl. Phys. B* 584 (2000), pp. 615–640. DOI: 10.1016/S0550-3213(00)00389-8. arXiv: hep-th/9910188.

[40] Shiin-Shen Chern and James Simons. "Characteristic forms and geometric invariants". In: *Annals Math.* 99 (1974), pp. 48–69. DOI: 10.2307/1971013.

[41] Sidney Coleman and Jeffrey Mandula. "All Possible Symmetries of the S Matrix". In: *Phys. Rev.* 159 (5 July 1967), pp. 1251–1256. DOI: 10.1103/PhysRev.159.1251. URL: https://link.aps.org/doi/10.1103/PhysRev.159.1251.

[42] Clay Cordova, Thomas T. Dumitrescu, and Kenneth Intriligator. *Multiplets of Superconformal Symmetry in Diverse Dimensions*. 2016. arXiv: 1612.00809 [hep-th]. URL: https://arxiv.org/abs/1612.00809.

[43] Miguel S. Costa, Vasco Gonçalves, and João Penedones. "Spinning AdS Propagators". In: *JHEP* 09 (2014), p. 064. DOI: 10.1007/JHEP09(2014)064. arXiv: 1404.5625 [hep-th].

[44] E. Cremmer, B. Julia, and Joel Scherk. "Supergravity Theory in 11 Dimensions". In: *Phys. Lett. B* 76 (1978), pp. 409–412. DOI: 10.1016/0370-2693(78)90894-8.

[45] M. Cvetič, H. Lü, and C.N. Pope. "Consistent warped-space Kaluza–Klein reductions, half-maximal gauged supergravities and constructions". In: *Nuclear Physics B* 597.1–3 (Mar. 2001), pp. 172–196. ISSN: 0550-3213. DOI: 10.1016/s0550-3213(00)00708-2. URL: http://dx.doi.org/10.1016/S0550-3213(00)00708-2.

[46] Eric D'Hoker et al. "Graviton and gauge boson propagators in AdS(d+1)". In: *Nucl. Phys. B* 562 (1999), pp. 330–352. DOI: 10.1016/S0550-3213(99)00524-6. arXiv: hep-th/9902042.

[47] F. A. Dolan. "On superconformal characters and partition functions in three dimensions". In: *Journal of Mathematical Physics* 51.2 (Feb. 2010). ISSN: 1089-7658. DOI: 10.1063/1.3211091. URL: http://dx.doi.org/10.1063/1.3211091.

[48] F. A. Dolan and H. Osborn. "On short and semi-short representations for four-dimensional superconformal symmetry". In: *Annals Phys.* 307 (2003), pp. 41–89. DOI: 10.1016/S0003-4916(03)00074-5. arXiv: hep-th/0209056.

[49] M Duff. "String solitons". In: *Physics Reports* 259.4–5 (Aug. 1995), pp. 213–326. ISSN: 0370-1573. DOI: 10.1016/0370-1573(95)00002-x. URL: http://dx.doi.org/10.1016/0370-1573(95)00002-X.

[50] M. J. Duff, B. E. W. Nilsson, and C. N. Pope. "Kaluza-Klein Supergravity". In: *Phys. Rept.* 130 (1986), pp. 1–142. DOI: 10.1016/0370-1573(86)90163-8.

[51] Gerald V. Dunne. *Aspects of Chern-Simons Theory*. 1999. arXiv: hep-th/9902115 [hep-th]. URL: https://arxiv.org/abs/hep-th/9902115.

[52] Simon Ekhammar, Joseph A. Minahan, and Charles Thull. "The ABJM Hagedorn Temperature from Integrability". In: *JHEP* 10 (2023), p. 066. DOI: 10.1007/JHEP10(2023)066. arXiv: 2307.02350 [hep-th].

[53] L. D. Faddeev. "How algebraic Bethe ansatz works for integrable model". In: *Les Houches School of Physics: Astrophysical Sources of Gravitational Radiation*. May 1996, pp. 149–219. arXiv: hep-th/9605187.

[54] Davide Forcella and Waldemar Schulgin. "Spin chain for the deformed ABJM theory". In: *JHEP* 11 (2009), p. 107. DOI: 10.1088/1126-6708/2009/11/107. arXiv: 0906.3301 [hep-th].

[55] Fabio Franchini. *An introduction to integrable techniques for one-dimensional quantum systems*. Vol. 940. Springer, 2017. DOI: 10.1007/978-3-319-48487-7. arXiv: 1609.02100 [cond-mat.stat-mech].

[56] D.Z. Freedman and A. Van Proeyen. *Supergravity*. Cambridge University Press, 2012. ISBN: 9780521194013. URL: https://books.google.dk/books?id=7ANrOHJLTjMC.

[57] Daniel Z. Freedman et al. "Correlation functions in the CFT(d) / AdS(d+1) correspondence". In: *Nucl. Phys. B* 546 (1999), pp. 96–118. DOI: 10.1016/S0550-3213(99)00053-X. arXiv: hep-th/9804058.

[58] Davide Gaiotto, Simone Giombi, and Xi Yin. "Spin chains in = 6 superconformal Chern-Simons-Matter theory". In: *Journal of High Energy Physics* 2009.04 (Apr. 2009), pp. 066–066. ISSN: 1029-8479. DOI: 10.1088/1126-6708/2009/04/066. URL: http://dx.doi.org/10.1088/1126-6708/2009/04/066.

[59] Davide Gaiotto and Xi Yin. "Notes on superconformal Chern-Simons-Matter theories". In: *Journal of High Energy Physics* 2007.08 (Aug. 2007), pp. 056–056. ISSN: 1029-8479. DOI: 10.1088/1126-6708/2007/08/056. URL: http://dx.doi.org/10.1088/1126-6708/2007/08/056.

[60] H. Georgi and S. L. Glashow. "Unity of All Elementary Particle Forces". In: *Phys. Rev. Lett.* 32 (1974), pp. 438–441. DOI: 10.1103/PhysRevLett.32.438.

[61] H. Georgi, Helen R. Quinn, and Steven Weinberg. "Hierarchy of Interactions in Unified Gauge Theories". In: *Phys. Rev. Lett.* 33 (1974), pp. 451–454. DOI: 10.1103/PhysRevLett.33.451.

[62] Simone Giombi, Stefan A. Kurlyand, and Arkady A. Tseytlin. "Non-planar corrections in ABJM theory from quantum M2 branes". In: *JHEP* 11 (2024), p. 056. DOI: 10.1007/JHEP11(2024)056. arXiv: 2408.10070 [hep-th].

[63] Jeffrey Goldstone, Abdus Salam, and Steven Weinberg. "Broken Symmetries". In: *Phys. Rev.* 127 (1962), pp. 965–970. DOI: 10.1103/PhysRev.127.965.

[64] Gianluca Grignani, Troels Harmark, and Marta Orselli. "The SU(2) x SU(2) sector in the string dual of N=6 superconformal Chern-Simons theory". In: *Nucl. Phys. B* 810 (2009), pp. 115–134. DOI: 10.1016/j.nuclphysb.2008.10.019. arXiv: 0806.4959 [hep-th].

[65] Gianluca Grignani et al. "Finite size Giant Magnons in the string dual of N=6 superconformal Chern-Simons theory". In: *JHEP* 12 (2008), p. 008. DOI: 10.1088/1126-6708/2008/12/008. arXiv: 0807.0205 [hep-th].

[66] Gianluca Grignani et al. "New Penrose limits and AdS/CFT". In: *Journal of High Energy Physics* 2010.6 (June 2010). ISSN: 1029-8479. DOI: 10.1007/jhep06(2010)034. URL: http://dx.doi.org/10.1007/JHEP06(2010)034.

[67] D. J. Gross and Frank Wilczek. "Asymptotically Free Gauge Theories - I". In: *Phys. Rev. D* 8 (1973), pp. 3633–3652. DOI: 10.1103/PhysRevD.8.3633.

[68] David J. Gross and Frank Wilczek. "Ultraviolet Behavior of Nonabelian Gauge Theories". In: *Phys. Rev. Lett.* 30 (1973). Ed. by J. C. Taylor, pp. 1343–1346. DOI: 10.1103/PhysRevLett.30.1343.

[69] S.S. Gubser, I.R. Klebanov, and A.M. Polyakov. "Gauge theory correlators from non-critical string theory". In: *Physics Letters B* 428.1–2 (May 1998), pp. 105–114. ISSN: 0370-2693. DOI: 10.1016/s0370-2693(98)00377-3. URL: http://dx.doi.org/10.1016/S0370-2693(98)00377-3.

[70] M. Gunaydin and N. Marcus. "The Spectrum of the s**5 Compactification of the Chiral N=2, D=10 Supergravity and the Unitary Supermultiplets of U(2, 2/4)". In: *Class. Quant. Grav.* 2 (1985), p. L11. DOI: 10.1088/0264-9381/2/2/001.

[71] Andreas Gustavsson. "Selfdual strings and loop space Nahm equations". In: *Journal of High Energy Physics* 2008.04 (Apr. 2008), pp. 083–083. ISSN: 1029-8479. DOI: 10.1088/1126-6708/2008/04/083. URL: http://dx.doi.org/10.1088/1126-6708/2008/04/083.

[72] Rudolf Haag, Jan T. Lopuszanski, and Martin Sohnius. "All Possible Generators of Supersymmetries of the s Matrix". In: *Nucl. Phys. B* 88 (1975), p. 257. DOI: 10.1016/0550-3213(75)90279-5.

[73] R. Hagedorn. “Statistical thermodynamics of strong interactions at high-energies”. In: *Nuovo Cim. Suppl.* 3 (1965), pp. 147–186.

[74] Troels Harmark, Kristjan R. Kristjansson, and Marta Orselli. “Magnetic Heisenberg-chain/pp-wave correspondence”. In: *JHEP* 02 (2007), p. 085. DOI: 10.1088/1126-6708/2007/02/085. arXiv: hep-th/0611242.

[75] Troels Harmark, Kristjan R. Kristjansson, and Marta Orselli. “Matching gauge theory and string theory in a decoupling limit of AdS/CFT”. In: *JHEP* 02 (2009), p. 027. DOI: 10.1088/1126-6708/2009/02/027. arXiv: 0806.3370 [hep-th].

[76] Troels Harmark, Kristjan R. Kristjansson, and Marta Orselli. “The Hagedorn temperature in a decoupled sector of AdS/CFT”. In: *Fortsch. Phys.* 55 (2007). Ed. by Dieter Lust and Franco Pezzella, pp. 754–759. DOI: 10.1002/prop.200610360. arXiv: hep-th/0701088.

[77] Troels Harmark and Marta Orselli. “Matching the Hagedorn temperature in AdS/CFT”. In: *Phys. Rev. D* 74 (2006), p. 126009. DOI: 10.1103/PhysRevD.74.126009. arXiv: hep-th/0608115.

[78] Troels Harmark and Marta Orselli. “Quantum mechanical sectors in thermal N=4 super Yang-Mills on R x S**3”. In: *Nucl. Phys. B* 757 (2006), pp. 117–145. DOI: 10.1016/j.nuclphysb.2006.08.022. arXiv: hep-th/0605234.

[79] Troels Harmark and Marta Orselli. “Spin Matrix theory: a quantum mechanical model of the AdS/CFT correspondence”. In: *Journal of High Energy Physics* 2014.11 (Nov. 2014). ISSN: 1029-8479. DOI: 10.1007/jhep11(2014)134. URL: http://dx.doi.org/10.1007/JHEP11(2014)134.

[80] Troels Harmark, Marta Orselli, and Kristján R Kristjánsson. “Decoupling limits of = 4 super Yang-Mills on ×S3”. In: *Journal of High Energy Physics* 2007.09 (Sept. 2007), pp. 115–115. ISSN: 1029-8479. DOI: 10.1088/1126-6708/2007/09/115. URL: http://dx.doi.org/10.1088/1126-6708/2007/09/115.

[81] Troels Harmark et al. “Finite size giant magnons in the string dual of = 6 superconformal Chern-Simons theory”. In: *Journal of High Energy Physics* 2008.12 (Dec. 2008), pp. 008–008. ISSN: 1029-8479. DOI: 10.1088/1126-6708/2008/12/008. URL: http://dx.doi.org/10.1088/1126-6708/2008/12/008.

[82] Troels Harmark et al. “Spin Matrix theory string backgrounds and Penrose limits of AdS/CFT”. In: *Journal of High Energy Physics* 2021.3 (Mar. 2021). ISSN: 1029-8479. DOI: 10.1007/jhep03(2021)129. URL: http://dx.doi.org/10.1007/JHEP03(2021)129.

[83] Troels Harmark et al. “Strings with non-relativistic conformal symmetry and limits of the AdS/CFT correspondence”. In: *Journal of High Energy Physics* 2018.11 (Nov. 2018). ISSN: 1029-8479. DOI: 10.1007/jhep11(2018)190. URL: http://dx.doi.org/10.1007/JHEP11(2018)190.

[84] Diego M Hofman and Juan Maldacena. “Giant magnons”. In: *Journal of Physics A: Mathematical and General* 39.41 (Sept. 2006), pp. 13095–13117. ISSN: 1361-6447. DOI: 10.1088/0305-4470/39/41/s17. URL: http://dx.doi.org/10.1088/0305-4470/39/41/S17.

[85] G. 't Hooft. *Dimensional Reduction in Quantum Gravity*. 2009. arXiv: gr-qc/9310026 [gr-qc]. URL: https://arxiv.org/abs/gr-qc/9310026.

[86] Seungjoon Hyun and Hyeonjoon Shin. “Script N = (4, 4) Type IIA String Theory on PP-Wave Background”. In: *Journal of High Energy Physics* 2002.10 (Oct. 2002), pp. 070–070. ISSN: 1029-8479. DOI: 10.1088/1126-6708/2002/10/070. URL: http://dx.doi.org/10.1088/1126-6708/2002/10/070.

[87] S. J. Gates Jr et al. *Superspace, or One thousand and one lessons in supersymmetry.* 2001. arXiv: hep-th/0108200 [hep-th]. URL: https://arxiv.org/abs/hep-th/0108200.

[88] Hsein-Chung Kao, Ki-Myeong Lee, and Taejin Lee. "The Chern-Simons coefficient in supersymmetric Yang-Mills Chern-Simons theories". In: *Phys. Lett. B* 373 (1996), pp. 94–99. DOI: 10.1016/0370-2693(96)00119-0. arXiv: hep-th/9506170.

[89] Hsien-Chung Kao. "Selfdual Yang-Mills Chern-Simons Higgs systems with an N=3 extended supersymmetry". In: *Phys. Rev. D* 50 (1994), pp. 2881–2892. DOI: 10.1103/PhysRevD.50.2881.

[90] Hsien-Chung Kao and Ki-Myeong Lee. "Selfdual Chern-Simons systems with an N=3 extended supersymmetry". In: *Phys. Rev. D* 46 (1992), pp. 4691–4697. DOI: 10.1103/PhysRevD.46.4691. arXiv: hep-th/9205115.

[91] Anton Kapustin, Brian Willett, and Itamar Yaakov. "Nonperturbative Tests of Three-Dimensional Dualities". In: *JHEP* 10 (2010), p. 013. DOI: 10.1007/JHEP10(2010)013. arXiv: 1003.5694 [hep-th].

[92] H. J. Kim, L. J. Romans, and P. van Nieuwenhuizen. "Mass spectrum of chiral ten-dimensional N=2 supergravity on S^5". In: *Phys. Rev. D* 32 (2 July 1985), pp. 389–399. DOI: 10.1103/PhysRevD.32.389. URL: https://link.aps.org/doi/10.1103/PhysRevD.32.389.

[93] Seok Kim. "The Complete superconformal index for N=6 Chern-Simons theory". In: *Nucl. Phys. B* 821 (2009). [Erratum: Nucl.Phys.B 864, 884 (2012)], pp. 241–284. DOI: 10.1016/j.nuclphysb.2009.06.025. arXiv: 0903.4172 [hep-th].

[94] Elias Kiritsis. *Introduction to superstring theory.* Vol. B9. Leuven notes in mathematical and theoretical physics. Leuven: Leuven U. Press, 1998. ISBN: 978-90-6186-894-1. arXiv: hep-th/9709062.

[95] Igor R. Klebanov and Giuseppe Torri. "M2-branes and AdS/CFT". In: *Int. J. Mod. Phys. A* 25 (2010). Ed. by Marco Peloso and Arkady Vainshtein, pp. 332–350. DOI: 10.1142/S0217751X10048652. arXiv: 0909.1580 [hep-th].

[96] Igor R. Klebanov and Edward Witten. "Superconformal field theory on three-branes at a Calabi-Yau singularity". In: *Nucl. Phys. B* 536 (1998), pp. 199–218. DOI: 10.1016/S0550-3213(98)00654-3. arXiv: hep-th/9807080.

[97] Thomas Klose. "Review of AdS/CFT Integrability, Chapter IV.3: N=6 Chern-Simons and Strings on AdS4xCP3". In: *Lett. Math. Phys.* 99 (2012), pp. 401–423. DOI: 10.1007/s11005-011-0520-y. arXiv: 1012.3999 [hep-th].

[98] M. Kruczenski, J. Russo, and Arkady A. Tseytlin. "Spiky strings and giant magnons on S**5". In: *JHEP* 10 (2006), p. 002. DOI: 10.1088/1126-6708/2006/10/002. arXiv: hep-th/0607044.

[99] Martin Kruczenski. "Spin Chains and String Theory". In: *Physical Review Letters* 93.16 (Oct. 2004). ISSN: 1079-7114. DOI: 10.1103/physrevlett.93.161602. URL: http://dx.doi.org/10.1103/PhysRevLett.93.161602.

[100] Kasper J Larsen and Niels A Obers. "Phases of thermal = 2 quiver gauge theories". In: *Journal of High Energy Physics* 2008.01 (Jan. 2008), pp. 057–057. ISSN: 1029-8479. DOI: 10.1088/1126-6708/2008/01/057. URL: http://dx.doi.org/10.1088/1126-6708/2008/01/057.

[101] Bum-Hoon Lee, Kamal L. Panigrahi, and Chanyong Park. "Spiky Strings on AdS(4) x CP**3". In: *JHEP* 11 (2008), p. 066. DOI: 10.1088/1126-6708/2008/11/066. arXiv: 0807.2559 [hep-th].

[102] Marius de Leeuw et al. “Introduction to integrability and one-point functions in $\mathcal{N} = 4$ supersymmetric Yang–Mills theory and its defect cousin”. In: (Aug. 2017). Ed. by Patrick Dorey et al. DOI: 10.1093/oso/9780198828150.003.0008. arXiv: 1708.02525 [hep-th].

[103] M. Leoni et al. “Superspace calculation of the four-loop spectrum in N=6 supersymmetric Chern-Simons theories”. In: *JHEP* 12 (2010), p. 074. DOI: 10.1007/JHEP12(2010)074. arXiv: 1010.1756 [hep-th].

[104] Matias Leoni and Andrea Mauri. “On the infrared behaviour of 3d Chern-Simons theories in N=2 superspace”. In: *JHEP* 11 (2010), p. 128. DOI: 10.1007/JHEP11(2010)128. arXiv: 1006.2341 [hep-th].

[105] Tomasz Lukowski and Olof Ohlsson Sax. “Finite size giant magnons in the SU(2) x SU(2) sector of AdS(4) x CP**3”. In: *JHEP* 12 (2008), p. 073. DOI: 10.1088/1126-6708/2008/12/073. arXiv: 0810.1246 [hep-th].

[106] Juan Maldacena. “AdS/CFT conjecture”. In: *International Journal of Theoretical Physics* 38.4 (1999), pp. 1113–1133. ISSN: 0020-7748. DOI: 10.1023/a:1026654312961. URL: http://dx.doi.org/10.1023/A:1026654312961.

[107] Andre Miemiec and Igor Schnakenburg. “Basics of M-theory”. In: *Fortsch. Phys.* 54 (2006), pp. 5–72. DOI: 10.1002/prop.200510256. arXiv: hep-th/0509137.

[108] Christopher M. Miller, Koenraad Schalm, and Erick J. Weinberg. “Nonextremal black holes are BPS”. In: *Physical Review D* 76.4 (Aug. 2007). ISSN: 1550-2368. DOI: 10.1103/physrevd.76.044001. URL: http://dx.doi.org/10.1103/PhysRevD.76.044001.

[109] J.A Minahan, W Schulgin, and K Zarembo. “Two loop integrability for Chern-Simons theories with N = 6 supersymmetry”. In: *Journal of High Energy Physics* 2009.03 (Mar. 2009), pp. 057–057. ISSN: 1029-8479. DOI: 10.1088/1126-6708/2009/03/057. URL: http://dx.doi.org/10.1088/1126-6708/2009/03/057.

[110] J.A Minahan and K Zarembo. “The Bethe ansatz for superconformal Chern-Simons”. In: *Journal of High Energy Physics* 2008.09 (Sept. 2008), pp. 040–040. ISSN: 1029-8479. DOI: 10.1088/1126-6708/2008/09/040. URL: http://dx.doi.org/10.1088/1126-6708/2008/09/040.

[111] Joseph A Minahan and Konstantin Zarembo. “The Bethe-ansatz for Script N = 4 super Yang-Mills”. In: *Journal of High Energy Physics* 2003.03 (Mar. 2003), pp. 013–013. ISSN: 1029-8479. DOI: 10.1088/1126-6708/2003/03/013. URL: http://dx.doi.org/10.1088/1126-6708/2003/03/013.

[112] Joseph A. Minahan. “Review of AdS/CFT Integrability, Chapter I.1: Spin Chains in N=4 Super Yang-Mills”. In: *Lett. Math. Phys.* 99 (2012), pp. 33–58. DOI: 10.1007/s11005-011-0522-9. arXiv: 1012.3983 [hep-th].

[113] C. Montonen and David I. Olive. “Magnetic Monopoles as Gauge Particles?” In: *Phys. Lett. B* 72 (1977), pp. 117–120. DOI: 10.1016/0370-2693(77)90076-4.

[114] Manikantt Mummalaneni. *AdS4/CFT3 : ABJM Theory, Brane Geometry, Correlators and Mellin Space*. 2024. arXiv: 2408.11835 [hep-th]. URL: https://arxiv.org/abs/2408.11835.

[115] Asad Naqvi. “Propagators for massive symmetric tensor and p forms in AdS(d+1)”. In: *JHEP* 12 (1999), p. 025. DOI: 10.1088/1126-6708/1999/12/025. arXiv: hep-th/9911182.

[116] Horatiu Nastase. *Introduction to AdS-CFT*. 2007. arXiv: 0712.0689 [hep-th]. URL: https://arxiv.org/abs/0712.0689.

[117] Peter van Nieuwenhuizen. “The compactification of IIB supergravity on S_5 revisted”. In: *Strings, gauge fields, and the geometry behind: The legacy of Maximilian Kreuzer*. Ed. by Anton Rebhan et al. June 2012, pp. 133–157. DOI: 10.1142/9789814412551_0005. arXiv: 1206.2667 [hep-th].

[118] B. E. W. Nilsson and C. N. Pope. “Hopf Fibration of Eleven-dimensional Supergravity”. In: *Class. Quant. Grav.* 1 (1984). Ed. by A. Salam and E. Sezgin, p. 499. DOI: 10.1088/0264-9381/1/5/005.

[119] Hitoshi Nishino and S. James Gates Jr. “Chern-Simons theories with supersymmetries in three-dimensions”. In: *Int. J. Mod. Phys. A* 8 (1993), pp. 3371–3422. DOI: 10.1142/S0217751X93001363.

[120] Tatsuma Nishioka and Tadashi Takayanagi. “On type IIA Penrose limit and = 6 Chern-Simons theories”. In: *Journal of High Energy Physics* 2008.08 (Aug. 2008), pp. 001–001. ISSN: 1029-8479. DOI: 10.1088/1126-6708/2008/08/001. URL: http://dx.doi.org/10.1088/1126-6708/2008/08/001.

[121] Gerben Oling and Ziqi Yan. “Aspects of Nonrelativistic Strings”. In: *Frontiers in Physics* 10 (June 2022). ISSN: 2296-424X. DOI: 10.3389/fphy.2022.832271. URL: http://dx.doi.org/10.3389/fphy.2022.832271.

[122] J.M. Overduin and P.S. Wesson. “Kaluza-Klein gravity”. In: *Physics Reports* 283.5–6 (Apr. 1997), pp. 303–378. ISSN: 0370-1573. DOI: 10.1016/s0370-1573(96)00046-4. URL: http://dx.doi.org/10.1016/S0370-1573(96)00046-4.

[123] Georgios Papathanasiou and Marcus Spradlin. “The morphology of = 6 Chern-Simons theory”. In: *Journal of High Energy Physics* 2009.07 (July 2009), pp. 036–036. ISSN: 1029-8479. DOI: 10.1088/1126-6708/2009/07/036. URL: http://dx.doi.org/10.1088/1126-6708/2009/07/036.

[124] Roger Penrose. “Any Space-Time has a Plane Wave as a Limit”. In: *Differential Geometry and Relativity: A Volume in Honour of Andr’e Lichnerowicz on His 60th Birthday*. Ed. by M. Cahen and M. Flato. Dordrecht: Springer Netherlands, 1976, pp. 271–275. ISBN: 978-94-010-1508-0. DOI: 10.1007/978-94-010-1508-0_23. URL: https://doi.org/10.1007/978-94-010-1508-0_23.

[125] Jan Plefka. “Spinning strings and integrable spin chains in the AdS/CFT correspondence”. In: *Living Rev. Rel.* 8 (2005), p. 9. DOI: 10.12942/lrr-2005-9. arXiv: hep-th/0507136.

[126] Jan Christoph Plefka. “Lectures on the plane wave string / gauge theory duality”. In: *Fortsch. Phys.* 52 (2004), pp. 264–301. DOI: 10.1002/prop.200310121. arXiv: hep-th/0307101.

[127] J. Polchinski. *String theory. Vol. 2: Superstring theory and beyond*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Dec. 2007. ISBN: 978-0-511-25228-0, 978-0-521-63304-8, 978-0-521-67228-3. DOI: 10.1017/CBO9780511618123.

[128] Joseph Polchinski. *Introduction to Gauge/Gravity Duality*. 2010. arXiv: 1010.6134 [hep-th]. URL: https://arxiv.org/abs/1010.6134.

[129] Pavel Putrov and Masahito Yamazaki. “Exact ABJM Partition Function from TBA”. In: *Mod. Phys. Lett. A* 27 (2012), p. 1250200. DOI: 10.1142/S0217732312502008. arXiv: 1207.5066 [hep-th].

[130] Ana L Retore. “Introduction to classical and quantum integrability”. In: *Journal of Physics A: Mathematical and Theoretical* 55.17 (Apr. 2022), p. 173001. ISSN: 1751-8121. DOI: 10.1088/1751-8121/ac5a8e. URL: http://dx.doi.org/10.1088/1751-8121/ac5a8e.

[131] Shijong Ryang. “Giant Magnon and Spike Solutions with Two Spins in AdS(4) x CP**3”. In: *JHEP* 11 (2008), p. 084. DOI: 10.1088/1126-6708/2008/11/084. arXiv: 0809.5106 [hep-th].

[132] Abdus Salam. “Weak and Electromagnetic Interactions”. In: *Conf. Proc. C* 680519 (1968), pp. 367–377. DOI: 10.1142/9789812795915_0034.

[133] John H Schwarz. “Superconformal Chern–Simons Theories”. In: *Journal of High Energy Physics* 2004.11 (Nov. 2004), pp. 078–078. ISSN: 1029-8479. DOI: 10.1088/1126-6708/2004/11/078. URL: http://dx.doi.org/10.1088/1126-6708/2004/11/078.

[134] Igor Shenderovich. “Giant magnons in AdS(4) / CFT(3): Dispersion, quantization and finite-size corrections”. In: (July 2008). arXiv: 0807.2861 [hep-th].

[135] D. Sorokin and L. Wulff. “Peculiarities of string theory on AdS4 × CP3”. In: *Fortschritte der Physik* 59.7–8 (Mar. 2011), pp. 775–784. ISSN: 1521-3978. DOI: 10.1002/prop.201100009. URL: http://dx.doi.org/10.1002/prop.201100009.

[136] Marcus Spradlin and Anastasia Volovich. “A Pendant for Polya: The One-loop partition function of N=4 SYM on R x S**3”. In: *Nucl. Phys. B* 711 (2005), pp. 199–230. DOI: 10.1016/j.nuclphysb.2005.01.007. arXiv: hep-th/0408178.

[137] B. Stefanski jr. “Green-Schwarz action for Type IIA strings on AdS(4) x CP**3”. In: *Nucl. Phys. B* 808 (2009), pp. 80–87. DOI: 10.1016/j.nuclphysb.2008.09.015. arXiv: 0806.4948 [hep-th].

[138] Andrej Stepanchuk. “Aspects of integrability in string sigma-models”. PhD thesis. Imperial Coll., London, 2015. DOI: 10.25560/28904.

[139] Bo Sundborg. “The Hagedorn transition, deconfinement and N=4 SYM theory”. In: *Nucl. Phys. B* 573 (2000), pp. 349–363. DOI: 10.1016/S0550-3213(00)00044-4. arXiv: hep-th/9908001.

[140] Leonard Susskind. “The world as a hologram”. In: *Journal of Mathematical Physics* 36.11 (Nov. 1995), pp. 6377–6396. ISSN: 1089-7658. DOI: 10.1063/1.531249. URL: http://dx.doi.org/10.1063/1.531249.

[141] David Tong. *Lectures on String Theory.* 2012. arXiv: 0908.0333 [hep-th]. URL: https://arxiv.org/abs/0908.0333.

[142] Mark Van Raamsdonk. “Comments on the Bagger-Lambert theory and multiple M2-branes”. In: *JHEP* 05 (2008), p. 105. DOI: 10.1088/1126-6708/2008/05/105. arXiv: 0803.3803 [hep-th].

[143] Bernard de Wit. “BPS black holes”. In: *Nucl. Phys. B Proc. Suppl.* 171 (2007). Ed. by Laurent Baulieu et al., pp. 16–38. DOI: 10.1016/j.nuclphysbps.2007.06.004. arXiv: 0704.1452 [hep-th].

[144] Edward Witten. *Anti De Sitter Space And Holography.* 1998. arXiv: hep-th/9802150 [hep-th]. URL: https://arxiv.org/abs/hep-th/9802150.

[145] Edward Witten. “String theory dynamics in various dimensions”. In: *Nucl. Phys. B* 443 (1995), pp. 85–126. DOI: 10.1201/9781482268737-32. arXiv: hep-th/9503124.

[146] Daisuke Yamada and Laurence G Yaffe. “Phase diagram of Script N = 4 super-Yang-Mills theory withR-symmetry chemical potentials”. In: *Journal of High Energy Physics* 2006.09 (Sept. 2006), pp. 027–027. ISSN: 1029-8479. DOI: 10.1088/1126-6708/2006/09/027. URL: http://dx.doi.org/10.1088/1126-6708/2006/09/027.

[147] Benjamin I Zwiebel. “Two-loop integrability of planar N=6 superconformal Chern–Simons theory”. In: *Journal of Physics A: Mathematical and Theoretical* 42.49 (Nov. 2009), p. 495402. ISSN: 1751-8121. DOI: 10.1088/1751-8113/42/49/495402. URL: http://dx.doi.org/10.1088/1751-8113/42/49/495402.