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MSc thesis

Tidal response of black holes

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Abstract

Tidal Love numbers encode gravitational response to external tidal fields generated by companions. These depend on the structure of the gravitating object, such as a black hole or a neutron star, and in a binary coalescence are measurable in the last stages of the inspiral before the merger. Quite strikingly, the black hole tidal Love numbers are zero. By now, they have been calculated for Schwarzschild, Reissner-Nordström, and Kerr black holes. Several of these calculations are reviewed here. Special emphasis is given to the case of Kerr black holes, where the Love numbers have been a matter of debate. Zero tidal and nonzero dissipative Love numbers for Kerr black holes have been obtained. This computation, however, relies on a specific regularisation scheme, namely analytic continuation in the harmonic quantum numbers. Here, the response of the Kerr-Newman black holes to charged scalar field perturbations is described. This is used to obtain the Kerr Love numbers for scalar field perturbations in the zero-charge limit. The black hole charge serves as a regularisation parameter and, unlike the analytic continuation approach, this procedure is physically well-defined. Zero tidal and nonzero dissipative Love numbers are obtained in full agreement with the method of analytic continuation.

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Contents

1 Introduction

1.1 Black hole detection and tidal imprints

Over the past century, several big leaps have been made in our understanding of the physical world. New theories have emerged, with general relativity [1] being one of the more noteworthy. It introduces the idea that spacetime is curved. Finding actual spacetime geometries that conform to general relativity is a formidable task, and only a handful have been found. However, associated with them is a variety of features, many of which initially sparked controversy, but with time have gained acceptance and even been observed.

The first exact curved spacetime [2] that was formulated was designed to represent the gravitational field around a star and describe interactions with it in the context of general relativity. A remarkable implication of this spacetime, which became clear only years later, is that a compact enough object creating the curvature would become hidden inside a region where gravity is so intense that nothing can escape it. This is a black hole.

The formation of black holes depends on whether such compact objects can be created at all. Even theoretically, this is not possible in Newtonian gravity [3], but it is in general relativity [4]. Stars collapse after their nuclear fuel has run out and the nuclear reactions cannot provide enough pressure to counteract their own gravity. Not all collapsing stars, however, turn into black holes. The lighter stars become white dwarfs, heavier stars end their life as neutron stars, and only even heavier stars form black holes [5]. Besides the collapse of self-gravitating objects, there are other channels for black hole formation. It is believed that black holes of very different sizes exist in nature. At the center of most galaxies, almost certainly, are supermassive black holes, with as much as ten billion solar masses. Then again, primordial black holes possibly many orders of magnitude lighter than stars can also exist. Primordial black holes could have formed in the early universe, these are among the candidates for dark matter [6–8].

Black holes are thought to be simple objects, characterized by very few parameters. This is known as the no-hair conjecture [9]. The first black hole spacetime was fully characterized by its mass. Since then, other black hole spacetimes have been described [10–12], nevertheless, even the most general one depends only on the mass, angular momentum, and charge. The most prominent feature of a black hole is its boundary - the event horizon. Once an object crosses the event horizon, it is forever trapped inside the region from which not even passing information to outside observers is possible.

Exact spacetimes depend on conditions that hold only approximately in nature, and they can be generically used to represent some particular regions of the actual spacetime only if they are stable. The stability of the exact spacetimes can be studied by exploring the dynamics of slight deviations in the spacetime geometry. Analytically this is done with perturbation theory. Studying black holes and their dynamics is important for a better grasp of general relativity's predictions about the nature of gravity and observing them is a way to test the theory [13–17].

Since black holes trap light, it is impossible to see them with telescopes, except for the indirect observations of the motion of luminous objects near them, or images of the black hole shadows on the distorted backgrounds of their accretion disks. Such images for the supermassive black holes at the centers of M87 [18] and Milky Way [19] were recently taken with the Event Horizon Telescope. While this already strongly indicates that black holes exist, it required the LIGO and VIRGO observation of gravitational waves, to definitively confirm black hole existence [20, 21].

Gravitational waves are a fundamentally relativistic concept, theorized already in the early days of general relativity. They are ripples in spacetime that emanate from accelerating massive objects, particularly ones in periodic motion. The properties of the sources, such as distribution of mass, angular momentum, and orbital parameters, are reflected in the shape of the gravitational waves they emit. Black holes, neutron stars, and white dwarfs are the most compact objects there can be. Therefore, they are the most efficient sources of gravitational waves. Strong gravitational waves are produced where spacetime curvature is large, which requires masses to be compressed into small regions. The emitted waves can traverse vast distances without much attenuation or scattering, making it possible to observe far-away cosmic events. The detection of gravitational waves from binary coalescences allows, for the first time, to explore the properties of black holes.

In a binary coalescence, two closely orbiting compact objects release gravitational waves causing them to spiral toward one another and eventually merge into a single object. The coalescence is broken down into three phases - inspiral, merger, and ringdown. This way the complex analysis of the dynamics of the coalescence can be treated with different modeling methods appropriate for each phase. During the initial inspiral, the two objects orbit each other, gradually losing energy to the emitted gravitational waves, causing their orbits to shrink, consequently increasing the orbital frequency. Analytically, this phase is treated in the post-Newtonian approximation. As the binary becomes tighter, the gravitational wave emission becomes more intense. The inspiral ends with the plunge, where the orbit rapidly shrinks, and the emitted gravitational waves reach their peak amplitude and frequency. This phase is followed by the merger where the objects collide. A gravitational wave burst is produced. During this period, quasi-circular binary motion transitions into quasinormal ringing of a single object. Due to the large curvature and the fast dynamics, the perturbative treatment is no longer valid. The analysis relies on numerics. In the ringdown phase, the object formed in the merger is vibrating in quasinormal modes. Gravitational waves are produced with decreasing intensity as the compact object settles down. This phase is described in the context of perturbation theory.

Gravitational waves from events such as binary coalescences have distinct waveforms corresponding to specific sources. The waveform analysis of gravitational wave detectors relies on perturbative methods to predict and confirm the detected signals. Through gravitational wave observations, the properties of the merging objects can be explored. Already the first gravitational wave detection, GW150914, allowed identifying the coalescing objects as black holes, proving their existence and also that of gravitational waves [22].

One way to study the properties of black holes is by analyzing quasinormal modes, probing, in particular, the light ring. The dynamics of the event horizon can be explored by considering tidal effects caused by external perturbations. These provide ways to test the stability of black holes and the no-hair conjecture, which are of tremendous theoretical significance. Tidal effects are the focus of this thesis.

Tidal deformations induced by the companions in binary coalescences would cause orbital changes and leave traces in the gravitational waves emitted from the binaries in the late stages of the inspiral. The fact that these orbital changes could be observable with Earth-based gravitational wave detectors was quantitatively established in a study on the tidal coupling of neutron stars [23]. Tidal deformations hold information about the intrinsic structure of objects and their equation of state. This, in particular, is important in the study of neutron stars as it could provide a way to determine, among other, their size. The imprints in gravitational waves from tidal deformations are present long enough before the merger to make their effect discernible on background inspiral models without tides. The GW170817 event [24] was the first detection of a neutron star coalescence. With both a gravitational and an electromagnetic observation, this event placed the first constraints on neutron star size and structure.

In the context of black holes, with the establishment of a relativistic framework in which to study tides [25], tidal deformations of various kinds of black holes have been explored [25–31]. The generally accepted conclusion is that neither non-rotating black holes nor rotating uncharged ones can be tidally deformed, though, dissipative effects can be expected in the rotating case. Studies exploring whether rotating black holes would have a dissipative response have been carried out and concluded that it is present [29]. However, there has been a debate about the robustness and meaning of this result and the method employed to derive it. By first describing the dynamics of charged scalar field perturbations for the most general charged and rotating black hole spacetimes, this thesis aims to then determine the response for the uncharged rotating black holes as a limit, using the charge as a physical regularization parameter, and thus provide an alternative physical method for finding the dissipative response.

1.2 Outline

After a short introduction to black holes, tidal deformations, and the prospects of their observation in chapter 1, the basic principles of general relativity, the Einstein equation, and black holes as exact solutions to the Einstein equation are discussed in chapter 2. A review of basic concepts in differential geometry, the highlights of the derivation of the Einstein equation from the Hilbert action, the formulation of general relativity in the tetrad formalism and in Newman-Penrose formalism, as well as the Goldberg-Sachs theorem, are given in appendix A.

Chapter 3 focuses on black hole perturbation theory, where the linearized Einstein equation, gauge choices, and unphysical perturbations are discussed. Then scalar field, vector field, and metric perturbations are described. The two conventional methods for treating perturbations on a background with spherical and axial symmetry are introduced together with the master equations including the Regge-Wheeler and Zerilli equations for the spherically symmetric case and the Teukolsky equation for the axially symmetric case. The dynamics of charged scalar fields on a dyonic Kerr-Newman background spacetime are described last. Key steps in the derivations of the master equations are summarized in appendix B.

In chapter 4, the relativistic formulation of tidal effects is introduced. The response of black holes is expressed in terms of tidal and dissipative Love numbers. The surprising results of zero Love numbers for the various black holes so far investigated are summarized. Some details of finding the Schwarzschild black hole Love numbers are presented. The method for finding the Kerr black hole love numbers is shown. The tidal response of charged scalar field perturbations on dyonic Kerr-Newman black hole backgrounds is described for the first time. The result of the analytic continuation approach conventionally used in the Kerr case is confirmed as a limit of the Kerr-Newman charged scalar field response calculation providing an alternative, physically motivated method.

Chapter 5 contains a summary of the results, conclusions, and outlook.

1.3 Conventions and notation

Throughout the thesis natural units $c = G = 1$ are adopted and exclusively four-dimensional spacetime with the mostly plus convention $(-+++)$ is used. Tetrad components are denoted with lowercase Latin letters from the beginning of the alphabet (abc), tensor components are denoted with lowercase Latin letters from the latter part of the alphabet (ijk), spacetime coordinate components are denoted with lowercase Greek letters from the latter part of the alphabet $(\mu\nu\rho)$, while the three-dimensional space coordinate components following the common convention are denoted with lowercase Latin letters from the latter part of the alphabet (ijk), to be distinguished from general tensor components based on context. For the treatment of spherically symmetric metrics, the components of the Lorentz part of the metric are denoted with uppercase Latin letters from the latter part of the alphabet (IJK), and components of the spherical part of the metric are denoted with uppercase calligraphic Latin letters $(\mathcal{IJK})$. To indicate particular coordinates, letters $(t, r, \vartheta, \varphi)$ are used, to indicate specific tetrad components, numbers $(1, 2, 3, 4)$ are used, Newman-Penrose tetrad basis vectors are named $(l, n, m, \bar{m})$, the naming of various other objects in Newman-Penrose formalism follows the convention established by Newman and Penrose [32]. Indices in equations imply the object type, the objects in text are often referred to without indices, abstract index notation is used when necessary. When a specification is required, perturbations are marked with $..^B$, and the background quantities with $..^A$, in particular, this distinction is used in the treatment of perturbations with the Newman-Penrose formalism. Regarding the naming of often used quantities, uppercase letters (M, C, J, G, R, S, T) are used for observable quantities and geometric objects that are not related to position. Lowercase letters are used for naming quantities like position x, velocity u, angular velocity a, metric g. Square brackets are used to indicate the variables of functions.

2 Black holes in general relativity

2.1 The Einstein equation

General relativity is the theory for gravity. It describes how matter content determines the geometry of spacetime and how, in this spacetime, the matter evolves and propagates [1, 33, 34]. This two-way interaction follows the fundamental equation of the theory - the Einstein equation

$$G_{\mu\nu}[g] = T_{\mu\nu}[g, \phi] \,. \tag{2.1.1}$$

Here, T is the energy-momentum tensor and G is the Einstein tensor. Both are represented in a coordinate basis. The energy-momentum tensor depends on the metric g and some matter field ϕ. The Einstein tensor is metric-dependent, it combines quantities describing spacetime curvature.

In developing the theory, Einstein incorporated several key physics concepts and mathematical tools he and others had worked on in recent years. In 1905, he presented special relativity [35], a theory that describes the physics of objects moving at constant speeds, particularly those moving at speeds comparable to the speed of light. This brought a new understanding of space and time. In 1908, Minkowski formulated spacetime as a four-dimensional continuum where time is treated analogously to the three spatial dimensions [36]. The equivalence principle, which states that in a small enough region of spacetime, the effects of gravity are indistinguishable from acceleration, was refined by Einstein in 1907 [37]. It led to the consideration of a curved spacetime making it possible to dispense with the treatment of gravity as a force. The mathematical formalism for curved spaces, which Einstein needed for his theory of gravity, was already well-established. By 1900, mathematicians like Riemann, Ricci, and Levi-Civita had developed the formalism of tensor calculus on Riemannian manifolds [38, 39]. Einstein's theory of general relativity heavily relies on the tensor formalism. Grossmann's work in collaboration with Einstein laid the groundwork for the development of general relativity [40]. A review of the differential geometry basics used in general relativity is given in appendix A.1. The synthesis of these ideas with Einstein's groundbreaking insight into the nature of gravity as the curvature of spacetime resulted in the theory of general relativity.

The Einstein equation was found independently with different methods by Hilbert and Einstein [41]. Einstein's approach was primarily based on the equivalence principle. He started with the idea that the laws of physics should be the same for all observers at rest or in uniform motion. The key insight was to consider gravity not as a force between masses, as described thus far, but as a curvature of spacetime. While the priority is often given to Einstein, who gave a more physically motivated derivation, almost simultaneously Hilbert presented an independent derivation of the Einstein equation using the variational principle [42]. His method is discussed in appendix A.2.

The study of gravity in the context of general relativity before all else entails finding solutions to the Einstein equation. Each solution represents a distinct spacetime geometry that relates the curvature of spacetime to the distribution of matter within it. Due to the nonlinear and coupled nature of its components, solving the Einstein equation is a formidable task, making exact analytical solutions difficult to find. Spacetime is viewed as a dynamic entity that curves in response to changing matter distribution, leading to different physical phenomena such as gravitational attraction, cosmological expansion, gravitational waves, and black holes.

The Einstein equation is typically solved under the constraints of some specific properties that the spacetime is assumed to have. Geometric constraints, such as spherical or axial symmetry, reduce the number of independent components of the metric tensor. Further algebraic constraints of the constituent parts of the equation can be used to find simpler forms of the Einstein equation components. Using such methods, over the last century, several exact solutions to the Einstein equation have been found.

2.2 Classification of spacetimes

With the evolution of general relativity, several frameworks for it have been developed. In some circumstances, instead of the standard coordinate treatment, the tetrad formalism [43] can be used. The tetrad formalism in more detail is introduced in appendix A.4. Another formalism, one that has been key in finding exact solutions to the Einstein equation as well as in studying gravitational waves in perturbation theory, is the Newman-Penrose formalism [44]. A tetrad basis of complex null vectors $(l^a, n^a, m^a, \bar{m}^a)$ is chosen, and the quantities such as the Weyl tensor and the Ricci tensor are projected onto this basis forming scalars. The equations of general relativity are split into components that can be treated individually. The Newman-Penrose quantities are introduced in appendix A.5 and the Newman-Penrose equations are available in the sources by Newman and Penrose [32] and Chandrasekhar [45].

The Petrov classification [46], developed in 1954, characterizes the Weyl tensor based on algebraic properties. Extending beyond gravitational radiation, commonly associated with the Weyl tensor, Petrov classification makes it possible to analyze and interpret the solutions to the Einstein equation by identifying specific characteristics and behaviors associated with the spacetimes of the different Petrov types. The Petrov classification is most easily described in the Newman-Penrose formalism. Some basic properties can be seen in the tensor variant, a more in-depth treatment can be done in the spinor variant of the formalism [32, 47]. The values of the Weyl scalars $\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4$ determine the Petrov type of the spacetime. The six different Petrov types (I, II, D, III, N, O) are distinguished by which combinations of the Weyl scalars are zero:

$$
\begin{aligned}
&\text{Type I:} && \Psi_0 = 0 \,, \\
&\text{Type II:} && \Psi_0 = \Psi_1 = 0 \,, \\
&\text{Type D:} && \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0 \,, \\
&\text{Type III:} && \Psi_0 = \Psi_1 = \Psi_2 = 0 \,, \\
&\text{Type N:} && \Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0 \,, \\
&\text{Type O:} && \Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0 \,.
\end{aligned}
\tag{2.2.1}
$$

These specific combinations of Weyl scalars are associated with a particular Newman-Penrose tetrad, however, the Lorentz symmetry mentioned in appendix A.4 makes choosing this tetrad always possible.

The algebraically different Petrov types carry distinct physical implications. Type D regions are linked to the gravitational fields surrounding localized massive objects. The tidal effects within a Type D region exhibit tension in one direction and compression in the orthogonal directions. Thus such gravitational fields closely mirror those described in Newtonian gravity. If the gravitating object is rotating, in addition to tidal effects, various gravitomagnetic effects, such as spin-spin forces, appear. Type III regions are linked to a form of longitudinal gravitational radiation. In these regions, tidal forces induce shear. Type N regions are associated with transverse gravitational radiation. The quadruple principal null direction leads to the wave vector determining the propagation direction. Type II regions intricately combine the effects of types D, III, and N. For type O regions, the Weyl tensor is zero. In such cases, the curvature is described solely by the Ricci scalar, corresponding to conformally flat geometry, where all gravitational effects arise from non-gravitational matter fields. Long-range influences are damped, and gravitational fields of distant regions cannot be detected.

Petrov type I spacetimes are algebraically general, the rest are algebraically special. The Goldberg-Sachs theorem [48], described in more detail in appendix A.5, relates specific geometric properties of spacetimes with the algebraic properties of the Weyl tensor, in particular, it states that an algebraically special vacuum spacetime contains at least one geodesic, shear-free null congruence, and the other way round. With this, algebraic properties can be incorporated as specific complementary constraints to those from symmetries for solving the Einstein equation. They have been used in deriving exact black hole solutions and black hole perturbation equations.

2.3 Black holes

Black hole spacetimes are the ones relevant to this thesis. They are solutions to the Einstein-Maxwell equations with a region from which no events can reach future null infinity. The boundary of this region is a null surface known as the event horizon.

In four dimensions with an asymptotically flat boundary, the most general solution is the Kerr-Newman black hole. Its line element in Boyer-Lindquist coordinates $(t, r, \vartheta, \varphi)$ is

$$ds^2_{\mathrm{KN}} = -\frac{\Delta - a^2\sin^2\vartheta}{\Sigma}dt^2 - 2a\sin^2\vartheta\frac{r^2+a^2-\Delta}{\Sigma}dtd\varphi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\vartheta^2 + \sin^2\vartheta\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\vartheta}{\Sigma}d\varphi^2 \ , \qquad (2.3.2)$$

where M is the mass, $J = aM$ is the angular momentum, Q and P are the electric and magnetic charges, and the abbreviations $\Sigma = r^2 + a^2\cos^2\vartheta$, $\Delta = r^2 + a^2 - 2Mr + Q^2 + P^2$ are used. In the intermediate gauge [49], the associated Maxwell field is

$$A = -\frac{Qr}{\Sigma}\left(dt - a\sin^2\vartheta d\varphi\right) + \frac{P\cos\vartheta}{\Sigma}\left(adt - (r^2+a^2)d\varphi\right) \ . \qquad (2.3.3)$$

The field strength is

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \ . \qquad (2.3.4)$$

Then the electric and magnetic charges are defined[1] as

$$Q = \frac{1}{4\pi}\int_{S^2}\star F \ , \qquad P = \frac{1}{4\pi}\int_{S^2} F \ , \qquad (2.3.5)$$

Magnetic charges are included as they are allowed by the Einstein-Maxwell equations. While in the intermediate gauge A is not regular everywhere, the physical F is. This geometry has two horizons defined by

$$\Delta = 0 \quad \Rightarrow \quad r_\pm = M \pm \sqrt{M^2 - a^2 - P^2 - Q^2} \ , \qquad (2.3.6)$$

the outer one of which is the event horizon.

The Kerr-Newman black hole was the last to be found. All the earlier black hole solutions are special cases of the Kerr-Newman black hole.

The Schwarzschild black hole solution was found in 1916 [2]. It is the first non-trivial exact solution to the Einstein equation. It describes an asymptotically flat vacuum spacetime geometry outside a spherically symmetric, non-rotating, uncharged mass. Spherical symmetry is the key constraint that leads to enough simplification in the Einstein equation components, that the solution can be found. Furthermore, the Birkhoff-Jebsen theorem states that any spherically symmetric solution to the vacuum Einstein equation is necessarily Schwarzschild in the region outside the mass [50, 51].

Each following black hole solution is linked to a reduction of the constraints used in solving the Einstein equation. The Reissner-Nordström black hole describes the spacetime outside a non-rotating, charged mass. The metric was derived independently by Reissner [10], Weyl [52], Nordström [53], and Jeffery [54]. Some complication comes from the introduction of an electromagnetic field - instead of the vacuum Einstein equation, the Einstein-Maxwell equations have to be solved. Nevertheless, the same assumption as in the Schwarzschild case, that the asymptotically flat spacetime has spherical symmetry holds.

Systems with angular momentum only retain axial symmetry. The Kerr metric is a solution to the Einstein equation that describes the geometry of spacetime around a rotating, uncharged mass. It was presented in 1963 [11], considerably later than the spherically symmetric black hole solutions. All attempts using the same approach as Schwarzschild did of restricting the form of the metric using symmetry arguments, and then solving the Einstein equation were largely unsuccessful [55]. While some stationary, axially symmetric solutions were found, the first

[1] The Hodge dual of a p-form $X_{a_1\dots a_p}$ is a $(d-p)$-form given by $\star X_{b_1\dots b_{d-p}} = \frac{1}{p!}\epsilon_{b_1\dots b_{d-p}a_1\dots a_p}X^{a_1\dots a_p}$, where d is the manifold's dimension.

one by Lewis [56] and others after that [57–59], these had singularities and were not asymptotically flat. What proved essential in finding a physically meaningful solution was Kerr's introduction of the condition for the spacetime to be algebraically special - Petrov type II. Symmetries of the metric are linked to algebraic properties of the Weyl tensor, which therefore complement one another. Petrov classification is not easily incorporated in the Schwarzschild-like way of finding solutions. Instead, a null tetrad can be used to start. The approach is to first identify a coordinate basis, then impose stationarity and axial symmetry of the spacetime and express the metric in terms of the associated conserved quantities - the mass and the angular momentum [60].

Finding a charged generalization of the Kerr solution by solving Einstein equations directly was difficult. However, in 1965 it was shown that with a complex coordinate transformation, the Kerr solution could be obtained from the Schwarzschild result, in what is known as the Newman-Janis algorithm [61]. The same transformation can be applied to the Reissner-Nordström solution. Then the Maxwell equation has to be solved to find the corresponding electromagnetic field. It is most straightforwardly done with Keane's extension of the Newman-Janis algorithm [62]. This leads to the full Kerr-Newman black hole solution, found in 1965 [12].

The no-hair conjecture states that black holes are characterized at most by their mass [63], charge [64], and angular momentum [65]. The static, uncharged Schwarzschild black holes are fully described by their mass, the static, charged Reissner-Nordström black holes - by mass and charge, the stationary, uncharged Kerr black holes - by mass and angular momentum, and the most general stationary, charged Kerr-Newman black holes - by mass, angular momentum, and charge.

Black holes are Petrov type D stationary solutions to the Einstein-Maxwell equations. This can be seen by choosing the Kinnersley frame B.2.131 [66] and calculating the spin coefficients. The ones that are zero with Goldberg-Sachs theorem imply zero Weyl scalars $\Psi_0, \Psi_1, \Psi_3, \Psi_4$ corresponding to Petrov type D. Black holes are instances of the Kerr-Schild class of metrics [67], which can be expressed as

$$g_{\mu\nu} = \eta_{\mu\nu} + F l_\mu l_\nu \ ,$$

where η is the flat Minkowski metric, F is a scalar, and l is a null vector with respect to η.

3 Relativistic theory of perturbations

3.1 Linearized Einstein equation

So far, only exact solutions of the Einstein equation have been considered. Finding exact solutions, however, is often not possible. In particular, this is the case when the spacetime lacks symmetry. In general, symmetries cannot be assumed in dynamical systems. However, small fluctuations can be described with perturbation theory. This approach can be used to compute the gravitational wave emission from sources and the gravitational fluctuations on black hole spacetimes.

When the spacetime has some small deviations h from a known background metric $\bar{g}$, these metric perturbations h can be found by perturbatively expanding and then solving the Einstein equation along with any other equations of motion for the coupled matter fields. The metric perturbations can be thought of as being induced by perturbations δT of the energy-momentum tensor $\bar{T}$, but perturbations without a source are also possible. Small field perturbations $\delta\phi$ of the background fields $\bar{\phi}$ induce small perturbations of the energy-momentum tensor. In general, fields appear to second order in the expressions for the energy-momentum tensor, as in equations A.3.5 and A.3.10. Therefore, if the background field value is $\bar{\phi} = 0$, to linear order, small field perturbations do not influence the energy-momentum tensor, and the field perturbations $\delta\phi$ are decoupled from the metric perturbations h. The fields with nonzero background values are unavoidably coupled. Taking $g = \bar{g} + h$ and $\phi = \bar{\phi} + \delta\phi$, with the known background variables $\bar{g}, \bar{\phi}$ solving the Einstein equation, and then expanding the Einstein equation to linear order leads to the linearized Einstein equation

$$\delta G_{\mu\nu}[h] = \delta T_{\mu\nu}[\delta\phi] \ . \tag{3.1.1}$$

When the background energy-momentum tensor is zero, the perturbations of the energy-momentum tensor are gauge invariant. The components of the metric, however, have gauge freedom. It may be possible to define a slightly perturbed metric in one coordinate system, that has the same form as only the unperturbed part of the metric in another coordinate system. It must be that in this case, the metric perturbations are unphysical since a small transformation of coordinates can produce them. True metric perturbations do not include contributions from coordinate transformations. Small changes in variables are characterized by the Lie derivative:

$$h_{\mu\nu} \to h_{\mu\nu} + \mathcal{L}_X \bar{g}_{\mu\nu} = h_{\mu\nu} + 2\nabla_{(\mu} X_{\nu)} \ , \tag{3.1.2}$$

$$\delta\phi \to \delta\phi + \mathcal{L}_X \bar{\phi} = \delta\phi + X^\mu \nabla_\mu \bar{\phi} \ , \tag{3.1.3}$$

and by the gauge freedom of a linear theory, variables $h, \delta\phi$ and $h + \mathcal{L}_X \bar{g}$, $\delta\phi + \mathcal{L}_X \bar{\phi}$ can be regarded as describing the same physical solution. If $\bar{\phi} = 0$, then $\delta\phi$ is gauge-invariant.

In general, the perturbation of the Einstein tensor is complicated, but for vacuum backgrounds, it is

$$\delta G_{\mu\nu} = \delta R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \delta R = \delta R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \delta R_{\rho\sigma} \ . \tag{3.1.4}$$

Indices in the perturbation equations are raised and lowered with the unperturbed metric and its inverse. In a locally inertial frame, the Ricci tensor involves the derivatives of the Christoffel connections. Then its perturbation consists only of the derivatives of the perturbed Christoffel connections. By promoting the derivatives to covariant derivatives a tensor is obtained:

$$\delta R_{\mu\nu} = \delta\Gamma^\lambda_{\mu\nu;\lambda} - \delta\Gamma^\lambda_{\mu\lambda;\nu} \ . \tag{3.1.5}$$

This is the Palatini identity [68] defining the perturbed Ricci tensor. The perturbed Christoffel connection is

$$\delta\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} (h_{\nu\rho;\mu} + h_{\rho\mu;\nu} - h_{\mu\nu;\rho}) \ . \tag{3.1.6}$$

With the above expressions the perturbation of the Einstein tensor for vacuum background cases can be written in terms of the metric perturbation as

$$\delta G_{\mu\nu} = \frac{1}{2} \left(-h_{\mu\nu;\rho}{}^{\rho} + h_{\rho\mu;\nu}{}^{\rho} + h_{\rho\nu;\mu}{}^{\rho} - h^\rho{}_{\rho;\mu\nu} + g_{\mu\nu} h^\rho{}_{\rho;\sigma}{}^{\sigma} - g_{\mu\nu} h^{\rho\sigma}{}_{;\rho\sigma} \right) \ . \tag{3.1.7}$$

3.2 Gravitational waves

Gravitational waves were first theoretically predicted by Einstein in 1916, shortly after he published his theory of general relativity. This precedes even the first exact solution of the Einstein equation - the Schwarzschild spacetime. Einstein's original approach to deriving gravitational waves involved linearizing his field equations and examining the propagation of small perturbations on a flat spacetime background. Analogously, gravitational waves can be considered on known curved backgrounds as well. He demonstrated that disturbances in the curvature of spacetime could propagate as waves with, to a first approximation, the speed of light. The solutions he derived suggested that massive accelerating bodies, such as binary systems of compact objects could emit gravitational waves. This work represented a significant step in understanding the theory's implications. Over the subsequent decades, other physicists built upon Einstein's foundational work, refining the theory of gravitational waves and exploring their implications.

The perturbed metric g assuming small perturbations h and a flat background spacetime η is

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \; . \tag{3.2.8}$$

In the following, the approach is to always retain quantities up to first order in small parameters. It is always possible to select the Lorentz gauge by an appropriate coordinate transformation:

$$\bar{h}^{\mu\nu},_{\nu} = 0 \; , \tag{3.2.9}$$

where $\bar{h}_{\mu\nu}$ is the trace-free part of the perturbation:

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^{\rho}{}_{\rho} \; . \tag{3.2.10}$$

With this choice, the linearized Einstein equation reduces to the wave equation

$$\Box\bar{h}^{\mu\nu} = -2\delta T^{\mu\nu} \; . \tag{3.2.11}$$

If the source is zero, a solution with $k^2 = 0$ is

$$\bar{h}^{\mu\nu} = \mathcal{R}e\{A^{\mu\nu}e^{ik\cdot x}\} \; . \tag{3.2.12}$$

This is a gravitational wave. The perturbations cannot be gauged away, gravitational waves are not coordinate waves, but they are physical. They have two degrees of freedom, in the traceless-transverse gauge they are h_+ and $h_\times$. The effects of passing gravitational waves can be seen by considering a set of measuring test particles. Gravitational waves move at about the speed of light. The relative motion of the test particles can be described by choosing a frame where the test particles are at rest and using the geodesic deviation equation. Checking for various directions in which the test particle can be offset, the quadrupolar behavior in both the h_+ and the $h_\times$ polarizations becomes apparent. The passing of a gravitational wave has the effect of stretching in one direction while simultaneously squeezing in the other.

The production of gravitational waves requires sources. These are most straightforwardly described in spherical coordinates. Correspondingly, in the spherically transverse gauge, the perturbation is related to the source through the quadrupole formula

$$\bar{h}_{ij} = \frac{2}{r}\ddot{I\!\!\!-}_{ij}(t-r) \; , \tag{3.2.13}$$

where $I\!\!\!-$ is the reduced quadrupole moment, which in terms of the energy density ρ is

$$I\!\!\!-_{ij} = \int d^3x\rho\left(x_i x_j - \frac{1}{3}\delta_{ij}r^2\right) \; . \tag{3.2.14}$$

It is calculated by defining a surface around the source and using Gauss' theorem to integrate the linearized Einstein equation. The flux of energy from the source to gravitational waves is

$$\frac{dE}{dt} = -\frac{1}{5}\left\langle \partial_t^3 I\!\!\!-_{ij}\partial_t^3 I\!\!\!-^{ij}\right\rangle \; , \tag{3.2.15}$$

and the flux of angular momentum correspondingly is

$$\frac{dL_i}{dt} = -\frac{2}{5}\epsilon_{ijk}\left\langle \partial_t^2 {I\!\!\!-}^{jl}\partial_t^3 {I\!\!\!-}_l{}^k \right\rangle \ , \tag{3.2.16}$$

where the time averaging is denoted with the angled brackets [69].

The quadrupole moment of a source can be estimated to be proportional to the moving mass M in a system and the square of its size R. Taking a period where the changes to the source are slight, the power radiated with the gravitational waves can be estimated as

$$\frac{dE}{dt} \sim \left(\frac{M}{R}\right)^5 \ . \tag{3.2.17}$$

The more compact the source the more power radiated. Black holes are the most compact objects there can be, and thus the most efficient gravitational wave sources. This makes the study of gravitational wave emission by black holes important.

3.3 Black hole perturbation theory

Black hole perturbation theory is a framework in which black hole stability and uniqueness can be studied [70]. The linearized Einstein equation describing the dynamics of the perturbations is a complicated equation with coupled components dependent on the coordinate variables in an intricate way. Nevertheless, in the case of Schwarzschild and Reissner-Nordström [71, 72], as well as Kerr black hole perturbations [73], using symmetries and algebraic properties, the components of the linearized Einstein equation can be decoupled, and the variables can be separated leading to single variable dependent master equations governing the perturbations of the Schwarzschild, Reissner-Nordström, and Kerr black holes. In the case of Kerr-Newman black hole perturbations, one is left with a system of two coupled differential equations [45]. While Schwarzschild and Reissner-Nordström black holes are static and spherically symmetric, the Kerr and Kerr-Newman black holes are stationary and axisymmetric.

Since the symmetries of the black holes are not the same, the methods used in obtaining the master equations by simplifying the Einstein equation or other equations of motion depending on the perturbation type are also different. These are the subject of this section with more details on the derivations in appendix B.

A choice of sources and boundary conditions specifies perturbations, but all perturbations obey the master equations. Therefore, the formalism presented here has a wide variety of applications. These include studying black hole oscillations described with quasinormal modes and the tidal effects of external perturbations. The latter are considered in this thesis.

3.3.1 Perturbations of spherically symmetric black holes

The spherical symmetry of the background can be used to decompose objects in tensor spherical harmonics. The method of decomposition is described by Martel and Poisson in [74, 75], and summarized in appendix B.1. The master equations described here and derived in appendix B.1 closely follow Berti's presentation [3].

Scalar field perturbations are discussed first, followed by vector field perturbations, and metric perturbations. The various perturbations of spherically symmetric spacetimes each follow one master equation that depends only on the radial coordinate. Although these equations are derived by considering the equation of motion of the specific field, which for scalar field perturbations is the Klein-Gordon equation, for vector perturbations the Maxwell equation, and for metric perturbations the Einstein equation, all resulting master equations have the same general form:

$$\partial_{r*}^2\psi_{s\pm} + \left(\omega^2 - V_{s\pm}\right)\psi_{s\pm} = S \ . \tag{3.3.1}$$

Here the master equations are written in the Fourier time-decomposed form with the frequency ω dependence for each master variable $\psi_{s\pm}$ implied. The radial tortoise coordinate $r*$ is introduced to remove the first-derivative term in the equations. $V_{s\pm}$ are the potentials and S represent sources. The considered perturbing field type is indicated with s, $s = 0$ for scalar, $s = 1$ vector, and $s = 2$ tensor perturbations. The even or odd sectors of the perturbations are indicated with $\pm$, they behave differently under coordinate transformations and are thus independent from one another. While in principle massive vector fields could be considered with the Proca equation [76] replacing the Maxwell equation, this would cause issues with considering interactions covering the entire spacetime, only interactions mediated by the massless vector fields are long-range. The master variable $\psi_{s\pm}$ relations to the corresponding field perturbations as well as the associated potentials $V_{s\pm}$ are described below.

The master equations can be considered with the source term set to zero $S = 0$. In this work, no other sources but the fields themselves are considered, and they are interpreted as being infinitely far away, by having boundary conditions to the master equations that allow the solutions to grow asymptotically.

There are two spherically symmetric black hole solutions, and the decomposition of the variables as described below can be used for both. However, while the Schwarzschild background is a vacuum solution, the Reissner-Nordström is not. As a result, finding the master equations in the latter case is a more elaborate process. Here the Schwarzschild case is considered first.

When perturbations are sourced by scalar and vector fields that are zero in the background, since they appear to second order in the expressions for the charged current and the energy-momentum tensor, to first order these perturbing fields do not influence the sources. This for the Schwarzschild black hole implies that there is no mixing between the scalar field, the vector field, and the metric perturbations, they can be considered separately.

The Klein-Gordon equation for the scalar field ϕ reduces to the scalar field master equation with the master variable ψ_0 related to the scalar field as

$$\phi[t, r, \vartheta, \varphi] = \int dw\, e^{-iwt} \frac{\psi_{0_{\ell m}}[r]}{r} Y^{\ell m} , \tag{3.3.2}$$

where the Fourier time decomposition and the spherical harmonic decomposition are performed. The potential for the scalar field perturbation master equation on a Schwarzschild background is

$$V_{0_{\mathrm{Sc}}} = \left(1 - \frac{2M}{r}\right) \left(\mu^2 + \frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3}\right) . \tag{3.3.3}$$

The derivation of the master equation for Schwarzschild scalar field perturbations is outlined in appendix B.1.2.

The Maxwell equation for the massless vector field A without source reduces to two decoupled vector field master equations that together fully specify the vector perturbations. The vector field is expanded in vector spherical harmonics that split it into an odd and an even sector respectively:

$$A_\mu[t, r, \vartheta, \varphi] = A_\mu^- + A_\mu^+ = \begin{pmatrix} 0 \\ 0 \\ \frac{a_{\ell m}[t,r]}{\sin\vartheta} Y^{\ell m}{}_{,\varphi} \\ -a_{\ell m}[t,r] \sin\vartheta Y^{\ell m}{}_{,\vartheta} \end{pmatrix} + \begin{pmatrix} b_{\ell m}[t,r] Y^{\ell m} \\ c_{\ell m}[t,r] Y^{\ell m} \\ d_{\ell m}[t,r] Y^{\ell m}{}_{,\vartheta} \\ d_{\ell m}[t,r] Y^{\ell m}{}_{,\varphi} \end{pmatrix} . \tag{3.3.4}$$

The coefficients a, b, c, d are coordinate t and r-dependent. In the absence of sources, the dynamics of the two sectors are independent. The master equation for the odd sector is obtained from the Maxwell equation with the variable ψ_{1-} related to the vector field A simply through the coefficient a:

$$\psi_{1-} = a . \tag{3.3.5}$$

The master equation for the even sector is obtained from the Maxwell equation with the variable ψ_{1+} related to the vector field A through coefficients b and c:

$$\psi_{1+} = r^2 \frac{\dot{c} - b'}{\ell(\ell+1)} . \tag{3.3.6}$$

The coefficient d can be expressed in terms of b and c. The Fourier time decomposition can then be used to treat the perturbations for each frequency mode independently. The master equation potentials for both odd and even cases, with this choice of variables, on a Schwarzschild background are

$$V_{1\pm_{\mathrm{Sc}}} = \left(1 - \frac{2M}{r}\right) \frac{\ell(\ell+1)}{r^2} , \tag{3.3.7}$$

The derivation of the master equations for vector field perturbations is outlined in appendix B.1.3.

The Einstein equation also reduces to two master equations - the Regge-Wheeler equation [71] for the odd metric perturbations and the Zerilli equation [72] for the even ones. The metric perturbation is expanded in tensor spherical harmonics, and the Regge-Wheeler gauge B.1.70 is adopted. Analogously to the vector case, the metric perturbation is split into an even and an odd sector respectively:

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & \frac{h_{0_{\ell m}}}{\sin\vartheta} Y^{\ell m}{}_{,\varphi} & -h_{0_{\ell m}} \sin\vartheta Y^{\ell m}{}_{,\vartheta} \\ 0 & 0 & \frac{h_{1_{\ell m}}}{\sin\vartheta} Y^{\ell m}{}_{,\varphi} & -h_{1_{\ell m}} \sin\vartheta Y^{\ell m}{}_{,\vartheta} \\ \frac{h_{0_{\ell m}}}{\sin\vartheta} Y^{\ell m}{}_{,\varphi} & \frac{h_{1_{\ell m}}}{\sin\vartheta} Y^{\ell m}{}_{,\varphi} & 0 & 0 \\ -h_{0_{\ell m}} \sin\vartheta Y^{\ell m}{}_{,\vartheta} & -h_{1_{\ell m}} \sin\vartheta Y^{\ell m}{}_{,\vartheta} & 0 & 0 \end{pmatrix} + \begin{pmatrix} -f H_{0_{\ell m}} Y^{\ell m} & -H_{1_{\ell m}} Y^{\ell m} & 0 & 0 \\ -H_{1_{\ell m}} Y^{\ell m} & -\frac{1}{f} H_{2_{\ell m}} Y^{\ell m} & 0 & 0 \\ 0 & 0 & -r^2 K_{\ell m} Y^{\ell m} & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\vartheta K_{\ell m} Y^{\ell m} \end{pmatrix} , \tag{3.3.8}$$

where $h_0, h_1, H_0, H_1, H_2, K$ are coordinate t and r-dependent coefficients, and f is the function that holds the specifics about the chosen Lorentz part of the background metric, which in the Schwarzschild case is $f_{\text{Sc}} = 1 - \frac{2M}{r}$. The dynamics of the odd and even sectors are independent.

The Regge-Wheeler equation is obtained for the variable ψ_{2-} related to the metric perturbation through the coefficient h_1. The other odd sector coefficient h_0 can be expressed in terms of h_1. The master variable in the Schwarzschild background case is

$$\psi_{2-_{\text{Sc}}}[t, r] = \left(1 - \frac{2M}{r}\right) \frac{h_1[t, r]}{r} \ . \tag{3.3.9}$$

The time dependence as before separates with the Fourier transform. The Regge-Wheeler potential for the Schwarzschild black hole background spacetime is

$$V_{2-_{\text{Sc}}} = \left(1 - \frac{2M}{r}\right) \left(\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3}\right) \ . \tag{3.3.10}$$

Analogously, the Zerilli equation is obtained for the variable ψ_{2+} related to the metric perturbation through coefficients H_1 and K :

$$\psi_{2+_{\text{Sc}}} = \zeta_1 H_1 + \zeta_2 K \ , \tag{3.3.11}$$

with ζ_1 and ζ_2 being

$$\zeta_1 = \frac{r - 2M}{i\omega(nr + 3M)} \ , \qquad \zeta_2 = \frac{r^2}{nr + 3M} \ , \tag{3.3.12}$$

where the abbreviation $n = \frac{(\ell-1)(\ell+2)}{2}$ is used. The coefficients H_0 and H_2 are functions of H_1 and K. The Zerilli potential is

$$V_{2+_{\text{Sc}}} = \left(1 - \frac{2M}{r}\right) \frac{n^3 r^3 + n^2 r^3 + 3n^2 M r^2 + 9nM^2 r + 9M^3}{(nr + 3M)^2 r^4} \ . \tag{3.3.13}$$

The Regge-Wheeler and the Zerilli equations fully describe metric perturbations. The more important steps in the derivations of these equations are given in appendix B.1.4. The Zerilli equation derivation is much more involved, it is not surprising that it was found only seven years after the Regge-Wheeler equation. The fact that this equation could be found at all is related to some hidden symmetries. Chandrasekhar in his 1984 book [45] demonstrated that the two metric perturbation master equations have the same eigenvalues, they are isospectral. With a coordinate transformation of the Zerilli variable ψ_{2+} the Zerilli equation can be written in an alternate form, which is the same as that of the Regge-Wheeler equation. These two seemingly different equations are in fact linked. The above are the Schwarzschild black hole perturbations master equation variables and potentials.

The procedure for obtaining the master equations for the Reissner-Nordström black hole perturbations is analogous. The scalar field perturbation master equation is found exactly in the same way. However, due to the nonzero background vector field, there is a mixing between the vector field and metric perturbations, and while master equations are obtained, their master variables are a combination of the metric and vector field perturbations. The perturbations can be expanded analogously to the Schwarzschild case. The scalar perturbations decouple as the background scalar field is zero. The same master variable 3.3.2, for the scalar field master equation, can be used. The coupled vector field and metric perturbations require solving the Einstein and

Maxwell equations together. However, for a particular choice of variables, as described by Chandrasekhar [45], the equations can be decoupled and put in the general master equation form 3.3.1. The variables are [45, 77]

$$\psi_{1\pm\mathrm{RN}} = q_1\,\chi_{1\pm} + \sqrt{-q_1 q_2}\,\chi_{2\pm}\ , \tag{3.3.14}$$
$$\psi_{2\pm\mathrm{RN}} = -\sqrt{-q_1 q_2}\,\chi_{1\pm} + q_1\,\chi_{2\pm}\ , \tag{3.3.15}$$

where the abbreviations $q_{1,2} = 3M \pm \sqrt{9M^2 + 4Q^2\mu^2}$ and $\mu^2 = 2n = (\ell-1)(\ell+2)$ are used. The coefficients χ depend on the specifics of the decompositions of the vector field and the metric perturbations, and they determine how the two types of perturbations are combined in the master variables. The Reissner-Nordström potentials are

$$V_{0\,\mathrm{RN}} = f\left(\mu^2 + \frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3} - \frac{Q^2}{r^4}\right)\ , \tag{3.3.16}$$
$$V_{1\pm\mathrm{RN}} = f\left(\mp q_2 W_1' + \frac{q_2^2}{f}W_1^2 + \frac{\ell(\ell+1)}{f}\mu^2 W_1\right)\ , \tag{3.3.17}$$
$$V_{2\pm\mathrm{RN}} = f\left(\mp q_1 W_2' + \frac{q_1^2}{f}W_2^2 + \frac{\ell(\ell+1)}{f}\mu^2 W_2\right)\ , \tag{3.3.18}$$

where $W_1 = \frac{f}{r(\mu^2 r + q_2)}$, $W_2 = \frac{f}{r(\mu^2 r + q_1)}$, and the function f in the Reissner-Nordström case is $f_{\mathrm{RN}} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$.

3.3.2 Perturbations of axially symmetric black holes

With only axial symmetry of the background, the method of expanding quantities in tensor spherical harmonics cannot be adopted. Since the Kerr spacetime has two Killing vectors associated with the t and φ coordinates, Fourier methods can be used to separate the time and azimuthal dependence. Radial and polar dependence is not easily separable. Nevertheless, at least in the case of Kerr background, the perturbation master equations were obtained by Teukolsky in 1973 [73, 78, 79]. He took an entirely different approach than that for perturbations of spherical spacetimes. As in the derivation of the Kerr metric itself, algebraic speciality was again the key constraint used in conjunction with axial symmetry. Naturally, the Newman-Penrose formalism was employed. After the Kerr solution was found, it was established that it is not only algebraically special but is of Petrov type D with two principal null directions, one of which is described at the end of appendix A.5 in connection with the Goldberg-Sachs theorem. Teukolsky expanded the variables in the Newman-Penrose formalism in a background and a perturbative part, and for the Weyl scalars used the implication of the background metric being Petrov type D, that only the background Ψ_2^A is nonzero. He then could solve the Newman-Penrose equations involving Ψ_0^B and, by making the coordinate choice B.2.131 for the tetrad vectors, the Teukolsky equation was obtained[1]:

$$\begin{aligned}\left(\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\vartheta\right)\partial_t^2\Psi_0^B + \frac{4Mar}{\Delta}\partial_t\partial_\varphi\Psi_0^B + \left(\frac{a^2}{\Delta} - \frac{1}{\sin^2\vartheta}\right)\partial_\varphi^2\Psi_0^B - \Delta^{-s}\partial_r\left[\Delta^{s+1}\partial_r\Psi_0^B\right] - \frac{1}{\sin\vartheta}\partial_\theta\left[\sin\vartheta\partial_\vartheta\Psi_0^B\right] \\ - 2s\left(\frac{a(r-M)}{\Delta} + \frac{i\cos\vartheta}{\sin^2\vartheta}\right)\partial_\varphi\Psi_0^B - 2s\left(\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\vartheta\right)\partial_t\Psi_0^B + (s^2\cot^2\vartheta - s)\Psi_0^B = \Sigma S\ ,\end{aligned} \tag{3.3.1}$$

where the conventional abbreviations $\Sigma = r^2 + a^2\cos^2\vartheta$ and $\Delta = r^2 + a^2 - 2Mr$ are used. In the above equation, metric perturbations are implied. These are associated with spin $s = 2$. The parameter s, which describes the spin of the field, is written explicitly for future convenience. With the same idea as in the previous section, the equation can be considered without the source term ΣS. Using the symmetry of the background the variable in a Fourier decomposed way can be written as

$$\Psi_0^B = e^{-i\omega t}R[r]\Theta[\vartheta]e^{im\varphi}\ . \tag{3.3.2}$$

[1] As Teukolsky recalled in a recent talk, he was first looking for Ψ_4^B, which is slightly more involved due to the particular choice of principal null directions, thus delaying finding the Teukolsky equation by a few months [80].

Then the Teukolsky equation can be decoupled into

$$\Delta^{-s}\partial_r\left[\Delta^{s+1}\partial_r R\right]+\left(\frac{K^2-2is(r-M)K}{\Delta}+4is\omega r-a^2\omega^2+2am\omega-\lambda\right)R=0\ ,\tag{3.3.3}$$

$$\frac{1}{\sin\vartheta}\partial_\vartheta\left[\sin\vartheta\partial_\vartheta\Theta\right]+\left(a^2\omega^2\cos^2\vartheta-\frac{m^2}{\sin^2\vartheta}-2a\omega s\cos\vartheta-\frac{2ms\cos\vartheta}{\sin^2\vartheta}-s^2\cot^2\vartheta+s+\lambda\right)\Theta=0\ ,\tag{3.3.4}$$

with $K=(r^2+a^2)\omega-am$. In general, both eigenvalue problems need to be solved simultaneously as both λ and ω appear in the equations. The situation is much simpler in the static case $\omega=0$. This is the only case considered in this work. Then the polar equation is treated as an eigenvalue problem, the solutions are the spin-weighted spherical harmonics. The eigenvalues $\lambda=(\ell-s)(\ell+s+1)$ are the separation constants that can be inserted in the radial equation.

The Teukolsky equation applies to other variables as well. These include $\Sigma^{-2}\Psi_4^B$ involving the Weyl scalar associated with the metric perturbation, ϕ_0 and $\Sigma^{-1}\phi_2$ involving the Maxwell scalars associated with the vector field perturbation, as well as ϕ of the scalar field perturbations. Thus the radial and polar Teukolsky equations can be used to describe scalar field, vector field, and metric perturbations. With these equations, the perturbations are described fully.

The derivation of the Teukolsky equation starts from some of the Newman-Penrose equations. While it is not always entirely transparent what physical properties the Newman-Penrose equations describe, the motivation for looking for perturbations of the Weyl scalars Ψ_0 and Ψ_4 has to do with the fact that these scalars are invariant under infinitesimal tetrad rotations of the Kinnersley frame [66], given in B.2.131, relative to which they are defined. The scalars are independent of gauge choices, and thus represent physical observables [73]. In particular, these Weyl scalars can be associated with gravitational waves. For asymptotically flat geometries, the tetrad at infinity can be identified with the flat spacetime spherical coordinates, the metric perturbations on a flat background can be related to the Riemann tensor components, and the Weyl scalar Ψ_4 can be expressed in terms of the metric perturbations as [45]

$$\Psi_4=\frac{\ddot h_{\vartheta\vartheta}-\ddot h_{\varphi\varphi}}{2}+i\ddot h_{\vartheta\varphi}=-\ddot h_{+}+i\ddot h_{\times}\ .\tag{3.3.5}$$

Thus the scalar Ψ_4 at infinity encodes all outgoing gravitational wave power.

The interpretation of the Weyl scalars and other variables is more convoluted in regions where curvature cannot be neglected. By the peeling theorem [32] gravitational as well as electromagnetic radiation can have intricate structure close to the source, but it progressively diminishes with increasing distance from it. Eventually, only type N radiation remains.

After a successful method for dealing with perturbations on axially symmetric backgrounds has been established and decoupled and separated master equations for the perturbations of the Kerr spacetime found, it could seem possible to apply the same procedure to find perturbations of the charged Kerr-Newman spacetime. However, this is not the case. In the presence of both charge and rotation, the coupling between metric and vector field perturbations is complex, no clear approaches to disentangle the perturbations are known. While Kerr spacetime metric and vector field perturbations can be excited independently they are intertwined in the Kerr-Newman case [81]. The only Kerr-Newman spacetime perturbations that have decoupled master equations are the scalar perturbations [82]. For metric perturbations of the Kerr-Newman spacetime decoupled master equations have not been discovered. Since in principle, there is no reason to assume such master equations have to exist, chances of finding them are slight. Two coupled equations [2] are used instead [45].

Studying the dynamics of the most general black hole of Einstein-Maxwell theory, the dyonic Kerr-Newman black hole, without encountering major technical complications, is feasible only for scalar fields. In this work, charged scalar field perturbations are studied. The associated perturbation master equations are described in the following section.

[2]For arbitrary type D spacetimes analogous coupled equations were described by Dudley and Finley in 1979 [83, 84].

3.3.3 Dynamics of charged scalar fields on a dyonic Kerr-Newman background

Here scalar field dynamics coupled to Einstein-Maxwell theory is considered. The action is [49]

$$S[g,\phi,A] = \frac{1}{16\pi}\int d^4x\sqrt{-g}(R - F_{\mu\nu}F^{\mu\nu}) - \frac{1}{2}\int d^4x\sqrt{-g}(D_\mu\bar{\phi}D^\mu\phi - V[\bar{\phi}\phi]) \,. \tag{3.3.6}$$

The action depends on the metric g, a complex scalar field ϕ, and a coupled vector field A. The field strength tensor of the vector field is

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \,. \tag{3.3.7}$$

With the symmetries of the fields under the simultaneous local transformations with real θ being

$$\phi \to e^{i\theta}\phi \,, \qquad A_\mu \to A_\mu - \frac{1}{q}\nabla_\mu\theta \,, \tag{3.3.8}$$

the scalar field is gauge coupled to electromagnetism with the gauge covariant derivative

$$D_\mu = \nabla_\mu + iqA_\mu \,. \tag{3.3.9}$$

The variation of the action with respect to the metric and the fields leads to the coupled equations of motion

$$G_{\mu\nu} = T^\phi_{\mu\nu} + T^A_{\mu\nu} \,, \tag{3.3.10}$$

$$D_\mu D^\mu\phi - V\phi = 0 \,, \tag{3.3.11}$$

$$\mathrm{d}\star F = -4\pi q \star J^\phi \,. \tag{3.3.12}$$

The scalar and vector field contributions to the energy-momentum tensor $T^\phi_{\mu\nu}$ and $T^A_{\mu\nu}$ and the current J^ϕ_μ are

$$T^\phi_{\mu\nu} = 4\pi\left(D_\mu\bar{\phi}D_\nu\phi + D_\mu\phi D_\nu\bar{\phi} - \left(D_\rho\bar{\phi}D^\rho\phi + V\right)g_{\mu\nu}\right) \,, \tag{3.3.13}$$

$$T^A_{\mu\nu} = F_{\mu\rho}F_\nu{}^\rho + \star F_{\mu\rho}\star F_\nu{}^\rho \,, \tag{3.3.14}$$

$$J^\phi_\mu = \frac{i}{2}\left(\bar{\phi}D_\mu\phi - \phi D_\mu\bar{\phi}\right) \,. \tag{3.3.15}$$

Scalar electrodynamics on a dyonic Kerr-Newman black hole background (2.3.2) permits perturbations of the scalar field, the vector field, and the metric. When the background scalar field is zero, the scalar perturbations at linear order decouple from the remaining two. The potential is reduced to only a mass term $V[\phi,\bar{\phi}] = \mu^2\bar{\phi}\phi$, as linear perturbations are not affected by any higher-order terms. The Kerr-Newman spacetime charged scalar field perturbation master equation is derived from the Klein-Gordon equation 3.3.11. With the vector field A in the intermediate gauge [49]

$$A_\mu = \left(-\frac{Qr + aP\cos\vartheta}{\Sigma}, 0, 0, \frac{Qra\sin^2\vartheta - P(r^2+a^2)a\cos\vartheta}{\Sigma}\right) \,, \tag{3.3.16}$$

and the ansatz for a particular mode of the scalar field

$$\phi = e^{-i\omega t}R[r]\Theta[\vartheta]e^{im\varphi} \,, \tag{3.3.17}$$

the master equation is obtained:

$$\begin{aligned}\partial_r\Big[\Delta\partial_r\phi\Big] + \frac{1}{\Delta}\Big(\omega^2(a^2+r^2)^2 - 2a\omega m(a^2+r^2) + a^2m^2 + 2qQ\omega r(a^2+r^2) - 2qQmar + q^2Q^2r^2\Big)\phi - \mu^2\Sigma\phi + q^2P^2\phi \\ + \frac{1}{\sin\vartheta}\partial_\vartheta\Big[\sin\vartheta\partial_\vartheta\phi\Big] + \frac{1}{\sin^2\vartheta}\Big(-m^2 - 2qPm\cos\vartheta - q^2P^2 - \omega^2a^2\sin^4\vartheta + 2m\omega a\sin^2\vartheta + 2qP\omega a\sin^2\vartheta\cos\vartheta\Big)\phi = 0 \,,\end{aligned} \tag{3.3.18}$$

where the abbreviations $\Sigma = r^2 + a^2\cos^2\vartheta$ and $\Delta = r^2 + a^2 - 2Mr + Q^2 + P^2$ are used. The master equation can be decoupled into the radial and polar equations for the variables R and Θ respectively [49]:

$$\partial_r\Big[\Delta\partial_r R\Big] + \left(\frac{((r^2+a^2)\omega - ma + eQr)^2}{\Delta} - \mu^2r^2 - \lambda - a^2\omega^2 + 2a\omega m\right)R = 0 \,, \tag{3.3.19}$$

$$\frac{1}{\sin\vartheta}\partial_\vartheta\Big[\sin\vartheta\partial_\vartheta\Theta\Big] - \left(\frac{(m + qP\cos\vartheta - a\omega\sin^2\vartheta)^2}{\sin^2\vartheta} + \mu^2a^2\cos^2\vartheta - \lambda - a^2\omega^2 + 2a\omega m\right)\Theta = 0 \,, \tag{3.3.20}$$

where in the static case when $\omega = 0$ and with the mass of the scalar field set to zero $\mu = 0$, the polar equation eigenfunctions reduce to the spherical harmonics and the corresponding eigenvalues λ become $\lambda = \ell(\ell+1)$.

4 Tidal response of black holes

4.1 Tidal deformations

Foundational work on describing tidal effects in Newtonian gravity was done by Love in 1911 [85–87]. He studied tidal effects in the context of the Earth-Moon system and described the elastic deformations of the Earth as its gravitational potential changes due to the presence of the Moon. He characterized the Earth's response with parameters now called Love numbers. With the prospects of gravitational wave astronomy growing, Flanagan and Hinderer studied the possibility of observing tidal effects in binary coalescences [23]. If tidal effects could be detected, the information could be used to infer some properties about neutron stars and their equation of state. In light of Flanagan's and Hinderer's 2008 analysis on the detection of tidal effects on neutron stars with Earth-based gravitational wave detectors, the framework for investigating tidal deformations was extended to general relativity by Binnington and Poisson in 2009 [25]. It is reviewed here.

Tidal distortion occurs when a massive body is influenced by an external gravitational field. The source of the external perturbation can be another massive body and tidal effects are commonly observed in binary systems, but to discuss tidal effects in general, the source can be left arbitrary. The deformations of the shapes of bodies depend on their equation of state and various other effects. In Newtonian gravity, tidal deformations are studied in terms of the full gravitational potential U. For a weakly externally perturbed object with mass M and equilibrium radius r_s, the total Newtonian potential outside it can be written in a spherical harmonic $Y_{\ell m}$ expansion with the primary potential and the response separated one from the other:

$$U = -\frac{M}{r} + \frac{(\ell-2)!}{\ell!} \sum_{\ell=2} \sum_{m=-\ell}^{\ell} Y_{\ell m} \mathcal{E}_{\ell m} r^{\ell} \left(1 + k_{\ell m} \left(\frac{r}{r_s} \right)^{-2\ell-1} \right) , \tag{4.1.1}$$

where the dipole term corresponds to shifts of the center of mass so it is not included, and the monopole term is explicitly separated. The monopole term corresponds to a potential created by a spherically symmetric self-gravitating body. In non-spherical cases, the body's potential is in general more complicated than just a monopole term, but provided that it is known, it can be separated analogously to the above. The coefficients $\mathcal{E}_{\ell m}$ can then be interpreted as describing the external tidal potential. The body's response is described with the dimensionless Love numbers $k_{\ell m}$. The Love numbers are entirely fixed by specifying a boundary condition at the surface of the deformed body, and they are independent of the magnitude of the tidal potential coefficients $\mathcal{E}_{\ell m}$. This means that the Love numbers hold information about intrinsic properties of objects, such as the equation of state, and therefore they are relevant physical observables. If the system is not static, the Love numbers $k_{\ell m}$ are in general complex [30]:

$$k_{\ell m} = \kappa_{\ell m} + i\nu_{\ell m} . \tag{4.1.2}$$

The real part $\kappa_{\ell m}$ describes tidal effects, while the imaginary part $\nu_{\ell m}$ represents dissipative effects. For static objects in static tidal environments, the dissipative Love numbers are zero, and the object response is described solely with tidal Love numbers.

Through the long-distance limit, the Newtonian gravitational potential U can be linked to the tt-component of the metric perturbation h:

$$U = -\frac{1}{2} h_{tt} . \tag{4.1.3}$$

This relation can be used to directly introduce Newtonian gravitational potentials into a general relativity problem with the weak-field approximation. The metric component h_{tt} can be analogously expanded and the perturbation part without the body's own contribution is

$$-2\tilde{h}_{tt} = \frac{(\ell-2)!}{\ell!} \sum_{\ell=2} \sum_{m=-\ell}^{\ell} Y_{\ell m} \mathcal{E}_{\ell m} r^{\ell} \left(\left(1 + \mathcal{O}\left[r^{-1} \right] \right) + k_{\ell m} \left(\frac{r}{r_s} \right)^{-2\ell-1} \left(1 + \mathcal{O}\left[r^{-1} \right] \right) \right) . \tag{4.1.4}$$

The analysis of Love numbers is directly extended to all other field variables:

$$\psi \propto r^{\ell}\left(1+k\left(\frac{r}{r_{+}+r_{-}}\right)^{-2\ell-1}\right) , \tag{4.1.5}$$

where r_+ and r_- represent the radial location of the outer and inner horizons.

With this, tidal deformations have a clear interpretation in general relativity. However, not all tidal effects of general relativity have counterparts in Newtonian mechanics. Perturbations can have two decoupled parts, the odd and even sectors described in the previous section, the Newtonian potential is related to one of them, but the other does not have an analog in Newtonian gravity. For black holes, the rotational contribution, subtracted in the above expression, can be found directly from the metric [88].

The source and the response can have terms with the same power of r entangling the two. Kol and Smolkin suggested using higher dimensions to remove the power overlap and therefore avoid the associated ambiguity between what is considered a source and what a response [89]. This effectively corresponds to allowing the multipole expansion parameter ℓ to be real, and not only integer-valued. The analytic continuation has become a popular method having been used by various authors to find Love numbers of the Kerr black hole [29–31]. In this work, an alternative, entirely physical method for disentangling these source and response effects is presented.

The way that Love numbers for black holes are found is by solving the perturbation master equations. The solutions are then matched to the expression 4.1.5. All master equations are second-order differential equations that require two boundary conditions. The perturbation sources can be considered to be infinitely far. This means that the external tidal perturbations are progressively growing with increasing distance r from the black hole, while the response is diminishing with increasing distance from the black hole. The boundary condition at infinity can be to fix the normalization of the growing mode of the solution. The horizon is a one-way membrane, on the horizon energy and momentum fluxes can be directed only into the black hole. In terms of boundary conditions, this requires that the solution is regular at the horizon.

4.2 Static response of the Schwarzschild black hole

In their influential article on the relativistic theory of tidal Love numbers, Binnington and Poisson arrive at the important conclusion that the static Love numbers of the Schwarzschild black hole are zero [25]. These are the first calculations of Love numbers for black holes, outlining the method widely used in the subsequent Love number calculations by numerous authors. As an example, the calculation of the odd sector Love numbers for metric perturbations is summarized here.

The odd metric perturbations obey the Regge-Wheeler equation B.1.86:

$$\nabla_I \nabla^I \psi_{2-} - \left(1-\frac{2M}{r}\right)\left(\frac{\lambda^2}{r^2}-\frac{6M}{r^3}\right)\psi_{2-} = 0 , \tag{4.2.1}$$

where $\lambda^2 = \ell(\ell+1)$. From now on, the variable label marking the type of perturbation is dropped: $\psi = \psi_{2-}$. For static perturbations, the equation reduces to

$$\psi''[r] + \frac{2M}{r(r-2M)}\psi'[r] - \frac{\lambda^2 r - 6M}{r^2(r-2M)}\psi[r] = 0 . \tag{4.2.2}$$

Inverting the independent variable $\psi[r] \to \psi[z]$, where $z = \frac{2M}{r}$, leads to the expression

$$\psi''[z] + \frac{(3z-2)}{(z-1)z}\psi'[z] + \frac{\lambda^2 - 3z}{(z-1)z^2}\psi[z] = 0 . \tag{4.2.3}$$

This is a Fuchsian equation [90] with three regular singular points at $z = 0, z = 1$, and $z = \infty$, where the regularity of the third can be inferred from the equation before the inversion of the variable. An equation with these singular points can be brought into hypergeometric form [91]:

$$z(1-z)u''[z] + (c-(a+b+1)z)u'[z] - ab u[z] = 0 , \tag{4.2.4}$$

with a, b, c being the coefficients that determine the solutions.

In this case, the equation is brought to the hypergeometric form for the variable

$$u[z] = z^{-l}\psi[z] \tag{4.2.5}$$

and with coefficients

$$a = \ell - 1\ , \qquad b = \ell + 3\ , \qquad c = 2(\ell + 1)\ . \tag{4.2.6}$$

The hypergeometric equations and the properties of their solutions are well studied. The solution to the static case Regge-Wheeler equation in terms of the hypergeometric functions is [26, 92]

$$\psi[z] = z^{\ell}\left(c_1 F[a, b, c; z] + c_2 z^{1-c} F[1 - a, b, b - a + 1; z^{-1}]\right)\ . \tag{4.2.7}$$

This is a degenerate case. The solution can be regular at the horizon $z = 1$ only if c_1 is zero. This is because the hypergeometric function in the first term, which behaves as $\sim \log[1 - z]$, is logarithmically divergent at the horizon [26, 90]. Then substituting the values of the coefficients, the solution is a polynomial [92]

$$\psi[z] = c_2 z^{-\ell-1} F[-\ell, \ell + 3, 5; z^{-1}] = c_2 z^{-\ell-1} \sum_{n=0}^{\ell} \frac{(-\ell)_n (\ell + 3)_n}{(5)_n} \frac{z^{-n}}{n!}\ . \tag{4.2.8}$$

The polynomial has negative powers of z, and therefore only positive powers of the radial variable r:

$$\psi_{2-\ell m}[r] \propto r^{\ell+1} \sum_{n=0}^{\ell} \frac{(-\ell)_n (\ell + 3)_n}{(5)_n\, n!} \left(\frac{r}{2M}\right)^n\ . \tag{4.2.9}$$

The solution is purely growing towards infinity and therefore represents the external perturbation. The absence of decaying modes implies that there is no response, the Love numbers are zero:

$$k_{2-\ell m} = 0\ . \tag{4.2.10}$$

4.3 Static response of the Kerr black hole

The significant feature in the calculation of the Love numbers for the Kerr black hole is that while the tidal Love numbers are zero, the dissipative Love numbers are not. This result was worked out by Le Tiec, Casals, and Franzin [29], and reproduced by Charalambous, Dubovsky, and Ivanov [30] soon after. It can be reasoned that a nonzero dissipative response should be expected, as this is in agreement with Newtonian analogs, where due to tidal torquing, rotation necessarily introduces a dissipative response in the perturbed system. However, to find this result for the Kerr black hole, analytic continuation was used. This is an unphysical renormalization scheme, and there has been a discussion about the validity of this approach. Here the main points of the Kerr black hole Love number calculation are reproduced.

The Teukolsky equation describes perturbations of the Kerr black hole. The equations for the separate pieces of the full variable $\psi[t, r, \vartheta, \varphi] = e^{-i\omega t} R[r]\Theta[\vartheta]e^{im\varphi}$ are the radial and polar equations 3.3.3 and 3.3.4:

$$\Delta^{-s}\partial_r\left[\Delta^{s+1}\partial_r R\right] + \left(\frac{K^2 - 2is(r - M)K}{\Delta} + 4is\omega r - a^2\omega^2 + 2am\omega - \lambda\right) R = 0\ , \tag{4.3.1}$$

$$\frac{1}{\sin\vartheta}\partial_\vartheta\left[\sin\vartheta\,\partial_\vartheta\Theta\right] + \left(a^2\omega^2\cos^2\vartheta - \frac{m^2}{\sin^2\vartheta} - 2a\omega s\cos\vartheta - \frac{2ms\cos\vartheta}{\sin^2\vartheta} - s^2\cot^2\vartheta + s + \lambda\right)\Theta = 0\ , \tag{4.3.2}$$

where $s = 0, \pm1, \pm2$ determines the perturbation type, $r_\pm = M \pm \sqrt{M^2 - a^2}$ are the inner and outer horizons of the Kerr black hole, $\Delta = (r - r_+)(r - r_-)$, $K = (r^2 + a^2)\omega - am$. For the polar equation, λ are the eigenvalues. Both the radial and the polar equations are confluent Heun equations, with one irregular and two regular singular points. If the irregular singularity can be turned into a regular one, the equations become hypergeometric. This happens in the static limit $\omega = 0$. In the static case, the polar equation has a reduced form and its eigenfunctions become the spin-weighted spherical harmonics [73]. Their eigenvalues are

$$\lambda = (\ell - s)(\ell + s + 1)\ . \tag{4.3.3}$$

Similarly, with $\omega = 0$, the radial equation becomes simpler:

$$\Delta^{-s}\partial_r\left[\Delta^{s+1}\partial_r R\right] + \left(\frac{a^2m^2 - 2isam(r-M)}{\Delta} - \lambda\right) R = 0\ . \tag{4.3.4}$$

This equation is put in hypergeometric form

$$z(1-z)\mathcal{R}''[z] + (c - (a+b+1)z)\mathcal{R}'[z] - ab\mathcal{R}[z] = 0\ , \tag{4.3.5}$$

with the following definitions:

$$a = \ell + 1 - s\ , \qquad b = \ell + 1 + 2im\gamma\ , \qquad c = 1 - s + 2im\gamma\ , \tag{4.3.6}$$

$$\frac{r - r_+}{r - r_-} = z\ , \qquad R = (r - r_-)^{-i\gamma m - s}(r - r_+)^{i\gamma m - s}\mathcal{R}\ , \qquad \gamma = \frac{a}{r_+ - r_-}\ . \tag{4.3.7}$$

The procedure for obtaining the hypergeometric form, albeit more involved, is analogous to the one described in the previous section. The radial solution R in terms of the hypergeometric functions is

$$R[z] = (1-z)^{-i\gamma m - s} z^{i\gamma m - s}(1-z)^a \left(c_1 F[a,b,c;z] + c_2 z^{1-c} F[a-c+1, b-c+1, 2-c; z]\right)\ , \tag{4.3.8}$$

where c_1 and c_2 are to be fixed using boundary conditions. Here with analytic continuation ℓ is allowed to be real. This means the coefficients a, b, c do not have to be integer-valued and the generic hypergeometric equation is used. Now the boundary conditions can be used. The radial solution R has to be regular across the horizon, therefore c_2 must be zero. The reduced solution is expressed for the large distance variable with the following relation [29, 30, 92]:

$$F[a,b,-c+a+b+1;z] = \frac{\Gamma[c]\Gamma[c-a-b]}{\Gamma[c-a]\Gamma[c-b]} F[a,b,a-b-c+1;1-z] + z^{c-a-b}\frac{\Gamma[c]\Gamma[a+b-c]}{\Gamma[a]\Gamma[b]} F[c-a,c-b,c-a-b+1;1-z]\ , \tag{4.3.9}$$

which implies an expansion in $\ell = \ell_0 + \epsilon$ taking ℓ_0 to be integer, and ϵ to be small. This then gives the radial solution at infinity $r \to \infty$

$$R\Big|_{r\to\infty} = \alpha_{\ell m} r^\ell \left(1 + \left(\frac{r}{r_+ - r_-}\right)^{-2\ell-1} \frac{\Gamma[-2\ell-1]\Gamma[\ell+1-s]\Gamma[1+\ell+2im\gamma]}{\Gamma[2\ell+1]\Gamma[-\ell-s]\Gamma[-\ell+2im\gamma]}\right)\ . \tag{4.3.10}$$

By comparing the form of the radial solution with the expansion for the metric perturbation 4.1.5, the static response coefficients can be found:

$$k_{\ell m} = \left(\frac{r_+ - r_-}{r_+ + r_-}\right)^{2\ell+1} \frac{\Gamma[-2\ell-1]\Gamma[\ell+1-s]\Gamma[1+\ell+2im\gamma]}{\Gamma[2\ell+1]\Gamma[-\ell-s]\Gamma[-\ell+2im\gamma]}\ . \tag{4.3.11}$$

Now ℓ can be restricted to integer values. Then with the gamma function identity [31, 90]

$$\frac{\Gamma[l+1+ia]}{\Gamma[-l+ia]} = (-1)^l ia \prod_{n=1}^{l}[n^2 + a^2]\ , \tag{4.3.12}$$

where n, l represent any integers, a is any real number, and z is any complex number, the above general expression 4.3.11 for Kerr black hole Love numbers is brought to its final form [29–31]:

$$\kappa_{\ell m} = 0\ , \tag{4.3.13}$$

$$\nu_{\ell m} = (-1)^{s+1} m\gamma \left(\frac{r_+ - r_-}{r_+ + r_-}\right)^{2\ell+1} \frac{(\ell+s)!(\ell-s)!}{(2\ell+1)!(2\ell)!} \prod_{n=1}^{\ell}\left[n^2 + 4m^2\gamma^2\right]\ . \tag{4.3.14}$$

Static tidal Love numbers $\kappa_{\ell m}$ are zero for Kerr black holes, static dissipative Love numbers are not. The dissipative response can be associated with frame dragging. This is true for scalar, vector, and metric perturbations.

Crucially, the result depends on analytic continuation. This is undesirable since the procedure is unphysical. In the following section a new and physical way of performing this computation is shown, without analytic continuation.

4.4 Dyonic Kerr-Newman black hole static response to charged scalar field perturbations

Here the main calculation of this work is described. The static response of a charged scalar field in the vicinity of a dyonic Kerr-Newman black hole excited by an external scalar field tide is derived. This is a new result.

The perturbation equations for the charged scalar field perturbations of the Kerr-Newman black hole [49] are derived in section 3.3.3. The master radial equation 3.3.19 and the master polar equation 3.3.20 respectively are

$$\partial_r\Big[\Delta\partial_r R\Big]+\left(\frac{((r^2+a^2)\omega-ma+eQr)^2}{\Delta}-\mu^2r^2-\lambda-a^2\omega^2+2a\omega m\right)R=0\ , \tag{4.4.1}$$

$$\frac{1}{\sin\vartheta}\partial_\vartheta\Big[\sin\vartheta\partial_\vartheta\Theta\Big]-\left(\frac{(m+qP\cos\vartheta-a\omega\sin^2\vartheta)^2}{\sin^2\vartheta}+\mu^2a^2\cos^2\vartheta-\lambda-a^2\omega^2+2a\omega m\right)\Theta=0\ . \tag{4.4.2}$$

The perturbing charged scalar field is

$$\phi=e^{-i\omega t}R[r]\Theta[\vartheta]e^{im\phi}\ . \tag{4.4.3}$$

The charged scalar field is coupled to the black hole via the electromagnetic gauge coupling, and the vector field in the intermediate gauge [49] is

$$A=-\frac{Qr}{\Sigma}\left(dt-a\sin^2\vartheta d\varphi\right)+\frac{P\cos\vartheta}{\Sigma}\left(adt-(r^2+a^2)d\varphi\right)\ . \tag{4.4.4}$$

The two master equations can always be brought to confluent Heun form [91]:

$$u''[z]+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\epsilon\right)u'[z]+\frac{\alpha z-\beta}{z(z-1)}u[z]=0\ . \tag{4.4.5}$$

For the polar equation, this requires redefining the dependent variable to θ:

$$\Theta[x]=e^{\frac{\epsilon}{2}x}x^{\frac{\gamma-1}{2}}(x-1)^{\frac{\delta-1}{2}}\theta[x]\ , \tag{4.4.6}$$

where the coordinate is changed according to

$$x=\frac{\cos\vartheta+1}{2}\ . \tag{4.4.7}$$

The polar case confluent Heun equation parameter definitions are

$$\begin{aligned}&\alpha=4aqP\omega-4a(qP-1)\sqrt{\omega^2-\mu^2}\ , &&\beta=a^2(\omega^2-\mu^2)+2a\sqrt{\omega^2-\mu^2}(1-m-2qP)+qP(2a\omega+1)+\lambda\ ,\\ &\gamma=1-m-2qP\ , \qquad \delta=1+m\ , &&\epsilon=4a\sqrt{\omega^2-\mu^2}\ .\end{aligned} \tag{4.4.8}$$

Analogously, the radial equation is in confluent Heun form for the complex variable $\mathcal{R}$:

$$R[z]=e^{\frac{\epsilon}{2}z}z^{\frac{\gamma-1}{2}}(z-1)^{\frac{\delta-1}{2}}\mathcal{R}[z]\ , \tag{4.4.9}$$

where the coordinate redefinition is

$$z=\frac{r-r_-}{r_+-r_-}\ , \tag{4.4.10}$$

and the radial case confluent Heun equation parameters are

$$\begin{aligned}&\alpha=2(r_+-r_-)\left(\sqrt{\omega^2-\mu^2}(qQ+2M\omega+i)+qQ\omega-M\mu^2+2M\omega^2\right)\ ,\\ &\beta=-2(a^2+r_-^2)\left(\omega\omega_-+(\omega_-+i\kappa_-)\sqrt{\omega^2-\mu^2}\right)+r_-^2\mu^2+a^2\omega^2-2a\omega(m+qP)+i(qQ+2M\omega)+\lambda\ ,\\ &\gamma=1-i\frac{\omega_-}{\kappa_-}\ , \qquad \delta=1-i\frac{\omega_+}{\kappa_+}\ , \qquad \epsilon=2i(r_+-r_-)\sqrt{\omega^2-\mu^2}\ ,\end{aligned} \tag{4.4.11}$$

where

$$\Omega_\pm = \frac{a}{r_\pm^2 + a^2} , \qquad \kappa_\pm = \frac{r_\pm - r_\mp}{2(a^2 + r_\mp^2)} ,$$
$$\Phi_\pm = -A_t - \Omega_\pm A_\varphi = \frac{r_\pm Q}{r_\pm^2 + a^2} , \qquad \omega_\pm = \omega - \Omega(m + qP) + q\Phi_\pm . \tag{4.4.12}$$

The confluent Heun form is relevant for general calculations. Where static perturbations are concerned, zero frequency $\omega = 0$ related simplifications occur and the radial equation singular points become those of the hypergeometric equation. For the calculation here, a further restriction is made, the scalar field is taken to be massless $\mu = 0$, thus avoiding any mass term-related associated regularity problems for the scalar field ϕ at infinity. This has the additional benefit of simplifying the eigenvalues of the polar master equation:

$$\lambda = \ell(\ell + 1) . \tag{4.4.13}$$

Then the radial master equation becomes

$$z(z-1)R''[z] + (2z-1)R'[z] + \left(\frac{(am + qQ((z-1)r_+ - zr_-))^2}{(r_- - r_+)^2(z-1)z} + q^2P^2 - \lambda\right) R[z] = 0 . \tag{4.4.14}$$

A complex variable $\mathcal{R}$ can be introduced, that is related to the radial part of the scalar field as

$$R[z] = (z+1)^d z^{-e} \mathcal{R}[z] , \tag{4.4.15}$$

where

$$d = \frac{i(qQr_- - am)}{r_+ - r_-} , \qquad e = \frac{i(qQr_+ - am)}{r_+ - r_-} , \tag{4.4.16}$$

and the coordinate relation is

$$z = \frac{r - r_+}{r_+ - r_-} . \tag{4.4.17}$$

The radial master equation expressed for this variable is

$$z(1-z)\mathcal{R}''[z] + \left(\frac{r_+ - r_- - 2iqQr_+ + 2iam}{r_+ - r_-} - (2 - 2iqQ)z\right)\mathcal{R}'[z] - (q^2P^2 - \lambda - iqQ)\mathcal{R}[z] = 0 , \tag{4.4.18}$$

which is the hypergeometric equation

$$z(1-z)\mathcal{R}''[z] + (c - (a+b+1)z)\mathcal{R}'[z] - ab\mathcal{R}[z] = 0 \tag{4.4.19}$$

with coefficients

$$a = -f - iqQ , \tag{4.4.20}$$
$$b = f + 1 - iqQ , \tag{4.4.21}$$
$$c = 1 + 2i(m\gamma - qQ) , \tag{4.4.22}$$
$$f = \frac{\sqrt{1 + 4(\ell(\ell+1) - q^2(Q^2 + P^2))}}{2} - \frac{1}{2} . \tag{4.4.23}$$

Importantly, in the zero charge limit $Q \to 0, P \to 0$, the coefficient f reduces to $f \to \ell$. The general solution to this hypergeometric equation in terms of the hypergeometric functions is

$$R[z] = (z+1)^d(z)^{-e}\left(c_1 F[a, b, c; z] + c_2 z^{1-c} F[a - c + 1, b - c + 1, 2 - c; z]\right) . \tag{4.4.24}$$

At the inner horizon $r = r_-$, the variable z is $z = 1$, at the event horizon $r = r_+$ the variable is $z = 0$, and at infinity $r \to \infty$ the variable is $z \to -\infty$. The solution can be regular at the horizon, only if c_2 is set to zero since

$z^{1-c}F[a-c+1,b-c+1,2-c;z]$ is not analytic for $z=0$. To evaluate the regular solution at infinity $r\to\infty$, where $z\to-\infty$ the following hypergeometric function connection formula for the inverse variable can be used [29, 92]:

$$\frac{\sin[\pi(b-a)]}{\pi}F[a,b,c,x]=\frac{(-x)^{-a}}{\Gamma[b]\Gamma[c-a]}F[a,a-c+1,a-b+1;x^{-1}]-\frac{(-x)^{-b}}{\Gamma[a]\Gamma[c-b]}F[b,b-c+1,b-a+1;x^{-1}]\,. \tag{4.4.25}$$

The hypergeometric functions at $x=\infty$ are $F[a,b,c;x^{-1}]=1$. Then

$$R_{\ell m}[r]\Big|_{\infty}=\frac{\pi c_1}{\sin[\pi(b-a)]}\frac{1}{\Gamma[b]\Gamma[c-a]}(r-r_-)^d(r-r_+)^{-e}\left(\frac{r}{r_+-r_-}\right)^{-a}\left(1-\left(-\frac{r}{r_+-r_-}\right)^{a-b}\frac{\Gamma[b]\Gamma[c-a]}{\Gamma[a]\Gamma[c-b]}\right)\,. \tag{4.4.26}$$

where $a-b=-2f-1$ and $\Gamma[a]=-f-iqQ$. This expression is not analytical for integer values of f. As the second term in the parentheses tends to zero as $f\to n$, the overall coefficient in front of this expression containing the sine function in the denominator approaches infinity.

The integer f case is important for reproducing the Kerr black hole Love numbers for scalar field perturbations as a zero-charge limit of the Kerr-Newman perturbations, thus the expression needs to be regularized. This can be done using the charges Q and P as physical regularization parameters. As noted previously, the coefficient f reduces to $f\to\ell$ as $Q\to 0\,,P\to 0$. Thus small charges can be chosen as expansion parameters, such that $f=\ell+\epsilon$. Expanding the gamma function near any of its poles then gives the expressions

$$\frac{\sin[\pi(2f+1)]}{\pi}=\frac{1}{\Gamma[2f+1]\Gamma[-2f]}=-2\epsilon+\mathcal{O}[\epsilon^2]\,, \tag{4.4.27}$$

$$\frac{1}{\Gamma[-f]}=(-1)^{l+1}\ell!\,\epsilon\,, \tag{4.4.28}$$

and the expansion formula 4.4.25, with a further specification of charges to ensure $a\to-f$ and $b\to f+1$, can be rewritten as [30]

$$F[a,b,c,x]=\frac{\Gamma[c]\Gamma[b-a]}{\Gamma[b]\Gamma[c-a]}(-x)^{-a}F[a,a-c+1,a-b+1;x^{-1}]+\frac{\Gamma[c]\Gamma[a-b]}{\Gamma[a]\Gamma[c-b]}(-x)^{-b}F[b,b-c+1,b-a+1;x^{-1}]\,. \tag{4.4.29}$$

Thus a regularized expression of the solution valid for $f=\ell+\epsilon$ is obtained:

$$R_{\ell m}[r]\Big|_{\infty}=c_1\frac{\Gamma[c]\Gamma[b-a]}{\Gamma[b]\Gamma[c-a]}(r-r_-)^d(r-r_+)^{-e}\left(-\frac{r}{r_+-r_-}\right)^{-a}\left(1+\left(-\frac{r}{r_+-r_-}\right)^{a-b}\frac{\Gamma[b]\Gamma[a-b]\Gamma[c-a]}{\Gamma[a]\Gamma[b-a]\Gamma[c-b]}\right)\,, \tag{4.4.30}$$

and corresponding to the decaying modes, the Love numbers are

$$k_{\ell m}=\left(-\frac{r_+-r_-}{r_++r_-}\right)^{b-a}\frac{\Gamma[b]\Gamma[a-b]\Gamma[c-a]}{\Gamma[a]\Gamma[b-a]\Gamma[c-b]}\,. \tag{4.4.31}$$

Substituting the values for $a,b,$ and c, this expression becomes

$$k_{\ell m}=\left(-\frac{r_+-r_-}{r_++r_-}\right)^{2f+1}\frac{\Gamma[-2f-1]}{\Gamma[2f+1]}\frac{\Gamma[f+1]}{\Gamma[-f]}\frac{\Gamma[f+1+2im\gamma]}{\Gamma[-f+2im\gamma]}\,. \tag{4.4.32}$$

Then using the gamma function identities the following asymptotic expression for $f\to\ell$ can be obtained

$$\frac{\Gamma[-2f-1]}{\Gamma[2f+1]}\frac{\Gamma[f+1]}{\Gamma[-f]}\to\frac{(\ell)!(\ell)!}{(2\ell+1)!(2\ell)!}\,. \tag{4.4.33}$$

In the uncharged limit $Q=0,P=0$, which implies that the real f reduces to exactly the integer multipole number $f=\ell$, using identity 4.3.12, the Love numbers become

$$k_{\ell m}=-im\gamma\left(\frac{r_+-r_-}{r_++r_-}\right)^{2\ell+1}\frac{(\ell)!(\ell)!}{(2\ell+1)!(2\ell)!}\prod_{n=1}^{\ell}\left[n^2+4m^2\gamma^2\right]\,. \tag{4.4.34}$$

Kerr Love numbers for scalar field perturbations are obtained. They are purely imaginary, thus only dissipative effects are present. Tidal Love numbers of the Kerr black hole for scalar perturbations are zero. This result precisely matches the expression 4.3.11 for the Kerr Love numbers in the scalar case obtained by Le Tiec using analytic continuation [29]. However, here the regularization parameters are physical.

5 Conclusion

In this thesis, the static response of black holes to external tides has been studied. First, the basic concepts of black holes in general relativity were reviewed, including the classification of solutions of the Einstein equation and the Kerr-Newman black hole. Afterwards, a summary of the perturbation theory framework in general relativity was presented. The black hole perturbation equations in the spherical as well as the axially symmetric case were discussed. Then the theory of tides in general relativity was introduced, revising some specific cases. For the Kerr black hole, the conclusion is that the dissipative response to external perturbations is nonzero. But the result crucially depends on analytic continuation. Charged scalar perturbations on a dyonic Kerr-Newman black hole background were considered next. A new calculation of Love numbers for this case was presented.

The dyonic Kerr-Newman black hole response to static charged scalar field perturbations represents the most general case of static scalar field perturbations of black holes in four dimensions with asymptotically flat boundaries, that are not charged under scalar fields themselves. It is described in this work for the first time. In this case, Love numbers are generally not zero. The Kerr black hole response to scalar perturbations was explored in this context as an asymptotic zero-charge case. The result is that the Love numbers are purely imaginary, and tidal Love numbers are zero. This is in full agreement with previously described findings, such as in the work by Le Tiec [29]. Importantly though, those results relied on the use of analytic continuation. While there can be some doubts about the physicality and uniqueness of the analytic continuation, the asymptotic zero-charge approach is completely physical. The matching results help validate both methods and establish an alternative renormalization scheme for finding Kerr Love numbers.

While only static response has been investigated in this work, the more general Heun form perturbation equations have been established. These can be used to investigate dynamical behavior as one of the most direct future perspectives. In the static case, the tidal Love numbers of Schwarzschild, Reissner-Nordström, and Kerr black holes are all zero. Recent studies [31] suggest that, perhaps, the dynamical tidal Love numbers are not zero, but have a logarithmic behavior. Some preliminary findings of the static response for non-zero charge cases indicate a similar trend. Studying charged cases in more detail is another direction in which this work could be expanded on.

Appendices

A Tools in general relativity

A.1 Review of differential geometry

The notion that leads to a quantitative description of the curvature of spacetime is that curvature affects the direction and length of vectors as they are moved from point to point. While vectors can be represented in any basis that spans the spacetime, using a coordinate basis provides a convenient way to track the changes in vectors. A coordinate basis is aligned with coordinate axes, the tangent space basis vectors are identified with the partial derivatives of the coordinates, and the cotangent basis vectors with coordinate differentials:

$$e_\mu = \partial_\mu \ , \tag{A.1.1}$$

$$e^\mu = dx^\mu \ . \tag{A.1.2}$$

The chain rule defines the transformations to new coordinate bases:

$$\partial_\mu = \frac{\partial x^\nu}{\partial x^\mu}\partial_\nu \ , \tag{A.1.3}$$

$$dx^\mu = \frac{\partial x^\mu}{\partial x^\nu}dx^\nu . \tag{A.1.4}$$

By the fundamental theorem of Riemannian geometry [93], a coordinate basis is associated with a unique, metric-compatible, torsion-free Christoffel connection Γ describing the change to the coordinate basis vectors:

$$\partial_\mu e_\nu = \Gamma^\rho{}_{\mu\nu} e_\rho \ , \tag{A.1.5}$$

$$\Gamma^\rho{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right) \ . \tag{A.1.6}$$

In the covariant derivative ∇ both changes of a vector v and the coordinate basis are considered together:

$$\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^\nu{}_{\mu\rho} v^\rho \ . \tag{A.1.7}$$

A shorthand notation "," for the derivative and ";" for the covariant derivative is often used:

$$v^\nu{}_{;\mu} = v^\nu{}_{,\mu} + \Gamma^\nu{}_{\mu\rho} v^\rho \ . \tag{A.1.8}$$

The covariant derivative along a curve $x[\tau]$ is

$$\frac{\nabla v^\mu}{d\tau} = \frac{dv^\mu}{d\tau} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{d\tau} v^\gamma \ . \tag{A.1.9}$$

The connection is chosen to make the covariant derivative annihilate the metric. The Christoffel connection is the most basic element capturing curved spacetime features, however, it is not a tensor. The commutation relations of the covariant derivatives provide a covariant way to quantify the curvature of spacetime [94]. For each type of tensor component, they are

$$\nabla_{[\mu}\nabla_{\nu]} s = -S^\rho{}_{\mu\nu}\nabla_\rho s \ , \tag{A.1.10}$$

$$\nabla_{[\mu}\nabla_{\nu]} v^\sigma = R^\sigma{}_{\rho\mu\nu} v^\rho - 2S^\rho{}_{\mu\nu}\nabla_\rho v^\sigma \ , \tag{A.1.11}$$

$$\nabla_{[\mu}\nabla_{\nu]} v_\sigma = -R^\rho{}_{\sigma\mu\nu} v_\rho - 2S^\rho{}_{\mu\nu}\nabla_\rho v_\sigma \ . \tag{A.1.12}$$

The rank-4 Riemann tensor is constructed from Christoffel connections and their derivatives:

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu{}_{\sigma\nu} - \partial_\sigma \Gamma^\mu{}_{\rho\nu} + \Gamma^\mu{}_{\rho\upsilon}\Gamma^\upsilon{}_{\sigma\nu} - \Gamma^\mu{}_{\sigma\upsilon}\Gamma^\upsilon{}_{\rho\nu} \ . \tag{A.1.13}$$

The contraction of the Riemann tensor in its first and third indices is the rank-2 Ricci tensor

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu} \ . \tag{A.1.14}$$

The contraction of the Ricci tensor is the rank-0 Ricci scalar

$$R = R^{\mu}{}_{\mu} \ . \tag{A.1.15}$$

The Weyl tensor C is the traceless part of the Riemann tensor

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{2}(g_{\mu\rho}R_{\nu\sigma} - g_{\mu\sigma}R_{\nu\rho} - g_{\nu\rho}R_{\mu\sigma} + g_{\nu\sigma}R_{\mu\rho}) + \frac{1}{6}R(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \ . \tag{A.1.16}$$

These comprise the family of curvature tensors R. The Ricci scalar provides a measure of the curvature, the Ricci tensor encapsulates the curvature associated with sources, and the Weyl tensor describes source-free effects of curvature, such as tidal effects and gravitational waves.

The skew-symmetric part of connections and the Lie bracket, which is the symmetric part of the covariant derivative, make up the torsion tensor S. In general relativity, it is taken to be zero. When a coordinate system is used, since partial derivatives commute, the Lie bracket is zero meaning that paths in loops close and the labeling of spacetime is unique. Then the torsion tensor is purely the skew-symmetric part of the Christoffel connection

$$S^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\mu\nu} - \Gamma^{\rho}{}_{\nu\mu} = 0 \ . \tag{A.1.17}$$

Being torsion-free, the Christoffel connection is symmetric in its lower indices. Consequently, the Riemann tensor also exhibits a set of symmetries and skew-symmetries. Its component relations are

$$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu} = -R_{\mu\nu\sigma\rho} = -R_{\nu\mu\rho\sigma} \ . \tag{A.1.18}$$

Additionally, the Riemann tensor satisfies the Bianchi identity

$$R_{\mu[\nu\rho\sigma]} = 0 \ , \tag{A.1.19}$$

and the contracted Bianchi identity

$$\nabla_{[\upsilon}R_{\mu\nu]\rho\sigma} = 0 \ , \tag{A.1.20}$$

which consider the tensor in a cyclic sum [95].

The Lie derivative of the metric being zero is the Killing equation:

$$\mathcal{L}_K g_{\mu\nu} = 0 \quad \Leftrightarrow \quad \nabla_\nu K_\mu + \nabla_\mu K_\nu = 0 \ , \tag{A.1.21}$$

For every Killing vector field K_μ there is a constant of motion K:

$$K = K_\mu \frac{dx^\mu}{d\lambda} \ , \tag{A.1.22}$$

$$\frac{dK}{d\lambda} = 0 \ . \tag{A.1.23}$$

If the metric does not depend on a particular coordinate, then the vector field related to the direction of the coordinate is a Killing vector field.

Spacetime curvature manifests in the emergence of gravity. Inertial motion is described by geodesics, which are intrinsically unchanging smooth vector fields. All alterations in the geodesics can then be attributed to curvature. Inertial motion follows the geodesic equation

$$\frac{\nabla u^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau} + \Gamma^{\mu}{}_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0 \ , \tag{A.1.24}$$

where u is the four-velocity associated with the motion. The geodesic deviation equation describes how in a curved spacetime nearby geodesics diverge from one another:

$$\frac{\nabla^2\xi^\mu}{d\tau^2} + R^{\mu}{}_{\nu\rho\sigma}\xi^\rho u^\nu u^\sigma = 0 \ . \tag{A.1.25}$$

The deviations of inertial paths lead to tidal effects, a hallmark of gravity.

The abstract treatment of curvature can be made physically meaningful only after a spacetime geometry and the complementary sources of the geometry are introduced. The two influence one another, and the Einstein equation 2.1.1 ensures their compatibility.

A.2 The Einstein equation from the Hilbert action

Hilbert's approach to finding the Einstein equation is outlined here. The presentation closely follows that of Caroll [94].

The Hilbert action is

$$S_H = \int d^4x \mathscr{L}_H \, , \tag{A.2.1}$$

where the Hilbert Lagrange density is

$$\mathscr{L}_H = \sqrt{-g} R \, . \tag{A.2.2}$$

The motivation for using this Lagrange density is the principle of general covariance. To maintain the laws of physics in a covariant form, the Lagrange density must be a scalar. Any equation of motion, compatible with gravity as a source of acceleration, must be a second-order equation. The Ricci scalar R is the simplest quantity that can be formed from the metric and its derivatives. Due to the presence of the metric, the Ricci scalar encodes the properties of spacetime curvature and leads to a second-order Euler-Lagrange equation. To preserve the invariance of the action under general coordinate transformations, the Lagrange density includes the Jacobian determinant of the metric

$$\sqrt{-g} = \det \left| \frac{\partial x^\mu}{\partial x^\nu} \right| \, . \tag{A.2.3}$$

The variation of the action with respect to the inverse metric is

$$\delta S_H = \int d^4x (\sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} + \sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} + R \delta \sqrt{-g}) \, . \tag{A.2.4}$$

The variation of the action involves the variations of the Christoffel connections. While the Christoffel connection is not a tensor, its variation, being a difference of two Christoffel connections, is a tensor since the non-tensor terms cancel. The covariant derivative of the variation of the Christoffel connection is

$$\nabla_\sigma \delta \Gamma^\rho_{\mu\nu} = \partial_\sigma \delta \Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\sigma\upsilon} \delta \Gamma^\upsilon_{\mu\nu} - \Gamma^\upsilon_{\sigma\mu} \delta \Gamma^\rho_{\upsilon\nu} - \Gamma^\upsilon_{\sigma\nu} \delta \Gamma^\rho_{\upsilon\mu} \, . \tag{A.2.5}$$

In turn, the variation of the Riemann tensor can be expressed as

$$\delta R^\rho{}_{\mu\sigma\nu} = \nabla_\sigma \delta \Gamma^\rho_{\mu\nu} - \nabla_\nu \delta \Gamma^\rho_{\mu\sigma} \, . \tag{A.2.6}$$

One of the terms in the variation of the action can be identified as the volume element of the covariant divergence. Using Stokes' theorem [96] which relates an integral over a region to an integral around the boundary of that region, this volume element term can be substituted by a boundary term at infinity. Requiring the variation of the action to be zero at infinity this term disappears from the action:

$$\int d^4x \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} = \int d^4x \sqrt{-g} g^{\mu\nu} (\nabla_\rho \delta \Gamma^\rho_{\mu\nu} - \nabla_\nu \delta \Gamma^\rho_{\mu\rho}) = \int d^4x \sqrt{-g} \nabla_\rho (g^{\mu\nu} \delta \Gamma^\rho_{\mu\nu} - g^{\mu\rho} \delta \Gamma^\nu_{\mu\nu}) = 0 \, . \tag{A.2.7}$$

The variation of the determinant can be expressed as

$$\delta \sqrt{-g} = \delta [-g^{-1}]^{-\frac{1}{2}} = -\frac{1}{2} (-g^{-1})^{-\frac{3}{2}} \delta [-g^{-1}] = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} \, . \tag{A.2.8}$$

Using the above expression, the variation of the action can be reduced to

$$\delta S_H = \int d^4x \left(\sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} - \frac{1}{2} R \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} \right) = \int d^4x \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \sqrt{-g} \delta g^{\mu\nu} \, . \tag{A.2.9}$$

Extremizing the Hilbert action, i.e. setting its variation to zero, leads to the Einstein equation without source:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \, . \tag{A.2.10}$$

The Hilbert action can be extended with an action S_ϕ for matter fields

$$S = \frac{S_H}{8\pi} + S_\phi \,. \tag{A.2.11}$$

Analogously, this leads to the expression

$$\frac{8\pi\delta S}{\sqrt{-g}\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \frac{8\pi\delta S_\phi}{\sqrt{-g}\delta g^{\mu\nu}} = 0 \,. \tag{A.2.12}$$

Identifying the energy-momentum tensor as

$$T_{\mu\nu} = -8\pi\frac{\delta S_\phi}{\sqrt{-g}\delta g^{\mu\nu}} \,, \tag{A.2.13}$$

the full Einstein equation is obtained:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu} \,. \tag{A.2.14}$$

Deriving the energy-momentum tensor from the action principle assures gauge invariance.

By the Lovelock theorem, which considers what form, in general, can a metric-dependent tensor have, in the case of a four-dimensional spacetime, the only possible equation of motion obtainable from an arbitrary metric-dependent scalar Lagrange density $\mathscr{L} = \mathscr{L}[g]$ is the Einstein equation [97]. This implies that in four dimensions the Hilbert action is the most general action that leads to a second-order equation of motion describing gravity.

A.3 Matter field sources of gravity

Gravitational coupling to a real scalar field ϕ with the Lagrangian

$$\mathscr{L}_\phi = -\nabla_\mu\phi\nabla^\mu\phi - V[\phi] \tag{A.3.1}$$

leads to the action

$$S[g,\phi] = S_H + S_\phi = \frac{1}{16\pi}\int d^4x\sqrt{-g}R - \frac{1}{2}\int d^4x\sqrt{-g}(\nabla_\mu\phi\nabla^\mu\phi + V[\phi]) \,, \tag{A.3.2}$$

and subsequently to the coupled equations of motion

$$G_{\mu\nu} = T_{\mu\nu} \,, \tag{A.3.3}$$

$$(\Box - V)\phi = 0 \,, \tag{A.3.4}$$

where $\Box = \nabla_\mu\nabla^\mu$ is the d'Alembertian, and the energy-momentum tensor in terms of the scalar field ϕ is

$$\begin{aligned} T_{\mu\nu}[\phi] &= -8\pi\frac{\delta[\sqrt{-g}\mathscr{L}_\phi]}{\sqrt{-g}\delta g^{\mu\nu}} = \frac{-8\pi}{\sqrt{-g}}\left(\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\mathscr{L}_\phi + \sqrt{-g}\frac{\delta\mathscr{L}_\phi}{\delta g^{\mu\nu}}\right) = \frac{-8\pi}{\sqrt{-g}}\left(\frac{\frac{1}{2}\sqrt{-g}g^{\rho\sigma}\delta g_{\rho\sigma}}{-g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma}}\mathscr{L}_\phi + \sqrt{-g}\frac{\delta\mathscr{L}_\phi}{\delta g^{\mu\nu}}\right) \\ &= 8\pi\left(\frac{1}{2}g_{\mu\nu}(-\nabla_\rho\phi\nabla^\rho\phi - V) + \nabla_\mu\phi\nabla_\nu\phi\right) \,. \end{aligned} \tag{A.3.5}$$

Analogously, the action for a Maxwell field is

$$S[g,\phi,A] = \frac{1}{16\pi}\int d^4x\sqrt{-g}(R - F_{\mu\nu}F^{\mu\nu}) \,, \tag{A.3.6}$$

where the field strength tensor of the vector field is

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \,. \tag{A.3.7}$$

This leads to the equations of motion

$$G_{\mu\nu} = T_{\mu\nu} \,, \tag{A.3.8}$$

$$\nabla_\nu F^{\mu\nu} = 0 \,. \tag{A.3.9}$$

The coupling of electromagnetism to gravity is through the energy-momentum tensor, which is

$$T_{\mu\nu} = F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \,. \tag{A.3.10}$$

A.4 Tetrad formalism and the Cartan structure equations

In the most widely recognized formulation of general relativity coordinate bases are used, but this is not the only possibility. Non-coordinate bases can be employed in what is known as the tetrad formalism [43, 98]. In some circumstances, using alternative bases may be advantageous, for instance, to reflect a symmetry that can be used to simplify a particular physical problem. The standard method is to adopt an orthonormal basis at each spacetime point. The rationale for this choice and the introduction of the tetrad formalism is that it provides a way to describe fermions in curved spacetime. In the well-established spinor formalism on flat space, the gamma matrices have anti-commutation relations that tie them to the Minkowski metric, as spinors transform according to the spin representation of the Lorentz group with generators proportional to the commutators of the gamma matrices. The Lorentz transformations preserve the Minkowski metric. Modifying the gamma matrices to account for curvature is conceptually challenging, to sidestep this issue, orthonormal bases can be used, effectively concealing the curvature with the choice of basis. This strategy allows to preserve the intrinsic properties of gamma matrices. Furthermore, in addition to the local coordinate invariance of general relativity, another symmetry is revealed - local Lorentz invariance. The spin connection one-forms encapsulate the geometric effects of curvature, that manifest in the derivative operator of the spinors. Beyond providing a method for introducing fermions in the theory, tetrad formalism provides a complementary interpretation of the curvature, in particular, via the Cartan structure equations [99]. The basic ingredients of the tetrad formalism and their relation to a coordinate basis are highlighted in the following summary based on [9, 43, 94, 100], where the topics are discussed in more detail. A concise introduction to the calculus of forms can be found in [45].

It is possible to locally choose an orthonormal basis by a coordinate transformation. Abstractly, an analog global transformation can be defined, but this transformation in a curved spacetime depends on the location and is no longer a coordinate transformation A.1.3. This is known as a tetrad. Tetrad basis vectors in terms of the coordinate basis vectors can be expressed as

$$e_a = {e_a}^{\mu}\partial_\mu \ , \tag{A.4.1}$$

$$e^a = {e^a}_{\mu} dx^\mu \ , \tag{A.4.2}$$

with ${e_a}^{\mu}$ and ${e^a}_{\mu}$ being the components of the tangent and cotangent tetrad basis vectors. The metrics of the two are related via

$$\eta_{ab} = {e_a}^{\mu}{e_b}^{\nu} g_{\mu\nu} \ , \tag{A.4.3}$$

$$g_{\mu\nu} = {e^a}_{\mu}{e^b}_{\nu}\eta_{ab} \ . \tag{A.4.4}$$

where $g_{\mu\nu}$ are the coordinate metric components, and η_{ab} are the the tetrad metric components. Since the tetrad basis vectors are chosen to be orthonormal, the latter are the Minkowski metric components. The above-mentioned Lorentz symmetry can be seen from the transformation

$$\eta_{ab} = {\Lambda^a}_c {\Lambda^b}_d \eta_{cd} = {\Lambda^a}_c {\Lambda^b}_d {e^\mu}_c {e^\nu}_d g_{\mu\nu} \ , \tag{A.4.5}$$

where Λ represents a Lorentz transformation. This is an equally valid transformation to the one defined in A.4.3. Unlike in the generic treatment of spinors, where the Lorentz symmetry is global, here the transformations are position-dependent, thus the Lorentz symmetry is local.

While here the tetrad formalism is introduced starting from a coordinate basis, in a broader approach no assumptions about the basis are needed, as long as the differentiation operation is defined to link adjacent spacetime regions. A general tensor basis is related to a tetrad basis through

$$e_a = {e_a}^{i} e_i \ , \tag{A.4.6}$$

$$e^a = {e^a}_{i} e^i \ , \tag{A.4.7}$$

The tensor and tetrad components of a tensor t are related as

$$t^a{}_b = {e^a}_i {e_b}^j {t^i}_j \ . \tag{A.4.8}$$

The covariant derivative of a tensor can be expressed with spin connections ${\omega_i}^a{}_b$ replacing the connections ${\omega^k}_{ij}$ associated with the tensor basis:

$$\nabla_i t^a{}_b = \partial_i t^a{}_b + {\omega_i}^a{}_c t^c{}_b - {\omega_i}^c{}_b t^a{}_c \ . \tag{A.4.9}$$

By comparison to the covariant derivative of the same tensor with tensor basis components, the connection $\omega_i{}^a{}_b$ in terms of the connection $\omega^k{}_{ij}$ and vice versa are

$$\omega_i{}^a{}_b = e^a{}_k e^j{}_b \omega^k{}_{ij} - e^j{}_b \partial_i e^a{}_j \,, \tag{A.4.10}$$
$$\omega^k{}_{ij} = e^k{}_a \partial_i e^a{}_j + e^k{}_a e^b{}_j \omega_i{}^a{}_b \,. \tag{A.4.11}$$

This implies that the covariant derivative of the tetrad is zero

$$\nabla_i e^a{}_j = 0 \,. \tag{A.4.12}$$

With the above, it is possible to project tensors involving covariant derivatives onto the tetrad basis. In particular, this is relevant to be able to express quantities describing curvature. Having set torsion to zero, the Riemann tensor as the commutator of the covariant derivatives A.1.11, generalized to an arbitrary tensor basis, is

$$R_{ijkl} v^j = \nabla_{[k} \nabla_{l]} v_i \,. \tag{A.4.13}$$

The projection of the Riemann tensor onto a tetrad basis is

$$\begin{aligned} R_{abcd} &= R_{ijkl} e^i{}_a e^j{}_b e^k{}_c e^l{}_d = (\nabla_{[k} \nabla_{l]} e^i{}_a) e^j{}_b e^k{}_c e^l{}_d = \\ &= \omega_{abd,c} - \omega_{abc,d} + \omega^e{}_{ac} \omega_{bed} - \omega^e{}_{ad} \omega_{bec} + \omega^e{}_{dc} \omega_{bae} - \omega^e{}_{cd} \omega_{bae} \,, \end{aligned} \tag{A.4.14}$$

where the connections are used to replace the covariant derivatives of tetrads, and the non-commutativity of the tetrads is expressed with a Lie bracket accounting for the additional terms compared to the coordinate definition of the Riemann tensor A.1.13. The Riemann tensor in terms of the Ricci scalar, Ricci tensor, and Weyl scalar in tetrad basis is:

$$R_{abcd} = C_{abcd} + \frac{1}{2} (\eta_{ac} R_{bd} - \eta_{bc} R_{ad} - \eta_{ad} R_{bc} + \eta_{bd} R_{ac}) - \frac{1}{6} (\eta_{ac} \eta_{bd} - \eta_{ad} \eta_{bc}) R \,. \tag{A.4.15}$$

The covariant derivative of the metric in terms of the tetrad components is

$$\nabla^i \eta_{ab} = \partial^i \eta_{ab} - \omega^i{}_{ca} \eta^c{}_b - \omega^i{}_{cb} \eta_a{}^c \,. \tag{A.4.16}$$

When the metric compatibility requirement holds:

$$\nabla^i \eta_{jk} = 0 \,, \tag{A.4.17}$$

and when the components of the tetrad metric are constant, the partial derivative term vanishes and it follows that the spin connection must be skew-symmetric in the tetrad components:

$$\omega^i{}_{ab} = -\omega^i{}_{ba} \,, \tag{A.4.18}$$

the spin connection therefore has six independent tetrad components. A tensor that is skew-symmetric in its covariant coordinate indices can be thought of as a tensor-valued differential form, whose exterior derivative transforms as a form under coordinate transformations, however, it does not transform as a tensor under Lorentz transformations. Covariant differentiation preserves tensor transformations, and to preserve all transformation properties, the exterior derivative should be complemented with a tetrad connection. For the tensor-valued form t the exterior derivative is

$$(\mathrm{d}t)^a{}_{ij} = \partial_i t^a{}_j - \partial_j t^a{}_i \,, \tag{A.4.19}$$

but the covariant object containing the exterior derivative is

$$(\mathrm{d}t + \omega \wedge t)^a{}_{ij} = \partial_i t^a{}_j - \partial_i t^a{}_j + \omega^a{}_{bi} t^b{}_j - \omega^a{}_{bj} t^b{}_i \,. \tag{A.4.20}$$

This can be applied to the two tensors made of the Christoffel connections - the torsion S and the curvature R. With the definitions A.1.17 and A.1.13 of the two together with the expression for the Christoffel connection in the tetrad basis A.4.11, these can be brought to a mixed form with two tetrad and two tensor indices. Considering them as forms the Cartan structure equations can be obtained. Leaving only the tetrad indices, they are

$$S^a = \mathrm{d}e^a + \omega^a{}_b \wedge e^b \,, \tag{A.4.21}$$
$$R^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \,. \tag{A.4.22}$$

Cartan structure equations hold information about the curvature of the spacetime. They relate the spin connection and the torsion to the derivatives of the dual basis. This provides a way of computing connection components.

A.5 Newman-Penrose formalism

The Newman-Penrose formalism is a variation of the tetrad formalism [32, 44]. It has been extensively used in gravity research since its development in 1962, crucially, in finding exact solutions to the Einstein equation and developing the perturbative framework used, among others, for describing Kerr black hole perturbations. Here the tensor variant of the formalism is introduced, based on the descriptions by Newman and Penrose [32] and Chandrasekhar [45], where more details and the Newman-Penrose equations can be found.

The innovation is in the use of a null basis. For this, the key is to allow the basis vectors to be complex, then all four basis vectors can be made null. Null vectors allow for a natural decomposition of four-component spinors into two-component spinors. This decomposition simplifies the representation of spinors, which motivates the introduction of the Newman-Penrose formalism. The objects in the Newman-Penrose formalism are given explicit names. The tensor indices are dispensed, and tensor equations are treated one component at a time.

The Newman - Penrose tetrad λ is

$$\lambda^a{}_i = (l^a, n^a, m^a, \bar{m}^a) \ . \tag{A.5.1}$$

The first two basis vectors l, n are real. The other two basis vectors $m, \bar{m}$ are complex and conjugate to one another. The real null vectors describe the lightcone. The tetrad vector null condition is

$$l^a l_a = n^a n_a = m^a m_a = \bar{m}^a \bar{m}_a = 0 \ , \tag{A.5.2}$$

the orthogonality condition is

$$l^a m_a = l^a \bar{m}_a = n^a \cdot m_a = n^a \cdot \bar{m}_a = 0 \ , \tag{A.5.3}$$

and the normalization convention is

$$-l^a n_a = m^a \bar{m}_a = 1 \ . \tag{A.5.4}$$

Then the metric of the tangent space is

$$\eta_{ab} = -l_a n_b - n_a l_b + m_a \bar{m}_b + \bar{m}_a m_b = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \ . \tag{A.5.5}$$

Raising and lowering indices is equivalent to the permutations $1, 2, 3, 4 \to 2, 1, -4, -3$, and complex conjugates are obtained by permuting $1, 2, 3, 4 \to 2, 1, 3, 4$. The directional derivatives are

$$\nabla_i = \lambda^a{}_i = (D, \Delta, \delta, \bar{\delta}) \ , \tag{A.5.6}$$

and the covariant derivative along x is

$$\nabla_x = x^i \nabla_i \ . \tag{A.5.7}$$

The spin connection definition is

$$\gamma^i{}_{jk} = \lambda^a{}_j \lambda^b{}_k \nabla_a \lambda^i{}_b = \eta^{il} \gamma_{ljk} = -\eta^{il} \gamma_{jlk} \ . \tag{A.5.8}$$

The spin connection components are referred to as spin coefficients. They are the basic objects with which other quantities and equations are expressed. Their naming convention is

$$\begin{aligned}
&\kappa = \gamma_{131} = -m^a D l_a, && \pi = \gamma_{421} = \bar{m}^a D n_a \ , && \epsilon = \tfrac{1}{2}(\gamma_{121} - \gamma_{341}) = \tfrac{1}{2}(\bar{m}^a D m_a - n^a D l_a) \ , \\
&\tau = \gamma_{132} = -m^a \Delta l_a \ , && \nu = \gamma_{422} = \bar{m}^a \Delta n_a \ , && \gamma = \tfrac{1}{2}(\gamma_{122} - \gamma_{342}) = \tfrac{1}{2}(\bar{m}^a \Delta m_a - n^a \Delta l_a) \ , \\
&\sigma = \gamma_{133} = -m^a \delta l_a \ , && \mu = \gamma_{423} = \bar{m}^a \delta n_a \ , && \beta = \tfrac{1}{2}(\gamma_{123} - \gamma_{343}) = \tfrac{1}{2}(\bar{m}^a \delta m_a - n^a \delta l_a) \ , \\
&\rho = \gamma_{134} = -m^a \bar{\delta} l_a \ , && \lambda = \gamma_{424} = \bar{m}^a \bar{\delta} n_a \ , && \alpha = \tfrac{1}{2}(\gamma_{124} - \gamma_{344}) = \tfrac{1}{2}(\bar{m}^a \bar{\delta} m_a - n^a \bar{\delta} l_a) \ .
\end{aligned} \tag{A.5.9}$$

The Riemann tensor is comprised of the Weyl tensor, Ricci tensors, and Ricci scalar as in equation A.4.15. The distinct Riemann tensor components in the Newman-Penrose tetrad basis are

$$\begin{aligned}
&R_{1212} = C_{1212} + R_{12} - \tfrac{1}{6}R\ ,\\
&R_{1213} = C_{1213} + \tfrac{1}{2}R_{13}\ ,\\
&R_{1223} = C_{1223} - \tfrac{1}{2}R_{23}\ ,\\
&R_{1234} = C_{1234}\ ,\\
&R_{1313} = C_{1313}\ ,\\
&R_{1314} = \tfrac{1}{2}R_{11}\ ,\\
&R_{1324} = C_{1324} + \tfrac{1}{12}R\ ,\\
&R_{1334} = C_{1334} + \tfrac{1}{2}R_{13}\ ,\\
&R_{2323} = C_{2323}\ ,\\
&R_{2324} = \tfrac{1}{2}R_{22}\ ,\\
&R_{2334} = C_{2334} + \tfrac{1}{2}R_{23}\ ,\\
&R_{3132} = -\tfrac{1}{2}R_{33}\ ,\\
&R_{3434} = C_{3434} - R_{34} - \tfrac{1}{6}R\ .
\end{aligned} \tag{A.5.10}$$

These can be complex, and the remainder of the nonzero components are the complex conjugates. The Weyl and Ricci tensors obey additional cyclic relations that relate some of their components. Each has ten degrees of freedom, that are represented using the Newman-Penrose scalars. The Weyl tensor is represented by five complex Weyl scalars Ψ:

$$\begin{aligned}
&\Psi_0 = C_{1313} = C_{abcd}l^a m^b l^c m^d\ ,\\
&\Psi_1 = C_{1213} = C_{abcd}l^a n^b l^c m^d\ ,\\
&\Psi_2 = C_{1342} = C_{abcd}l^a m^b \bar{m}^c n^d\ ,\\
&\Psi_3 = C_{1242} = C_{abcd}l^a n^b \bar{m}^c n^d\ ,\\
&\Psi_4 = C_{2424} = C_{abcd}n^a \bar{m}^b n^c \bar{m}^d\ .
\end{aligned} \tag{A.5.11}$$

The Ricci tensor is represented with the Ricci scalars Φ and Λ:

$$\begin{aligned}
&\Phi_{00} = \tfrac{1}{2}R_{11} = \tfrac{1}{2}R_{ab}l^a l^b\ ,\\
&\Phi_{01} = \tfrac{1}{2}R_{13} = \tfrac{1}{2}R_{ab}l^a m^b\ ,\\
&\Phi_{02} = \tfrac{1}{2}R_{33} = \tfrac{1}{2}R_{ab}m^a m^b\ ,\\
&\Phi_{11} = \tfrac{1}{4}(R_{12} + R_{34}) = \tfrac{1}{4}R_{ab}(l^a n^b + m^a \bar{m}^b)\ ,\\
&\Phi_{12} = \tfrac{1}{2}R_{32} = \tfrac{1}{2}R_{ab}m^a n^b\ ,\\
&\Phi_{22} = \tfrac{1}{2}R_{22} = \tfrac{1}{2}R_{ab}n^a n^b\ ,\\
&\Lambda = \tfrac{1}{24}R\ .
\end{aligned} \tag{A.5.12}$$

Three scalars $\Phi_{01}, \Phi_{02}, \Phi_{12}$ are complex, and four scalars $\Phi_{00}, \Phi_{11}, \Phi_{22}, \Lambda$ are real. In the Newman-Penrose formalism, the choice of basis sets the Ricci tensor equal to the Einstein tensor G. The electromagnetic field strength F is expressed with three Maxwell scalars ϕ:

$$\begin{aligned}
&\phi_0 = F_{13} = F_{ab}l^a m^b\ ,\\
&\phi_1 = \tfrac{1}{2}(F_{12} + F_{43}) = \tfrac{1}{2}F_{ab}(l^a n^b + \bar{m}^a m^b)\ ,\\
&\phi_2 = F_{42} = F_{ab}\bar{m}^a n^b\ .
\end{aligned} \tag{A.5.13}$$

Through the energy-momentum tensor T the Ricci and Maxwell scalars are related as $\Phi_{ij} = 2\phi_i \phi_j$.

The equations of general relativity can be grouped into the transportation equations that describe parallel transport, the spin coefficient equations that encode the information of the Ricci identity with the Cartan structure equations, and the Bianchi identity equations.

The Newman-Penrose tetrad vector null length A.5.2 can be viewed as the directional derivatives of the corresponding tetrad vectors being zero. The connection is chosen to annihilate the metric, consequently, the covariant derivatives along the direction of the basis vectors are zero. This gives the transportation equations. The metric compatibility can also be expressed in terms of commutators of the directional derivatives. The Riemann tensor components expressed in terms of the commutator of the covariant derivatives, as in equation A.4.14, relate the Weyl scalars to the spin coefficients in the spin coefficient equations. The contracted Bianchi identity A.1.20 can be recast into the Bianchi identity equations. In addition to the Einstein equation, the Maxwell equation can be written as a set of Newman-Penrose equations.

The transportation equations are available in the book on spinors by O'Donnel [101]. The remainder of the equations are available in the Newman-Penrose Scholarpedia article [32] with the additional nonzero Ricci scalar contributions in the Bianchi identities available in Chandrasekhar's book [45].

The Newman-Penrose quantities can be greatly simplified if the lightcone structure integrated into the Newman-Penrose formalism is exploited. Many of the Newman-Penrose quantities disappear if the spacetime is algebraically special. Black holes are of Petrov type D. In terms of the Weyl scalars, that means that only Ψ_2 is nonzero. The zero Weyl scalars are associated geometrically with particular tetrad vectors being principal null directions. This is frame-dependent, but Lorentz symmetry allows performing tetrad transformations to obtain the above identification of the Weyl scalars. The Lorentz transformations are divided into class I rotations that leave vector l unchanged, class II rotations that leave n unchanged, and class III rotations that leave both l and n unchanged [45].

A principal null direction at a point in spacetime is a null vector tangent to a null geodesic such that the expansion, rotation, and shear of the congruence of null geodesics containing this vector are simultaneously zero. A null vector v is tangent to an affinely parametrized null geodesic when the geodesic equation is satisfied:

$$v^a \nabla_a v^b = 0 \ . \tag{A.5.14}$$

From the Newman-Penrose transportation equations [101], the covariant derivative of the vector l along its direction, expressed with the directional derivatives and spin coefficients, is

$$Dl^a = (\epsilon + \bar{\epsilon}) l^a - \bar{\kappa} m^a - \kappa \bar{m}^a \ . \tag{A.5.15}$$

If the spin coefficient $\kappa = 0$, the tetrad vector l is tangent to a geodesic. If $\epsilon + \bar{\epsilon} = 0$, the geodesic has an affine parameter. Similarly, for the vector n, the covariant derivative in its direction is

$$\Delta n^a = -(\gamma + \bar{\gamma}) n^a + \nu m^a + \bar{\nu} \bar{m}^a \ . \tag{A.5.16}$$

The tetrad vector is tangent to an affinely parametrized null geodesic if $\nu = 0$ and $\gamma + \bar{\gamma} = 0$.

The lightcone is most straightforwardly associated with the two real tetrad vectors. This is why these particular Weyl scalars are identified with the Petrov types. The complex vectors can also be used for this purpose if they are split into components, which can be done after performing a basis change to represent them on a sphere, but then Petrov classification would involve different combinations of Weyl scalars. The algebraic properties of the Weyl tensor and the geometric properties of spacetime are linked through The Goldberg-Sachs theorem formulated in 1962. It states that a vacuum spacetime is algebraically special if it contains at least one geodesic, shear-free null congruence and vice versa [48]. This can be seen from the Bianchi identity equations [32] of the Newman-Penrose formalism. If $\Psi_0 = \Psi_1 = 0$, then l is the principal null direction. The geodesic property $k = 0$ follows from the equation

$$\bar{\delta}\Psi_0 - D\Psi_1 = (4\alpha - \pi)\Psi_0 - 2(2\rho + \epsilon)\Psi_1 + 3\kappa\Psi_2 \quad \Rightarrow \tag{A.5.17}$$

$$0 = 3\kappa\Psi_2 \ . \tag{A.5.18}$$

The zero shear $\sigma = 0$ follows from the equation

$$\Delta\Psi_0 - \delta\Psi_1 = (4\gamma - \mu)\Psi_0 - 2(2\tau + \beta)\Psi_1 + 3\sigma\Psi_2 \quad \Rightarrow \tag{A.5.19}$$

$$0 = 3\sigma\Psi_2 \ . \tag{A.5.20}$$

With a class III rotation, the tetrad basis can be chosen to also have zero expansion $\tau = 0$.

B Black hole perturbation master equations

B.1 Spherically symmetric black hole perturbation master equations

In this section, the highlights of the derivations of the master equations for background black hole spacetimes with spherical symmetry are presented. The conventions here are analogous to the ones established by Martel and Poisson [74, 75], the definitions of the various objects are as described by Pereñiguez [77], and the derivations of the master equations of Schwarzschild black hole perturbations follow closely the presentation by Berti [3].

The line element of spherically symmetric black hole spacetimes can be expressed with the Lorentz part and the spherical part explicitly separated [75]:

$$ds^2 = g_{\mu\nu}[x]dx^\mu dx^\nu = \hat{g}_{IJ}[y]dy^I dy^J + r[y]^2 \tilde{g}_{\mathcal{K}\mathcal{L}}[z]dz^{\mathcal{K}} dz^{\mathcal{L}} , \tag{B.1.1}$$

where $r[y]$ is a real function, $\hat{g}$ is the two-dimensional Lorentz metric with capital Latin letters used for the corresponding Lorentz indices, and $\tilde{g}$ is the spherical metric with calligraphic capital Latin letters used for the corresponding spherical indices. The Lorentz indices are lowered and raised with the Lorentz metric, and the spherical indices are lowered and raised with the spherical metric. In Schwarzschild coordinates $(t, r, \vartheta, \varphi)$ the Lorentz metric is

$$\hat{g}_{IJ}[y]dy^I dy^J = -f dt^2 + \frac{1}{f} dr^2 , \tag{B.1.2}$$

$$f = r_{;I} r_{;}{}^{I} , \tag{B.1.3}$$

and the spherical metric is

$$\tilde{g}_{\mathcal{K}\mathcal{L}}[z]dz^{\mathcal{K}} dz^{\mathcal{L}} = d\vartheta^2 + \sin^2\vartheta d\varphi^2 . \tag{B.1.4}$$

Both the uncharged and the charged spherical black holes are represented with the above, with the expressions defining f for the Schwarzschild black hole and the Reissner-Nordström respectively being

$$\begin{aligned} f_{\mathrm{Sc}} &= 1 - \frac{2M}{r} , \\ f_{\mathrm{RN}} &= 1 - \frac{2M}{r} + \frac{Q^2}{r^2} . \end{aligned} \tag{B.1.5}$$

The relations between various full curvature objects and their Lorentz and spherical parts are given below.

The full Christoffel connections are related to the Lorentz and spherical metric Christoffel connections through

$$\Gamma^{I}{}_{JK} = \hat{\Gamma}^{I}{}_{JK} , \tag{B.1.6}$$

$$\Gamma^{I}{}_{\mathcal{J}\mathcal{K}} = -r r_{;}{}^{I} \tilde{g}_{\mathcal{J}\mathcal{K}} , \tag{B.1.7}$$

$$\Gamma^{\mathcal{I}}{}_{J\mathcal{K}} = \frac{1}{r} r_{;J} \delta^{\mathcal{I}}{}_{\mathcal{K}} , \tag{B.1.8}$$

$$\Gamma^{\mathcal{I}}{}_{\mathcal{J}\mathcal{K}} = \tilde{\Gamma}^{\mathcal{I}}{}_{\mathcal{J}\mathcal{K}} . \tag{B.1.9}$$

Similarly, the full Riemann tensor components have the following relations to the Riemann tensor components of the Lorentz and spherical metrics:

$$R^{I}{}_{JKL} = \hat{R}^{I}{}_{JKL} , \tag{B.1.10}$$

$$R^{I}{}_{\mathcal{J}K\mathcal{L}} = -r r_{;K}{}^{I} \tilde{g}_{\mathcal{J}\mathcal{L}} , \tag{B.1.11}$$

$$R^{\mathcal{I}}{}_{\mathcal{J}\mathcal{K}\mathcal{L}} = (1 - f) \left(\delta^{\mathcal{I}}{}_{\mathcal{K}} \tilde{g}_{\mathcal{J}\mathcal{L}} - \delta^{\mathcal{I}}{}_{\mathcal{L}} \tilde{g}_{\mathcal{J}\mathcal{K}} \right) . \tag{B.1.12}$$

The expressions for the full Ricci tensor components are

$$R_{IJ} = \hat{R}_{IJ} - \frac{2}{r} r_{;JI} \, , \tag{B.1.13}$$

$$R_{\mathcal{I}\mathcal{J}} = \left((1-f) - r r_{;K}{}^{K}\right) \tilde{g}_{\mathcal{I}\mathcal{J}} \, . \tag{B.1.14}$$

The expressions for the full Einstein tensor components are

$$G_{IJ} = \hat{G}_{IJ} + \frac{f-1}{r^2}\hat{g}_{IJ} - \frac{2}{r}\left(r_{;JI} - r_{;K}{}^{K}\hat{g}_{IJ}\right) \, , \tag{B.1.15}$$

$$G_{\mathcal{I}\mathcal{J}} = \left(r r_{;K}{}^{K} - \frac{r^2}{2} R\right) \tilde{g}_{\mathcal{I}\mathcal{J}} \, . \tag{B.1.16}$$

The energy-momentum tensor split into the Lorentz and spherical parts is

$$T = \hat{T}_{IJ}[y] dy^I dy^J + r^2 \tilde{T}[y] \tilde{g}_{\mathcal{K}\mathcal{L}} dz^{\mathcal{K}} dz^{\mathcal{L}} \, , \tag{B.1.17}$$

with $\tilde{T}[y]$ describing the scalar degree of freedom of the energy-momentum tensor on the sphere. The conservation of the energy-momentum tensor is

$$r^2 \hat{T}_{JI;}{}^{J} + r^2{}_{;}{}^{J} \hat{T}_{JI} - 2r\tilde{T} r_{;I} = 0 \, . \tag{B.1.18}$$

The Einstein equation components as functions of the Lorentz and spherical coordinates are

$$\hat{G}_{IJ} + \hat{g}_{IJ}\frac{f-1}{r^2} - \frac{2}{r}\left(r_{;JI} - \hat{g}_{IJ} r_{;K}{}^{K}\right) = \hat{T}_{IJ} \, , \tag{B.1.19}$$

$$\frac{r_{;I}{}^{I}}{r} - \frac{R}{2} = \tilde{T} \, . \tag{B.1.20}$$

B.1.1 Tensorial spherical harmonics

The idea in decoupling the scalar, vector, and tensor sectors is to use the spherical symmetry and expand all fields using basis functions that are eigenfunctions of the Laplace operator, making it possible to treat each expansion mode separately. On a sphere, the spherical harmonics satisfy the Legendre equation

$$\nabla_{\mathcal{I}} \nabla^{\mathcal{I}} Y^{\ell m} = -\ell(\ell+1) Y^{\ell m} \, , \tag{B.1.21}$$

where $Y^{\ell m}$ are the spherical harmonics defined with Legendre polynomials $P^{\ell m}$ as

$$Y^{\ell m}[\vartheta, \varphi] = P^{\ell m}[\vartheta] e^{i m \varphi} \, , \tag{B.1.22}$$

and ℓ and m are integers satisfying $\ell \geq 0$ and $-\ell \leq m \leq \ell$. The spherical harmonics are a complete set of orthonormal functions. They are a natural choice for the basis on which to expand the spherical parts of the fields. To expand not only scalars, but objects up to rank-2 tensors, vector, and tensor harmonics have to be introduced. There are two orthogonal directions on a sphere, thus two sets of harmonics are required. A vector can be expanded using the scalar spherical harmonics for the components t and r, that are normal to the sphere. Then the derivatives of the spherical harmonics provide the first independent set for one of the components along the sphere. This set transforms as vectors and is known as the even vector harmonics. The set of harmonics for the other component along the sphere can be obtained by taking the cross-product of the components normal to the sphere and the even vector harmonics. These transform as pseudovectors and are known as odd vector harmonics. Analogously, taking more derivatives, higher rank tensor harmonics can be obtained. The conventions used here are based on the definitions given by Martel [74].

The even vector harmonics are

$$Z^{lm}_{\mathcal{I}} = \nabla_{\mathcal{I}} Y^{lm} \; . \tag{B.1.23}$$

The odd vector harmonics are

$$X^{lm}_{\mathcal{I}} = \epsilon_{\mathcal{I}}{}^{\mathcal{J}} Z^{lm}_{\mathcal{I}} = \epsilon_{\mathcal{I}}{}^{\mathcal{J}} \nabla_{\mathcal{J}} Y^{lm} \; . \tag{B.1.24}$$

The even rank-2 tensor harmonics are

$$U^{lm}_{\mathcal{I}\mathcal{J}} = Y^{lm} \tilde{g}_{\mathcal{I}\mathcal{J}} \; , \tag{B.1.25}$$

$$V^{lm}_{\mathcal{I}\mathcal{J}} = \left(\nabla_{\mathcal{I}} \nabla_{\mathcal{J}} + \frac{l(l+1)}{2} \tilde{g}_{\mathcal{I}\mathcal{J}} \right) Y^{lm} \; . \tag{B.1.26}$$

The odd rank-2 tensor harmonics are

$$W^{lm}_{\mathcal{I}\mathcal{J}} = \nabla_{(\mathcal{I}} X^{lm}_{\mathcal{J})} \; . \tag{B.1.27}$$

The rank-2 tensor spherical harmonics are constructed to be traceless to impose their orthogonality. The normalization of all spherical harmonics is determined from that of Y^{lm}, which here is taken to be

$$\int d\Omega \, Y^{l'm'*} Y^{lm} = \delta^{ll'} \delta^{mm'} \; . \tag{B.1.28}$$

A scalar field with respect to the spherical part of the background metric is then expanded as

$$S = s_{lm} Y^{lm} \; . \tag{B.1.29}$$

A vector field is expanded as

$$V_{\mathcal{I}} = u_{lm} Z^{lm}_{\mathcal{I}} + v_{lm} X^{lm}_{\mathcal{I}} \; . \tag{B.1.30}$$

A rank-2 tensor field is expanded as

$$T_{\mathcal{I}\mathcal{J}} = K_{lm} U^{lm}_{\mathcal{I}\mathcal{J}} + G_{lm} V^{lm}_{\mathcal{I}\mathcal{J}} + H_{lm} W^{lm}_{\mathcal{I}\mathcal{J}} \; . \tag{B.1.31}$$

The phase space completeness conditions for the coefficients $s_{lm}, u_{lm}, v_{lm}, K_{lm}, G_{lm}, H_{lm}$ are

$$s_{lm} = \int d\Omega \, S Y^{*}_{lm} \; , \tag{B.1.32}$$

$$u_{lm} = \frac{1}{l(l+1)} \int d\Omega \, V_{\mathcal{I}} Z^{\mathcal{I}*}_{lm} \; , \tag{B.1.33}$$

$$v_{lm} = \frac{1}{l(l+1)} \int d\Omega \, V_{\mathcal{I}} X^{\mathcal{I}*}_{lm} \; , \tag{B.1.34}$$

$$K_{lm} = \frac{1}{2} \int d\Omega \, T_{\mathcal{I}\mathcal{J}} U^{\mathcal{I}\mathcal{J}*}_{lm} \; , \tag{B.1.35}$$

$$G_{lm} = \frac{(l-2)!}{(l+2)!} \int d\Omega \, T_{\mathcal{I}\mathcal{J}} V^{\mathcal{I}\mathcal{J}*}_{lm} \; , \tag{B.1.36}$$

$$H_{lm} = \frac{(l-2)!}{(l+2)!} \int d\Omega \, T_{\mathcal{I}\mathcal{J}} W^{\mathcal{I}\mathcal{J}*}_{lm} \; . \tag{B.1.37}$$

B.1.2 The master equation for scalar field perturbations

Here scalar field perturbations of the Schwarzschild black hole are described. Small perturbations of the background fields induce small perturbations of the energy-momentum tensor $T_{\mu\nu}[g, \phi]$. Scalar fields appear to second order in the expression for the energy-momentum tensor, therefore to linear order, small perturbations of a scalar field ϕ do not influence the energy-momentum tensor if the background field is zero, as it is in the Schwarzschild case. Thus in linear perturbation theory, the scalar field evolution can be considered separately from other perturbations. The scalar field equation of motion is

$$(\Box - V)\phi = 0 \; . \tag{B.1.38}$$

The potential V can be reduced to only the mass term in the potential. Any higher-order terms do not contribute when linear perturbations are considered:

$$V[\phi] = \frac{1}{2}\mu^2\phi^2 \; , \tag{B.1.39}$$

where μ is the mass. Thus the equation describing scalar field perturbations is the Klein-Gordon equation

$$(\Box - \mu^2)\phi = 0 \; , \tag{B.1.40}$$

which can be expanded as

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) - \mu^2\phi = 0 \; . \tag{B.1.41}$$

Employing the spherical symmetry of the background, the scalar field is expanded in the spherical harmonics:

$$\phi[t, r, \vartheta, \varphi] = \psi_{\ell m}[t, r]Y^{\ell m} \; . \tag{B.1.42}$$

Since the background is also static, the time dependence can be treated separately. The scalar field can be Fourier decomposed:

$$\phi[t, r, \vartheta, \varphi] = \int dw \; e^{-iwt}\frac{\psi_{\ell m}[r]}{r}Y^{\ell m} \; . \tag{B.1.43}$$

Inserting a particular mode of the time decomposition of the scalar field ϕ into the Klein-Gordon equation leads to the radial equation

$$f^2\psi + ff'\psi' + \left(\omega^2 - V_0\right)\psi = 0 \; , \tag{B.1.44}$$

where the potential is

$$V_0[\mu] = f\left(\mu^2 + \frac{\ell(\ell+1)}{r^2} + \frac{f'}{r}\right) \; . \tag{B.1.45}$$

and the labels ℓ, m are implied without explicitly writing them. The first derivative term disappears with the coordinate transformation to the radial tortoise coordinate r_*, defined through $\frac{dr}{dr_*} = f$, leading to the master equation for the scalar field

$$\partial^2_{r*}\psi_0 + (\omega^2 - V_0)\psi_0 = 0 \; , \tag{B.1.46}$$

where the convention $\psi = \psi_0$ is used to indicate that the variable refers to a scalar field with zero spin. Written with the d'Alembert operator this equation is

$$\nabla_I\nabla^I\psi_0 - V_0\psi_0 = 0 \; . \tag{B.1.47}$$

With the Schwarzschild value $f = 1 - \frac{2M}{r}$ the scalar field potential is

$$V_0 = \left(1 - \frac{2M}{r}\right)\left(\mu^2 + \frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3}\right) \; . \tag{B.1.48}$$

B.1.3 The master equations for perturbations of vector fields

Here vector field perturbations of the Schwarzschild black hole are described. Vector fields are commonly introduced as gauge fields mediating the interactions of other charged fields. A consistent treatment of interactions throughout spacetime is possible only for long-range interactions. Long-range interactions are mediated by massless vector fields that obey the Maxwell equation. Similarly to scalar fields, vector fields also appear to second order in the expression for the energy-momentum tensor 3.3.14. If the background field is zero, as is the case for the Schwarzschild black hole, to linear order, small perturbations of the vector field do not influence the energy-momentum tensor. The Maxwell equation for the field strength tensor F of the vector field A is

$$\nabla_\mu F^{\mu\nu} = 0 \,. \tag{B.1.49}$$

The vector field can be expanded in vector spherical harmonics:

$$A = A_{\ell m_I} Y^{\ell m} dy^I + \left(a_{\ell m} Z^{\ell m}_{\mathcal{I}} + d_{\ell m} X^{\ell m}_{\mathcal{I}}\right) dz^{\mathcal{I}} \,. \tag{B.1.50}$$

Dropping the labels ℓ, m and with the definition for A_I:

$$A_I = \begin{pmatrix} b \\ c \end{pmatrix} \,, \tag{B.1.51}$$

the vector field can be written in manifestly decoupled odd A^-_μ and even A^+_μ parts:

$$A_\mu = A^-_\mu + A^+_\mu = \begin{pmatrix} 0 \\ 0 \\ \dfrac{a[t,r]}{\sin\theta} Y_{,\varphi} \\ -a[t,r]\sin\theta Y_{,\theta} \end{pmatrix} + \begin{pmatrix} b[t,r]Y \\ c[t,r]Y \\ d[t,r]Y_{,\theta} \\ d[t,r]Y_{,\varphi} \end{pmatrix} \,, \tag{B.1.52}$$

where a, b, c, and d are coordinate t and r-dependent coefficients. The vector field is split into an odd and an even parity part, with the names referring to their transformation properties with respect to how the spherical harmonics transform. Since their transformation properties differ, in the absence of sources, the two parts do not mix. Their evolution is independent and they can be treated separately. With this decomposition of the vector field, the Maxwell equation can be recast as a set of four equations by taking its t-component, r-component, the sum of θ and ϕ-components, and the difference of θ and ϕ-components respectively:

$$\ell(\ell+1)(b - \dot{d}) - rf(2b' + rb'' - 2\dot{c} - r\dot{c}') = 0 \,, \tag{B.1.53}$$

$$\ell(\ell+1)(c - d') + \frac{r^2}{b}(-\dot{b}' + \ddot{c}) = 0 \,, \tag{B.1.54}$$

$$2f^2(c' - d'') + ff'(c - d') - \dot{b} + d'' \,, \tag{B.1.55}$$

$$f\partial_r f(h - \partial_r k) + f^2(\partial_r h - \partial_r^2 k) - \partial_t f + \partial_t^2 k = 0 \,, \tag{B.1.56}$$

$$\ell(\ell+1)\frac{f}{r^2} a - ff'a' - f^2 a'' + \ddot{a} = 0 \,. \tag{B.1.57}$$

The last equation is an equation of the function a only, this function defines the odd sector of the vector field. Using the tortoise coordinate r_*, the first-derivative term in the last equation disappears leading to the equation

$$\partial^2_{r*} a - \ddot{a} - \frac{f}{r^2}\ell(\ell+1)a = 0 \,. \tag{B.1.58}$$

With the conventions $\psi_{1-} = a$ for the variable, and $V_{1-} = \frac{f}{r^2}\ell(\ell+1)$ for the potential, the master equation governing the spin-one vector field odd perturbations is

$$\partial^2_{r*}\psi_{1-} + (\omega^2 - V_{1-})\psi_{1-} = 0 \,, \tag{B.1.59}$$

where time Fourier decomposition has been done. Alternatively, this equation can be written with the d'Alembert operator:

$$\nabla_I \nabla^I \psi_{1-} - V_{1-}\psi_{1-} = 0 \,. \tag{B.1.60}$$

An equation for the even sector can be obtained by considering equations B.1.53 and B.1.54. The scalars b, c, d are related. The time derivative of the second can be subtracted from the radial derivative of the first eliminating d and giving the equation

$$(\ell(\ell+1)-2f)\left(b'-\dot{c}\right)-2rf'\left(b'-\dot{c}\right)+4rf\left(\dot{c}'-b''\right)+r^2f'\left(\dot{c}'-b''\right)+r^2f\left(\dot{c}''-b^{(0,3)}\right)+\frac{r^2}{f}\left(\ddot{b}'-c^{(3,0)}\right)=0\,. \tag{B.1.61}$$

Defining a variable for the even sector of the vector perturbations as

$$\psi_{1+}=-r^2\frac{b'-\dot{c}}{\ell(\ell+1)} \tag{B.1.62}$$

the above equation becomes

$$f^2\psi''_{1+}-\ddot{\psi}_{1+}+ff'\psi'_{1+}-\frac{f}{r^2}\ell(\ell+1)\psi_{1+}=0\,, \tag{B.1.63}$$

which has a simpler form using the tortoise radial coordinate $r*$:

$$\partial^2_{r*}\psi_{1+}-\ddot{\psi}_{1+}-V_{1+}\psi_{1+}=0\,, \tag{B.1.64}$$

where the potential is $V_{1+}=\frac{f}{r^2}\ell(\ell+1)$. This can be Fourier decomposed to

$$\partial^2_{r*}\psi_{1+}+(\omega^2-V_{1+})\psi_{1+}=0\,. \tag{B.1.65}$$

With the d'Alembert operator, this equation is

$$\nabla_I\nabla^I\psi_{1+}-V_{1+}\psi_{1+}=0\,. \tag{B.1.66}$$

The above is the master equation for the even vector field perturbations.

B.1.4 Regge-Wheeler and Zerilli equations

The topic of this section is small spacetime deviations from the Schwarzschild black hole geometry. As described in section 3.1, the linearized Einstein equation 3.1.1 describes the metric perturbations h. The Schwarzschild metric is a vacuum solution, therefore $T_{\mu\nu}=0$. The perturbation of the energy-momentum tensor is not necessarily zero $\delta T_{\mu\nu}\neq 0$, but as was described in the previous sections, when scalar and vector fields that vanish in the background geometry are the sources since they appear to second order in the expression for the energy-momentum tensor, their perturbations do not contribute to linear order to the energy-momentum tensor. This is one instance when the linearized Einstein equation reduces to

$$\delta G_{\mu\nu}=0\,. \tag{B.1.67}$$

Writing this explicitly in terms of the metric perturbation using equation 3.1.7 gives:

$$\delta G_{\mu\nu}=\frac{1}{2}\left(g_{\mu\nu}h_{\kappa}{}^{\kappa}{}_{;\lambda}{}^{\lambda}-g_{\mu\nu}h_{\kappa\lambda;}{}^{\kappa\lambda}-h_{\lambda}{}^{\lambda}{}_{;\mu\nu}+h_{\lambda\mu;\nu}{}^{\lambda}+h_{\lambda\nu;\mu}{}^{\lambda}-h_{\mu\nu;\lambda}{}^{\lambda}\right)=0\,. \tag{B.1.68}$$

It can be seen that the linearized Einstein equation contains terms that all involve taking two covariant derivatives of the metric perturbation. As before, spherical symmetry can be used to reduce these second-order differential terms on the sphere by expanding quantities with tensor spherical harmonics. The metric perturbation expansion in tensor spherical harmonics is

$$h=h_{\ell m IJ}Y^{\ell m}dy^Idy^J+2\left(h_{\ell m I}Z^{\ell m}_{\mathcal{J}}+j_{\ell m I}X^{\ell m}_{\mathcal{J}}\right)dy^Idz^{\mathcal{J}}+\left(K_{\ell m}U^{\ell m}_{\mathcal{IJ}}+G_{\ell m}V^{\ell m}_{\mathcal{IJ}}+H_{\ell m}W^{\ell m}_{\mathcal{IJ}}\right)dz^{\mathcal{I}}dz^{\mathcal{J}}\,. \tag{B.1.69}$$

From now on the labels ℓ, m are assumed, but not explicitly written. As described in section 3.1, there is some redundancy in describing the perturbations due to gauge freedom. The gauge freedom is associated with coordinate transformations. There are four coordinates, so four independent coordinate conditions would fix the gauge. A convenient choice is to use the Regge-Wheeler gauge [71]:

$$h_I = 0\,, \quad G = 0\,, \quad H = 0\,. \tag{B.1.70}$$

In this gauge, the expression for the perturbation is

$$h = h_{IJ} Y dy^I dy^J + 2 j_I X_{\mathcal{J}} dy^I dz^{\mathcal{J}} + K U_{\mathcal{IJ}} dz^{\mathcal{I}} dz^{\mathcal{J}}\,. \tag{B.1.71}$$

With the expansions in terms of scalars K and h_0, h_1, H_0, H_1, H_2, where the latter are defined as the components of j_I and h_{IJ}:

$$j_I = \begin{pmatrix} h_0 \\ h_1 \end{pmatrix}\,, \qquad h_{IJ} = \begin{pmatrix} H_0 & H_1 \\ H_1 & H_2 \end{pmatrix}\,, \tag{B.1.72}$$

the metric perturbation can be explicitly written in terms of manifestly decoupled odd $h^-_{\mu\nu}$ and even $h^+_{\mu\nu}$ parts:

$$h_{\mu\nu} = h^-_{\mu\nu} + h^+_{\mu\nu} = \begin{pmatrix} 0 & 0 & \frac{h_0}{\sin\theta} Y_{,\phi} & -h_0 \sin\theta Y_{,\theta} \\ 0 & 0 & \frac{h_1}{\sin\theta} Y_{,\phi} & -h_1 \sin\theta Y_{,\theta} \\ \frac{h_0}{\sin\theta} Y_{,\phi} & \frac{h_1}{\sin\theta} Y_{,\phi} & 0 & 0 \\ -h_0 \sin\theta Y_{,\theta} & -h_1 \sin\theta Y_{,\theta} & 0 & 0 \end{pmatrix} + \begin{pmatrix} -f H_0 Y & -H_1 Y & 0 & 0 \\ -H_1 Y & -\frac{1}{f} H_2 Y & 0 & 0 \\ 0 & 0 & -r^2 K Y & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta K Y \end{pmatrix} \tag{B.1.73}$$

These can be considered separately. The odd metric perturbation involves scalars h_0 and h_1. The time dependence separates with time Fourier transform as the background spacetime is static. The nonzero components of the Einstein perturbation in terms of the scalars h_0 and h_1 are

$$\delta G_{tt} = -\frac{f(rf'+f-1)}{r^2}\,, \tag{B.1.74}$$

$$\delta G_{t\varphi} = \frac{\left(r^2 f''+2rf'+2f+\ell^2+\ell-2\right)h_0 - irf\left(-irh_0''+r\omega h_1'+2\omega h_1\right)}{2r^2} \sin^2\vartheta e^{-i\omega t} Y'\,, \tag{B.1.75}$$

$$\delta G_{rr} = \frac{rf'+f-1}{r^2 f}\,, \tag{B.1.76}$$

$$\delta G_{r\varphi} = \frac{\left(r^2 ff''+2rff'+f(\ell^2+\ell-2)-r^2\omega^2\right)h_1 + ir^2\omega h_0' - 2ir\omega h_0}{2r^2 f} \sin^2\vartheta e^{-i\omega t} Y'\,, \tag{B.1.77}$$

$$\delta G_{\vartheta\vartheta} = \frac{r(rf''+2f')}{2\sin^2\vartheta}\,, \tag{B.1.78}$$

$$\delta G_{\vartheta\varphi} = -\frac{f\left(h_1 f' + f h_1'\right) + i\omega h_0}{2f} e^{-i\omega t}\left(\ell(\ell+1)Y - 2\cos\vartheta Y'\right)\,, \tag{B.1.79}$$

$$\delta G_{\varphi\varphi} = \frac{r sin^2\vartheta}{2}\left(rf'' + 2f'\right)\,. \tag{B.1.80}$$

In particular the components $\delta G_{r\varphi}, \delta G_{\vartheta\varphi}$ are relevant. By considering the $\vartheta\varphi$-component of the Einstein equation, h_0 can be expressed as a function of h_1:

$$f\left(h_1 f' + f h_1'\right) + i\omega h_0 = 0 \quad \Rightarrow \quad h_0 = \frac{ff'h_1 + f^2 h_1'}{-i\omega}\,. \tag{B.1.81}$$

Using this in the $r\varphi$-component of the Einstein equation leads to the second-order differential equation

$$r^2 f^2 h_1'' - rf\left(2f - 3rf'\right) h_1' + \left(r^2\omega^2 + 2f^2 + r^2 ff'' + r^2 f'^2 - f\ell(\ell+1)\right) h_1 = 0\,. \tag{B.1.82}$$

Defining a variable for the odd metric perturbations as

$$\psi_{2-} = \frac{f h_1}{r}\,, \tag{B.1.83}$$

the above equation becomes

$$f^2 \psi_{2-}'' + ff'\psi_{2-}' + \left(\omega^2 + \frac{3rff' - f\ell(\ell+1)}{r^2}\right)\psi_{2-} = 0\,. \tag{B.1.84}$$

Using the tortoise coordinate $r*$ the master equation for the even metric perturbations is obtained:

$$\partial_{r*}^2\psi_{2-} + \left(\omega^2 - V_{2-}\right)\psi_{2-} = 0 \,, \tag{B.1.85}$$

where the potential is $V_{2-} = f\frac{\ell(\ell+1)-3rf'}{r^2}$. In terms of the d'Alembert operator, the equation is

$$\nabla_I\nabla^I\psi_{2-} - V_{2-}\psi_{2-} = 0 \,. \tag{B.1.86}$$

This is the Regge-Wheeler equation [71]. The potential for the Schwarzschild background spacetime is

$$V_{2-} = \left(1 - \frac{2M}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3}\right) \,. \tag{B.1.87}$$

The procedure for finding a master equation for the even metric perturbations is analogous, albeit more mathematically involved. The nonzero Einstein perturbation components in terms of the even metric perturbation scalars H_0, H_1, H_2, K are

$$\delta G_{tt} = \left(\left(1-f-rf'\right)H_0 - \left(f + rf' + \frac{\ell^2+\ell}{2}\right)H_2 - rfH_2' - \frac{\ell^2+\ell-2}{2}K + \frac{r(f'+6f)}{2}K' + fr^2K'' + \frac{1-f-rf'}{2Y}\right)\frac{fY}{r^2} \,, \tag{B.1.88}$$

$$\delta G_{tr} = \left(-\left(rf' + f + \tfrac{\ell^2+\ell-2}{2}\right)H_1 - r\dot{H}_2 + \left(r - \tfrac{r^2f'}{2f}\right)\dot{K} + r^2\dot{K}'\right)\tfrac{Y}{r^2} \,, \tag{B.1.89}$$

$$\delta G_{t\vartheta} = \left(-f'H_1 - fH_1' + \dot{H}_2 + \dot{K}\right)\tfrac{Y'}{2} \,, \tag{B.1.90}$$

$$\delta G_{rr} = \left(-\tfrac{\ell^2+\ell}{2r}H_0 + fH_0' - 2\dot{H}_1 + \tfrac{1}{r}H_2 + \tfrac{\ell^2+\ell-2}{2r}K - \left(1 + \tfrac{rf'}{2}\right)K' + \tfrac{r}{f}\ddot{K} - \tfrac{1-f-rf'}{rY}\right)\tfrac{Y}{fr} \,, \tag{B.1.91}$$

$$\delta G_{r\vartheta} = \left(\tfrac{2f-rf'}{2rf}H_0 - H_0' + \tfrac{1}{f}\dot{H}_1 - \tfrac{2f+rf'}{2rf}H_2 + K'\right)\tfrac{1}{2Y'} \,, \tag{B.1.92}$$

$$\begin{aligned}\delta G_{\vartheta\vartheta} = \Bigg(&-\tfrac{\cos\vartheta Y'}{Yr}H_0 + \left(f - \tfrac{3rf'}{2}\right)H_0' + rfH_0'' - \left(\tfrac{rf'}{f} + 2\right)\dot{H}_1 - 2r\dot{H}_1' - \left(+2f' - rf'' - \tfrac{\cos\vartheta Y'}{Yr}\right)H_2 \\ &+ \tfrac{r}{f}\ddot{H}_2 + \tfrac{f+rf'}{2}H_2' - \left(2f' + rf''\right)K + \tfrac{r}{f}\ddot{K} - \left(2f + rf'\right)K' - rfK'' + \tfrac{2f'}{Y} + \tfrac{r^2f''}{Y}\Bigg)\tfrac{Yr}{2\sin^2\vartheta} \,,\end{aligned} \tag{B.1.93}$$

$$\begin{aligned}\delta G_{\varphi\varphi} = \Bigg(&\left(\frac{cos\vartheta Y'}{rY} - \frac{\ell^2+\ell}{r}\right)H_0 + \left(f + \frac{3r}{2}\right)H_0' + rfH_0'' - \left(2 + \frac{rf'}{f}\right)\dot{H}_1 - 2r\dot{H}_1' + \left(rf'' + 2f' + \frac{\ell^2+\ell}{r} - \frac{cos\vartheta Y'}{rY}\right)H_2 \\ &+ \frac{r}{f}\ddot{H}_2 + \left(f + \frac{rf'}{2}\right)H_2' - \left(2f' + rf''\right)K + \frac{r}{f}\ddot{K} - \left(2f + rf'\right)K' - rfK'' + \frac{2f' - r}{Y}\Bigg)\frac{\sin^2\vartheta rY}{2} \,.\end{aligned} \tag{B.1.94}$$

Assuming the perturbations are sourced by scalar and vector fields, the energy-momentum perturbation is zero. Since the background is time-independent, the time differentials can be simplified using the Fourier transform. The $\varphi\varphi$-component can be subtracted from $\vartheta\vartheta$-component to obtain

$$H_0 = H_2 \,. \tag{B.1.95}$$

Then the tr, $t\vartheta$, and $r\vartheta$-components of the Einstein equation can be chosen to provide a set of equations for the three respective derivatives of perturbation scalars:

$$K' + f_1K + f_2H_1 + f_3H_2 = 0 \,, \tag{B.1.96}$$
$$H_1' + f_4K + f_5H_1 + f_4H_2 = 0 \,, \tag{B.1.97}$$
$$H_2' + f_1K + f_6H_1 + f_7H_2 = 0 \,. \tag{B.1.98}$$

The identity B.1.95 is used to remove the perturbation scalar H_0 from the equations. A further relation between the perturbation scalars K, H_1, H_2 can be found with the rr-component:

$$f_8K + f_9H_1 + f_{10}H_2 = f_{11} \,. \tag{B.1.99}$$

With this, H_2 can also be removed from the equations, leading to two equations for the derivatives K' and H_1':

$$K' = -\left(f_1 - \frac{f_3 f_8}{f_{10}}\right) K - \left(f_2 - \frac{f_3 f_9}{f_{10}}\right) H_1 - \frac{f_3 f_{11}}{f_{10}} , \tag{B.1.100}$$

$$H_1' = -\left(f_4 - \frac{f_4 f_8}{f_{10}}\right) K - \left(f_5 - \frac{f_4 f_9}{f_{10}}\right) H_1 - \frac{f_4 f_{11}}{f_{10}} . \tag{B.1.101}$$

In the above, $f_1 - f_{11}$ are r-dependent functions that can be identified from the Einstein equation components. The perturbation scalars K and H_1 can be combined into the variable ψ:

$$\psi = \zeta_1 K + \zeta_2 H_1 . \tag{B.1.102}$$

The derivative ψ' can be expressed in terms of K and H_1 using equations B.1.100 and B.1.101:

$$\psi' = \eta_1 K + \eta_2 H_1 + \sigma , \tag{B.1.103}$$

where

$$\eta_1 = \zeta_1' - \zeta_1 \left(f_1 - \frac{f_3 f_8}{f_{10}}\right) - \zeta_2 \left(f_4 - \frac{f_4 f_8}{f_{10}}\right) , \tag{B.1.104}$$

$$\eta_2 = \zeta_2' - \zeta_2 \left(f_5 - \frac{f_4 f_9}{f_{10}}\right) - \zeta_1 \left(f_2 - \frac{f_3 f_9}{f_{10}}\right) , \tag{B.1.105}$$

$$\sigma = -\frac{\zeta_1 f_3 + \zeta_2 f_4}{f_{10}} f_{11} . \tag{B.1.106}$$

The same applies also to the second derivative:

$$\psi'' = \iota_1 K + \iota_2 H_1 + \Sigma , \tag{B.1.107}$$

where

$$\iota_1 = \eta_1' - \eta_1 \left(f_1 - \frac{f_3 f_8}{f_{10}}\right) - \eta_2 \left(f_4 - \frac{f_4 f_8}{f_{10}}\right) , \tag{B.1.108}$$

$$\iota_2 = \eta_2' - \eta_2 \left(f_5 - \frac{f_4 f_9}{f_{10}}\right) - \eta_1 \left(f_2 - \frac{f_3 f_9}{f_{10}}\right) , \tag{B.1.109}$$

$$\Sigma = \sigma' + \sigma . \tag{B.1.110}$$

Using the equations for ψ and ψ', the perturbation scalars can also be expressed in terms of ψ and ψ' as

$$K = \frac{\zeta_2 Z' - \eta_2 Z + \zeta_2 \sigma}{\zeta_2 \eta_1 - \zeta_1 \eta_2} , \qquad H_1 = \frac{\zeta_1 Z' - \eta_1 Z + \zeta_1 \sigma}{\zeta_1 \eta_2 - \zeta_2 \eta_1} . \tag{B.1.111}$$

Substituting this back into the expression for the second derivative ψ'' gives an expression purely in terms of ψ, ψ', and ψ''. The second derivative of ψ with respect to the tortoise coordinate $r*$ is related to the second derivative with respect to the radial coordinate r through

$$\partial_{r*}^2 \psi = f \partial_r [f \partial_r \psi] = f^2 \psi'' + f f' \psi' . \tag{B.1.112}$$

In the previously found master equations, the transformation to the radial tortoise coordinate leads to the disappearance of the first-order derivative term. An analogous master equation in terms of the radial tortoise coordinate can also be formulated by a choice of ζ_1 and ζ_2. This is known as the Zerilli equation [72]:

$$\partial_{r*}^2 \psi_{2+} + (\omega^2 - V_{2+}) \psi_{2+} = 0 , \tag{B.1.113}$$

here specifically identifying that the Zerilli variable as the variable for the spin-two even metric perturbations: $\psi = \psi_{2+}$. The Zerilli variable relation to the metric perturbation scalars is $\psi_{2+} = \zeta_1 K + \zeta_2 H_1$, and in the Schwarzschild case, the functions ζ_1 and ζ_2 are

$$\zeta_1 = \frac{r^2}{nr + 3M} , \qquad \zeta_2 = \frac{r - 2M}{i\omega(nr + 3M)} . \tag{B.1.114}$$

The corresponding potential is [3]

$$V_{2+} = \left(1 - \frac{2M}{r}\right) \frac{n^3 r^3 + n^2 r^3 + 3n^2 M r^2 + 9nM^2 r + 9M^3}{(nr + 3M)^2 r^4} , \tag{B.1.115}$$

where, following Zerilli's notation, $n = \frac{(\ell-1)(\ell+2)}{2}$.

B.2 Axially symmetric black hole perturbation master equations

B.2.1 Teukolsky equation

The master equation for the perturbations on a Kerr background was derived by Teukolsky [73] using the Newman-Penrose formalism. Here a summary of key considerations described by Teukolsky in his article is given.

The idea is to use the property of the Kerr spacetime, that, being Petrov type D, it has two principal null directions. By the Goldberg-Sachs theorem if l and n are chosen to align with these two directions, the background Weyl scalars $\Psi_0, \Psi_1, \Psi_3, \Psi_4$ are zero, and since the principal null directions are shear-free and geodesic, in the background spacetime the spin coefficients $\kappa, \sigma, \nu, \lambda$ are also zero. Instead of the Weyl scalar Ψ_4, here Ψ_0 is considered, for which the derivation of the Teukolsky equation is more straightforward. Among the Newman-Penrose spin coefficient equations, only one involves Ψ_0 :

$$D\sigma - \delta\kappa = (\rho + \bar{\rho})\sigma + (3\epsilon - \bar{\epsilon})\sigma - (\tau - \bar{\pi} + \bar{\alpha} + 3\beta)\kappa + \Psi_0 \; . \tag{B.2.116}$$

The Newman-Penrose Bianchi identity nonvacuum equations, that involve the derivatives of the Weyl scalar Ψ_0 are

$$\bar{\delta}\Psi_0 - D\Psi_1 - (4\alpha - \pi)\Psi_0 + 2(2\rho + \epsilon)\Psi_1 - 3\kappa\Psi_2 = \delta\Phi_{00} + D\Phi_{01} + 2(\epsilon + \bar{\rho})\Phi_{01} + 2\sigma\Phi_{10} - 2\kappa\Phi_{11} - \bar{\kappa}\Phi_{02} + (\bar{\pi} - 2\bar{\alpha} - 2\bar{\beta})\Phi_{00} \; , \tag{B.2.117}$$

$$\Delta\Psi_0 - \delta\Psi_1 - (4\gamma - \mu)\Psi_0 + 2(2\tau + \beta)\Psi_1 - 3\sigma\Psi_2 = \delta\Phi_{01} - D\Phi_{02} + 2(\bar{\pi} - \beta)\Phi_{01} - 2\kappa\Phi_{12} - \bar{\lambda}\Phi_{00} + 2\sigma\Phi_{11} + (\bar{\rho} + 2\epsilon - 2\bar{\epsilon})\Phi_{02} \; , \tag{B.2.118}$$

where the Ricci tensor is equated with the energy-momentum tensor. As the Kerr spacetime is a vacuum solution, the background energy-momentum tensor is zero. The above equations can be turned into linear perturbation equations, dropping all terms that are second order in smallness, then they are

$$(D - \rho - \bar{\rho} - 3\epsilon + \bar{\epsilon})^A \sigma^B - (\delta - \tau + \bar{\pi} - \bar{\alpha} - 3\beta)^A \kappa^B - {\Psi_0}^B = 0 \; , \tag{B.2.119}$$

$$(\bar{\delta} - 4\alpha + \pi)^A {\Psi_0}^B - (D - 4\rho - 2\epsilon)^A {\Psi_1}^B - 3\kappa^B {\Psi_2}^A = (\delta + \bar{\pi} - 2\bar{\alpha} - 2\beta)^A {\Phi_{00}}^B - (D - 2\epsilon - 2\bar{\rho})^A {\Phi_{01}}^B \; , \tag{B.2.120}$$

$$(\Delta - 4\gamma + \mu)^A {\Psi_0}^B - (\delta - 4\tau - 2\beta)^A {\Psi_1}^B - 3\sigma^B {\Psi_2}^A = (\delta + 2\bar{\pi} - 2\beta)^A {\Phi_{01}}^B - (D - 2\epsilon + 2\bar{\epsilon} - \bar{\rho})^A {\Phi_{02}}^B \; . \tag{B.2.121}$$

The Newman-Penrose Bianchi identities [45]

$$\bar{\delta}\Psi_1 - D\Psi_2 - \lambda\Psi_0 + 2(\alpha - \pi)\Psi_1 + 3\rho\Psi_2 - 2\kappa\Psi_3 = \bar{\delta}\Phi_{01} - \Delta\Phi_{00} - 2(\alpha + \bar{\tau})\Phi_{01} + 2\rho\Phi_{11} + \bar{\sigma}\Phi_{02} - (\bar{\mu} - 2\gamma - 2\bar{\gamma})\Phi_{00} - 2\tau\Phi_{10} - 2D\Lambda \; , \tag{B.2.122}$$

$$\Delta\Psi_2 - \delta\Psi_3 - 2\nu\Psi_1 + 3\mu\Psi_2 - 2(\beta - \tau)\Psi_3 - \sigma\Psi_4 = \delta\Phi_{21} - D\Phi_{22} + 2(\bar{\pi} + \beta)\Phi_{21} - 2\mu\Phi_{11} - \bar{\lambda}\Phi_{20} + 2\pi\Phi_{12} + (\bar{\rho} - 2\epsilon - 2\bar{\epsilon})\Phi_{22} - 2\Delta\Lambda \; , \tag{B.2.123}$$

for the background metric, reduce to

$$D\Psi_2 = 3\rho\Psi_2 \; , \tag{B.2.124}$$

$$\delta\Psi_2 = 3\tau\Psi_2 \; . \tag{B.2.125}$$

Multiplying the Ψ_0 spin coefficient equation with Ψ_2 from the left, commuting Ψ_2 with the directional derivatives D and δ, and then using equations B.2.124 and B.2.125 to replace the directional derivatives of Ψ_2 leads to the expression

$$(D - 4\rho - \bar{\rho} - 3\epsilon + \bar{\epsilon})\Psi_2\sigma^B - (\delta - 4\tau + \bar{\pi} - \bar{\alpha} - 3\beta)\Psi_2\kappa^B - \Psi_2{\Psi_0}^B = 0 \; . \tag{B.2.126}$$

Multiplying the Bianchi equation B.2.120 with $(\delta - 4\tau + \bar{\pi} - \bar{\alpha} - 3\beta)$ and the Bianchi equation B.2.121 with $(D - 4\rho - \bar{\rho} - 3\epsilon + \bar{\epsilon})$ and then subtracting gives

$$\left((D - 4\rho - \bar{\rho} - 3\epsilon + \bar{\epsilon})(\delta - 2\beta - 4\tau) - (\delta - 4\tau + \bar{\pi} - \bar{\alpha} + 3\beta)(D - 2\epsilon - 4\rho)\right)\Psi_1^B \; . \tag{B.2.127}$$

The metric compatibility equations [45] have the form of commutation relations and can be used in finding

$$(D + q\rho - \bar{\rho} - (p+1)\epsilon + \bar{\epsilon})(\delta - p\beta + q\tau) - (\delta + q\tau + \bar{\pi} - \bar{\alpha} + (p+1)\beta)(D - p\epsilon + q\rho) = 0 \,. \tag{B.2.128}$$

Setting $p = 2, q = -4$ the two equations above can be identified with one another and the commutation relation can be used to remove the Ψ_1 term leading to

$$\big((D - 4\rho - \bar{\rho} - 3\epsilon + \bar{\epsilon})(\Delta - 4\gamma + \mu) - (\delta - 4\tau + \bar{\pi} - \bar{\alpha} - 3\beta)(\bar{\delta} + \pi - 4\alpha) - 3\Psi_2\big)\,{\Psi_0}^B = T_0 \,. \tag{B.2.129}$$

This is a decoupled equation for the perturbation ${\Psi_0}^B$. With the Einstein equation, the Ricci scalars are identified with the corresponding energy-momentum tensor tetrad projections, which describe the properties of sources. These are put together in the source term T_0:

$$\begin{aligned} T_0 =& \frac{1}{2}(D - 4\rho - \bar{\rho} - 3\epsilon + \bar{\epsilon})\Big((\delta + 2\bar{\pi} - 2\beta){T_{01}}^B - (D - 2\epsilon + 2\bar{\epsilon} - \bar{\rho}){T_{02}}^B\Big) \\ &+ \frac{1}{2}(\delta - 4\tau + \bar{\pi} - \bar{\alpha} - 3\beta)\Big((D - 2\epsilon - 2\bar{\rho}){T_{00}}^B - (\delta + \bar{\pi} - 2\bar{\alpha} - 2\beta){T_{01}}^B\Big) \,. \end{aligned} \tag{B.2.130}$$

Making the coordinate choice

$$l^\mu = \left(\frac{r^2 + a^2}{\Delta}, 1, 0, \frac{a}{\Delta}\right) \,, \qquad n^\mu = \left(r^2 + a^2, -\Delta, 0, a\right)\frac{1}{2\Sigma} \,, \qquad m^\mu = \left(ia \sin\vartheta, 0, 1, \frac{1}{i \sin\vartheta}\right)\frac{\sqrt{2}}{2(r + ia\cos\vartheta)} \,, \tag{B.2.131}$$

the nonzero spin coefficients are

$$\rho = \frac{-1}{r - ia\cos\vartheta} \,, \quad \beta = \frac{-\sqrt{2}\cot\vartheta\bar{\rho}}{4} \,, \quad \pi = \frac{\sqrt{2}ia\sin^2\vartheta\rho^2}{2} \,, \quad \tau = \frac{-\sqrt{2}ia\sin\vartheta\rho\bar{\rho}}{2} \,, \quad \mu = \frac{\rho^2\bar{\rho}}{2\Delta} \,, \quad \gamma = \mu + \frac{(r-M)\rho\bar{\rho}}{2} \,, \quad \alpha = \pi - \bar{\beta} \,. \tag{B.2.132}$$

The only nonzero background Weyl scalar is Ψ_2, which obeys

$$\Psi_2 = M\rho^3. \tag{B.2.133}$$

With this, the dynamics of the perturbation $\psi = {\Psi_0}^B$ can be described with a single equation:

$$\begin{aligned} &\left(\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\vartheta\right)\partial_t^2\psi + \frac{4Mar}{\Delta}\partial_t\partial_\varphi\psi + \left(\frac{a^2}{\Delta} - \frac{1}{\sin^2\vartheta}\right)\partial_\varphi^2\psi - \Delta^{-s}\partial_r\left[\Delta^{s+1}\partial_r\psi\right] - \frac{1}{\sin\vartheta}\partial_\theta\left[\sin\vartheta\partial_\vartheta\psi\right] \\ &\quad - 2s\left(\frac{a(r-M)}{\Delta} + \frac{i\cos\vartheta}{\sin^2\vartheta}\right)\partial_\varphi\psi - 2s\left(\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\vartheta\right)\partial_t\psi + (s^2\cot^2\vartheta - s)\psi = \Sigma T \,. \end{aligned} \tag{B.2.134}$$

This is the Teukolsky equation. The variable ψ can be identified with scalar field, vector field, metric, and spin-half field perturbation variables described by Teukolsky [73]. Using the symmetry of the background, the variable can be written in a Fourier decomposed way:

$$\psi = e^{-i\omega t}R[r]\Theta[\vartheta]e^{im\varphi} \,. \tag{B.2.135}$$

Considering the vacuum case $T = 0$, the Teukolsky equation can be decoupled into

$$\Delta^{-s}\partial_r\left[\Delta^{s+1}\partial_r R\right] + \left(\frac{K^2 - 2is(r-M)K}{\Delta} + 4is\omega r - a^2\omega^2 + 2am\omega - \lambda\right)R = 0 \,, \tag{B.2.136}$$

$$\frac{1}{\sin\vartheta}\partial_\vartheta\left[\sin\vartheta\partial_\vartheta\Theta\right] + \left(a^2\omega^2\cos^2\vartheta - \frac{m^2}{\sin^2\vartheta} - 2a\omega s\cos\vartheta - \frac{2ms\cos\vartheta}{\sin^2\vartheta} - s^2\cot^2\vartheta + s + \lambda\right)\Theta = 0 \,, \tag{B.2.137}$$

where $K = (r^2 + a^2)\omega - am$.

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