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Subsectors of ABJM Theory and Spin Matrix Theory Limits on $AdS_4 \times \mathbb{CP}^3$

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ABSTRACT

This thesis firstly reviews the underlying physics that set the framework for the AdS/CFT correspondence, including conformal field theories, anti-de Sitter space, supergravity theories, and the holographic principle. Specifically, we review literature of computations done on the $AdS_5 \times S^5$ geometry. We study the metrics of the complex projective space, and exhibit the field contents, subsectors and decoupling limits of $\mathcal{N} = 4$ super Yang-Mills and $\mathcal{N} = 6$ super Chern-Simons theory. This is followed by a theoretical introduction of Penrose limits and description of Spin Matrix theory. Lastly, the corresponding Spin Matrix and Penrose limits and flat gauge string actions for various BPS-bounds for ABJM theory with $AdS_4 \times \mathbb{CP}^3$ background are computed, finding also the respective TNC data. These findings generally compare well to other limits and procedures in the literature.

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1

Introduction

As stated in any introduction found in any thesis in theoretical physics, the great dream of modern theoretical physics is to formulate a theory that models the nature of the Universe in the form of a grand unification of all fundamental forces. Being a promising and hopeful yet perhaps seemingly far-fetched idea for the great physicists who laid the groundwork and foundation many decades before us, this idea is now more studied and better captivated than ever. The big problem has always been the attempt to unify the theory of General Relativity (GR) with the theory of Quantum Mechanics (QM). While the unification of QM and Special Relativity (quantum field theory) has been seen to produce the greatest model of science to ever describe the underlying mechanics of the Universe, the Standard Model, GR apparently does not possess the same inclination to play nice in its role in a grand unified theory of everything. However, as time progresses and more and more research was and is done in this field of study, some promising and interesting ideas started and still start to arise. One great achievement was accomplished by Juan Maldacena in the late 90's [1]. Through the scope of the holographic principle he formulated a correspondence, or duality, between a special type of quantum field theory called conformal field theories (CFT's) and string theory on an anti-de Sitter spacetime (AdS). While string theory already is a theory whose spectrum produces both gravitons and other fundamental bosons, the AdS/CFT correspondence provides a strong computational tool to calculate something non-perturbative using a perturbative framework for example. In this thesis, we start by reviewing the underlying foundational principles that lead up to the AdS/CFT correspondence. Thereafter and throughout we compare computations already done in the duality discovered by Maldacena, $AdS_5 \times S^5 \leftrightarrow \mathcal{N} = 4$ super Yang-Mills theory with another duality, $AdS_4 \times \mathbb{CP}^3 \leftrightarrow \mathcal{N} = 6$ super Chern-Simons theory (ABJM theory). We walk through the subsectors and decoupling limits of both CFT's, introduce the concept of Penrose limits and Spin Matrix, then take various Spin Matrix and Penrose limits on the subsectors, showing computations in the framework of ABJM that have not been carried out before. The aim is to end up with results that make physical and intuitive sense comparatively.

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Gauge-Gravity Duality

As stated, one of the greatest achievements in the history of string theory is the so-called AdS/CFT correspondence, which simply put is the use of holography to investigate strongly coupled quantum field theories. This correspondence posits a computational duality between strongly coupled quantum field theories and classical gravitational theories [2]. In this chapter, we review the underlying foundation for this correspondence, starting with conformal field theories, and end up arriving at the correspondence between $\mathrm{AdS}_5 \times S_5$ and $\mathcal{N} = 4$ super Yang-Mills Theory.

2.1 Conformal Field Theories

The behavior of quantum field theories can vary drastically at different energy scales, and a small change in the energy scale of a theory changes the coupling constants of that theory, name according to beta function

$$\frac{\partial g}{\partial \log(\mu)} = \beta(g). \tag{2.1}$$

This determines a trajectory in the space of the coupling constants which is known as the renormalization group flow. However, in our case, we are interested in theories where the beta function vanishes, meaning a change in the energy scale no longer affects the theory. An example of this is a scalar field with a quartic interaction:

$$S = \int dx^4 \left((\partial\phi)^2 + \frac{\lambda}{4!}\phi^4 \right). \tag{2.2}$$

By rescaling the space-time coordinates and the field with a scaling dimension Δ

$$\phi(x) \to \lambda^{-\Delta}\phi(\lambda x), \tag{2.3}$$

the action remains invariant provided, in this case, $\Delta = 1$. This theory would not be invariant with a mass term, however. Generally, this scale invariance is enhanced into what is called *conformal symmetry*, which allows us to put further constraints on the theory. Quantum field theories invariant under conformal transformrations are called conformal field theories (CFTs). What constitutes a conformal transformation is its angle preservation, which we can represent as

$$g_{\alpha\beta}(x) \to \tilde{g}_{\alpha\beta}(x) = e^{2\sigma(x)} g_{\alpha\beta}(x), \tag{2.4a}$$

$$g_{\mu\nu}(x) \to g'_{\mu\nu} = \Omega^2(x) g_{\mu\nu}(x). \tag{2.4b}$$

On top of this it is natural to let fields transform under the Weyl transformation

$$\phi(x) = \Omega(x)^{-\Delta}\phi(x). \tag{2.5}$$

Combining this with (2.4) we thus require that

$$\phi(x) \to \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/d} \phi(x). \tag{2.6}$$

Hence, this scalar number Δ is also referred to as the *conformal weight* of ϕ.

If we want to know the infinitesimal transformations in flat space, we should take the Lie derivative of the Minkowski metric and solve the killing equation to get the isometries:

$$\mathcal{L}_\epsilon \eta_{\mu\nu} = \eta_{\mu\nu} + \partial_\mu \epsilon \nu + \partial_\nu \epsilon \mu \quad \text{and} \quad e^{2\sigma(x)}\eta_{\mu\nu}(x) \approx (1 + 2\sigma(x))\eta_{\mu\nu}. \tag{2.7}$$

After tracing the metric to relate σ to ϵ we get

$$\partial^\mu \partial_\mu \epsilon_\nu = \frac{1}{d}(2-d)\partial_\nu \partial_\lambda \epsilon^\lambda. \tag{2.8}$$

We see that there are two different cases to consider, $d < 2$ and $d > 2$. The case for $d < 2$ is known for an infinitesimal conformal transformation given by

$$\epsilon^\mu(x) = a^\mu + \sigma^\mu_\nu x^\nu + \lambda x^\mu + b^\mu x^2 - 2x^\mu b \cdot x \tag{2.9}$$

We can link each term to an operator corresponding to a generator of the conformal group. The first term we recognize as $a^\mu \leftrightarrow p_\mu$ which is nothing but translation, and secondly we have the Lorentz transformations $\sigma^\mu_\nu x^\nu \leftrightarrow J_{\mu\nu}$. The third term $\lambda x^\mu \leftrightarrow$ is linked to the dilatation operator D which has the property of scaling the coordinates. Lastly we have what we call special conformal transformations $(b^\mu x^2 - 2x^\mu b \cdot x) \leftrightarrow K_\mu$. These objects are the building blocks for conformal field theory. This will be the starting point for the algebra. D and $J_{\mu\nu}$ correspond to the subgroup $SO(1,1) \times SO(1,3)$ of $SO(2,4)$, though sometimes it is convenient to use the maximal compact subgroup $SO(2)\times SU(2)\times SU(2) \subset SO(2,4)$. The quantum numbers defining a state are (Δ, j_1, j_2), viewed as eigenvalues of the aforementioned maximal compact subgroup. The generator $H = (P_0 + K_0)/2$ of $SO(2)$ is called the conformal energy [2]. Since the Poincaré algebra is a subalgebra of the conformal group, we have all the same commutators, but with our new operators, we get non-vanishing commutation relations

$$[D, P_\mu] = -iP_\mu, \tag{2.10a}$$
$$[J_{\mu\nu}, K_\rho] = -i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu), \tag{2.10b}$$
$$[D, K_\mu] = iK_\mu, \tag{2.10c}$$
$$[P_\mu, K_\nu] = 2i(J_{\mu\nu} - \eta_{\mu\nu}D). \tag{2.10d}$$

The conformal algebra is isomorphic to $ISO(d,2)$ with signature $\{-,+,...,+,-\}$. One can construct elements of the Lorentz matrix consisting of the other generators to manifest the isomorphism such that

$$J_{\mu d} = \frac{K_\mu - P_\mu}{2}, \quad J_{\mu(d+1)} = \frac{K_\mu - P_\mu}{2} \quad J_{d(d+1)} = D. \tag{2.11}$$

This can be used to derive the correlator between two or more conformal fields and what kind of function one might expect. First we start by expecting states of the form

$\phi(x) = e^{ix^\mu P_\mu}\phi(0)$. One can find that the commutator is now a field at $x = 0$ where the dilatation operator obeys $[D, \phi(0)] = -i\Delta\phi(0)$. This implies that

$$[D, \phi(x)] = [D, e^{ix^\mu P_\mu}\phi(0)] = ([D, e^{ix^\mu P_\mu}] + e^{ix^\mu P_\mu} D)\phi(0) + e^{ix^\mu P_\mu}\phi(0)D. \tag{2.12}$$

We can expand the exponential and get

$$[D, e^{ix^\mu P_\mu}] = \sum_{i=0}^{\infty} \frac{i^n}{n!} x^{\mu_1}...x^{\mu_n}[D, P_{\mu_1}...P_{\mu_n}], \tag{2.13}$$

where we define $[D, P_{\mu_1}...P_{\mu_n}] = [D, P^n]$. Since $[D, P_\mu] = -iP_\mu$, one can prove that $[D, P^n] = inP^n$ through mathematical induction: Consider for $n + 1$

$$[D, P^{n+1}] = [D, P^n]P + P^n[D, P] = inP^n + iP^n = i(n + 1)P^n. \tag{2.14}$$

With these identities, eq. (2.12) becomes

$$\begin{aligned}[D, \phi(x)] &= \sum_{i=0}^{\infty} \frac{i^{n+1}n}{n!}(x^\mu P_\mu)^n \phi(0) + e^{ix^\mu P_\mu}[D, \phi(0)] \\ &= i^2 x^\mu P_\mu \sum_{i=1}^{\infty} \frac{i^{n-1}}{(n-1)!}(x^\mu P_\mu)^{n-1}\phi(0) - i\Delta\phi(x) \\ &= i(x^\mu\partial_\mu - \Delta)\phi(x).\end{aligned} \tag{2.15}$$

Now we can turn ourselves to look at the two-point function for scalar operators and see what we might expect. From rotational and translational invariance we get

$$\langle \phi_1(x)\phi_2(y)\rangle = f(|x - y|). \tag{2.16}$$

To find the function, we are aided by a general Ward identity concerning dilatation, namely

$$0 = \sum_{i=1}^{n} \left(x_i \frac{\partial_i}{\partial x_i^\mu} - \Delta_i \right) \langle \phi(x)...\phi_i(x_i)...\phi_n(x_n)\rangle . \tag{2.17}$$

It reads easily for the two-point case by considering the commutator between the dilatation operator and the fields at hand

$$\begin{aligned}0 = \langle 0| [D, \phi_1(x)\phi_2(y)] |0\rangle &= \langle 0| \phi_1(x)[D, \phi_2(y)] - [D, \phi_1(x)]\phi_2(y) |0\rangle \\ &= (x^\mu\partial_\mu^{(x)} - \Delta_1 + y^\mu\partial_\mu^{(y)} - \Delta_2) \langle \phi_1(x)\phi_2(y)\rangle .\end{aligned} \tag{2.18}$$

The superscripts refer to the on which variable the derivatives act. One finds that the solutions of the differential equation $(x^\mu\partial_\mu^{(x)} - \Delta_1 + y^\mu\partial_\mu^{(y)} - \Delta_2)f(|x - y|) = 0$ take the form

$$f(|x - y|) = \frac{C}{|x - y|^{2\Delta}}. \tag{2.19}$$

The exponent is actually $\Delta_1 + \Delta_2$ but by special conformal transformations we can fix them to be $\Delta_1 = \Delta_2$. This procedure can also be done for 3-point functions, but the story does change for the 4-point. Lastly we mention primary operators and how one can lower and raise the conformal dimension from the commutations. If we consider the following

$$[D, K_\mu\phi(0)] = K_\mu[D, \phi(0)] - [D, K_\mu]\phi(0) = -i(\Delta - 1)K_\mu\phi(0), \tag{2.20}$$

then one can deduce that by applying an arbitrary number of K_μ operators on an operator, this process must eventually terminate giving us $[K_\mu, \phi(0)] = 0$, meaning that $\phi(0)$ is a primary operator. From primary operators, it is then possible to construct what is called descendants, which can be obtained by applying consecutive momentum operators on such primaries $P_{\mu_1}...P_{\mu_n}\phi(0)$ giving us a conformal weight of $\Delta + n$. This can be stated in commutator language as $[D, P_\mu\phi(0)] = -i(\Delta + 1)P_\mu\phi(0)$. So, for an operator to be primary it must meet these conditions:

$$[D, \phi(0)] = -i\Delta\phi(0), \quad [J_{\mu\nu}, \phi(0)] = \mathcal{J}_{\mu\nu}\phi(0), \quad [K_\mu, \phi(0)] = 0. \tag{2.21}$$

We see that conformal invariance gives many constraints on the theory, namely the Ward identities giving constraints on the Green functions and also the possible dimensions of the primary fields. However, there is one more important object that is affected in the conformal space. Consider the translation

$$x^\mu \to x^\mu + \epsilon^\mu(x). \tag{2.22}$$

Normally we can identify the energy-momentum tensor as the associated Noether currrent (assuming flat spacetime)

$$\delta S = -\int d^d x T^\mu_\nu \partial_\mu \epsilon^\nu. \tag{2.23}$$

If we assume the tensor is symmetric, then eq. (2.23) can be written in a general spacetime as

$$\delta S = -\frac{1}{2}\int d^d x\sqrt{-g}T^{\mu\nu}\delta_\epsilon g_{\mu\nu}, \tag{2.24}$$

where $g = \det(g_{\mu\nu})$ and the variation $\delta_\epsilon g_{\mu\nu}$ corrresponds to the Lie deriative $\mathcal{L}_\epsilon g_{\mu\nu}$ which standardly is defined as

$$\mathcal{L}_\epsilon g_{\mu\nu} = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu. \tag{2.25}$$

This leads us to the definition of the energy-momentum tensor

$$T^{\mu\nu} = -\frac{2}{\sqrt{g}}\frac{\delta S}{\delta g_{\mu\nu}}. \tag{2.26}$$

Plugging the Lie derivative (2.25) into eq. (2.24) gives the known conservation law $\nabla_\mu T^{\mu\nu}$, meaning the energy-momentum tensor is conserved under diffeomorphisms. However, in a theory with Weyl invariance, the transformation (2.4b) is a symmetry of the action, and thus eq. (2.26) implies that

$$0 = \delta S = -\frac{1}{2}\int d^d x\sqrt{-g}T^\mu_\nu\Omega^2 \implies T^\mu_\nu = 0, \tag{2.27}$$

since $\Omega(x)$ is arbitrary. In a classical theory, this just means the energy-momentum tensor is traceless in theories with conformal symmetry. In quantum theories, however, this traceless condition gets modified by quantum effects in even spacetime dimensions [3, 4, 5, 6].

2.2 Anti-de Sitter Spacetime

We now turn to the foundation of the other part of the duality, namely gravity. As described by Einstein in 1915 [7] gravity in empty space is described by the following field equations:

$$R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{2.28}$$

By taking the trace we obtain the relation

$$R = 2\Lambda\frac{d+1}{d-1} \tag{2.29}$$

and see that the sign of the cosmological constant Λ determines the sign of the Ricci scalar R. Now, for this project we are interested in maximally symmetric solutions to the field equations (2.28). This means they have the property that the Riemann tensor, normally defined as [8]

$$R^{\rho}_{\sigma\mu\nu} \equiv \partial_\mu\Gamma^{\rho}_{\nu\sigma} - \partial_\nu\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma},$$

now becomes fully expressed in terms of the Ricci scalar such that it can be written as [9]

$$R_{\mu\nu\rho\sigma} = \frac{R}{d(d-1)}(g_{\nu\sigma}g_{\mu\rho} - g_{\nu\rho}g_{\mu\sigma}). \tag{2.30}$$

It turns out such geometries can be described conveniently as an embedding in a higher dimensional geometry, which gives us a convenient way of deriving the present isometries. In our case we will be interested in solutions where $\Lambda < 0$ such that we get geometries described by negative curvature. Spacetimes that are maximally symmetric have the property that they look the same at every point and in every direction at every point, and the metric of interest in this project is the so-called anti-de Sitter spacetime (anti meaning negative curvature), abbreviated as AdS. The metric can be expressed in different ways, but the probably most often used and the one that will be used in future calculations of this paper is the one with Lorentzian signature in $d+1$ dimensions:

$$ds^2 = L^2(-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega^2_{d-1}), \tag{2.31}$$

where L, ρ, and t come from the embedding coordinates

$$x = L\cosh\rho\cos t, \tag{2.32a}$$
$$y = L\cosh\rho\sin t \tag{2.32b}$$
$$z = L\sinh\rho, \tag{2.32c}$$

used in the embedding equation

$$-x^2 - y^2 + z^2 = -L^2, \tag{2.33}$$

but where the z-direction in (2.31) has been replaced by the $(d-1)$-sphere $d\Omega^2_{d-1}$. One can consider the $(d+1)$ dimensional anti-de Sitter space (AdS_{d+1}) being embedded into an $\mathbb{R}^{d,2}$ which is just $(d+2)$-dimensional Minkowski space. The metric signature is $\eta = \text{diag}(-,+,+,...,+,-)$ and is given by

$$ds^2 = (dx^0)^2 + (dx^1)^2 + (dx^2)^2 + ... + (dx^d+1)^2 = \eta_{MN}dx^M dx^N. \tag{2.34}$$

AdS_{d+1} can also be written in coordinates as a hypersurface

$$\eta_{MN}x^M x^N = -(x^0)^2 + \sum_{i=1}^{d}(x^i)^2 - (x^{d+1})^2 = -L^2, \tag{2.35}$$

where L is the radius of curvature of AdS_{d+1}. The embedding is clearly invariant under the Lorentz group for $\mathbb{R}^{d,2}$, $SO(d,2)$, which has dimension $\frac{1}{2}(d+1)(d+2)$. This is the number of Killing vectors associated to AdS_{d+1}, leading us to conclude that the space is maximally symmetric. $SO(d,2)$ is the conformal group of d-dimensional Minkowski space, pointing in the right direction with regards to the symmetries of the dulaity.

One can parametrize the coordinates in multiple ways. Let us introduce the coordinates $t \in \mathbb{R}, \vec{x} = (x_1, ..., x^{d-1}) \in \mathbb{R}^{d-1}$ and $r \in \mathbb{R}_+$. The parametrization in these coordinates is given by

$$\begin{aligned} X^0 &= \frac{L^2}{2r}\left(1 + \frac{r^2}{L^4}(\vec{x}^2 - t^2 + L^2)\right), \qquad X^i = \frac{rx^i}{L}, \quad i \in \{1, ..., d-1\}, \\ X^d &= \frac{2r}{L^2}\left(1 + \frac{r^2}{L^4}(\vec{x}^2 - t^2 + L^2)\right), \qquad X^{d+1} = \frac{rt}{L}. \end{aligned} \tag{2.36}$$

Due to the restriction that r be positive, we cover only half of the AdS_{d+1} spacetime. These local coordinates are referred to as Poincaré patch coordinates. In the Poincaré patch, the metric of the space reads

$$ds^2 = \frac{L^2}{r^2}dr^2 + \frac{r^2}{L^2}(d\vec{x}^2 - dt^2) = \frac{L^2}{r^2}dr^2 + \frac{r^2}{L^2}(\eta_{\mu\nu}dx^\mu dx^\nu), \tag{2.37}$$

where we recognized the metric of d-dimensional Minkowski space. Using this metric, we can compute the Ricci scalar, which becomes $R = -d(d+1)/L^2$, implying that L^2 is indeed the radius of curvature. Another useful form of the Poincaré metric is obtained by inverting the radial coordinate, $z = L^2/r$, thus yielding the metric in Poincaré z-coordinates,

$$ds^2 = \frac{L^2}{z^2}(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu). \tag{2.38}$$

Note that the boundary in these coordinates is located at $z = 0$

Another possibility is to introduce global coordinates τ, ρ, θ_i, and describe the spacetime through hyperbolic functions

$$X^0 = L\cosh\rho\cos\tau, \tag{2.39a}$$

$$X^{d+1} = L\cosh\rho\cos\tau, \tag{2.39b}$$

$$X^i = L\Omega_i\sinh\rho. \tag{2.39c}$$

Here, Ω_i $(i = 1, ..., d)$ are angular coordinates satisfying $\sum_i \Omega_i^2 = 1$. In other words, Ω_i parametrize a $(d-1)$ dimensional sphere. These coordinates are referred to as global coordinates of AdS_{d+1} since all points of the hypersurface of the eq. (2.35) are taken into account exactly once. From this one finds that the induced metric then becomes

$$ds^2 = L^2(-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_{d-1}^2). \tag{2.40}$$

Since the metric above does not depend on τ, we infer the existence of a timelike Killing vector ∂_τ, and since this Killing vector is defined globally on the manifold, τ acts as a sensible global time coordinate. Near the center $\rho = 0$, the metric assumes the form $ds^2 = -L^2(d\tau^2 + d\rho^2 + \rho^2 d\Omega^2_{S^{d-1}})$, implying that the spacetime has topology, since τ is periodic of $S^1 \times \mathbb{R}^d$, where S^1 is the periodic time; in particular, since ∂_τ is everywhere timelike, keeping ρ and θ_i fixed while varying τ will produce closed time-like curves. This is, however, not an intrinsic property of this spacetime but merely a consequence of our embedding: $\mathbb{R}^{d,2}$ has two timelike directions, so the appearance of closed timelike curves is not so surprising after all.

2.3 Supersymmetry

Before diving into (super) strings we will quickly review a symmetry which plays an important role in the gauge/gravity duality and the realm of M-theory. As of today the Standard Model unites all fundamental forces of Nature except gravity in which the typical electroweak scale is $M_{\text{ew}} \sim 250$ GeV, and at which the model is tested very well. Since the gravitational force is so much weaker than the other fundamental forces, the energy scale for which gravity would be expected to become non-negligeble is at the Planck scale $M_{\text{Pl}} \sim 10^{19}$ GeV. Since this scale is so much greater than the electroweak, it would be reasonable to except new interactions between them. After renormalization in perturbation theory, masses of scalar particles usualy diverge in a quadratic manner, for example a fermion coupled to a Higgs boson through the Yukawa interaction $-\lambda_f H\bar{f}f$, whose one-loop correction is $\Delta m_H^2 \sim -2\lambda_f^2\Lambda^2$, with Λ being the UV cutoff. This means that to conserve the experimentally found Higgs mass at $m_H \sim 125$ GeV, the UV cutoff would have to be in the TeV scale, rendering the Standard Model an effective theory at energy scales less then TeV. Going beyond this scale would induce new interactions that fit to protect otherwise divergent perturbation corrections to the Higgs mass, resulting in new fermionic and bosonic degrees of freedom, which means more couplings to the Higgs boson. These will induce other pertubartive corrections of the Planck scale order (the new UV cutoff), and this would entail a lot of work done simply to keep the Higss mass to the original value. This hierarchy problem could be solved by assuming that the model admits couplings of a form that cancel the UV divergences. The introduction of *supersymmetry* [10] manifests such a solution by being represented with a generator Q, called supercharge, that has the property

$$Q\,|f\rangle = |b\rangle\,, \quad Q\,|b\rangle = |f\rangle\,, \tag{2.41}$$

where f is for fermion and b is for boson. Note that it changes the spin of a particle, and this matches the level of bosonic and fermionic degrees of freedom.

In 1967, Coleman and Mandula [11] proved that it is impossible to combine spacetime and internal symmetries of the S-matrix in any but a trivial way, a theorem known as a no-go theorem. This means that an internal symmetry generator G has to commute with the Poincaré generators:

$$[G, P_\mu] = [G, J_{\mu\nu}] = 0. \tag{2.42}$$

It turns out that by including anti-commutators in the algebra, thus softening the assumptions of the no-go theorem, we can evade the theorem. In 1975 Haag, Lopuszanski, and Sonhius [12] showed that the possible combinations of symmetries in the no-go theorem are enhanced to super-Poincaré and internal symmetries. Recall the standard translation and Lorentz group algebras

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \quad [K_i, K_j] = -i\epsilon_{ijk}J_k, \quad [J_i, K_j] = i\epsilon_{ijk}K_j, \tag{2.43}$$

$$[P_\mu, P_\nu] = 0, \tag{2.44}$$

$$[J_i, P_j] = i\epsilon_{ijk}P_k,\ [J_i, P_0] = 0,\ [K_i, P_j] = -iP_0,\ [K_i, P_0] = -iP_j, \tag{2.45}$$

with J_i being rotations, K_i being boosts, and P_μ translations. We want to enlarge this Poincaré algebra with generators that transform under the Lorentz group and commute with translations. These will be the spinors Q^I_α and $\tilde{Q}^I_{\dot{\alpha}}$. The index I runs from 1 to $\mathcal{N}$ and labels the number of copies of pairs of supersymmetric generators. The number is constrained by the fact that a supermultiplet contains particles that have spin $\geq \mathcal{N}/4$. In four dimensions this means in particular that a theory with $\mathcal{N} \leq 4$ describes a local gauge theory with maximum spin one particles. In general, the supersymmetric algebra looks like

$$\begin{aligned} \{Q^A_\alpha, Q^B_\beta\} &= \epsilon_{\alpha\beta}Z^{AB}, \quad \{\tilde{Q}_{\dot{\alpha}A}, \tilde{Q}_{\dot{\beta}B}\} = \epsilon_{\dot{\alpha}\dot{\beta}}(Z^{AB})^* \\ \{Q^A_\alpha, \tilde{Q}_{\dot{\alpha}B}\} &= 2(\sigma^\mu)_{\alpha\dot{\alpha}}P_\mu\delta^A_B, \\ [Q^A_\alpha, J^{\mu\nu}] &= (\sigma^{\mu\nu})^\beta_\alpha Q^A_\beta, \quad [Q^A_\alpha, P^\mu] = 0. \end{aligned} \tag{2.46}$$

Here, $Z^{AB} = -Z^{BA}$ is the central charge (we elaborate on this term in the next section), σ are the Pauli matrices. For the case $\mathcal{N} = 1$, the centrally extended algebra (2.46) looks simpler due to the fact that now $Z = 0$ by its anti-symmetric nature (though this is not necessarily the case [13, 14]).

Generically, if a theory is supersymmetric it means that its action is invariant under some supersymmetric spacetime transformations that, as per (2.41), relate bosons and fermions. An example is the Wess-Zumino model (see [15] for a more in-depth review)

$$\mathcal{L} = -|\partial\phi|^2 - i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + |F^2| - m\left(\frac{1}{2}\psi\psi + \frac{1}{2}\bar{\psi}\bar{\psi} + F\phi + \tilde{F}\tilde{\phi}\right),$$

which is invariant under the transformations

$$\begin{aligned} \delta_\epsilon\phi &= \sqrt{2}\epsilon^\alpha\psi_\alpha, \\ \delta_\epsilon\psi_\alpha &= \sqrt{2}\epsilon_\alpha F + \sqrt{2}i(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\epsilon}^{\dot{\beta}}\partial_\mu\phi, \\ \delta_\epsilon F &= \sqrt{2}i\bar{\epsilon}_{\dot{\beta}}(\bar{\sigma}^\mu)^{\dot{\beta}\alpha}\partial_\mu\psi_\alpha. \end{aligned} \tag{2.47}$$

One can obtain the algebra by defining the transformations through $\delta_\epsilon\phi = i[\epsilon Q + \bar{\epsilon}\tilde{Q}, \phi]$ and likewise for ψ. In the $\mathcal{N} = 1$ case, the algebra is invariant under phase rotations $Q_\alpha \to e^{i\lambda}Q_\alpha$. This symmetry is called R-symmetry, and in the general case this symmetry is a $U(\mathcal{N})$ global symmetry.

2.3.1 Representations and Multiplets

Obtaining the irreducible representations of the supersymmetry algebra is done by looking at the (anti)commutators (2.46) and defining the Casimir operators. In the standard Poincaré group these Casimir operators are $P^2 = P^\mu P_\mu$ and $W^2 = W^\mu W_\mu$, where

$$W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}, \quad M_{ij} = \epsilon_{ijk}J_k, M_{\mu\nu} = -M_{\nu\mu}, M_{0i} = K_i. \tag{2.48}$$

But now in the supersymmetric case, W^2 is no longer a Casimir operator since $M_{\mu\nu}$ does not commute with the supercharges $Q, \tilde{Q}$. So, in the $\mathcal{N}=1$ case, it is replaced by [16]

$$\begin{aligned} C^2 &= C_{\mu\nu}C^{\mu\nu}, \\ C_{\mu\nu} &= B_\mu P_\nu - B_\nu P_\mu, \\ B_\mu &= W_\mu - \frac{1}{4}\tilde{Q}_{\dot{\alpha}}\tilde{\sigma}_\mu^{\dot{\alpha}\alpha}Q_\alpha. \end{aligned} \tag{2.49}$$

To get the irreducible representations one can employ Wigner's technique of induced representations (see e.g. [17, 18]). Note that, as stated, every irreducible representation contains the same number of bosonic states as fermionic states. This can be seen be defining the fermion number operator $(-)^{N_f}$ with the properties

$$(-)^{N_f}\left|b\right\rangle = +\left|b\right\rangle, \quad (-)^{N_f}\left|f\right\rangle = -\left|f\right\rangle. \tag{2.50}$$

From this it follows that

$$(-)^{N_f}Q_\alpha = -Q_\alpha(-)^{N_f}, \quad (-)^{N_f}\tilde{Q}_{\dot{\alpha}} = -\tilde{Q}_{\dot{\alpha}}(-)^{N_f}, \tag{2.51}$$

which imply that

$$0 = \mathrm{Tr}\Big[(-)^{N_f}\{Q_\alpha, \tilde{Q}_{\dot{\alpha}}\}\Big] = 2\sigma^\mu_{\alpha\dot{\alpha}}\delta^{ij}P_\mu\,\mathrm{Tr}(-)^{N_f}. \tag{2.52}$$

For non-zero P_μ this means that $\mathrm{Tr}(-)^{N_f} = 0$. The interpretation of the equal number of bosons and fermions is naturally that every particle has a superpartner, which should be detectable at high enough energies. The superpartner for e.g. the graviton is called the gravitino[1] (which is fermionic).

Consider the rest frame of a particle with mass m, $P_\mu = (m, 0, 0, 0)$. If we act with particle states $|p_\mu, s, s_3\rangle$ on the general supersymmetry algebr, and assuming vanishing central charges, we get

$$\{Q^A_\alpha, \bar{Q}_{\dot{\beta}B}\} = 2m\delta^A_B(\sigma_0)_{\alpha\dot{\beta}} = 2m\delta^A_B\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \tag{2.53}$$

We expect more states to exist since it is not given that $Q^A_2\left|p_\mu, s, s_3\right\rangle = 0$ for all A. Thus we define a pair of creation an annihilation operators

$$a^B_\alpha = \frac{Q^B_\alpha}{\sqrt{2m}}, \quad (a^\dagger)^A_{\dot{\alpha}} = \frac{\bar{Q}^A_{\dot{\alpha}}}{\sqrt{2m}}. \tag{2.54}$$

Here, a lowers and $a^\dagger$ raises as usual. One can thus raise states and make combinations of products of these in $2\mathcal{N}$ ways, since $\dot{\alpha} \in \{1, 2\}$ and $A, B \in \{1, 2, ..., \mathcal{N}\}$. Hence, all in all we get $2^{2\mathcal{N}}$ states compared to half as many in the exponent as one would get for the massless states. Now, assume on the contrary that we have central charges that do not vanish. Then we know that they commute with all generators of the SUSY algebra. So what we want to do then is choose a basis where the central charges are diagonal and

[1] Some even suggest this as a candidate for dark matter (see e.g. [19, 20]). This goes to show the possible powerful extend to which supersymmetry goes to solve problems of nature!

have eigenvalues q_i. This can be arranged in the anti-symmetric matrix Z^{AB} giving us for $\mathcal{N} = 2$

$$Z^{AB} = \begin{pmatrix} 0 & q_1 \\ -q_1 & 0 \end{pmatrix}. \tag{2.55}$$

The same construction follows for $\mathcal{N} > 2$. Here onw would just build block diagonals consisting of Lego blocks for the $\mathcal{N} = 2$ matrix

$$Z^{AB} = \begin{pmatrix} 0 & q_1 & 0 & 0 & 0 & ... & & \\ -q_1 & 0 & 0 & 0 & 0 & ... & & \\ 0 & 0 & 0 & q_2 & 0 & ... & & \\ 0 & 0 & -q_2 & 0 & 0 & ... & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & & & \\ & & & & & \ddots & & \\ & & & & & & 0 & q_{\frac{\mathcal{N}}{2}} \\ & & & & & & -q_{\frac{\mathcal{N}}{2}} & 0 \end{pmatrix}. \tag{2.56}$$

Thus if we want to make a corresponding set of raising a lowering operators, the only ones that are non-zero will take the additional term of the eigenvalues for the non-vanishing central charges. Using a linear combination we then construct

$$\tilde{Q}^j_{\alpha\pm} = Q^{2j-1}_\alpha \pm (Q^{2j}_\alpha)^\dagger, \quad j \in \left\{1, ..., \frac{\mathcal{N}}{2}\right\}. \tag{2.57}$$

Hence taking the anti-commutator we can find that all the non-zero terms goes on the following form

$$\{\tilde{Q}^i_{\alpha_+}, (\tilde{Q}^j_{\beta_+})^\dagger\} = \delta^j_i \delta^\beta_\alpha (2m + q_i), \quad \{\tilde{Q}^i_{\alpha_-}, (\tilde{Q}^j_{\beta_-})^\dagger\} = \delta^j_i \delta^\beta_\alpha (2m - q_i). \tag{2.58}$$

For unitary particle representations we must insist that both right-hand sides stay positive, leading to $|q_j| \leq 2m$ for all j. But precisely when equality holds $|q_j| = 2m$ we get the so-called BPS (Bogomolnyi-Prasad-Sommerfield [21, 22]) bound. In the event that k of the q_j are fulfilling the BPS-bound, we see that $2\mathcal{N} - 2k$ of these operators satisfy the equality such that we now have $2^{2\mathcal{N}-2k}$ states. These are referred to as $1/2^k$ BPS-multiplets. Possible BPS-multiplets are

$$k = 0 \quad \longleftrightarrow \quad 2^{2\mathcal{N}} \text{ States} \quad \text{Long Multiplet}, \tag{2.59}$$
$$0 < k < \frac{\mathcal{N}}{2} \quad \longleftrightarrow \quad 2^{2(\mathcal{N}-k)} \text{ States} \quad \text{Short Multiplet}, \tag{2.60}$$
$$k = \frac{\mathcal{N}}{2} \quad \longleftrightarrow \quad 2^{\mathcal{N}} \text{ States} \quad \text{Ultra Short Multiplet}. \tag{2.61}$$

For the massless case, $P_\mu = (-E, 0, 0, E)$, the algebra reduces to

$$\{Q^A_\alpha, \tilde{Q}_{\dot{\beta}B}\} = 2 \begin{pmatrix} 2E & 0 \\ 0 & 0 \end{pmatrix} \delta^A_B, \tag{2.62}$$

and the creation and annihilation operators are defined as

$$a^A = \frac{1}{2\sqrt{E}} Q^A_1, \quad a^\dagger_B = \frac{1}{2\sqrt{E}} \tilde{Q}^A_1. \tag{2.63}$$

These generate the algebra

$$\begin{aligned}\{a^A, a^\dagger_B\} &= \delta^A_B,\\ \{a^A, a^B\} &= \{a^\dagger_A, a^\dagger_B\} = 0.\end{aligned} \tag{2.64}$$

The vacuum is per usual defined as

$$a^A \left|\Omega_\lambda\right\rangle = 0, \tag{2.65}$$

with λ labeling helicity. The multiplets are then defined as

$$\left|\Omega^{(n)}_{\lambda+\frac{n}{2};i_1,\ldots i_n}\right\rangle = \frac{1}{\sqrt{n!}} a^\dagger_{i_n} \ldots a^\dagger_{i_1} \left|\Omega_\lambda\right\rangle. \tag{2.66}$$

This state has helicity $\lambda + n/2$ and due to the antisymmetry of the exchange of indices, the total number of states in the irreducible representation is

$$\sum_{k=0}^{\mathcal{N}} \binom{\mathcal{N}}{k} = 2^{\mathcal{N}},$$

half/half of fermionic and bosonic states. Note that CPT invariance changes the sign of the helicity, and so if the helicity is not distributed symmetrically around 0, we get double the amount of states.

2.3.2 $\mathcal{N} = 4$ super Yang-Mills Theory

A model with great importance for the discovery of the correspondence found by Maldecena is the $\mathcal{N} = 4$ super Yang-Mills theory (SYM), which as the name indicates is the supersymmetric extension of Yang-Mills theory. The new constraints imposed by supersymmetry at $\mathcal{N} = 4$ makes this theory perturbatively finite meaning there is no renormalization of both the wave function and coupling constants, and there is no UV divergence either in the computation of correlation functions. On top of that, $\mathcal{N} = 4$ SYM is a superconformal theory; it is supersymmetrically and conformally invariant at all loops, and as a consequence it has vansishing beta functions. Here we will introduce the action and in the next section show how it practically can be derived from string theory.

The field content corresponds to the $\mathcal{N} = 4$ case of the massless vector multiplets. It has six real scalar fields ϕ_i, which transform in the anti-symmetric representation of the R-symmetry group $SO(6) \simeq SU(4)$ (these can be combined into three complex fields). It contains four Weyl fermions ψ^A, with $A = 1 \ldots 4$ which transform in the fundamental representation of $SU(4)$. Lastly it has one gauge field A_μ being a singlet under the R-symmetry group. The action can be obtained in different ways but here we simply state it. With Grassmann variable θ the Lagrangian is

$$\begin{aligned}\mathcal{L} = \mathrm{Tr}\Big\{ &-\frac{1}{2g^2} F^2_{\mu\nu} + \frac{\theta}{16\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} - (D_\mu\phi_i)^2 - i\psi^\dagger_A \bar{\sigma}^\mu D_\mu \psi^A \\ &+ gC^i_{AB}\psi^A[\phi_i, \psi^B] + g\bar{C}^{iAB}\psi^\dagger_A[\phi_i, \psi^\dagger_B] + \frac{g^2}{2}([\phi_i, \phi_j])^2\Big\},\end{aligned} \tag{2.67}$$

with the field strengths

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu], \tag{2.68}$$

and the covariant derivatives

$$D_\mu \phi_i = \partial_\mu \phi_i + i[A_\mu, \phi_i], \tag{2.69}$$
$$D_\mu \psi^A = \partial_\mu \psi^A + i[A_\mu, \psi^A], , \tag{2.70}$$

and the dual field strength

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}. \tag{2.71}$$

The C^i_{AB} are Clebsch-Gordan coefficients that couple to the R-symmetry group $SU(4)$, and the fields in the action transform in the ajdoint representation of the gauge group $SU(N)$. The action (2.67) enjoys Poincaré invariance, $\mathcal{N} = 4$ supersymmetry, global $SU(4)$ R-symmetry, and conformal invariance. All of these symmetries can be formulated through the so-called superconformal group $PSU(2, 2|4)$, though we will only comment briefly on some of the properties of this (see [23] for a thorough review).
One can define the operator

$$\mathcal{O}(x) = \mathrm{Str}\left(\phi^i(x)\dots\phi^j(x)\right), \quad \mathrm{Str}(T_{a_1}\dots T_{a_n}) = \sum_\sigma \mathrm{Tr}\left(T_{\sigma(a_1)}\dots T_{\sigma(a_n)}\right), \tag{2.72}$$

where the sum is over the permutations σ. This forms a completely symmetric object that makes up an irreducible representation of the so-called superconformal algebra. This is also a chiral primary operator corresponding to a 1/2 BPS-state. The operator has dimension $\Delta = n$ and is protected even at the quantum level.

2.4 Supergravity Theories

So far we have gone through most of the gauge side of the gauge/gravity duality, and now we turn to the gravity side, more precisely, supergravity (SUGRA). Firstly we will briefly look at string theory, a theory for which the addition of supersymmetry helps solving some complications that arise naturally without it. We go through Dp-branes, in particular the D3-brane which we will see constitutes the gravity dual side of its correspondence with $\mathcal{N} = 4$ SYM on the gauge side. We will see how this gauge theory can be derived from string theory, and we establish how SUGRA theories arise as low energy limits of superstring theories for both type IIA and IIB.

2.4.1 String Theory

There is a lot to be said about string theory as it encompasses an enormous field of study of theoretical physics, but here we will look only briefly at relevant concepts and properties for this thesis. Many (introductory) papers and notes on string theory can be found on the internet, the following captures the essence of many of them (warmly recommend [24]).

An infinitely thin relativistic string spans a two-dimensional surface in a D-dimensional spacetime. It is parametrized by its so-called worldsheet (τ, σ) which is mapped onto the target space $X^\mu(\tau, \sigma)$. Two types of strings are distinguished from each other; the closed and open strings. A closed string satisfies the condition

$$X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau, \sigma), \tag{2.73}$$

and for an open string there exists at least one μ for which $X^\mu(\tau,0) \neq X^\mu(\tau,\pi)$ with $0 \leq \sigma \leq \pi$ is true (otherwise it would be closed). These strings generically move according to the Polyakov action:

$$S_{\text{pol}} = -\frac{T}{2}\int d^2\xi\sqrt{-g}g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu}, \tag{2.74}$$

where $T(= 1/(2\pi l_s^2))$ is the tension of the string, l_s the length, and the coordinates $\xi^0 = \tau$, $\xi^1 = \sigma$. $g_{\alpha\beta}(\tau,\sigma)$ is a symmetric 2×2 metric on the worldsheet. Also, $\partial_\alpha = \partial/\partial\xi^\alpha$. From the equations of motion one actually finds that $g_{\alpha\beta} = \lambda(\tau,\sigma)^2\gamma_{\alpha\beta}$ where

$$\gamma_{\alpha\beta} = \eta_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu \tag{2.75}$$

is called the induced metric on the worldsheet. A constraint equation that arises for the Polyakov action is that the energy-momentum tensor

$$T_{\alpha\beta} = 0. \tag{2.76}$$

The Euler-Lagrange equations of the corresponding Polyakov Lagrangian in the flat gauge $g_{\alpha\beta} = \eta_{\alpha\beta}$,

$$\mathcal{L}_{\text{pol}} = -\frac{T}{2}\partial_\alpha X^\mu\partial_\beta X^\nu, \tag{2.77}$$

gives the equations of motion in the form of the wave equation:

$$\partial_\alpha\partial^\alpha X^\mu = 0. \tag{2.78}$$

By introducing the lightcone coordinates

$$\xi^\pm = \tau \pm \sigma = \xi^0 \pm \xi^1. \tag{2.79}$$

This results in the breakdown of the solution to be the sum of a right and left moving sector:

$$X^\mu(\xi) = X_R^\mu(\xi^-) + X_L^\mu(\xi^+). \tag{2.80}$$

The solutions can be written as Fourier expansions. For the open string one gets two possible boundary conditions know as

$$\text{Neumann}: \quad X'^\mu(\tau,0) = 0, \tag{2.81a}$$

$$\text{Dirichlet}: \quad \dot{X}^\mu(\tau,0) = 0. \tag{2.81b}$$

Here a dot (prime) denotes differentiation with respect to τ (σ).

Now, when quantizing the action and strings by imposing canonical commutation relations between the operators x^μ, p^μ and the Fourier modes $\alpha_n^\mu, n \in \mathbb{Z}$ and defining α_n^μ to be annihilation operators and α_{-n}^μ the creation operators, one stumbles into ghost states. By the relations

$$[\alpha_m^\mu, \alpha_n^\nu] = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\delta_{m+n,0}\eta^{\mu\nu} \tag{2.82}$$

(tilde denoting left moving sector), one sees that the state

$$|g\rangle = \frac{1}{\sqrt{n}}\alpha_{-n}^0|0;k\rangle \tag{2.83}$$

produces a negative norm, thus they are known as ghost states. However, Fourier modes of the energy-momentum tensor, which can be found to be

$$L_n = \frac{1}{2}\sum_{k\in\mathbb{Z}} \alpha_{n-k}\cdot\alpha_k, \quad \tilde{L}_n = \frac{1}{2}\sum_{k\in\mathbb{Z}} \tilde{\alpha}_{n-k}\cdot\tilde{\alpha}_k, \tag{2.84}$$

give the quantum analog of (2.76):

$$(L_n - a\delta_{n,0})\,|\phi\rangle = (\tilde{L}_n - a\delta_{n,0})\,|\phi\rangle = 0 \text{ for } n \geq 0. \tag{2.85}$$

A theorem known as the no-ghost theorem [25] combined with the adding of conformal symmetry states that no ghost states exists provided that $a = 1$ and $D = 26$. Interestingly, one can compute the algebra of the modes L_n to be generally

$$[L_m, L_n] = (m-n)L_{m+n} + \left(\frac{c}{12}m^3 + km\right)\delta_{m+n,0}. \tag{2.86}$$

Here, c and k are constants and c is known as the central charge. The above relation is known as the centrally extended Virasoro algebra, with the second term being the central extension. With the relation being non-zero the conformal symmetry algebra is said to be anomalous. This is because we need the local symmetries of the Polyakov action to remain symmetries after quantization in order to keep the physics consistent. One can resolve this by introducing Fadeev-Popov ghost fields as done in quantum field theory. When introducting supersymmetry, the algebra (2.87) looks like

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{2}(cm^2 + qm)\delta_{m+n,0}, \tag{2.87}$$

To avoid anomalies, after introducing ghost fields, it turns out the necessary conditions are that q and the central charge must be zero and the dimension reduces to $D = 10$. So going from stringtheory to superstring theory reduces the number of dimensions by 16. Another thing supersymmetry helps with is to get rid of the so-called tachyon state. In ordinary string theory one finds a state with negative mass, implying it travels beyond the speed of light. In superstring theory, this state does not exist.

More generally than strings we have the dymnamical p-branes. A 0-brane is a particle, 1-brane is a string and so on. A p-brane is thus parametrized by a $(p+1)$-dimensional worldvolume mapped to D-dimensional target space. The endpoints of the open string can be thought of as lying on a p-dimensional hyperplane defined by $x^I = c^I$ with $I = p+1, \ldots, D-1$. A special type of brane which is defined by the open strings that live on it is the Dp-brane, where D is for Dirichlet (2.81b). These types branes can be seen as being made out of strings that live on the brane. If we look at a single D-brane in 10 dimensions and split the space directions in to $\mu \in \{0,1,2,3\}$ and $I \in \{4,\ldots 9\}$, then the dynamics are described by the Dirac-Born-Infield (DBI) action [26]:

$$S_{\text{DBI}} = -\frac{1}{(2\pi)^3 g_s l_s}\int d^4\xi \sqrt{-\det(\gamma_{ab} + 2\pi\alpha' F_{ab})}, \tag{2.88}$$

where have the γ_{ab} is (2.75) and F is the field strength 2-form given as $F = dA$ (following the standard form notation used in e.g. [27]) of the $U(1)$ gauge field. From the splitting of directions we must choose our coordinates appropriately. First we specify the embedding and choose $X^a(\xi) = \xi^a$ with $a \in \{0,1,2,3\}$. The transverse directions, which we center at the origin for convenience, can be described by six scalars that are fluctuations in the position of the brane on the worldvolume:

$$X^{i+3}(\xi) = 2\pi\alpha'\phi_i(\xi), \quad i \in \{4, ..., 9\}. \tag{2.89}$$

Applying this to the induced metric we get the Minkowski metric with some fluctuations that we interpret as the scalars in the theory:

$$\gamma_{ab} = \eta_{ab} + (2\pi\alpha')^2\partial_a\phi^i\partial_b\phi_i. \tag{2.90}$$

Using these relations, we find that the determinant can be written as

$$\det\big[\eta_{ab} + 2\pi\alpha' F_{ab} + (2\pi\alpha')^2\partial_a\phi^i\partial_b\phi_i\big]. \tag{2.91}$$

In the low energy limit where $\alpha' \to 0$ we can expand around the parameter. For convenience we can look at the determinant and write it as $\eta_{ab} + \epsilon\Lambda_{ab} = \eta_{ac}(\delta^c_b + \epsilon\Lambda^c_b)$. Using this, we can use the homomorphism property of determinants to split it up:

$$\det\big[\eta_{ac}(\delta^c_b + \epsilon\Lambda^c_b)\big] = -\det(\delta^c_b + \epsilon\Lambda^c_b). \tag{2.92}$$

This suggest that we should use the identity $\det(\Gamma) = \exp(\mathrm{Tr}[\log(\Gamma)]$ where Γ is an $n \times n$ matrix. Using that $\Gamma = \mathbb{I} + \epsilon\Lambda$, we get

$$\begin{aligned}\det(\mathbb{I} + \epsilon\Lambda) &= \exp(\mathrm{Tr}[\log(\mathbb{I} + \epsilon\Lambda)]) = \exp\left(\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\epsilon\mathrm{Tr}[\Lambda^n]\right)\\ &= 1 + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\epsilon\mathrm{Tr}[\Lambda^n] - \frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\epsilon\mathrm{Tr}[\Lambda^n]\right)\left(\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\epsilon\mathrm{Tr}[\Lambda^m]\right)\\ &= 1 + \epsilon\mathrm{Tr}[\Lambda].\end{aligned} \tag{2.93}$$

Finally, we have to deal with the square root in the action, so we approximate $\sqrt{1+x} \simeq 1 + \frac{1}{2}x + \mathcal{O}(x^2)$. With this in mind, we can write the determinant in the desired form:

$$\sqrt{-\det} = 1 + (2\pi\alpha')^2\,\mathrm{Tr}\left(\frac{1}{2}F_{ab}F^{ab} + \frac{1}{2}\partial_a\phi^i\partial_b\phi_i\right). \tag{2.94}$$

Ignoring the identity which integrates to the worldvolume, we find the action

$$S_{\mathrm{DBI}} = \frac{1}{4\pi g_s}\int d^4\xi\,\mathrm{Tr}\left(\frac{1}{4}F_{ab}F^{ab} + \partial_a\phi^i\partial_b\phi_i\right) + \mathrm{Fermions} \tag{2.95}$$

This corresponds to the action of $\mathcal{N} = 4$ SYM with gauge group $U(1)$ given that we make the identification between the Yang-Mills coupling and string coupling $g^2_{\mathrm{ym}} = 4\pi g_s$.

When considering a stack on N D-branes on top of each other, one gets that a non-abelian gauge theory with $U(N)$ gauge symmetry appears, living on the stack of D-branes. This is the open string picture of D-branes, and it is described by a weak coupling. However, D-branes as strings also have a tension, meaning they are massive objects. So naturally, with many stacked up against each other, they react by curving the spacetime around them. This is the strong-coupling picture. This dual description are solutions of the low-energy effective action of string theory. The strings in this picture are closed.

2.4.2 Type II Supergravity

There generally exists five superstring theories (see **Table 2.1**). The most common of these are the type IIA and type IIB strings. By the so-called GSO projection one finds that what distinguishes the two is an eigenvalue s of the states in the closed superstring Fock space. By definition $s = -1$ describes type IIA string theory and $s = 1$ described type IIB. For type IIA the massless spectrum is non-chiral which is not the case for type IIB. Normally type II is a theory of closed strings while type I is a theoy of both open and closed strings. However, the end points of open strings in type II can be seen to end on the so-called D-branes. Another related theory of supergravity is the type II supergravity which can be related to string theory. To begin with we briefly mention Kaluza-Klein compactification, which can be used to obtain relevant actions for SUGRA theories. Usually in supergravity theories the dimension of the action far exceeds the dimension for which we should make phenomenological models in particle physics. For this reason w could consider what could happen if we shrink the redundant dimensions to be so small that their effect becomes negligible in the observable theory. One way manifest this is through Kaluza-klein compactification [28] on certain geometries (Calabi-Yau manifolds, a torus or spheres/circles). Thus we wish to compactify one space dimension on a circle S^1_R of radius R. This means we turn one of our x^μ coordinates into a y-coordinate on a circle and let the remaining ones be called $x^{\bar{\mu}}$. Hence for example our wave operator can be written as

$$\Box_D = \Box_{D-1} + \frac{\partial^2}{\partial y^2}. \tag{2.96}$$

Now if we want to investigate how fields transform in the vanishing radius limit $R \to 0$ (dimensional reduction), we start with a scalar field $\phi(x^\mu)$ obeying periodic boundary conditions on S^1_R so that it can be expressed through its Fourier decomposition:

$$\phi(x^{\bar{\mu}}, y) = \sum_{n \in \mathrm{Z}} \phi_n(x^{\bar{\mu}}) e^{\frac{2\pi i n y}{R}}. \tag{2.97}$$

Looking at the standard kinetic term of the Klein-Gordon action in d dimensions, we can use dimensional reduction to get

$$\begin{aligned} \int d^d x \phi(-\Box_D + m^2)\phi &= \sum_{n \in \mathrm{Z}} 2\pi R \int d^d x \phi_n \left(-\Box_{D-1} + \frac{\partial^2}{\partial y^2} + m^2\right) \phi_n e^{\frac{2\pi i n y}{R}} \\ &= \sum_{n \in \mathrm{Z}} 2\pi R \int d^d x \phi_n \left(-\Box_{D-1} + \frac{4\pi^2 n^2}{R^2} + m^2\right) \phi_n. \end{aligned} \tag{2.98}$$

One defines the mass of the nth mode as $m_n = \frac{n^2}{R^2}$. Thus we see that as R goes to zero the only mode that contributes is the $n = 0$ mode since all others acquire an infinitely heavy mass and thus decouples. Due to the infinity of every increasing mass, this became known as the Kaluza-Klein tower of states. When $L \ll 1$, the non-zero modes will be immensely heavy and can be safely neglected. These heavy masses truncates the Kaluza-Klein spectrum and is known as the Kaluza-Klein reduction ansatz.

Starting off with the maybe most classical type IIB SUGRA, one needs to consider superstrings to capture all of the dynamics. When adding spinors (fermions) to the theory, one needs periodicity constraints on the left and right moving sectors from the equations of motion. This leads to what is called the Ramond (R) and Neveu-Schwarz (NS) conditions. Thus one can have four type of periodic conditions to work with namely NS-NS, R-R, NS-R, R-NS. To help motivate the content for the SUGRA actions we look at the closed string. According to the periodic conditions chosen, looking at the mass spectrum results in the decomposition of the representation. The mass spectrum transforms as a vector in $SO(D-2)$. In 10 dimensions this becomes $SO(8)$ with 56 generators. Thus, for the choice of sector, one has either the fundamental representation in the NS-sector or the chiral represenations in the R-sector. For $SO(N)$ one can decompose a tensor product into a direct sum $N \otimes N = (\frac{1}{2}N(N+1)-1) \oplus (\frac{1}{2}N(N-1) \oplus 1$. For the $(NS\oplus, NS\oplus)$ one obtains $8 \otimes 8 = 35 \oplus 28 \oplus 1$. This procedure can be played to find fields, symmetric and anti-symmetric tensors and correspondingly the particles that will be used for the SUGRA action. To summarize we give the complete direct sum for both types of string theories, which will help us associate the components in the SUGRA action:

$$\begin{aligned} &\text{Type IIA:} \quad 1 \oplus 8_V \oplus 28 \oplus 56_t \oplus 35 \oplus 8 \oplus 8' \oplus 56 \oplus 56', \\ &\text{Type IIB:} \quad 1^2 \oplus 28^2 \oplus 35 \oplus 35_+ 8 \oplus 8'^2 \oplus 56^2. \end{aligned} \tag{2.99}$$

So the low-energy action for type IIB superstrings can now be obtained in the string frame, using the direct sum

$$\begin{aligned} \mathcal{S}_{\text{IIB}} = \frac{1}{2\kappa_{10}^2}\Bigg[\int d^{10}X\sqrt{-g}\Big(e^{-2\phi}(R + 4\partial_M\phi\partial^M\phi - \frac{1}{2}|H_{(3)}|^2 - \frac{1}{2}|F_{(1)}|^2 - \\ \frac{1}{2}|\tilde{F}_{(3)}|^2 - \frac{1}{4}|\tilde{F}_{(5)}|^2) - \frac{1}{2}\int C_{(4)} \wedge H_{(3)} \wedge F_{(3)}\Bigg]. \end{aligned} \tag{2.100}$$

The prefactor is the 10-dimensional gravitational constant $2\kappa_{10}^2 = (2\pi)^7\alpha'^4$. To get the Newton constant one simply modifies with the coupling constant. The final part is to introduce the fields composed in one-forms

$$\begin{aligned} F_{(p)} = dC_{(p-1)}, \quad H_{(3)} = dB_{(2)}, \quad \tilde{F}_{(3)} = F_{(3)} - C_{(0)}H_{(3)}, \\ \tilde{F}_{(5)} = F_{(5)} - \frac{1}{2}C_{(2)} \wedge H_{(3)} + \frac{1}{2}B_{(2)} \wedge F_{(3)}, \end{aligned} \tag{2.101}$$

and further one must impose a self-duality constraint for $*\tilde{F}_{(5)} = \tilde{F}_{(5)}$. Similarly, one can

open	closed	String theory	Low energy limit
	x	IIA	$\mathcal{N}=2$ IIA SUGRA
	x	IIB	$\mathcal{N}=2$ IIB SUGRA
x	x	Type I	$\mathcal{N}=1$ $SO(32)$ YM
	x	Heterotic $SO(32)$	$\mathcal{N}=1$ $SO(32)$ YM
	x	Heterotic $E_8 \times E_8$	$\mathcal{N}=1$ $E_8 \times E_8$ YM

Table 2.1: Table of the five different string theories and their low energy limits.

construct an action for type IIA supergravity that is build from the grounds of product representations as we wrote in eq. 3.5. The SUGRA action reads

$$\begin{aligned} \mathcal{S}_{\text{IIA}} = \frac{1}{2\kappa_{10}^2}\Bigg[\int d^{10}X\sqrt{-g}\Big(e^{-2\phi}(R + 4\partial_M\phi\partial^M\phi - \frac{1}{2}|H_{(3)}|^2) \\ -\frac{1}{2}|F_{(2)}|^2 - \frac{1}{2}|\tilde{F}_{(4)}|^2 - \frac{1}{2}\int B \wedge F_{(4)} \wedge F_{(4)}\Bigg]. \end{aligned} \tag{2.102}$$

Here we define $\tilde{F}_{(4)} = dA_{(3)} - A_{(1)} \wedge F_{(3)}$. We will see below how the action can be obtained by dimensional reduction of a eleven-dimensional supergravity that is unique in the sense that it is the only (local) supersymmetric theory in eleven dimensions containing only massless particles of spin ≤ 2. In particular, it contains two bosonic fields, the metric G_{MN} and a three-form potential $A_{(3)} = A_{MNR} dx^M \wedge dx^N \wedge dx^R$. We will see below how this arises in the context of M-theory and branes, and also how the near-horizon limit emerges in a certain regime given the type IIA SUGRA.

2.4.3 M-Theory and M-Branes

Let us say we compactify the x^9 coordinate on a circle with radius R. Then the fields on this circle can be expanded in terms of their eigenvalues and each term is proportional to $\exp(im/R)$ where $m \in \mathbb{Z}$ is the m'th term. In the limit $R \to \infty$ we simply obtain the uncompactified theory. In the limit $R \to 0$ however, the momentum will be either zero of infinitely large, thus this is a decoupling limit. In the case of closed strings, the strings can wrap around the circle, and this changes the periodicity condition to

$$X^9(\tau, \sigma + 2\pi) = X^9(\tau, \sigma) + 2\pi m R. \tag{2.103}$$

m is called the winding number of the string as it counts how many times it wraps around the circle. When imposing the condition (2.103), one gets that the mass M^2 of the spectrum is

$$M^2 = \left(\frac{k}{R} - \frac{mR}{l_s^2} \right)^2 , \tag{2.104}$$

where k is the momentum. We see here that exchanging k and m while simultaneously exchanging R with l_s^2/R leaves the mass unchanged. This symmetry is called the T-duality. The transformations result in the chance of the sign of s in type II string theory, thus the T-duality maps the two different type II string theories to each other. We say that type IIA and type IIB are T-dual to each other. They describe the same physics but with inverse radii and exchanged values of k and m. The T stands for target space, as target space clearly is not a fundamental property in string theory since the radius of the metric does not seem to affect the physics it describes. This duality is part of the concept that the five different superstring theories should be able to be unified if they aim to describe the same physics. The big unification that relates them all to each other was first proposed by Horava and Witten in [29] as M-theory. They show that heterotic $E_8 \times E_8$ string theory is related to an eleven-dimensional supergravity theory by the so-called S-duality. This is a strong-weak duality that maps a strongly coupled regime in one theory to a weakly coupled regime in another theory. This makes it useful for obtaining non-perturbative information in one theory by the means of perturbative theory in the other theory. There also exists an S-duality between type I and heterotic $SO(32)$ string theory. And type IIB turns out to be self-dual. Further, type IIA is S-dual to eleven-dimensional SUGRA, and the two heterotic theories are T-dual to each other. Thus we see all five string theories are related to each other with dualities with the connecting branch SUGRA in $D = 11$. This eleven-dimensional theory is said to be the low energy effective description of M-theory. It is also the only possible theory for supergravity in eleven dimensions, suggesting M-theory is unique. It is not known what M-theory is, though suggestions have been proposed [30].

M-theory can be described as originally being an 11D supergravity theory that was found to spit out a relation between the radius and the string coupling constant after a Kaluza-Klein reduction [31];

$$R_s/l_p = g_s^{2/3}, \quad l_p^3 = g_s l_s^3. \tag{2.105}$$

Here, M-theory is seen as the strong coupling limit of type IIA superstring theory at $E \ll 1/l_p$, which is also what happens with type IIA superstring and supergravity. which can be obtained from a dimensional reduction on a circle from 11D SUGRA. Hence M-theory reduces to 11D SUGRA while type IIA superstring theory reduces to 10D type IIA SUGRA at low energy. By compactification M-theory reduces to type IIA superstring theory while 11D SUGRA reduces to 10D type IIA SUGRA. The action of the bosonic sector of 11D supergravity is given by

$$S_{11} = \frac{1}{l_p^9} \int \mathrm{d}^{11}x \left\{ \sqrt{-g} \left(R - \frac{l_p^6}{48} F_4^2 \right) + \frac{1}{6} F_4 \wedge F_4 \wedge A_3 \right\}. \tag{2.106}$$

Here, the 3-form antisymmetric gauge potential A_3 comes with the gauge transformation $\delta A_3 = d\Lambda_2$, and the field strength is given by $F_4 = dA_3$. If we consider for the metric the ansatz

$$ds_{11}^2 = R_s^2 (dx^s + \mathcal{A}_\mu dx^\mu)^2 + ds_{10}^2, \tag{2.107}$$

where R_s is the fluctuating radius of compatification measured in the 11D metric, and $\mathcal{A}$ is the Kaluze-Klein $U(1)$ gauge field coming from the isometry and x^s (s denoting string theory), then we can perform a dimensional reduction of the action (2.106). Dimensionally, the scalar curvature of this is

$$R(g_{MN}) = R(g_{\mu\nu}) + \left(\frac{\partial R_s}{R_s} \right)^2 + R_s^2 (d\mathcal{A})^2. \tag{2.108}$$

From here, the action now reads

$$\begin{aligned} S_{10} = \frac{1}{l_p^9} \int \mathrm{d}^{10}x R_s \sqrt{-g} \bigg\{ R + \left(\frac{\partial R_s}{R_s} \right)^2 + R_s^2 (d\mathcal{A})^2 + l_p^6 F_4^2 + \frac{l_p^6}{R_s^2} (dB)^2 \bigg\} \\ + \int B \wedge F_4 \wedge F_4. \end{aligned} \tag{2.109}$$

Compare this now to the action of the low energy limit of type IIA string theory which can be written as

$$\begin{aligned} S_{\mathrm{IIA}} = \frac{1}{l_s^8} \int \mathrm{d}^{10}x \sqrt{-g} \bigg\{ e^{-2\phi} \left(R + 4(\partial\phi)^2 - \frac{l_s^4}{12} (dB)^2 \right) - \frac{l_s^2}{4} (d\mathcal{A})^2 - \frac{l_s^6}{48} F_4^2 \bigg\} \\ + \int B \wedge F_4 \wedge F_4. \end{aligned} \tag{2.110}$$

Identifying the dilaton field ϕ with $\ln(R_S)$ up to numerical factors and matching the two actions (2.109) and (2.110) one gets the following relations:

$$\frac{R_s}{l_p^9} = \frac{1}{g_s^2 l_s^8}, \quad \frac{1}{R_s l_p^3} = \frac{1}{g_s^2 l_s^4}, \quad \frac{R_s^3}{l_p^9} = \frac{1}{l_s^6}, \tag{2.111}$$

with $g_s^2 = e^{2\phi}$.

Looking again at action (2.106), the equation of motion for A_3 is

$$d^* F_4 + \frac{1}{2} F_4 \wedge F_4 = 0, \tag{2.112}$$

and this EOM leads to the conserved charge

$$U = \int_{\partial \mathcal{M}_s} ({}^* F_4 + \frac{1}{2} A_3 \wedge F_4), \tag{2.113}$$

which is of "electric" type. The integral over the second term which is 7-form is over the boundary at infinity of an infinite spacelike 8-dimensional subspace of the 11-dimensional spacetime. The Bianchi identity $dF_4 = 0$ leads to another conserved current,

$$V = \int_{\partial \tilde{\mathcal{M}}_s} F_4, \tag{2.114}$$

where the integral is the same but over a 5-dimensional subspace of the 11-dimensional spacetime. This charge is of "magnetic" type. The charges U and V are 2- and 5-form charges which can be seen by the fact that the supersymmetry algebra $\{Q, Q\}$ can be written in terms of the one-form momentum vector as $\Gamma^A P_A$, and the two charges as $\Gamma^{AB} U_{AB}$ and $\Gamma^{ABCDE} V_{ABCDE}$. Topologically speaking, one might validly say that the 2 and 5 indices come from the ways that the 8 and 5 dimensional integration volumes might be embedded into a 10-dimensional surface. We now turn to solutions in supergravity that namely carry the charges (2.113) and (2.114). These are of course the p-branes. To make the system easier to study, we consider the action

$$S = \int D^D x \sqrt{-g} \left[R - \frac{1}{2} \nabla_M \phi \nabla^M \phi - \frac{1}{2n!} e^{a\phi} F_n^2 \right], \tag{2.115}$$

where the field strength is $F_n = dA_{n-1}$. This action we call a consistent truncation from a full D-dimensional supergravity theory. The solutions of this consistent truncation are (definitionally) also solutions of the original theory. Notice here the action (2.115) is described by (g_{MN}, ϕ, A_{n-1}). We ignore here the inconsistent solutions for which $n = D/2$. Varying (2.115) one gets the following equations of motion:

$$S_{MN} = \frac{1}{2(n-1)!} e^{a\phi} (F_{M \dots} F_{\ddot{N}}^{\dots} - \frac{n-1}{n(D-2)} F^2 g_{MN}), \tag{2.116}$$

$$R_{MN} = \frac{1}{2} \partial_M \phi \partial_N \phi + S_{MN}, \tag{2.117}$$

$$\nabla_{M_1} (e^{a\phi} F^{M_1 \cdots M_n}) = 0, \tag{2.118}$$

$$\Box \phi = \frac{a}{2n!} e^{a\phi} F^2. \tag{2.119}$$

To solve these we make an ansatz requiring (Poincaré)$_d \times$ $SO(D-d)$ symmetry. We can then view the solutions we are looking for as flat $d = p + 1$ dimensional hyperplanes embedded in the D-dimensional spacetime, and these hyperplanes can then be viewed as the worldvolumes of the p-dimensional surfaces. We split the spacetime coordinates into two parts: $x^M = (x^\mu, y^m)$, with $\mu = 0, 1, \dots, d-1$ and $m = d, \dots, D-1$. Here,

x^μ correspond to the Poincaré isometries on the worldvolume and y^m correspond to the coordinates transverse to the worldvolume. A viable ansatz is then

$$ds^2 = e^{2A(r)} dx^\mu dx^\nu \eta_{\mu\nu} + e^{2B(r)} dy^m dy^n \delta_{mn}, \tag{2.120}$$

where $\mu = 0, 1, \ldots p$ and $m = p+1, \ldots D-1$, and $r = \sqrt{y^m y^m}$ is the isotropic radial coordinate in the transverse space. The components of this metric depend only on r we are guaranteed to have translational invariance in the x^μ directions and $\mathrm{SO}(D-d)$ symmetry in the y^m directions. From here one computes the Ricci tensor by introducing vielbeins with tangent-space indices and constructing the corresponding 1-forms. After computations one gets that the source is given by

$$\rho = \begin{cases} C' e^{\frac{1}{2}a\phi - dA + C} & \text{electric: } d = n-1,\ \zeta = +1 \\ \lambda r^{-\tilde{d}-1} e^{\frac{1}{2}a\phi - \tilde{d}B} & \text{magnetic: } d = D-n-1,\ \zeta = -1 \end{cases}. \tag{2.121}$$

One finds further that

$$e^{\frac{\zeta\Delta}{2a}\phi} \equiv H(y) = 1 + \frac{k}{r^{\tilde{d}}}, \quad k > 0. \tag{2.122}$$

Here and before, $\tilde{d} = D - d - 2$. Returning now to the original action (2.106), which is absent of any scalar fields. To make this absence consistent with our truncated solutions, we simply identify the scalar coupling parameter a with zero. This implies $\Delta = 4$ in $D = 11$. Dropping the last term in (2.106), we identify $n = 4$, which leads to the electric solutions with $d = 3$, that is, a 2 membrane. Using also $D = 11$, the magnetic solutions becomes of dimension $d = 11 - 4 - 1 = 6$, that is, a 5-membrane. In both of these cases the last term of the action vanishes and thus this term does not destroy our previous study and ansatz. All in all, the two M-branes we get are the $M2$- and $M5$-branes:

$$M2\text{-brane: } ds^2 = \left(1 + \frac{k}{r^6}\right)^{-2/3} dx^\mu dx^\nu \eta_{\mu\nu} + \left(1 + \frac{k}{r^6}\right)^{1/3} dy^m dy^m, \tag{2.123}$$

$$A_{\mu\nu\lambda} = \epsilon_{\mu\nu\lambda} \left(1 + \frac{k}{r^6}\right)^{-1}, \tag{2.124}$$

where $k = \kappa^2 T/(3\Omega_7)$, with T being the tension of the action, and Ω_7 being the volume of the unit 7-sphere $\mathcal{S}^7$. Further:

$$M5\text{-brane: } ds^2 = \left(1 + \frac{k}{r^3}\right)^{-1/3} dx^\mu dx^\nu \eta_{\mu\nu} + \left(1 + \frac{k}{r^3}\right)^{2/3} dy^m dy^m, \tag{2.125}$$

$$F_{m_1 \ldots m_4} = 3k \epsilon_{m_1 \ldots m_4 p} \frac{y^p}{r^5}. \tag{2.126}$$

In general we have

$$H_p(r) = 1 + \left(\frac{L_p}{r}\right)^{7-p}, \tag{2.127}$$

where

$$L_p^{7-p} = (4\pi)^{(5-p)/2} \Gamma\left(\frac{7-p}{2}\right) g_s N \alpha'^{(7-p)/2}. \tag{2.128}$$

Using this and writing a stack of N coincident $M2$-branes in flat spacetime we can write the $M2$-brane metric as

$$ds^2 = \left(1 + \frac{L^6}{r^6}\right)^{-2/3} dx^\mu dx^\nu \eta_{\mu\nu} + \left(1 + \frac{L^6}{r^6}\right)^{1/3} (dr^2 + r^2 d\Omega_7^2). \tag{2.129}$$

The gauge field is $A_3 = (1 + L^6/r^6)^{-1} dx^0 \wedge dx^1 \wedge dx^2$. In the near-horizon limit where $r \ll L$, we have that the $H(r)$ factors reduce to

$$\left(1 + \frac{L^6}{r^6}\right)^{-2/3} \simeq \frac{r^4}{L^4}, \quad \text{and} \quad \left(1 + \frac{L^6}{r^6}\right)^{1/3} \simeq \frac{L^2}{r^2},$$

and (2.129) reduces to

$$ds^2 = \frac{r^4}{L^4} dx^\mu dx^\nu \eta_{\mu\nu} + \frac{L^2}{r^2}(dr^2 + r^2 d\Omega_7^2). \tag{2.130}$$

Recalling that the Anti-de Sitter metric in Poincaré coordinates can be written

$$ds^2_{AdS_4} = \frac{L^2}{r^2} dr^2 + \frac{r^2}{L^2} dx^\mu dx^\nu \eta_{\mu\nu}$$

and using the transformation $z = L^3/2r^2 \implies dz^2 = L^6/r^6 dr^2$, we can write the M2-brane in the near-horizon limit as

$$\begin{aligned} ds^2 &= \frac{L^2}{4z^2} dx^\mu dx^\nu \eta_{\mu\nu} + \frac{L^2}{4z^2} dz^2 + L^2 d\Omega_7^2 \\ &= L^2 \left(\frac{1}{4} ds^2_{Ads_4} + ds^2_{S^7} \right). \end{aligned} \tag{2.131}$$

Perhaps interesting to note that one can also derive the relation $L = 32\pi^2 N l_p^6$ [32] by requiring g_{00} to be related to the Newtonian potential in the asymptotic limit. In this limit the M2-brane can be thought of as a source in eight dimensions, and the Schwarzschild solution can be generalized to

$$\lim_{r\to\infty} g_{00} \simeq -1 + \frac{2L^6}{3r^6} = -1 + \frac{16\pi G_{11} N T_{M2}}{9\Omega_7 r^6},$$

where T_{M2} is the tension of the $M2$-brane and G_{11} is Newton's constant in eleven dimensions.

2.5 AdS/CFT Correspondence and the Holographic Principle

With the knowledge of conformal field theories and gravity we are now ready to make the link between them. The principle of this link originates from the holographic principle [33, 34]. It goes something like the following: the degrees of freedom of any given quantum theory can be related to its entropy through the third law of thermodynamics. The dimension of the Hilbert space is the exponent of the entropy $e^{\mathcal{S}} = \mathcal{N}$. By the so-called Bekenstein bound [35], the entropy must be smaller than that of a black hole: $\mathcal{S} < \mathcal{S}_{\text{BH}} = A/(4G)$, where A is the surface area of the black hole. This implies that $\mathcal{N} \leq e^{A/(4G)}$. Suppose now that the universe is a lattice with Planck length scaling and with each site having either spin up or down. The total number of states in a volume V is $\mathcal{N}(V) = 2^n$ with $n = V/l_p^d$. The limit on the entropy is $\mathcal{S} \leq \ln(\mathcal{N}(V)) = V \ln(2)/l_p^d$. The entropy grows with volume. Above the Planck threshold the volume is larger than the area, resulting in a larger entropy bound. By the first relation, we have that $\mathcal{N} \sim e^V$. However, if the region collapsed to a black hole, the entropy would have decreased to $e^{A/(4G)}$, meaning the number of states would have decreased as well. This violates unitarity, so we must assume that the Hilbert space had dimension $e^{A/(4G)}$ from the beginning. This leads us to conclude that a quantum gravity theory on some manifold will be determined by another theory living on the boundary of that manifold, and we have a duality. With duality we mean a relation between two different theories that predict the same values for the same physical observables. Here, we will refer to the fields living in AdS_{d+1} space as the bulk fields, and we assume that they interact with each other according to some effecive action $S_{\mathrm{AdS}_{d+1}}(g_{\mu\nu}, A_\mu, \phi, \psi, \dots)$, where the vacuum is AdS_{d+1}. The fields living on the corresponding CFT will be called the boundary fields, and these will be described with the d-dimensional action S_{CFT}. Now we associate a field h in AdS with an operator $\mathcal{O}$ in the CFT with same quantum numbers, and they are related to each other through behavior near the boundary. We can write this from the CFT perspective as

$$S_{\mathrm{CFT}} + \int d^d x h(x)\mathcal{O}(x). \tag{2.132}$$

This leads us to the correlation function

$$\left\langle e^{\int h\mathcal{O}} \right\rangle_{\mathrm{QFT}} = e^{W(h)}, \tag{2.133}$$

with

$$\langle \mathcal{O} \dots \mathcal{O} \rangle = \frac{\delta^n W}{\delta h^n}\Big|_{h=0}. \tag{2.134}$$

But if we now change the perspective to the Ads side, h is the boundary value of the higher dimensional bulk field $\hat{h}(x, x_{d+1})$, which is a solution to the effective AdS_{d+1} action. With boundary condition this bulk field is unique for every boundary field $h(x)$. Thus we have

$$\left\langle e^{\int h\mathcal{O}} \right\rangle_{\mathrm{QFT}} = e^{W(h)} = e^{-S_{\mathrm{AdS}_d+1}(\hat{h})}. \tag{2.135}$$

This states an equivalence between a conformal field theory and a gravitational theory. Note that we assume the gravity theory is weakly coupled and assuming a UV completion (2.135) can be interpreted at quantum level. The boundary theory is off-shell, while the gravity side is on-shell as we stated.

The most common example first found by Maldecena is the correspondence between $\mathcal{N} = 4$ SYM with coupling constant g_{YM} and type IIB string theory on $\mathrm{AdS}_5 \times S^5$ with coupling g_s and string length $l_s = \sqrt{\alpha'}$. The correspondence states that dynamics of the two theories are equivalent. The five-sphere on the gravity side comes from Kaluza-Klein reduction of the 10-dimensional type IIB action. The duality also relates

$$g^2_{\mathrm{YM}} = 2\pi g_s, \quad 2g^2_{\mathrm{YM}} N = 2\lambda = \frac{L^4}{l_s^4}. \tag{2.136}$$

Here $\lambda = g^2_{\mathrm{YM}} N$ is the t'Hooft coupling, L is the radius, and N is the number of units of F_5 flux on S^5.

The radial coordinate in AdS plays the role of an energy scale, $r \sim \mu$. Further it leads to the mass-conformal weight relation for scalars

$$ml^2 = \Delta(\Delta - d). \tag{2.137}$$

Different relations exist for other types of fields. When $g_{\mathrm{YM}} \ll 1$ and $\lambda \to \infty$, we get $g_s \ll 1$ and $L/l_s \to \infty$, which means the strirngs become pointlike and the only states to practically exist are the massless ones. This resembles type II SUGRA. This is only one example of an AdS/CFT correspondence, and later we will turn to the one example that is the focus of this project.

3

ABJM Theory and $\text{AdS}_4 \times \mathbb{CP}^3$

In this section we study the correspondence that lies the foundation for this thesis. ABJM theory, named after Aharony, Bergman, Jafferis, and Maldecena [36], is the dual between M-theory on $\text{AdS}_4 \times S^7$ and $\mathcal{N} = 6$ super Chern-Simons theory. However, for some calculations in this paper the seven-sphere can be reduced to $\mathbb{CP}^3$. We first look at what this means.

3.1 Complex Projective Space

$\mathbb{CP}^n$ is loosely defined as the set of all lines in $\mathbb{C}^{n+1}$. A point in the projective plane has coordinates $(u^1, u^2, \ldots, u^n)$ with equation $u^1 X_1 + u^2 X_2 + \cdots u^n X_n = 0$. The points $(u^1, u^2, \ldots, u^n)$ and $(u^1, u^2, \ldots, u^n)k$ where k is a real number represent the same point. An inclusion $\mathbb{C}^{m+1} \subset \mathbb{C}^{n+1}$ induces an inclusion $\mathbb{CP}^m \subset \mathbb{CP}^n$, the image of which is a linear subspace. One can consider $\mathbb{CP}^n$ as a compactification of $\mathbb{C}^n$. The hyperplane H at infinity is added to $\mathbb{C}^n$, so we have [37]

$$\mathbb{CP}^n = \mathbb{C} \cup \mathbb{C}^{n-1} \cup \cdots \cup \mathbb{C}^0. \tag{3.1}$$

We have that $\mathbb{CP}^1$ is diffeomorphic to S^2, as S^2 can be described via stereographic projection from the north and south pole $(0,0,1)$ and $(0,0,-1)$ through [38]

$$\phi_1(x^1, x^2, x^3) = \left(\frac{x^1}{1 - x^3}, \frac{x^2}{1 - x^3} \right), \tag{3.2}$$

$$\phi_2(x^1, x^2, x^3) = \left(\frac{x^1}{1 + x^3}, \frac{x^2}{1 + x^3} \right), \tag{3.3}$$

with transition map $z \to 1/z$. But this is the same as $[1, z] \to [1/z, 1]$ of $\mathbb{CP}^1$.
We can write the projective plane as $\mathbb{CP}^n \cong S^{2n+1}/\mathbb{S}^1$; they are isomorphic. Each line in $\mathbb{C}^{n+1}$ intersects S^{2n+1} in a circle (S^1), and this line defines the point of $\mathbb{CP}^n$. $\mathbb{CP}^n$ is also a homogeneous $U(n+1)$-space (the Lie group).
We can define a **Hopf map** by the projection

$$\pi : S^{2n+1} \to \mathbb{CP}^n. \tag{3.4}$$

By realising $\mathbb{CP}^1$ is isomorphic to S^2, we obtain the known $\pi : S^3 \to S^2$ projection with fiber S^1.
Hopf fibration can be dfined as $H : \mathbb{C}^2 \setminus \{0\} \to S^2$,

$$H : (u, v) \to \left(\frac{|v|^2 - |u|^2}{|u|^2 + |v|^2}, \frac{2u\overline{v}}{|u|^2 + |v|^2} \right). \tag{3.5}$$

3.1.1 Metric Interpretation of $\mathbb{CP}^{m+n+1}$

One can write the metric of the $(p+q+1)$ unit sphere in terms of the foliation of $S^p \times S^q$ for integers p and q as

$$d\Omega_{p+q+1}^2 = d\xi^2 + \cos(\xi)^s d\Omega_p^2 + \sin(\xi)^2 d\Omega_q^2. \tag{3.6}$$

Here, the angle is bounded $0 \leq \xi \leq \pi/2$. If we consider the specific case where both p and q are odd such that $p = 2m+1$ and $q = 2n+1$ for some integers m and n, then we can write the metrics $d\Omega_p^2$ and $d\Omega_q^2$ of the S^p and S^q unit spheres and the Fubini study metrics for $\mathbb{CP}^m$ and $\mathbb{CP}^n$, $d\Sigma_m^2$ and $d\Sigma_n^2$, as [39]

$$d\Omega_p^2 = (d\tau_1 + A)^2 + d\Sigma_m^2, \quad d\Omega_q^2 = (d\tau_2 + \tilde{A})^2 + d\Sigma_n^2. \tag{3.7}$$

The A terms are usually called a connection, and they are related to something called the Kähler form defined as $dA = 2J$ and $d\tilde{A} = 2\tilde{J}$ which are related to $\mathbb{CP}^m$ and $\mathbb{CP}^n$ respectively. Thus if one starts with the unit sphere in an odd dimension, one can generally write this as a $U(1)$ fiber connection or Hopf fibration with the addition of a Fubini study metric. In [39] a closed form for any choice of index of m and n of the Fubini study metric is given on $\mathbb{CP}^{m+n+1}$

$$d\Sigma_{m+n+1}^2 = d\xi^2 + \cos(\xi)^2 d\Sigma_m^2 + \sin(\xi)^2 d\Sigma_n^2 + \sin(\xi)^2 \cos(\xi)^2 (d\psi + A - \tilde{A}). \tag{3.8}$$

This will come in handy since working in ABJM theory gives rise to the $AdS_4 \times S^7/\mathbb{Z}_k$ geometry. Here we have an orbifolding singularity [40] which will affect the coordinates. But the point is that when we have S^7, we can employ the foliation of the spheres and get a geometry of the form $AdS_4 \times \mathbb{C}P^3$. Moreover, the use of the $\mathbb{CP}^2$ turns out to be useful when considering Spin Matrix Theory (see section hej) in the geometries of non-relativistic strings for SYM. Considering the cases when $m = 0, n = 1$ and $m = n = 1$ we get

$$d\Sigma_2^2 = d\xi^2 + \sin(\xi)^2 (d\theta + \sin^2\theta d\phi)^2 + \frac{1}{4}\sin(\xi)^2 \cos(\xi)^2 (d\psi + \cos\theta)^2, \tag{3.9}$$

$$\begin{aligned} d\Sigma_3^2 = d\xi^2 &+ \cos(\xi)^2 (d\theta_1 + \sin^2\theta_1 d\phi_1)^2 + \sin(\xi)^2 (d\theta_2 + \sin^2\theta_2 d\phi_2)^2 \\ &+ \frac{1}{4}\sin(\xi)^2 \cos(\xi)^2 (d\psi - \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2. \end{aligned} \tag{3.10}$$

This way of parameterizing the geometry of the gravity side of the correspondence will prove to be quite helpful later on, but we must consider the fact that there exist other ways to express the metric in cases of different BPS-bounds.

3.2 The Correspondence

The AdS_4/CFT_3 correspondence conjectures the following [9]: $\mathcal{N} = 6$ superconformal Chern-Simons matter (CSM) theory in 2+1 dimensions with gauge group $U(N) \times U(N)$ and Chern-Simons levels $(k, -k)$, referred to as ABJM theory is dynamically equivalent to M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$ with N units of R-R four-form flux $F_{(4)}$ through AdS_4. The 't Hooft coupling is given by $\lambda = \frac{N}{k}$ and is related to the AdS_4 radius L and the eleven-dimensional Planck length l_p by

$$\frac{L^3}{l_p^3} = 4\pi\sqrt{2kN} = 4\pi k\sqrt{2\lambda}, \quad g_s \sim \left(\frac{N}{k^5}\right)^{1/4} = \frac{\lambda^{5/4}}{N}, \quad \frac{R^2}{\alpha'} = 4\pi\sqrt{2\lambda}. \tag{3.11}$$

The two dualities, ABJM and SYM, contain two free parameters each. In the SYM case one finds on the AdS part the string coupling g_s amd the dimensionless parameter L^2/α'. In ABJM this conversely corresponds to the Chern-Simons Levels $(k, -k)$ since we are now dealing with a so-called a quiver gauge theory [41] living in two different representations. We are led to the conclusion that the two sides of the duality describe the same physics. This is a bit surprising since we have a gravitational theory on one side, but no gravitational degrees of freedom on the other. Further, the holographic principle is in some way satisfied. In SYM, the information of the five-dimensional theory obtained from KaluzaKlein reduction of type IIB string theory on S^5 is mapped to a four-dimensional theory that lives on the conformal boundary of the five-dimensional spacetime. In ABJM we take a low energy limit of M-theory and obtain eleven-dimensional type IIA SUGRA, which via KaluzaKlein reduction reduces to a ten-dimensional type IIA SUGRA. This in despite all its glory unfortunately not very practical.

To apply the framework, we must consider weaker regimes instead of assuming arbitrary values of the 't Hooft coupling and the rank of the gauge group. On the string side for SYM we usually keep to tree-level computations and restrict higher genus expansions. This is the weak coupling regime $g_s \ll 1$ while keeping L^2/α' fixed. We call it the strong form of the duality. On the CFT side this corresponds to $g_{\text{YM}} \ll 1$ while keeping $g_{\text{YM}}N$ fixed. This indicates that $N \to \infty$ for a fixed λ which is known as the 't Hooft limit and corresponds to the planar limit of the gauge theory. 't Hooft might have been right when he said that the planar limit of a quantum field theory is a string theory [42]. Conversely for ABJM, in order to approximate M-theory by weakly coupled type IIA string theory on $AdS_4 \times \mathbb{CP}^3$, we must take the limit where $k^5 \gg N$. This is because the $S^7/\mathbb{Z}_k$ manifold is equivalent to an S^1 Hopf fibration over $\mathbb{CP}^3$ with the periodicity of S^1 going from $2\pi L$ to $2\pi L/k$. As the supergravity regime holds validity in the large N limit, we have that the radius of the S^1 circle in M-theory is L/k, but this is small for large k and therefore the theory reduces to type IIA supergravity on $\text{AdS}_4 \times \mathbb{CP}^3$.

3.2.1 $\mathcal{N} = 6$ CSM vs $\mathcal{N} = 4$ SYM

We now consider the actions for the ABJM and SYM and compare their structures.

The AdS_5/CFT_4 Action: We already stated the action of $\mathcal{N} = 4$ SYM in the previous, but there are different ways to derive the action other than the route of going through the DBI-action and splitting up your directions into transverse and perpendicular coordinates. This produces keeps both the gauge field and the scalars in the bosonic sector. Historically it seems that there have been two ways of obtaining this action, however. Either one starts from $\mathcal{N} = 1$ superspace for $\mathcal{N} = 4$ SYM and express the action in terms of chiral superfields Φ, as well as a gauge superfield $\mathcal{V}$ with associated field strength $\mathcal{W}$

$$S_{\mathcal{N}=4} = \int d^4x \Big[\int d^4\theta \Phi^{i\dagger} e^{\mathcal{V}} \Phi^i e^{-\mathcal{V}} + \frac{1}{8\pi} \text{Im}\Big(\tau \int d^2\theta \mathcal{W}_\alpha \mathcal{W}^\alpha\Big) + \Big(i g_{\text{YM}} \frac{\sqrt{2}}{3!} \int d^2\theta \epsilon_{ijk} \Phi^i [\Phi^j, \Phi^k] + \text{h.c}\Big)\Big]. \tag{3.12}$$

Or one considers dimensional reduction of the $\mathcal{N} = 1$ SYM in ten dimensions:

$$S_{10D} = \int d^{10}x \text{Tr}\Big(-\frac{1}{2} F_{mn}F^{mn} + i\bar{\Psi}\Gamma^m D_m \Psi \Big), \tag{3.13}$$

where the Γ_m are the Dirac matrices in ten dimensions. The field strength tensor has got additional structure in terms of a coupling constant glued to the commutators, resulting in a manifest non-abelian gauge theory. This can be compared to the derivation of the bosonic part from DBI. We define it as $F_{mn} = \partial_m A_n - \partial_n A_m + ig[A_m, A_n]$. Ψ represents a Majorana-Weyl fermion which is a spinor and has 16 real independent components. Both Ψ and F_{mn} transform in the adjoint representation of the gauge group and thus the covariant derivative D_m on Ψ reads $D_m\Psi = \partial_m \Psi_m + ig[A_m, \Psi]$. To obtain the final action, one must in Kaluza-Klein style do dimensional reduction on the six-dimensional torus T^6. The idea is the same as for the DBI-action. We split the space into two ranges for $\mu \in \{0, 1, 2, 3\}$ and $\phi_{i+3} \in \{1, ..., 6\}$, which decompose the gauge field as $A_m = (A_\mu(x^\nu), \phi^i(x^\nu)$. Following this line of thought for the fermions and calculating the contributions will lead us to the same destination as for the superspace route. With the prescription down, we get

$$\begin{aligned} S_{\text{SYM}} = \int d^4\xi \text{Tr}\Big\{ &- \frac{1}{g^2_{\text{YM}}} F_{\mu\nu}F^{\mu\nu} + \frac{\theta_I}{8\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} - \sum_a i\bar{\lambda}^a \bar{\sigma}^m u D_\mu \lambda - \sum_i D_\mu X^i D^\mu X^i \\ &+ \sum_{a,b,i} g C^{ab}_i \lambda_a [X^i, \lambda_b] + \sum_{a,b,i} g \bar{C}_{i,ab} \bar{\lambda}^a [X^i, \bar{\lambda}^b] + \frac{g}{2} \sum_{i,j} [X^i, X^j]^2 \Big\}. \end{aligned} \tag{3.14}$$

The constants C^{ab}_i and $\bar{C}_{iab}$ are related to the Clifford Dirac matrices for $SO(6)_R = SU(4)_R$. g is the gauge coupling and θ_I is the instanton angle. Further, λ arises as the gaugino fields from the $\mathcal{N} = 1$ superspace expansion one gets from supersymmetry considerations.

The AdS_4/CFT_3 **Action:** Things turn out to be more complicated in this instance. First of all, the amount of preserved supersymmetry is not maximal in ABJM. Whereas one has 32 generators in SYM, we only have 24 generators in ABJM where the global $SO(8)$ R-symmetry only has a $U(1)_R \times SU(4)$ subgroup manifesting. Secondly, as mentioned, we have a quiver gauge theory, so we get contributions from fields that live in the bifundamental and anti-bifundamental representations. Lastly, Chern-Simons theories are topological quantum field theories ordinarily [43], but when introducing a coupling to matter, topological contributions are dismissed. Nevertheless, superconformal symmetry is still preserved. We first write up the components for the full action $S = S_{CS} + S_{mat} + S_{pot}$. The superspace actions are given by

$$S_{CS} = -iK \int d^3x d^4\theta \int_0^1 dt \text{Tr}\big[\mathcal{V}\bar{D}^\alpha(e^{t\mathcal{V}} D_\alpha e^{-t\mathcal{V}}) - \hat{\mathcal{V}}\bar{D}^\alpha(e^{t\hat{\mathcal{V}}} D_\alpha e^{-t\hat{\mathcal{V}}})\big], \tag{3.15}$$

$$S_{mat} = - \int d^3x d^4\theta \text{Tr}\bar{\mathcal{Z}}_A e^{-\mathcal{V}} \mathcal{Z}^A e^{\hat{\mathcal{V}}}, \tag{3.16}$$

$$S_{pot} = L \int d^3x d^2\theta \text{W}(\mathcal{Z}) + L \int d^3x d^2\bar{\theta} \bar{\text{W}}(\bar{\mathcal{Z}}). \tag{3.17}$$

These will be the building blocks for the full action. It is quite long and hairy to get to the final expression, but in its full we have

$$\begin{aligned} S = \frac{k}{4\pi} \int d^3x \Big[&\epsilon^{\mu\nu\lambda} \text{Tr}(A_\mu \partial_\nu A_\lambda + \frac{2i}{3} A_\mu A_\nu A_\lambda - \hat{A}_\mu \partial_\nu \hat{A}_\lambda - \frac{2i}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda) \\ &- \text{Tr}(D_\mu Y^\dagger) D^\mu Y - i\text{Tr}(\psi^\dagger \not{D} \psi) - V_{\text{ferm}} - V_{\text{bos}} \Big]. \end{aligned} \tag{3.18}$$

Here we have obtained fermionic and bosonic potentials that are expressions containing combinations of scalar fields coupled to each other. Writing out the sextic bosonic and quartic mixed potentials one obtains

$$\begin{aligned} V_{\text{bos}} = -\frac{1}{12}\text{Tr}(&Y^A Y_A^\dagger Y^B Y_B^\dagger Y^C Y_C^\dagger + Y_A^\dagger Y^A Y_B^\dagger Y^B Y_C^\dagger Y^C \\ &+ 4Y^A Y_B^\dagger Y^C Y_A^\dagger Y^B Y_C^\dagger - 6Y^A Y_B^\dagger Y^B Y_A^\dagger Y^C Y_C^\dagger), \end{aligned} \tag{3.19}$$

$$\begin{aligned} V_{\text{ferm}} = -\frac{i}{2}\text{Tr}(&Y_A^\dagger Y^A \psi^{\dagger B}\psi_B + Y^A Y_A^\dagger \psi_B \psi^{\dagger B} + 2Y^A Y_B^\dagger \psi_A \psi^{\dagger B} - 2Y_A^\dagger Y^B \psi^{\dagger A}\psi_B \\ &- \epsilon^{ABCD} Y_A^\dagger \psi_B Y_C^\dagger \psi_D + \epsilon^{ABCD} Y^A \psi^{\dagger B} Y^C \psi^{\dagger D}). \end{aligned} \tag{3.20}$$

First note that the gauge fields come in a copy with one of them being hatted. This comes from the two gauge vector superfields $\mathcal{V}$ and $\hat{\mathcal{V}}$ which is written in the so-called Wess-Zumino gauge (see [44]). The Y and their conjugate fields are composed of scalars (Z^A, W^A) that are complex combinations of the bifundamental fields that appear from $SU(4)$ R-symmetry. Another interesting thing to note is that while in the SYM case we have through the Chern-Simons action gained an almost fully topological quantity that has to be quantized and only take integers values. This is referred to as the level.

3.2.2 Field Content and Global Symmetries

To get a better hang of the differences, we look how the two dualities are split by the amount of supersymmetry that is preserved by the algebra and overall global symmetry, and what corresponding matter turns out to be present in each case.

Operators, charges and global symmetries of SYM

The three main components of the superconformal algebra in SYM are the supersymmetry, conformal symmetry ,and R-symmetry. Together they are part of the bigger Lie supergroup $PSU(2,2|4)$. By considering the bosonic subalgebra of the supergroup $SU(2,2) \times SU(4) \simeq SO(4,2) \times SO(6)$, one can through the similarity explicitly see the appearance of both the R-symmetry, manifesting as the $SO(6)$, and the conformal group manifesting as $SO(4,2)$. Ten of the generators belong to Poincaré group of $SO(3,1)$, whereof four generators are spacetime translations, and the last 6 are Lorentz transformations. The remaining generators are devoted to dilatations and special conformal transformations For completeness, one also finds 32 supercharges $(Q_{\alpha a}, \tilde{Q}^a_{\dot{\alpha}}, S^a_\alpha, \tilde{S}_{\dot{\alpha}a})$, and also R-symmetry generators R_{IJ}.

To get the full algebra, we must consider all the commutators (see [45] for more). Now, we state the field content and matter. Readily from both the action and global symmetry, it is given that there is one gauge field A_μ in the singlet **1** representation of $SU(4)$. One also finds the Weyl fermions $\lambda^a_\alpha, a \in \{1,2,3,4\}$ transforming in the fundamental **4** representation and scalars $\phi^i, i \in \{1,...,6\}$ transforming in the antisymmetric **6** representation. For the operators, we require that they must be gauge invariant as well, since all the matter content in the theory is gauge invariant. The way we build them is by taking the trace of a product of such covariant fields evaluated at the same spacetime point. However, we note that since these fields all lie in the adjoint representation, they transformunder a gauge transformation according to $\chi(x) \to \chi(x) + [\mathcal{E}(x), \chi(x)]$ where $\chi(x)$ is one of the covariant fields and $\mathcal{E}(x)$ is a generator of gauge transformations. The local single trace operator then takes the form as $\mathcal{O}(x) = \mathrm{Tr}(\chi_1(x)\chi_2(x)...\chi_L(x))$. A specific class of operators only contains scalars defined as $\mathcal{O}(x) = \mathrm{Str}(\phi^{\{i_1}\phi^{i_2}...\phi^{i_k\}})$, where Str stands for the symmetrised trace for the gauge algebra defined in (2.72). This ensures that operators are totally symmetric. We want to construct the simplest operators now in terms of the scalars ϕ^i that combine into three complex scalars defined as

$$Z = \frac{1}{\sqrt{2}}(\phi^1 + i\phi^2), \quad W = \frac{1}{\sqrt{2}}(\phi^3 + i\phi^4), \quad X = \frac{1}{\sqrt{2}}(\phi^5 + i\phi^6). \tag{3.21}$$

Now we can start building operators, but before we look at a specific example, we might also want to know the dimensions of these. Given the bosonic subgroup, we know that it has rank six and thus has six Cartan generators (or charges) $(\Delta, S_1, S_2, J_1, J_2, J_3)$. Here, Δ is the conformal dimension, $S_1,\ S_2$ are the two charges of the $SO(1,3)$ Lorentz group which we call spin, and $J_1,\ J_2,\ J_3$ are the R-symmetry generators. These will become important when we study subsectors of SYM (and correspondingly ABJM). The scalars in $SU(4)$ transformed as $[0,1,0]$, so the dimension was $\mathrm{Dim}(0,L,0) = \frac{1}{12}(L+1)(L+2)^2(L+3)$. For concreteness, in the case $L = 2$ we get the **20** representation of $SU(4)$. Generally, in cases when the planar limit is taken, one can interpret the trace operators as spin chains. Doing this makes it natural to construct a general ground state $|0\rangle = \mathrm{Tr}Z^L$, which is just a spin chain with no excitations on the sites. These are the 1/2 BPS-operators. One can also make 1/4 and 1/8 BPS-operators, by considering different configurations of the dimensions of operators. They turn out to also be products of single trace operators.

Operators, charges and global symmetries of Chern-Simons

Gazing at ABJM theory, one might find this duality to contain more peculiarities than SYM due to the lack of supersymmetry. We have the same kind of symmetries, but as mentioned ABJM is a quiver gauge theory with a non-semisimple group $U(N) \times U(M)$ where in this case $M = N$. We get a superalgebra $OSp(6|4)$ that has bosonic subalgebra $SU(4)_R \times SO(3,2) \simeq SO(6)_R \times Sp(4)$. One sees the manifest three-dimensional conformal algebra as $SO(3,2)$ which is just a dimension smaller than for SYM, reducing the amount of generators from 15 to 10. Otherwise the R-symmetry is the same as in SYM. In terms of supercharges we now have $2 \times 12 = 24$ supercharges, indicating that as mentioned supersymmetry has been partially reduced or broken from the maximal case of 32. But again the same amount of R-symmetry generators are present. One can obtain the full algebra again by considering all the commutators between elements. Turning to the action it can be seen that the two Chern-Simons terms exist at levels $(k, -k)$. The gauge fields A_μ and $\hat{A}_\mu$ respectively transform as a connection under the $U(N) \times U(N)$ subgroups. In the matter sector we have four complex scalars accompanied by the same amount of fermions given by $Y^A, \psi_A, A \in \{1,2,3,4\}$. The bifundamental matter fields transform in the representation $(N, \bar{N})$ while the conjugate fields transforms in the anti-bifundamental $(\bar{N}, N)$. It is worth noticing that one can similarly construct a $SU(N) \times SU(N)$ group, but then one must take into account altercations to conditions on the moduli space, which is beyond the scope here. Most important will be the scalars which can be grouped in pairs of two complex scalars given by A_1 and A_2 in the $N \times \bar{N}$ representation, and B_1 and B_2 in the $\bar{N} \times N$ representation. We can group them into multiplets of the R-symmetry group as [46]

$$Y^a = (A_1, A_2, B_1^\dagger, B_2^\dagger) \quad Y_a^\dagger = (A_1^\dagger, A_2^\dagger, B_1, B_2). \tag{3.22}$$

Now a significant difference appears when looking at what type of operators we can build. As a consequence of the gauge theory, we have an alternation for the matter fields on the odd and even spacings in the trace operators. This mean that the general class of gauge ivariant operators are

$$\mathcal{O} = \text{Tr}(Y^{A_1}Y^\dagger_{B_1}Y^{A_2}Y^\dagger_{B_2}...Y^{A_L}Y^\dagger_{B_L})\chi^{B_1...B_L}_{A_1...A_L}. \tag{3.23}$$

The bare dimension of the operators is L and it is considered a chiral primary if χ is symmetric in all A_i, B_i indices and all traces are zero. What will be interesting is when this is not the case, in which case the operators will pick up quantum loop contributions from the anomalous dimension. This first appears at two-loop in ABJM compared to one-loop in SYM.
For the dimension of operators we have the Cartan charges $[\Delta, S, J_1, J_2, J_3]$. From the geometry side, one gets a restriction on J_4 such that due to orbifolding when considering a type IIA background we have [47]

$$J_1 + J_2 + J_3 + J_4 = 0. \tag{3.24}$$

Using the same expression for the dimension as for SYM one finds that the trivial representation will have $\dim(0,0,0) = \mathbf{1}$ and fundamental and anti-fundamental will have $\dim(1,0,0) = \mathbf{4}$ and $\dim(0,0,1) = \bar{\mathbf{4}}$. From the way we construct operators, we must consider the dimension of the vacuum to be of the form $\dim(L,0,L)$ since the bi- and anti-bifundamentals transform like this. The usual choice and convention for ground state operators is $|0\rangle = \text{Tr}(Y^1Y^\dagger_4)^L$. From here building more complicated operators is just a matter of changing respectively on odd and even sites scalars or fermions that coincide in the same representation.

3.3 Super Chern-Simons Theories

To get a better grasp of ABJM theory it might be helpful to look at Cherm-Simons theory in general. In this section we review some general theory and different supersymmetrical versions, and also briefly take a look at superspace formalism. We finish it off by stating the algebra of ABJM.

3.3.1 Chern-Simons Action, Invariance and Quantization

Originally the notion of topological manifolds with boundary terms was introduced by Chern and Simons [48]. This led to a surprising use of this manifold action in theoretical physics, in particular quantum field theories. For the sake of the AdS/CFT correspondence we will focus solely on the non-abelian case of the Chern-Simons(CS)-Lagranigan. It has the following form

$$\mathcal{L}_{cs} = \kappa\epsilon^{\mu\nu\rho}\text{Tr}\left(A_\mu\partial_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho\right). \tag{3.25}$$

The aim from here is to reach a point where we can argue that under a gauge transformation of the Lagrangian, the introduction of a quantization condition is needed for the path integral of the Lagrangian, such that it behaves as a quantum theory and does not contain ambiguities. The first step is to consider an infinitesimal variation δA_μ. This implies $\delta\mathcal{L}_{cs} = \kappa\epsilon^{\mu\nu\rho}\text{Tr}(\delta A_\mu F_{\nu\rho})$ with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. We see the equations of motion are reduced to a familiar looking term of $\kappa\epsilon^{\mu\nu\rho}F_{\mu\nu} = j^\mu$. The source-free equation will then give us $F_{\mu\nu} = 0$, and the solutions give a pure gauge or rather flat connections $A_\mu = g^{-1}\partial_\mu g$. This was sort of an interlude between abelian and non-abelian elements, but now we see what happens to the Lagrangian under a gauge transformation associated to the group in the non-abelian case (usually $SU(N)$). Such a transformation is given by $A_\mu \to A_\mu^g = g^{-1}A_\mu g + g^{-1}\partial_\mu g$. The Lagrangian then transforms as

$$\begin{aligned}\mathcal{L}_{cs} = \kappa\epsilon^{\mu\nu\rho}(&\mathrm{Tr}((g^{-1}A_\mu g + g^{-1}\partial_\mu g)\partial_\nu(g^{-1}A_\rho g + g^{-1}\partial_\rho g)) \\ &+ \frac{2}{3}\mathrm{Tr}((g^{-1}A_\mu g + g^{-1}\partial_\mu g)(g^{-1}A_\nu g + g^{-1}\partial_\nu g)(g^{-1}A_\rho g + g^{-1}\partial_\rho g))) \end{aligned} \tag{3.26}$$

We can write the result of this gauge transformation as the Lagrangian itself plus some boundary terms:

$$\mathcal{L}_{cs} \to \mathcal{L}_{cs} - \kappa\epsilon^{\mu\nu\rho}\partial_\mu \mathrm{Tr}(\partial_\nu g g^{-1}A_\rho) - \frac{\kappa}{24\pi^2}\epsilon^{\mu\nu\rho}\mathrm{Tr}(g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\rho g) \tag{3.27}$$

The total spacetime derivative vanishes under some appropriate boundary conditions as in the case for abelian Chern-Simons theories. However, the last term is a new contribution which we denote as the winding number density

$$w(g) = \frac{\kappa}{24\pi^2}\epsilon^{\mu\nu\rho}\mathrm{Tr}(g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\rho g). \tag{3.28}$$

With appropriate boundary conditions the integral of the winding number density will be an integer. Thus, considering the path integral, the action will transform into itself plus an additive constant $\mathcal{S}_{cs} \to \mathcal{S}_{cs} - 8\pi^2\kappa N$. Thus for the path integral $\exp(i\mathcal{S})$ to stay gauge invariant, we demand that the parameter in the CS-Lagrangian must take discrete values: $\kappa = \frac{k}{4\pi}$, $k \in \mathrm{N}$.

3.3.2 Superspace Formalism

We briefly comment on the formalism of superspace as we have to introduce the notion of superfields and superspaces in ABJM. In a usual sense, when one is not dealing with supersymmetry, the input in functions one might deal with has input x^μ; coordinates of Minkowski space. The idea is now to extend Minkowski such that we also get anti-commuting fields. Thus the set of variables will now be a triplet $(x^\mu, \theta_\alpha, \bar{\theta}^{\dot{\alpha}})$, which consists of the Minkowski coordinates and Grassmann spinors. A brief mentioning of the geometry of superspace may be convenient now. Usually when given a Lie group we want to know what manifold would be useful in that situation when knowing the action of the group G. So it would be natural to just equate the group manifold to the manifold itself. However, one can also create something called a coset space, which means letting the manifold be a quotient between the Lie group and a subgroup of G. Formally this is written as $\mathcal{M} = G/H$ with $H \subset G$. This will be helpful in formulating a general element of the superspace. The coset that generates the superspace is the quotient between the super-Poincaré group and the Lorentz group. For the Poincaré group we know that the algebra gives generators for Lorentz boosts $M^{\mu\nu}$ and translations P^μ. Now we only need to add the supersymmetry generators accompanied by the Grassmann spinors, and this will constitute the generator of the superspace:

$$g(\omega, x^\mu, \theta, \bar{\theta}) = \exp\left(-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu} + ix_\mu P^\mu + i\theta^\alpha Q_\alpha + i\bar{\theta}_{\dot{\alpha}}Q^\alpha\right). \tag{3.29}$$

What we are interested in now is figuring out how the generators work on the coordinates. As in Poincaré we know that momentum generates translation, and this is still the case. However, what about the Grassmann valued spinors? It is relatively straightforward to show how they transform. First we split the generator into two pieces:

$$g(\omega, x^\mu, \theta, \bar{\theta}) = \tilde{g}(x^\mu, \theta, \bar{\theta}) h(\omega). \tag{3.30}$$

Here, $h(\omega)$ is the Lorentz transformation while the rest is the coset structure with the form $\tilde{g}(x^\mu, \theta, \bar{\theta}) = \exp(ix_\mu P^\mu + i\theta^\alpha Q_\alpha + i\bar{\theta}_{\dot{\alpha}} Q^\alpha)$. This gives us points in superspace. To see how our coordinates are now affected by suiting generators, we can try to act with the following objects $U(a) = \exp(ia_\mu p^\mu)$ and $V(\epsilon, \bar{\epsilon})$. It is an easy exercise to see that acting with the momentum generator gives $U(a)\tilde{g}(x, \theta, \bar{\theta}) = \tilde{g}(x+a, \theta, \bar{\theta})$. It is a bit more involved but still simple to show what happens for $V(\epsilon, \bar{\epsilon})\tilde{g}(x, \theta, \bar{\theta})$. Due to the anti-commutative nature of the supersymmetry generators, we have to use the BCH-formula to first order when calculating this. One gets

$$V(\epsilon, \bar{\epsilon})\tilde{g}(x, \theta, \bar{\theta}) = \tilde{g}(x + i\theta\sigma^\mu\bar{\epsilon} - i\epsilon\sigma^\mu\bar{\theta}, \theta + \epsilon, \bar{\theta} + \bar{\epsilon}). \tag{3.31}$$

This asserts how the generators act on the coordinates. The Grassmann spinor not only affects the spinor part of the space but also the normal spacetime coordinates. The last thing worth mentioning before moving on to the actual Chern-Simons action are the superfields. Grassmann valued objects truncate after quadratic order, that is to say that the highest order terms are of the form $\theta^2 = \theta^\alpha\theta_\alpha$. This will help when investigating a superfield $Y(x, \theta, \bar{\theta})$. If we Taylor expand in θ and $\bar{\theta}$, we the find a finite expression for the superfield containing a mix of particle multiplets:

$$\begin{aligned} Y(x, \theta, \bar{\theta}) = \phi(x) + \theta^\alpha\psi_\alpha(x) + \bar{\theta}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}}(x) + \theta^2 M(x) + \bar{\theta}^2 N(x) \\ \theta^\alpha\bar{\theta}_{\dot{\alpha}} V_{\alpha\dot{\alpha}}(x) + \theta^2\bar{\theta}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}(x) + \bar{\theta}^2\theta^\alpha\rho_\alpha(x) + \theta^2\bar{\theta}^2 D(x). \end{aligned} \tag{3.32}$$

It becomes apparent that what comes out of this, such that all indices match, is a total of four complex scalars, ϕ, M, N, and D. Then there are two left-handed spinors ψ, ρ and two right-handed spinors $\bar{\chi}$ and $\bar{\lambda}$. At last there will also be a vector $V_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}} V_\mu$. Now we have a basic understanding of the matter content in the superspace formulation. Before we conclude the section there is still some things left to be noted. One can, as with the generators, formulate the same transformations for superfields in a manner such that

$$VY(x, \theta, \bar{\theta})V^\dagger = Y(x + i\theta\sigma^\mu\bar{\epsilon} - i\epsilon\sigma^\mu\bar{\theta}, \theta + \epsilon, \bar{\theta} + \bar{\epsilon}). \tag{3.33}$$

Through the commutation with the supersymmetry generators and treating ϵ_α as an infinitesimal spinor, one can find the following relations

$$[Q_\alpha, Y] = \left(-i\frac{\partial}{\partial\theta^\alpha} - \sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu\right)Y, \qquad [\bar{Q}_{\dot{\alpha}}, Y] = \left(-i\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - \theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\right)Y. \tag{3.34}$$

3.3.3 $\mathcal{N} = 2$ Chern-Simons Action in Superspace

To go from pure CS theory, we need to add fermionic degrees of freedom, which will come through the vector and chiral multiplets:

$$\mathbf{V} : \{A_\mu, \chi, \sigma, D\}, \quad \Phi : \{\phi, \psi, F\}. \tag{3.35}$$

For the vector multiplet, A_μ is the gauge field, χ is the two Majorana spinors combined into one complex spinor, σ is a real scalar, and D is a real auxiliary scalar. For the chiral superfield we have that ϕ is a complex scalar, ψ is the two Majorana spinors combined into a complex spinor, and F is a complex auxiliary scalar. The Lorentz group in $2+1$ dimensions is $SO(2,1)$, and its covering group for fermions is $Spin(2,1) \cong SL(2,\mathbb{R})$. Hence, the spinor representations correspond to 2-component Majorana spinors, and so therefore we start from $\mathcal{N}=2$, as this superspace consists of four fermionic degrees of freedom. To proceed from here, we introduce the super-covariant derivatives and charges, and also the chiral and corresponding vector superfields:

$$\begin{aligned}\bar{D}_\alpha &= -\frac{\partial}{\partial\bar{\theta}^\alpha} - \theta^\beta\gamma^\mu_{\beta\alpha}\partial_\mu, \quad D_\alpha = \frac{\partial}{\partial\bar{\theta}^\alpha} + \bar{\theta}^\beta\gamma^\mu_{\alpha\beta}\partial_\mu, \\ \bar{Q}_\alpha &= -\frac{\partial}{\partial\bar{\theta}^\alpha} + \theta^\beta\gamma^\mu_{\beta\alpha}\partial_\mu, \quad Q_\alpha = \frac{\partial}{\partial\bar{\theta}^\alpha} - \bar{\theta}^\beta\gamma^\mu_{\alpha\beta}\partial_\mu.\end{aligned} \tag{3.36}$$

Here, $\{\gamma_0,\gamma_1,\gamma_2\} = \{i\sigma_2,\sigma_1,\sigma_3\}$, where σ_i are the usual Pauli matrices. To lower and raise indices we use the two-dimensional Levi-Civita symbol $\epsilon^{\alpha\beta},\epsilon_{\alpha\beta} : \epsilon^{12} = \epsilon_{21} = 1$. As mentioned in the original paper, the form of the replaced kinetic term in the vector multiplet, which is replaced by supersymmetric Chern-Simons term, will look akward in superspaace, but will take a nice form in the Wess-Zumino Gauge in component form. The chrial superfield will then read $\Phi : \bar{D}_\alpha\Phi = 0$ and $D_\alpha\bar{\Phi} = 0$:

$$V(x,\theta,\bar{\theta}) = -\theta\gamma^\mu\bar{\theta}A_\mu - \theta\bar{\theta}\sigma(x) + i\theta^2\bar{\theta}\bar{\chi}(x) - i\bar{\theta}^2\theta\chi(x) + \frac{1}{2}\bar{\theta}^2\theta^2 D(x), \tag{3.37}$$

$$\Phi(x,\theta,\bar{\theta}) = \phi(x) + \sqrt{2}\theta\psi(x) + \theta^2 F(x) + i\theta\gamma^\mu\bar{\theta}\partial_\mu\phi(x) - \frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi(x)\gamma^\mu\bar{\theta} - \frac{1}{4}\theta^2\bar{\theta}^2\partial^2\phi(x), \tag{3.38}$$

$$\bar{\Phi}(x,\theta,\bar{\theta}) = \bar{\phi}(x) + \sqrt{2}\bar{\theta}\bar{\psi}(x) + \bar{\theta}^2\bar{F}(x) + i\theta\gamma^\mu\bar{\theta}\partial_\mu\bar{\phi}(x) - \frac{i}{\sqrt{2}}\bar{\theta}^2\gamma^\mu\theta\partial_\mu\bar{\psi}(x) - \frac{1}{4}\bar{\theta}^2\theta^2\partial^2\bar{\phi}(x). \tag{3.39}$$

Here the notation is $\theta^2 = \theta^\alpha\theta_\alpha$, $\bar{\theta}^2 = \bar{\theta}^\alpha\bar{\theta}_\alpha$, $\partial^2 = \partial^\mu\partial_\mu$. An interesting feature of all this is that the $d=3$ $\mathcal{N}=2$ vector superfield can be obtained by dimensional reduction from $d=4$ $\mathcal{N}=1$ vector superfield. Thus it can be seen that the $D=3$ $\mathcal{N}=2$ Chern-Simons matter theory can be obtained by dimensional reduction of SYM, with the exception that the kinetic part of the vector multiplet is replaced the supersymmetric version in the pure CS-Lagrangian. One can ponder what superspace action would do this, and in a very non-trivial way it can be chosen to be [49]

$$\mathcal{S}^{\mathcal{N}=2}_{CSM} = \int d^3x \int d^4\theta \left\{ \frac{k}{2\pi}\int_0^1 dt \mathrm{Tr}[V\bar{D}^\alpha(e^{-tV}D_\alpha e^{tV})] + \sum_{i=1}^{N_f}\bar{\Phi}^i e^V \Phi^i \right\}. \tag{3.40}$$

The index i is a global $U(N_f)$ flavor symmetry acting on Φ. The trace is in the fundamental representation for $U(N)$ and $SU(N)$. Thus the generators T^a obey $\mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$. Φ^i is a vector which is acted upon by the representation R_i of the group. Using what has been established by the superfields and the super-covariant derivatives, we can proceed for each term and calculate how this reduces in components after integrating out the superfields. Starting with the kinetic part we get

$$
\begin{aligned}
\mathcal{S}_{CS}^{\mathcal{N}=2} &= \int d^3x \int d^4\theta \left\{ \frac{k}{2\pi} \int_0^1 dt \mathrm{Tr}[V\bar{D}^\alpha(e^{-tV}D_\alpha e^{tV})] \right\} \\
&= \frac{k}{4\pi}\int \mathrm{Tr}(A^aT^a \wedge dA^aT^a + \frac{2}{3}A^aT^a \wedge A^bT^b \wedge A^cT^c) + \bar{\chi}^a\chi^b\delta^{ab} + D^a\sigma^b\delta^{ab} \quad (3.41) \\
&= \frac{k}{4\pi}\int \mathrm{Tr}(A \wedge dA + \frac{2}{3}A^3 + \bar{\chi}\chi + 2D\sigma).
\end{aligned}
$$

Here $a \in \{1, 2, ...dim(\mathcal{G})\}$ and is the index related to the fundamental generators, such as $A_\mu = A_\mu^a T^a$, and the same for the rest of the components in the multiplet. Now, if we focus on the chiral field, we note that the covariant derivative goes as follows

$$
D_\mu\{\phi^i, \psi^i\} = \partial_\mu\{\phi^i, \psi^i\} + \frac{i}{2}A_\mu^a T_{R_i}^a\{\phi^i, \psi^i\}. \tag{3.42}
$$

Using this we can split up the superspace integral and get

$$
\begin{aligned}
\int d^4\theta \sum_{i=1}^{N_f} \bar{\Phi}^i e^V \Phi^i \ &= \sum_{i=1}^{N_f}(D_\mu\bar{\phi}^i D^\mu\phi^i - i\bar{\psi}^i\gamma^\mu D_\mu\psi^i - \frac{1}{4}\bar{\phi}^i\sigma^a\sigma^b T_{R_i}^a T_{R_i}^b \Phi^i + \frac{1}{2}\bar{\phi}^i D^a T_{R_i}^a \Phi^i \\
&\quad - \frac{1}{2}\bar{\psi}^i\sigma^a T_{R_i}^a\psi^i + \frac{i}{\sqrt{2}}\bar{\phi}^i\chi^a T_{R_i}^a\psi^i - \frac{i}{\sqrt{2}}\bar{\psi}^i T_{R_i}^a\bar{\chi}^a\phi^i + \bar{F}^i F^i). \quad (3.43)
\end{aligned}
$$

Here, $T_{R_i}^a, a \in \{1, 2, \ldots \dim(\mathcal{G})\}$ are the generators of gauge group $\mathcal{G}$ in the R_i representation, and $A_\mu^i = A_\mu^a T_{R_i}^a$ etc. Now going on to solve the equations of motion we find

$$
\begin{aligned}
D^\alpha : \sigma^a = -\frac{2\pi}{k}\bar{\phi}^i T_{R_i}^a \phi^i, \quad F, \bar{F} : F = 0, \bar{F} = 0, \\
\chi^a : \bar{\chi}^a = \frac{4\pi i}{\sqrt{2}k}\bar{\phi}^i T_{R_i}^a\psi^i, \quad \bar{\chi}^a : \chi^a = -\frac{4\pi i}{\sqrt{2}k}\bar{\psi}^i T_{R_i}^a\phi^i.
\end{aligned} \tag{3.44}
$$

To get the full action, after all the parts have been manipulated, we can combine them and integrate out the fields. This will in the end yield

$$
\begin{aligned}
\mathcal{S}_{CSM}^{\mathcal{N}=2} = \mathcal{S}_{CS} + \int d^3x \Big\{ & D_\mu\bar{\phi}^i D^\mu\phi^i - i\bar{\psi}^i\gamma^\mu D_\mu\psi^i + \frac{\pi^2}{k^2}(\bar{\phi}^i T_{R_i}^a\phi^i)(\bar{\phi}^j T_{R_j}^b\phi^j)(\bar{\phi}^k T_{R_k}^a T_{R_k}^b\phi^k) \\
& + \frac{\pi}{k}(\bar{\phi}^i T_{R_i}^a\phi^i)(\bar{\psi}^j T_{R_j}^a\psi^j) + \frac{2\pi}{k}(\bar{\psi}^i T_{R_i}^a\phi^i)(\bar{\phi}^j T_{R_j}^a\psi^j)\Big\}. \quad (3.45)
\end{aligned}
$$

This concludes the case for $\mathcal{N} = 2$ superspace. What we will proceed with is to look at how we can construct the ABJM action in broad terms by enhancing the supersymmetry from $\mathcal{N} = 2$ to $\mathcal{N} = 4$ and then decrease the symmetry to $\mathcal{N} = 3$.

3.3.4 $\mathcal{N} = 3$ Chern-Simons Action in Superspace

As mentioned, there are several ways to attack the problem of the breaking of supersymmetry so we get an $\mathcal{N} = 3$ Chern-Simons matter theory. One way which is feasible is to start from $\mathcal{N} = 4$ SYM and replace the kinetic term with the Chern-Simons one which precisely breaks the symmetry to the desired one [50]. So to proceed as in the case from last section, we organize our three-dimensional $\mathcal{N}=3$ multiplet in the language of $\mathcal{N}=2$ superspace in the following manner:

$$\mathbf{V}:\{A_\mu,\chi,\sigma,D\},\quad \mathbf{Q}:\{q,\lambda,S\}, \tag{3.46}$$

$$\Phi^i:\{\phi^i,\psi^i,F^i\},\quad \tilde{\Phi}^i:\{\tilde{\phi}^i,\tilde{\psi}^i,\tilde{F}^i\}. \tag{3.47}$$

$\mathbf{V}$ and $\mathbf{Q}$ are respectively $\mathcal{N}=2$ vector and chiral multiplets in the adjoint representation of $\mathcal{G}$, and Φ^i and $\tilde{\Phi}^i$ are $\mathcal{N}=2$ chiral multiplets, transforming under the representation R_i and conjugate representation $\bar{R}_i$ of $\mathcal{G}$, respectively. By comparison we see that an auxillary chiral multiplet $\mathbf{Q}$ has been added to the matter content. So if we want to write up the corresponding CS-action for the $\mathcal{N}=3$ we get [51]

$$\mathcal{S}^{\mathcal{N}=3}_{CSM}=\mathcal{S}^{\mathcal{N}=2}_{CS}+\int d^3x\int d^4\theta\sum_{i=1}^{N_f}\left(\bar{\Phi}^i e^V\Phi^i+\tilde{\Phi}^i e^{-V}\bar{\tilde{\Phi}}^i\right) \tag{3.48}$$

$$+\left[\int d^3x\int d^4\theta\left(\frac{k}{2\pi}\mathrm{Tr}Q^2-\sum_{i=1}^{N_f}\tilde{\Phi}^iQ\Phi^i\right)+c.c\right]. \tag{3.49}$$

Here c.c stands for complex conjugation. To symmetrize the V|Q multiplet with the Chern-Simons term, the $\{q,\lambda,S\}$ have to introduce terms in the action similar to how the original chiral multiplet did in the $\mathcal{N}=2$ case. Here, the addition of $\mathrm{Tr}Q^2+c.c$ does this for us with its coefficient fixed by supersymmetry and its form fixed by the requirement of holomorphicity of a superpotential. This can be seen by writing the Taylor superspace expansion for Q:

$$\left[\int d^3x\int d^4\theta\left(\frac{k}{2\pi}\mathrm{Tr}Q^2\right)\right]+c.c=\frac{k}{4\pi}(-\lambda^a_\alpha\lambda^{b\alpha}-\bar{\lambda}^a_\alpha\bar{\lambda}^{b\alpha}+S^aq^b+\bar{S}^a\bar{q}^b)\delta^a_b. \tag{3.50}$$

This term is exactly what will break the $\mathcal{N}=4$ to $\mathcal{N}=3$. Now we can proceed to integrate out Q since it is an auxillary field with no dynamical degrees of freedom:

$$Q:Q^a=\frac{2\pi}{k}(\tilde{\Phi}^iT^a_{R_i}\Phi^i) \tag{3.51}$$

$$W=\frac{k}{2\pi}\mathrm{Tr}Q^2-\sum_{i=1}^{N_f}\tilde{\Phi}^iQ\Phi^i=-\frac{\pi}{k}(\tilde{\Phi}^iT^a_{R_i}\Phi^i)(\tilde{\Phi}^jT^a_{R_i}\Phi^j) \tag{3.52}$$

From this we can conclude that the $\mathcal{N}=3$ Chern-Simons matter theory is nothing but an additional superpotential term added to the $\mathcal{N}=2$ theory where matter is organized in a hypermultiplet now $(\Phi,\tilde{\Phi})$. By procedure as for the terms in the previous section, we expand $(\Phi,\tilde{\Phi},Q)$ in superspace and insert them into the matter action to obtain

$$\begin{aligned}\int d^4\theta\sum_{i=1}^{N_f}\left(\bar{\Phi}^ie^V\Phi^i+\tilde{\Phi}^ie^{-V}\bar{\tilde{\Phi}}^i\right)=\sum_{i=1}^{N_f}\Big(&D_\mu\bar{\phi}^iD^\mu\phi^i+D_\mu\tilde{\phi}^iD^\mu\bar{\tilde{\phi}}^i-i\bar{\psi}^i\gamma^\mu D_\mu\psi^i\\&-i\tilde{\psi}^i\gamma^\mu D_\mu\bar{\tilde{\psi}}^i+\bar{F}^iF^i-\frac{1}{4}\bar{\phi}^i\sigma^a\sigma^bT^a_{R_i}T^b_{R_i}\phi^i-\frac{1}{4}\tilde{\phi}^i\sigma^a\sigma^bT^a_{R_i}T^b_{R_i}\bar{\tilde{\phi}}^i\\&+\frac{1}{2}\bar{\phi}^iD^aT^a_{R_i}\phi^i+\frac{1}{2}\tilde{\phi}^iD^aT^a_{R_i}\bar{\tilde{\phi}}^i-\frac{1}{2}\bar{\psi}^i\sigma^aT^a_{R_i}\psi^i-\frac{1}{2}\tilde{\psi}^i\sigma^aT^a_{R_i}\bar{\tilde{\psi}}^i\\&+\frac{i}{\sqrt{2}}\bar{\phi}^i\chi^aT^a_{R_i}\psi^i-\frac{i}{\sqrt{2}}\tilde{\psi}^i\chi^aT^a_{R_i}\bar{\tilde{\phi}}^i-\frac{i}{\sqrt{2}}\bar{\psi}^iT^a_{R_i}\bar{\chi}^a\phi^i+\frac{i}{\sqrt{2}}\tilde{\phi}^iT^a_{R_i}\bar{\chi}^a\bar{\tilde{\psi}}^i\Big)\\&+\tilde{F}^i\bar{\tilde{F}}^i.\end{aligned} \tag{3.53}$$

3.3.5 The Superconformal Group $Osp(4|6)$

Considering both global bosonic and fermionic symmetries of the ABJM theory one finds that this will constitute the Lie superalgebra $OSp(4|6)$. If we first consider the bosonic subalgebra we get $SO(6) \times Sp(4) \cong SU(4) \times SO(3,2)$. $SU(4) \times SO(3,2)$ is the $3D$ conformal algebra and has ten components. Six of these belong to the Poincaré algebra which contains the Lorentz algebra $so(2,1) \cong sl(2,\mathbb{R})$ with generators $M_{\mu\nu}$. Additionally we have spacetime generators P_μ. The remainder is the dilatation operator D and special conformal transformations K_μ. It is a standard exercise to derive the commutation relations, but we just state them here:

$$\begin{aligned} [P_\mu, K_\nu] = 2\delta^\mu_\nu + 2M_{\mu\nu}, \quad [D, P_\mu] = P_\mu, \quad [D, K_\mu] = -K_\mu, \\ [M_{\mu\nu}, M_{\rho\sigma}] = \delta_{\mu[\nu} M_{\rho]\mu}, \quad [P_\mu, M_{\nu\rho}] = \delta_{\mu[\nu} P_{\rho]}, \quad [K_\mu, M_{\nu\rho}] = \delta_{\mu[\nu} K_{\rho]}. \end{aligned} \tag{3.54}$$

Then we consider the $su(4)$ part of the algebra which contains the R-symmetry generators. We denote them as R^I_J, where $I, J = \{1,2,3,4\}$ and $R^I_I = 0$. Thus the R-symmetry generators have commutation relations

$$[R^I_J, R^L_K] = \delta^L_I R^J_K - \delta^J_K R^L_I. \tag{3.55}$$

Lastly we have the supercharges Q^{IJ}_α and S^{IJ}_α. Since they have fermionic nature they obey anti-commutation relations:

$$\begin{aligned} \{Q^{IJ}_\alpha, Q^{KL\beta}\} = 2\epsilon^{IJKL}(\gamma^\mu)^\beta_\alpha P_\mu, \quad \{S^{IJ}_\alpha, S^{KL\beta}\} = 2\epsilon^{IJKL}(\gamma^\mu)^\beta_\alpha K_\mu, \\ \{Q^{IJ}_\alpha, S^{KL\beta}\} = \epsilon^{IJKL}(\gamma^\mu)^\beta_\alpha M_{\mu\nu} + 2\delta^\beta_\alpha \left(\epsilon^{IJKL} D - \epsilon^{NJKL} J^I_N - \epsilon^{INKL} J^J_N\right). \end{aligned} \tag{3.56}$$

The only remaining commutators to look at are the bosonic ones from the conformal group and the supercharges. They will finally give us

$$\begin{aligned} [D, Q^{IJ}_\alpha] = \frac{1}{2} Q^{IJ}_\alpha, \quad [M_{\mu\nu}, Q^{IJ}_\alpha] = -\frac{1}{2}(\gamma_{\mu\nu})^\beta_\alpha Q^{IJ}_\beta, \quad [K_\mu, Q^{IJ}_\alpha] = (\gamma_\mu)^\beta_\alpha S^{IJ}_\beta, \\ [D, S^{IJ}_\alpha] = -\frac{1}{2} S^{IJ}_\alpha, \quad [M_{\mu\nu}, S^{IJ}_\alpha] = -\frac{1}{2}(\gamma_{\mu\nu})^\beta_\alpha S^{IJ}_\beta, \quad [K_\mu, S^{IJ}_\alpha] = (\gamma_\mu)^\beta_\alpha Q^{IJ}_\beta, \\ [J^J_I, Q^{KL}_\alpha] = \delta^K_I Q^{JL}_\alpha + \delta^L_I Q^{KJ}_\alpha - \frac{1}{2}\delta^J_I Q^{KL}_\alpha, \quad [J^J_I, S^{KL}_\alpha] = \delta^K_I S^{JL}_\alpha + \delta^L_I S^{KJ}_\alpha - \frac{1}{2}\delta^J_I S^{KL}_\alpha. \end{aligned} \tag{3.57}$$

4

Subsectors and Decoupling Limits

To motivate why we are interested in subsectors is best done by example. Through the framework of Spin Matrix Theory (see chapter 6), one considers near BPS-bounds or, in a thermodynamic sense, a near critical point in temperature. This realizes decoupling limits in the whole theory to the full symmetry group. This is well described in [52], where it is stated that a subsector is a set of fields that are closed under the action of the spin-chain Hamiltonian , i.e. there is no overlap between spin-chains from within a subsector with spin-chains from outside. One defines a semi-definite charge from the eigenvalues of all operators (Cartan generators) that commute with the spin-chain Hamiltonian $E = \Delta - J$.

$$0 \leq k\Delta + \sum_i n_i J_i + \sum_a n_a S_a + \text{other} = P. \tag{4.1}$$

Here, we are just reformulating the condition of energy from above, where J is a specific choice of Cartan generators that constitutes the exact subsector. Here the coefficients in front are related to the weights and Dynkin Labels of the specific bosonic subalgebra of the correspondence.

The use of this has been explored in various contexts, both in the contexts of gravity and spacetime manifolds but also free partition functions, confinement/deconfinement transition related to the Hagedorn temperature and more [53, 54, 55, 56]. We proceed by analyzing the subsectors for both SYM and Chern-Simons, where we find decoupling limits and subsectors that will be relevant in later calculations.

4.1 Subsectors and Decoupling Limits for $\mathcal{N} = 4$ SYM

We consider N = 4 SYM on $\mathbb{R} \times S^3$ with gauge group $SU(N)$. We will consider the thermal partition function in the grand canonical ensemble. This means non-zero chemical potentials, which are given by the generators of the $SU(4)$ R-symmetry and Cartan generators of $SO(4)$ of the S^3. We will find that only a subset of subsectors will survive given contributions from tree-level and one-loop interactions. To start we have the 't Hooft coupling $\lambda = \frac{g_{ym}^2 N}{4\pi^2}$, where g_{YM} is the Yang-Mills coupling. The set of letters for the theory is given by the full $psu(2,2|4)$. It is important to know what kind of operators we will be dealing with, since they will generate the crucial quantum numbers associated to the symmetry. Such operators can be constructed from linear combination of multi-trace operators of the form

$$\prod_{i=1}^{k} \text{Tr}\left(A_1^{(1)} A_2^{(2)} ... A_{L_k}^{(k)}\right). \tag{4.2}$$

The quantum numbers associated is the energy E, two angular momenta S_1 and S_2 given by the $SO(4)$ of S^3, three R-symmetry charges J_1, J_2, J_3 corresponding to the Cartan generators of $SU(4)$, and lastly a dilatation operator D. If we wish to construct the partition function in the grand canonical ensemble for $SU(N)$ we can write

$$Z_{\lambda,N}(\beta,\omega_1,\omega_1,\Omega_1,\Omega_2,\Omega_3) = \mathrm{Tr}_M\left[\exp\left(-\beta D + \beta\sum_{a=1}^{2}\omega_a S_a + \beta\sum_{i=1}^{3}\Omega_i J_i\right)\right]. \tag{4.3}$$

The pre-factors in front of the generators will be the chemical potentials. To get to the point of decoupled theories, we set the chemical potentials to be equal to the same parameter such that $(\omega_1,\omega_2,\Omega_1,\Omega_2,\Omega_3) = \Omega(n_1,n_2,n_3,n_4,n_5)$, where n_i are real numbers and $\Omega \in [0,1]$. Apparently, it is the case that for $\Omega \to 1$ we approach critical values of the set of chemical potentials. Employing this and defining $J = n_1 S_1 + n_2 S_2 + n_3 J_1 + n_4 J_2 + n_5 J_3$, we have

$$Z_{\lambda,N}(\beta,\omega_1,\omega_1,\Omega_1,\Omega_2,\Omega_3) = \mathrm{Tr}_M\left[e^{-\beta D+\beta\Omega J}\right] = \mathrm{Tr}_M\left[e^{-\beta(D-J)-\beta(1-\Omega)J}\right]. \tag{4.4}$$

Here it becomes clear that in the limit $\Omega \to 1$, we get a decoupled theory, where the contribution purely comes from $D-J$. In general, the dilatation operator can be expanded in powers for small λ such that $D = D_0 + \lambda D_2 + \lambda^{\frac{3}{2}} D_3 + ...$ We see that the coupling enters through only the dilatation, thus for each term we take into consideration, this will be another loop-order considered. We will look at two cases:

$$Z_{\lambda=0,N}(\beta,\omega_1,\omega_1,\Omega_1,\Omega_2,\Omega_3) = \mathrm{Tr}_M\left[e^{-\beta(D_0-J)-\beta(1-\Omega)J}\right], \tag{4.5a}$$

$$Z_{\lambda,N}(\beta,\omega_1,\omega_1,\Omega_1,\Omega_2,\Omega_3) = \mathrm{Tr}_M\left[e^{-\beta(D_0-J)-\beta\lambda D_2-\beta(1-\Omega)J+\beta\mathcal{O}(\lambda^{\frac{3}{2}})}\right]. \tag{4.5b}$$

In the case of no interactions, we restrict ourselves to choices of (n_1,n_2,n_3,n_4,n_5) that satisfy $D_0 \geq J$. If we let $\beta \to \infty$ then all states with our chosen condition will decouple from the partition function. However, this would not be useful, so we need to furthermore demand that our choice of the integers or half-integers must fulfill that some states obey $D_0 = J$. This means that to get a non-trivial partition function, we keep $\beta(1-\Omega)$ fixed in the $\beta \to \infty$ limit. Thus we can write $Z_N(\tilde{\beta}) = \mathrm{Tr}_M[e^{-\tilde{\beta}D_0}]$, $\tilde{\beta} = \beta(1-\Omega)$. Considering the other case where we include the one-loop dilatation operator, we still demand the same from the $\lambda = 0$ case, now we only need to include the $\beta\lambda$ term to get a non-trivial interaction. Thus in the large β limit we find

$$\beta \to \infty, \quad \tilde{\beta} = \beta(1-\Omega)\text{fixed}, \quad \tilde{\lambda} = \frac{\lambda}{1-\Omega}\text{fixed}, \quad N \quad \text{fixed}. \tag{4.6}$$

This will in the end give us $Z_N(\tilde{\beta}) = \mathrm{Tr}_M[e^{-\tilde{\beta}D_0+\tilde{\lambda}D_2}]$, bringing us close to the zero temperature, $\Omega = 1$ and zero coupling. Higher loop terms for $n \geq 3$ in the dilatation operator will be negligible in the considered limit. Also, no assumption on N has been made, hence it works in finite cases, and in the decoupled theory the partition function will depend on $\tilde{\lambda}, N, \tilde{\beta}$. Lastly, from our choices of n_i this will mean that $(T,\Omega) = (0,1)$ is a critical point, or rather $(T,\omega_1,\omega_2,\Omega_1,\Omega_2,\Omega_3) = (0,n_1,n_2,n_3,n_4,n_5)$.

We now list the respective weights of letters in the theory for both the R-symmetry $SU(4)$ and S^3 $SO(4)$ that we use to construct the subsectors. Following [57] we can organize that data into tables through the reprsentations.

	F_+	F_0	F_-	$\tilde{F}_+$	$\tilde{F}_0$	$\tilde{F}_-$
SO(4)	(1,-1)	(0,0)	(-1,1)	(1,1)	(0,0)	(-1,-1)
SU(4)	(0,0,0)	(0,0,0)	(0,0,0)	(0,0,0)	(0,0,0)	(0,0,0)

Table 4.1: Weight of gauge field strength components.

	Z	X	W	$\bar{Z}$	$\bar{X}$	$\bar{W}$
SO(4)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
SU(4)	(1,0,0)	(0,1,0)	(0,0,1)	(-1,0,0)	(0,-1,0)	(0,0,-1)

Table 4.2: Complex scalars in $\mathcal{N} = 4$ SYM.

	χ_1,χ_3,χ_5,χ_7	χ_2,χ_4,χ_6,χ_8	$\bar{\chi}_1$,$\bar{\chi}_3$,$\bar{\chi}_5$,$\bar{\chi}_7$	$\bar{\chi}_2$,$\bar{\chi}_4$,$\bar{\chi}_6$,$\bar{\chi}_8$
SO(4)	$(\frac{1}{2},-\frac{1}{2})$	$(-\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$(-\frac{1}{2},-\frac{1}{2})$

Table 4.3: $SO(4)$ weights of fermions in $\mathcal{N} = 4$ SYM.

	χ_1,χ_2	,χ_3,χ_4	χ_5,χ_6	χ_7,χ_8
SU(4)	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$	$(-\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	$(-\frac{1}{2},-\frac{1}{2},\frac{1}{2})$

Table 4.4: $SO(4)$ weights for $\chi_1, ..., \chi_8$ fermions in $\mathcal{N} = 4$ SYM.

	$\bar{\chi}_1$,$\bar{\chi}_2$	$\bar{\chi}_3$,$\bar{\chi}_4$	$\bar{\chi}_5$,$\bar{\chi}_6$	$\bar{\chi}_7$,$\bar{\chi}_8$
SU(4)	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$	$(-\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	$(-\frac{1}{2},-\frac{1}{2},\frac{1}{2})$

Table 4.5: $SO(4)$ weights for $\chi_1, ..., \chi_8$ fermions in $\mathcal{N} = 4$ SYM.

	d_1	d_2	$\bar{d}_1$	$\bar{d}_2$
SO(4)	(1,0)	(0,1)	(-1,0)	(0,-1)
SU(4)	(0,0,0)	(0,0,0)	(0,0,0)	(0,0,0)

Table 4.6: Derivative operators of $\mathcal{N} = 4$ SYM.

The way to read these tables is for the $SU(4)$ one has a vector with entries (J_1, J_2, J_3) and for $SO(4)$ one has (S_1, S_2). From the decoupling prescription, we see that the choices of coefficients will pick out the letters for the specific subsector by matching it with $\Delta_0 = J$. We illustrate by example: Take the vector $(0, 0, 1, 1, 0)$. This corresponds to $J = J_1 + J_2$ which is the classic $SU(2)$ sector. Once we have this, we go through all the letters and see which are the ones that satisfy $\Delta_0 = J_1 + J_2$. We only take use of the $SU(4)$, since there is no dependence on spin in this case. Looking at the gauge fields, we see that all the components equate to 0 while $\Delta_{0_F} = 2$. Thus no gauge fields are found. It is important to note that the conformal dimension is different for different letters due to the different representations they occupy. For the scalars one finds that Z and X satisfy our condition since $\Delta_{0_{(x,z)}} = 1$ and $J_1 + J_2 = 1$ for both. In the end, after going through all of this, these remain as the only letters contained in this subsector. This is the general fashion of how to determine the complete landscape. One could do a more thorough analysis by considering inequalities of the coefficients and thereby determine different numbers of fermions present in each case.

4.2 Subsectors and Decoupling Limits for $\mathcal{N} = 6$ Chern-Simons

Consider the letters in $\mathcal{N} = 6$ Chern-Simons. The procedure is the same, but the weights and Dynkin labels change. In the literature, different weights can be used, but they will amount to the same physics, even though there may be a difference in the values of the coefficients on the chemical potentials. Using data from [52], tables can be arranged.

	D_-	D_0	D_+
SO(3)	-1	0	1
SU(4)	(0,0,0)	(0,0,0)	(0,0,0)

Table 4.7: Weight of derivative operators in $SO(3)$ and $SU(4)$ representation.

	Y_1 $\psi^\dagger_{1\pm}$	Y_2,$\psi^\dagger_{2\pm}$	Y_3,$\psi^\dagger_{3\pm}$	Y_4,$\psi^\dagger_{4\pm}$
SU(4)	(1,0,0)	(-1,1,0)	(0,1,-1)	(0,0,-1)

Table 4.8: Weight of scalars in bifundamental representation.

	$Y^\dagger_1$,$\psi_{1\pm}$	$Y^\dagger_2$,$\psi_{2\pm}$	$Y^\dagger_3$,$\psi_{3\pm}$	$\psi_{4\pm}$,$Y^\dagger_4$
SU(4)	(-1,0,0)	(1,-1,0)	(0,1,-1)	(0,0,1)

Table 4.9: Weight of scalars in anti-bifundamental representation.

Lastly, the $SO(3)$ weights are 0 for all the scalars, and $\pm\frac{1}{2}$ for fermions. The supergroup $OSp(4|2)$ will be the main frame, that is the maximal subgroup of the full $OSp(6|4)$. From the geometry side of ABJM, a thorough analysis was done relating angular momentum generators to R-symmetry generators of $SU(4)$ through orbifolding of S^7. The inequality for operators takes the form

$$\Delta_0 \geq m_1 R_1 + m_2 R_2 + m_3 R_3 + m_4 S. \tag{4.7}$$

The Cartan generators manifest themselves as R_i for $SU(4)$ while S is the Cartan generator for $SO(3)$. This can alternatively be written in the language of the angular momenta as

$$\Delta_0 \geq n_1 J_1 + n_2 J_2 + n_3 J_3 + n_4 J_4 + n_5 S. \tag{4.8}$$

But this comes with the restriction $\sum_{i=1}^4 J_i = 0$. Using the same prescription as for SYM, it becomes a matter of systematically obtaining the subsectors. The problem can be solved in two ways, either by constructing a matrix with Dynkin labels as columns and weights as rows, after which diagonalizing the matrix then gives the BPS-vector $(m_1R_1, m_2R_2, m_3R_3, m_4S)$ that can be used to determine the letter content and spin group. Or one can reverse engineer by putting restrictions on coefficients and generators, this will also work. An example is determining all sectors with derivatives, which immediately gives $S = \pm 1$. We summarize the result in the table below, where BPS-vector, spin group and letter content is presented.

BPS-vector Letter Content	$(m_1R_1, m_2R_2, m_3R_3, m_4S)$	G_s
$(1/2, q, 1/2, 0)$	Vacuum	$Y_1, Y_4^\dagger$
$(1/2, 0, 1/2, 0)$	$SU(2) \times SU(2)$	$Y_1, Y_4^\dagger, Y_2, Y_3^\dagger$
$(1/2, 1, 1/2, 0)$	$SU(2)$	$Y_1, Y_4^\dagger, Y_2$
$(1/2, 0, 1/2, 1/2)$	$SU(1,1)$	$Y_1, Y_4^\dagger, \psi_{4+}$
$(1/2, 1, 1/2, 1)$	$SU(2\|1)$	$Y_1, Y_4^\dagger, Y_2, \psi_{4+}$
$(1/2, 1/2, 1/2, 1/2)$	$OSp(4\|2)$	$Y_1, Y_4^\dagger, Y_2, Y_3^\dagger, \psi_{4+}, \psi_+^{\dagger 1}, \psi_{3+}, \psi_+^{\dagger 2}, D_+$
$(1/2, 0, 1, 1/2)$	$OSp(2\|2)$	$Y_1, Y_4^\dagger, \psi_{4+}, \psi_+^{\dagger 1}, D_+$
	$SU(3\|2)$	$Y_1, Y_4^\dagger, Y_2, Y_3^\dagger, , \psi_{4+}, \psi_+^{\dagger 1}$

One can look at the type of operators appearing for different cases in the subsectors. An example is the $SU(2) \times SU(2)$ sector which has been studied extensively. From the content we found one can build single trace operators $\mathcal{O} = W^{j_1 j_2 \dots j_J}_{i_1 i_2 \dots i_J} \mathrm{Tr}(A_{i_1} B_{j_1} \dots A_{i_J} B_{j_J})$ [58] (here the A's and B's correspond to the Y's). The interpretation is that one has two decoupled ferromagnetic $XXX_{1/2}$ Heisenberg spin-chains [59] living on odd and even sites respectively:

$$\Delta - J = \lambda^2 \sum_{l=1}^{2J} (1 - P_{l,l+2}) = \lambda^2 \sum_{l=1}^{2J} (1 - P_{2l-1,2l+1} + 1 - P_{2l,2l+2}). \tag{4.9}$$

Other sectors that might be interesting could be the $SU(3)$ sector. Operators that saturate this condition on even sites is $Y_4^\dagger$ and on the odd it is $(Z^1, Z^2, Z^3) = (A_1.A_2, B_1^\dagger)$. Single trace operators will be of the form

$$\mathcal{O} = W_{a_1 a_2 \dots a_n} \mathrm{Tr}(Z^{a_1} B_2 \dots Z^{a_n} B_2), \quad a_j = 1, 2, 3. \tag{4.10}$$

Adding derivatives is also an option. This sets $n_5 \in \{-1, 1\}$ corresponding to $OSp(2|2)$ sector. This sector just extends the previous $SU(2) \times SU(2)$ to include superpartners of the scalars and the D_+. We will return to subsectors later.

5

Penrose Limits

In 1976 Sir Roger Penrose showed that every spacetime has a limit at which a neighbourhood of a null geodesic becomes a so-called pp-wave spacetime [60]. These are plane waves with parallel propagation. In the context of this project, it was shown [61] that eleven-dimensional supergravity admits a maximally supersymmetric Hpp-wave background, where the Hpp-waves describes solutions where the geometry is a Lorentzian symmetric one and the four-form field strengths are parallel and null. The weird thing about these solutions is that their transverse geometry is not asymptotically flat, but they have to be treated the way wave as flat space solutions and $AdS \times S$ solutions. However, the Hpp-waves can be shown to be obtained as Penrose limits of the maximally supersymmetric AdS solutions of M-theory (11D SUGRA) [62]. In this chapter we go through the standard formalism of pp-waves and look at how they can be obtained by Penrose limits.

5.1 pp-waves

Let the pair of M and g constitute a Lorentzian spacetime. If γ is a null geodesic that contains no conjugate points, it turns out that it is possible to introduce some local coordinates U, V, and Y^i so that the metric g takes the form [63]

$$ds^2 = dV \left(dU + \alpha dV + \sum_i \beta_i dY^i \right) + \sum_{i,j} C_{ij} dY^i dY^j. \tag{5.1}$$

Here, α, β, and C_{ij} are functions of the coordinates and C is a symmetric and positive matrix. Now, at the moment the determinant of C vanishes, the metric breaks down as this suggests that the number of conjugate points is non-zero [64]. The U-coordinate is the affine parameter along a congruence of null geodesics which are labelled by the other two coordinates; we see that γ is null when $V = Y^i = 0$. In supergravity theories we need to take into account other fields than only the metric, such as the dilaton field and field strengths. For a gauge invariant field strength $F_{p+1} = dA_p$ the gauge potentials transform as $A_p \to A_p + d\lambda_{p-1}$. We can choose a local gauge where

$$i\frac{\partial A}{\partial U} = 0, \tag{5.2}$$

and obtain similar conditions for interactive field strengths. We can now rescale the coordinates:

$$U = u, \quad V = \Omega^2 v, \quad Y^i = \Omega y^i, \tag{5.3}$$

with $\Omega \in \mathbb{R}^+$. If we act with this rescaling on the tensor fields of the theory we then get a family of fields g, Φ, and A_p which all depend on Ω. The coordinates and choice of gauge then makes sure that the following limits are well-defined:

$$\begin{aligned} ds^2 \to d\bar{s}^2 &= \lim_{\Omega\to 0} \Omega^{-2} g, \\ \Phi \to \bar{\Phi} &= \lim_{\Omega\to 0} \Phi, \\ A_p \to \bar{A}_p &= \lim_{\Omega\to 0} \Omega^{-p} A_p. \end{aligned} \tag{5.4}$$

This is the Penrose limit. The limiting fields now only depend on the u-coordinate being the affine parameter along the null geodesic. We get that the metric now takes the form

$$d\bar{s}^2 = dudv + \sum_{i,j} \bar{C}_{ij} dy^i dy^j. \tag{5.5}$$

The gauge forms $\bar{A}_p$ only have non-vanishing components in the transverse directions y^i, so their derivatives with respect to u or v vanishes. The metric in (5.5) is written in Rosen coordinates. There are both advantages and disadvantages by writing the pp-wave in Rosen coordinates. On the one hand, it is manifest what the killing vector fields/symmetries are, but on the other hand, one encounters spurious coordinate singularities. Historically this led to the mistaken belief in the past that there are no non-singular plane wave solutions of the non-linear Einstein equations, which turned out to be wrong. Therefore one can turn to other coordinates, and in this case we turn to Brinkmann coordinates.

The way to get to the Brinkmann coordinates is by assuming that there is a parallel null vector Z of the Lorentzian metric which implies $\nabla_\mu Z^\mu = 0$. This is the same as Z being a killing and gradient vector field:

$$\begin{aligned} \nabla_\mu Z_\nu + \nabla_\mu Z_\nu = 0, \\ \nabla_\mu Z_\nu - \nabla_\mu Z_\nu = 0. \end{aligned} \tag{5.6}$$

We can assume that since Z is non-zero everywhere, it can be set equal to $Z = \partial_v$. In terms of coordinates, this means that $Z_\mu = g_{\mu\nu}$. The fact that Z is null means that $Z_v = g_{vv} = 0$, and from the killing equation it can be seen that the metric is independent of the v-coordinate; $\partial_v g_{\mu\nu} = 0$. We can now change the previous condition to the following: $\nabla_\mu Z_\nu + \nabla_\mu Z_\nu = 0 \to \partial_\mu Z_\nu + \partial_\mu Z_\nu = 0$. Locally this means that there is a function $u = u(x^\mu)$ such that $Z_\mu = g_{v\mu} = \partial_\mu u$. There are now no other constraints, and changing x^μ to $\{u, v, x^a\}, a \in 1, ...d$ we find [65]

$$ds^2 = 2dudv + K(u, x^c)du^2 + 2A_a(u, x^c)dx^a du + g_{ab}(u, x^c)dx^a dx^b. \tag{5.7}$$

This metric is expressed in what is usually called the Brinkmann coordinates. There are still transformations left which leave the metric invariant; one could e.g. eliminate K and A_a in favor of g_{ab}. The special class of metrics where $g_{ab} = \delta_{ab}$ are called pp-waves. They are plane-fronted in the sense the wave fronts are planar at constant u. They are parallel rays in the sense that there exists a parallel null vector. We also note that plane waves are a special kind of pp-waves. Plane waves are pp-waves with $A_a = 0$ and K quadratic in x^a, thus their metric has the form

$$ds^2 = 2dudv + A_{ab}(u)x^a x^b du^2 + d\vec{x}^2. \tag{5.8}$$

This is a plane wave expressed in Brinkmann coordinates. The reason we are interested in these plane waves is as mentioned due to the fact that they are emerging from any spacetime in the Penrose limit. Besides this, we see very little redundancy left in terms of data and coordinate transformations that leave the metric invariant, since we have acquired a symmetric matrix-valued function $A_{ab}(u)$ that contains all of this. This indeed contrasts the Rosen coordinates making the Brinkman ones more applicable in computations.

5.2 Geodesics and String Mode Plane Waves

Starting from the plane wave metric, one can solve the geodesic equation and find $x^\mu(\tau)$ to get a sense of the trajectory for particles in this particular spacetime. However, this involves computing Christoffel symbols which can be quite tedious as everyone who has ever taken a GR course can testify to. Instead, we can try to see what might be provided by the Euler-Lagrange equation, supplemented by a constraint:

$$\mathcal{L} = \frac{1}{2}\bar{g}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \dot{u}\dot{v} + A_{ab}(u)x^a x^b \dot{u}^2 + \frac{1}{2}\dot{\vec{x}}^2, \quad 2\mathcal{L} = \epsilon, \epsilon \in \{0, 1\}. \tag{5.9}$$

The constraint is for massless and massive particles taking either the values 0 or 1. Next, we define the light-cone momentum, observing that nothing depends on v, such that $p_v = \frac{\partial \mathcal{L}}{\partial \dot{v}} = \dot{u}$ is conserved. In the case where the geodesic are not straight lines meaning $p_v \neq 0$ we choose light-cone gauge $u = p_v\tau$. This makes us translate the geodesic equation for the transverse coordinates into the Euler-Lagange equations:

$$\ddot{x}^a(\tau) = A_{ab}(p_v\tau)x^b(\tau)p_v^2 = -\omega_{ab}^2(\tau)x^b(\tau). \tag{5.10}$$

We recognize obviously as the equations of motion for a harmonic oscillator with a (possibly time-dependent) frequency matrix ω_{ab}. Now, the constraint for null geodesics,

$$p_v\dot{v}(\tau) + A_{ab}(p_v\tau)x^a(\tau)x^b(\tau)p_v^2 + \frac{1}{2}\dot{x}^a(\tau)\dot{x}^a(\tau) = 0, \tag{5.11}$$

implies the equation of motion for v. By multiplying the harmonic oscillator equation by $x^a(\tau)$ and inserting it into the constraint we find by integration that

$$p_v v(\tau) = p_v v_0 - \frac{1}{2}x^a(\tau)\dot{x}^a(\tau). \tag{5.12}$$

There exists a specific solution $x^\mu(\tau)$ of the null geodesic equation with the properties $u = p_v\tau, v = v_0, x^a = 0$; this has vanishing Christoffel symbols in Brinkmann coordinates. This motivates us to extend the formalism to strings, where we might hope that splitting the coordinates as we have done in Brinkmann can help us quantize easier when considering the equations of motion for the Polyakov action. For a curved background described by $g_{\mu\nu}$ we can write the usual action for a string in conformal gauge as

$$S(X, h) = \frac{1}{2\pi}\int d^2z g_{\mu\nu}(X)\eta^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x^\nu. \tag{5.13}$$

To find the equations of motion of the embedding coordinates $X^\mu(\tau, \sigma)$, one gets

$$(\partial_\tau^2 - \partial_\sigma^2)X^\mu(\tau, \sigma) = -\Gamma^\mu_{\nu\lambda}(X)(\partial_\tau X^\nu \partial_\tau X^\lambda - \partial_\sigma X^\nu \partial_\sigma X^\lambda). \tag{5.14}$$

These equations need to be supplemented by the equations of motion for the two-dimensional worldsheet metric, that is by the condition that $T_{\alpha\beta} = \frac{\delta S}{\delta h^{\alpha\beta}} = 0$. Since the action is conformally invariant, $T_{\alpha\beta}$ enjoys the tracesless condition

$$T^{\alpha}_{\alpha} = h^{\alpha\beta} T_{\alpha\beta} = 0, \tag{5.15}$$

hence there are only two independent conditions:

$$g_{\mu\nu}(\partial_\tau X^\mu \partial_\tau X^\nu - \partial_\sigma X^\mu \partial_\sigma X^\nu) = 0, \tag{5.16}$$

$$g_{\mu\nu}\partial_\tau X^\mu \partial_\sigma X^\nu = 0. \tag{5.17}$$

One would think that these equations are not the ideal starting point since they exhibit non-linear coupled differential equations for the embedding. But with our new coordinates (U, V, X^a), this simplifies enormously. For U we get

$$(\partial_\tau^2 - \partial_\sigma^2) U(\tau, \sigma) = 0, \tag{5.18}$$

We can here again choose the light-cone gauge $U(\tau, \sigma) = p_v \tau$. For the transverse coordinates we simply get the linear equations

$$(\partial_\tau^2 - \partial_\sigma^2) X^\mu(\tau, \sigma) = A_{ab}(p_v \tau) x^b(\tau) p_v^2. \tag{5.19}$$

Expanding in Fourier modes for the transverse embedding $X^a(\tau, \sigma)$,

$$X^a(\tau, \sigma) = \sum_n x_n^a(\tau) e^{in\sigma}, \tag{5.20}$$

leads one to obtain

$$\ddot{X}^a_n = (p_v^2 A_{ab}(p_v \tau) - \delta_{ab} n^2) X^b_n(\tau) \tag{5.21}$$

One can also find the equation for v, though this is a bit more complicated and we leave it out here. The point was to show explicitly that all modes expand in terms of a complete set of solutions to the classical equations of motion. Then one can take these modes as a starting point for the canonical quantization of strings in the light-cone gauge. Lastly we quickly mention curvature of space-time.

We saw for Brinkmann coordinates that all data about our spacetime was contained in the matrix-valued function $A_{ab}(u)$. If we relate this to the Riemann tensor, we get one non-vanishing component. The null structure of the metric also provides that there is one non-trivial term for the Ricci tensor. One also finds that Ricci scalar is zero, and additionally that there is only one non-trivial term for the Einstein tensor:

$$R_{uaub} = A_{ab} \quad R_{uu} = -\delta^{ab} A_{ab} = -\mathrm{Tr} A, \quad R = 0, \quad G_{\mu\nu} = R_{uu}. \tag{5.22}$$

5.3 Penrose Limits of AdS Spacetimes

Given a metric $g_{\mu\nu}$ or a line element $ds^2 = g_{\mu\nu} x^\mu x^\nu$, we can consider the Penrose Limit for a choice of null geodesic γ that amounts to a plane wave metric. We first write the coordinates adapted to γ as

$$ds^2 = 2dUdV + a(U, V, Y^k) dV^2 + 2b_i(U, V, Y^k) dV dY^i + g_{ij}(U, V, Y^k) dY^i dY^j. \tag{5.23}$$

We then perform a set of coordinate transformations with rescaling $(U, V, Y^k) = (u, \lambda^2\tilde{v}, \lambda y_k)$. The Penrose Limit of the metric is then

$$d\bar{s}^2 = \lim_{\lambda\to 0} \lambda^{-2} ds^2_{\gamma,\lambda} = 2dud\tilde{v} + g_{ij}(U)dy^i dy^j. \tag{5.24}$$

To narrow in on the the problem related to spacetimes in AdS we consider different pp-waves emerging from two different types of AdS spacetimes.

5.3.1 pp-waves of $AdS_5 \times S^5$

Let us first take a look at the case of $AdS_5 \times S^5$. The metric can be written as [66]

$$ds^2 = R^2\big[-dt^2\cosh^2\rho + d\rho^2 + \sinh^2\rho d\Omega_3^2 + d\psi^2\cos^2\theta + d\theta^2 + \sin^2\theta d\Omega_3'^2\big]. \tag{5.25}$$

We look at a particle moving along the ψ direction and sitting at $\rho = 0$ and $\theta = 0$. We will focus on the geometry near this trajectory. We can do this systematically by introducing light-cone coordinates $\tilde{x}^\pm = (t \pm \psi)/2$ and then performing the rescaling of the coordinates:

$$x^+ = \tilde{x}^+, \quad x^- = R^2\tilde{x}^-, \quad \rho = \frac{r}{R^2}, \quad \theta = \frac{y}{R^2}. \tag{5.26}$$

Taking the $R \to \infty$ limit is exactly the Penrose limit. Expanding around the parameters in the rescaled variables yields

$$ds^2 = R^2\big[-dt^2(1-\rho^2) + dr^2 + r^2 d\Omega_3^2 + d\psi^2(1-\theta^2) + dy^2 + y^2 d\Omega_3'^2\big]. \tag{5.27}$$

By the new coordinate $\tilde{x}^\pm$ we see that $-dt^2 + d\psi^2 = -4d\tilde{x}^+ d\tilde{x}^-$. The squared terms will contribute through the light-cone coordinates which can easily be seen by

$$dt = dx^+ - \frac{dx^-}{R^2}, \quad d\psi = dx^+ + \frac{dx^-}{R^2}. \tag{5.28}$$

Inserting this and keeping to the order of $\mathcal{O}(\frac{1}{R^2})$ exactly gives

$$ds^2 = R^2\big[-4d\tilde{x}^+ d\tilde{x}^- - (y^2 + r^2)(dx^+)^2 + d\vec{r}^2 + d\vec{y}^2\big]. \tag{5.29}$$

This is following the geodesic around $\rho = 0$ and $\theta = 0$ where y and r parametrize points on R^4.

5.3.2 pp-waves of $AdS_4 \times \mathbb{CP}^3$

Here we look at a case we will return to later in the project from a different perspective. We consider the metric of $AdS_4 \times \mathbb{CP}^3$ [47],

$$ds^2 = \frac{R^2}{4}\left(-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\hat{\Omega}_2^2\right) + R^2 ds^2_{\mathbb{CP}^3}, \tag{5.30}$$

where

$$ds^2_{\mathbb{CP}^3} = d\theta^2 + \frac{\cos^2\theta}{4}d\Omega_2^2 + \frac{\sin^2\theta}{4}d\Omega_2'^2 + 4\cos^2\theta\sin^2\theta(d\delta + \omega)^2, \tag{5.31}$$

with

$$\omega = \frac{1}{4}\sin\theta_1 d\varphi_1 + \frac{1}{4}\sin\theta_2 d\varphi_2. \tag{5.32}$$

We now introduce the coordinates

$$t' = t, \quad \chi = \delta - \frac{1}{2}t, \tag{5.33}$$

which will be a common thing we will do in later computations as well. In these coordinates, the metric (5.30) takes the form

$$\begin{aligned} ds^2 = &\frac{R^2}{4}dt'^2(1 - 4\cos^2\theta\sin^2\theta + \sinh^2\rho) + \frac{R^2}{4}(d\rho^2 + \sinh^2\rho d\hat{\Omega}_2^2) \\ &R^2\left[d\theta^2 + \frac{\cos^2\theta}{4}d\Omega_2^2 + \frac{\sin^2\theta}{4}d\Omega_2'^2 + 4\cos^2\theta\sin^2\theta(dt' + d\chi + \omega)(d\chi + \omega)\right]. \end{aligned} \tag{5.34}$$

Now, introducing the rescaled coordinates

$$v = R^2\chi, \quad u_4 = R(\theta - \frac{\pi}{4}), \quad r = \frac{R}{2}\rho, \quad x_a = R\phi_a, \quad y_a = R\theta_a, \quad a = 1, 2. \tag{5.35}$$

The Penrose limit is again realized when the $R \to \infty$ limit is performed which exactly corresponds to zooming in on the null geodesic. The metric will reduce down to a type IIA pp-wave background when the terms have been expanded around the respective coordinates. This reduces the metric to

$$\begin{aligned} ds^2 = &\frac{R^2}{4}dt'^2(4u_4 + 4r^2)\frac{1}{R^2} + \frac{R^2}{4}(4dr^2 + 4r^2d\hat{\Omega}_2^2)\frac{1}{R^2} \\ &R^2[du_4^2 + dy_1^2 + dx_1^2 + dy_2^2 + dx_2^2 + 2dt'(y_1dx_1 + y_2dx_2) + dt'dv]\frac{1}{R^2}. \end{aligned} \tag{5.36}$$

The full calculation is a little tedious, but essentially it boils down to keeping terms that have a $\frac{1}{R^2}$ dependence such that they do not vanish in the limit. If we define $r^2 = \sum_{i=1}^{3} u_i^2$ and $dr^2 + r^2 d\hat{\Omega}_2^2 = \sum_{i=1}^{3} du_i^2$ as done in [47], the metric can compactly be written as

$$ds^2 = dt'dv + \sum_{i=1}^{4}(du_i^2 - u_i^2dt'^2) + \frac{1}{8}\sum_{a=1}^{2}(dx_a^2 + dy_a^2 + 2dt'y_adx_a). \tag{5.37}$$

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5.3.3 pp-waves of $AdS_p \times S^q$

One can also look at general AdS spacetimes. In general, the radius of AdS_p is not necessarily going to be the same as the radius of the sphere. For $AdS_3 \times S^3$ though the radii are equal and the computation is the same as for $AdS_5 \times S^5$. In the context of ABJM, the near horizon geometry of $M2$- and $M5$-(and $D3$-)brane solutions take the form $AdS_{p+2} \times S^{D-p-2}$. For the three values of p this means we have

$$\begin{aligned} p = 2 &\implies AdS_4 \times S^7 \quad (D = 11), \\ p = 3 &\implies AdS_5 \times S^5 \quad (D = 10), \\ p = 5 &\implies AdS_7 \times S^4 \quad (D = 11). \end{aligned} \tag{5.38}$$

It turns out the Penrose limit for $AdS_4 \times S^7$ is the same for the limit on $AdS_7 \times S^4$. To see this, consider the general metric of $AdS_{p+2} \times S^{D-p-2}$ in the form [62]

$$ds^2/R^2 = \rho^2\left[-d\tau^2 + \sin^2\tau\left(\frac{dr^2}{1 + r^2} + r^2d\Omega_p^2\right)\right] + d\phi^2 + \sin^2\phi d\Omega_{D-p-3}^2, \tag{5.39}$$

where ρ is the ratio between the radius of the AdS spacetime and the radius of the sphere denoted R. The last two terms in (5.39) correspond to the round metric on the sphere. By changing coordinates to

$$u = \psi + \rho\tau, \quad v = \psi - \rho\tau, \tag{5.40}$$

the metric becomes

$$ds^2/R^2 = dudv + \rho^2 \sin^2\left(\frac{u-v}{2\rho}\right)\left[\frac{dr^2}{1+r^2} + r^2 d\Omega_p^2\right] + \sin^2\left(\frac{u-v}{2\rho}\right) d\Omega_{D-p-3}^2. \tag{5.41}$$

Practically speaking, taking the Penrose limit corresponds to dropping the dependence on all coordinates other than u, and when doing it one finds

$$d\bar{s}^2/R^2 = dudv + \rho^2 \sin^2(u/2\rho) ds^2(\mathbb{E}^{p+1}) + \sin^2(u/2) ds^2(\mathbb{E}^{D-p-3}), \tag{5.42}$$

where $\mathbb{E}$ is Euclidean space. We see that for the $AdS_4 \times S^7$ case we have $p = 2$, and the metric becomes

$$p = 2: \quad d\bar{s}^2/R^2 = dudv + \sin^2(u/2)[ds^2(\mathbb{E}^3) + ds^2(\mathbb{E}^6)]. \tag{5.43}$$

For the $AdS_7 \times S^4$ case we have $p = 5$ and the metric becomes

$$p = 5: \quad d\bar{s}^2/R^2 = dudv + \sin^2(u/2)[ds^2(\mathbb{E}^6) + ds^2(\mathbb{E}^3)]. \tag{5.44}$$

See that (5.43) and (5.44) are the same. For more on this see [62, 61, 67, 68].

6

Spin Matrix Theory

So far we have been discussing the gauge/gravity duality in the Maldacena case, where one takes two opposite limits and gets that massive strings curving the AdS space background is conjectured to be compatible with $\mathcal{N} = 4$ SYM at two different limits of the 't Hooft coupling, usually taken in the planar limit $N \to \infty$. Usually the link between this duality is interpreted in terms of integrable spin-chains, so the integrability is the interpolation from going from one side of the duality to the other. However, in the case when $N < \infty$ we need to start revise our strategy. To interpolate between strong and weak 't Hooft coupling at finite N, we must consider new possibilities. And the motivations for doing this are plenty. Non-perturbative effects on black holes [69], emergence of D-branes as giant gravitons [70] etc. are just a few examples. Hence the idea is to consider non-relativistic limits of the AdS/CFT correspondence in the grand canonical ensemble, such that it corresponds to the approach of critical temperatures $T = 0$. Here we will let $\vec{\Omega}$ denote the respective chemical potentials conjugate to the global symmetry charges for the certain theory. If we consider for starters $\mathcal{N} = 4$ SYM, we can take limit of the form

$$(T, \vec{\Omega}) \to (0, \vec{\Omega}^{(c)}), \quad \lambda = 0, \quad \text{with} \quad \frac{\lambda}{T}, \frac{\vec{\Omega} - \vec{\Omega}^{(c)}}{T} \quad \text{fixed.} \tag{6.1}$$

This will give a simple spin-chain with nearest-neighbour interaction apparently, based on the fact that we build our Hilbert space out of harmonic oscillators giving us the non-relativistic quantum mechanical theory that we want. Having introduced this, we go through the construction and some consequences. The idea is to take the case for $\mathcal{N} = 4$ SYM and see if the same procedure can be applied to $\mathcal{N} = 6$ Chern-Simons theory.

6.1 Definitions and Construction

We build the Spin Matrix theory (SMT) [71] on representation R_s of a semi-simple graded Lie group G_s, also called the spin group. Furthermore, we also have matrix indices that are in the adjoint representation of R_m of the $U(N)$ group (could also be generalized to others). If we first look at the bosonic part of the Hilbert space, it is natural to formulate it in terms of raising and lowering operators $(a_s^\dagger)^i_j$. Here, $s \in R_s$ and $i, j \in R_m$ where $i, j = (1, ..., N)$ and i labels the fundamental while i labels the anti-fundamental represenations of $U(N)$. To complete the construction we need a vacuum and a lowering operator that satisfies $(a_s)^j_i |0\rangle = 0$ and commutation relation $[(a^s)^j_i, (a_s^\dagger)^k_l] = \delta^s_s \delta^k_i \delta^l_j$. Thus for each s and i, j we can have a new harmonic oscillator, making it natural to construct a Hilbert space as the symmetric product between the two representations:

$$\bar{\mathcal{H}} = \sum_{L=1}^{\infty} \text{sym}[(R_s \otimes R_m)]^L. \tag{6.2}$$

The basis for $\bar{\mathcal{H}}$ can be written as

$$\prod_{k=1}^{L}(a^\dagger_{s_k})^{i_k}_{j_k}|0\rangle\,. \tag{6.3}$$

The Hilbert space $\mathcal{H}$ that we will use for SMT will be a linear subspace of $\bar{\mathcal{H}}$ with a singlet condition on R_m for a state $|\phi\rangle$

$$\Phi^i_j|\phi\rangle = 0, \quad \Phi^i_j \equiv \sum_{s\in R_s}\sum_{k=1}^{N}[(a^\dagger_s)^i_k(a^s)^k_j - (a^\dagger_s)^k_j(a^s)^i_k]. \tag{6.4}$$

Apparently $\mathcal{H}$ is spanned by states of the form

$$\sum_{i_1,i_2,\ldots,i_L=1}^{N}\prod_{k=1}^{L}(a^\dagger_{s_k})^{i_k}_{i_{\sigma(k)}}|0\rangle\,. \tag{6.5}$$

Here, $\sigma \in S(L)$ are elements of the permutation group of L elements. This can also be written in terms of product of traces

$$\mathrm{Tr}(a^\dagger_{s1}a^\dagger_{s2}...a^\dagger_{sl})\mathrm{Tr}(a^\dagger_{sl+1}...)\mathrm{Tr}(a^\dagger_{sk+1},...a^\dagger_{L})|0\rangle\,, \quad L = 1,2,... \tag{6.6}$$

Note that the traces run over the R_m indices. The connection to previous equations is that the individual cycles of the permutation elements correspond to single traces, and one can establish a linear relation in the $L > N$ case. Last thing to note is that we can extend the bosonic language to also include fermionic excitations. We simply split $R_s = B_s \otimes F_s$ of the spin group represenation, so the rules of the framework we have established is relevant for $s \in B_s$. The only difference now is that for $s \in F_s$ we have

$$(a_s)^j_i|0\rangle = 0, \quad \{(a^s)^j_i,(a^\dagger)^k_l\} = \delta^s_s\delta^k_i\delta^l_j, \tag{6.7}$$

asserting anti-commutation relations. Note that by doing this split, we can now work in the framework of Lie supergroups of the type $SU(p,q|r)$ with $p+q$ and r non-zero. The generators of the respective $su(p,q)$ and $su(r)$ algebras are bosonic, while the remaining part will be fermionic. Now we focus our attention to what kind of interactions and thus Hamiltonian constructions we can do, and then in the end consider the content of the partition function

6.2 Hamiltonian of Spin Matrix Theory

We now walk through the construction of the Hamiltonian. One considers a 2 to 2 creation and annihilation of states, and further demands that such interactions are commuting with generators G_s, and that spin and matrix part separate such that the Hamiltonian will look like

$$H_{\mathrm{int}} = \frac{1}{N}U^{s'r'}_{sr}\sum_{\sigma\in S(4)}T_\sigma(a^\dagger_{s\prime})^{i_{\sigma(1)}}_{i_3}(a^\dagger_{r\prime})^{i_{\sigma(2)}}_{i_4}(a^s)^{i_{\sigma(3)}}_{i_1}(a^r)^{i_{\sigma(4)}}_{i_2}. \tag{6.8}$$

Here, $T_\sigma, \sigma \in S(4)$ are coefficients, and there are implicit sums over $r, s, r', s', i_1, i_2, i_3, i_4$. The Hamiltonian preserves singlet conditions such that we know it is within our desired Hilbert space. For the spin part, U is a linear operator, and taking an element from $U : R_s \otimes R_s \to R_s \otimes R_s$, which can be expanded into a sum of irreducible representations

labelled by $\mathcal{J}$, gives $R_s \otimes R_s = \sum_{\mathcal{J}} V_{\mathcal{J}}$. Imposing that H_{int} commutes with G_s makes U proportional to $\mathbf{I}$ in all $\mathcal{V}_j$.
In general, a diagonal term is added, giving the length of a spin-chain state with the same properties as U:

$$L = \sum_s \text{Tr}(a_s^\dagger a^s). \tag{6.9}$$

Finally, Cartan generators are also included (K_p). The Hamiltonian is then

$$H = gH_{\text{int}} + \mu_0 L - \sum_p \mu_p K_p. \tag{6.10}$$

But we notice that the structure of the partition function is invariant under a rescaling of a parameter such that $T, g, \mu_0, \mu_0 \to \alpha T, \alpha g, \alpha\mu_0, \alpha\mu_0$, hence we can remove a parameter. Different choices can be made, but here we chose $\mu_0 = 1$. In this way one connects high (low) temperature to long (short) average lengths of the states. At large N, non-planar effects set in, but for $T \ll 1$ the theory is effectively planar. The opposite holds for $T \gg 1$. Thus the modified version of the Hamiltonian is

$$H = gH_{\text{int}} + L - \sum_p \mu_p K_p. \tag{6.11}$$

Considering μ_p as chemical potentials gives an interpretation in the language of statistical mechanics to construct a partition function

$$Z(\beta, \mu_p) = \text{Tr}(e^{-\beta H}) = \text{Tr}(e^{-\beta(gH_{\text{int}} + L - \sum_p \mu_p K_p)}), \tag{6.12}$$

where the trace is over the Hilbert space $\mathcal{H}$.

6.3 Spin Matrix Theory for $\mathcal{N} = 4$ SYM

The framework developed has a gateway to $\mathcal{N} = 4$ SYM when considered at close to zero temperature critical points in the grand canonical ensemble. In this ensemble, a partition function can be constructed with chemical potentials present, given by the bosonic subalgebra for the field theory. It was previously established what generators are present given by the $SU(4)$ R-symmetry and $SO(4,2)$ conformal group. The partition function is $Z(\beta, \vec{\Omega}) = \text{Tr}(e^{-\beta D + \beta \vec{\Omega} \cdot \vec{J}})$, with $T = \frac{1}{\beta}$ and the dot product for chemical potentials given by a weight $\vec{\Omega} = (\omega_1, \omega_2, \Omega_1, \Omega_2, \Omega_3)$ dotted with the generators $\vec{J} = (S_1, S_2, J_1, J_2, J_3)$. This gives normally $\vec{\Omega} \cdot \vec{J} = \omega_1 S_1 + \omega_2 S_2 + \Omega_1 R_1 + \Omega_2 R_2 + \Omega_3 R_3$. The theory is considered on $\mathbb{R} \times S^3$ due to isometries of the geometry and groups as well. As in the case of renormalizing the conformal two-point function, we expand the dilatation operator in powers of the 't Hooft coupling λ such that $D = D_0 + \delta D$. Doing this only up to one-loop order gives $\delta D = \lambda D_2 + \mathcal{O}(\lambda^{3/2})$. To obtain one-loop corrections, an explicit form of D_2 is then needed, which luckily has been studied intensively for $PSU(2,2|4)$ [72]. Acting with D_2 hits two letters at a time in the singleton representation $\mathcal{A}$ of $PSU(2,2|4)$. The product of two singletons are the irreducible represenations $\mathcal{V}_j$, labelled uniquely by the quadratic Casimir. This gives

$$\mathcal{A} \otimes \mathcal{A} = \sum_{j=0}^{\infty} \mathcal{V}_j. \tag{6.13}$$

Having asserted this, D_2 can then be found as [73]

$$D_2 = -\frac{1}{8\pi^2 N} \sum_{j=0}^{\infty} h(j)(p_j)^{AB}_{CD} : \mathrm{Tr}[W_A, \partial_{W_c}][W_B, \partial_{W_D}] : . \tag{6.14}$$

Here, $h(j) = \sum_{k=1}^{j} \frac{1}{k}, h(0) = 0$ are the harmonic numbers, P_j is the projection operator from the product of singletons to the irreducible representations, and W_A, $A \in \mathcal{A}$ represent all letters in SYM while maintaining normal ordering [74]. Amazingly, one can identify components between SMT and SYM. Raising operators become letters $a_s^\dagger \leftrightarrow W_s$, D_2 can be interchanged with H_{int} if $\mathcal{J} \leftrightarrow j$, and $V_{\mathcal{J}} \leftrightarrow \mathcal{V}_j$ such that $C_j = \frac{1}{8\pi^2} h(j), j = 0, 1, 2.$ The caveat to this story though is that it only holds in a non-relativistic limit when $\lambda = 0$. This restricts us to subsectors of the space of operators, simplifying matters. Using this, SMT has found many applications in non-relativistic string theory, which we demonstrate later on.

6.4 Near BPS-limit for Subsectors and Zero-Temperature Critical Points

We now briefly consider the essential features of near BPS-limits. From the definition of the partition function it becomes apparent that confinement/deconfinement might happen considering certain bounds, such that the system undergoes a phase transition. The zero-temperature critical points is defined as a continuation of a submanifold of phase transitions to zero temperature, meaning $(T, \vec{\Omega}) \to (0, \vec{\Omega}^{(c)})$. The critical points exactly correspond to choices of weight-vectors for Cartan generators when obtaining subsectors for a theory. By specific choices, the BPS-bound considered is given by $D \geq \vec{\Omega}^{(c)} \cdot \vec{J}$ for all operators while there should still be some that saturate the bound. This will be the most crucial feature. The discussion of SMT is long and can be extended to find SMT theories for subsectors by appropriate use of the translation between D_2, H_{int} and the respective representations. Particularly interesting was the connection to the $XXX_{1/2}$ ferromagnetic spin chain in the planar limit that was found from $SU(2)$ SMT theory [71]. By raising the temperature, one encounters a singularity in $Z(\beta, \vec{\Omega})$ at $T_H(g)$. This is known as the Hagedorn temperature [75]. This had already been analyzed for both ends of the coupling regimes in earlier works. The resemblance between this and AdS/CFT is evident in the planar limit connecting to the integrable spin-chain interpretation. With these results already on the table, we can become hopeful and want to see if this can be continued for $\mathcal{N} = 6$ Chern-Simons theories.

6.5 Spin Matrix Theory for $\mathcal{N} = 6$ Chern-Simons

We proceed as for SYM and list the immediate differences. Constructing the grand canonical partition function will be identical up to differences in the bosonic subalgebra. The same R-symmetry is present, but the conformal theory goes down a dimension to $SO(3, 2)$. The weights and generators are shortened to $\vec{\Omega} = (\omega, \Omega_1, \Omega_2, \Omega_3, \Omega_4)$ times the generators $\vec{J} = (S_1, J_1, J_2, J_3, J_4)$. This is of course $\vec{\Omega} \cdot \vec{J} = \omega S + \Omega_1 J_1 + \Omega_2 J_2 + \Omega_3 J_3 + \Omega_4 J_4$. One can reformulate this in terms of R-generators, but due to orbifolding, we have a translation between J_i and R_i by a linear set of equations. This theory is now considered on $\mathbb{R} \times S^2$, due to one less degree of freedom of S. The major differences occur when we consider loop-order of D, since the first contribution D_2 comes at two-loop. Another major difference occurs when translating the singleton representations of the theory. The global symmetry is now $OSp(6|4)$ which tied to $\mathcal{N}=6$ CS has both a fundamental and anti-fundamental representation $(\mathbf{N},\bar{\mathbf{N}})$ and $(\bar{\mathbf{N}},\mathbf{N})$. This will all in all give three different types of ways D_2 can act on two letters (or modules) at a time. Following [76], the tensor product of a conjugate pair of modules $\mathcal{V}_\phi$ and $\mathcal{V}_{\bar{\phi}}$ has one highest-weight state for each nonnegative integer spin j. Similarly, a like pair of modules has one highest-weight state with spin $(j-1/2)$ for each nonnegative integer j. This gives the combinations

$$\mathcal{V}_\phi \otimes \mathcal{V}_{\bar{\phi}} = \sum_{j=0}^{\infty} \mathcal{V}_j, \quad \mathcal{V}_\phi \otimes \mathcal{V}_\phi = \sum_{j=0}^{\infty} \mathcal{V}_{j-1/2}, \quad \mathcal{V}_{\bar{\phi}} \otimes \mathcal{V}_{\bar{\phi}} = \sum_{j=0}^{\infty} \mathcal{V}_{j-1/2}. \tag{6.15}$$

The structure of the tensor products both accounts for nearest and next to nearest neighbour interactions since the vector space is composed as $(\mathcal{V}\otimes\bar{\mathcal{V}})^L$. The irreducible representation is again labelled uniquely by the quadratic Casimir which in $OSp(6|4)$ takes the form

$$J^2 = \frac{1}{8}([Q_{ij,\alpha}, S^{ij,\alpha}] - 2R^i_j R^j_i + 2M^\alpha_\beta M^\beta_\alpha + 4D^2 - \{P_{\alpha\beta}, K^{\alpha\beta}\}). \tag{6.16}$$

Acting with J^2 on highest weight states (HWS), this reduces to an expression in terms of Dynkin labels

$$\begin{aligned} J^2 &= \frac{1}{2}\left(D(D+3) + s(s+2) + 3J^1_1 + 2J^2_2 + J^3_3 + \frac{1}{2}\sum_{i=1}^{4}(J^i_i)^2\right) = \frac{1}{2}\Big(D(D+3) \\ &+ s(s+2) + \frac{1}{4}q_1(q_1+2) + \frac{1}{4}q_2(q_2+2) + \frac{1}{8}(2p+q_1+q_2)^2 - (2p+q_1+q_2)\Big). \end{aligned} \tag{6.17}$$

Here D is the dimension and s is the Lorentz spin. The first expression uses eigenvalues of all diagonal entries of the traceless matrix of R-symmetry generators, while the second uses the standard $SU(4)$ Dynkin labels

$$q_1 = J^2_2 - J^1_1, \quad q_2 = J^3_3 - J^2_2, \quad q_3 = J^4_4 - J^3_3, \tag{6.18}$$

which satisfy the relation $j(j+1) = J^2$. With this in mind, we can write the full $OSp(6|4)$ two-loop dilatation operator as

$$\begin{aligned} D_2 = \sum_{i=0}^{2L}\Bigg(&2\log 2 + \sum_{j=0}^{\infty} h(j)\mathcal{P}^{(j)}_{i,i+1} \\ &+ \sum_{j_1,j_2,j_3=0}^{\infty} (-1)^{j_1+j_3}\frac{1}{2}h(j_2-1/2)\left(\mathcal{P}^{(j_1)}_{i,i+1}\mathcal{P}^{(j_2-1/2)}_{i,i+2}\mathcal{P}^{(j_3)}_{i,i+1} + \mathcal{P}^{(j_1)}_{i+1,i+2}\mathcal{P}^{(j_2-1/2)}_{i,i+2}\mathcal{P}^{(j_3)}_{i+1,i+2}\right)\Bigg). \end{aligned} \tag{6.19}$$

We see to some extent the same structure as in SYM, however this time the structure seems to be more rich. $h(j)$ is still the harmonic numbers, and the projectors $\mathcal{P}$ come from one of the tensor products combinations. But the same type of one-to-one mapping is not as trivial. The terms in D_2 can be summarized as nearest and next to nearest types of interactions, thus the coefficient C_j has to be split up into two pieces:

$$C_j = C_0 + C^{\text{Near}}_j + C^{\text{Next Near}}_j = (2\log 2 + h(j)) + (-1)^{j_1+j_3}\frac{1}{2}h(j_2-1/2). \tag{6.20}$$

This seems to be the only natural way of describing the coefficients of the interactions in terms of SMT language.

7

Spin Matrix Theory String Backgrounds and Penrose Limits

As we saw in chapter 6, Spin Matrix theory (SMT) provides a way for one to evaluate near BPS-bounds in the AdS/CFT correspondence. It has already been established for SYM with super Lie algebra $PSU(2|2,4)$ that in certain BPS-bounds one can establish connections to spin-chains as an example. SMT is a theory that is described by a Hamiltonian consisting of harmonic oscillator operators. They transform in both the adjoint representation of SU(N) and also in particular spin subgroups G_s of $PSU(2|2,4)$ that is determined by the choice of Cartan generators. Apparently it appears that SMT can take the form of a non-relativistic string theory with a non-relativistic target space described by a $U(1)$ Galilean geometry. One direct way to observe it is to look at magnon dispersions [77] which exhibit non-relativistic features in the SMT limit. The starting point in the string theory side starts from the torsional Newton-Cartan (TNC) string which led to the SMT string which will be considered. In this chapter we consider this non-relativistic approach and establish a manifold that will be useful when going to BPS-bounds to describe the apparent emerging $U(1)$ geometry. Continuing on we look at the already known $\mathcal{N} = 4$ SYM and review some cases and calculations, and then turn to $\mathcal{N} = 6$ Chern-Simons theory and use the needed geometry to get backgrounds for ABJM theory. Here we consider $OSp(4|6)$ and look at the spin subgroups in certain limits which we will use to parametrize the $AdS_4 \times S^7/\mathbb{Z}_k$ which we reduce to $AdS_4 \times \mathbb{CP}^3$. In these instances we will perhaps find non-trivial background geometries as for $\mathcal{N} = 4$ SYM.

7.1 Brief review of TNC strings and BPS-bounds in SMT limit

The starting point is to consider a relativistic string that couples to a non-relativistic TNC geometry. To this mean it will be convenient to consider a $(d+1)$-dimensional Lorentzian geometry with null isometry ∂_u as

$$ds^2 = 2\tau_\mu dx^\mu(du - m_\mu dx^\mu) + h_{\mu\nu}dx^\mu dx^\nu. \tag{7.1}$$

By our null reduction along u, we see an emerging TNC geometry characterized by a clock one-form τ_μ, a symmetric tensor $h_{\mu\nu}$ of rank $(d-1)$ and the $U(1)$ connection m_μ. Without going into details, one can write up gauge transformations for the TNC data which makes the decomposition non-unique. For further elaboration see [78]. The important thing to note however is that this will make the Galilean boost and $U(1)$ transformations visible. The trick that was considered was to make the constant momentum P_u off-shell by exchanging a single winding mode in a direction η dual to u. But the question is now how to ensure that when we pick a specific BPS-bound that u will be null on the background geometry. If we look at the SMT limit we establish the following and introduce new coordinates that will ensure that we face no trouble. Consider the BPS-bound

$$g_s = 0, \quad N = \text{fixed}, \quad \frac{E-Q}{g_s} = \text{fixed}. \tag{7.2}$$

Depending on the specific duality one can define Q in various ways, but for ABJM and SYM the cases for the Cartan generators will be the same, such that $Q = S + J$. From the time coordinate one can extract $E = i\partial_t$ and also $S = i\partial_{\bar{\gamma}}$ and $J = i\partial_\gamma$. Now we make the coordinate change that will give a non-relativistic string on the world sheet. Introduce x_0 and u such that

$$i\partial_{x_0} = E - Q = E - S - J, \quad -i\partial_u = \frac{1}{2}(E - S + J). \tag{7.3}$$

Then one can rescale x_0 such that the conserved charge scales as g_s when the limit $g_s \to 0$. So we introduce $x_0 = \frac{\tilde{x_0}}{4\pi g_s N}$. In the SMT limit one can then obtain

$$c \to \infty, \quad x_0 = c^2 \tilde{x_0}, \quad c = \frac{1}{4\pi g_s N}, \quad \text{N and } \tilde{x_0} \text{ fixed.} \tag{7.4}$$

So using these coordinates, we will, when taking a certain BPS-bound, transform our global coordinates of the geometry to something depending on x_0 and u. With all these components, we can outline the general procedure for each correspondence in the following sections.

7.2 SMT Limits of $\mathcal{N} = 4$ SYM

The starting point from here is to consider a parametrization of $AdS_5 \times S^5$, in the following way

$$\begin{aligned}
z_0 &= R\cosh(\rho)\mathrm{e}^{it}, & w_1 &= R\sin(\beta_1/2)\sin(\beta_2/2))\mathrm{e}^{i\alpha_1}, \\
z_1 &= R\sinh(\rho)\sin(\bar{\beta}/2)\mathrm{e}^{i\bar{\alpha}_1}, & w_2 &= R\sin(\beta_1/2)\cos(\beta_2/2))\mathrm{e}^{i\alpha_2}, \\
z_2 &= R\sinh(\rho)\sin(\bar{\beta}/2)\mathrm{e}^{i\bar{\alpha}_2}, & w_3 &= R\cos(\beta_1/2))\mathrm{e}^{i\alpha_3}.
\end{aligned} \tag{7.5}$$

The geometry exhibits both features from S^5 which is associated to the angular momentum $J_j = -i\partial_{\alpha_j}$ and also $S^3 \subset AdS_5$ which associates to spin $S_i = -i\partial_{\bar{\alpha}_i}$. By appropiate combination of angles we can define γ and $\bar{\gamma}$ from α_j and $\bar{\alpha}_i$. Additionally, if we consider the global time coordinate, we can define new coordinates as per the discussion of null isometries of the non-relativistic strings:

$$\begin{pmatrix} t \\ \bar{\gamma} \\ \gamma \end{pmatrix} = \begin{pmatrix} 1 & 1/2 & 0 \\ 1 & -1/2 & c_1 \\ 1 & 1/2 & c_2 \end{pmatrix} \begin{pmatrix} x^0 \\ u \\ w \end{pmatrix}. \tag{7.6}$$

This matrix equation precisely leads to the relations established in the previous section. The only addition is introducing the parameter w which is aligned along S^3 and is controlled by c_1 and c_2. It turns out that the parameters can be gauge fixed to $c_1 = 1$ and $c_2 = 0$ such that $s = -i\partial_w$. From here we should be able to study specific subsectors employing what we have established so far. This leads to reviewing some calculations for specific cases of $PSU(2|2,4)$.

7.2.1 The $SU(2)$ Background

The maybe simplest example is to consider the BPS-bound $E \geq Q = J_1 + J_2$. Since we only concern ourselves with Angular momentum generators we can focus solely on the S^5 part. One can decompose it in terms of a Fubini-Study metric and a fibration over one of the directions on the sphere. This means we can write

$$\begin{aligned} d\Omega_5^2 &= d\alpha^2 + \sin^2\alpha d\beta^2 + \cos^2\alpha[d\Sigma_1^2 + (d\gamma + A)^2], \\ A &= \frac{1}{2}\cos\theta d\phi, \\ d\Sigma_1^2 &= \frac{1}{4}(d\theta^2 + \sin^2\theta d\phi^2). \end{aligned} \quad . \tag{7.7}$$

Thus we will focus on t and γ. If we write them as linear combination of the coordinates x_0 and u we get

$$t = x^0 - \frac{1}{2}u, \quad \gamma = x^0 + \frac{1}{2}u. \tag{7.8}$$

If we want to get conditions for when we exactly have a manifold being null, we impose conditions on the full metric such that $g_{uu} = 0$ when $\rho = \alpha = 0$. Hence we insert our transformations in the metric and try to reformulate it in terms of TNC variables:

$$\begin{aligned} ds^2/R^2 = -\cosh^2\rho \left(dx^0 - \frac{1}{2}du\right)^2 + d\rho^2 + \sinh^2 d\Omega_3^2 + d\alpha^2 + \sin^2\alpha d\beta^2 \\ + \cos^2\alpha \left[d\Sigma_1^2 + \left(dx^0 + \frac{1}{2}du\right)^2 + A^2 + 2A\left(dx^0 + \frac{1}{2}du\right)\right]. \end{aligned} \tag{7.9}$$

We describe in detail how we might identify The TNC variable in this case, as this will be the same procedure used in the other cases as well. Firstly, if we group terms that have a factor of du attached we can group terms and get $du(dx^0 + A)$, meaning we can identify $\tau = dx^0 + \frac{1}{4}\cos\theta d\phi$. Then we look at the rest of the terms left. Since the structure of the TNC variable is of the form $2\tau(du - m)$ we look at terms that fit with τ when factorized. Terms that are left are $A^2 + 2Adx^0$, so we need to satisfy the equation $2\tau(du - m) = dx^0 du + A^2 + 2Adx^0 + Adu$. The choice can easily be seen to be $m = -\frac{1}{2}\cos\theta d\phi$. Lastly, we have the term $h_{\mu\nu}dx^\mu dx^\nu$. We look for squared elements in the range of μ, ν meaning that our transformed coordinates are out of question. It can easily be seen that the Fubini-Study metric precisely has the structure needed, meaning $h_{\mu\nu}dx^\mu dx^\nu = \frac{1}{4}(d\theta^2 + \sin^2\theta d\phi^2)$. We can also group the $\cosh^2\rho$ term and $\cos^2\alpha$ and using standard trigonometry identities to get $-(\sinh^2\rho + \sin^2\alpha)(dx^0 + \frac{1}{2}du)^2$. Thus assembling it all we get the metric in terms of the TNC variables

$$ds^2/R^2 = 2\tau(du - m) + h_{\mu\nu}dx^\mu dx^\nu. \tag{7.10}$$

A more elaborate continuation is given in [78], but here we just show how one can get this type of metric to begin with. The only thing that could be missing is to take the SMT limit now and obtain $\tau = d\tilde{x}^0$ when combining the BPS-bound with the coordinate transformation. Further one can gauge fix the worldsheet by fixing the zweibeins and and taking a gauge choice on η. This will reduce the sigma-model Lagrangian to a Landau-Lifshitz model describing spin chains.

7.2.2 The $SU(2|3)$ Background

From the previous example, we extend the BPS-bounds and now consider the maximal choice of S^5 where $E \geq Q = J_1 + J_2 + J_3$. This subsector is a $SU(2|3)$ theory with the largest possible compact spin group of $\mathcal{N} = 4$ SYM. Hence we are zooming in on all the commuting generators of S^5 seen from the bulk perspective. There will be an emergence of $\mathbb{CP}^2$ as the compact spatial section parametrized by a Fubini-Study metric gives the $U(1)$-Galilean background. The strategy is to perform a Hopf fibration such that the S^5 is described as a circle fibration (parametrized by χ) over the $\mathbb{CP}^2$ space. The way we want to define the fibration coordinate is through $Q = J_1+J_2+J_3 = -i\partial_\chi$. The reasoning leads back to this vector being of constant length on particular submanifolds on the geometry. When defining u, this will ensure that ∂_u will be null on specific submanifolds as well. Doing this we perform a set of linear transformations of the α_i's by the following matrix

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 & -1/2 \\ 1 & -1/2 & 1/2 \\ 1 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} \chi \\ \psi \\ \phi \end{pmatrix}. \tag{7.11}$$

Since only considering angles on S^5, the AdS_5 can be disregarded and this leads us to the background

$$\begin{aligned} ds^2/R^2 &= d\xi^2 + \sin^2(\xi)(d\theta + \sin^2(\theta/2)d\alpha_1^2 + \cos^2(\theta/2)d\alpha_2^2) + \cos^2(\xi)d\alpha_3^2 \\ &= d\xi^2 + \sin^2(\xi)(d\theta + \sin^2(\theta/2)\left(d\chi + \frac{1}{2}d\psi - \frac{1}{2}d\phi\right)^2 \\ &+ \cos^2(\theta/2)\left(d\chi - \frac{1}{2}d\psi + \frac{1}{2}d\phi\right)^2 + \cos^2(\xi)\left(d\chi + \frac{1}{2}d\psi\right)^2. \end{aligned} \tag{7.12}$$

The angles lie in the ranges $\xi \in (0, \pi/2)$ and $\theta \in (0, \pi)$. Expanding and gathering terms in a way such that we have the circle fibration over χ, we can rewrite the full metric in the following form:

$$ds^2/R^2 = -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\bar{\Omega}_3^2 + (d\chi + B)^2 + d\Sigma_2^2. \tag{7.13}$$

The metric can be expressed in terms of a Fubini-Study metric and potentials defined as

$$\begin{aligned} &B = \sin^2\xi(d\psi + A), \quad A = \frac{1}{2}\cos\theta d\phi, \\ &d\Sigma_2^2 = d\xi^2 + \sin^2\xi d\Sigma_1^2 + \cos^2\xi\sin^2\xi(d\psi + A), \quad d\Sigma_1^2 = \frac{1}{4}(d\theta^2 + \sin^2\theta d\phi^2). \end{aligned} \tag{7.14}$$

To obtain a $U(1)$ Galilean background, we relate the coordinates considered to a pair of new ones related to a submanifold where the null are geodesics along the isometry of the considered subsector. Introducing x^0 and u we get

$$\begin{pmatrix} t \\ \chi \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 1 & -1/2 \end{pmatrix} \begin{pmatrix} x^0 \\ u \end{pmatrix}. \tag{7.15}$$

Since u is of constant length across $\mathbb{CP}^2 \subset S^5$, we need to have

$$4(\partial_u)^2/R^2 = -\cosh^2\rho + 1 \leq 0. \tag{7.16}$$

From this we see that u will be null if and only if $\rho = 0$. This six-dimensional manifold is now described by coordinates $\{x^0, u, \theta, \phi, \xi, \psi\}$, where the last angle is part of the $\mathbb{CP}^2$. One obtains a metric that can be written using the condition on ρ

$$\begin{aligned} ds^2/R^2 &= -\left(dx^0 - \frac{1}{2}du\right)^2 + \left(dx^0 + \frac{1}{2}du + B\right)^2 + d\Sigma_2^2 \\ &= du(2dx^0 + B) + B^2 + 2Bdx^0 + d\Sigma_2^2 \\ &= 2\tau(du - m) + h_{ij}dx^i dx^j. \end{aligned} \tag{7.17}$$

It is easy to read off what the three different TNC-variables are:

$$\tau = dx^0 + \frac{1}{2}B, \quad m = -B, \quad h = d\Sigma_2^2. \tag{7.18}$$

To check to see if we are not completely off, we can look for a structure of a subsector which in this case would be $SU(2) \subset SU(2|3)$. This seems plausible since it has been engineered via a Hopf fibration of $\mathbb{CP}^1$ inside the S^3, corresponding to the previous BPS-bound we considered. Setting $\xi = \pi/2$ and fixing ψ realizes the same potentials and Fubini-Study metrics. This is a general trend that can be derived starting from the maximal $PSU(1,2|3)$ background and reducing on it.

7.2.3 The $SU(1,1)$ Background

The last real subsector we will look at before the full is a background mixed between the spin and angular momentum mixing both the S^5 and AdS_5. This will correspond to a $SU(1,1)$ background with the particular choice $Q = S_1 + J_1$ with the definitions $S_1 = -i\partial_{\bar{\alpha}_1}$ and $J_1 = -i\partial_{\alpha_1}$. Taking the embedding coordinates defined in the (7.5), one finds the induced metric can be written as

$$\begin{aligned} ds^2/R^2 = &-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho\left(\frac{d\bar{\beta}^2}{4} + \sin^2(\bar{\beta}/2)d\bar{\alpha}_1^2 + \cos^2(\bar{\beta}/2)d\bar{\alpha}_2^2\right) \\ &+ \frac{d\beta_1^2}{4} + \sin^2(\beta_1/2)\left(\frac{d\beta_2^2}{4} + \sin^2(\beta_2/2)d\alpha_1^2 + \cos^2(\beta_2/2)d\alpha_2^2\right) \\ &+ \cos^2(\beta_1/2)d\alpha_3^2. \end{aligned} \tag{7.19}$$

Given this, performing the transformation x^0, u, w to the choices that correspond to our choice in the near BPS-limit gives

$$\begin{pmatrix} t \\ \bar{\alpha}_1 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} 1 & -1/2 & 0 \\ 1 & -1/2 & c_1 \\ 1 & 1/2 & c_2 \end{pmatrix} \begin{pmatrix} x^0 \\ u \\ w \end{pmatrix}. \tag{7.20}$$

Reading off from the second column and gathering the pre-factors in front of our original coordinates gives the null condition on u:

$$4(\partial_u)^2/R^2 = -\cosh^2\rho + \sinh^2\rho\sin^2(\bar{\beta}/2) + \sin^2(\beta_1/2)\sin^2(\beta_2/2) \leq 0. \tag{7.21}$$

The condition for u being null is exactly satisfied when $\bar{\beta} = \beta_1 = \beta_2 = \pi$. Inserting this into the metric and using the conditions we find

$$\begin{aligned}
ds^2/R^2 &= -\cosh^2\rho\left(dx^0 - \frac{1}{2}du\right)^2 + d\rho^2 + \sinh^2\rho\left(dx^0 - \frac{1}{2}du + c_1 dw\right)^2 \\
&+ \left(dx^0 + \frac{1}{2}du + c_2 dw\right)^2 \\
&= 2dx^0 du + d\rho^2 + \sinh^2\rho\left(c_1^2 dw^2 + 2c_1 dw\left(dx^0 - \frac{1}{2}du\right)\right) \\
&+ c_2^2 dw^2 + 2c_2 dw\left(dx^0 + \frac{1}{2}du\right) \\
&= du(2dx^0 - (c_1\sinh^2\rho - c_2)dw) + (c_1^2\sinh^2\rho + c_2^2)dw^2 + 2dx^0 dw(c_1\sinh^2\rho + c_2).
\end{aligned} \tag{7.22}$$

Now it can easily be read off what the TNC data are on the submanifold considered:

$$\begin{aligned}
\tau &= dx^0 - \frac{1}{2}(c_1\sinh^2\rho - c_2)dw, \\
m/R^2 &= -(c_1\sinh^2\rho + c_2)dw, \\
h/R^2 &= d\rho^2 + c_1^2\sinh^2\rho\cosh^2\rho dw^2.
\end{aligned} \tag{7.23}$$

Apparently the spatial slices of the geometry parametrized by ρ, w are non-compact compared to the $SU(3|2)$. Also it can be shown that the constants can be fixed such that they have the values $c_1 = 1$ and $c_2 = 0$. If we insert this into the gauge-fixed action on this background we arrive at the result

$$\begin{aligned}
S_{\text{flat,gf}} &= -\frac{J}{2\pi}\int d^2\sigma(m_\mu x^\mu + \frac{1}{2}h_{\mu\nu}dx^\mu dx^\nu) \\
&= -\frac{J}{2\pi}\int d^2\sigma\Big[\sinh^2\rho\dot{w} - \frac{1}{2}\left((\rho')^2 + \sinh^2\rho\cosh^2\rho(w')^2\right)\Big].
\end{aligned} \tag{7.24}$$

To compare, one can make the correct coordinate choices and reproduce the action obtained from coherent states in the $sl(2)$ spin chain and spinning strings on $\text{AdS}_5 \times \text{S}^5$.

7.2.4 All backgrounds from $PSU(1,2|3)$ Background

We now look at the final great BPS-bound $E \geq Q = S_1 + S_2 + J_1 + J_2 + J_3$. This bound leads to $PSU(1,2|3)$ SMT, and this has the property of course that it can be restricted to obtain the other theories associated to the other bounds. We here use Hopf coordinates for $S^3 \subset AdS_5$ and parametrize the five-sphere using a one-sphere fibration over $\mathbb{CP}^2$ in Fubini-Study coordinates:

$$\begin{aligned}
z_0 &= R\cosh\rho e^{it}, & w_1 &= R\sin\xi\sin(\theta/2)e^{i(\chi+\psi/2-\varphi/2)}, \\
z_1 &= R\sinh\rho\sin\big(\bar{\theta}/2\big)e^{i(\bar{\psi}-\bar{\varphi}/2)}, & w_2 &= R\sin\xi\cos(\theta/2)e^{i(\chi+\psi/2+\varphi/2)}, \\
z_2 &= R\sinh\rho\cos\big(\bar{\theta}/2\big)e^{i(\bar{\psi}+\bar{\varphi}/2)}, & w_3 &= R\cos\xi e^{i(\chi-\psi/2)}.
\end{aligned} \tag{7.25}$$

This means we have $-i\partial_{\bar{\psi}} = S_1 + S_2$ and $-i\partial_\chi = J_1 + J_2 + J_3$. The total metric is given by

$$ds^2/R^2 = -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho\left[d\bar{\Sigma}_1^2 + \big(d\bar{\psi} + \bar{A}\big)^2\right] + d\Sigma_2^2 + (d\chi + B)^2. \tag{7.26}$$

By defining

$$\begin{pmatrix} t \\ \bar{\psi} \\ \chi \end{pmatrix} = \begin{pmatrix} 1 & -1/2 & 0 \\ 1 & -1/2 & c_1 \\ 1 & 1/2 & c_2 \end{pmatrix} \begin{pmatrix} x^0 \\ u \\ w \end{pmatrix}, \tag{7.27}$$

we get the following TNC data:

$$\begin{aligned} \tau &= dx^0 - \frac{1}{2}\left(c_1 \sinh^2\rho - c_2\right) dw + \frac{1}{2}\sinh^2\rho \bar{A} + \frac{1}{2}B, \\ m/R^2 &= -\left(c_1 \sinh^2\rho + c_2\right) dw - \sinh^2\rho \bar{A} - B, \\ h/R^2 &= d\rho^2 + \sinh^2\rho d\bar{\Sigma}_1^2 + \sinh^2\rho \cosh^2\rho \left(c_1 dw + \bar{A}\right)^2 + d\Sigma_2^2. \end{aligned} \tag{7.28}$$

The resulting big SMT string action of this is

$$\begin{aligned} S_{\text{flat,gf}} = \frac{J}{4\pi}\int d^2\sigma \Big[& 2\sinh^2\rho \dot{w} + \sinh^2\rho \cos\bar{\theta}\dot{\bar{\varphi}} - \cos(2\xi)\dot{\psi} + \sin^2\xi \cos\theta\dot{\varphi} \\ & - (\rho')^2 - \frac{1}{4}\sinh^2\rho \left((\bar{\theta}')^2 + \sin^2\bar{\theta}(\bar{\varphi}')^2\right) \\ & - \sinh^2\rho\cosh^2\rho \left(w' + \frac{1}{2}\cos\bar{\theta}\bar{\varphi}'\right)^2 \\ & - (\xi')^2 - \frac{1}{4}\sin^2\xi \left((\theta')^2 + \sin^2\theta(\varphi')^2\right) \\ & - \sin^2\xi\cos^2\xi \left(\psi' + \frac{1}{2}\cos\theta\varphi'\right) \Big]. \end{aligned} \tag{7.29}$$

7.3 SMT Limits of $\mathcal{N} = 6$ Chern-Simons Theory and ABJM

Now we shift the scope and consider what has not been considered before. As for the $\mathcal{N} = 4$ SYM case, the vast landscape has been explored and all subsectors and backgrounds can be considered starting from the most general BPS-bound, namely the $PSU(2|2,4)$. Furthermore, the Penrose limits have also been studied and are known now. Here, we set out to do the same but in the context of ABJM theory, where we consider now an $AdS_4 \times S^7/\mathbb{Z}_k$ or rather $AdS_4 \times \mathbb{CP}^3$ geometry for large k. Thus the starting point is to establish a general metric depending on the Cartan generators. The same formalism will be used in this correspondence where we instead now have an S^7 which is associated to the angular momentum $J_j = -i\partial_{\alpha_j}$, and also $S^2 \subset AdS_4$ which is associated to spin $S = -i\partial_\phi$. In contrast to SYM, we only have one spin degree of freedom from the AdS part, but a further addition of angular momentum freedom. But this is not to say that we are not without restrictions. From the orbifolding condition one finds that the $\sum_{i=1}^4 J_i = 0$ [47]. So when considering BPS bounds, the maximal number of angular momenta generators that can go into a subsector is 3. With this in mind we define in complex coordinates, following [79], the S^7 as

$$\begin{aligned} X_1 = \cos(\xi)\cos\left(\frac{\theta_1}{2}\right)e^{i\alpha_1}, \quad X_2 = \cos(\xi)\sin\left(\frac{\theta_1}{2}\right)e^{i\alpha_2}, \\ X_3 = \sin(\xi)\cos\left(\frac{\theta_2}{2}\right)e^{i\alpha_3}, \quad X_4 = \sin(\xi)\sin\left(\frac{\theta_2}{2}\right)e^{i\alpha_4}. \end{aligned} \tag{7.30}$$

For the AdS part, we proceed as follows:

$$
\begin{aligned}
&Z_0 = R\cosh(\rho)\cos(t), \quad Z_1 = R\cosh(\rho)\sin(t), \\
&Z_2 = R\sinh(\rho)\cos\left(\frac{\theta}{2}\right), \quad Z_3 = R\sinh(\rho)\sin\left(\frac{\theta}{2}\right)\cos(\phi), \\
&Z_4 = R\sinh(\rho)\sin\left(\frac{\theta}{2}\right)\sin(\phi).
\end{aligned}
\tag{7.31}
$$

From this point onward, it is straightforward obtaining the metric element from the parametrization of the geometry. It is done most easily by using a software such as Mathematica, but it can be done in hand too. The overall expression for the metric can be found by

$$
ds^2 = \sum_{i=0}^{4} |dz_i|^2 + \sum_{i=1}^{4} |dx_i|^2. \tag{7.32}
$$

After finding all the infinitesimal elements and adding all the contributions together we arrive at the metric

$$
\begin{aligned}
ds^2 = &-\cosh(\rho)dt^2 + d\rho^2 + \sinh^2(\rho)\left(\frac{1}{4}d\theta^2 + \sin\left(\frac{\theta}{2}\right)d\phi^2\right) + d\xi^2 \\
&+\cos^2(\xi)\left(\frac{1}{4}d\theta_1^2 + \sin^2\left(\frac{\theta_1}{2}\right)d\alpha_1^2 + \cos^2\left(\frac{\theta_1}{2}\right)d\alpha_2^2\right) \\
&+\sin^2(\xi)\left(\frac{1}{4}d\theta_2^2 + \sin^2\left(\frac{\theta_2}{2}\right)d\alpha_3^2 + \cos^2\left(\frac{\theta_2}{2}\right)d\alpha_4^2\right).
\end{aligned}
\tag{7.33}
$$

Now that we have established this, we can proceed as for the case of SYM. We turn our attention to the case of submanifolds having the TNC structure. The idea is to again identify the metric in specific subsectors, such that it factorizes to a $U(1)$-Galilean geometry. We proceed by first analyzing the "simplest case", namely the $SU(2)\times SU(2)$ sector.

7.3.1 The $SU(2)\times SU(2)$ Background and Penrose Limit

We start by considering a double-copy of the cousin from SYM, namely the $SU(2)\times SU(2)$ case. In ABJM theory this subsector has the same BPS-bound as for SYM $Q = J_1 + J_2$, which purely consists of the $S^7/\mathbb{CP}^3$ part of the metric. From previous calculations and review, our starting point is writing S^7 as two 3-spheres and then further to two two-spheres. But instead of starting here, one can just consider the type IIA background described by a 10D metric. This will not pose any trouble, since doing it as was done in [47] introduces an 11D metric with a fibration term $(d\gamma + A)^2$. As was considered this can not be fixed such that the one-form disappears, leading to the needs of relations between the M-theory metric and type IIA metric along with relations such as $l_p^3 = g_s l_s^3$ and $R_{11} = g_s l_s$ such that we can obtain the type IIA background

$$
\begin{aligned}
ds^2/R^2 = &-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\hat{\Omega}_2^2 + ds^2_{\mathbb{CP}^3} = -\cosh^2\rho dt^2 + d\rho^2 \\
&+\sinh^2\rho d\hat{\Omega}_2^2 + d\theta^2 + \frac{1}{4}(\cos^2\theta d\Omega_2^2 + \sin^2\theta d\Omega_2'^2) + 4\cos^2\theta\sin^2\theta(d\delta+\omega)^2.
\end{aligned}
\tag{7.34}
$$

The two two-spheres and the 1-form ω are given by

$$\omega = \frac{1}{4}\sum_{i=1}^{2} \sin\theta_i d\phi_i, \quad d\Omega_2'^2 = d\theta_2^2 + \cos^2\theta_2 d\phi_2^2, \quad d\Omega_2^2 = d\theta_1^2 + \cos^2\theta_1 d\phi_1^2. \tag{7.35}$$

Having established the metric one may introduce and transform the global AdS time and the fibration over δ via the coordinates x^0 and u as follows:

$$\begin{pmatrix} t \\ \delta \end{pmatrix} = \begin{pmatrix} 1 & -1/2 \\ 1 & 1/2 \end{pmatrix} \begin{pmatrix} x^0 \\ u \end{pmatrix}. \tag{7.36}$$

Reading off from the second column, we get the same type of condition as for $\mathcal{N} = 4$ SYM such that ∂_u is null

$$4(\partial_u)^2/R^2 = -\cosh^2\rho + 4\cos^2\theta \sin^2\theta \le 0. \tag{7.37}$$

This is exactly met when $\rho = 0$ and $\theta = \pi/4$. Using these conditions and transforming to the new coordinates yields

$$\begin{aligned} ds^2/R^2 &= -\left(dx^0 - \frac{1}{2}du\right)^2 + (dx^0 + \frac{1}{2}du + \omega)^2 + \frac{1}{8}(d\Omega_2^2 + d\Omega_2'^2) \\ &= du(2dx^0 + \omega) + \omega^2 + 2dx^0\omega + \frac{1}{8}(d\Omega_2^2 + d\Omega_2'^2) \\ &= 2\tau(du - m) + h_{ij}dx^i dx^j. \end{aligned} \tag{7.38}$$

One can see that the same structure appears as for $SU(2)$ case in SYM, but with a copy such that we have two S^2's instead of a single one. On top of this, the 1-form has a richer structure. Moving forward, we can identify the TNC variables

$$\tau = dx^0 + \frac{1}{2}\omega, \quad m = -\omega, \quad h_{ij}dx^i dx^j = \frac{1}{8}(d\Omega_2^2 + d\Omega_2'^2). \tag{7.39}$$

The addition of a two-sphere can be seen as a consequence of the ABJM theory. It builds around a bifundamental and an anti-bifundamental representation, each having their own multiplets with scalars. This would suggest that the additional sphere represents the splitting between the two representations. As was shown in [47], in a certain limit one finds a spin-chain description expressed as two separate Landau-Lifshitz models living on odd and even sites on the spin-chain, corresponding to the two different representations that can affect each other through momentum constraints. Hence we would expect this to be our result as well. If we proceed to consider the flat gauge fixed action for the non-relativistic SMT string using the TNC-variables we get

$$\begin{aligned} S_{\text{flat,gf}} &= -\frac{J}{2\pi}\int d^2\sigma (m_\mu x^\mu + \frac{1}{2}h_{\mu\nu}dx^\mu dx^\nu) \\ &\frac{J}{8\pi}\sum_{i=1}^{2}\int d^2\sigma \left(\sin\theta_i \dot{\phi}_i - \frac{1}{2}[\theta_i'^2 + \cos^2\theta_i \phi'^2]\right) \end{aligned} \tag{7.40}$$

We see that one exactly retrieves the Landau-Lifshitz model on odd and even sites in this regime considering an SMT limit on the BPS-bounds $SU(2) \times SU(2)$ for the ABJM theory.

7.3.2 The OSp(2|2) Background

The subsector defined for the OSp(2|2) background is given by the BPS bound $Q = J_1 + J_2 + S$, that is, we extend the $SU(2) \times SU(2)$ sector by introducing spin. Our starting point will be the general metric we wrote and performing a transformation as follows

$$\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{pmatrix} = \begin{pmatrix}1 & 1 & 0 & 0\\ 1 & -1 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 1 & -1\end{pmatrix}\begin{pmatrix}\chi_1\\ \phi_1\\ \chi_2\\ \phi_2\end{pmatrix}. \tag{7.41}$$

This linear transforamtion will result in the following new metric:

$$\begin{aligned}ds^2_{S^7} = d\xi^2 &+ \cos^2(\xi)\left(\frac{1}{4}d\theta_1^2 + \sin^2\left(\frac{\theta_1}{2}\right)(d\chi_1 + d\phi_1)^2 + \cos^2\left(\frac{\theta_1}{2}\right)(d\chi_1 - d\phi_1)^2\right)\\ &+ \sin^2(\xi)\left(\frac{1}{4}d\theta_2^2 + \sin^2\left(\frac{\theta_2}{2}\right)(d\chi_2 + d\phi_2)^2 + \cos^2\left(\frac{\theta_2}{2}\right)(d\chi_2 - d\phi_2)^2\right)\end{aligned} \tag{7.42}$$

If we expand everything and manipulate it we can rewrite this in such a way that the half-angles disappear using the standard $\sin^2(\frac{\theta_2}{2}) + \cos^2(\frac{\theta_2}{2}) = 1$ and $\sin^2(\frac{\theta}{2}) - \cos^2(\frac{\theta}{2}) = \cos\theta$. The idea is that we want terms that parameterize two-spheres for ϕ_1 and ϕ_2. So in the end we can rewrite the metric as

$$\begin{aligned}ds^2_{S^7} = d\xi^2 &+ \frac{1}{4}\cos^2(\xi)\left[\left(d\chi_1 + \cos\left(\frac{\theta_1}{2}\right)d\phi_1\right)^2 + d\theta_1^2 + \sin^2\theta_1 d\phi_1^2\right]\\ &+ \frac{1}{4}\sin^2(\xi)\left[(d\chi_2 + \cos\theta_2 d\phi_2)^2 + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2\right].\end{aligned} \tag{7.43}$$

Next we introduce coordinates which will act as the circle fibration to construct a $\mathbb{CP}^3$ metric such that we can obtain a type IIA background

$$\chi_1 = 2y + \psi, \quad \chi_2 = 2y - \psi. \tag{7.44}$$

It is through these transformations that we make the connection to J_1 and J_2 by defining that $-i\partial_{\chi_1} - i\partial_{\chi_2} = -i\partial_y$. This will exactly be the fibration coordinate that we look for. A thing to note is that one could have performed the transformations from the beginning instead of first defining χ_1 and χ_2 and just inserted their respective definition in the 4×4 matrix. Now the Z_k orbifold also becomes $y \sim y + 2\pi/k$, and in the large limit of k this reduces to y. Then we can write the S^7 metric in the following way:

$$ds^2 = ds^2_{\mathbb{CP}^3} + (dy + A)^2. \tag{7.45}$$

Here we will not state what the fibration terms involve (these can be found in [79]), but the $\mathbb{CP}^3$ metric reads [80]

$$\begin{aligned}ds^2_{\mathbb{CP}^3} = d\xi^2 &+ 4\cos^2\xi\sin^2\xi\left(d\psi + \frac{\cos\theta_1}{2}d\phi_1 - \frac{\cos\theta_2}{2}d\psi_2\right)^2\\ &+ \frac{1}{4}\cos^2\xi(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \frac{1}{4}\sin^2\xi(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2).\end{aligned} \tag{7.46}$$

As for the case of the $SU(2) \times SU(2)$ sector, we can make the same considerations resulting in the same structure of the metric where the fibration term is dropped through M-theory considerations:

$$\begin{aligned} ds^2/R^2 &= ds^2_{\text{AdS}_4} + ds^2_{\mathbb{CP}^3} \\ &= -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\phi^2 + 4\cos^2\xi\sin^2\xi(d\psi + P)^2 \\ &+ \frac{1}{4}\cos^2\xi(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \frac{1}{4}\sin^2\xi(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2), \end{aligned} \tag{7.47}$$

where we set $P = \frac{\cos\theta_1}{2}d\phi_1 - \frac{\cos\theta_2}{2}d\psi_2$. We are now ready to introduce the coordinates that will describe the $U(1)$ Galilean geometry given by the following matrix:

$$\begin{pmatrix} t \\ \phi \\ \psi \end{pmatrix} = \begin{pmatrix} 1 & -1/2 & 0 \\ 1 & -1/2 & c_1 \\ 1 & 1/2 & c_2 \end{pmatrix} \begin{pmatrix} x^0 \\ u \\ w \end{pmatrix}, \tag{7.48}$$

Reading off from the second column, we get ∂_u is null under the following condition

$$4(\partial_u)^2/R^2 = -\cosh^2\rho + \sinh^2\rho\sin^2\theta + 4\cos^2\xi\sin^2\xi \leq 0. \tag{7.49}$$

This condition will exactly be met when $\theta = \pi/2$ and $\xi = \pi/4$. This will reduce the metric to

$$\begin{aligned} ds^2/R^2 &= ds^2_{\text{AdS}_4} + ds^2_{\mathbb{CP}^3} \\ &= -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\phi^2 \\ &\quad + (d\psi + P)^2 + \frac{1}{8}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2), \end{aligned} \tag{7.50}$$

and immediately our metric explicitly becomes

$$\begin{aligned} ds^2/R^2 &= d\rho^2 + \frac{1}{8}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) \\ &+ \frac{du^2}{4} + (dx^0)^2 + \left(\frac{1}{4}du^2 + (dx^0)^2\right)(\sinh^2\rho - \cosh^2\rho) \\ &+ dudx^0(1 + \cosh^2\rho - \sinh^2\rho) + c_2 dudw + 2c_2 dwdx^0 + c_2^2 dw^2 \\ &+ Pdu + 2c_2 Pdw + 2Pdx^0 + P^2 + 2c_1\sinh^2\rho dwdx^0 \\ &+ c_1^2\sinh^2\rho dw^2 - c_1\sinh^2\rho dudw. \end{aligned} \tag{7.51}$$

By employing the identity $\cosh^2\rho - \sinh^2\rho = 1$, we get the reduced form

$$\begin{aligned} ds^2/R^2 &= d\rho^2 + \frac{1}{8}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) \\ &+ 2dw(c_2 dx^0 + c_2 P + c_1\sinh^2\rho dx^0) + 2Pdx^0 + c_2^2 dw^2 + P^2 \\ &+ du(2dx^0 + c_2 dw + P - c_1\sinh^2\rho dw) + c_1^2\sinh^2\rho dw^2. \end{aligned} \tag{7.52}$$

Quite readily we determine the clock one-form TNC coordinate by looking at the last line in the metric and get

$$\begin{aligned} \tau &= dx^0 + \frac{1}{2}\left(P + (c_2 - c_1\sinh^2\rho)dw\right) \\ &= dx^0 + \frac{1}{4}\left(\cos\theta_1 d\phi_1 - \cos\theta_2 d\phi_2 + 2(c_2 - c_1\sinh^2\rho)dw\right), \end{aligned} \tag{7.53}$$

and we define the m coordinate to be

$$\begin{aligned} m &= 2(\tau - dx^0) \\ &= \frac{1}{2}\left(\cos\theta_1 d\phi_1 - \cos\theta_2 d\phi_2 + 2(c_2 - c_1\sinh^2\rho)dw\right). \end{aligned} \tag{7.54}$$

We can now determine our $h_{ij}dx^i dx^j$ term by subtracting our TNC data from the reduced metric, which gives us (using P again)

$$\begin{aligned} ds^2 - 2\tau(du - m) = d\rho^2 &+ \frac{1}{8}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) \\ &+ 2c_2^2 dw^2 + 4c_2 dw dx^0 + 4c_2 P dw + 4P dx^0 + c_1^2 \sinh^2\rho dw^2 \\ &- 2c_1c_2\sinh^2\rho dw^2 - 2c_1 P\sinh^2\rho dw + c_1^2\sinh^4\rho dw^2 + 2P^2. \end{aligned} \tag{7.55}$$

Now we again choose the gauge $c_1 = 0$ and $c_2 = 0$ and the flat gauge η for our metric in the calculation of the action, this means in our choice of h we drop non-diagonal terms and terms involving the factor c_2. In (7.55) we have a $\sinh^4$ term, but this can be combined with the $c_1^2\sinh^2$ term to get

$$c_1^2 dw^2(\sinh^2\rho + \sinh^4\rho) = c_1^2\sinh^2(\rho)dw^2(1 + \sinh^2\rho) = c_1^2\cosh^2\rho\sinh^2\rho dw^2$$

as per the identity, and thus we have (in flat gauge with $c_1 = 1$ and $c_2 = 0$)

$$\begin{aligned} \tau &= dx^0 + \frac{1}{4}\left(\cos\theta_1 d\phi_1 - \cos\theta_2 d\phi_2 - 2\sinh^2\rho dw\right) \\ m &= \frac{1}{2}\left(\cos\theta_1 d\phi_1 - \cos\theta_2 d\phi_2 - 2\sinh^2\rho dw\right) \\ h &= d\rho^2 + \frac{1}{8}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) + \cosh^2\rho\sinh^2\rho dw^2. \end{aligned} \tag{7.56}$$

Using now

$$S_{\text{flat,gf}} = -\frac{J}{2\pi}\int d^2\sigma(m_\mu x^\mu + \frac{1}{2}h_{\mu\nu}dx^\mu dx^\nu)$$

we get the 'extended' Landau-Lifshitz model:

$$\begin{aligned} S_{\text{flat,gf}} = -\frac{J}{2\pi}\int d^2\sigma\Big[&\dot{w}\sinh^2\rho - \frac{1}{2}\left((\rho')^2 + (w')^2\sinh^2\rho\cosh^2\rho\right) \\ &+ \frac{1}{2}\dot{\phi}_1\cos\theta_1 - \frac{1}{2}\dot{\phi}_2\cos\theta_2 + \frac{1}{16}\sum_{i=1}^{2}\left((\theta_i')^2 + (\phi_i')^2\sin^2\theta_i\right)\Big]. \end{aligned} \tag{7.57}$$

7.4 The $SU(3|2)$ Background

The last example before going to the all background case, we consider the maximal R-symmetry admitted by the BPS-sectors, namely $Q = J_1 + J_2 + J_3$. This sector admits $SU(3)$ symmetry and is purely connected to the S^7, thus most of the AdS_4 geometry will be redundant. In section 3.1, we saw that there are different ways of writing the metric of $\mathbb{C}P^3$. For the particular choice of Q, one can show, using the previous metric, that a type IIA background cannot be obtained. With three angular momenta and four complex coordinates to parametrize S^7, one falls short of obtaining a null isometry. This leads to the shift in metric. It can be seen as for the structure in the $SU(3|2)$ case of SYM that the metric decomposes into $\mathbb{CP}^3 \sim \mathbb{CP}^2 + (d\chi + A)^2$, such that we can obtain the Fubini-Study for $\mathbb{CP}^2$ with a non-dynamic angle $d\chi$ which can be factored with Kähler potential. Using the metric, we employ a transformation from angular coordinates on $\mathbb{C}P^3$, namley the α_j's into three new coordinates

$$\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix} = \begin{pmatrix}1 & -1/2 & 0\\ 1 & 1/2 & -1/2\\ 1 & 1/2 & 1/2\end{pmatrix}\begin{pmatrix}\chi\\ \psi\\ \phi\end{pmatrix} \quad \to \quad \begin{aligned} z_1/z_4 &= \tan\xi\cos\alpha e^{i\chi-\psi/2}\\ z_2/z_4 &= \tan\xi\sin\alpha\sin\frac{\theta}{2}e^{i\chi}e^{i(\psi-\phi)/2}\\ z_3/z_4 &= \tan\xi\cos\alpha\cos\frac{\theta}{2}e^{i\chi}e^{i(\psi+\phi)/2}\end{aligned} . \tag{7.58}$$

The resulting metric is then

$$\begin{aligned} ds^2_{\mathbb{CP}^3} &= 4d\xi^2 + 4\sin^2\xi\cos^2\xi(d\chi + \frac{1}{2}(\sin^2\alpha(d\psi + \cos\theta d\phi) - d\psi))^2\\ &\quad + 4\sin^2\xi\left[d\alpha^2 + \frac{1}{4}\sin^2\alpha\left(d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\alpha(d\psi + \cos\theta d\phi\right)^2\right]. \end{aligned} \tag{7.59}$$

Note that the factor of 4 is a matter of convention and is sometimes omitted. In this case, we take advantage of it for the null conditions. The metric can be put in a nicer form following [78] by defining the quantities corresponding to the Kähler potential

$$B = \sin^2\alpha(d\psi + \cos\theta d\phi) - d\psi, \quad A = \cos\theta d\phi. \tag{7.60}$$

Looking at the second line of the metric, we can recognize this exactly as the Fubini-Study metric over $\mathbb{CP}^2$ defined as

$$d\Sigma_2^2 = d\alpha^2 + \sin^2\alpha d\Sigma_1^2 + \sin^2\alpha\cos^2\alpha(d\psi + A)^2, \quad d\Sigma_1^2 = \frac{1}{4}(d\theta^2 + \sin^2\theta d\phi^2). \tag{7.61}$$

Putting all this together we obtain for the full $\mathrm{AdS}_4 \times \mathbb{CP}^3$ metric on the $SU(3|2)$ background

$$ds^2/R^2 = -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_2^2 + 4d\xi^2 + 4\sin^2\xi\cos^2\xi(d\chi + \frac{1}{2}B)^2 + 4\sin^2\xi d\Sigma_2^2. \tag{7.62}$$

Using the same type of transformation as was done for both SYM and the $SU(2)\times SU(2)$ background,

$$\begin{pmatrix}t\\ \chi\end{pmatrix} = \begin{pmatrix}1 & -1/2\\ 1 & 1/2\end{pmatrix}\begin{pmatrix}x^0\\ u\end{pmatrix}, \tag{7.63}$$

we see that the null condition becomes

$$4(\partial_u)^2/R^2 = -\cosh^2\rho + 4\sin^2\xi\cos^2\xi \leq 0, \tag{7.64}$$

so the null condition is fulfilled when $\rho = 0$ and $\xi = \pi/4$. This means our metric is now

$$ds^2/R^2 = -dt^2 + \left(d\chi + \frac{1}{2}B\right)^2 + 2d\Sigma_2^2. \tag{7.65}$$

Inserting this into the metric, the reduced metric in terms of the new coordinates becomes

$$ds^2/R^2 = 2dx^0du + \frac{1}{2}Bdu + Bdx^0 + \frac{1}{4}B^2 + 2d\Sigma_2^2, \tag{7.66}$$

which can readily be written in the TNC form

$$ds^2/R^2 = 2\tau(du - m) + h, \tag{7.67}$$

with

$$\begin{aligned} \tau &= dx^0 + \frac{1}{4}B \\ m &= -\frac{1}{2}B \\ h &= 2d\Sigma_2^2. \end{aligned} \tag{7.68}$$

It can be seen that the difference from the Yang-Mills case lies in the definition of A and $d\Sigma_2^2$ up to some numerical factors. The last step is of course to find the flat gauge fixed action, given by

$$\begin{aligned} S_{\text{flat,gf}} &= -\frac{J}{2\pi}\int d^2\sigma(m_\mu x^\mu + \frac{1}{2}h_{\mu\nu}dx^\mu dx^\nu) \\ &= \frac{J}{4\pi}\int d^2\sigma(\sin^2\alpha\cos\theta\dot{\phi} - \cos^2(\alpha)\dot{\psi} - 2[(\alpha')^2 \\ &+ \frac{1}{4}\sin^2\alpha(\theta'^2 + \sin^2\theta\phi'^2) + \frac{1}{4}\cos^2\alpha(\psi' + \cos\theta\phi')^2]). \end{aligned} \tag{7.69}$$

7.5 All Backgrounds From the $OSp(4|2)$ Background

For the grand background, we obtain the $U(1)$ Galilean geometry for $S+J_1+J_2+J_3 = Q \leq E$. This BPS-bound leads to the $OSp(4|2)$ spin matrix theory which can be used to obtain the other bounds in SMT previously considered by considering different manipulations for angles that gives the different backgrounds. The full geometry will be parametrized through a Hopf coordinate for the $S^2 \subset \text{AdS}_4$ and the $\mathbb{CP}^3$ using an S^1-fibration over $\mathbb{CP}^2$ for the full in the Fubini-Study coordinates. This will result in the isometries $-i\partial_\phi = S$ and $-i\partial_\chi = J_1 + J_2 + J_3$. In terms of the coordinates we write the metric in terms of Fubini-Study potentials

$$\begin{aligned} ds^2/R^2 = -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho(d\bar{\theta}^2 + \sin^2\bar{\theta}d\phi^2) + 4d\xi^2 \\ + 4\sin^2\xi\cos^2\xi\left(d\chi + \frac{1}{2}B\right)^2 + 4\sin^2\xi d\Sigma_2^2. \end{aligned} \tag{7.70}$$

Following the transformations we have used before, we get

$$\begin{pmatrix} t \\ \phi \\ \chi \end{pmatrix} = \begin{pmatrix} 1 & -1/2 & 0 \\ 1 & -1/2 & c_1 \\ 1 & 1/2 & c_2 \end{pmatrix} \begin{pmatrix} x^0 \\ u \\ w \end{pmatrix}. \tag{7.71}$$

Compared to the $PSU(1,2|3)$ case, we do not have that $-i\partial_\phi$ and $-i\partial_\chi$ are of constant length in ABJM, so the null condition we get this time is

$$4(\partial_u)^2/R^2 = -\cosh^2\rho + \sinh^2\rho\sin^2\bar{\theta} + 4\sin^2\xi\cos^2\xi \leq 0. \tag{7.72}$$

This is satisfied when $\xi = \pi/4$ and $\bar{\theta} = \pi/2$. Using the transformation one thus arrives at

$$\begin{aligned} ds^2/R^2 &= 2dx^0du + d\rho^2 + \sinh^2\rho\left(dw^2 + 2dw\left(dx^0 - \frac{1}{2}du\right)\right) \\ &+ \frac{1}{4}B^2 + B\left(dx^0 + \frac{1}{2}du\right) + 2d\Sigma_2^2 \\ &= 2\tau(du - m) + h, \end{aligned} \tag{7.73}$$

where the TNC variables are given by

$$\begin{aligned} \tau &= dx^0 + \frac{1}{4}B + \frac{1}{2}\sinh^2\rho dw, \\ m &= -\left(\frac{1}{2}B + \sinh^2\rho dw\right), \\ h &= d\rho^2 + \sinh^2\rho\cosh^2\rho dw + 2d\Sigma_2^2, \end{aligned} \tag{7.74}$$

On this background the flat gauge-fixed SMT string action gives

$$\begin{aligned} S_{\text{flat,gf}} = \frac{J}{4\pi}\int d^2\sigma\Big(&2\sinh^2\rho\dot{w} + \sin^2\alpha\cos\theta\dot{\phi} - \cos^2(\alpha)\dot{\psi} - (\rho')^2 - \sinh^2\rho\cosh^2\rho(w')^2 \\ &- 2\Big[(\alpha')^2 + \frac{1}{4}\sin^2\alpha((\bar{\theta}')^2 + \frac{1}{4}\sin^2\bar{\theta}(\bar{\phi}')^2) + \sin^2\alpha\cos^2\alpha(\psi' + \cos\theta\phi')^2\Big]\Big). \end{aligned} \tag{7.75}$$

8

Flat Spin Matrix Theory String Backgrounds and Penrose Limits

As was done for $\mathcal{N} = 4$ SYM, in this project we have also been able to succeed in finding all the $U(1)$-Galilean backgrounds corresponding to all of the Spin Matrix limits. However, we do face some complicated and non-linear theories when considering the flat gauge-fixed action. But as was considered [78], one can take a large charge limit for J and zoom in on excitations around specific angles. It is then expected that we might obtain free theories that will resemble Penrose limits where we zoom in on the geometry around the null geodesic. Starting from $AdS_5 \times S^7$ and zooming in on the null geodesic along AdS_4 and S^7, the resulting geometry should become the 10D maximally symmetric pp-wave (see chapter 5). However, it is important to remember that the expression will depend on the coordinates we start with for our null geodesic since it depends on the coordinate pair (x^0, u). When zooming in on a neighbourhood around ∂_u on a submanifold where the vector is null we can expect a background of the form

$$ds^2/R^2 = dx^0(du0 + x_i dy_i) + dx_i dx^i + dy_i dy^i + dx_a dx^a - x_a x^a (dx^0)^2, \tag{8.1}$$

where $i = 1, ..., n$ and $a = 1, ..., 8 - 2n$.
The pp-wave can be written in the form of the above metric and has the $2n$ flat directions (x^i, y^i). There will also be a quadratic potential contained in the $(8 - 2n)$ remaining transverse directions x^a. In the rescaling $x^0 = \frac{\tilde{x^0}}{c^2}$ one gets that the slope diverges in the SMT-limit, hence we get dynamics that are suppressed and only the flat directions contribute. In terms of the TNC-data we expect the following structure to appear:

$$\begin{aligned} \tilde{\tau} &= d\tilde{x^0}, \\ m &= -\sum_{i=1}^{n} x^i dy^i, \\ h &= \sum_{i=1}^{n} (dx^i)^2 + (dy^i)^2. \end{aligned} \tag{8.2}$$

This is what originally has been dubbed the flat fluxed (FF) backgrounds, since they contain a mass flux term $x_i dy^i$ that supports the dynamics in the flat directions of both components. The goal is now to show that the SMT and large charge limit compared with the Penrose limit should correspond to the same FF $U(1)$-Galilean Geometry. We do this for $SU(2) \times SU(2)$, $OSp(2|2)$, $SU(3|2)$, and $OSp(4|2)$. Remaining subsectors can be obtained in a similar way.

8.1 The $SU(2) \times SU(2)$ flat background

We consider the simplest of cases, namely the $SU(2) \times SU(2)$ sector. As we found in the previous chapter, we can write the TNC data as

$$\tilde{\tau} = d\tilde{x}^0, \quad m = -\omega, \quad h_{ij}dx^i dx^j = \frac{1}{8}(d\Omega_2^2 + d\Omega_2'^2). \tag{8.3}$$

For the large charge limit of $J \to \infty$, we define the following coordinate transformations:

$$\theta_i = x_i/\sqrt{J} - \pi/2, \quad \phi_i = y_i/\sqrt{J}. \tag{8.4}$$

Using this for m and h we get

$$\begin{aligned} m_0 &= \lim_{J\to\infty} Jm = \frac{1}{4}\sum_{i=1}^{2} x_i dy_i, \\ h_0 &= \lim_{J\to\infty} Jm = \frac{1}{8}\sum_{i=1}^{2} dx_i^2 + dy_i^2. \end{aligned} \tag{8.5}$$

To finish off we obtain the SMT action

$$S_{\text{flat,gf}} = \frac{J}{4\pi}\sum_{i_1}^{2}\int d^2\sigma\Big[\frac{1}{2}x_i\dot{y}_i - \frac{1}{4}[(x_i')^2 + (y_i')^2]\Big]. \tag{8.6}$$

This is the same action up to some factors as was found in [47].

Now we take the Penrose limit and show that we retrieve the same geometry from a SMT limit of the corresponding pp-wave. To get the pp-wave, we write the metric in the adapted coordinates we used previously, and then zoom in on a null geodesic on the submanifold corresponding to $\rho = 0$ and $\xi = \pi/4$ by defining

$$R = R'/\epsilon, u = U\epsilon^2, \phi_a = y_a\epsilon, \theta_a = x_a\epsilon, \rho = r\epsilon, \xi = \pi/4 + q\epsilon. \tag{8.7}$$

The metric we want to transform is

$$\begin{aligned} ds^2/R^2 = &-\cosh^2\rho(dx^0 - \frac{1}{2}du)^2 + d\rho^2\sinh^2\rho(d\bar{\theta}_2^2 + \sin^2\bar{\theta}(dx^0 - \frac{1}{2}du + dw)^2) \\ &+ d\xi^2 + \frac{1}{4}(\cos^2\xi d\Omega_2^2 + \sin^2\xi d\Omega_2'^2) + 4\cos^2\xi\sin^2\xi(dx^0 + \frac{1}{2}du + \omega)^2. \end{aligned} \tag{8.8}$$

When taking the $\epsilon \to 0$ we use the following appropriate expansions in ϵ in the limit to write the final metric:

$$\begin{aligned} &\cos^2(\pi/4 + q\epsilon) = \frac{1 - \sin(2q\epsilon)}{2} \approx \frac{1}{2}(1 - 2q), \quad \sin^2(\pi/4 + q\epsilon) = \frac{1 + \sin(2q\epsilon)}{2} \approx \frac{1}{2}(1 + 2q) \\ &\cos^2(\pi/4 + q\epsilon)\sin^2(\pi/4 + q\epsilon) = \frac{\cos^2(2q\epsilon)}{4} \approx \frac{1}{4}(1 + 4q^2) \\ &\cosh^2 r\epsilon \approx 1, \quad \sinh^2 r\epsilon \approx r^2 \quad d\rho^2 \approx dr^2. \end{aligned} \tag{8.9}$$

Using these we can in the end collect terms that goes as ϵ^2 and ignore higher orders, since this is the global factor defined on the radius of the geometry. Ultimately, we obtain

$$\begin{aligned} ds^2/R^2 &= 2dx^0 dU + dr^2 + r^2 d\hat{\Omega}_2^2 + 4du_4^2 + \frac{1}{8}dx_1^2 + \frac{1}{8}dx_2^2 \\ &\quad + \frac{1}{2}dx^0(x_1 dy_1 + x_2 dy_2) + \frac{1}{8}dy_1^2 + \frac{1}{8}dy_2^2 - u_4^2(dx^0)^2 \\ &= 2dx^0(dU - m_0) + h_0 - u_4^2(dx^0)^2 + 4du_4^2 + dr^2 + r^2 d\hat{\Omega}_2^2. \end{aligned} \tag{8.10}$$

In the SMT limit, the quadratic potential r^2 in the directions parametrized by $dr^2+r^2d\hat{\Omega}_2^2$ becomes infinitely steep, rendering these directions suppressed, and with a vanishing u_4 we obtain the $U(1)$-Galilean form.

8.2 The $OSp(2|2)$ flat background

Next we turn to the case where we keep the same number of generators over the S^7, but additionally we add a spin degree of freedom from the $\mathbb{CP}^1 \subset AdS_4$. We again look at the TNC data which in this case is

$$\begin{aligned} \tau &= d\tilde{x}^0, \\ m &= \frac{1}{2}\left(\cos\theta_1 d\phi_1 - \cos\theta_2 d\phi_2\right) - \sinh^2\rho dw, \\ h &= d\rho^2 + \frac{1}{8}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) + \cosh^2\rho\sinh^2\rho dw^2. \end{aligned} \tag{8.11}$$

Defining in the large charge limit for $J \to \infty$ the following new coordinates we find

$$r = \sqrt{J}\rho, \quad x_1 = \sqrt{J}(\theta_1 - \pi/2), \quad x_2 = \sqrt{J}(\theta_2 + \pi/2), \quad y_a = \sqrt{J}\phi_a, \tag{8.12}$$

$$\Longrightarrow$$

$$\begin{aligned} m_0 &= \lim_{J\to\infty} Jm = \frac{1}{4}\sum_{i=1}^{2} x_i dy_i + r^2 dw, \\ h_0 &= \lim_{J\to\infty} Jh = \frac{1}{8}\sum_{i=1}^{2}(dx_i^2 + dy_i^2) + dr^2 + r^2 dw^2. \end{aligned} \tag{8.13}$$

The corresponding Penrose limit can be obtained from the $AdS_4 \times \mathbb{CP}^3$ coordinates by introducing

$$\begin{aligned} &R = R'/\epsilon, \quad u = U\epsilon^2, \quad \phi_a = y_a\epsilon, \quad \theta_1 = x_1\epsilon - \pi/2, \\ &\theta_2 = x_2\epsilon + \pi/2, \quad \rho = r\epsilon, \quad \xi = \pi/4 + q\epsilon. \end{aligned} \tag{8.14}$$

In the $\epsilon \to 0$ limit, we obtain the metric

$$\begin{aligned} ds^2/R^2 = 2dx^0(dU + r^2\sin^2\bar{\theta}dw + \sum_{i=1}^{2} x_i dy_i) + \sum_{i=1}^{2}(dx_i^2 + dy_i^2) \\ + dr^2 + r^2(d\bar{\theta}^2 + \sin^2\bar{\theta}dw^2) + (q^2 + r^2\sin^2\bar{\theta})(dx^0)^2. \end{aligned} \tag{8.15}$$

We see that when $q^2 = -r^2$ and $\bar{\theta} = \pi/2$, we exactly retrieve the $U(1)$ Galilean data we derived from the large J limit above. One does indeed get the quadratic potential in this SMT limit.

8.3 The $SU(3|2)$ flat background

Instead of adding a spin degree of freedom this time, we extend to the maximal amount of generators for S^7 giving us the $SU(3)$ sector from the BPS-bound $Q = J_1 + J_2 + J_3 \leq E$. Our TNC data is

$$\tilde{\tau} = d\tilde{x}^0, \quad m = -\frac{1}{2}B, \quad h = 2d\Sigma_2^2. \tag{8.16}$$

One can write coordinates in the $J \to \infty$ limit as

$$\alpha = \frac{\pi}{4} + \frac{q}{\sqrt{j}}, \quad \theta = \frac{x}{\sqrt{j}} + \frac{\pi}{2}, \quad \phi = y/\sqrt{J}, \quad \psi = p/\sqrt{J}, \tag{8.17}$$

and from this obtain

$$\begin{aligned} m_0 &= \lim_{J\to\infty} Jm = \frac{1}{4}xdy - \frac{1}{2}qdp \\ h_0 &= \lim_{J\to\infty} Jh = 2dq^2 + \frac{1}{4}(dx^2 + dy^2) + \frac{1}{8}dp^2. \end{aligned} \tag{8.18}$$

The action subsequently takes the form

$$S_{\text{flat,gf}} = \frac{J}{4\pi} \oint d\sigma \Big[\frac{1}{2}x\dot{y} - q\dot{p} - \frac{1}{2}(x')^2 - \frac{1}{2}(y')^2 - (q')^2 - \frac{1}{4}(p')^2\Big]. \tag{8.19}$$

Similarly the coordinates that are chosen for the specific Penrose limit will be

$$\begin{aligned} &R = R'/\epsilon, \quad u = U\epsilon^2, \quad \phi_a = y_a\epsilon, \quad \theta_i = x_i\epsilon + \pi/2 \\ &\psi = p\epsilon, \quad \rho = r\epsilon, \quad \alpha = \pi/4 + q\epsilon, \quad \xi = \pi/4 + z\epsilon. \end{aligned} \tag{8.20}$$

After painstakingly expanding and bookkeeping powers of ϵ^2, the terms that are left after using the (x^0, u) in (7.62) are

$$ds^2 = 2dx^0(dU - m_0) + h_0 + \frac{1}{2}dz^2 + 2dq^2 + dr^2 + r^2 d\Omega_2^2 - (r^2 + 4z^2)(dx^0)^2. \tag{8.21}$$

The same kind of phenomena occurs in the three -charge case as for SYM, where the relativistic string experiences a quadratic potential r^2 in the now three transverse directions $dr^2 + r^2 d\Omega_2^2$. In the SMT limit with $x^0 = \tilde{x^0}/c^2, c \to \infty$, the potential becomes infinitely steep as well. Hence, the geometry is restricted to a $U(1)$-Galilean geometry described by the coordinates in the limit $J \to \infty$. The extra term of $2dq^2$ is a problem, as it will mess up the $U(1)$-Galilean structure in its corresponding limit. Due to time limitations, this has not been resolved yet.

8.4 The $OSp(4|2)$ flat background

The last background which corresponds to the $OSp(4|2)$ sector has to be considered as the maximal one, given the type IIA condition [47]. The TNC data for the BPS-bound $\Delta - J_1 - J_2 - J_3 - S$ is

$$\begin{aligned} \tilde{\tau} &= d\tilde{x^0}, \\ m &= -(\frac{1}{2}B + \sinh^2 \rho dw), \\ h &= d\rho^2 + \sinh^2 \rho \cosh^2 \rho dw + \frac{1}{2}d\Sigma_2^2. \end{aligned} \tag{8.22}$$

Again we define the coordinates that will go into play when taking the large charge limit

$$\alpha = \frac{\pi}{4} + \frac{q}{\sqrt{j}}, \quad \theta = \frac{x}{\sqrt{j}} + \frac{\pi}{2}, \quad \phi = y/\sqrt{J}, \quad \psi = p/\sqrt{J}, \quad \rho = r/\sqrt{J}. \tag{8.23}$$

Inserting into m and h, we find in the $J \to \infty$ limit

$$\begin{aligned} m_0 &= \lim_{J\to\infty} Jm = \frac{1}{4}xdy - \frac{1}{2}qdp + r^2 dw, \\ h_0 &= \lim_{J\to\infty} Jh = dr^2 + r^2 dw^2 + \frac{1}{2}dq^2 + \frac{1}{4}(dx^2 + dy^2) + \frac{1}{8}dp^2. \end{aligned} \tag{8.24}$$

In the large charge limit, we find the maximal extension of the action which hints towards having an interpretation in action-angle variables and symplectic potential in phase space:

$$\begin{aligned} S_{\text{flat,gf}} = \frac{J}{8\pi}\oint d\sigma \Bigg[& q\dot{p} - \frac{1}{2}x\dot{y} + r^2\dot{w} - (r')^2 - r^2(w')^2 \\ & - \frac{1}{2}\left((q')^2 + \frac{1}{2}\Big((x')^2 + (y')^2 \Big) + \frac{1}{4}(p')^2 \right) \Bigg]. \end{aligned} \tag{8.25}$$

Going to the Penrose limit we define as previously all the coordinates in the same manner:

$$\begin{aligned} R = R'/\epsilon, \quad u = U\epsilon^2, \quad \phi_a = y_a\epsilon, \quad \theta_i = x_i\epsilon + \pi/2 \\ \psi = p\epsilon, \quad \rho = r\epsilon, \quad \alpha = \pi/4 + q\epsilon, \quad \xi = \pi/4 + z\epsilon. \end{aligned} \tag{8.26}$$

Taking $\epsilon \to 0$, we finally obtain the final piece

$$\begin{aligned} ds^2/R^2 &= 2dx^0(dU + r^2\sin^2\bar{\theta}dw - \frac{1}{4}xdy + \frac{1}{2}qdp) + dr^2 + \frac{1}{2}dz^2 + 4dq^2 \\ &\quad + r^2(d\bar{\theta}^2 + \sin^2\bar{\theta}dw^2) + \frac{1}{4}(dx^2 + dy^2) + \frac{1}{8}dp^2 + (r^2\sin^2\bar{\theta} - z^2)(dx^0)^2 \\ &= 2dx^0(dU - m_0) + h_0 + dr^2 + \frac{1}{2}dz^2 + 2dq^2 + (r^2\sin^2\bar{\theta} - z^2)(dx^0)^2. \end{aligned} \tag{8.27}$$

In the "maximal" case we are restricted to submanifolds in which setting $\bar{\theta} = \pi/2$ and $z = \frac{1}{2}r$ exactly reproduces the result obtained in the large charge limit.

9

Conclusion and Outlook

In this thesis we have presented some of the current work and formalism in the field of AdS/CFT dualities, focusing on ABJM theory while comparing it to the first discovered and well known case of $AdS_5 \times S^5$ dual to $\mathcal{N} = 4$ SYM. By proceding as done in the literature on SMT and Penrose limits of subsectors of SYM, we have computed the corresponding limits in the case of the ABJM subsectors. These have shown to compare interestingly and in some cases intuitively enough to SYM in the form of extra terms that make sense given the additional features of various spheres and angles that appear in the geometry of the supergravity dual of ABJM.

On top of this, we have shown that the computation of spin matrix theory limits in the form of torsional Newton-Cartan geometry give rise to the same flat gauge action models as the corresponding computations in the framework of sigma models do. This is great and an important verification/suggestion that the SMT framework and thereby the computations done in this project seem to be on the right track, not yielding something completely different results but giving something that is comprehensive and compatible with already known literature.

In the future it seems promising to try to extend the work done for SYM in [81, 82, 83] to ABJM theory. This also includes computing the Penrose limit in the case of $OSp(2|2)$ and avoiding the extra $2dq^2$ term for $SU(3|2)$ which the time of the deadline of this thesis unfortunately did not allow for, though conceptually this should be straight forward. What could also be done is to generate the new giant magnon. The spin element of this could be implemented in various corrections to both the string, magnons, etc. It would also be interesting to look at theory of how magnetic monopoles would contribute to partition functions in the framework of $\mathcal{N} = 6$ CS [84].

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