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Relativistic theory of tidal Love numbers

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In Newtonian gravitational theory, a tidal Love number relates the mass multipole moment created by tidal forces on a spherical body to the applied tidal field. The Love number is dimensionless, and it encodes information about the body's internal structure. We present a relativistic theory of Love numbers, which applies to compact bodies with strong internal gravities; the theory extends and completes a recent work by Flanagan and Hinderer, which revealed that the tidal Love number of a neutron star can be measured by Earth-based gravitational-wave detectors. We consider a spherical body deformed by an external tidal field, and provide precise and meaningful definitions for electric-type and magnetic-type Love numbers; and these are computed for polytropic equations of state. The theory applies to black holes as well, and we find that the relativistic Love numbers of a nonrotating black hole are all zero.

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I. INTRODUCTION AND SUMMARY

A. Context of this work

The exciting prospect of using gravitational-wave detectors to measure the tidal coupling of two neutron stars during the inspiral phase of their orbital evolution was recently articulated by Flanagan and Hinderer [1,2]. The idea is as follows. The orbital motion of a binary system of neutron stars produces the emission of gravitational waves, which remove energy and angular momentum from the system. This causes the orbits to decrease in radius and increase in frequency, and leads to the inspiraling motion of the compact bodies. Late in the inspiral the gravitational waves enter the frequency band of the detector, and detailed features of the orbital motion are revealed in the shape and phasing of the wave. At the large orbital separations that correspond to the low-frequency threshold of the instrument, the tidal interaction between the bodies is negligible, and the bodies behave as point masses. As the frequency increases, however, the orbital separation decreases sufficiently that the influence of the tidal interaction becomes important. The bodies acquire a tidal deformation, and this affects their gravitational field and orbital motion; the effect is revealed in the shape and phasing of the gravitational waves.

Flanagan and Hinderer have provided a quantitative analysis of this story, and they have shown that the tidal coupling between neutron stars is accessible to measurement by the current generation of Earth-based gravitational-wave detectors (such as Enhanced LIGO). This prospect is exciting, because the details of the tidal interaction depend on the internal structure of each body, and the measurement can thus reveal important information regarding the compactness of each body, as well as its equation of state; and this information is released cleanly, during the inspiral phase of the orbital evolution, well before the messy merger of the two companions.

B. Newtonian theory of tidal Love numbers

The effect of the tidal interaction on the orbital motion and gravitational-wave signal is measured by a quantity known as the *tidal Love number* of each companion [3]. In Newtonian gravity (see, for example, Ref. [4]), the tidal Love number is a constant of proportionality between the tidal field applied to the body and the resulting multipole moment of its mass distribution. In the quadrupolar case, the tidal field is characterized by the *tidal moment* $\mathcal{E}_{ab}(t) := -\partial_{ab}U_{\rm ext}$, in which the external Newtonian potential $U_{\rm ext}$ is sourced by the companion body and evaluated (after differentiation with respect to the spatial coordinates) at the body's center of mass. Because the external potential satisfies Laplace's equation in the body's neighborhood, the tidal-moment tensor is not only symmetric but also tracefree; it is a symmetric-tracefree (STF) tensor.

The quadrupole moment is $Q^{ab} := \int \rho(x^a x^b - \frac{1}{3}\delta^{ab} r^2)d^3x$, where ρ is the mass density inside the body, x^a is a Cartesian coordinate system whose origin is at the center of mass, and $r := (\delta_{ab}x^a x^b)^{1/2}$ is the distance to the center of mass; the quadrupole moment is another STF tensor. In the absence of a tidal field, the body would be spherical, and its quadrupole moment would vanish. In the presence of a (weak) tidal field, the quadrupole moment is proportional to the tidal field, and dimensional analysis requires an expression of the form $Q_{ab} = -\frac{2}{3}k_2 R^5 \mathcal{E}_{ab}$. (We use relativistic units and set $G = c = 1$.) Here R is the body's radius, and the factor of $\frac{2}{3}$ is conventional; the dimensionless constant k_2 is the tidal Love number for a quadrupolar deformation. Using these expressions, the Newtonian potential outside the body can be written as a sum of body and external potentials, and we have

$$U = \frac{M}{r} - \frac{1}{2}[1 + 2k_2(R/r)^5]\mathcal{E}_{ab}(t)x^a x^b. \tag{1.1}$$

The first term is evidently the monopole piece of the potential, which depends on the body's mass M. Within the square brackets, the first term represents the applied tidal field, and the second term is the body's response, measured in terms of the Love number k_2.

In Eq. (1.1) the total potential was truncated to the leading, quadrupole order in a Taylor expansion of the external potential; additional terms would involve tidal moments of higher multipole orders, and higher powers of the coordinates x^a. When the tidal field is a pure multipole of order l, Eq. (1.1) generalizes to

$$U = \frac{M}{r} - \frac{1}{(l-1)l}[1 + 2k_l(R/r)^{2l+1}]\mathcal{E}_L(t)x^L. \tag{1.2}$$

Here k_l is the Love number for this multipolar configuration, and $L := a_1 a_2 \cdots a_l$ is a multi-index that contains a number l of individual indices. The tidal moment is now defined by $\mathcal{E}_L(t) := -\partial_L U_{\rm ext}/(l-2)!$, and it is symmetric and tracefree in all pairs of indices. We also introduced $x^L := x^{a_1}x^{a_2}\cdots x^{a_l}$. In this generalized case the l-pole moment of the mass distribution is the STF tensor $Q^L := \int \rho x^{\langle L\rangle} d^3x$, where the angular brackets indicate that all traces must be removed from the tensor x^L; it is related to the tidal moment by $Q_L = -[2(l-2)!/(2l-1)!!]k_l R^{2l+1}\mathcal{E}_L$.

C. Purpose of this work

Our purpose in this paper is to introduce a precise notion of tidal Love numbers in general relativity, something that was not pursued in the original work by Flanagan and Hinderer [1,2]. In fact, we provide precise definitions for two types of tidal Love numbers: an electric-type Love number $k_{\rm el}$ that has a direct analogy with the Newtonian Love number introduced previously, and a magnetic-type Love number $k_{\rm mag}$ that has no analogue in Newtonian gravity. Magnetic-type Love numbers were introduced in post-Newtonian theory in the works of Damour, Soffel, and Xu [5] and Favata [6]. Our definitions apply to gravitational fields that are arbitrarily strong, and to (weak) tidal deformations of any multipolar order.

Our relativistic Love numbers are defined within the context of linear perturbation theory, in which an initially spherical body is perturbed slightly by an applied tidal field. Our definitions are restricted to slowly changing tidal fields; this means that while a tidal moment such as $\mathcal{E}_L(t)$ does depend on time, to reflect the changes in the external distribution of matter, the dependence is sufficiently slow that the body's response presents only a *parametric dependence* upon time. This allows us to ignore time-derivative terms in the field equations, because they are much smaller than the spatial-derivative terms. For all practical purposes the perturbation is stationary, and t appears as an adiabatic parameter.

Gravitational perturbations of spherically symmetric bodies are described by a metric perturbation $p_{\alpha\beta}$ that can be decomposed into tensorial spherical harmonics; each multipole can be considered separately. The complete spacetime metric is $g_{\alpha\beta} = g^0_{\alpha\beta} + p_{\alpha\beta}$, with $g^0_{\alpha\beta}$ denoting the (spherically symmetric) metric of the unperturbed body. We work in the body's immediate neighborhood, and the external bodies that create the (multipolar) tidal field are assumed to live outside this neighborhood. To *define* the relativistic Love numbers it is sufficient to consider the vacuum region external to the body, and to construct $g_{\alpha\beta}$ in this region only; this metric will be a solution to the vacuum field equations, and will represent the relativistic generalization of Eq. (1.2). To *compute* the Love numbers it is necessary to construct $g_{\alpha\beta}$ in the body's interior also, and this requires the formulation of a stellar model. The external problem therefore applies to any type of body, while the internal problem refers to a specific choice of equation of state.

D. External problem

We review the external problem first. We erect a coordinate system (v, r, θ, ϕ) that is intimately tied to the behavior of light cones: The advanced-time coordinate v is constant on past light cones that converge toward the center at $r = 0$, r is both an areal radius and an affine-parameter distance along the null generators of each light cone, and the angular coordinates $\theta^A = (\theta, \phi)$ are constant on each generator. This choice of coordinates is inherited from previous work on the tidal deformation of black holes [7].

In these coordinates the external metric of the *unperturbed body* is given by $ds_0^2 = -f dv^2 + 2dvdr + r^2 d\Omega^2$, in which $f := 1 - 2M/r$ and $d\Omega^2 := d\theta^2 + \sin^2\theta d\phi^2$; this is the Schwarzschild metric presented in Eddington-Finkelstein coordinates. To construct the perturbation we impose the *light-cone gauge conditions* $p_{vr} = p_{rr} = p_{r\theta} = p_{r\phi} = 0$ to ensure that the coordinates keep their geometrical meaning in the perturbed spacetime [8]. (This property makes the light-cone gauge superior to the popular Regge-Wheeler gauge, which does not provide the coordinates with any geometrical meaning.) A perturbation of multipole order l can be decomposed into even-parity and odd-parity sectors, and each sector must be a solution to the Einstein field equations linearized about the Schwarzschild metric.

The even-parity sector is generated by the electric-type tidal moment $\mathcal{E}_L(v)$, an STF tensor defined in a quasi-Cartesian system x^a related in the usual way to the spherical coordinates (r, θ^A). The $(2l+1)$ independent components of this tensor can be encoded in the functions $\mathcal{E}_m^{(l)}(v)$, in which the azimuthal index m is an integer within the interval $-l \leq m \leq l$; the encoding is described by $\mathcal{E}_L x^L = r^l \sum_m \mathcal{E}_m^{(l)} Y^{lm}(\theta^A)$, in which Y^{lm} are the usual spherical-harmonic functions. We define the tidal potentials

$$\mathcal{E}^{(l)}(v, \theta^A) = \sum_m \mathcal{E}_m^{(l)}(v) Y^{lm}(\theta^A), \tag{1.3a}$$

$$\mathcal{E}_A^{(l)}(v, \theta^A) = \frac{1}{l} \sum_m \mathcal{E}_m^{(l)}(v) Y_A^{lm}(\theta^A), \tag{1.3b}$$

$$\mathcal{E}_{AB}^{(l)}(v, \theta^A) = \frac{2}{l(l-1)} \sum_m \mathcal{E}_m^{(l)}(v) Y_{AB}^{lm}(\theta^A), \tag{1.3c}$$

in which Y_A^{lm} and Y_{AB}^{lm} are vector and tensor spherical harmonics of even parity; these are defined in Sec. II.

The odd-parity sector is generated by the magnetic-type tidal moment $\mathcal{B}_L(v)$, another STF tensor whose independent components can be encoded (as previously) in the functions $\mathcal{B}_m^{(l)}(v)$. The odd-parity tidal potentials are

$$\mathcal{B}_A^{(l)}(v, \theta^A) = \frac{1}{l} \sum_m \mathcal{B}_m^{(l)}(v) X_A^{lm}(\theta^A), \tag{1.4a}$$

$$\mathcal{B}_{AB}^{(l)}(v, \theta^A) = \frac{2}{l(l-1)} \sum_m \mathcal{B}_m^{(l)}(v) X_{AB}^{lm}(\theta^A), \tag{1.4b}$$

in which X_A^{lm} and X_{AB}^{lm} are vector and tensor spherical harmonics of odd parity; these also are defined in Sec. II. There is no scalar potential $\mathcal{B}^{(l)}$ in the odd-parity sector.

The metric outside any spherical body deformed by a tidal environment characterized by the tidal moments $\mathcal{E}_L$ and $\mathcal{B}_L$ is calculated in Sec. III. It is given by

$$g_{vv} = -f - \frac{2}{(l-1)l} r^l e_1(r) \mathcal{E}^{(l)}, \tag{1.5a}$$

$$g_{vr} = 1, \tag{1.5b}$$

$$g_{vA} = -\frac{2}{(l-1)(l+1)} r^{l+1} e_4(r) \mathcal{E}_A^{(l)} + \frac{2}{3(l-1)} r^{l+1} b_4(r) \mathcal{B}_A^{(l)}, \tag{1.5c}$$

$$g_{AB} = r^2 \Omega_{AB} - \frac{2}{l(l+1)} r^{l+2} e_7(r) \mathcal{E}_{AB}^{(l)} + \frac{2}{3l} r^{l+2} b_7(r) \mathcal{B}_{AB}^{(l)}. \tag{1.5d}$$

The radial functions are

$$e_1 = A_1 + 2k_{\rm el}(R/r)^{2l+1} B_1, \tag{1.6a}$$

$$e_4 = A_4 - 2\frac{l+1}{l} k_{\rm el}(R/r)^{2l+1} B_4, \tag{1.6b}$$

$$e_7 = A_7 + 2k_{\rm el}(R/r)^{2l+1} B_7, \tag{1.6c}$$

$$b_4 = A_4 - 2\frac{l+1}{l} k_{\rm mag}(R/r)^{2l+1} B_4, \tag{1.6d}$$

$$b_7 = A_7 + 2k_{\rm mag}(R/r)^{2l+1} B_7, \tag{1.6e}$$

with

$$A_1 := f^2 F(-l+2, -l; -2l; 2M/r), \tag{1.7a}$$

$$B_1 := f^2 F(l+1, l+3; 2l+2; 2M/r), \tag{1.7b}$$

$$A_4 := F(-l+1, -l-2; -2l; 2M/r), \tag{1.7c}$$

$$B_4 := F(l-1, l+2; 2l+2; 2M/r), \tag{1.7d}$$

$$A_7 := \frac{l+1}{l-1} F(-l, -l; -2l; 2M/r) - \frac{2}{l-1} F(-l, -l-1; -2l; 2M/r), \tag{1.7e}$$

$$B_7 := \frac{l}{l+2} F(l+1, l+1; 2l+2; 2M/r) + \frac{2}{l+2} F(l, l+1; 2l+2; 2M/r). \tag{1.7f}$$

Here R is the body's radius, and $F(a, b; c; z)$ is the hypergeometric function. The functions A_n are finite polynomials in $2M/r$, while the functions B_n have nonterminating expansions in powers of $2M/r$; for selected values of l they can be expressed in terms of elementary functions such as $\ln(1 - 2M/r)$ and finite polynomials (see Table I in Sec. III). Each one of these functions goes to 1 as r goes to infinity. And while A_n is finite at $r = 2M$, we observe that B_n diverges logarithmically when $r \rightarrow 2M$.

The metric of Eqs. (1.5) is valid in a neighborhood of the deformed body, and it provides a definition for the electric-type Love numbers $k_{\rm el}$ and the magnetic-type Love numbers $k_{\rm mag}$; these refer to the multipole order l, but we suppress the use of this label to keep the notation clean. While the definitions seem to rely on a specific choice of gauge for the metric perturbation, we prove in Sec. III that our Love numbers are gauge invariant.

When the tidal moments are switched off, the metric reduces to the Schwarzschild metric expressed in the light-cone coordinates (v, r, θ^A). When the mass parameter M is set equal to zero, the metric describes the neighborhood of a geodesic world line in a Ricci-flat spacetime. In this limit the tidal moments can be related to the derivatives of the Weyl tensor evaluated at $r = 0$. According to Eq. (1.3) of Ref. [9], we have that $\mathcal{E}_L = [(l-2)!]^{-1} (C_{ta_1 ta_2; a_3 \cdots a_l})^{\rm STF}$ and $\mathcal{B}_L = [\frac{2}{3}(l+1)(l-2)!]^{-1} (\epsilon_{a_1 bc} C^{bc}{}_{a_2 t; a_3 \cdots a_l})^{\rm STF}$, where ϵ_{abc} is the permutation symbol and the tensor components are listed in the quasi-Lorentzian coordinates ($t := v - r, x^a$); the STF superscript indicates that the a_n indices are symmetrized and all traces are removed. In the spacetime of Eq. (1.5) the tidal moments $\mathcal{E}_L$ and $\mathcal{B}_L$ retain a similar relationship with the Weyl tensor, with the understanding that the relations are now approximate and refer to the asymptotic behavior of the Weyl tensor for $r \gg M$.

The perturbed metric of Eq. (1.5) can be compared with the Newtonian potential of Eq. (1.2). We define an effective Newtonian potential $U_{\rm eff}$ by $g_{vv} = -(1 - 2U_{\rm eff})$, and our expression for g_{vv} implies that, in general relativity,

$$U_{\rm eff} = -\frac{M}{r} - \frac{1}{(l-1)l} [A_1 + 2k_{\rm el}(R/r)^{2l+1} B_1] \mathcal{E}_L(v) x^L. \tag{1.8}$$

In the nonrelativistic limit, A_1 and B_1 are both approximately equal to unity, and we recover Eq. (1.2); the electric-type Love number k_{el} reduces to the Newtonian number k_l. In the strong-field regime we still recognize the A_1 term as coming from the applied tidal field, while the B_1 term is clearly associated with the body's response. There is no confusion between these terms, because the structure of A_1 is that of the finite polynomial $1 + \cdots + \lambda(2M/r)^l$, which does not contain a term of order $(2M/r)^{2l+1}$; λ is a numerical factor that can be determined by expanding the hypergeometric function. Because r is geometrically well defined, we can always distinguish the tidal terms from the body terms in the metric.

The light-cone coordinates (v, r, θ^A) are well behaved across an eventual event horizon of the perturbed spacetime, and our formalism is capable of handling black holes as well as material bodies. In general, however, the metric of Eqs. (1.5) is not regular at the event horizon, because of the presence of the B_n functions, which diverge logarithmically in the limit $r \to 2M$. To represent a perturbed black hole the metric must be devoid of these terms, and this can be accomplished by assigning $k_{el} = k_{mag} = 0$ to a black hole. This is one of the major conclusions of this work: *The relativistic Love numbers of a nonrotating black hole are all zero.* This result is contained implicitly in Ref. [7], but the formalism of this paper permits a much clearer articulation of this property.

E. Internal problem

To *compute* the relativistic Love numbers for a selected stellar model requires the construction of the internal metric (also expressed as a sum of unperturbed solution and linear perturbation) and its matching with the external metric at the perturbed boundary of the matter distribution. We carry out this exercise in Secs. IV and V, adapting the formalism of Thorne and Campolattaro [10] to our light-cone coordinates. We take the body to consist of a perfect fluid with a polytropic equation of state

$$p = K\rho^{1+1/n}. \tag{1.9}$$

Here p is the fluid's pressure, ρ its proper energy density, K is a constant, and n is the polytropic index (another constant).

Our results are presented in Figs. 1–8, and tables of values are provided in the Appendix (Tables III–XXX). In each figure we plot the Love number for a selected multipole order (from $l = 2$ to $l = 5$), and for selected values of the polytropic index n (from $n = 0.5$ to $n = 2.0$), as a function of the stellar compactness parameter $C := 2M/R$; this ranges from $C = 0$—a weak-field, Newtonian configuration—to $C = C_{max}$, with C_{max} representing the compactness of the maximum-mass configuration for the selected equation of state.

For the electric-type Love numbers we observe the following features. (i) At $C = 0$ we recover the Newtonian values for polytropes, as tabulated by Brooker and Olle [11]. (ii) For a constant C, k_{el} decreases as the polytropic index increases; this reflects the fact that as n increases, the matter distribution becomes increasingly concentrated near the center, which inhibits the development of large multipole moments. (iii) For a constant n, k_{el} decreases as the compactness parameter increases; this reflects the fact that as C increases, the strength of the internal gravity increases, which produces an increased resistance to tidal deformations.

For the magnetic-type Love numbers we observe the following features. (i) At $C = 0$ the Love numbers are all zero; this reflects the fact that the magnetic-type tidal coupling is a purely relativistic effect that has a vanishing

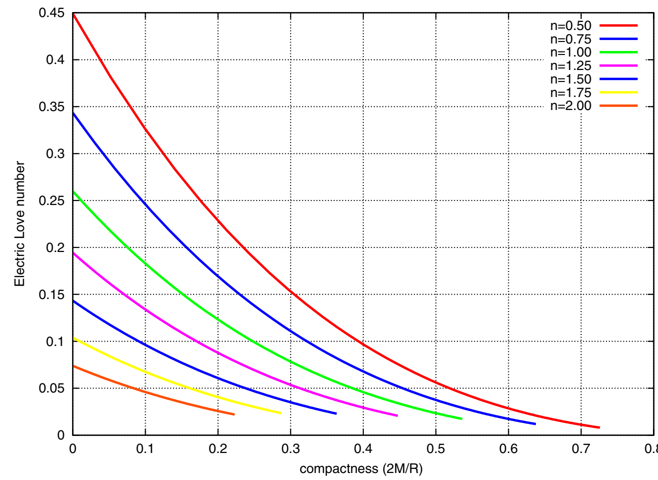

FIG. 1 (color online). Electric-type Love numbers for $l = 2$, plotted as functions of the compactness parameter $2M/R$. The uppermost curve corresponds to $n = 0.5$ and the stiffest equation of state. The lowermost curve corresponds to $n = 2.0$ and the softest equation of state. The curves in between are ordered by the value of n. The arrangement is the same in all other figures.

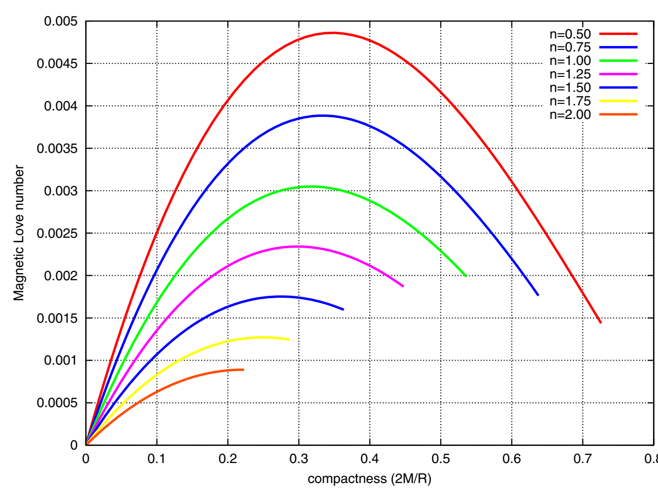

FIG. 2 (color online). Magnetic-type Love numbers for $l = 2$, plotted as functions of the compactness parameter $2M/R$.

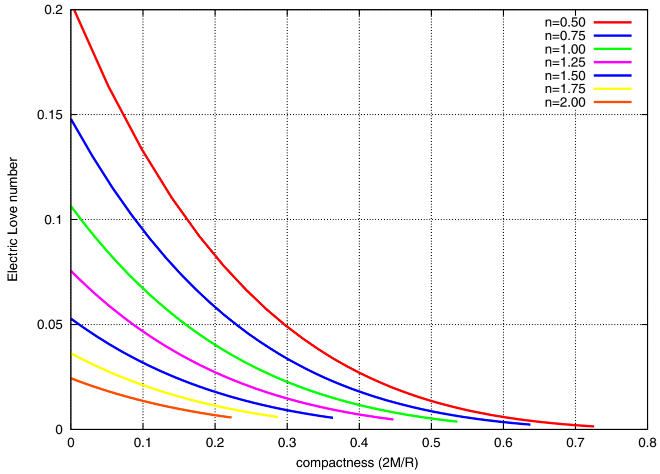

FIG. 3 (color online). Electric-type Love numbers for $l = 3$.

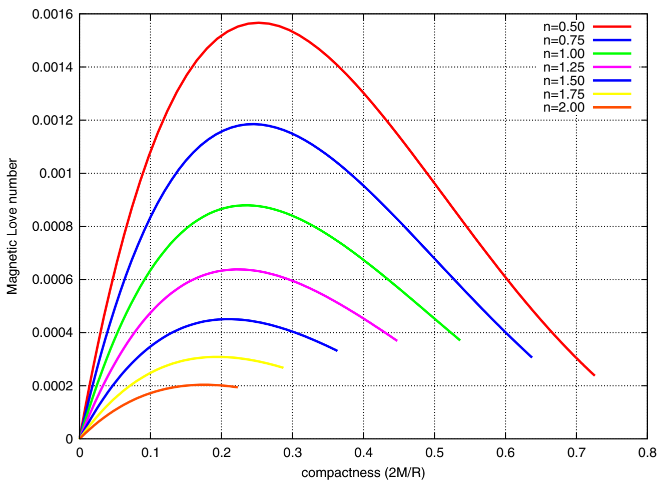

FIG. 4 (color online). Magnetic-type Love numbers for $l = 3$.

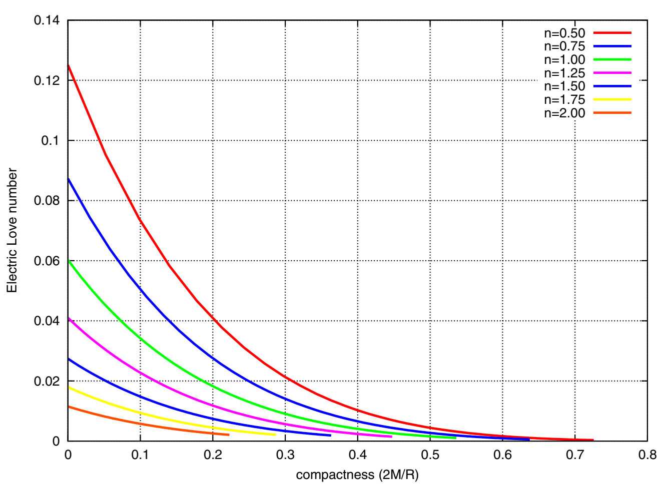

FIG. 5 (color online). Electric-type Love numbers for $l = 4$.

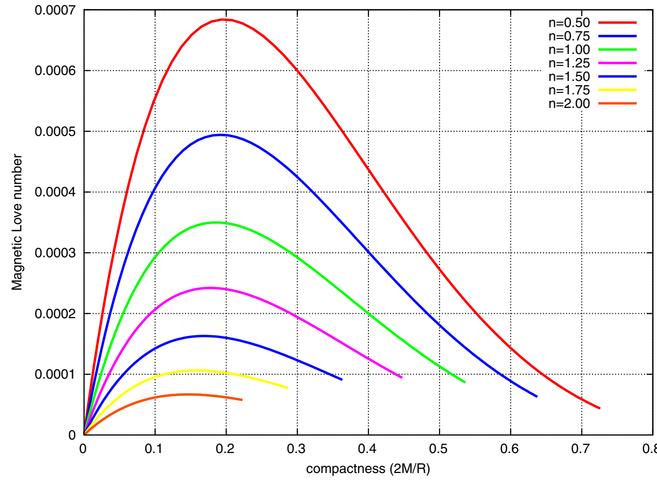

FIG. 6 (color online). Magnetic-type Love numbers for $l = 4$.

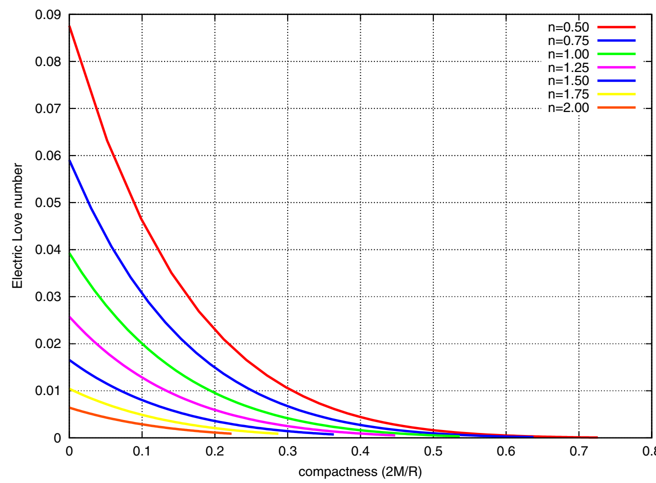

FIG. 7 (color online). Electric-type Love numbers for $l = 5$.

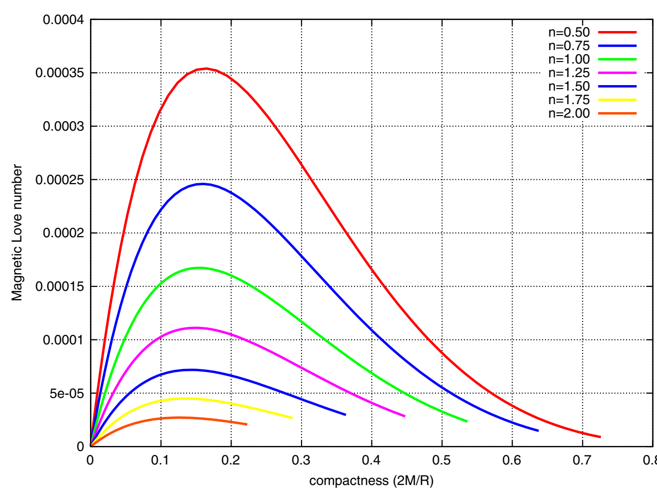

FIG. 8 (color online). Magnetic-type Love numbers for $l = 5$.

Newtonian limit. (ii) For a constant C, $k_{\rm mag}$ decreases as the polytropic index increases; this is explained as in the preceding paragraph. (iii) For a constant n, $k_{\rm mag}$ first increases as C increases, but then it decreases after reaching a maximum; this reflects the fact that the magnetic-type tidal coupling is the result of an internal competition: A strong field is required to produce an effect in the first place, but it eventually causes a large resistance to tidal deformation.

F. Damour and Nagar

After this work was completed we witnessed the appearance of an article by Damour and Nagar [12] in which almost identical work is presented. Their paper, like ours, is concerned with the tidal deformation of compact bodies in full general relativity, and presents precise definitions for electric-type and magnetic-type Love numbers. And their paper, like ours, presents computations of Love numbers for selected matter models. Their coverage of the parameter space is wider: Damour and Nagar examine two types of polytropic equations of state, and two tabulated equations of state for realistic nuclear matter. In addition, Damour and Nagar define and compute ‘‘shape Love numbers,’’ something that we did not pursue in this work.

There are superficial differences between our treatments. One concerns the choice of coordinates: Damour and Nagar work in Schwarzschild coordinates and adopt the Regge-Wheeler gauge for the metric perturbation; we work in Eddington-Finkelstein coordinates and the light-cone gauge. Another concerns notation: We adopt different normalization conditions for the Love numbers and the tidal moments. These differences are not important.

A more significant difference concerns the conclusion that the tidal Love numbers of a black hole must be zero. In this paper we boldly proclaim this conclusion, which we firmly believe to be a correct interpretation of our results. Damour and Nagar, however, shy away from the conclusion, although they agree with us on the basic results. We do not understand the reasons behind this reluctance. Damour and Nagar comment on the need to understand ‘‘diverging diagrams that enter the computation of interacting black holes at the five-loop (or 5PN) level’’ before reaching a conclusion. But since the results presented here do not rely at all on a post-Newtonian expansion of the field equations, the fate of 5PN terms in a post-Newtonian representation of interacting black holes seems to us to be irrelevant. We point out, also, that the Damour-Nagar work does not provide a very clean foundation for the tidal deformation of black holes, because their coordinate system is ill behaved on the event horizon. Our light-cone coordinates were selected precisely because they permit a unified treatment of material bodies and black holes.

Aside from this issue of interpretation, and as far as we can judge, the results presented here are in complete agreement with the Damour-Nagar results. The Damour-Nagar work was carried out in complete independence from us, and our work was carried out in complete independence from them. The near-simultaneous completion of our works provides evidence that the problem is interesting and timely, and the agreement is a reassuring confirmation that each team performed their calculations without error.

G. Fang and Lovelace

The deformation of a black hole produced by an applied tidal field was previously examined by Fang and Lovelace [13], who concluded that $Q_{ab} = 0$ when the perturbation is expressed in Regge-Wheeler gauge. Fang and Lovelace therefore anticipated our result that the quadrupole, electric-type Love number of a black hole is zero. These authors, however, qualified their conclusion by raising doubts about the gauge invariance of the result, and claiming that the induced quadrupole moment of a tidally deformed black hole is inherently ambiguous. We do not share these reservations.

We first discuss the issue of gauge invariance. The argument advanced by Fang and Lovelace in favor of a gauge dependence of the tidal Love number goes as follows. In Newtonian theory, the coordinate transformation $r = \bar{r}[1 + 2\chi(R/\bar{r})^5]^{1/2}$, where χ is an arbitrary constant, turns a pure tidal potential $\mathcal{E}_{ab}x^a x^b$ into $[1 + 2\chi(R/\bar{r})^5]\mathcal{E}_{ab}\bar{x}^a\bar{x}^b$, which appears to describe a sum of tidal and body potentials; the transformation shifts the Love number by χ. Fang and Lovelace correctly dismiss this coordinate dependence as irrelevant in Newtonian theory, because r has a well-defined meaning, but they point out that in a relativistic context, the coordinate transformation could be viewed as a change of gauge. The implication, then, is that the relativistic Love number can be altered by a gauge transformation. Notice that the argument applies to all types of compact bodies: material bodies and black holes.

We do not accept the validity of this argument. The coordinate transformation considered by Fang and Lovelace is not of a type that can be associated with a gauge transformation of the perturbation theory. A gauge transformation necessarily involves coordinate displacements that are of the same order of magnitude as the perturbation field. But the transformation from r to $\bar{r}$ does not involve the perturbation at all, and represents a large change of the background coordinates. The new coordinate $\bar{r}$ does not share the geometrical properties of the original r, and one would easily be able to distinguish the two coordinate systems. The argument, therefore, does not make a case for the gauge dependence of the Love numbers. And in fact, the *gauge invariance* of $k_{\rm el}$ and $k_{\rm mag}$ for all types of compact bodies (material bodies and black holes) is established in Sec. III.

We next discuss the issue of ambiguity. Unlike Fang and Lovelace, we believe that the relativistic Love numbers of compact bodies, as defined in this paper, are well defined and completely devoid of ambiguity. The reason is that the metric of Eqs. (1.5), which is presented in coordinates that have clear geometrical properties, defines a perfectly well-defined spacetime geometry. Given this spacetime, one could, in principle, monitor the motion of test masses and light rays and thereby measure its detailed features, including the mass M, the tidal moments $\mathcal{E}_L$ and $\mathcal{B}_L$, and the Love numbers. These measurements would contain no ambiguities.

The ambiguity identified by Fang and Lovelace concerns the coupling of Q_{ab}, the induced quadrupole moment, to $\mathcal{E}_{abc}$, the octupole moment of the applied tidal field. According to Newtonian ideas, this coupling should lead to a force $F^a = -\frac{1}{2}\mathcal{E}^a{}_{bc}Q^{bc}$ acting on the compact body. (Once more, the argument applies to all types of compact bodies.) Fang and Lovelace associate F^a with $\dot{P}^a(r)$, the rate of change of three-momentum contained within a world tube of radius r that surrounds the compact body; this is calculated by integrating the flux of Landau-Lifshitz energy-momentum pseudotensor across the world tube. They observe that the result is indeed proportional to $\mathcal{E}^a{}_{bc}Q^{bc}$, but that the coefficient in front depends on r. They interpret this as a statement that the force is ambiguous, assign the ambiguity to Q_{ab}, and conclude that the induced quadrupole moment of a tidally deformed compact body is inherently ambiguous.

We believe that the ambiguity in $\dot{P}^a(r)$ is genuine—the result does depend on the world tube's radius. It is hasty, however, to conclude from this that F^a itself is ambiguous, because force calculations that rely on techniques of matched asymptotic expansions [14,15] must involve a limiting procedure in which both M and r are taken to approach zero. Although ambiguities remain in this procedure, they are much smaller than those claimed by Fang and Lovelace. At the accuracy level of our calculations, the induced quadrupole moment of a tidally deformed compact body is not ambiguous.

H. Suen

An earlier determination of the induced quadrupole moment of a tidally deformed black hole was made by Suen [16], who examined the specific case of a black hole perturbed by an axisymmetric ring of matter. Suen found that the black-hole quadrupole moment is $Q_{ab} = +\frac{4}{21}M^5\mathcal{E}_{ab}$, so that it gives rise to a negative Love number, $k_{\rm el} = -\frac{1}{122}$. This result contradicts our own results.

Suen's result is wrong. The starting point of Suen's analysis is the perturbed metric presented in Eq. (2.6) of his paper. It is easy to show that while the metric does indeed satisfy the Einstein field equations (up to terms that are quadratic in the small parameter A), it fails to be regular at the event horizon. The metric does not, therefore, represent a perturbed black hole, and the nonzero result for $k_{\rm el}$ is a consequence of this fact. The regularity of the metric perturbation $p_{\alpha\beta}$ at $r = 2M$ can be judged by examining its components in the light-cone coordinates (v, r, θ, ϕ), which are regular on the event horizon. A simple calculation reveals that in Suen's notation, $p_{rr} = -2(2U - V)/f$, where $f = 1 - 2M/r$. This is singular at $r = 2M$ unless $2U - V$ vanishes there, but Eqs. (2.7) of Suen's paper show instead that $2U - V \to AM^2$ in the limit. The perturbation is singular.

I. Organization of the paper

In the remaining sections of this paper we present the details of our analysis, and describe how the results reviewed previously were obtained. We begin in Sec. II with a discussion of tidal moments and tidal potentials, and motivate the definitions presented in Eqs. (1.3) and (1.4). In Sec. III we solve the external problem, and show that the metric of Eqs. (1.5) is a solution to the vacuum field equations linearized about the Schwarzschild metric. In Sec. IV we formulate the internal problem for general stellar models, and we specialize this to polytropes in Sec. V. In Sec. VI we review the numerical techniques that were employed to generate the figures and the tables displayed in the Appendix.

II. TIDAL MOMENTS AND POTENTIALS

A spherical stellar model is perturbed by an external tidal field characterized by the electric-type tidal moments $\mathcal{E}_L(v)$ and the magnetic-type tidal moments $\mathcal{B}_L(v)$. These are STF tensors, and L is a multi-index that contains a number l of individual indices. The tidal moments depend on v (and not on the spatial coordinates), but this time dependence is taken to be so slow that all v derivatives will be ignored in the Einstein field equations.

We begin our discussion of tidal potentials by adopting quasi-Cartesian coordinates x^a related in the usual way to our spherical coordinates (r, θ^A). We write the transformation as $x^a = r\Omega^a(\theta^A)$, with $\Omega^a = [\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta]$ denoting the unit radial vector. We introduce

$$\gamma_{ab} := \delta_{ab} - \Omega_a\Omega_b \tag{2.1}$$

as the projector to the transverse space orthogonal to Ω^a, and we let $\Omega^a_A := \partial\Omega^a/\partial\theta^A$. We note the helpful identities

$$\Omega_a\Omega^a_A = 0, \tag{2.2a}$$

$$\Omega_{AB} = \gamma_{ab}\Omega^a_A\Omega^b_B = \delta_{ab}\Omega^a_A\Omega^b_B, \tag{2.2b}$$

$$\Omega^{AB}\Omega^a_A\Omega^b_B = \gamma^{ab}. \tag{2.2c}$$

Here $\Omega_{AB} = \mathrm{diag}[1, \sin^2\theta]$ is the metric on the unit two-sphere, and Ω^{AB} is its inverse. We introduce D_A as the covariant-derivative operator compatible with Ω_{AB}, and ϵ_{AB} as the Levi-Civita tensor on the unit two-sphere (with nonvanishing components $\epsilon_{\theta\phi} = -\epsilon_{\phi\theta} = \sin\theta$). In addition to Eqs. (2.2) we also have

$$\epsilon_{AB} = \epsilon_{abc}\Omega^a_A\Omega^b_B\Omega^c, \tag{2.3a}$$
$$\epsilon_A{}^B\Omega^b_B = -\Omega^a_A\epsilon_{ap}{}^b\Omega^p, \tag{2.3b}$$
$$D_AD_B\Omega^a = D_BD_A\Omega^a = -\Omega^a\Omega_{AB}. \tag{2.3c}$$

Here and below, uppercase Latin indices are raised and lowered with Ω^{AB} and Ω_{AB}, respectively. Finally, we note that $D_C\Omega_{AB} = D_C\epsilon_{AB} = 0$.

For an electric-type tidal moment $\mathcal{E}_L$ of degree $l \geq 2$, the Cartesian versions of the tidal potentials are defined by

$$\mathcal{E}^{(l)} := \mathcal{E}_L\Omega^L, \tag{2.4a}$$
$$\mathcal{E}^{(l)}_a := \gamma_a{}^c\mathcal{E}_{cL-1}\Omega^{L-1}, \tag{2.4b}$$
$$\mathcal{E}^{(l)}_{ab} := 2\gamma_a{}^c\gamma_b{}^d\mathcal{E}_{cdL-2}\Omega^{L-2} + \gamma_{ab}\mathcal{E}^{(l)}. \tag{2.4c}$$

Here $\mathcal{E}^{(l)}$ is a scalar potential, $\mathcal{E}^{(l)}_a$ is a transverse vector potential, and $\mathcal{E}^{(l)}_{ab}$ is a transverse-tracefree tensor potential. The angular versions of the tidal potentials are

$$\mathcal{E}^{(l)} = \mathcal{E}_L\Omega^L, \tag{2.5a}$$
$$\mathcal{E}^{(l)}_A := \mathcal{E}^{(l)}_a\Omega^a_A = \Omega^a_A\mathcal{E}_{aL-1}\Omega^{L-1}, \tag{2.5b}$$
$$\mathcal{E}^{(l)}_{AB} := \mathcal{E}^{(l)}_{ab}\Omega^a_A\Omega^b_B = 2\Omega^a_A\Omega^b_B\mathcal{E}_{abL-2}\Omega^{L-2} + \Omega_{AB}\mathcal{E}^{(l)}. \tag{2.5c}$$

For a magnetic-type tidal moment $\mathcal{B}_L$ of degree $l \geq 2$, the Cartesian versions of the tidal potentials are defined by

$$\mathcal{B}^{(l)}_a := \epsilon_{apq}\Omega^p\mathcal{B}^q{}_{L-1}\Omega^{L-1}, \tag{2.6a}$$
$$\mathcal{B}^{(l)}_{ab} := (\epsilon_{apq}\Omega^p\mathcal{B}^q{}_{dL-2}\gamma^d{}_b + \epsilon_{bpq}\Omega^p\mathcal{B}^q{}_{cL-2}\gamma^c{}_a)\Omega^{L-2}. \tag{2.6b}$$

Here $\mathcal{B}^{(l)}_a$ is a transverse vector potential, and $\mathcal{B}^{(l)}_{ab}$ is a transverse-tracefree tensor potential; there is no scalar potential in the magnetic case. The angular versions of the tidal potentials are

$$\mathcal{B}^{(l)}_A := \mathcal{B}^{(l)}_a\Omega^a_A = \Omega^a_A\epsilon_{apq}\Omega^p\mathcal{B}^q{}_{L-1}\Omega^{L-1}, \tag{2.7a}$$
$$\mathcal{B}^{(l)}_{AB} := \mathcal{B}^{(l)}_{ab}\Omega^a_A\Omega^b_B = (\Omega^a_A\epsilon_{apq}\Omega^p\mathcal{B}^q{}_{bL-2}\Omega^b_B + \Omega^b_B\epsilon_{bpq}\Omega^p\mathcal{B}^q{}_{aL-2}\Omega^a_A)\Omega^{L-2}. \tag{2.7b}$$

The tidal potentials can all be expressed in terms of (scalar, vector, and tensor) spherical harmonics. Let Y^{lm} be the standard (scalar) spherical-harmonic functions. The vector and tensor harmonics of even parity are $Y^{lm}_A := D_AY^{lm}$, $\Omega_{AB}Y^{lm}$, and $Y^{lm}_{AB} := [D_AD_B + \frac{1}{2}l(l+1)\Omega_{AB}]Y^{lm}$; notice that $\Omega^{AB}Y^{lm}_{AB} = 0$ by virtue of the eigenvalue equation satisfied by the spherical harmonics. The vector and tensor harmonics of odd parity are $X^{lm}_A := -\epsilon_A{}^BD_BY^{lm}$ and $X^{lm}_{AB} := -\frac{1}{2}(\epsilon_A{}^CD_B + \epsilon_B{}^CD_A)D_CY^{lm}$; X^{lm}_{AB} also is tracefree: $\Omega^{AB}X^{lm}_{AB} = 0$.

We first express the electric-type tidal potentials in terms of the even-parity spherical harmonics. We begin with $\mathcal{E}^{(l)}$, which we decompose as

$$\mathcal{E}^{(l)}(v, \theta^A) = \sum_m \mathcal{E}^{(l)}_m(v)Y^{lm}(\theta^A), \tag{2.8}$$

in terms of harmonic components $\mathcal{E}^{(l)}_m(v)$. There are $2l+1$ terms in the sum, and the $2l+1$ independent components of $\mathcal{E}_L$ are in a one-to-one correspondence with the $2l+1$ coefficients $\mathcal{E}^{(l)}_m$. Returning to the original representation of Eq. (2.4), we find after differentiation that $D_A\mathcal{E}^{(l)} = l\Omega^a_A\mathcal{E}_{aL-1}\Omega^{L-1}$, and we conclude that

$$\mathcal{E}^{(l)}_A = \frac{1}{l}D_A\mathcal{E}^{(l)} = \frac{1}{l}\sum_m \mathcal{E}^{(l)}_m Y^{lm}_A. \tag{2.9}$$

An additional differentiation using the last of Eqs. (2.3) reveals that $D_AD_B\mathcal{E}^{(l)} = -l\Omega_{AB}\mathcal{E}^{(l)} + l(l-1)\Omega^a_A\Omega^b_B\mathcal{E}_{abL-2}\Omega^{L-2}$. From this we conclude that

$$\mathcal{E}^{(l)}_{AB} = \frac{2}{l(l-1)}\left[D_AD_B + \frac{1}{2}l(l+1)\Omega_{AB}\right]\mathcal{E}^{(l)} = \frac{2}{l(l-1)}\sum_m \mathcal{E}^{(l)}_m Y^{lm}_{AB}. \tag{2.10}$$

We next express the magnetic-type potentials in terms of the odd-parity spherical harmonics. We begin with $\mathcal{B}^{(l)} := \mathcal{B}_L\Omega^L$ and its decomposition $\mathcal{B}^{(l)} = \sum_m \mathcal{B}^{(l)}_m Y^{lm}(\theta^A)$. Differentiating the first expression, multiplying this by the Levi-Civita tensor, and involving the second of Eqs. (2.3) returns $\epsilon_A{}^BD_B\mathcal{B}^{(l)} = -l\Omega^a_A\epsilon_{apq}\Omega^p\mathcal{B}^q{}_{L-1}\Omega^{L-1}$. From this we conclude that

$$\mathcal{B}^{(l)}_A = \frac{1}{l}(-\epsilon_A{}^BD_B)\mathcal{B}^{(l)} = \frac{1}{l}\sum_m \mathcal{B}^{(l)}_m X^{lm}_A. \tag{2.11}$$

A second differentiation yields $-\epsilon_A{}^CD_BD_C\mathcal{B}^{(l)} = l\epsilon_{AB}\mathcal{B}^{(l)} + l(l-1)\Omega^a_A\epsilon_{apq}\Omega^p\mathcal{B}^q{}_{bL-2}\Omega^b_B\Omega^{L-2}$, and after symmetrization we obtain

$$\mathcal{B}^{(l)}_{AB} = -\frac{1}{l(l-1)}(\epsilon_A{}^CD_B + \epsilon_B{}^CD_A)D_C\mathcal{B}^{(l)} = \frac{2}{l(l-1)}\sum_m \mathcal{B}^{(l)}_m X^{lm}_{AB}. \tag{2.12}$$

III. EXTERNAL PROBLEM

A. Even-parity sector

In this subsection we determine the tidal deformation of the metric outside the matter distribution, in the even-parity sector. The unperturbed external solution is the Schwarzschild metric

$$ds^2_0 = -fdv^2 + 2dvdr + r^2d\Omega^2, \tag{3.1}$$

with $f := 1 - 2M/r$ and M denoting the body's mass; the metric is valid for $r > R$, where R is the body's radius. We employ the perturbation formalism of Martel and Poisson

TABLE I. Functions A_n and B_n for selected values of l, expressed in terms of $z := 2M/r$. The numbers μ_l and λ_l are given by $\lambda_l = (2l)!(2l+1)!/[(l-2)!(l-1)!(l+1)!(l+2)!]$ and $\mu_l = (l+1)\lambda_l/l$.

$l = 2$	$\mu_2 = 30,\ \lambda_2 = 20$
$A_1 = (1-z)^2$	$z^5 B_1 = -\mu_2 A_1 \ln(1-z) - \frac{5}{2} z(2-z)(6-6z-z^2)$
$A_4 = 1-z$	$z^5 B_4 = \lambda_2 A_4 \ln(1-z) + \frac{5}{3} z(12-6z-2z^2-z^3)$
$A_7 = 1-\frac{1}{2}z^2$	$z^5 B_7 = -\mu_2 A_7 \ln(1-z) - 5z(6+3z-z^2)$
$l = 3$	$\mu_3 = 840,\ \lambda_3 = 630$
$A_1 = \frac{1}{2}(1-z)^2(2-z)$	$z^7 B_1 = -\mu_3 A_1 \ln(1-z) - 7z(120-240z+130z^2-10z^3-z^4)$
$A_4 = \frac{1}{3}(1-z)(3-2z)$	$z^7 B_4 = \lambda_3 A_4 \ln(1-z) + \frac{7}{2} z(180-210z+30z^2+5z^3+z^4)$
$A_7 = 1-z+\frac{1}{10}z^3$	$z^7 B_7 = -\mu_3 A_7 \ln(1-z) - 14z(60-30z-10z^2+z^3)$
$l = 4$	$\mu_4 = 17\,640,\ \lambda_4 = 14\,112$
$A_1 = \frac{1}{14}(1-z)^2(14-14z+3z^2)$	$z^9 B_1 = -\mu_4 A_1 \ln(1-z) - 21z(2-z)(420-840z+440z^2-20z^3-z^4)$
$A_4 = \frac{1}{28}(1-z)(28-35z+10z^2)$	$z^9 B_4 = \lambda_4 A_4 \ln(1-z) + \frac{42}{5} z(1680-2940z+1370z^2-90z^3-9z^4-z^5)$
$A_7 = 1-\frac{5}{3}z+\frac{5}{7}z^2-\frac{1}{42}z^4$	$z^9 B_7 = -\mu_4 A_7 \ln(1-z) - 14z(1260-1470z+270z^2+65z^3-3z^4)$
$l = 5$	$\mu_5 = 332\,640,\ \lambda_5 = 277\,200$
$A_1 = \frac{1}{12}(1-z)^2(2-z)(6-6z+z^2)$	$z^{11} B_1 = -\mu_5 A_1 \ln(1-z) - 66z(5040-15\,120z+16\,380z^2-7560z^3 + 1288z^4-28z^5-z^6)$
$A_4 = \frac{1}{30}(1-z)(30-54z+30z^2-5z^3)$	$z^{11} B_4 = \lambda_5 A_4 \ln(1-z) + 22z(12\,600-28\,980z+21\,840z^2-5670z^3 + 210z^4+14z^5+z^6)$
$A_7 = 1-\frac{9}{4}z+\frac{5}{3}z^2-\frac{5}{12}z^3+\frac{1}{168}z^5$	$z^{11} B_7 = -\mu_5 A_7 \ln(1-z) - 66z(5040-8820z+4410z^2-420z^3-77z^4+2z^5)$

[17], and implement the light-cone gauge of Preston and Poisson [8].

In the light-cone gauge the even-parity metric perturbation is given by

$$p_{vv} = \sum_m h^{lm}_{vv}(r) Y^{lm}(\theta^A), \tag{3.2a}$$

$$p_{vA} = \sum_m j^{lm}_v(r) Y^{lm}_A(\theta^A), \tag{3.2b}$$

$$p_{AB} = r^2 \sum_m K^{lm}(r)\Omega_{AB} Y^{lm}(\theta^A) + r^2 \sum_m G^{lm}(r) Y^{lm}_{AB}(\theta^A). \tag{3.2c}$$

We consider each l mode separately, and we henceforth omit the label lm on the perturbation variables h_{vv}, j_v, K, and G, which depend on r only. As discussed by Preston and Poisson, K^{lm} can always be set equal to zero when the perturbation satisfies the vacuum field equations; this represents a refinement of the light-cone gauge, and we shall make this choice here.

To simplify the task of solving the field equations, we set

$$h_{vv} = -\frac{2}{(l-1)l} r^l e_1(r) \mathcal{E}^{(l)}_m, \tag{3.3a}$$

$$j_v = -\frac{2}{(l-1)l(l+1)} r^{l+1} e_4(r) \mathcal{E}^{(l)}_m, \tag{3.3b}$$

$$G = -\frac{4}{(l-1)l^2(l+1)} r^l e_7(r) \mathcal{E}^{(l)}_m, \tag{3.3c}$$

where the functions $e_1(r)$, $e_4(r)$, and $e_7(r)$ are to be determined. Substitution of Eqs. (3.3) into Eq. (3.2) produces

$$p_{vv} = -\frac{2}{(l-1)l} r^l e_1(r) \mathcal{E}^{(l)}, \tag{3.4a}$$

$$p_{vA} = -\frac{2}{(l-1)(l+1)} r^{l+1} e_4(r) \mathcal{E}^{(l)}_A, \tag{3.4b}$$

$$p_{AB} = -\frac{2}{l(l+1)} r^{l+2} e_7(r) \mathcal{E}^{(l)}_{AB}, \tag{3.4c}$$

where $\mathcal{E}^{(l)}$, $\mathcal{E}^{(l)}_A$, and $\mathcal{E}^{(l)}_{AB}$ are the tidal potentials introduced in Eq. (2.5).

The motivation behind the introduction of the functions e_1, e_4, and e_7 goes as follows. We first observe that when we set $e_1 = e_4 = e_7 = 1$, the perturbation defined by Eqs. (3.3) or Eqs. (3.4) satisfies the equations of linearized theory for a perturbation of Minkowski spacetime. This exercise reveals that h_{vv} must be proportional to r^l, j_v to r^{l+1}, and G to r^l; the relative numerical coefficients between these fields are also determined by solving the perturbation equations in flat spacetime. The remaining absolute numerical coefficient that relates the perturbation to the tidal moment $\mathcal{E}_L$ is determined by the definition of the tidal moment in terms of the Weyl tensor of the perturbed spacetime; this coefficient—the factor $-2/[(l-1)l]$ in h_{vv}—can be read off Eq. (3.26a) of Ref. [9].

Inserting the functions e_1, e_4, and e_7 in Eqs. (3.3) allows the perturbation to be a solution to the Einstein field equations linearized about the Schwarzschild metric instead of the Minkowski metric. We impose the boundary conditions

$$e_1(r\to\infty) = e_4(r\to\infty) = e_7(r\to\infty) = 1. \tag{3.5}$$

The field equations do not determine these functions uniquely. The light-cone gauge comes with a class of residual gauge transformations that preserve the light-cone nature of the coordinate system (see Preston and Poisson [8]). In the even-parity sector, and for static perturbations, the residual gauge freedom that keeps $K = 0$ is a one-parameter family described by

$$e_1 \to e_1 - la_1(2M/r)^{l+2}, \tag{3.6a}$$

$$e_4 \to e_4 + a_1[(l-1)(l+2) + 4M/r](2M/r)^{l+1}, \tag{3.6b}$$

$$e_7 \to e_7 + 2la_1(2M/r)^{l+1}, \tag{3.6c}$$

in which a_1 is the (dimensionless) parameter. The residual gauge freedom does not interfere with the boundary conditions of Eq. (3.5).

When K is allowed to change, the residual gauge freedom becomes a three-parameter family. In this case we have

$$e_1 \to e_1 - la_1(2M/r)^{l+2} + a_3(2M/r)^{l+2}, \tag{3.7a}$$

$$e_4 \to e_4 + a_1[(l-1)(l+2) + 4M/r](2M/r)^{l+1} - (l+1)a_3(2M/r)^{l+1}, \tag{3.7b}$$

$$e_7 \to e_7 + 2la_1(2M/r)^{l+1} + 2a_2(2M/r)^{l}, \tag{3.7c}$$

and K becomes

$$K = \frac{4(2M)^l}{(l-1)l}[a_2 + a_3(2M/r)]\mathcal{E}_m^{(l)}. \tag{3.8}$$

Here a_2 and a_3 are two additional gauge parameters.

The differential equations satisfied by e_1, e_4, and e_7 can be extracted from the perturbation equations. These equations are coupled, and some effort must be devoted to their decoupling before an attempt is made to find solutions. We shall not describe these routine steps here. We state simply that the solutions are the ones that were displayed in Eqs. (1.6) and (1.7). These are given in a minimal implementation of the light-cone gauge, in which all constants of integration are set equal to zero. The most general form of the solution is obtained from this by effecting the shifts described by Eqs. (3.7) and (3.8). The functions A_n and B_n are displayed for selected values of l in Table I.

The metric perturbation can be represented in terms of gauge-invariant variables. We employ the set defined by Eqs. (4.10)–(4.12) of Martel and Poisson [17]. According to these equations, and as can be directly verified from Eq. (3.7), the variables

$$\tilde{h}_{vv} := h_{vv} + \frac{2M}{r^2}j_v - MfG', \tag{3.9a}$$

$$\tilde{h}_{vr} := MG' - j_v', \tag{3.9b}$$

$$\tilde{h}_{rr} := 2rG' + r^2G'', \tag{3.9c}$$

$$\tilde{K} := -\frac{2}{r}j_v + \frac{1}{2}l(l+1)G + rfG' \tag{3.9d}$$

are gauge invariant; a prime indicates differentiation with respect to r. We express them as

$$\tilde{h}_{vv} := -\frac{2}{(l-1)l}r^l e_{vv}(r)\mathcal{E}_m^{(l)}, \tag{3.10a}$$

$$\tilde{h}_{vr} := \frac{2}{(l-1)l}r^l e_{vr}(r)\mathcal{E}_m^{(l)}, \tag{3.10b}$$

$$\tilde{h}_{rr} := -\frac{4}{(l-1)l}r^l e_{rr}(r)\mathcal{E}_m^{(l)}, \tag{3.10c}$$

$$\tilde{K} := -\frac{2}{(l-1)l}r^l e_K(r)\mathcal{E}_m^{(l)}, \tag{3.10d}$$

in terms of new radial functions e_{vv}, e_{vr}, e_{rr}, and e_K. Calculation reveals that these are given in terms of the old ones by

$$e_{vv} = e_1 + \frac{1}{l+1}\frac{2M}{r}e_4 - \frac{1}{l+1}\frac{2M}{r}fe_7 - \frac{1}{l(l+1)}2Mfe_7', \tag{3.11a}$$

$$e_{vr} = e_4 + \frac{1}{l+1}re_4' - \frac{1}{l+1}\frac{2M}{r}e_7 - \frac{1}{l(l+1)}2Me_7', \tag{3.11b}$$

$$e_{rr} = e_7 + \frac{2}{l}re_7' + \frac{1}{l(l+1)}r^2e_7'', \tag{3.11c}$$

$$e_K = -\frac{2}{l+1}e_4 + \frac{1}{l+1}(l+3-4M/r)e_7 + \frac{2}{l(l+1)}rfe_7'. \tag{3.11d}$$

It is easy to see that these functions, like the old ones, all go to 1 as r goes to infinity.

Substitution of our expressions for e_1, e_4, and e_7 into Eqs. (3.11) and repeated use of the properties of hypergeometric functions reveal that

$$e_{vv} = fe_{vr} = f^2e_{rr} = A_1 + 2k_{\rm el}(R/r)^{2l+1}B_1 \tag{3.12}$$

and

$$e_K = A_7 + 2k_{\rm el}(R/r)^{2l+1}B_7. \tag{3.13}$$

Notice that e_{vv}, fe_{vr}, and f^2e_{rr} are all equal to the minimal implementation of e_1, and e_K is equal to the minimal implementation of e_7. All this shows that the relativistic Love numbers $k_{\rm el}$ possess gauge-invariant significance.

B. Odd-parity sector

In the light-cone gauge the odd-parity metric perturbation is given by

$$p_{vA} = \sum_m h_v^{lm}(r)X_A^{lm}(\theta^A), \tag{3.14a}$$

$$p_{AB} = \sum_m h_2^{lm}(r)X_{AB}^{lm}(\theta^A). \tag{3.14b}$$

We consider each l mode separately, and we henceforth omit the label lm on the perturbation variables h_v and h_2, which depend on r only. To simplify the task of solving the field equations, we set

$$h_v = \frac{2}{3(l-1)l} r^{l+1} b_4(r) \mathcal{B}_m^{(l)}, \tag{3.15a}$$

$$h_2 = \frac{4}{3(l-1)l^2} r^{l+2} b_7(r) \mathcal{B}_m^{(l)}, \tag{3.15b}$$

where the functions $b_4(r)$ and $b_7(r)$ are to be determined. Substitution of Eqs. (3.15) into Eq. (3.14) produces

$$p_{vA} = \frac{2}{3(l-1)} r^{l+1} b_4(r) \mathcal{B}_A^{(l)}, \tag{3.16a}$$

$$p_{AB} = \frac{2}{3l} r^{l+2} b_7(r) \mathcal{B}_{AB}^{(l)}, \tag{3.16b}$$

where $\mathcal{B}_A^{(l)}$ and $\mathcal{B}_{AB}^{(l)}$ are the tidal potentials first introduced in Eq. (2.7).

The motivation behind the introduction of the functions b_4 and b_7 is identical to what was done in the even-parity sector. When we set $b_4 = b_7 = 1$, the perturbation defined by Eqs. (3.15) or Eqs. (3.16) satisfies the equations of linearized theory for a perturbation of Minkowski spacetime. This exercise reveals the relative numerical coefficients between h_v and h_2. The remaining absolute numerical coefficient that relates the perturbation to the tidal moment $\mathcal{B}_L$ is determined by the definition of the tidal moment in terms of the Weyl tensor of the perturbed spacetime; this coefficient—the factor $2/[3(l-1)l]$ in h_v—can be read off Eq. (3.26b) of Ref. [9].

Inserting the functions b_4 and b_7 in Eqs. (3.15) allows the perturbation to be a solution to the Einstein field equations linearized about the Schwarzschild metric instead of the Minkowski metric. We impose the boundary conditions

$$b_4(r \to \infty) = b_7(r \to \infty) = 1. \tag{3.17}$$

The field equations do not determine these functions uniquely. As in the even-parity case, we have a residual gauge freedom that preserves the nature of the light-cone coordinates. It is described by

$$b_4 \to b_4, \tag{3.18a}$$

$$b_7 \to b_7 + \alpha\left(\frac{2M}{r}\right)^l, \tag{3.18b}$$

in which α is a (dimensionless) parameter. The residual gauge freedom does not interfere with the boundary conditions of Eq. (3.17).

The differential equations satisfied by b_4 and b_7 can be extracted from the perturbation equations. The solutions are displayed in Eqs. (1.6) and (1.7). They are given in a minimal implementation of the light-cone gauge, in which all constants of integrations are set equal to zero. The most general form of the solution is obtained from this by effecting the shifts described by Eqs. (3.18).

The metric perturbation can be represented in terms of gauge-invariant variables. We employ the set defined by Eq. (5.7) of Martel and Poisson [17]. According to this, and as can be directly verified from Eq. (3.18), the variables

$$\tilde{h}_v := h_v, \tag{3.19a}$$

$$\tilde{h}_r := \frac{1}{r} h_2 - \frac{1}{2} h_2' \tag{3.19b}$$

are gauge invariant. We express them as

$$\tilde{h}_v := \frac{2}{3(l-1)l} r^{l+1} b_v(r) \mathcal{B}_m^{(l)}, \tag{3.20a}$$

$$\tilde{h}_r := -\frac{2}{3(l-1)l} r^{l+1} b_r(r) \mathcal{B}_m^{(l)}, \tag{3.20b}$$

in terms of new radial functions b_v and b_r. Calculation reveals that these are given in terms of the old ones by

$$b_v = b_4, \tag{3.21a}$$

$$b_r = b_7 + \frac{r}{l} b_7'. \tag{3.21b}$$

It is easy to see that these functions, like the old ones, all go to 1 as r goes to infinity.

Substitution of our expressions for b_4 and b_7 into Eqs. (3.21) and repeated use of the properties of the hypergeometric functions reveal that

$$b_v = f b_r = A_4 - 2\frac{l+1}{l} k_{\rm mag} (R/r)^{2l+1} B_4. \tag{3.22}$$

Notice that b_v and $f b_r$ are both equal to b_4, which is gauge invariant. This shows that the relativistic Love numbers $k_{\rm mag}$ possess gauge-invariant significance.

IV. INTERNAL PROBLEM

A. Background metric for relativistic stellar models

We begin with an examination of the internal gravitational field of a body that is not yet perturbed by an external tidal field. The body is spherically symmetric, and the matter consists of a perfect fluid. In light-cone coordinates (v, r, θ^A), the metric is expressed as

$$ds_0^2 = -e^{2\psi} f dv^2 + 2e^{\psi} dv dr + r^2 d\Omega^2, \tag{4.1}$$

with $f = 1 - 2m(r)/r$ and $\psi = \psi(r)$. The Einstein field equations are

$$m' = 4\pi r^2 \rho, \qquad \psi' = \frac{4\pi r}{f}(\rho + p), \tag{4.2}$$

and the equation of hydrostatic equilibrium is

$$p' = -\frac{m + 4\pi r^3 p}{r^2 f}(\rho + p). \tag{4.3}$$

Here ρ is the fluid's proper energy density, and p is the pressure.

These equations can be integrated once an equation of state is specified. The boundary conditions are $m(r=0) = 0$ and $\psi(r=0) = \psi_0$, where ψ_0 is chosen so that ψ vanishes at the stellar surface: $\psi(r=R) = 0$.

B. Light-cone gauge

The internal light-cone gauge is a modified version of the external gauge constructed by Preston and Poisson [8]. We define it properly in this section.

The metric of Eq. (4.1) reveals the meaning of the coordinates (v, r, θ^A) in the background spacetime. We note first that $l_\alpha = -\partial_\alpha v$ is a null vector, so that the surfaces $v =$ constant are null hypersurfaces; they describe light cones that converge toward $r = 0$. The vector

$$l^\alpha = (0, -e^{-\psi}, 0, 0) \tag{4.4}$$

is tangent to the null generators of these light cones, and the expression reveals that θ^A is constant along the generators. In addition, the affine parameter λ that runs along the generators is related to r by $d\lambda = -e^{\psi}dr$. In the interior portion of the spacetime, r is no longer an affine parameter on the null generators; but it still possesses the property of being an areal radius, in the sense that the area of a surface of constant (v, r) is given by $4\pi r^2$.

In the *internal light-cone gauge*, the metric of the perturbed spacetime is presented in coordinates (v, r, θ^A) that possess the same geometrical meaning as in the background spacetime. In particular, v continues to label null hypersurfaces, θ^A continues to be constant along the null generators, and r continues to be related to the affine parameter by $d\lambda = -e^{\psi}dr$. It is easy to show that these statements imply the same conditions,

$$p_{vr} = p_{rr} = p_{rA} = 0, \tag{4.5}$$

that were employed in the external problem. The nonvanishing components of the metric perturbation are therefore p_{vv}, p_{vA}, and p_{AB}. The radial coordinate, however, will lose its meaning as an areal radius in the stellar interior.

In the even-parity sector the perturbation is decomposed as in Eq. (3.2), and the fields h^{lm}_{vv}, j^{lm}_v, K^{lm}, G^{lm} depend (in general) on the coordinates (v, r). An even-parity gauge transformation is generated by the vector field Ξ_α, with components

$$\Xi_v = \sum_{lm} \xi^{lm}_v(v, r) Y^{lm}(\theta^A), \tag{4.6a}$$

$$\Xi_r = \sum_{lm} \xi^{lm}_r(v, r) Y^{lm}(\theta^A), \tag{4.6b}$$

$$\Xi_A = \sum_{lm} \xi^{lm}(v, r) Y^{lm}_A(\theta^A). \tag{4.6c}$$

It can be shown that the condition $h_{vr} = 0$ determines ξ_v, that $h_{rr} = 0$ determines ξ_r, and that $j_r = 0$ determines ξ. The gauge, however, is not determined uniquely. There exists a residual gauge freedom that preserves the geometrical meaning of the coordinates. In the case of v-independent perturbations, the residual gauge freedom is a three-parameter family described by

$$\xi_v = -a_1 e^{2\psi} f + a_2, \tag{4.7a}$$

$$\xi_r = a_1 e^{\psi}, \tag{4.7b}$$

$$\xi = -a_1 r^2 \int^r r'^{-2} e^{\psi(r')} dr' + a_3 r^2. \tag{4.7c}$$

Here we suppressed the lm labels on ξ_v, ξ_r, and ξ, as well as the constants a_1, a_2, and a_3.

In the odd-parity sector the perturbation is decomposed as in Eq. (3.14), and the fields h^{lm}_v, h^{lm}_2 depend (in general) on the coordinates (v, r). An odd-parity gauge transformation is generated by the vector field Ξ_α, with components

$$\Xi_v = \Xi_r = 0, \qquad \Xi_A = \sum_{lm} \xi^{lm}(v, r) X^{lm}_A(\theta^A). \tag{4.8}$$

It can be shown that the condition $h_r = 0$ determines ξ. In this case also there exists a residual gauge freedom that preserves the geometrical meaning of the coordinates. In the case of v-independent perturbations, the residual gauge freedom is a one-parameter family described by

$$\xi = \alpha r^2. \tag{4.9}$$

Here also we suppressed the lm labels on ξ and the constant α.

The decompositions of Eqs. (3.2) and (3.14) can be used to compute $\delta G_{\alpha\beta}$, the perturbation of the Einstein tensor inside the body. The even-parity sector decouples from the odd-parity sector, and the perturbation takes the form of

$$\delta G_{vv} = \sum_{lm} A^{lm}_{vv} Y^{lm}, \tag{4.10a}$$

$$\delta G_{vr} = \sum_{lm} A^{lm}_{vr} Y^{lm}, \tag{4.10b}$$

$$\delta G_{rr} = \sum_{lm} A^{lm}_{rr} Y^{lm}, \tag{4.10c}$$

$$\delta G_{vA} = \sum_{lm} (A^{lm}_v Y^{lm}_A + B^{lm}_v X^{lm}_A), \tag{4.10d}$$

$$\delta G_{rA} = \sum_{lm} (A^{lm}_r Y^{lm}_A + B^{lm}_r X^{lm}_A), \tag{4.10e}$$

$$\delta G_{AB} = \sum_{lm} (A^{lm}_\flat \Omega_{AB} Y^{lm} + A^{lm}_\# Y_{AB} + B^{lm} X^{lm}_{AB}). \tag{4.10f}$$

Here the even-parity fields $A_{vv}, A_{vr}, A_{rr}, A_v, A_r, A_\flat, A_\#$ and the odd-parity fields B_v, B_r, B depend on v and r only. In the case of a stationary perturbation, they depend on r only.

The expressions are too long to be displayed here. In practice, they are easily generated with GRTENSORII [18] by specializing the perturbation to an axisymmetric mode $m = 0$ with a specific multipole order l. With $Y^{lm} = Y(\theta)$ we have $Y_\theta = Y'$, $Y_\phi = 0$, $Y_{\theta\theta} = -\cos\theta Y'/\sin\theta - \frac{1}{2}l(l+1)Y$, $Y_{\theta\phi} = 0$, and $Y_{\phi\phi} = \sin\theta\cos\theta Y' + \frac{1}{2}l(l+1)\sin^2\theta Y$ in the even-parity case, and $X_\theta = 0$, $X_\phi = \sin\theta Y'$, $X_{\theta\theta} = 0$, $X_{\theta\phi} = -\cos\theta Y' - \frac{1}{2}l(l+1)\sin\theta Y$, and $X_{\phi\phi} = 0$ in the odd-parity case. The definition of the metric implements the constraint $Y'' = -\cos\theta Y'/\sin\theta - l(l+1)Y$ on the spherical-harmonic functions, and this simplifies the final expression for the perturbed Einstein tensor.

C. Energy-momentum tensor

We consider *stationary tides* raised by a tidal environment characterized by an electric-type tidal moment $\mathcal{E}_L$ and a magnetic-type tidal moment $\mathcal{B}_L$; these are actually time dependent, but the dependence is sufficiently slow that it can be neglected in the process of integrating the Einstein field equations. The perturbed metric will therefore carry a parametric dependence upon v.

The fluid's velocity vector in the background configuration is given by $u^\alpha = (e^{-\psi} f^{-1/2}, 0, 0, 0)$. In the perturbed configuration it becomes $\hat{u}^\alpha = (\hat{u}^v, 0, 0, 0)$, reflecting the fact that the tide is stationary and does not create motion within the fluid. The time component of the vector changes by virtue of the fact that the metric changes; we have that $\hat{u}^v = e^{-\psi} f^{-1/2} + \delta u^v$, with $\delta u^v = \frac{1}{2} e^{-3\psi} f^{-3/2} p_{vv}$.

After lowering the index on $\hat{u}^\alpha$ with the perturbed metric $g^0_{\alpha\beta} + p_{\alpha\beta}$, we find that $\hat{u}_v = -e^{\psi} f^{1/2} + \delta u_v$, $\hat{u}_r = f^{-1/2} + \delta u_r$, and $\hat{u}_A = \delta u_A$, with

$$\delta u_v = \tfrac{1}{2} e^{-\psi} f^{-1/2} p_{vv}, \tag{4.11a}$$

$$\delta u_r = \tfrac{1}{2} e^{-2\psi} f^{-3/2} p_{vv}, \tag{4.11b}$$

$$\delta u_A = e^{-\psi} f^{-1/2} p_{vA}. \tag{4.11c}$$

These expressions are valid in the light-cone gauge. The perturbation δu_A can be decomposed into even-parity and odd-parity components; the perturbations δu_v and δu_r are necessarily of even parity.

The perturbation in the energy-momentum tensor is generated by the perturbation in u_α, but also by a perturbation in the density ρ and pressure p created by the tide; these are related by the equation of state. We have

$$\delta T_{\alpha\beta} = (\rho + p)(u_\alpha \delta u_\beta + u_\beta \delta u_\alpha) + p p_{\alpha\beta} + (\delta\rho + \delta p) u_\alpha u_\beta + (\delta p) g_{\alpha\beta}, \tag{4.12}$$

and in the light-cone gauge this reads

$$\delta T_{vv} = -\rho p_{vv} + e^{2\psi} f \delta\rho, \tag{4.13a}$$

$$\delta T_{vr} = -e^{\psi} \delta\rho, \tag{4.13b}$$

$$\delta T_{vA} = -\rho p_{vA}, \tag{4.13c}$$

$$\delta T_{rr} = (\rho + p) e^{-2\psi} f^{-2} p_{vv} + f^{-1}(\delta\rho + \delta p), \tag{4.13d}$$

$$\delta T_{rA} = e^{-\psi} f^{-1} (\rho + p) p_{vA}, \tag{4.13e}$$

$$\delta T_{AB} = p p_{AB} + r^2 \delta p \Omega_{AB}. \tag{4.13f}$$

The perturbations δT_{vA}, δT_{rA}, and δT_{AB} can be decomposed into even-parity and odd-parity components; the perturbations δT_{vv}, δT_{vr}, and δT_{rr} are necessarily of even parity.

From Eqs. (4.13) we find that $\delta T_{\alpha\beta}$ is given by

$$\delta T_{vv} = \sum_{lm} Q^{lm}_{vv} Y^{lm}, \tag{4.14a}$$

$$\delta T_{vr} = \sum_{lm} Q^{lm}_{vr} Y^{lm}, \tag{4.14b}$$

$$\delta T_{rr} = \sum_{lm} Q^{lm}_{rr} Y^{lm}, \tag{4.14c}$$

$$\delta T_{vA} = \sum_{lm} (Q^{lm}_v Y^{lm}_A + P^{lm}_v X^{lm}_A), \tag{4.14d}$$

$$\delta T_{rA} = \sum_{lm} (Q^{lm}_r Y^{lm}_A + P^{lm}_r X^{lm}_A), \tag{4.14e}$$

$$\delta T_{AB} = \sum_{lm} (Q^{lm}_\flat \Omega_{AB} Y^{lm} + Q^{lm}_\# Y^{lm}_{AB} + P^{lm} X^{lm}_{AB}). \tag{4.14f}$$

The even-parity fields are

$$Q_{vv} = -\rho h_{vv} + e^{2\psi} f \sigma, \tag{4.15a}$$

$$Q_{vr} = -e^{\psi} \sigma, \tag{4.15b}$$

$$Q_{rr} = (\rho + p) e^{-2\psi} f^{-2} h_{vv} + f^{-1}(\sigma + q), \tag{4.15c}$$

$$Q_v = -\rho j_v, \tag{4.15d}$$

$$Q_r = e^{-\psi} f^{-1} (\rho + p) j_v, \tag{4.15e}$$

$$Q_\flat = r^2 (pK + q), \tag{4.15f}$$

$$Q_\# = r^2 p G, \tag{4.15g}$$

and the perturbations in the density and pressure were also decomposed into spherical harmonics:

$$\delta\rho = \sum_{lm} \sigma^{lm} Y^{lm}, \qquad \delta p = \sum_{lm} q^{lm} Y^{lm}. \tag{4.16}$$

The odd-parity fields are

$$P_v = -\rho h_v, \tag{4.17a}$$

$$P_r = e^{-\psi} f^{-1} (\rho + p) h_v, \tag{4.17b}$$

$$P = p h_2. \tag{4.17c}$$

Information about $\delta\rho$ and δp, or σ and q, can be obtained from the equation of hydrostatic equilibrium. In the perturbed spacetime the equation states that $(\hat{\rho} + \hat{p})\hat{a}_\alpha + \partial_\alpha \hat{p} = 0$, where $\hat{\rho} = \rho + \delta\rho$ is the perturbed density, $\hat{p} = p + \delta p$ is the perturbed pressure, and $\hat{a}_\alpha$ is the perturbed acceleration of the fluid elements. The equation becomes

$$(\rho + p)\delta a_\alpha + (\delta\rho + \delta p) a_\alpha + \partial_\alpha \delta p = 0 \tag{4.18}$$

when expressed in terms of the perturbations $\delta\rho$, δp, and δa_α. The unperturbed acceleration has $a_r = \frac{1}{2} e^{-2\psi} f^{-1} (e^{\psi} f)'$ as its only nonvanishing component, and the perturbation has components

$$\delta a_v = 0, \tag{4.19a}$$

$$\delta a_r = -\tfrac{1}{2} e^{-2\psi} f^{-1} \partial_r p_{vv} + \tfrac{1}{2} e^{-4\psi} f^{-2} (e^{2\psi} f)' p_{vv}, \tag{4.19b}$$

$$\delta a_A = -\tfrac{1}{2} e^{-2\psi} f^{-1} \partial_A p_{vv}. \tag{4.19c}$$

Substitution of Eqs. (4.16) and (4.19), as well as $p_{vv} = \sum_{lm} h^{lm}_{vv} Y^{lm}$, into Eq. (4.18) reveals that

$$q' = \tfrac{1}{2}(\rho + p)e^{-2\psi} f^{-1} h'_{vv} - \tfrac{1}{2}(\rho + p)e^{-4\psi} f^{-2}(e^{2\psi} f)' h_{vv} - \tfrac{1}{2} e^{-2\psi} f^{-1}(e^{2\psi} f)'(\sigma + q) \tag{4.20}$$

and

$$q = \tfrac{1}{2}(\rho + p)e^{-2\psi} f^{-1} h_{vv}. \tag{4.21}$$

If we next differentiate Eq. (4.21) and insert the result within Eq. (4.20), we discover that

$$(\rho + p)' h_{vv} = -(e^{2\psi} f)'(\sigma + q). \tag{4.22}$$

The last two equations allow us to express σ^{lm} and q^{lm} directly in terms of h^{lm}_{vv}; hydrostatic equilibrium implies that these are not independent variables.

D. Perturbation equations: Even-parity sector

The useful combinations of Einstein field equations are

$$E_1 := (A_{vv} - 8\pi Q_{vv}) + e^{\psi} f(A_{vr} - 8\pi Q_{vr}) = 0, \tag{4.23a}$$

$$E_2 := (A_{vv} - 8\pi Q_{vv}) + 2e^{\psi} f(A_{vr} - 8\pi Q_{vr}) + e^{2\psi} f^2(A_{rr} - 8\pi Q_{rr}) = 0, \tag{4.23b}$$

$$E_3 := (A_{rr} - 8\pi Q_{rr}) = 0, \tag{4.23c}$$

$$E_4 := e^{-\psi} r E_2 + 2f(A_v - 8\pi Q_v) + 2e^{\psi} f^2(A_r - 8\pi Q_r) = 0. \tag{4.23d}$$

These are a set of coupled differential equations for the variables $h_{vv}(r)$, $j_v(r)$, $K(r)$, and $G(r)$; the remaining field equations are redundant by virtue of the Bianchi identities. The explicit forms reveal that $E_1 = 0$ is a first-order differential equation for j_v, $E_2 = 0$ is a first-order differential equation for h_{vv}, $E_3 = 0$ is a second-order differential equation for K, and $E_4 = 0$ is a first-order differential equation for G.

The field equations can be manipulated to yield a decoupled equation for the master function

$$\tilde{h}_{vv} := h_{vv} + e^{-\psi}(e^{2\psi} f)' j_v - \frac{1}{2} r^2 f (e^{2\psi} f)' G' = h_{vv} + \frac{2e^{\psi}}{r^2}(m + 4\pi r^3 p) j_v - e^{2\psi} f (m + 4\pi r^3 p) G'. \tag{4.24}$$

This function is gauge invariant, and it joins smoothly with the external version of Eq. (3.9) at $r = R$. The master equation is

$$r^2 \tilde{h}''_{vv} + A r \tilde{h}'_{vv} - B \tilde{h}_{vv} = 0, \tag{4.25}$$

where

$$A = \frac{2}{f}\left[1 - \frac{3m}{r} - 2\pi r^2(\rho + 3p)\right], \tag{4.26a}$$

$$B = \frac{1}{f}\left[l(l+1) - 4\pi r^2(\rho + p)\left(3 + \frac{d\rho}{dp}\right)\right]. \tag{4.26b}$$

The master equation is equivalent to Eq. (27) of Ref. [12], in which $H := e^{-2\psi} f^{-1} \tilde{h}_{vv}$ is used as an alternative choice of dependent variable.

The master equation can be derived by the following procedure. First, integrate the field equation $E_\# := A_\# - 8\pi Q_\# = 0$ and obtain $j_v = \frac{1}{2} r^2 f e^{\psi} G'$. This implies that $\tilde{h}_{vv} = h_{vv}$. Second, make the substitution in the other field equations. The result is that E_1 now involves h_{vv}, G', and G''; E_2 involves h_{vv}, h'_{vv}, K, K', and G'; E_3 involves h_{vv}, K', and K''; and E_4 involves h_{vv}, K, K', G, and G'. Third, differentiate E_2 with respect to r, and use E_1 to eliminate the terms in G'', and E_3 to eliminate the terms in K''. The result is that E'_2 now involves h_{vv}, h'_{vv}, h''_{vv}, K, K', G, and G'. Fourth, construct the linear combination $rE'_2 + aE_2 + bE_4$ and determine the functions a and b that eliminate all terms involving K, K', G, G'. The solution is unique, and the final result is Eq. (4.25).

For numerical integration it is advantageous to make the same substitution as in Eq. (3.10),

$$\tilde{h}_{vv} = -\frac{2}{(l-1)l} r^l e_{vv}(r) \mathcal{E}^{(l)}_m, \tag{4.27}$$

and to rewrite Eq. (4.25) as a second-order differential equation for $e_{vv}(r)$. This function joins smoothly with the external version of Eq. (3.12), and $k_{\rm el}$ is determined by matching the values of the internal and external functions (along with their first derivatives) at $r = R$.

E. Perturbation equations: Odd-parity sector

The useful combinations of field equations are

$$O_1 := (B_v - 8\pi P_v) = 0, \tag{4.28a}$$

$$O_2 := (B_v - 8\pi P_v) + e^{\psi} f(B_r - 8\pi P_r) = 0. \tag{4.28b}$$

The first is a second-order differential equation for h_v, while the second is a first-order differential equation for h_2.

The equation $O_1 = 0$ is fully decoupled, and the perturbation variable h_v is easily shown to be gauge invariant, as it was in the external problem. The master variable for the odd-parity sector is therefore $\tilde{h}_v := h_v$, and the master equation is

$$r^2 \tilde{h}''_v - F r \tilde{h}'_v - G \tilde{h}_v = 0, \tag{4.29}$$

where

$$F = \frac{4\pi r^2}{f}(\rho + p), \tag{4.30a}$$

$$G = \frac{1}{f}\left[l(l+1) - \frac{4m}{r} + 8\pi r^2(\rho + p)\right]. \tag{4.30b}$$

This equation is equivalent to Eq. (31) of Ref. [12], in which $\psi := r\tilde{h}_v' - 2\tilde{h}_v$ is used as an alternative choice of dependent variable. The function $\tilde{h}_v$ joins smoothly with the external version of Eq. (3.19) at $r = R$.

For numerical integration it is advantageous to make the same substitution as in Eq. (3.20),

$$\tilde{h}_v = \frac{2}{3(l-1)l} r^{l+1} b_v(r) \mathcal{B}_m^{(l)}, \tag{4.31}$$

and to rewrite Eq. (4.29) as a second-order differential equation for $b_v(r)$. This function joins smoothly with the external version of Eq. (3.22), and $k_{\rm mag}$ is determined by matching the values of the internal and external functions (along with their first derivatives) at $r = R$.

V. IMPLEMENTATION FOR POLYTROPES

The relativistic Love numbers $k_{\rm el}$ and $k_{\rm mag}$ are determined by the numerical integration of Eqs. (4.25) and (4.29) and matching with the external solutions at $r = R$. This defines a simple computational procedure that can be implemented for any choice of equation of state. In this section we describe the steps that are involved when the polytropic form

$$p = K\rho^{1+1/n} \tag{5.1}$$

is adopted; here K and the polytropic index n are constants. We choose, however, to deviate from the procedure just outlined: Instead of integrating the master equations for the variables $\tilde{h}_{vv}$ and $\tilde{h}_v$, we integrate *the complete set of independent field equations*. This allows us to calculate all components of the metric perturbation, and matching them across $r = R$ determines, in addition to the Love numbers, the gauge parameters a_1, a_2, a_3, and α that are automatically selected by the internal solution.[1]

A. Unperturbed stellar model

The numerical integration of Eqs. (4.2) and (4.3) is conveniently accomplished by introducing the dimensionless variables θ, μ, and ξ defined by

$$\rho = \rho_c\theta^n, \qquad p = p_c\theta^{n+1}, \qquad m = m_0\mu, \qquad r = r_0\xi. \tag{5.2}$$

Here $\rho_c := \rho(r=0)$ is the central density, and $p_c := K\rho_c^{1+1/n}$ is the central pressure. The units of mass and radius are chosen to be

$$m_0 := 4\pi r_0^3\rho_c, \qquad r_0^2 := \frac{(n+1)p_c}{4\pi\rho_c^2}, \tag{5.3}$$

so as to simplify the form of the field equations.

It is useful to introduce also a "relativistic factor"

$$b := p_c/\rho_c, \tag{5.4}$$

which determines the degree with which the stellar model is relativistic. In terms of this we have $\rho_c = b^n/K^n$, $p_c = b^{n+1}/K^n$, and b can be used in place of ρ_c to label a stellar model, given a choice (K, n) of equation of state. We also note the relation $m_0/r_0 = (n+1)b$. We find that the units m_0 and r_0 vary with b even when the equation of state is fixed. To eliminate this dependence it is useful to define the alternative units

$$M_0 = \frac{(n+1)^{3/2}}{\sqrt{4\pi}} K^{n/2}, \qquad R_0 = \sqrt{\frac{n+1}{4\pi}} K^{n/2}, \tag{5.5}$$

which do not depend on b. We have that $m_0 = M_0 b^{(3-n)/2}$ and $r_0 = R_0 b^{(1-n)/2}$.

In terms of the dimensionless variables, the field equations (4.2) and (4.3) become

$$\frac{d\mu}{d\xi} = \xi^2\theta^n, \tag{5.6a}$$

$$\frac{d\psi}{d\xi} = (n+1)b\frac{\xi\theta^n(1+b\theta)}{f}, \tag{5.6b}$$

$$\frac{d\theta}{d\xi} = -\frac{(\mu + b\xi^3\theta^{n+1})(1+b\theta)}{\xi^2 f}, \tag{5.6c}$$

with $f = 1 - 2(n+1)b\mu/\xi$. The boundary conditions are $\theta(\xi=0) = 1$, $\mu(\xi=0) = 0$, and $\psi(\xi=0) = \psi_0$. In the limit $b \to 0$ the model becomes nonrelativistic, and the equations for μ and θ can be combined into the well-known Lane-Emden equation; in the limit the equation for ψ becomes irrelevant.

The formulation of Eq. (5.6) is not optimal from a numerical point of view. For accurate integrations it is better to use the variable $\nu := \mu/\xi^3$ instead of μ, and $x := \ln\xi$ instead of ξ. The system of equations becomes

$$\frac{d\nu}{dx} = \theta^n - 3\nu, \tag{5.7a}$$

$$\frac{d\psi}{dx} = (n+1)b\xi^2 f^{-1}\theta^n(1+b\theta), \tag{5.7b}$$

$$\frac{d\theta}{dx} = -\xi^2 f^{-1}(\nu + b\theta^{n+1})(1+b\theta), \tag{5.7c}$$

with $f = 1 - 2(n+1)b\xi^2\nu$. The integration begins at a large and negative value of x, so that $\xi = e^x$ is small, with the starting values

[1]There is no strong rationale for proceeding in this way. The honest truth is that we became aware of Eq. (4.25) only after completing the numerical work. We derived the master equation after noticing its appearance in Refs. [2,12] and wondering why our formulation was more complicated than theirs.

$$\nu = \frac{1}{3} - \frac{n}{30}(1+b)(1+3b)\xi^2 + \frac{n}{2520}(1+b)(1+3b)[8n-5+(18n-20)b+(15+30n)b^2]\xi^4 + O(\xi^6), \quad (5.8a)$$

$$\theta = 1 - \frac{1}{6}(1+b)(1+3b)\xi^2 + \frac{1}{360}(1+b)(1+3b)[3n-2nb+(30+15n)b^2]\xi^4 + O(\xi^6), \quad (5.8b)$$

$$\psi = \psi_0 + \frac{1}{2}(n+1)b(1+b)\xi^2 - \frac{1}{24}(n+1)b(1+b)[n-3b+(3+3n)b^2]\xi^4 + O(\xi^6). \quad (5.8c)$$

The integration stops at $\xi = \xi_1$, where θ goes to zero, and ψ_0 is chosen so that $\psi(\xi_1) = 0$. The stellar mass and radius are then given by

$$M = M_0 b^{(3-n)/2}\xi_1^3\nu(\xi_1), \qquad R = R_0 b^{(1-n)/2}\xi_1, \quad (5.9)$$

in the units of Eq. (5.5). The compactness of the body is measured by $C := 2M/R = 2(n+1)b\xi_1^2\nu(\xi_1)$; this is dimensionless, and therefore independent of the units M_0 and R_0.

B. Perturbation: Even-parity sector

The perturbation equations (4.23) are simplified by involving the background field equations (4.2) and (4.3). They are also simplified by making the substitutions of Eqs. (5.2), (5.3), and (5.4); we therefore write $\rho = \rho_c\theta^n$, $p = p_c\theta^{n+1}$, $r = r_0\xi$, and $m = m_0\xi^3\nu$, where $\rho_c = (n+1)b/(4\pi r_0^2)$, $p_c = (n+1)b^2/(4\pi r_0^2)$, and $m_0 = (n+1)br_0$, with θ and ν (as well as ψ) depending on ξ. Finally, we use the fact that a term ρ' in the perturbation equations can be related to p' by the equation $\rho' = (d\rho/dp)p'$, with $d\rho/dp$ determined by the equation of state.

Another useful set of substitutions is the one displayed in Eqs. (3.3), along with

$$r_0^2 K = \frac{2}{(l-1)l(l+2)(l+3)} r^{l+2} e_{10}(\xi)\mathcal{E}_m^{(l)}, \quad (5.10)$$

in which we replace the original variables with the radial functions e_1, e_4, e_7, and e_{10}. These replacements are motivated by an analysis of the perturbation equations for small values of r, which reveals that h_{vv} behaves as r^l, j_v as r^{l+1}, G as r^l, and K as r^{l+2}. The numerical factor in front of e_{10} is inserted to simplify the form of the small-r expansion of K, as we shall see below.

The final expression of the perturbation equations is

$$0 = E_1 = -\xi e_4' + (l+1)e^{-\psi}f^{-1}e_1 - f^{-1}A_1e_4, \quad (5.11a)$$

$$0 = E_2 = -\xi e_1' + \frac{1}{2}f^{-1}A_2e_1 + le^{\psi}f^{-1}B_2e_4 - \frac{1}{2}(l-1)(l+2)e^{2\psi}e_7 + \frac{1}{2(l+3)}e^{2\psi}C_2\xi^2e_{10} - \frac{1}{(l+2)(l+3)}e^{2\psi}B_2\xi^3e_{10}', \quad (5.11b)$$

$$0 = E_3 = -\xi^2 e_{10}'' + (l+2)(l+3)e^{-2\psi}f^{-2}A_3e_1 - (l+2)f^{-1}B_3e_{10} - f^{-1}C_3\xi e_{10}', \quad (5.11c)$$

$$0 = E_4 = -\xi e_7' + \frac{l(l+1)}{2(l-1)(l+2)}e^{-2\psi}f^{-2}A_4e_1 + \frac{l}{(l-1)(l+2)}e^{-\psi}f^{-2}B_4e_4 - \frac{1}{2}l[l+3-4(n+1)b\xi^2\nu]f^{-1}e_7 + \frac{l(l+1)}{2(l-1)(l+2)(l+3)}f^{-1}C_4\xi^2e_{10} - \frac{l(l+1)}{(l-1)(l+2)^2(l+3)}[(n+1)b(\nu+b\theta^{n+1})]f^{-1}\xi^5e_{10}', \quad (5.11d)$$

where a prime indicates differentiation with respect to ξ, and

$$A_1 = l+1-2(n+1)b\xi^2[(l+2)\nu + b\theta^{n+1}], \quad (5.12a)$$

$$A_2 = (l-2)(l+1) + 2(n+1)b\xi^2[2(l+1)\nu - \theta^n(1+b\theta)], \quad (5.12b)$$

$$B_2 = 1-(n+1)b\xi^2(\nu - b\theta^{n+1}), \quad (5.12c)$$

$$C_2 = l-3+2(n+1)b\xi^2(\nu - b\theta^{n+1}), \quad (5.12d)$$

$$A_3 = n\theta^{n-1} + (4n+3)b\theta^n + 3(n+1)b^2\theta^{n+1}, \quad (5.12e)$$

$$B_3 = l+3-(n+1)b\xi^2[2(l+3)\nu + \theta^n(1+b\theta)], \quad (5.12f)$$

$$C_3 = 2(l+3)-(n+1)b\xi^2[4(l+3)\nu + \theta^n(1+b\theta)], \quad (5.12g)$$

$$A_4 = (l-1)(l+2) + 2(n+1)b\xi^2[2\nu - \theta^n(1+b\theta)], \quad (5.12h)$$

$$B_4 = (l-1)(l+2) - (n+1)b\xi^2[(l^2+l-4)\nu - l(l+1)b\theta^{n+1}], \quad (5.12i)$$

$$C_4 = l-1-2(n+1)b\xi^2(\nu + b\theta^{n+1}). \quad (5.12j)$$

A small-ξ expansion of these equations, using Eqs. (5.8), reveals that $e_1 = a_0 + O(\xi^2)$, $e_4 = a_0 e^{-\psi_0} + O(\xi^2)$, $e_7 = a_0 e^{-2\psi_0} + O(\xi^2)$, and $e_{10} = a_0 e^{-2\psi_0}(1+b)[3(n+1)b + n] + O(\xi^2)$, where a_0 is a parameter that must be determined by matching the internal and external perturbations at the stellar boundary.

The perturbation equations are easily written as a first-order dynamical system for the variables $u_1 := e_1$, $u_2 := e_4$, $u_3 = e_7$, $u_4 := e_{10}$, and $u_5 := \xi e_{10}'$. The numerical integration is carried out with $x := \ln\xi$ as the independent variable, and the differential equations are integrated simultaneously with Eqs. (5.7) to determine the unperturbed stellar model. The integration proceeds from a large and negative value of x, for which $\xi = e^x$ is small, and it stops at $\xi = \xi_1$ where θ goes to zero.

The term $n\theta^{n-1}$ in A_3 originates from a term involving $d\rho/dp \propto \theta^{-1}$ that multiplies $\rho \propto \theta^n$ in the field equation for K (or e_{10}). This term diverges at the stellar boundary when $n < 1$. The singularity is integrable, however, and it can be shown that the solution for $K(r)$ (or e_{10}) is actually well behaved at the boundary. The divergence of A_3 nevertheless causes issues in the numerical integration of the perturbation equations. For this reason, the accuracy achieved for $n < 1$ is limited compared with the accuracy obtained for $n > 1$.

The internal perturbation must match the external perturbation at $\xi = \xi_1$, or $r = R$, the position of the stellar boundary. The five internal functions e_1, e_4, e_7, e_{10}, and $\xi e_{10}'$ depend on one free parameter a_0. The external functions, on the other hand, depend on three gauge parameters a_1, a_2, and a_3, as well as the electric-type Love number $k_{\rm el}$. The five matching conditions determine the five parameters uniquely, including the Love number.

We suppose that the internal functions $u_1, \cdots, u_5$ are determined by setting $a_0 \equiv 1$ in the numerical integrations. The desired functions $e_1, \cdots, e_{10}$ then differ from these by an overall multiplicative factor that we denote λ^{-1}. We have

$$e_1^{\rm in} = \lambda^{-1} u_1, \qquad (5.13a)$$
$$e_4^{\rm in} = \lambda^{-1} u_2, \qquad (5.13b)$$
$$e_7^{\rm in} = \lambda^{-1} u_3, \qquad (5.13c)$$
$$e_{10}^{\rm in} = \lambda^{-1} u_4, \qquad (5.13d)$$
$$\xi \frac{de_{10}^{\rm in}}{d\xi} = \lambda^{-1} u_5, \qquad (5.13e)$$

and the matching conditions are

$$e_1^{\rm in} = e_1^{\rm out}, \qquad (5.14a)$$
$$e_4^{\rm in} = e_4^{\rm out}, \qquad (5.14b)$$
$$e_7^{\rm in} = e_7^{\rm out}, \qquad (5.14c)$$
$$e_{10}^{\rm in} = e_{10}^{\rm out}, \qquad (5.14d)$$
$$\xi \frac{de_{10}^{\rm in}}{d\xi} = \xi \frac{de_{10}^{\rm out}}{d\xi}, \qquad (5.14e)$$

where each side of the equation is evaluated at $\xi = \xi_1$. The external expressions for e_1, e_4, and e_7 are presented in Eqs. (1.6) and (1.7), and these must be modified by the gauge adjustments of Eqs. (3.7).

The function e_{10} is related to K by Eq. (5.10), and the external expression for K is given by Eq. (3.8). This equation and its derivative with respect to r imply that at $\xi = \xi_1$,

$$e_{10}^{\rm out} = 2(l+2)(l+3)C^l \xi_1^{-2}[a_2 + a_3(2M/R)], \qquad (5.15a)$$
$$\xi \frac{de_{10}^{\rm out}}{d\xi} = -2(l+2)(l+3)C^l \xi_1^{-2} \times [(l+2)a_2 + (l+3)a_3(2M/R)], \qquad (5.15b)$$

where $C := 2M/R$ is the compactness factor. These equations can be solved for a_2 and a_3. Involving also the matching equations and Eqs. (5.13), we arrive at

$$\lambda a_2 = \frac{\xi_1^2}{2(l+2)(l+3)C^l}[(l+3)u_4 + u_5], \qquad (5.16a)$$
$$\lambda a_3 = -\frac{\xi_1^2}{2(l+2)(l+3)C^{l+1}}[(l+2)u_4 + u_5]. \qquad (5.16b)$$

We see that the gauge parameters a_2 and a_3, rescaled by the unknown coefficient λ, are determined by the numerical values obtained for u_4 and u_5.

To solve the remaining matching equations we transfer the a_2 and a_3 terms from the right-hand side of Eqs. (3.7) to the left-hand side. Taking Eqs. (5.16) into account, we form the combinations

$$w_1 := u_1 + \frac{C\xi_1^2}{2(l+2)(l+3)}[(l+2)u_4 + u_5], \qquad (5.17a)$$
$$w_2 := u_2 - \frac{(l+1)\xi_1^2}{2(l+2)(l+3)}[(l+2)u_4 + u_5], \qquad (5.17b)$$
$$w_3 := u_3 - \frac{\xi_1^2}{(l+2)(l+3)}[(l+3)u_4 + u_5], \qquad (5.17c)$$

which can be determined numerically. Involving now Eqs. (1.6), the matching conditions take the explicit form

$$w_1 = A_1 \cdot \lambda + 2B_1 \cdot (\lambda k_{\rm el}) - lC \cdot (\lambda C^{l+1} a_1), \qquad (5.18a)$$
$$w_2 = A_4 \cdot \lambda - 2\frac{l+1}{l} B_4 \cdot (\lambda k_{\rm el}) + [(l-1)(l+2) + 2C] \cdot (\lambda C^{l+1} a_1), \qquad (5.18b)$$
$$w_3 = A_7 \cdot \lambda + 2B_7 \cdot (\lambda k_{\rm el}) + 2l \cdot (\lambda C^{l+1} a_1); \qquad (5.18c)$$

in these expressions the functions A_n and B_n are evaluated at $r = R$, or $2M/r = C$.

If we define a vector $\boldsymbol{w} = (w_1, w_2, w_3)$ of numerical quantities, and another vector $\boldsymbol{p} = (\lambda, \lambda k_{\rm rel}, \lambda C^{l+1} a_1)$ of unknown parameters, these equations take the form of the matrix equation $\boldsymbol{w} = \mathsf{M}\boldsymbol{p}$, with a matrix M that is known analytically. Solving for $\boldsymbol{p}$, the Love number is finally determined by $k_{\rm el} = p_2/p_1$.

C. Perturbation: Odd-parity sector

To arrive at the final form of the perturbation equations (4.28), we follow the same steps as in the even-parity sector. These include making the substitutions of Eqs. (3.15), to replace the original variables h_v and h_2 with the radial functions b_4 and b_7.

The perturbation equations are

$$0 = O_1 = -\xi^2 b_4'' - f^{-1}F_1\xi b_4' + f^{-1}G_1 b_4, \tag{5.19a}$$

$$0 = O_2 = -\xi b_7' - l b_7 + l e^{-\psi} f^{-1} b_4, \tag{5.19b}$$

with

$$F_1 = 2(l+1) - (n+1)b\xi^2[4(l+1)\nu + \theta^n(1+b\theta)], \tag{5.20a}$$

$$G_1 = (n+1)b\xi^2[2(l-1)(l+2)\nu + (l+3)\theta^n(1+b\theta)]. \tag{5.20b}$$

A small-ξ expansion of these equations reveals that $b_4 = \alpha_0 + O(\xi^2)$ and $b_7 = \alpha_0 e^{-\psi_0} + O(\xi^2)$, where α_0 is a parameter that must be determined by matching the internal and external perturbations at the stellar boundary.

The perturbation equations are easily written as a first-order dynamical system for the variables $v_1 := b_4$, $v_2 := \xi b_4'$, and $v_3 := b_7$.

The internal perturbation must match the external perturbation at $\xi = \xi_1$, or $r = R$, the position of the stellar boundary. The three internal functions b_4, $\xi b_4'$, and b_7 depend on one free parameter, α_0. The external functions, on the other hand, depend on one gauge parameter, α, as well as the magnetic-type Love number $k_{\rm mag}$. The three matching conditions determine the three parameters uniquely, including the Love number.

We suppose that the perturbation equations for v_1, v_2, and v_3 are integrated with $\alpha_0 \equiv 1$. The desired internal functions b_4 and b_7 are then given by

$$b_4^{\rm in} = \lambda^{-1} v_1, \tag{5.21a}$$

$$\xi \frac{db_4^{\rm in}}{d\xi} = \lambda^{-1} v_2, \tag{5.21b}$$

$$b_7^{\rm in} = \lambda^{-1} v_3, \tag{5.21c}$$

where λ is an unknown constant. The matching conditions are

$$b_4^{\rm in} = b_4^{\rm out}, \tag{5.22a}$$

$$\xi \frac{db_4^{\rm in}}{d\xi} = \xi \frac{db_4^{\rm out}}{d\xi}, \tag{5.22b}$$

$$b_7^{\rm in} = b_7^{\rm out}, \tag{5.22c}$$

where each side of the equation is evaluated at $\xi = \xi_1$. The external expressions for b_4 and b_7 are presented in Eqs. (1.6), together with the gauge adjustment of Eq. (3.18). We observe that b_4 is gauge invariant, and that the purpose of the matching equation for b_7 is to determine the (uninteresting) gauge parameter α.

We focus on the two equations involving b_4. Using Eqs. (1.6), we find that the explicit form of the matching conditions is

$$v_1 = A_4 \cdot \lambda - 2\frac{l+1}{l} B_4 \cdot (\lambda k_{\rm mag}), \tag{5.23a}$$

$$v_2 = -CA_4' \cdot \lambda + 2\frac{l+1}{l}[CB_4' + (2l+1)B_4] \cdot (\lambda k_{\rm mag}). \tag{5.23b}$$

In these expressions the functions A_4, $A_4' := dA_4/dz$, B_4, and $B_4' := dB_4/dz$ are evaluated at $z := 2M/r = C$.

If we define a vector $\boldsymbol{v} = (v_1, v_2)$ of numerical quantities, and another vector $\boldsymbol{p} = (\lambda, \lambda k_{\rm mag})$ of unknown parameters, these equations take the form of the matrix equation $\boldsymbol{v} = \mathsf{M}\boldsymbol{p}$, with a matrix M that is known analytically. Solving for $\boldsymbol{p}$, the Love number is finally determined by $k_{\rm mag} = p_2/p_1$.

To evaluate the derivatives of A_4 and B_4 with respect to z, we use the well-known property of hypergeometric functions that $(d/dz)F(a, b; c; z) = (ab/c)F(a+1, b+1; c+1; z)$.

VI. NUMERICAL RESULTS

The computations presented in this work were generated with two independent codes, one written by each author. Consistency between our results provides evidence that each set of computations was carried our correctly, and the comparison allows us to estimate the numerical accuracy of our results.

The background spacetime is constructed by solving the Einstein field equations for a spherical matter configuration with a polytropic equation of state. The equations were formulated in Sec. VA, and the system of equations (5.7) is integrated numerically for selected values of the polytropic index n. The integration begins at a large and negative value of the radial variable $x = \ln\xi$, using the starting values listed in Eqs. (5.8). It proceeds until θ changes sign at the stellar boundary, $x = x_1$. In the first code, the integration is performed using the Bulirsh-Stoer method as implemented in the *Numerical Recipes* routine `bsstep`, which is embedded within `odeint`; we use the Second Edition of *Numerical Recipes* [19], and the code is written in C++. In the second code, the integration is performed using the embedded Runge-Kutta Prince-Dormand method as implemented in the *GNU Scientific Library* routine `rk8pd`, which is embedded within `odeiv`; we use version 1.9 of the libraries [20], and the code is written in C. In each code all floating-point operations are carried out with double precision. The accuracy of the integration is determined by the integrator's tolerance ϵ and the errors of order ξ^6 that are incorporated in the starting values. As Eqs. (5.7) are exceptionally well conditioned toward numerical integration, a high degree of accuracy can easily be achieved. We estimate that our stellar configurations are computed accurately to at least 12 significant digits.

The stellar boundary is identified with the help of a bisection search for the solution to $\theta(x) = 0$. In the first code this is carried out with the *Numerical Recipe* routine `zbrent`; the search is loosely bracketed between the values $x_0 < x_1$ (where θ is positive) and $x_2 > x_1$ (where θ is negative). In the second code this is carried out with the *GNU Scientific Library* routine `brent`, using a similar bracketing method. The search is carried out with high accuracy, again of the order of 12 significant digits.

The even-parity perturbation equations (5.11) are next integrated for selected values of n and l, simultaneously with the background field equations (5.7). Once more, the integration begins at a large and negative value of x, using the starting values derived in Sec. V B, and it proceeds up to $x = x_1$. In the first code we continue to use `bsstep` and `odeint`, and the caption of Table II discusses the accuracy of these integrations. In the second code we continue to use `rk8pd` and `odeiv`; the tolerance of the integrator is set uniformly to $\epsilon = 1.0\text{e} - 12$, and all integrations begin at $x = -10.0$. Each code returns the values of u_1, u_2, u_3, u_4, and u_5 at the stellar boundary.

The odd-parity equations (5.19) are integrated in exactly the same way. Here the codes return the values of v_1, v_2, and v_3 at the stellar boundary.

The matching problem of Eqs. (5.18) requires the numerical solution of the matrix equation $\boldsymbol{w} = \mathsf{M}\boldsymbol{p}$, where $\boldsymbol{w}$ is constructed from the perturbations, M is known analytically, and $\boldsymbol{p}$ is the vector of unknown parameters, which include the electric-type Love number $k_{\rm el}$. In the first code the system of equations is solved by performing an LU decomposition of the matrix M, and this is handled by the *Numerical Recipes* routines `ludcmp` and `lubksb`. In the second code the LU decomposition is handled by the *GNU Scientific Library* routines $\texttt{gsl}_\texttt{l}\texttt{inalg}_\texttt{L}\texttt{U}_\texttt{d}\texttt{ecomp}$ and $\texttt{gsl}_\texttt{l}\texttt{inalg}_\texttt{L}\texttt{U}_\texttt{s}\texttt{olve}$. In view of the small number of equations involved (three), this task is essentially carried out at machine precision. The final output is $k_{\rm el}$.

The matching problem of Eqs. (5.23) is handled in exactly the same way. Here the final output is the magnetic-type Love number $k_{\rm mag}$.

Our results are presented in the figures displayed in Sec. I and in the tables provided in the Appendix. The electric-type and magnetic-type Love numbers are computed for selected values of n and l, as functions of the relativistic parameter $b := p_c/\rho_c$ and the compactness

TABLE II. Integration errors for even-parity perturbations. For each selected value of n, the first row shows the value of ϵ, the integrator's tolerance. When $\epsilon = 1.0\text{e} - 12$ the integrations are started at $x = -7.0$, so that the errors in the starting values are of the order of $1.0\text{e} - 12$. When $\epsilon > 1.0\text{e} - 12$ the integrations are started at $x = -6.5$, so that the errors in the starting values are of the order of $1.0\text{e} - 11$. For the odd-parity equations the tolerance of the integrator is set uniformly to $\epsilon = 1.0\text{e} - 12$, and all integrations begin at $x = -7.0$. The second column shows δ, an intrinsic measure of the accuracy of our results. This is defined as $\delta := |\nu_{\rm model} - \nu_{\rm pert}|/\nu_{\rm model}$, where $\nu_{\rm model}$ is the value of ν at the stellar boundary $\xi = \xi_1$ as determined with exquisite precision by integrating the stellar-model equations only, while $\nu_{\rm pert}$ is the value as determined by also integrating the perturbation equations. The least accurate determinations are for small values of b; the accuracy typically improves by 2 orders of magnitude at larger values of b. For reasons that were explained in Sec. V B, when $n < 1$ the accuracy that can be achieved for the even-parity perturbations is more limited than what is achieved for ν; for these cases δ gives an overestimate of the true accuracy. For $n > 1$, and for the odd-parity perturbations, δ should be an accurate measure of our accuracy.

	$l = 2$	$l = 3$	$l = 4$	$l = 5$
$n = 0.50$	$\epsilon = 1.0\text{e} - 10$	$\epsilon = 1.0\text{e} - 10$	$\epsilon = 1.0\text{e} - 10$	$\epsilon = 1.0\text{e} - 10$
	$\delta < 1.2\text{e} - 10$	$\delta < 1.2\text{e} - 10$	$\delta < 1.2\text{e} - 10$	$\delta < 1.2\text{e} - 10$
$n = 0.75$	$\epsilon = 3.0\text{e} - 11$	$\epsilon = 3.0\text{e} - 11$	$\epsilon = 3.0\text{e} - 11$	$\epsilon = 3.0\text{e} - 11$
	$\delta < 8.6\text{e} - 11$	$\delta < 6.6\text{e} - 11$	$\delta < 8.6\text{e} - 11$	$\delta < 8.6\text{e} - 11$
$n = 1.00$	$\epsilon = 1.0\text{e} - 12$	$\epsilon = 3.0\text{e} - 11$	$\epsilon = 3.0\text{e} - 11$	$\epsilon = 3.0\text{e} - 11$
	$\delta < 1.6\text{e} - 09$	$\delta < 1.7\text{e} - 09$	$\delta < 2.7\text{e} - 10$	$\delta < 4.0\text{e} - 11$
$n = 1.25$	$\epsilon = 1.0\text{e} - 12$	$\epsilon = 3.0\text{e} - 11$	$\epsilon = 3.0\text{e} - 11$	$\epsilon = 3.0\text{e} - 11$
	$\delta < 9.5\text{e} - 11$	$\delta < 9.6\text{e} - 11$	$\delta < 9.5\text{e} - 11$	$\delta < 9.5\text{e} - 11$
$n = 1.50$	$\epsilon = 1.0\text{e} - 12$	$\epsilon = 3.0\text{e} - 11$	$\epsilon = 3.0\text{e} - 11$	$\epsilon = 3.0\text{e} - 11$
	$\delta < 7.2\text{e} - 11$	$\delta < 7.2\text{e} - 11$	$\delta < 7.2\text{e} - 11$	$\delta < 7.2\text{e} - 11$
$n = 1.75$	$\epsilon = 1.0\text{e} - 12$	$\epsilon = 1.0\text{e} - 12$	$\epsilon = 7.0\text{e} - 11$	$\epsilon = 7.0\text{e} - 11$
	$\delta < 9.2\text{e} - 11$	$\delta < 9.2\text{e} - 11$	$\delta < 9.2\text{e} - 11$	$\delta < 9.2\text{e} - 11$
$n = 2.00$	$\epsilon = 1.0\text{e} - 12$	$\epsilon = 1.0\text{e} - 12$	$\epsilon = 1.0\text{e} - 12$	$\epsilon = 3.0\text{e} - 11$
	$\delta < 2.4\text{e} - 12$	$\delta < 2.4\text{e} - 12$	$\delta < 2.4\text{e} - 12$	$\delta < 2.4\text{e} - 12$

$C := 2M/R$. The allowed interval begins at $b = 0$ and $C = 0$, where the equations reduce to their Newtonian limit, and ends at $b = b_{\max}$ and $C = C_{\max}$, where the stellar configuration achieves its maximum mass. Each table caption discusses the estimated accuracy of our results. Overall, we claim an approximate accuracy of nine significant digits for the Love numbers (with some exceptions, as detailed in the table captions).

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APPENDIX: TABLES OF RELATIVISTIC LOVE NUMBERS

TABLE III. Love numbers for $n = 0.50$ and $l = 2$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{\rm el} = 4.491\,539\,995\,415\text{e} - 01$. This provides evidence that our results for the electric-type Love numbers are accurate to five significant digits. We believe that our results for the magnetic-type Love numbers are accurate to nine significant digits.

b	$2M/R$	$k_{\rm el}$	$k_{\rm mag}$
0.000 000 000 0	0.000 000 000 0	4.491 529 558 4e − 01	0.000 000 000 0e + 00
0.016 296 296 3	0.062 785 986 5	3.685 759 957 3e − 01	1.689 683 155 6e − 03
0.065 185 185 2	0.208 540 613 2	2.211 710 364 3e − 01	4.159 652 537 2e − 03
0.146 666 666 7	0.363 616 545 4	1.152 849 348 4e − 01	4.852 293 159 3e − 03
0.260 740 740 7	0.488 341 406 6	6.019 568 639 3e − 02	4.258 349 339 5e − 03
0.407 407 407 4	0.577 286 792 3	3.382 566 057 1e − 02	3.375 910 374 8e − 03
0.586 666 666 7	0.637 953 773 6	2.087 983 094 9e − 02	2.627 166 653 9e − 03
0.798 518 518 5	0.678 953 959 1	1.410 789 275 2e − 02	2.080 927 163 2e − 03
1.042 962 963 0	0.706 817 126 4	1.031 160 579 8e − 02	1.700 416 110 3e − 03
1.320 000 000 0	0.725 950 238 2	8.045 374 229 2e − 03	1.437 217 362 2e − 03

TABLE IV. Love numbers for $n = 0.75$ and $l = 2$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{\rm el} = 3.434\,291\,771\,770\text{e} - 01$. This provides evidence that our results for the electric-type Love numbers are accurate to nine significant digits. We believe that our results for the magnetic-type Love numbers are also accurate to nine significant digits.

b	$2M/R$	$k_{\rm el}$	$k_{\rm mag}$
0.000 000 000 0	0.000 000 000 0	3.434 291 776 1e − 01	0.000 000 000 0e + 00
0.009 259 259 3	0.036 329 314 4	3.052 845 667 2e − 01	8.495 821 704 4e − 04
0.037 037 037 0	0.129 439 333 2	2.211 467 798 4e − 01	2.515 174 743 9e − 03
0.083 333 333 3	0.245 587 844 2	1.405 570 291 8e − 01	3.645 594 442 6e − 03
0.148 148 148 1	0.356 460 354 9	8.489 011 906 4e − 02	3.870 055 810 6e − 03
0.231 481 481 5	0.448 520 088 7	5.170 838 371 9e − 02	3.528 278 134 8e − 03
0.333 333 333 3	0.519 439 341 0	3.285 745 402 5e − 02	3.002 696 363 3e − 03
0.453 703 703 7	0.571 983 982 7	2.209 137 445 5e − 02	2.497 930 521 5e − 03
0.592 592 592 6	0.610 158 941 0	1.575 596 721 5e − 02	2.082 769 608 7e − 03
0.750 000 000 0	0.637 610 726 0	1.188 236 781 2e − 02	1.762 906 267 7e − 03

TABLE V. Love numbers for $n = 1.00$ and $l = 2$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{el} = 2.599\,088\,771\,480e - 01$. This provides evidence that our results for the electric-type Love numbers are accurate to nine significant digits. We believe that our results for the magnetic-type Love numbers are also accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.000 000 000 0	0.000 000 000 0	2.599 088 773 2e − 01	0.000 000 000 0e + 00
0.005 432 098 8	0.021 188 776 0	2.419 893 748 6e − 01	4.183 250 050 0e − 04
0.021 728 395 1	0.078 832 645 9	1.976 136 279 0e − 01	1.387 454 444 4e − 03
0.048 888 888 9	0.158 617 817 3	1.459 460 111 7e − 01	2.341 856 264 3e − 03
0.086 913 580 2	0.244 994 075 7	1.013 520 147 0e − 01	2.909 202 515 2e − 03
0.135 802 469 1	0.326 497 763 8	6.865 673 891 1e − 02	3.047 077 816 8e − 03
0.195 555 555 6	0.397 110 035 6	4.667 256 471 3e − 02	2.893 214 667 8e − 03
0.266 172 839 5	0.455 036 029 6	3.243 879 869 4e − 02	2.603 059 766 1e − 03
0.347 654 321 0	0.500 890 569 3	2.329 306 332 1e − 02	2.282 421 004 8e − 03
0.440 000 000 0	0.536 309 247 3	1.736 010 515 1e − 02	1.985 444 548 1e − 03

TABLE VI. Love numbers for $n = 1.25$ and $l = 2$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{el} = 1.943\,393\,766\,752e - 01$. This provides evidence that our results for the electric-type Love numbers are accurate to nine significant digits. We believe that our results for the magnetic-type Love numbers are also accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.000 000 000 0	0.000 000 000 0	1.943 393 766 5e − 01	0.000 000 000 0e + 00
0.003 703 703 7	0.014 091 088 1	1.848 704 632 3e − 01	2.290 853 886 0e − 04
0.014 814 814 8	0.053 521 847 7	1.600 756 489 2e − 01	8.022 688 031 6e − 04
0.033 333 333 3	0.110 978 273 3	1.281 855 820 3e − 01	1.464 663 439 8e − 03
0.059 259 259 3	0.177 467 002 7	9.702 996 948 1e − 02	1.989 458 244 0e − 03
0.092 592 592 6	0.244 978 923 6	7.104 924 782 7e − 02	2.276 223 272 3e − 03
0.133 333 333 3	0.307 905 147 2	5.137 647 658 6e − 02	2.339 329 319 8e − 03
0.181 481 481 5	0.363 176 478 1	3.728 843 898 0e − 02	2.247 229 145 6e − 03
0.237 037 037 0	0.409 693 436 0	2.747 749 791 5e − 02	2.072 270 384 8e − 03
0.300 000 000 0	0.447 597 273 6	2.070 876 832 5e − 02	1.868 293 101 2e − 03

TABLE VII. Love numbers for $n = 1.50$ and $l = 2$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{el} = 1.432\,787\,706\,403e - 01$. This provides evidence that our results for the electric-type Love numbers are accurate to nine significant digits. We believe that our results for the magnetic-type Love numbers are also accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.000 000 000 0	0.000 000 000 0	1.432 787 705 8e − 01	0.000 000 000 0e + 00
0.002 592 592 6	0.009 506 130 9	1.382 451 947 2e − 01	1.251 687 977 8e − 04
0.010 370 370 4	0.036 614 471 7	1.245 572 300 4e − 01	4.546 186 196 0e − 04
0.023 333 333 3	0.077 538 708 8	1.056 854 102 8e − 01	8.766 843 745 2e − 04
0.041 481 481 5	0.127 214 130 7	8.549 182 672 0e − 02	1.271 904 978 7e − 03
0.064 814 814 8	0.180 526 269 3	6.686 412 265 4e − 02	1.560 333 479 4e − 03
0.093 333 333 3	0.233 212 456 9	5.126 690 679 2e − 02	1.715 439 047 2e − 03
0.127 037 037 0	0.282 262 783 9	3.901 159 369 7e − 02	1.751 046 409 4e − 03
0.165 925 925 9	0.325 904 247 4	2.975 854 425 7e − 02	1.699 807 886 0e − 03
0.210 000 000 0	0.363 356 780 7	2.292 914 204 5e − 02	1.596 344 970 3e − 03

TABLE VIII. Love numbers for $n = 1.75$ and $l = 2$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{el} = 1.039\,154\,459\,896e - 01$. This provides evidence that our results for the electric-type Love numbers are accurate to nine significant digits. We believe that our results for the magnetic-type Love numbers are also accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.000 000 000 0	0.000 000 000 0	1.039 154 459 6e − 01	0.000 000 000 0e + 00
0.001 851 851 9	0.006 474 329 8	1.012 358 250 5e − 01	6.788 254 152 8e − 05
0.007 407 407 4	0.025 178 838 6	9.375 592 481 6e − 02	2.527 929 253 9e − 04
0.016 666 666 7	0.054 125 998 6	8.292 310 898 6e − 02	5.066 719 366 1e − 04
0.029 629 629 6	0.090 496 283 8	7.053 127 485 7e − 02	7.714 759 206 4e − 04
0.046 296 296 3	0.131 183 213 1	5.817 778 731 1e − 02	9.986 275 228 2e − 04
0.066 666 666 7	0.173 277 121 2	4.695 224 744 7e − 02	1.159 905 733 8e − 03
0.090 740 740 7	0.214 382 563 5	3.739 378 315 5e − 02	1.248 082 969 4e − 03
0.118 518 518 5	0.252 748 994 8	2.961 606 994 9e − 02	1.271 009 109 4e − 03
0.150 000 000 0	0.287 254 858 4	2.347 851 025 4e − 02	1.244 021 476 2e − 03

TABLE IX. Love numbers for $n = 2.00$ and $l = 2$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{el} = 7.393\,839\,192\,094e - 02$. This provides evidence that our results for the electric-type Love numbers are accurate to nine significant digits. We believe that our results for the magnetic-type Love numbers are also accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.000 000 000 0	0.000 000 000 0	7.393 839 192 5e − 02	0.000 000 000 0e + 00
0.001 358 024 7	0.004 480 612 1	7.250 092 856 0e − 02	3.672 250 461 6e − 05
0.005 432 098 8	0.017 540 397 1	6.841 585 734 2e − 02	1.390 859 297 2e − 04
0.012 222 222 2	0.038 100 582 2	6.229 361 543 1e − 02	2.862 898 229 5e − 04
0.021 728 395 1	0.064 567 369 3	5.494 870 567 9e − 02	4.511 008 982 7e − 04
0.033 950 617 3	0.095 075 847 2	4.719 501 810 4e − 02	6.073 882 584 0e − 04
0.048 888 888 9	0.127 734 411 1	3.969 206 230 6e − 02	7.357 497 711 4e − 04
0.066 543 209 9	0.160 820 644 0	3.287 626 634 7e − 02	8.258 435 716 6e − 04
0.086 913 580 2	0.192 904 403 5	2.696 736 695 2e − 02	8.757 289 764 0e − 04
0.110 000 000 0	0.222 898 218 1	2.201 766 463 2e − 02	8.894 797 762 6e − 04

TABLE X. Love numbers for $n = 0.50$ and $l = 3$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{el} = 2.033\,844\,048\,605e - 01$. This provides evidence that our results for the electric-type Love numbers are accurate to five significant digits. We believe that our results for the magnetic-type Love numbers are accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.000 000 000 0	0.000 000 000 0	2.033 839 942 0e − 01	0.000 000 000 0e + 00
0.016 296 296 3	0.062 785 986 5	1.561 309 576 4e − 01	7.680 669 586 8e − 04
0.065 185 185 2	0.208 540 613 2	7.929 849 887 2e − 02	1.533 480 218 0e − 03
0.146 666 666 7	0.363 616 545 4	3.387 663 551 8e − 02	1.405 348 459 5e − 03
0.260 740 740 7	0.488 341 406 6	1.480 314 098 1e − 02	1.004 020 760 6e − 03
0.407 407 407 4	0.577 286 792 3	7.269 055 978 4e − 03	6.862 886 728 2e − 04
0.586 666 666 7	0.637 953 773 6	4.097 004 365 9e − 03	4.849 568 168 5e − 04
0.798 518 518 5	0.678 953 959 1	2.618 202 843 0e − 03	3.622 635 482 8e − 04
1.042 962 963 0	0.706 817 126 4	1.855 561 019 4e − 03	2.862 330 783 1e − 04
1.320 000 000 0	0.725 950 238 2	1.426 624 820 0e − 03	2.375 722 634 1e − 04

TABLE XI. Love numbers for $n = 0.75$ and $l = 3$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{el} = 1.479\,565\,910\,794e-01$, and this value was copied in the first row of the Table. [We were not able to accurately compute the electric-type Love number for $b = 0$ for these specific values of n and l. The reason has to do with the fact that for these values, $\xi e'_{10} = O(\xi^4)$ instead of being of order ξ^2 near $\xi = 0$; the integrator then has difficulty moving out of the small-ξ region and the number of steps required exceeds the set limit.] We believe that our results for the electric-type Love numbers are accurate to nine significant digits, and that our results for the magnetic-type Love numbers are also accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.000 000 000 0	0.000 000 000 0	1.479 565 910 8e − 01	0.000 000 000 0e + 00
0.009 259 259 3	0.036 329 314 4	1.265 691 247 4e − 01	3.760 600 660 9e − 04
0.037 037 037 0	0.129 439 333 2	8.272 911 834 2e − 02	9.744 265 704 9e − 04
0.083 333 333 3	0.245 587 844 2	4.580 295 440 7e − 02	1.184 710 480 7e − 03
0.148 148 148 1	0.356 460 354 9	2.398 066 130 1e − 02	1.053 366 537 2e − 03
0.231 481 481 5	0.448 520 088 7	1.287 007 278 2e − 02	8.245 103 634 5e − 04
0.333 333 333 3	0.519 439 341 0	7.395 819 627 2e − 03	6.235 520 903 4e − 04
0.453 703 703 7	0.571 983 982 7	4.621 915 449 7e − 03	4.765 400 972 2e − 04
0.592 592 592 6	0.610 158 941 0	3.138 800 981 3e − 03	3.751 039 711 2e − 04
0.750 000 000 0	0.637 610 726 0	2.297 160 362 0e − 03	3.058 957 387 7e − 04

TABLE XII. Love numbers for $n = 1.00$ and $l = 3$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{el} = 1.064\,540\,469\,774e-01$. This provides evidence that our results for the electric-type Love numbers are accurate to nine significant digits. We believe that our results for the magnetic-type Love numbers are also accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.000 000 000 0	0.000 000 000 0	1.064 540 470 7e − 01	0.000 000 000 0e + 00
0.005 432 098 8	0.021 188 776 0	9.692 031 509 0e − 02	1.771 243 610 5e − 04
0.021 728 395 1	0.078 832 645 9	7.435 415 738 5e − 02	5.404 697 362 5e − 04
0.048 888 888 9	0.158 617 817 3	5.016 462 201 6e − 02	8.092 719 728 9e − 04
0.086 913 580 2	0.244 994 075 7	3.141 281 410 0e − 02	8.780 579 415 0e − 04
0.135 802 469 1	0.326 497 763 8	1.919 779 936 2e − 02	8.053 659 877 9e − 04
0.195 555 555 6	0.397 110 035 6	1.189 429 694 7e − 02	6.796 113 267 0e − 04
0.266 172 839 5	0.455 036 029 6	7.652 030 873 8e − 03	5.546 617 288 4e − 04
0.347 654 321 0	0.500 890 569 3	5.174 209 592 1e − 03	4.506 937 018 3e − 04
0.440 000 000 0	0.536 309 247 3	3.691 595 210 7e − 03	3.704 317 655 9e − 04

TABLE XIII. Love numbers for $n = 1.25$ and $l = 3$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{el} = 7.558\,993\,098\,406e-02$. This provides evidence that our results for the electric-type Love numbers are accurate to eight significant digits. We believe that our results for the magnetic-type Love numbers are accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.000 000 000 0	0.000 000 000 0	7.558 993 071 3e − 02	0.000 000 000 0e + 00
0.003 703 703 7	0.014 091 088 1	7.084 100 030 7e − 02	9.148 443 889 5e − 05
0.014 814 814 8	0.053 521 847 7	5.878 845 653 0e − 02	3.022 057 521 3e − 04
0.033 333 333 3	0.110 978 273 3	4.416 768 436 2e − 02	5.055 394 016 6e − 04
0.059 259 259 3	0.177 467 002 7	3.096 510 712 8e − 02	6.184 860 942 9e − 04
0.092 592 592 6	0.244 978 923 6	2.091 033 115 4e − 02	6.340 227 072 4e − 04
0.133 333 333 3	0.307 905 147 2	1.398 588 846 6e − 02	5.864 435 224 2e − 04
0.181 481 481 5	0.363 176 478 1	9.466 065 677 3e − 03	5.126 460 042 6e − 04
0.237 037 037 0	0.409 693 436 0	6.578 065 620 2e − 03	4.364 191 129 8e − 04
0.300 000 000 0	0.447 597 273 6	4.733 176 758 1e − 03	3.688 159 898 2e − 04

TABLE XIV. Love numbers for $n = 1.50$ and $l = 3$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{el} = 5.284\,852\,444\,148e-02$. This provides evidence that our results for the electric-type Love numbers are accurate to eight significant digits. We believe that our results for the magnetic-type Love numbers are accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.000 000 000 0	0.000 000 000 0	5.284 852 412 7e − 02	0.000 000 000 0e + 00
0.002 592 592 6	0.009 506 130 9	5.047 832 843 4e − 02	4.681 860 275 7e − 05
0.010 370 370 4	0.036 614 471 7	4.417 267 636 7e − 02	1.631 454 447 3e − 04
0.023 333 333 3	0.077 538 708 8	3.583 378 452 7e − 02	2.951 680 648 2e − 04
0.041 481 481 5	0.127 214 130 7	2.740 992 807 1e − 02	3.955 317 753 6e − 04
0.064 814 814 8	0.180 526 269 3	2.015 452 354 7e − 02	4.444 848 279 0e − 04
0.093 333 333 3	0.233 212 456 9	1.451 641 881 4e − 02	4.470 235 809 6e − 04
0.127 037 037 0	0.282 262 783 9	1.041 115 380 1e − 02	4.191 650 813 1e − 04
0.165 925 925 9	0.325 904 247 4	7.532 674 992 6e − 03	3.768 044 717 6e − 04
0.210 000 000 0	0.363 356 780 7	5.550 301 817 7e − 03	3.310 750 874 7e − 04

TABLE XV. Love numbers for $n = 1.75$ and $l = 3$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{el} = 3.628\,620\,386\,492e-02$. This provides evidence that our results for the electric-type Love numbers are accurate to nine significant digits. We believe that our results for the magnetic-type Love numbers are also accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.000 000 000 0	0.000 000 000 0	3.628 620 385 1e − 02	0.000 000 000 0e + 00
0.001 851 851 9	0.006 474 329 8	3.510 619 626 3e − 02	2.363 764 362 0e − 05
0.007 407 407 4	0.025 178 838 6	3.186 306 826 1e − 02	8.544 896 729 7e − 05
0.016 666 666 7	0.054 125 998 6	2.730 445 066 7e − 02	1.634 575 774 4e − 04
0.029 629 629 6	0.090 496 283 8	2.230 460 678 2e − 02	2.344 383 005 4e − 04
0.046 296 296 3	0.131 183 213 1	1.757 012 016 9e − 02	2.833 811 659 5e − 04
0.066 666 666 7	0.173 277 121 2	1.350 906 398 1e − 02	3.060 968 476 4e − 04
0.090 740 740 7	0.214 382 563 5	1.025 535 745 5e − 02	3.062 994 276 2e − 04
0.118 518 518 5	0.252 748 994 8	7.765 417 591 1e − 03	2.910 571 080 4e − 04
0.150 000 000 0	0.287 254 858 4	5.914 327 324 6e − 03	2.673 704 758 5e − 04

TABLE XVI. Love numbers for $n = 2.00$ and $l = 3$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{el} = 2.439\,399\,851\,849e-02$. This provides evidence that our results for the electric-type Love numbers are accurate to nine significant digits. We believe that our results for the magnetic-type Love numbers are also accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.000 000 000 0	0.000 000 000 0	2.439 399 852 1e − 02	0.000 000 000 0e + 00
0.001 358 024 7	0.004 480 612 1	2.380 440 936 3e − 02	1.183 417 301 3e − 05
0.005 432 098 8	0.017 540 397 1	2.214 756 685 3e − 02	4.385 841 814 1e − 05
0.012 222 222 2	0.038 100 582 2	1.971 794 867 6e − 02	8.720 717 087 4e − 05
0.021 728 395 1	0.064 567 369 3	1.689 230 143 1e − 02	1.313 322 271 6e − 04
0.033 950 617 3	0.095 075 847 2	1.402 263 041 1e − 02	1.676 766 088 5e − 04
0.048 888 888 9	0.127 734 411 1	1.136 632 462 3e − 02	1.916 362 256 2e − 04
0.066 543 209 9	0.160 820 644 0	9.066 549 926 7e − 03	2.025 206 213 4e − 04
0.086 913 580 2	0.192 904 403 5	7.169 706 432 6e − 03	2.022 902 204 0e − 04
0.110 000 000 0	0.222 898 218 1	5.658 121 305 0e − 03	1.940 548 540 2e − 04

TABLE XVII. Love numbers for $n = 0.50$ and $l = 4$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{el} = 1.250\,625\,809\,919\text{e} - 01$. This provides evidence that our results for the electric-type Love numbers are accurate to six significant digits. We believe that our results for the magnetic-type Love numbers are accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.000 000 000 0	0.000 000 000 0	1.250 623 275 2e − 01	0.000 000 000 0e + 00
0.016 296 296 3	0.062 785 986 5	8.988 009 903 5e − 02	4.125 971 341 7e − 04
0.065 185 185 2	0.208 540 613 2	3.867 081 937 5e − 02	6.828 549 095 1e − 04
0.146 666 666 7	0.363 616 545 4	1.351 197 383 4e − 02	4.994 894 209 9e − 04
0.260 740 740 7	0.488 341 406 6	4.903 879 670 5e − 03	2.904 588 518 5e − 04
0.407 407 407 4	0.577 286 792 3	2.075 126 315 1e − 03	1.687 809 338 6e − 04
0.586 666 666 7	0.637 953 773 6	1.047 676 252 6e − 03	1.060 079 304 8e − 04
0.798 518 518 5	0.678 953 959 1	6.194 225 935 1e − 04	7.298 038 311 0e − 05
1.042 962 963 0	0.706 817 126 4	4.161 097 667 0e − 04	5.456 503 299 2e − 05
1.320 000 000 0	0.725 950 238 2	3.085 049 690 3e − 04	4.364 309 303 2e − 05

TABLE XVIII. Love numbers for $n = 0.75$ and $l = 4$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{el} = 8.731\,859\,904\,775\text{e} - 02$. This provides evidence that our results for the electric-type Love numbers are accurate to eight significant digits. We believe that our results for the magnetic-type Love numbers are accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.000 000 000 0	0.000 000 000 0	8.731 859 914 7e − 02	0.000 000 000 0e + 00
0.009 259 259 3	0.036 329 314 4	7.191 615 134 0e − 02	1.982 054 055 1e − 04
0.037 037 037 0	0.129 439 333 2	4.244 873 655 7e − 02	4.568 235 862 9e − 04
0.083 333 333 3	0.245 587 844 2	2.046 764 817 4e − 02	4.741 821 173 2e − 04
0.148 148 148 1	0.356 460 354 9	9.263 622 910 8e − 03	3.571 879 844 9e − 04
0.231 481 481 5	0.448 520 088 7	4.351 789 099 2e − 03	2.404 565 342 2e − 04
0.333 333 333 3	0.519 439 341 0	2.237 488 002 3e − 03	1.604 667 011 0e − 04
0.453 703 703 7	0.571 983 982 7	1.281 223 438 2e − 03	1.112 320 244 6e − 04
0.592 592 592 6	0.610 158 941 0	8.146 858 195 5e − 04	8.139 489 461 4e − 05
0.750 000 000 0	0.637 610 726 0	5.683 382 483 1e − 04	6.295 493 247 6e − 05

TABLE XIX. Love numbers for $n = 1.00$ and $l = 4$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{el} = 6.024\,125\,532\,418\text{e} - 02$. This provides evidence that our results for the electric-type Love numbers are accurate to nine significant digits. We believe that our results for the magnetic-type Love numbers are also accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.000 000 000 0	0.000 000 000 0	6.024 125 539 5e − 02	0.000 000 000 0e + 00
0.005 432 098 8	0.021 188 776 0	5.364 691 367 1e − 02	9.016 974 230 7e − 05
0.021 728 395 1	0.078 832 645 9	3.868 687 672 1e − 02	2.559 908 457 2e − 04
0.048 888 888 9	0.158 617 817 3	2.385 064 923 6e − 02	3.451 160 951 8e − 04
0.086 913 580 2	0.244 994 075 7	1.345 637 102 6e − 02	3.317 117 092 6e − 04
0.135 802 469 1	0.326 497 763 8	7.399 494 497 6e − 03	2.691 383 165 8e − 04
0.195 555 555 6	0.397 110 035 6	4.156 914 457 1e − 03	2.027 287 550 3e − 04
0.266 172 839 5	0.455 036 029 6	2.456 133 232 9e − 03	1.499 053 528 3e − 04
0.347 654 321 0	0.500 890 569 3	1.547 888 358 1e − 03	1.122 476 794 8e − 04
0.440 000 000 0	0.536 309 247 3	1.044 239 881 6e − 03	8.644 010 426 7e − 05

TABLE XX. Love numbers for $n = 1.25$ and $l = 4$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{el} = 4.096\,746\,123\,839e - 02$. This provides evidence that our results for the electric-type Love numbers are accurate to eight significant digits. We believe that our results for the magnetic-type Love numbers are accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.000 000 000 0	0.000 000 000 0	4.096 746 112 0e − 02	0.000 000 000 0e + 00
0.003 703 703 7	0.014 091 088 1	3.783 153 265 8e − 02	4.444 860 934 5e − 05
0.014 814 814 8	0.053 521 847 7	3.010 200 852 5e − 02	1.396 932 686 8e − 04
0.033 333 333 3	0.110 978 273 3	2.122 482 430 6e − 02	2.167 615 429 1e − 04
0.059 259 259 3	0.177 467 002 7	1.377 827 719 7e − 02	2.420 876 623 7e − 04
0.092 592 592 6	0.244 978 923 6	8.568 826 133 0e − 03	2.250 840 231 3e − 04
0.133 333 333 3	0.307 905 147 2	5.285 702 775 8e − 03	1.891 061 529 7e − 04
0.181 481 481 5	0.363 176 478 1	3.320 225 825 7e − 03	1.512 354 966 7e − 04
0.237 037 037 0	0.409 693 436 0	2.161 115 379 3e − 03	1.190 391 357 3e − 04
0.300 000 000 0	0.447 597 273 6	1.471 838 069 0e − 03	9.414 154 774 9e − 05

TABLE XXI. Love numbers for $n = 1.50$ and $l = 4$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{el} = 2.739\,306\,738\,271e - 02$. This provides evidence that our results for the electric-type Love numbers are accurate to eight significant digits. We believe that our results for the magnetic-type Love numbers are accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.000 000 000 0	0.000 000 000 0	2.739 306 729 4e − 02	0.000 000 000 0e + 00
0.002 592 592 6	0.009 506 130 9	2.590 413 686 4e − 02	2.159 090 574 6e − 05
0.010 370 370 4	0.036 614 471 7	2.202 286 012 7e − 02	7.266 190 073 3e − 05
0.023 333 333 3	0.077 538 708 8	1.708 510 329 8e − 02	1.245 684 841 5e − 04
0.041 481 481 5	0.127 214 130 7	1.235 691 723 3e − 02	1.560 079 641 7e − 04
0.064 814 814 8	0.180 526 269 3	8.536 069 494 7e − 03	1.625 552 653 8e − 04
0.093 333 333 3	0.233 212 456 9	5.765 665 383 7e − 03	1.512 135 065 1e − 04
0.127 037 037 0	0.282 262 783 9	3.885 879 808 6e − 03	1.314 252 653 8e − 04
0.165 925 925 9	0.325 904 247 4	2.655 024 195 4e − 03	1.101 088 200 7e − 04
0.210 000 000 0	0.363 356 780 7	1.859 857 288 9e − 03	9.085 820 027 2e − 05

TABLE XXII. Love numbers for $n = 1.75$ and $l = 4$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{el} = 1.795\,919\,608\,352e - 02$. This provides evidence that our results for the electric-type Love numbers are accurate to eight significant digits. We believe that our results for the magnetic-type Love numbers are accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.000 000 000 0	0.000 000 000 0	1.795 919 579 8e − 02	0.000 000 000 0e + 00
0.001 851 851 9	0.006 474 329 8	1.725 641 489 1e − 02	1.029 949 246 2e − 05
0.007 407 407 4	0.025 178 838 6	1.535 237 230 3e − 02	3.632 903 887 1e − 05
0.016 666 666 7	0.054 125 998 6	1.274 878 813 4e − 02	6.685 849 630 7e − 05
0.029 629 629 6	0.090 496 283 8	1.000 190 946 8e − 02	9.123 268 303 5e − 05
0.046 296 296 3	0.131 183 213 1	7.521 289 301 5e − 03	1.041 245 608 6e − 04
0.066 666 666 7	0.173 277 121 2	5.503 619 345 3e − 03	1.057 662 957 0e − 04
0.090 740 740 7	0.214 382 563 5	3.975 120 344 7e − 03	9.945 255 506 9e − 05
0.118 518 518 5	0.252 748 994 8	2.869 530 874 3e − 03	8.898 226 899 2e − 05
0.150 000 000 0	0.287 254 858 4	2.091 368 507 6e − 03	7.7283983050e − 05

TABLE XXIII. Love numbers for $n = 2.00$ and $l = 4$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{\rm el} = 1.150\,774\,963\,254\text{e} - 02$. This provides evidence that our results for the electric-type Love numbers are accurate to nine significant digits. We believe that our results for the magnetic-type Love numbers are also accurate to nine significant digits.

b	$2M/R$	$k_{\rm el}$	$k_{\rm mag}$
0.000 000 000 0	0.000 000 000 0	1.150 774 963 4e − 02	0.000 000 000 0e + 00
0.001 358 024 7	0.004 480 612 1	1.117 598 624 2e − 02	4.851 734 617 8e − 06
0.005 432 098 8	0.017 540 397 1	1.025 319 916 6e − 02	1.766 687 749 2e − 05
0.012 222 222 2	0.038 100 582 2	8.926 703 037 4e − 03	3.415 521 223 3e − 05
0.021 728 395 1	0.064 567 369 3	7.427 231 982 1e − 03	4.957 678 143 7e − 05
0.033 950 617 3	0.095 075 847 2	5.957 350 631 4e − 03	6.060 538 419 3e − 05
0.048 888 888 9	0.127 734 411 1	4.650 843 277 7e − 03	6.603 541 988 0e − 05
0.066 543 209 9	0.160 820 644 0	3.568 265 716 6e − 03	6.639 610 716 1e − 05
0.086 913 580 2	0.192 904 403 5	2.715 040 067 0e − 03	6.310 084 938 4e − 05
0.110 000 000 0	0.222 898 218 1	2.065 340 446 3e − 03	5.769 693 677 6e − 05

TABLE XXIV. Love numbers for $n = 0.50$ and $l = 5$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{\rm el} = 8.758\,378\,097\,872\text{e} - 02$. This provides evidence that our results for the electric-type Love numbers are accurate to five significant digits. We believe that our results for the magnetic-type Love numbers are accurate to nine significant digits.

b	$2M/R$	$k_{\rm el}$	$k_{\rm mag}$
0.000 000 000 0	0.000 000 000 0	8.758 359 747 7e − 02	0.000 000 000 0e + 00
0.016 296 296 3	0.062 785 986 5	5.895 372 692 3e − 02	2.456 662 541 2e − 04
0.065 185 185 2	0.208 540 613 2	2.150 213 523 3e − 02	3.401 764 953 9e − 04
0.146 666 666 7	0.363 616 545 4	6.143 863 801 6e − 03	2.004 014 503 5e − 04
0.260 740 740 7	0.488 341 406 6	1.848 447 587 8e − 03	9.530 920 570 3e − 05
0.407 407 407 4	0.577 286 792 3	6.714 663 382 5e − 04	4.705 529 018 8e − 05
0.586 666 666 7	0.637 953 773 6	3.020 177 090 0e − 04	2.614 355 243 6e − 05
0.798 518 518 5	0.678 953 959 1	1.641 505 397 5e − 04	1.647 065 169 3e − 05
1.042 962 963 0	0.706 817 126 4	1.038 369 302 7e − 04	1.156 391 300 4e − 05
1.320 000 000 0	0.725 950 238 2	7.377 700 961 8e − 05	8.849 041 163 0e − 06

TABLE XXV. Love numbers for $n = 0.75$ and $l = 5$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{\rm el} = 5.904\,211\,079\,675\text{e} - 02$. This provides evidence that our results for the electric-type Love numbers are accurate to nine significant digits. We believe that our results for the magnetic-type Love numbers are also accurate to nine significant digits.

b	$2M/R$	$k_{\rm el}$	$k_{\rm mag}$
0.000 000 000 0	0.000 000 000 0	5.904 211 083 0e − 02	0.000 000 000 0e + 00
0.009 259 259 3	0.036 329 314 4	4.683 027 075 3e − 02	1.165 380 927 3e − 04
0.037 037 037 0	0.129 439 333 2	2.497 615 876 2e − 02	2.405 109 713 9e − 04
0.083 333 333 3	0.245 587 844 2	1.049 166 476 8e − 02	2.148 876 029 2e − 04
0.148 148 148 1	0.356 460 354 9	4.103 059 826 6e − 03	1.381 230 645 0e − 04
0.231 481 481 5	0.448 520 088 7	1.684 538 307 3e − 03	8.034 031 504 7e − 05
0.333 333 333 3	0.519 439 341 0	7.727 961 295 7e − 04	4.738 252 082 7e − 05
0.453 703 703 7	0.571 983 982 7	4.039 434 759 1e − 04	2.975 515 817 6e − 05
0.592 592 592 6	0.610 158 941 0	2.394 304 756 3e − 04	2.017 943 102 0e − 05
0.750 000 000 0	0.637 610 726 0	1.584 662 983 6e − 04	1.474 343 322 3e − 05

TABLE XXVI. Love numbers for $n = 1.00$ and $l = 5$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{el} = 3.929\,250\,022\,713e - 02$. This provides evidence that our results for the electric-type Love numbers are accurate to nine significant digits. We believe that our results for the magnetic-type Love numbers are also accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.000 000 000 0	0.000 000 000 0	3.929 250 028 3e − 02	0.000 000 000 0e + 00
0.005 432 098 8	0.021 188 776 0	3.423 218 679 8e − 02	5.159 527 917 9e − 05
0.021 728 395 1	0.078 832 645 9	2.321 494 691 0e − 02	1.369 134 786 8e − 04
0.048 888 888 9	0.158 617 817 3	1.308 382 167 7e − 02	1.672 578 936 4e − 04
0.086 913 580 2	0.244 994 075 7	6.651 765 835 9e − 03	1.433 945 121 2e − 04
0.135 802 469 1	0.326 497 763 8	3.289 704 373 4e − 03	1.035 579 104 5e − 04
0.195 555 555 6	0.397 110 035 6	1.673 858 664 5e − 03	6.996 284 441 1e − 05
0.266 172 839 5	0.455 036 029 6	9.066 386 282 0e − 04	4.701 213 896 9e − 05
0.347 654 321 0	0.500 890 569 3	5.311 874 279 7e − 04	3.248 140 667 5e − 05
0.440 000 000 0	0.536 309 247 3	3.378 225 320 9e − 04	2.343 282 134 6e − 05

TABLE XXVII. Love numbers for $n = 1.25$ and $l = 5$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{el} = 2.574\,776\,897\,544e - 02$. This provides evidence that our results for the electric-type Love numbers are accurate to nine significant digits. We believe that our results for the magnetic-type Love numbers are also accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.000 000 000 0	0.000 000 000 0	2.574 776 889 2e − 02	0.000 000 000 0e + 00
0.003 703 703 7	0.014 091 088 1	2.343 173 228 9e − 02	2.448 459 439 1e − 05
0.014 814 814 8	0.053 521 847 7	1.788 234 304 5e − 02	7.346 991 367 1e − 05
0.033 333 333 3	0.110 978 273 3	1.183 806 730 0e − 02	1.063 002 581 4e − 04
0.059 259 259 3	0.177 467 002 7	7.117 592 985 6e − 03	1.090 395 836 8e − 04
0.092 592 592 6	0.244 978 923 6	4.076 478 525 6e − 03	9.252 083 099 8e − 05
0.133 333 333 3	0.307 905 147 2	2.318 012 236 7e − 03	7.100 609 930 7e − 05
0.181 481 481 5	0.363 176 478 1	1.350 020 554 2e − 03	5.219 665 186 0e − 05
0.237 037 037 0	0.409 693 436 0	8.218 299 205 9e − 04	3.812 141 741 1e − 05
0.300 000 000 0	0.447 597 273 6	5.287 394 632 7e − 04	2.828 014 368 8e − 05

TABLE XXVIII. Love numbers for $n = 1.50$ and $l = 5$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{el} = 1.656\,876\,321\,404e - 02$. This provides evidence that our results for the electric-type Love numbers are accurate to eight significant digits. We believe that our results for the magnetic-type Love numbers are accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.000 000 000 0	0.000 000 000 0	1.656 876 313 5e − 02	0.000 000 000 0e + 00
0.002 592 592 6	0.009 506 130 9	1.551 374 792 3e − 02	1.139 676 932 1e − 05
0.010 370 370 4	0.036 614 471 7	1.281 710 282 7e − 02	3.714 425 213 6e − 05
0.023 333 333 3	0.077 538 708 8	9.512 335 412 3e − 03	6.059 352 981 5e − 05
0.041 481 481 5	0.127 214 130 7	6.507 168 725 7e − 03	7.129 259 956 6e − 05
0.064 814 814 8	0.180 526 269 3	4.223 689 023 3e − 03	6.926 862 619 7e − 05
0.093 333 333 3	0.233 212 456 9	2.675 124 564 1e − 03	5.993 842 983 1e − 05
0.127 037 037 0	0.282 262 783 9	1.693 534 704 8e − 03	4.854 363 740 2e − 05
0.165 925 925 9	0.325 904 247 4	1.091 813 480 5e − 03	3.808 219 174 3e − 05
0.210 000 000 0	0.363 356 780 7	7.262 529 035 3e − 04	2.962 761 140 2e − 05

TABLE XXIX. Love numbers for $n = 1.75$ and $l = 5$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{el} = 1.043\,995\,446\,810e - 02$. This provides evidence that our results for the electric-type Love numbers are accurate to nine significant digits. We believe that our results for the magnetic-type Love numbers are also accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.000 000 000 0	0.000 000 000 0	1.043 995 438 7e − 02	0.000 000 000 0e + 00
0.001 851 851 9	0.006 474 329 8	9.963 588 236 4e − 03	5.190 736 449 1e − 06
0.007 407 407 4	0.025 178 838 6	8.690 617 185 4e − 03	1.790 364 460 2e − 05
0.016 666 666 7	0.054 125 998 6	6.995 465 820 2e − 03	3.180 685 268 2e − 05
0.029 629 629 6	0.090 496 283 8	5.272 476 720 9e − 03	4.147 331 115 9e − 05
0.046 296 296 3	0.131 183 213 1	3.785 783 640 3e − 03	4.491 454 107 0e − 05
0.066 666 666 7	0.173 277 121 2	2.636 680 028 2e − 03	4.312 938 484 1e − 05
0.090 740 740 7	0.214 382 563 5	1.811 692 183 6e − 03	3.831 010 337 3e − 05
0.118 518 518 5	0.252 748 994 8	1.246 318 421 9e − 03	3.243 746 131 1e − 05
0.150 000 000 0	0.287 254 858 4	8.686 485 307 3e − 04	2.676 073 276 8e − 05

TABLE XXX. Love numbers for $n = 2.00$ and $l = 5$. Integration of the Newtonian Clairaut equation (for $b = 0$) returns $k_{el} = 6.419\,966\,834\,096e - 03$. This provides evidence that our results for the electric-type Love numbers are accurate to nine significant digits. We believe that our results for the magnetic-type Love numbers are also accurate to nine significant digits.

b	$2M/R$	k_{el}	k_{mag}
0.000 000 000 0	0.000 000 000 0	6.419 966 835 0e − 03	0.000 000 000 0e + 00
0.001 358 024 7	0.004 480 612 1	6.205 468 381 3e − 03	2.327 325 318 0e − 06
0.005 432 098 8	0.017 540 397 1	5.614 673 660 3e − 03	8.341 058 191 8e − 06
0.012 222 222 2	0.038 100 582 2	4.781 425 083 4e − 03	1.572 236 249 1e − 05
0.021 728 395 1	0.064 567 369 3	3.864 754 655 6e − 03	2.207 582 357 5e − 05
0.033 950 617 3	0.095 075 847 2	2.996 023 856 8e − 03	2.595 023 418 2e − 05
0.048 888 888 9	0.127 734 411 1	2.253 155 724 0e − 03	2.708 417 921 3e − 05
0.066 543 209 9	0.160 820 644 0	1.662 818 798 5e − 03	2.603 743 207 0e − 05
0.086 913 580 2	0.192 904 403 5	1.217 237 365 1e − 03	2.366 085 625 0e − 05
0.110 000 000 0	0.222 898 218 1	8.922 788 920 0e − 04	2.072 082 377 4e − 05

The following tables display our results for the tidal Love numbers of relativistic polytropes. Each table caption contains a discussion of the estimated numerical accuracy of our results.

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