The Spinor Description of General Relativity and Applications to Black Hole Physics

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Contents

Preface

These lecture notes were prepared originally for a six-hour course for the Ph.D School *"Advanced Topics in Theoretical Physics 2025"* held in April 2025 at the University of Perugia, Italy. The course is intended to give a concise yet self-contained introduction to the spinor description of general relativity, oriented towards applications to black hole and gravitational wave physics. The first and second two-hour sessions were dedicated to Sections 2 and 3 of the notes, while the last two-hour session discussed Section 4 together with a demonstration using the appended `mathematica` notebook.

Conventions. We use geometric units where $c = G = 1$. The spacetime signature is $(+,-,-,-)$, as is customary in the spinor literature, to ensure that the rising and lowering operations with the spinor and spacetime metrics are consistent with one another. Following likewise standard conventions, normal-font upper (lower) case latin characters are abstract spinor (tensor) indices, while their boldface counterparts label components relative to a dyad (vector basis). The sign of the Riemann tensor is chosen to match the convention in the original paper by Newman and Penrose [1],

$$\nabla_a \nabla_b X_c - \nabla_b \nabla_a X_c = R_{abcd} X^d \,, \tag{0.1}$$

which is unfortunately the opposite one chosen in the original work by Penrose [2] as well as the main reference on the topic, the monograph by Penrose and Rindler [3]. The convention taken here follows that of many established references in spinor applications to gravitational physics (besides [1] see also Chandrasekar's monograph [4] and Valiente Kroon's one [5]), but has the undesirable issue of endowing de Sitter space with negative curvature. It is what it is.

References. For Section 2 I found Chapter 3 of Valiente Kroon's book [5] particularly useful, as well as Chapter 13 of Wald's book [6]. Section 3 follows closest the discussions in Penrose and Rindler's book [3]. Stewart's book [7] was also helpful. More specific references are indicated throughout the text.

1 Introduction

The spinor approach to general relativity started being developed around the 1960s, with an influential work by Penrose [2]. One of its most important advantages, noticed slightly earlier [8], is that spacetime curvature is described in remarkably simple terms. This point of view has lead to groundbreaking advancements in the description of gravity, that go from exploring the space of solutions of Einstein's theory to understanding aspects of gravitational radiation that are crucial for gravitational wave science. Furthermore, besides providing a deep and useful formulation of general relativity, spinor techniques have also been instrumental in establishing other important results in gravity, such as the positivity of energy [9], and continue to be used actively in today's research.

An outstanding application of spinors concerns some problems in gravitational radiation, especially about gravitational waves in Kerr's spacetime. The latter turns out to be relatively simple when written in terms of its spinor variables, and this has made it possible to study in a very precise manner what are the characteristics of gravitational fluctuations that generate and propagate in the vicinity of black holes. In order to write spinor equations in a way that is suitable for some computations (e.g., solving them numerically), one translates them into scalar ones via the so-called Newman–Penrose (NP) formalism [1] or its "refined" version, the Geroch–Held–Penrose (GHP) formalism [10]. These scalar equations are often taken as the starting point in both research applications and textbooks (Chandrasekar's monograph [4] is one example, although Chapter 10 contains a brief introduction to spinors). The advantage of this approach is that only prior knowledge of tensor calculus is required, but it has some drawbacks, in our view. First, the NP and GHP variables are only natural from a spinor perspective. When these are introduced instead as contractions of tensors and null frames, they seem rather arbitrary, and it is far from clear why such an approach should be useful. This is arguably undesirable from an educational perspective. A second disadvantage is that some times dealing with NP or GHP equations directly requires more guesswork than if working with spinors first, benefiting form their higher simplicity, and translating the equations to NP or GHP variables at the end of the day. We will exemplify this by providing an alternative derivation of Teukolsky's equations [11]. Although it might be counter-argued (and justifiably so) that translating spinor equations into NP or GHP ones is a computationally demanding task, currently available software for efficient symbolic manipulation allows one to perform those operations automatically.

These lecture notes are intended to provide a concise yet self-contained introduction to the spinor description of general relativity, oriented towards applications to black hole and gravitational wave physics. They come with a `mathematica` notebook based on `SpinFrames` (one of the `xAct` packages) where some examples are given on how to manipulate spinor equations and their NP and GHP projections efficiently. The notes are organised as follows. In Section 2 we introduce spinors as elements of a complex vector space, we review their most important properties and show how Lorentzian spaces emerge naturally from spinor space. In Section 3 we extend those notions to the entire spacetime manifold, and introduce the spinor covariant derivative. The main result is the decomposition of the Riemann curvature tensor in its irreducible spinor components. This allows us to derive Petrov's classification in a natural way, and obtain the wave equation satisfied by Weyl's spinor in vacuum. The section is concluded by introducing the GHP formalism, and discussing some theorems rephrased in terms of GHP variables (the NP formalism is introduced as a restriction of the GHP one). In Section 4 we apply the previous notions to black holes and gravitational waves. First, we characterise the class of vacuum type D spacetimes in terms of their GHP variables, with an emphasis on Kerr's solution. Next, starting from the wave equation satisfied by Weyl's spinor in vacuum, we derive non-perturbative wave equations describing the propagation of GHP curvature scalars in empty space (these were first obtained by Stewart and Walker [12], although we correct a typo that has propagated to some works in the literature). The notes only report the results, while the derivation is given in detail in the example notebook. Finally, we obtain Teukolsky's equations on any vacuum type D space as an immediate linearisation of the curvature wave equations derived previously, and conclude by discussing their implications for gravitational waves in Kerr's spacetime.

2 Spinors at a Point

Spinors provide a decomposition of the gravitational field into "irreducible pieces". Working in terms of these smaller elements, some apparently complicated properties of spacetime can be described in strikingly simple terms. However, before applying these ideas to describe the spacetime geometry it is necessary to establish a link between spinors and Lorentzian metric spaces. Spinors are elements in the vector representation of $SL(2,\mathbb{C})$, the group of unimodular 2×2 complex matrices, and this is related to the proper Lorentz group $\Lambda(1,3)$ by a two-to-one map

$$\Lambda(1,3) \cong SL(2,\mathbb{C})/\{\mathbb{I} \sim -\mathbb{I}\}\,, \tag{2.1}$$

where $\mathbb{I}$ is the identity in $SL(2,\mathbb{C})$. Mathematically, we say $SL(2,\mathbb{C})$ is the universal covering of $\Lambda(1,3)$, and this is the main reason underlying the spinor description of Lorentzian spaces. Reviewing these facts in some detail is the goal of the present section.

2.1 Spinors and Index Notation

A *spinor* ξ is an element of a 2-dimensional vector space W over the complex numbers $\mathbb{C}$. Besides the dual space W^* of linear maps from W to $\mathbb{C}$, W also has associated a *complex conjugate dual space* $\bar{W}^*$.

This is the set of maps ψ from W to $\mathbb{C}$ which are *antilinear*, that is, satisfying

$$\psi\left(a\xi+b\eta\right)=\bar{a}\psi\left(\xi\right)+\bar{b}\psi\left(\eta\right)\,, \tag{2.2}$$

for all $\xi,\eta\in W$, and $a,b\in\mathbb{C}$. The *complex conjugate space of* W, denoted $\bar{W}$, is simply the dual of $\bar{W}^*$. Every element ξ in W has associated a *complex conjugate*, the element $\bar{\xi}$ in $\bar{W}$ given by $\bar{\xi}\left(\psi\right)\coloneqq\psi(\xi)$ for all $\psi\in\bar{W}^*$. Similarly, the complex conjugate of $\alpha\in W^*$ is the element $\bar{\alpha}\in\bar{W}^*$ acting on $\xi\in W$ as $\bar{\alpha}(\xi)\coloneqq\overline{\alpha(\xi)}$. These maps are natural bijections that we call (as well as their inverses) *complex conjugation*. We notice that these notions apply to any vector space over $\mathbb{C}$, so in particular they extend to the tensor product space $W^{n_1}\otimes\bar{W}^{n_2}\otimes W^{*n_3}\otimes\bar{W}^{*n_4}$ (with n_i natural numbers).

In order to make things more explicit, it is convenient to use abstract index notation, where a spinor ξ is represented by ξ^A, the index refering to the vector space it belongs to (W in this case). It is customary to use primed indices to refer to elements in the complex conjugate space, $\phi^{A'}\in\bar{W}$, and denote the complex conjugate of a spinor ξ^A by $\bar{\xi}^{A'}$, that is,

$$\bar{\xi}^{A'}\coloneqq\overline{\xi^A}\,. \tag{2.3}$$

A basis of spinors is called a *dyad*, and consists of a pair of spinors $\epsilon_{\mathbf{0}}{}^A,\epsilon_{\mathbf{1}}{}^A$ that generate W. We will use boldface upper-case latin characters to label the basis elements as $\epsilon_{\mathbf{A}}{}^A$, and also to label spinor components. For example, the dual basis of $\epsilon_{\mathbf{A}}{}^A$, denoted $\epsilon_A{}^{\mathbf{A}}$, satisfies $\epsilon_{\mathbf{B}}{}^A\epsilon_A{}^{\mathbf{A}}=\delta_{\mathbf{B}}{}^{\mathbf{A}}$ and the components of $S^{AB'}{}_C$ are $S^{\mathbf{AB'}}{}_{\mathbf{C}}=\epsilon_A{}^{\mathbf{A}}\epsilon_{B'}{}^{\mathbf{B'}}\epsilon_{\mathbf{C}}{}^CS^{AB'}{}_C$.[1]

Remark 2.1. Primed and unprimed indices refer to different vector spaces. Therefore, their ordering is irrelevant and their contraction is ill-defined. Taking a spinor $S^{AB'}{}_C\in W\otimes\bar{W}\otimes W^*$ as example, one has that $S^{AB'}{}_C=S^{B'A}{}_C$, and $S^{AB'}{}_A\in\bar{W}$, but a contraction between its $\bar{W}$-upper index and W^*-lower index is not defined. We may often write $S^{AA'}{}_C$ to save index characters, since the presence of the prime avoids any confusion. Likewise, sometimes we may ommit the bar to denote complex conjugation, since the index structure again avoids any confusion, e.g. both $\bar{S}^{A'A}{}_{C'}$ and $S^{A'A}{}_{C'}$ denote the conjugate of $S^{AA'}{}_C$. An exception are the spinors with the same number of primed and unprimed indices, in that case the bar is necessary to avoid confusion.

Exercise 2.1. Let $\epsilon_{\mathbf{A}}{}^A$ be a dyad in W, $\epsilon_A{}^{\mathbf{A}}$ its dual, together with their conjugates $\bar{\epsilon}_{\mathbf{A}'}{}^{A'}\coloneqq\overline{\epsilon_{\mathbf{A}}{}^A}$ and $\bar{\epsilon}_{A'}{}^{\mathbf{A}'}\coloneqq\overline{\epsilon_A{}^{\mathbf{A}}}$. ($i$) Show that $\bar{\epsilon}_{\mathbf{A}'}{}^{A'}$ is a basis in $\bar{W}$ and $\bar{\epsilon}_{A'}{}^{\mathbf{A}'}$ its dual. (ii) Show that if $\xi^{\mathbf{A}}$ and $\alpha_{\mathbf{A}}$ are the components of ξ^A and α_A, then the components of $\bar{\xi}^{A'}$ and $\bar{\alpha}_{A'}$ are $\bar{\xi}^{\mathbf{A}'}=\overline{\xi^{\mathbf{A}}}$ and $\bar{\alpha}_{\mathbf{A}'}=\overline{\alpha_{\mathbf{A}}}$ (that is, the components of the complex conjugate of a spinor are simply the complex conjugate of its components).

[1] Our convention in the position of indices in a dyad is that of Penrose and Rindler [3], since in that case one has the convenient property explained below in part (ii) of Exercise 2.3.

2.2 Spinor Space

The fact that W has a low dimension will be crucial. In particular, the space of skew-symmetric, valence-2 spinors is 1-dimensional, and the choice of a representative gives raise to the notion of *spinor space*.

Definition 2.1. A *spinor space* is a pair (W, ϵ_{AB}) where W is a 2-dimensional complex vector space and ϵ_{AB} a non-vanishing, skew-symmetric valence-2 spinor ($\epsilon_{AB} = -\epsilon_{BA}$).

Given that the ϵ-spinor ϵ_{AB} is non-degenerate, we can use it to correspond W with W^* uniquely by lowering and raising spinor indices, just as we do in (pseudo-)Riemannian metric spaces. However, since ϵ_{AB} is skew-symmetric it is necessary to establish a convention in the index positionings. We choose to correspond a spinor $\xi^A \in W$ to one $\xi_A \in W^*$ by the relation

$$\xi_A = \epsilon_{BA}\xi^B \,. \tag{2.4}$$

We will write the inverse of this map in terms of ϵ^{AB}, defined as (minus) the inverse of ϵ_{AB} in the sense that it satisfies

$$\epsilon_{BC}\epsilon^{AC} = \delta_B{}^A \,, \tag{2.5}$$

where $\delta_B{}^A$ is the identity in W. Thus, we can correspond a spinor $\xi_A \in W^*$ to one $\xi^A \in W$ via

$$\xi^A = \epsilon^{AB}\xi_B \,. \tag{2.6}$$

It is customary to introduce the following notation for the complex conjugate of the ϵ-spinors,

$$\epsilon_{A'B'} \coloneqq \bar{\epsilon}_{A'B'} \,, \quad \epsilon^{A'B'} \coloneqq \bar{\epsilon}^{A'B'} \,, \tag{2.7}$$

where the bars are omitted for convenience. Using ϵ_{AB}, $\epsilon_{A'B'}$ and their inverses the operations of raising and lowering indices can be extended to spinors of arbitrary structure. It should be stressed that the index positions in the relations above are crucial in order to make the operations well-defined, as shown in the following exercise.

Exercise 2.2. (i) Show that given a spinor $\xi^A \in W$, then $\xi^A = \epsilon^{AB}\xi_B$ where ξ_B is as in (2.4). This shows that the operations of raising and lowering indices (equations (2.4) and (2.6)) are the inverse of each other. (ii) Show that for all $\xi^A \in W$ one has $\xi^A\xi_A = 0$. (iii) Show that $\epsilon_A{}^A = \epsilon^{AB}\epsilon_{AB} = 2$, $\epsilon_C{}^A = -\epsilon^A{}_C = \delta_C{}^A$ and $\delta_B{}^A = -\delta^A{}_B$. (iv) Show that for any spinor $S^A{}_B \in W \otimes W^*$, one has $S^A{}_A = -S_A{}^A$. This is known as the *see-saw rule*.

Finally, a spinor space has a family of distinguished basis, the orthonormal ones with respect to ϵ_{AB}. A *orthonormal spinor dyad* is a spinor basis $\{o^A, \iota^A\}$ satisfying

$$o_A\iota^A = 1 \,. \tag{2.8}$$

Exercise 2.3. (*i*) Denoting an orthonormal dyad as $\epsilon_{\mathbf{A}}{}^{A} = \{o^A, \iota^A\}$, and its dual by $\epsilon_A{}^{\mathbf{A}}$, show that the components of ϵ_{AB} and ϵ^{AB} are

$$\epsilon_{\mathbf{AB}} = \epsilon^{\mathbf{AB}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \tag{2.9}$$

where we are assuming that the first index labels rows and the second columns. (*ii*) Show that the basis and its dual are related by $\epsilon_A{}^{\mathbf{A}} = -\epsilon^{\mathbf{AB}}\epsilon_{BA}\epsilon_{\mathbf{B}}{}^{B}$, so one can write $\epsilon_{A\mathbf{A}} = -\epsilon_{\mathbf{A}A}$. (*iii*) Show that

$$\delta_A{}^B = o_A\iota^B - \iota_A o^B , \quad \epsilon_{AB} = o_A\iota_B - \iota_A o_B , \quad \epsilon^{AB} = o^A\iota^B - \iota^A o^B . \tag{2.10}$$

2.3 Irreducible Decomposition of Spinors

The decomposition of spinors into its "irreducible pieces" plays a central role in most of the things that will be discussed later. It can be formulated in the following way.

Proposition 2.1. *Any spinor $\xi_{A\ldots B}$ can be decomposed as the sum of the spinor $\xi_{(A\ldots B)}$ and products of ϵ-spinors with symmetrised contractions of $\xi_{A\ldots B}$.*

Proof. That this is true for a skew-symmetric valence-2 spinor $\xi_{AB} = \xi_{[AB]}$ is clear: the space of skew-symmetric 2-spinors is one dimensional, so $\xi_{[AB]} = f\epsilon_{AB}$ for some scalar f, and contracting both sides with ϵ^{AB} gives $f = (1/2)\epsilon^{AB}\xi_{AB}$. To extend the proof to a spinor of valence n, $\xi_{A\ldots B}$, we first note that

$$n!\xi_{(AB\ldots CD)} = (n-1)!\left(\xi_{A(B\ldots CD)} + \xi_{B(A\ldots CD)} + \ldots + \xi_{D(AB\ldots C)}\right) , \tag{2.11}$$

and apply the result of skew-symmetric valence-2 spinors to write the second term in the brackets as

$$\xi_{B(A\ldots CD)} = \xi_{A(B\ldots CD)} - 2\xi_{[A(B]\ldots CD)} = \xi_{A(B\ldots CD)} - \epsilon^{EF}\xi_{E(F\ldots CD)}\epsilon_{AB} = \xi_{A(B\ldots CD)} + \xi^{E}{}_{(E\ldots CD)}\epsilon_{AB} . \tag{2.12}$$

Applying this to all terms in the brackets of (2.11) but the first gives

$$\xi_{(AB\ldots CD)} = \xi_{A(B\ldots CD)} + \frac{1}{n}\left(\xi^{E}{}_{(E\ldots CD)}\epsilon_{AB} + \ldots + \xi^{E}{}_{(B\ldots CE)}\epsilon_{AD}\right) . \tag{2.13}$$

This formula can now be applied on $\xi_{A(B\ldots CD)}$, to write it as $\xi_{AB(\ldots CD)}$ plus ϵ-spinors and symmetrised contractions of $\xi_{A\ldots B}$. Doing this recursively one obtains an expression for $\xi_{A\ldots B}$ only in terms of products of its symmetrised contractions and ϵ-spinors. The proof extends in the exact same way to spinors carrying primed indices. □

The *irreducible components of a spinor* are the symmetrised contractions that appear in the irreducible decomposition, and they are independent in the sense that the spinor vanishes if and only if all of its irreducible components vanish. In addition, the number of independent components of each irreducible piece is obtained immediately after noticing the following.

Proposition 2.2. *The number of independent components of a valence-n, fully symmetric spinor $\xi_{(A\ldots B)}$ is $n+1$.*

Proof. Taking a dyad $\{o_A, \iota_A\}$, a basis of the space of fully-symmetric spinors of valence n is

$$\left\{o_{(A}o_B\ldots o_{C)}, \iota_{(A}o_B\ldots o_{C)}, \iota_{(A}\iota_B\ldots o_{C)}, \ldots, \iota_{(A}\iota_B\ldots\iota_{C)}\right\}, \tag{2.14}$$

which contains $n+1$ elements. ☐

Exercise 2.4. Show that the irreducible decomposition of a valence-4 spinor ξ_{ABCD} is

$$\begin{aligned}\xi_{ABCD} =&\xi_{(ABCD)} + \frac{1}{2}\xi_{(AB)P}{}^{P}\epsilon_{CD} + \frac{1}{2}\xi_{P}{}^{P}{}_{(CD)}\epsilon_{AB} + \frac{1}{4}\xi_{P}{}^{P}{}_{Q}{}^{Q}\epsilon_{AB}\epsilon_{CD}\\ &+\frac{1}{2}\epsilon_{A(C}\xi_{D)B} + \frac{1}{2}\epsilon_{B(C}\xi_{D)A} - \frac{1}{3}\epsilon_{A(C}\epsilon_{D)B}\xi\,,\end{aligned} \tag{2.15}$$

where $\xi_{AB} \coloneqq \xi_{Q(AB)}{}^{Q}$, and $\xi \coloneqq \xi_{PQ}{}^{PQ}$.

2.4 Relation to Lorentzian Spaces

We are now in conditions to establish the relation between spinor spaces and Lorentzian ones (in our context, these are 4-dimensional real vector spaces with a non-degenerate metric with signature $(+,-,-,-)$). For that, the notion of real spinor is crucial and we introduce it next.

2.4.1 Real Spinors and Lorentizan Spaces

A spinor $\xi^{AA'} \in W \otimes \bar{W}$ is *real* if it is equal to its complex conjugate,

$$\bar{\xi}^{AA'} = \xi^{AA'}\,. \tag{2.16}$$

We shall denote by $V \subset W \otimes \bar{W}$ the subset of all real spinors. Since complex conjugation is a linear opeartion, it follows that V has the structure of vector space over the field of *real* numbers. This notion extends naturally to spinors of higher valences, as long as these have the same number of primed and unprimed indices, but is ill-defined otherwise. In other words, real spinors are labelled by pairs of indices of the form AA', and may be thought of as a single index labeling a real vector space. The following proposition shows that Lorentzian spaces emerge naturally from the notions of spinor space and real spinors.

Proposition 2.3. *The vector space V of real spinors satisfying (2.16), together with the metric spinor*

$$g_{AA'BB'} = \epsilon_{AB}\epsilon_{A'B'}\,, \tag{2.17}$$

form a real four-dimensional Lorentzian space with signature $(+,-,-,-)$.

Proof. We will prove this using the following basis of $W \otimes \bar{W}$, constructed out of an orthonormal dyad $\{o_A, \iota_A\}$ of (W, ϵ_{AB}):

$$\begin{aligned} T^{AA'} &= \frac{1}{\sqrt{2}} \left(o^A o^{A'} + \iota^A \iota^{A'} \right) , \\ X^{AA'} &= \frac{1}{\sqrt{2}} \left(o^A \iota^{A'} + \iota^A o^{A'} \right) , \\ Y^{AA'} &= \frac{i}{\sqrt{2}} \left(o^A \iota^{A'} - \iota^A o^{A'} \right) , \\ Z^{AA'} &= \frac{1}{\sqrt{2}} \left(o^A o^{A'} - \iota^A \iota^{A'} \right) , \end{aligned} \tag{2.18}$$

That this is a basis of $W \otimes \bar{W}$ is manifest by its relation to the natural basis $\{o^A o^{A'}, o^A \iota^{A'}, \iota^A o^{A'}, \iota^A o^{A'}\}$. However, observe that all elements in (2.18) are real, that is, $\bar{T}^{AA'} = T^{AA'}, \bar{X}^{AA'} = X^{AA'}$..., so they also form a basis of V. In particular, any $\xi^{AA'} \in V$ is expressed as $\xi^{AA'} = tT^{AA'} + xX^{AA'} + yY^{AA'} + zZ^{AA'}$ with $t, x, y, z \in \mathbb{R}$. Next, we first notice that the metric (2.17) is manifestly real and symmetric in the pairs AA' and BB'. A direct computation (recalling that $o_A \iota^A = 1$) then shows that its components on the basis (2.18) are diag$[+1, -1, -1, -1]$. □

As we will see in the following sections, the description of Lorentzian spaces in terms of spinors is very powerful. The reason is that it allows one to trade tensors by their constituent spinors, which are simpler objects that often make certain proofs and computations more manageable. As illustrative examples we can take the cases of symmetric and skew-symmetric tensors, such as the energy-momentum tensor T_{ab} and the Maxwell field strength F_{ab}, respectively. Their spinor counterparts are $T_{AA'BB'}$ and $F_{AA'BB'}$ with the properties

$$T_{AA'BB'} = T_{BB'AA'} \, , \quad F_{AA'BB'} = -F_{BB'AA'} \, . \tag{2.19}$$

To obtain their irreducible decomposition, we first notice that (2.19) imply

$$T_{AB(A'B')} = T_{(AB)A'B'} = T_{(AB)(A'B')} \, , \quad T_{AB[A'B']} = T_{[AB]A'B'} = T_{[AB][A'B']} \, , \tag{2.20}$$

and

$$F_{AB(A'B')} = F_{[AB]A'B'} = F_{[AB](A'B')} \, , \quad F_{AB[A'B']} = F_{(AB)A'B'} = F_{(AB)[A'B']} \, . \tag{2.21}$$

Then, recalling that the ordering between primed and unprimed indices is irrelevant, we can write

$$\begin{aligned} T_{ABA'B'} =& T_{AB(A'B')} + T_{AB[A'B']} \\ =& T_{(AB)(A'B')} + T_{[AB][A'B']} \\ =& T_{(AB)(A'B')} + \frac{1}{4} \left(\epsilon^{CD} \epsilon^{C'D'} T_{CDC'D'} \right) \epsilon_{AB} \epsilon_{A'B'} \\ \equiv& S_{ABA'B'} + \tau \epsilon_{AB} \epsilon_{A'B'} \, , \end{aligned} \tag{2.22}$$

where in the third equality we applied the irreducible decomposition of a skew-symmetric 2-spinor (recall the proof in Proposition 2.1) relative to both AB and $A'B'$ indices, and in the last equality we introduced the fully symmetric spinor $S_{ABA'B'} = T_{(AB)(A'B')}$ and $\tau = (1/4)T_C{}^C{}_{C'}{}^{C'}$. These are the spinor irreducible components of a symmetric tensor. For a skew-symmetric tensor, we have

$$\begin{aligned} F_{ABA'B'} =& F_{AB(A'B')} + F_{AB[A'B']} \\ =& F_{[AB](A'B')} + F_{(AB)[A'B']} \\ \equiv& \psi_{A'B'}\epsilon_{AB} + \phi_{AB}\epsilon_{A'B'} \,, \end{aligned} \tag{2.23}$$

where we used again the irreducible decomposition of a skew-symmetric 2-spinor and introduced the symmetric spinors

$$\phi_{AB} = \frac{1}{2} F_{ABC'}{}^{C'} \,, \quad \psi_{A'B'} = \frac{1}{2} F_{A'B'C}{}^{C} \,, \tag{2.24}$$

whose symmetry follows by virtue of (2.19) and the see-saw rule (see Exercise 2.2). It will also prove useful to have an expression for the volume form of V in terms of ϵ-spinors.

Exercise 2.5. Show that the metric volume form in V is

$$\epsilon_{AA'BB'CC'DD'} = i\left(\epsilon_{AB}\epsilon_{CD}\epsilon_{A'C'}\epsilon_{B'D'} - \epsilon_{AC}\epsilon_{BD}\epsilon_{A'B'}\epsilon_{C'D'}\right) \,. \tag{2.25}$$

Hint: use the identity $\epsilon_{A[B}\epsilon_{CD]} = 0$, which follows immediately from the fact that W is two-dimensional.

We will employ all of these results later on for the more complicated task of decomposing the curvature tensor. Before moving on, we notice another important example, that of null vector fields, which turn out to be describable in terms of a single spinor.

Exercise 2.6. Let $k^{AA'}$ be a null vector in $(V, g_{AA'BB'})$. Show that it can be expressed as

$$k^{AA'} = \pm \kappa^A \bar{\kappa}^{A'} \,. \tag{2.26}$$

Remark 2.2. Notice that, although a single spinor defines two null vectors through (2.26), a null vector has associated a 1-parameter family of spinors, $e^{i\theta}\kappa^A$, with θ real. A pictorial interpretation of this can be given in terms of Penrose's *flags* and *poles*, that we will not discuss here. We refer the reader to [3].

2.4.2 The Infeld-van der Waerden Symbols

Above we have seen that out of an orthonormal dyad in spinor space $\epsilon_{\mathbf{A}}{}^A$ it is possible to construct an orthonormal basis in the Lorentzian space $(V, g_{AA'BB'})$. We did so identifying the basis (2.18). This construction can be generalisation in terms of the so-called *Infeld-van der Waerden symbols* (IvW). These are a collection of complex numbers, denoted $\sigma_{\mathbf{a}}{}^{\mathbf{AA'}}$, which relate a spinor basis $\epsilon_{\mathbf{A}}{}^{A}$ in (W, ϵ_{AB}) to an orthonormal basis in $(V, g_{AA'BB'})$, via

$$\sigma_{\mathbf{a}}{}^{AA'} = \sigma_{\mathbf{a}}{}^{\mathbf{AA'}} \epsilon_{\mathbf{A}}{}^{A} \epsilon_{\mathbf{A'}}{}^{A'} . \tag{2.27}$$

Here, the index $\mathbf{a}$ takes values from $\mathbf{0}$ to $\mathbf{3}$, and labels the orthonormal basis $\sigma_{\mathbf{a}}{}^{AA'}$. Since (2.27) is a basis of V by assumption, it must be real, and from $\bar{\sigma}_{\mathbf{a}}{}^{AA'} = \sigma_{\mathbf{a}}{}^{AA'}$ it follows that IvW symbols can be seen as a collection of Hermitian matrices,

$$\overline{\sigma_{\mathbf{a}}{}^{\mathbf{AA'}}} = \sigma_{\mathbf{a}}{}^{\mathbf{A'A}} . \tag{2.28}$$

The basis (2.18) corresponds to a particular choice of IvW symbols, as stated in the following exercise.

Exercise 2.7. Given an orthonormal dyad $\epsilon_{\mathbf{A}}{}^{A} = \{o^A, \iota^A\}$, show that the basis (2.18) emerges from (2.27) with the following choice of IvW symbols:

$$\begin{aligned} \sigma_{\mathbf{0}}{}^{\mathbf{AA'}} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad \sigma_{\mathbf{1}}{}^{\mathbf{AA'}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \\ \sigma_{\mathbf{2}}{}^{\mathbf{AA'}} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} , \quad \sigma_{\mathbf{3}}{}^{\mathbf{AA'}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \end{aligned} \tag{2.29}$$

Notice these are, up to global factors, the Pauli matrices. Recall, though, that this is a particular choice of IvW symbols, and that other possibilities exist.

From the assumption that (2.27) is orthonormal with respect to $g_{AA'BB'}$, it follows immediately that

$$\eta_{\mathbf{ab}} = \sigma_{\mathbf{a}}{}^{\mathbf{AA'}} \sigma_{\mathbf{b}}{}^{\mathbf{BB'}} \epsilon_{\mathbf{AB}} \epsilon_{\mathbf{A'B'}} , \tag{2.30}$$

where $\epsilon_{\mathbf{AB}}, \epsilon_{\mathbf{A'B'}}$ are given by (2.9). The following exercise introduces the dual of basis of $\sigma_{\mathbf{a}}{}^{AA'}$, and the inverse IvW symbols.

Exercise 2.8. Show that $\sigma^{\mathbf{a}}{}_{AA'} := \eta^{\mathbf{ab}} \epsilon_{BA} \epsilon_{B'A'} \sigma_{\mathbf{b}}{}^{BB'}$ is the dual of (2.27), that is,

$$\sigma_{\mathbf{b}}{}^{AA'} \sigma^{\mathbf{a}}{}_{AA'} = \delta_{\mathbf{b}}{}^{\mathbf{a}} . \tag{2.31}$$

Show that one can write $\sigma^{\mathbf{a}}{}_{AA'} = \sigma^{\mathbf{a}}{}_{\mathbf{AA'}} \epsilon_{A}{}^{\mathbf{A}} \epsilon_{A'}{}^{\mathbf{A'}}$, where the *inverse IvW* symbols $\sigma^{\mathbf{a}}{}_{\mathbf{AA'}}$ are

$$\sigma^{\mathbf{a}}{}_{\mathbf{AA'}} := \eta^{\mathbf{ab}} \epsilon_{\mathbf{BA}} \epsilon_{\mathbf{B'A'}} \sigma_{\mathbf{b}}{}^{\mathbf{BB'}} . \tag{2.32}$$

Given that $\sigma_{\mathbf{a}}{}^{AA'}$ and $\sigma^{\mathbf{a}}{}_{AA'}$ are the dual of each other, they realise the identity operator in V as

$$\delta_{BB'}{}^{AA'} = \sigma^{\mathbf{a}}{}_{BB'} \sigma_{\mathbf{a}}{}^{AA'} , \tag{2.33}$$

and from this one obtains an analogue of the completeness relation (2.30), satisfied by the inverse IvW symbols,

$$\epsilon_{\mathbf{AB}} \epsilon_{\mathbf{A'B'}} = \sigma^{\mathbf{a}}{}_{\mathbf{AA'}} \sigma^{\mathbf{b}}{}_{\mathbf{BB'}} \eta_{\mathbf{ab}} . \tag{2.34}$$

Exercise 2.9. Obtain the inverse of the IvW symbols given in (2.29), and show they satisfy the completeness relations (2.30) and (2.34).

2.4.3 Lorentz Transformations From Spinor Transformations

At the beginning of this section we advanced that the proper Lorentz and spinor groups are intimately related. We are now in conditions of making that relation more precise. Our starting point is the notion of spinor transformation.

Definition 2.2. A *spinor transformation* is a map relating two orthonormal dyads.

Spinor transformations can be written explicitly as

$$\tilde{\epsilon}_{\mathbf{A}}{}^{A} = L_{\mathbf{A}}{}^{\mathbf{B}}\epsilon_{\mathbf{B}}{}^{A}\,, \quad \tilde{\epsilon}_{A}{}^{\mathbf{A}} = L^{-1\mathbf{A}}{}_{\mathbf{B}}\epsilon_{A}{}^{\mathbf{B}}\,, \tag{2.35}$$

where $L_{\mathbf{A}}{}^{\mathbf{B}}$ and $L^{-1\mathbf{A}}{}_{\mathbf{B}}$ denote a complex invertible 2×2 matrix and its inverse, respectively. The condition that both basis $\tilde{\epsilon}_{\mathbf{A}}{}^{A}$ and $\epsilon_{\mathbf{A}}{}^{A}$ are orthonormal implies that spinor transformations preserve the canonical form (2.9) of the ϵ-spinor,

$$\epsilon_{\mathbf{AB}} = L_{\mathbf{A}}{}^{\mathbf{C}}L_{\mathbf{B}}{}^{\mathbf{D}}\epsilon_{\mathbf{CD}}\,. \tag{2.36}$$

Proposition 2.4. *The group of spinor transformations is $SL(2,\mathbb{C})$, that is, the group of complex 2×2 matrices with unit determinant.*

Proof. By definition,

$$\det(L) = \frac{1}{2}\epsilon_{\mathbf{AB}}\epsilon^{\mathbf{CD}}L_{\mathbf{C}}{}^{\mathbf{A}}L_{\mathbf{D}}{}^{\mathbf{B}}\,. \tag{2.37}$$

It follows that if $\det(L) = 1$, then $\left(\epsilon_{\mathbf{AB}}L_{\mathbf{C}}{}^{\mathbf{A}}L_{\mathbf{D}}{}^{\mathbf{B}}\right)\epsilon^{\mathbf{CD}} = 2$, where the term in brackets is skew-symmetric in $\mathbf{CD}$ so one has $\epsilon_{\mathbf{AB}}L_{\mathbf{C}}{}^{\mathbf{A}}L_{\mathbf{D}}{}^{\mathbf{B}} = \epsilon_{\mathbf{CD}}$, that is, $L_{\mathbf{A}}{}^{\mathbf{B}}$ satisfies (2.36). Conversely, from (2.36) it follows that $(\det(L))^2 = 1$, so either $\det(L) = \pm 1$. However, -1 is excluded because from (2.37) we would be lead to conclude $\epsilon_{\mathbf{AB}} = -L_{\mathbf{A}}{}^{\mathbf{C}}L_{\mathbf{B}}{}^{\mathbf{D}}\epsilon_{\mathbf{CD}}$, contradicting our initial assumption (2.36). □

Consider now two orthonormal spin dyads related by a spinor transformation (2.35). Through the IvW symbols they define two orthonormal basis of V, $\tilde{\sigma}_{\mathbf{a}}{}^{AA'} = \sigma_{\mathbf{a}}{}^{\mathbf{AA'}}\tilde{\epsilon}_{\mathbf{A}}{}^{A}\tilde{\epsilon}_{\mathbf{A'}}{}^{A'}$ and $\sigma_{\mathbf{a}}{}^{AA'} = \sigma_{\mathbf{a}}{}^{\mathbf{AA'}}\epsilon_{\mathbf{A}}{}^{A}\epsilon_{\mathbf{A'}}{}^{A'}$, related by

$$\tilde{\sigma}_{\mathbf{a}}{}^{AA'} = \sigma_{\mathbf{a}}{}^{\mathbf{AA'}}\tilde{\epsilon}_{\mathbf{A}}{}^{A}\tilde{\epsilon}_{\mathbf{A'}}{}^{A'} = \sigma_{\mathbf{a}}{}^{\mathbf{AA'}}L_{\mathbf{A}}{}^{\mathbf{B}}\bar{L}_{\mathbf{A'}}{}^{\mathbf{B'}}\epsilon_{\mathbf{B}}{}^{A}\epsilon_{\mathbf{B'}}{}^{A'} = \Lambda_{\mathbf{a}}{}^{\mathbf{b}}\sigma_{\mathbf{b}}{}^{AA'}\,, \tag{2.38}$$

where in the last equality we employed the completeness relation (2.34) and introduced

$$\Lambda_{\mathbf{a}}{}^{\mathbf{b}} := \sigma_{\mathbf{a}}{}^{\mathbf{AA'}}L_{\mathbf{A}}{}^{\mathbf{B}}\bar{L}_{\mathbf{A'}}{}^{\mathbf{B'}}\sigma^{\mathbf{b}}{}_{\mathbf{BB'}}\,. \tag{2.39}$$

Since $\tilde{\sigma}_{\mathbf{a}}{}^{AA'}$ and $\sigma_{\mathbf{a}}{}^{AA'}$ are both orthonormal with respect to $g_{AA'BB'}$, it follows that

$$\eta_{\mathbf{ab}} = g_{AA'BB'}\tilde{\sigma}_{\mathbf{a}}{}^{AA'}\tilde{\sigma}_{\mathbf{b}}{}^{BB'} = \Lambda_{\mathbf{a}}{}^{\mathbf{c}}\Lambda_{\mathbf{b}}{}^{\mathbf{d}}g_{AA'BB'}\sigma_{\mathbf{c}}{}^{AA'}\sigma_{\mathbf{d}}{}^{BB'} = \Lambda_{\mathbf{a}}{}^{\mathbf{c}}\Lambda_{\mathbf{b}}{}^{\mathbf{d}}\eta_{\mathbf{cd}}\,. \tag{2.40}$$

That is, we have shown that a spinor transformation in W induces a Lorentz transformation in V through (2.39). This makes explicit the relation between spinor and Lorentz transformations. However, this is not a one-to-one correspondance, since the two elements $\pm L_{\mathbf{A}}{}^{\mathbf{B}} \in SL(2,\mathbb{C})$ induce the same Lorentz transformation. More precisely, it can be shown that $L_{\mathbf{A}}{}^{\mathbf{B}}, M_{\mathbf{A}}{}^{\mathbf{B}} \in SL(2,\mathbb{C})$ induce the same Lorentz transformation if and only if $L_{\mathbf{A}}{}^{\mathbf{B}} = \pm M_{\mathbf{A}}{}^{\mathbf{B}}$, so (2.39) is exactly a two-to-one map from $SL(2,\mathbb{C})$ into the proper Lorentz group.

2.4.4 The Soldering Form

So far we have restricted our discussion to spinor space (W, ϵ_{AB}), and the relation to Lorentzian spaces has been established in terms of $(V, g_{AA'BB'})$, with $V \subset W \otimes \bar{W}$ the vector subspace of real spinors and $g_{AA'BB'} = \epsilon_{AB}\epsilon_{A'B'}$. It is also convenient to establish a formal connection between $(V, g_{AA'BB'})$ and any Lorentzian space $(\mathcal{V}, g_{ab})$ (where lower-case latin indices denote abstract indices in $\mathcal{V}$).

Let $e_{\mathbf{a}}{}^{a}$ be an orthonormal basis of $\mathcal{V}$ and $\epsilon_{\mathbf{A}}{}^{A}$ an orthonormal dyad in W, and denote their duals by $e_{a}{}^{\mathbf{a}}$ and $\epsilon_{A}{}^{\mathbf{A}}$. The dyad defines an orthonormal basis of V through $\sigma_{\mathbf{a}}{}^{AA'} = \sigma_{\mathbf{a}}{}^{\mathbf{AA'}}\epsilon_{\mathbf{A}}{}^{A}\epsilon_{\mathbf{A'}}{}^{A'}$. We introduce the *soldering form* $\sigma_{a}{}^{AA'}$ *and its inverse* $\sigma^{a}{}_{AA'}$ *relative to* $e_{\mathbf{a}}{}^{a}$ *and* $\epsilon_{\mathbf{A}}{}^{A}$ as the following tensors in $\mathcal{V}^* \otimes V$ and $\mathcal{V} \otimes V^*$,

$$\sigma_{a}{}^{AA'} := e_{a}{}^{\mathbf{a}}\sigma_{\mathbf{a}}{}^{AA'} = \sigma_{\mathbf{a}}{}^{\mathbf{AA'}}e_{a}{}^{\mathbf{a}}\epsilon_{\mathbf{A}}{}^{A}\epsilon_{\mathbf{A'}}{}^{A'}\,, \quad \sigma^{a}{}_{AA'} := e_{\mathbf{a}}{}^{a}\sigma^{\mathbf{a}}{}_{AA'} = \sigma^{\mathbf{a}}{}_{\mathbf{AA'}}e_{\mathbf{a}}{}^{a}\epsilon_{A}{}^{\mathbf{A}}\epsilon_{A'}{}^{\mathbf{A'}}\,. \tag{2.41}$$

From its definition, it is clear that the soldering form satisfies the completeness relations

$$\delta_{b}{}^{a} = \sigma_{b}{}^{AA'}\sigma^{a}{}_{AA'}\,, \quad \delta_{BB'}{}^{AA'} = \sigma^{a}{}_{BB'}\sigma_{a}{}^{AA'}\,, \tag{2.42}$$

so it can be used to correspond tensors between $\mathcal{V}$ and V. For example, a real spinor $T^{AA'}{}_{BB'}$ in V and its tensorial counterpart $T^{a}_{\ b}$ in $\mathcal{V}$ are related via

$$T^{a}_{\ b} = \sigma^{a}{}_{AA'}\sigma_{b}{}^{BB'}T^{AA'}{}_{BB'}\,, \quad T^{AA'}{}_{BB'} = \sigma_{a}{}^{AA'}\sigma^{b}{}_{BB'}T^{a}_{\ b}\,. \tag{2.43}$$

The relation between metrics, given in the following exercise, is particularly important.

Exercise 2.10. Show that the metrics in V and $\mathcal{V}$ correspond through the soldering form, that is,

$$g_{AA'BB'} = \sigma^{a}{}_{AA'}\sigma^{b}{}_{BB'}g_{ab}\,, \quad g_{ab} = \sigma_{a}{}^{AA'}\sigma_{b}{}^{BB'}g_{AA'BB'}. \tag{2.44}$$

From that, show that the soldering form and its inverse are related through the usual notion of raising and lowering indices with respect to g_{ab} and $g_{AA'BB'}$, that is,

$$\sigma_{a}{}^{AA'} = g^{AA'BB'}g_{ab}\sigma^{b}{}_{BB'}\,. \tag{2.45}$$

There is certain arbitrariness in the relation between tensors in V and $\mathcal{V}$. This is due to the fact that there is no natural way to make a unique choice of orthonormal frame and dyad with which one constructs the soldering form. However, some choices yield equivalent answers, and some relations are universal, in the sense that they do not depend on the choice of basis. This stated in the following exercise.

Exercise 2.11. Let $(\tilde{e}_{\mathbf{a}}{}^{a}, \tilde{\epsilon}_{\mathbf{A}}{}^{A})$ and $(e_{\mathbf{a}}{}^{a}, \epsilon_{\mathbf{A}}{}^{A})$ be two pairs of orthonormal basis and dyads. Assume they are related by a spinor transformation $L_{\mathbf{B}}{}^{\mathbf{A}}$ and the corresponding Lorentz transformation $\Lambda_{\mathbf{b}}{}^{\mathbf{a}}$ (see (2.39)). Show that they define the same soldering form. Show that the relations (2.44) do not depend on the choice of basis employed to construct the solder form.

Remark 2.3. It is customary that, once a solder form is constructed, the tensors in $\mathcal{V}$ and corresponding spinors in V are identified, and regarded as one and the same. From that perspective, one can fearlessly write equations of the form

$$g_{ab} = g_{AA'BB'}\,, \quad T_{ab} = T_{AA'BB'}\,, \quad k^a = k^{AA'}\,, \quad \ldots \tag{2.46}$$

Both conventions will be employed extensively throughout the rest of the notes.

2.4.5 Null Frames

In the previous sections we have related orthonormal dyads in spinor space to orthonormal frames in Lorentz space. Another notion that will be very useful is that of a *null frame* relative to an orthonormal dyad $\{o^A, \iota^A\}$. It is defined as the following set of vectors constructed out of $\{o^A, \iota^A\}$,

$$l^a := \sigma^a{}_{AA'}o^A\bar{o}^{A'}\,, \quad n^a := \sigma^a{}_{AA'}\iota^A\bar{\iota}^{A'}\,, \quad m^a := \sigma^a{}_{AA'}o^A\bar{\iota}^{A'}\,, \quad \bar{m}^a := \sigma^a{}_{AA'}\iota^A\bar{o}^{A'}\,. \tag{2.47}$$

These constitute an orthonormal null basis, in the sense that the only non-vanishing metric products among them are

$$l^a n_a = -m^a\bar{m}_a = 1\,, \tag{2.48}$$

as can be verified explicitly, so one has

$$g_{ab} = 2l_{(a}n_{b)} - 2m_{(a}\bar{m}_{b)}\,, \quad g^{ab} = 2l^{(a}n^{b)} - 2m^{(a}\bar{m}^{b)}\,. \tag{2.49}$$

In particular, l^a and n^a are real null vectors, while m^a and its conjugate $\bar{m}^a$ are also null, but complex-valued. Strictly, (2.47) forms a basis of the complexification of Lorentzian space $\mathcal{V}$, which contains the real space as a subspace.

Exercise 2.12. Choosing the Pauli matrices (2.29) as IvW symbols to construct the soldering form, show that the null tetrad is related to the orthonormal basis through

$$l^a = \frac{1}{\sqrt{2}}\left(e_0{}^a + e_3{}^a\right)\,, \quad n^a = \frac{1}{\sqrt{2}}\left(e_0{}^a - e_3{}^a\right)\,, \quad m^a = \frac{1}{\sqrt{2}}\left(e_1{}^a - ie_2{}^a\right)\,. \tag{2.50}$$

3 Spinors in Spacetime

So far we have considered spinors as elements in an abstract vector space. Now we wish to associate a spinor space to each point of the spacetime manifold M, in a similar way to how we define tangent space (abstractly just a four-dimensional vector space) as the vector space of derivations at each point. This is done most naturally in the language of principal bundles, which is not assumed as a pre-requisite in these lecture notes. However, introducing such notions would yield a long digression that departs from our main objectives. Thus, here we will simply assume that such construction can be made consistently, so that each point in spacetime x has associated a spinor space, $\mathfrak{S}(M)|_x$. The collection of all such spinor spaces accross spacetime forms the spinor bundle, denoted $\mathfrak{S}(M)$, similarly to how the collection of all tangent spaces $T(M)|_x$ accross spacetime forms the tangent bundle $T(M)$. Then, a spinor in spacetime is a smooth map

$$\begin{aligned} \xi^A\colon M &\to \mathfrak{S}(M) \\ x &\mapsto \left(x, \xi^A|_x\right) \end{aligned} \tag{3.1}$$

where $\xi^A|_x \in \mathfrak{S}(M)|_x$. Notice that this is in complete analogy to how we define vector fields in spacetime. In addition, the construction of $\mathfrak{S}(M)$ is such that the action of a Lorentz transformation of frames in tangent space at each point is accompanied by the action of a spinor transformation of dyads in spinor space,

$$e_{\mathbf{a}}{}^a|_x \mapsto \Lambda_{\mathbf{a}}{}^{\mathbf{b}}\left(e_{\mathbf{b}}{}^a|_x\right) \,, \quad \epsilon_{\mathbf{A}}{}^A|_x \mapsto L_{\mathbf{A}}{}^{\mathbf{B}}\left(\epsilon_{\mathbf{B}}{}^A|_x\right) \,, \tag{3.2}$$

where $\Lambda_{\mathbf{a}}{}^{\mathbf{b}}$ and $L_{\mathbf{A}}{}^{\mathbf{B}}$ correspond according to (2.39). However, it should be stressed that there are some global (topological) obstructions in associating tensors to spinors consistently everywhere in spacetime. Roughly, one must require spacetime to be suitably orientable in order to admit a spinor structure. Here we will simply note that every globally hyperbolic spacetime has a spinor structure [13] and that this is unique if, in addition, M is simply connected [14]. We will not elaborate more on this here, but we refer the reader to Chapter 13 of Wald's book [6] for a detailed introduction to the spinor bundle, and [3] for a more complete discussion.

Under these assumptions, all the concepts introduced above have a smooth extension to the entire manifold M and apply at each point. However, there is still no natural way of relating spinors at different spacetime points. Just as in the case of ordinary tensors, such relation can be defined in terms of a spin connection, or spinor covariant derivative, that we introduce next.

3.1 Spinor Covariant Derivative

Similarly to the case of ordinary tensor fields, the spinor covariant derivative of a spinor $\xi^{A...B'}{}_{C...D'}$ along a vector field X is another spinor field with the same structure, denoted $\nabla_X \xi^{A...B'}{}_{C...D'}$, which provides a well defined notion of propagation of $\xi^{A...B'}{}_{C...D'}$ along X. Furthermore, by using either the tensor or spinor representation of X, this can be written as

$$\nabla_X \xi^{A...B'}{}_{C...D'} = X^a \nabla_a \xi^{A...B'}{}_{C...D'} = X^{EE'} \nabla_{EE'} \xi^{A...B'}{}_{C...D'} \,, \tag{3.3}$$

where $\nabla \xi^{A...B'}{}_{C...D'}$ is the covariant derivative of $\xi^{A...B'}{}_{C...D'}$. It is customary to formalise the previous concepts axiomatically as follows.

Definition 3.1. A spinor covariant derivative $\nabla_{AA'}$ is a map

$$\begin{aligned} \nabla_{AA'} : \mathfrak{S}^{B'...C'}{}_{D...E'} &\to \mathfrak{S}^{B'...C'}{}_{AD...A'E'} \\ \xi^{B...C'}{}_{D...E'} &\mapsto \nabla_{AA'} \xi^{B...C'}{}_{D...E'} \end{aligned} \tag{3.4}$$

that for any spinors $\xi^{B...C'}{}_{D...E'}, \chi^{B...C'}{}_{D...E'} \in \mathfrak{S}^{B'...C'}{}_{D...E'}$ satisfies:

(i) *Linearity*:

$$\nabla_{AA'} \left(\xi^{B...C'}{}_{D...E'} + \chi^{B...C'}{}_{D...E'} \right) = \nabla_{AA'} \xi^{B...C'}{}_{D...E'} + \nabla_{AA'} \chi^{B...C'}{}_{D...E'} \tag{3.5}$$

(ii) *Leibniz rule*:

$$\nabla_{AA'} \left(\xi^{B...C'}{}_{D...E'} \chi^{E...F'}{}_{G...H'} \right) = \chi^{E...F'}{}_{G...H'} \nabla_{AA'} \xi^{B...C'}{}_{D...E'} + \xi^{E...F'}{}_{G...H'} \nabla_{AA'} \chi^{B...C'}{}_{D...E'} \tag{3.6}$$

(iii) *Hermiticity*:

$$\overline{\nabla_{AA'} \xi^{B...C'}{}_{D...E'}} = \nabla_{AA'} \bar{\xi}^{B'...C}{}_{D'...E} \tag{3.7}$$

(iv) *Action on scalars*: $\nabla_{AA'}\phi$ is the spinor counterpart of $\nabla_a \phi$ for any function ϕ in M.

(v) *Representation of derivations*: the action of any derivation D on spinors can be written as $D\xi^{B'...C}{}_{D'...E} = \zeta^{AA'} \nabla_{AA'} \xi^{B'...C}{}_{D'...E}$ for some spinor $\zeta^{AA'} \in \mathfrak{S}^{AA'}$.

This definition does not single out a unique derivative operator. However, in a way analogous to the covariant derivative of tensors, if one further requires that ∇ has vanishing torsion and that ϵ_{AB} is covariantly constant,

$$\nabla_{AA'} \epsilon_{BC} = 0 \,, \tag{3.8}$$

then ∇ is uniquely determined and, when acting on tensor fields, reduces to the Levi-Civita connection associated to g_{ab}. This is the connection that will be used in the remainder of the text.

Remark 3.1. Relating the tensorial Levi-Civita connection to the torsion-free spinor connection satisfying (3.8) is not entirely trivial, and can be approached from either an abstract perspective or via component analysis. We will not elaborate further on this here, but we refer the reader to [3] for an extensive discussion (Section 4.4 for the abstract approach, and Section 4.5 for the component derivation).

Let $\epsilon_{\mathbf{A}}{}^{A}$ be a spinor dyad, and consider the associated basis on $\mathfrak{S}^{AA'}$, denoted $e_{\mathbf{AA'}}{}^{AA'} := \epsilon_{\mathbf{A}}{}^{A}\epsilon_{\mathbf{A'}}{}^{A'}$. It is possible to specify $\nabla_{AA'}$ in terms of its components relative to the dyad, also called the *spin connection coefficients*, given by

$$\Gamma_{\mathbf{AA'}}{}^{\mathbf{B}}{}_{\mathbf{C}} := \epsilon_{B}{}^{\mathbf{B}} e_{\mathbf{AA'}}{}^{AA'} \nabla_{AA'} \epsilon_{\mathbf{C}}{}^{B} \,. \tag{3.9}$$

For instance, the components of $\nabla_{AA'}\xi^{B}$ are

$$\begin{aligned}
\nabla_{\mathbf{AA'}}\xi^{\mathbf{B}} =& \epsilon_{B}{}^{\mathbf{B}} e_{\mathbf{AA'}}{}^{AA'} \nabla_{AA'}\xi^{B} \\
=& e_{\mathbf{AA'}}{}^{AA'}\nabla_{AA'}\left(\xi^{B}\epsilon_{B}{}^{\mathbf{B}}\right) - e_{\mathbf{AA'}}{}^{AA'}\xi^{B}\nabla_{AA'}\left(\epsilon_{B}{}^{\mathbf{B}}\right) \\
=& e_{\mathbf{AA'}}{}^{AA'}\nabla_{AA'}\left(\xi^{\mathbf{B}}\right) - e_{\mathbf{AA'}}{}^{AA'}\xi^{\mathbf{C}}\epsilon_{\mathbf{C}}{}^{B}\nabla_{AA'}\left(\epsilon_{B}{}^{\mathbf{B}}\right) \\
=& e_{\mathbf{AA'}}{}^{AA'}\nabla_{AA'}\left(\xi^{\mathbf{B}}\right) + e_{\mathbf{AA'}}{}^{AA'}\xi^{\mathbf{C}}\epsilon_{B}{}^{\mathbf{B}}\nabla_{AA'}\left(\epsilon_{\mathbf{C}}{}^{B}\right) \\
=& e_{\mathbf{AA'}}{}^{AA'}\nabla_{AA'}\left(\xi^{\mathbf{B}}\right) + \Gamma_{\mathbf{AA'}}{}^{\mathbf{B}}{}_{\mathbf{C}}\xi^{\mathbf{C}} \,,
\end{aligned} \tag{3.10}$$

where in the fourth equality we used that $\nabla_{AA'}(\delta_{\mathbf{C}}{}^{\mathbf{B}}) = 0$. Another important example is that of a spinor with structure $\xi^{AA'}$, which includes the case of real vectors. A computation similar to (3.10) yields

$$\nabla_{\mathbf{AA'}}\xi^{\mathbf{BB'}} = e_{\mathbf{AA'}}{}^{AA'}\nabla_{AA'}\left(\xi^{\mathbf{BB'}}\right) + \Gamma_{\mathbf{AA'}}{}^{\mathbf{BB'}}{}_{\mathbf{CC'}}\xi^{\mathbf{CC'}} \,, \tag{3.11}$$

where we introduced

$$\Gamma_{\mathbf{AA'}}{}^{\mathbf{BB'}}{}_{\mathbf{CC'}} = \Gamma_{\mathbf{AA'}}{}^{\mathbf{B}}{}_{\mathbf{C}}\delta_{\mathbf{C'}}{}^{\mathbf{B'}} + \bar{\Gamma}_{\mathbf{AA'}}{}^{\mathbf{B'}}{}_{\mathbf{C'}}\delta_{\mathbf{C}}{}^{\mathbf{B}} \,. \tag{3.12}$$

The last expression can be seen as the spinor counterpart of the connection components of ∇ when restricted to tensor fields.

Exercise 3.1. Obtain the components of $\nabla_{AA'}\xi_{B}$. Use this and (3.10) to obtain the components of the covariant derivative of a general spinor field.

Exercise 3.2. Let $\epsilon_{A}{}^{\mathbf{A}}$ be a *rigid dyad*, meaning that the associated components of ϵ_{AB} are constants. Show that its spin coefficients satisfy

$$\Gamma_{\mathbf{AA'(BC)}} = \Gamma_{\mathbf{AA'BC}} \,. \tag{3.13}$$

3.2 Irreducible Decomposition of the Riemann Tensor

The spinor version of the Riemann tensor is $R_{AA'BB'CC'DD'}$. It satisfies the spinor equivalent of all algebraic and differential identities of the Riemann. One of the most powerful results of the spinor formalism is the irreducible decomposition of the curvature tensor, given in the following proposition.

Proposition 3.1. *Let $R_{AA'BB'CC'DD'}$ be the spinor counterpart of the Riemann tensor. Its irreducible decomposition is*

$$\begin{aligned} R_{AA'BB'CC'DD'} =& -\Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} - \bar{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} \\ & - \Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} - \Phi_{CDA'B'}\epsilon_{AB}\epsilon_{C'D'} \\ & + 2\Lambda\left(\epsilon_{AC}\epsilon_{BD}\epsilon_{A'C'}\epsilon_{B'D'} - \epsilon_{AD}\epsilon_{BC}\epsilon_{A'D'}\epsilon_{B'C'}\right) \end{aligned} \tag{3.14}$$

where Ψ_{ABCD} is a fully symmetric spinor, $\Phi_{ABC'D'}$ is a real and fully symmetric spinor, and Λ is a real function the are given in terms of the Riemann tensor by

$$\begin{aligned} \Psi_{ABCD} &= -\frac{1}{4}R_{(ABCD)X'}{}^{X'}{}_{Y'}{}^{Y'}\,, \\ \Phi_{ABC'D'} &= -\frac{1}{4}R_{ABX}{}^{X}{}_{C'D'Y'}{}^{Y'}\,, \\ \Lambda &= \frac{1}{24}R_{AB}{}^{AB}{}_{X'}{}^{X'}{}_{Y'}{}^{Y'}\,. \end{aligned} \tag{3.15}$$

Proof. This can be shown by implementing on $R_{AA'BB'CC'DD'}$ the algebraic symmetries of the Riemann, namely (i) $R_{abcd} = -R_{bacd} = -R_{abdc}$, (ii) $R_{abcd} = R_{cdab}$ and (iii) $R_{a[bcd]} = 0$. To implement (i) we will use the decomposition of a skew-symmetric tensor deduced previously in (2.23). Applying it first on the pair of indices AA' and BB' and subsequently on CC' and DD' gives

$$\begin{aligned} R_{AA'BB'CC'DD'} =& \frac{1}{2}R_{X}{}^{X}{}_{CDA'B'C'D'}\epsilon_{AB} + \frac{1}{2}R_{ABCDX'}{}^{X'}{}_{C'D'}\epsilon_{A'B'} \\ =& \frac{1}{4}R_{X}{}^{X}{}_{Y}{}^{Y}{}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} + \frac{1}{4}R_{X}{}^{X}{}_{CDA'B'X'X'}\epsilon_{AB}\epsilon_{C'D'} \\ & + \frac{1}{4}R_{ABX}{}^{X}{}_{X'}{}^{X'}{}_{C'D'}\epsilon_{A'B'}\epsilon_{CD} + \frac{1}{4}R_{ABCDX'}{}^{X'}{}_{Y'}{}^{Y'}\epsilon_{A'B'}\epsilon_{C'D'} \\ =& - X_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} - \Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} \\ & - \bar{X}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} - \bar{\Phi}_{A'B'CD}\epsilon_{AB}\epsilon_{C'D'}\,, \end{aligned} \tag{3.16}$$

where we introduced the so called *curvature spinors*,

$$X_{ABCD} \coloneqq -\frac{1}{4}R_{ABCDX'}{}^{X'}{}_{Y'}{}^{Y'}\,, \quad \Phi_{ABC'D'} \coloneqq -\frac{1}{4}R_{ABX}{}^{X}{}_{X'}{}^{X'}{}_{C'D'}\,. \tag{3.17}$$

The Riemann symmetries (i) imply

$$X_{ABCD} = -\frac{1}{4}R_{ABCDX'}{}^{X'}{}_{Y'}{}^{Y'} = \frac{1}{4}R_{BACD}{}^{X'}{}_{X'Y'}{}^{Y'} = -\frac{1}{4}R_{BACDX'}{}^{X'}{}_{Y'}{}^{Y'} = X_{BACD}\,, \tag{3.18}$$

and, similarly, $X_{ABCD} = X_{ABDC}$. An analogous argument applies to $\Phi_{ABC'D'}$ so, in sum, we have

$$X_{ABCD} = X_{(AB)(CD)}\,, \quad \Phi_{ABC'D'} = \Phi_{(AB)(C'D')}\,. \tag{3.19}$$

On the other hand, from the symmetries (ii) one finds

$$\begin{aligned} X_{ABCD} &= -\frac{1}{4}R_{ABCDX'}{}^{X'}{}_{Y'}{}^{Y'} = -\frac{1}{4}R_{CDABY'}{}^{Y'}{}_{X'}{}^{X'} = X_{CDAB}\,, \\ \Phi_{ABC'D'} &= -\frac{1}{4}R_{ABX}{}^{X}{}_{X'}{}^{X'}{}_{C'D'} = -\frac{1}{4}R_{X}{}^{X}{}_{ABC'D'X'}{}^{X'} = \bar{\Phi}_{ABC'D'}\,. \end{aligned} \tag{3.20}$$

In words, we have found that $\Phi_{ABC'D'}$ is a real, symmetric (and therefore trace-free) spinor. By the symmetries of (3.19), it also follows that the only trace of X_{ABCD} is $X_{ABC}{}^{A}$. Moreover, this must be skew-symmetric in BC, since

$$2X_{A(BC)}{}^{A} = X_{ABC}{}^{A} + X_{ACB}{}^{A} = X_{BA}{}^{A}{}_{C} + X_{B}{}^{A}{}_{AC} = 0\,, \tag{3.21}$$

where we used (3.19) and (3.20) in the second equality, and the see-saw rule in the third. Thus, one has

$$X_{ABC}{}^{A} = -3\Lambda\epsilon_{BC}\,, \qquad \Lambda \coloneqq -(1/6)X_{AB}{}^{AB}\,. \tag{3.22}$$

In order to implement the last algebraic symmetry, (iii), it is convenient to rewrite it in terms of its dual, which reads $R^{\star ab}{}_{bc} = 0$, where $R^{\star}{}_{abcd} = (1/2!)R_{ab}{}^{ef}\epsilon_{cdef}$ is the right dual of the Riemann tensor (remind yourself why $R^{\star ab}{}_{bc} = 0$ is equivalent to the cyclic identity (iii)!). The spinor version of $R^{\star ab}{}_{bc} = 0$ is, using the spinor analogue of the volume form (2.25),

$$\begin{aligned}
0 &= R^{\star AA'BB'}{}_{BB'CC'} \\
&= \frac{1}{2}R^{AA'BB'DD'EE'}\epsilon_{BB'CC'DD'EE'} \\
&= \frac{1}{2}\left(-X^{ABDE}\epsilon^{A'B'}\epsilon^{D'E'} + \Phi^{ABD'E'}\epsilon^{A'B'}\epsilon^{DE}\right)\epsilon_{BB'CC'DD'EE'} + c.c. \\
&= i\Phi^{A}{}_{C}{}^{A'}{}_{C'} + i\delta_{C'}{}^{A'}X^{A}{}_{D}{}^{D}{}_{C} + c.c. \\
&= i\delta_{C'}{}^{A'}X^{A}{}_{D}{}^{D}{}_{C} - i\delta_{C}{}^{A}\bar{X}^{A'}{}_{D'}{}^{D'}{}_{C'} \\
&= 3i\delta_{C}{}^{A}\delta_{C'}{}^{A'}\left(\Lambda - \bar{\Lambda}\right)
\end{aligned} \tag{3.23}$$

where in the third equality we used (3.16), in the fourth (2.25), in the fifth the reality condition for Φ coming from (3.20), and in the last (3.22). We thus conclude that Λ is real. Using the decomposition of a valence-4 spinor, given in (2.15), taking into account the symmetries of X_{ABCD} from (3.19) and (3.20), we arrive at

$$X_{ABCD} = \Psi_{ABCD} + 2\Lambda\epsilon_{A(C}\epsilon_{D)B} \tag{3.24}$$

where we introduced

$$\Psi_{ABCD} \coloneqq X_{(ABCD)}\,. \tag{3.25}$$

Plugging (3.24) into (3.16) we arrive at

$$\begin{aligned}
R_{AA'BB'CC'DD'} = &- \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} - \bar{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} \\
&- \Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} - \Phi_{A'B'CD}\epsilon_{AB}\epsilon_{C'D'} \\
&- 2\Lambda\left(\epsilon_{A(C}\epsilon_{D)B}\epsilon_{A'B'}\epsilon_{C'D'} + \epsilon_{A'(C'}\epsilon_{D')B'}\epsilon_{AB}\epsilon_{CD}\right)
\end{aligned} \tag{3.26}$$

Finally, the last term above can be imrpoved to make the symmetries (i) and (ii) of the Riemann manifest. Using that $\epsilon_{A[B}\epsilon_{CD]} = 0$ (from the 2-dimensionality of W) or, equivalently,

$$\epsilon_{AB}\epsilon_{CD} - \epsilon_{AC}\epsilon_{BD} + \epsilon_{AD}\epsilon_{BC} = 0 \tag{3.27}$$

and the complex conjugate version of this equation, the bracket in the last term of (3.26) can be recast in the form

$$\begin{aligned}&\Big(\epsilon_{A(C}\epsilon_{D)B}\epsilon_{A'B'}\epsilon_{C'D'}+\epsilon_{A'(C'}\epsilon_{D')B'}\epsilon_{AB}\epsilon_{CD}\Big)=\\ =&\frac{1}{2}\left(\epsilon_{AC}\epsilon_{DB}+\epsilon_{AD}\epsilon_{CB}\right)\left(\epsilon_{A'C'}\epsilon_{B'D'}-\epsilon_{A'D'}\epsilon_{B'C'}\right)+\frac{1}{2}\left(\epsilon_{A'C'}\epsilon_{D'B'}+\epsilon_{A'D'}\epsilon_{C'B'}\right)\left(\epsilon_{AC}\epsilon_{BD}-\epsilon_{AD}\epsilon_{BC}\right)\\ =&-\epsilon_{AC}\epsilon_{BD}\epsilon_{A'C'}\epsilon_{B'D'}+\epsilon_{AD}\epsilon_{BC}\epsilon_{A'D'}\epsilon_{B'C'}\,,\end{aligned} \tag{3.28}$$

and this completes the proof. □

Remark 3.2. The artificial minus signs in the definitions of $\Psi_{ABCD}, \Phi_{ABC'D'}$ are there to match their definitions in [3], who use the opposite sign for the Riemann. Thus, the spacetime spinors $\Psi_{ABCD}, \Phi_{ABC'D'}$ constructed from either convention are guaranteed to coincide. The sign of Λ has been chosen so that it is related to the trace of the Ricci tensor as in [1].

Exercise 3.3. The left and right duals of the Riemann tensors are defined respectively as ${}^{\star}R_{abcd} = (1/2)\epsilon_{ab}{}^{ef}R_{efcd}$ and $R^{\star}{}_{abcd} = (1/2)\epsilon_{cd}{}^{ef}R_{abef}$. Show that, in terms of the curvature spinors (3.17), their spinorial versions are

$$\begin{aligned}{}^{\star}R_{AA'BB'CC'DD'} =& iX_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'}+i\Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD}\\ &-i\bar{X}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD}-i\Phi_{A'B'CD}\epsilon_{AB}\epsilon_{C'D'}\,,\\ R^{\star}{}_{AA'BB'CC'DD'} =& iX_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'}-i\Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD}\\ &-i\bar{X}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD}+i\Phi_{A'B'CD}\epsilon_{AB}\epsilon_{C'D'}\,.\end{aligned} \tag{3.29}$$

From Proposition 3.1, it is straightforward to obtain the irreducible decomposition of the Ricci tensor,

$$R_{AA'BB'} = R^{CC'}{}_{AA'CC'BB'} = 2\Phi_{ABA'B'} + 6\Lambda\epsilon_{AB}\epsilon_{A'B'}\,, \tag{3.30}$$

from which we conclude that $\Phi_{ABA'B'}$ and Λ are its irreducible components ($\Phi_{ABA'B'}$ is the traceless part and Λ gives its trace by $R = R^{CC'}{}_{CC'} = 24\Lambda$). It follows that the spinorial version of Einstein's equations in vacuum is

$$\Phi_{ABA'B'} = 0\,, \quad \Lambda = 0\,, \quad \textit{(vacuum Einstein's equations)}\,. \tag{3.31}$$

The spinor Ψ_{ABCD} is the *Weyl spinor*, since it turns out to be the only irreducible component of the Weyl tensor, as you are encouraged to verify next.

Exercise 3.4. Recall that the Weyl tensor is defined as the traceless part of the Riemann tensor via

$$C_{ab}{}^{cd} \coloneqq R_{ab}{}^{cd} - 2R_{[a}{}^{[c}g_{b]}{}^{d]} + \frac{1}{3}R\delta_{[a}{}^{c}\delta_{b]}{}^{d}\,. \tag{3.32}$$

Show that its spinorial counterpart is

$$C_{AA'BB'CC'DD'} = -\Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} - \bar{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} \,. \tag{3.33}$$

From this, and Exercise 3.3, conclude that the left and right duals of the Weyl tensor coincide.

Exercise 3.5. Show that the Weyl and Riemann tensors have ten and twenty real independent components, respectively. (*Hint:* recall Proposition 2.2).

The irreducible decomposition of the curvature (Propostion 3.1) together with the vacuum Einstein's equations yield the most important lesson of this section: *according to general relativity, the curvature of spacetime in the absence of matter is described by a single, fully symmetric spinor Ψ_{ABCD}.* In other words, we have learned that the curvature of spacetime in vacuum is unexpectedly simple (that Ψ_{ABCD} is, in many respects, a simple object will be even more clear in the next sections). The spinorial description of gravity is a natural one precisely because it makes this simplicity manifest. Note that deriving, or even writing, this result by tensorial methods would be extremely involved and not natural. Next, in Sections 3.3 and 3.4, we will derive two relatively immediate consequences of this fact that had groundbreaking implications in gravitational physics.

3.3 Petrov's Classification of the Weyl Tensor

We start by stating the following property of fully symmetric spinors, such as Ψ_{ABCD}.

Proposition 3.2. *Let $\xi_{AB..C} = \xi_{(AB...C)}$ be a non-vanishing, fully symmetric spinor of valence n. Then*

$$\xi_{AB...C} = \alpha_{(A}\beta_B...\gamma_{C)} \,, \tag{3.34}$$

where the n spinors $\alpha_A, \beta_A, ...\gamma_A$ are unique up to proportionality.

Proof. This follows as a simple consequence of the fundamental theorem of algebra, see Proposition 3.5.18 of [3]. □

In the case of the Weyl spinor, we shall write this as

$$\Psi_{ABCD} = \kappa^{(1)}_{(A}\kappa^{(2)}_B\kappa^{(3)}_C\kappa^{(4)}_{D)} \,. \tag{3.35}$$

This is sometimes called the canonical decomposition of the Weyl spinor, and $\kappa^{(i)}_A$ are the associated *principal spinors*. Recall from Exercise 2.6 that any valence-1 spinor ξ_A yields a real null vector through $\pm\xi_A\bar{\xi}_{A'}$. Thus, the principal spinors of the Weyl tensor define four null directions, consisting on the proportionality classes of

$$k^{(i)}_{AA'} := \kappa^{(i)}_A\bar{\kappa}^{(i)}_{A'} \,, \quad (i = 1, ..., 4) \,. \tag{3.36}$$

These are called the *principal null directions of the Weyl tensor* (or PNDs for short). If two PNDs coincide, we say that is a *repeated PND*. Therefore, the Weyl tensor at each point (if nonvanishing) belongs to one of the following five types, depending on the degeneracy structure of its PNDs:

$$\begin{aligned} \textit{Type I}: \quad & \Psi_{ABCD} = \kappa^{(1)}_{(A}\kappa^{(2)}_{B}\kappa^{(3)}_{C}\kappa^{(4)}_{D)} \\ \textit{Type II}: \quad & \Psi_{ABCD} = \kappa^{(1)}_{(A}\kappa^{(1)}_{B}\kappa^{(2)}_{C}\kappa^{(3)}_{D)} \\ \textit{Type D or II-II}: \quad & \Psi_{ABCD} = \kappa^{(1)}_{(A}\kappa^{(1)}_{B}\kappa^{(2)}_{C}\kappa^{(2)}_{D)} \\ \textit{Type III}: \quad & \Psi_{ABCD} = \kappa^{(1)}_{(A}\kappa^{(1)}_{B}\kappa^{(1)}_{C}\kappa^{(2)}_{D)} \\ \textit{Type N or IV}: \quad & \Psi_{ABCD} = \kappa^{(1)}_{(A}\kappa^{(1)}_{B}\kappa^{(1)}_{C}\kappa^{(1)}_{D)} \end{aligned} \tag{3.37}$$

where $\kappa^{(i)}_A$ and $\kappa^{(j)}_A$ are assumed not to be aligned if $i \neq j$. A spacetime is *algebraically general* if its Weyl tensor is everywhere type I, *algebraically special of type II* if its Weyl tensor is everywhere type II, etc. This classification of spacetimes based on the algebraic structure of the Weyl tensor is known as *Petrov's Classification*, and types I to N defined above are called the *Petrov types*. The latter are sometimes represented with arrow diagrams as shown in Figure 1.

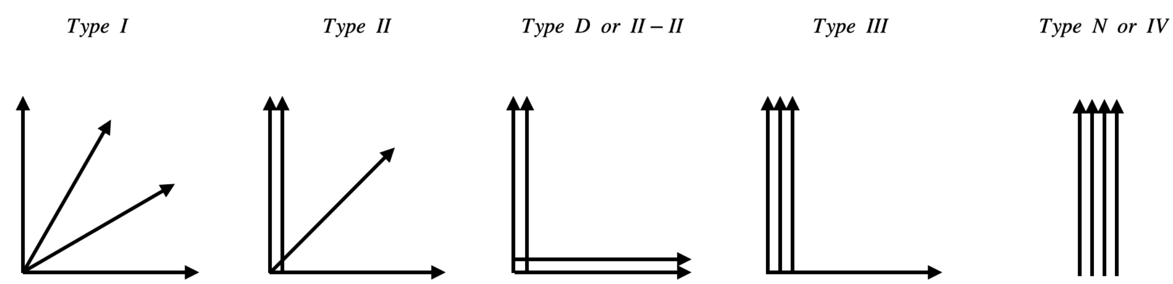

Figure 1: *Represeantion of Petrov types, where each PND is associated to the direction of an arrow.*

While Petrov's classification emerges naturally in the language of spinors, establishing it from a tensor perspective (the way it was originally found) is drastically more complicated. This entails that, when an algebraically special spacetime is expressed in terms of its spinorial quantities (in the way we will explain in Section 3.5) it has a simple form. This simplicity is otherwise blurred from the tensor perspective. It turns out that some of the most important vacuum solutions of general relativity are algebraically special of some type. A remarkable example is Kerr's black hole, as we will see later, which is of type D. Thus, the spinorial description is very useful in studying solutions of general relativity.

Exercise 3.6. Show that in a type D spacetime, choosing $\{o^A, \iota^A\}$ aligned with the repeated principal spinors, the Weyl spinor is

$$\Psi_{ABCD} = 6\Psi_2 o_{(A}o_B\iota_C\iota_{D)}\,, \quad \Psi_2 := \Psi_{ABCD}o^Ao^B\iota^C\iota^D\,. \tag{3.38}$$

Use this to show that if the spacetime is, in addition, vacuum (Ricci-flat), then the Kretschmann scalar is

$$R^{abcd}R_{abcd} = 24\left(\Psi_2^2 + c.c.\right) . \tag{3.39}$$

Conclude that Ψ_2 is a well-defined physical invariant. In particular, its divergences indicate the existence of true curvature singularities. Below we will see that Ψ_2 is one of the so-called GHP quantities, which give the description of a spacetime in terms of its spinorial components relative to a dyad.

3.4 The Vacuum Wave Equation for the Weyl Spinor

The physical degrees of freedom of gravity are encoded in the spacetime curvature. Given that, in vacuum, curvature is described by Weyl's spinor, it is natural to wonder what equation describes its propagation. Our starting point is translating the Bianchi identity of the Riemann tensor $\nabla_{[a}R_{bcd]e} = 0$ into its spinorial counterpart. Such identity can be written equivalently as (again, remind yourselves why!)

$$\nabla^{a\star}R_{abcd} = 0\,. \tag{3.40}$$

Using (3.29), its spinorial version gives

$$0 = \nabla^{AA'\star}R_{AA'BB'CC'DD'} = i\epsilon_{C'D'}\nabla^{A}{}_{B'}X_{ABCD} + i\epsilon_{CD}\nabla^{A}{}_{B'}\Phi_{ABC'D'} + c.c.\,, \tag{3.41}$$

and anti-symmetrising in $C'D'$, one arrives at

$$\nabla^{A}{}_{B'}X_{ABCD} = \nabla^{A'}{}_{B}\Phi_{CDA'B'}\,, \tag{3.42}$$

while symmetrising in $C'D'$ one simply gets the conjugate of this equation. Finally, trading X by Ψ and Λ according to (3.24), we arrive to the spinorial form of the Bianchi identity,

$$\nabla^{A}{}_{B'}\Psi_{ABCD} = \nabla^{A'}{}_{B}\Phi_{CDA'B'} + 2\epsilon_{B(C}\nabla_{D)B'}\Lambda\,. \tag{3.43}$$

Exercise 3.7. Contract (3.43) with ϵ^{CB} and obtain

$$\nabla^{CA'}\Phi_{CDA'B'} - 3\nabla_{DB'}\Lambda = 0\,. \tag{3.44}$$

Show that this is the spinorial version of the contracted Bianchi identity, that is, the conservation of Einstein's tensor $\nabla^a G_{ab} = 0$.

Restricting the identity (3.43) to a spacetime where the vacuum Einstein equations (3.31) hold, one obtains the propagation equation of Ψ_{ABCD} in empty space,

$$\nabla^{AA'}\Psi_{ABCD} = 0\,. \tag{3.45}$$

This equation can be regarded as a higher-spin generalisation of the usual Dirac equation for a massless spinor. However, it is written in covariant terms and holds in a fully general (yet vacuum) spacetime. Thus, unlike in flat space, covariant derivatives do not commute so the formal analogy with a Dirac spinor in flat space is lost in the second or higher derivative equations that Ψ_{ABCD} satisfies. In general, (3.45) does not lead to the wave equation for a free field $\Box\Psi_{ABCD} = 0$, but to one that includes a self-interacting term. To derive it, we need a spinorial version of Ricci's identity. We start by writing the commutator of two covariant derivatives in irreducible pieces using (2.23),

$$[\nabla_{AA'}, \nabla_{BB'}] = \epsilon_{A'B'}\Box_{AB} + \epsilon_{AB}\Box_{A'B'}\,, \tag{3.46}$$

where

$$\Box_{AB} \coloneqq \nabla_{X'(A}\nabla_{B)}{}^{X'}\,, \tag{3.47}$$

and $\Box_{A'B'} = \overline{\Box_{AB}}$ is simply its complex conjugate. Then, the action of $\Box_{AB}$ on a general spinor can be written purely in terms of the curvature spinors, as stated in the following proposition.

Proposition 3.3. *The action of $\Box_{AB}$ on a spinor $\chi^C{}_D{}^{E'}{}_{F'}$ is*

$$\Box_{AB}\chi^C{}_D{}^{E'}{}_{F'} = X_{ABQ}{}^C\chi^Q{}_D{}^{E'}{}_{F'} - X_{ABD}{}^Q\chi^C{}_Q{}^{E'}{}_{F'} + \Phi_{ABQ'}{}^{E'}\chi^C{}_D{}^{Q'}{}_{F'} - \Phi_{ABF'}{}^{Q'}\chi^C{}_D{}^{E'}{}_{Q'}\,, \tag{3.48}$$

and it extends in the natural way to spinors of general valence.

Proof. This is left as an exercise (you can follow Section 4.9 of [3]). □

It is now straightforward to obtain the wave equation satisfied by the Weyl spinor in vacuum. From (3.45) we have

$$0 = \nabla_{AA'}\nabla_E{}^{A'}\Psi^E{}_{BCD} = \nabla_{A'(A}\nabla_{E)}{}^{A'}\Psi^E{}_{BCD} + \nabla_{A'[A}\nabla_{E]}{}^{A'}\Psi^E{}_{BCD} = \Box_{AE}\Psi^E{}_{BCD} - \frac{1}{2}\Box\Psi_{ABCD} \tag{3.49}$$

where $\Box \coloneqq \nabla_{AA'}\nabla^{AA'}$. Now using Proposition 3.3 and the vacuum Einstein equations (3.31) one has

$$\Box_{AE}\Psi^E{}_{BCD} = \Psi^{EQ}{}_{CD}\Psi_{ABEQ} + \Psi^{EQ}{}_{BD}\Psi_{ACEQ} + \Psi^{EQ}{}_{BC}\Psi_{ADEQ} = 3\Psi^{EQ}{}_{(AB}\Psi_{CD)EQ}\,, \tag{3.50}$$

where we used the basic symmetries of Ψ_{ABCD}. Thus, one arrives at

$$\Box\Psi_{ABCD} = 6\Psi^{EQ}{}_{(AB}\Psi_{CD)EQ}\,. \tag{3.51}$$

As advanced earlier, we have found that the Weyl spinor in any vacuum spacetime satisfies a sourced wave equation, where the source term arises as a quadratic self-interaction. The existence of such an equation in terms of a single spinor is, again, very remarkable.

Equation (3.51) is instrumental in studying the propagation of gravitational waves. For small fluctuations about flat spacetime, where Ψ_{ABCD} is a small perturbation about zero of the form $\Psi_{ABCD} = \epsilon\dot{\Psi}_{ABCD} + O(\epsilon^2)$ (with ϵ a small parameter and $\dot{\Psi}_{ABCD}$ a fully symmetric spinor), equation (3.51) at order ϵ gives

$$\Box\dot{\Psi}_{ABCD} = 0\,. \tag{3.52}$$

That is, the gravitational degrees of freedom satisfy a massless wave equation, as is well known.

If instead one considers gravitational fluctuations about a spacetime with a nontrivial Weyl spinor $\mathring{\Psi}_{ABCD}$, as is the case of black holes, then (3.52) is no more true, since now $\Psi_{ABCD} = \mathring{\Psi}_{ABCD} + \epsilon\dot{\Psi}_{ABCD} + O(\epsilon^2)$ and (3.51) at first order contains terms $\sim \mathring{\Psi}\dot{\Psi}$ (among other additional terms). Such an equation is, in general, rather complicated and difficult to solve. However, above we have seen that algebraically special spacetimes possess simple Weyl spinors. It turns out that if the background spacetime is algebraically special the linear equations obtained from (3.51) undergo drastic simplifications. An important example of this are vacuum type D backgrounds (which include Kerr's solution) where, as we will see later on, one is lead to only one equation for a single scalar variable.

3.5 From Spinors to Scalars: the Geroch–Held–Penrose Formalism

The Geroch–Held–Penrose (GHP) formalism [10] consists in using an orthonormal dyad to translate spinorial equations into scalar ones in terms of a number of functions called *GHP scalars*. This has the advantage of making equations more explicit and easier to manipulate in some computations, but the price to pay is that the spinorial structure becomes less manifest. However, the connection to spinors is maintained in a clear way, since GHP scalars are simply the components of the spinors introduced above.

The starting point is an orthonormal dyad $\{o^A, \iota^A\}$, its associated basis in conjugate space $\{\bar{o}^{A'}, \bar{\iota}^{A'}\}$, and the corresponding null frame (see Section 2.4.5),

$$l^a = o^A\bar{o}^{A'}\,, \quad n^a = \iota^A\bar{\iota}^{A'}\,, \quad m^a = o^A\bar{\iota}^{A'}\,, \quad \bar{m}^a = \iota^A\bar{o}^{A'}\,. \tag{3.53}$$

To such orthonormal dyad we associate two other ones, their *primed* and *starred* counterparts, defined as (you can check that they are indeed orthonormal)

$$\begin{aligned} &o'^A := i\iota^A\,, \quad \iota'^A := io^A\,, \quad o'^{A'} := -i\bar{\iota}^{A'}\,, \quad \iota'^{A'} := -i\bar{o}^{A'}\,, \\ &o^{*A} := o^A\,, \quad \iota^{*A} := \iota^A\,, \quad o^{*A'} := \bar{\iota}^{A'}\,, \quad \iota^{*A'} := -\bar{o}^{A'}\,. \end{aligned} \tag{3.54}$$

The operations of *priming* and *starring* consist in the following formal replacements of orthonormal dyads,

$$\begin{aligned} &\textit{Priming:} \quad \{o^A, \iota^A\} \mapsto \{o'^A, \iota'^A\}\,, \quad \{\bar{o}^{A'}, \bar{\iota}^{A'}\} \mapsto \{o'^{A'}, \iota'^{A'}\}\,, \\ &\textit{Starring:} \quad \{o^A, \iota^A\} \mapsto \{o^{*A}, \iota^{*A}\}\,, \quad \{\bar{o}^{A'}, \bar{\iota}^{A'}\} \mapsto \{o^{*A'}, \iota^{*A'}\}\,. \end{aligned} \tag{3.55}$$

Thus, the primed and starred null frames are related to the original one by

$$l'^a = o'^A o'^{A'} = n^a\,, \quad n'^a = \iota'^A \iota'^{A'} = l^a\,, \quad m'^a = o'^A \iota'^{A'} = \bar{m}^a\,, \quad \bar{m}'^a = \iota'^A o'^{A'} = m^a\,, \tag{3.56}$$

and

$$l^{*a} = o^{*A} o^{*A'} = m^a\,, \quad n^{*a} = \iota^{*A} \iota^{*A'} = -\bar{m}^a\,, \quad m^{*a} = o^{*A} \iota^{*A'} = -l^a\,, \quad \bar{m}^{*a} = \iota^{*A} o^{*A'} = n^a\,. \tag{3.57}$$

The reason for introducing the primed and starred versions of a dyad is that, as we will see next, some GHP scalars and equations correspond under the priming and starring operations. This allows one to work with a smaller number of equations, and give the remaining ones simply as their complex conjugate, primed and starred versions.

To introduce the GHP quantities, we start by associating Greek characters to the spin coefficients defined previously in (3.9),

$$\Gamma_{\mathbf{AA'BC}} = \left(\begin{array}{c|c|c|c} \mathbf{AA'}\backslash\mathbf{BC} & \mathbf{00} & \mathbf{10} \text{ or } \mathbf{01} & \mathbf{11} \\ \hline \mathbf{00'} & \kappa & \epsilon & -\tau' \\ \mathbf{01'} & \sigma & \beta & -\rho' \\ \mathbf{10'} & \rho & -\beta' & -\sigma' \\ \mathbf{11'} & \tau & -\epsilon' & -\kappa' \end{array}\right), \tag{3.58}$$

where, since we work with an orthonomral dyad, we have the symmetry $\Gamma_{\mathbf{AA'(BC)}} = \Gamma_{\mathbf{AA'BC}}$ (see Exercise 3.2). We observe that, thanks to the notion of primed dyad, half of the spin coefficients can be written as primed versions of other ones. Focusing on κ as an example, we have

$$\kappa \coloneqq \Gamma_{\mathbf{00'00}} = -\Gamma_{\mathbf{00'}}{}^{\mathbf{1}}{}_{\mathbf{0}} = -o^A o^{A'} o_B \nabla_{AA'} o^B\,. \tag{3.59}$$

Then, we can check that κ' indeed follows by priming κ,

$$\kappa' \coloneqq -\Gamma_{\mathbf{11'11}} = -\Gamma_{\mathbf{11'}}{}^{\mathbf{0}}{}_{\mathbf{1}} = \iota^A \iota^{A'} \iota_B \nabla_{AA'} \iota^B = \left(-o^A o^{A'} o_B \nabla_{AA'} o^B\right)' = (\kappa)'\,, \tag{3.60}$$

and similarly for the other spin coefficients. Next, we introduce the GHP scalars that encode the independent components of the curvature spinors. The Weyl spinor Ψ_{ABCD} is described by five complex scalars,

$$\begin{aligned} &\Psi_0 \coloneqq \Psi_{ABCD} o^A o^B o^C o^D\,, \quad \Psi_1 \coloneqq \Psi_{ABCD} o^A o^B o^C \iota^D\,, \quad \Psi_2 \coloneqq \Psi_{ABCD} o^A o^B \iota^C \iota^D\,, \\ &\Psi_3 \coloneqq \Psi_{ABCD} o^A \iota^B \iota^C \iota^D\,, \quad \Psi_4 \coloneqq \Psi_{ABCD} \iota^A \iota^B \iota^C \iota^D\,, \end{aligned} \tag{3.61}$$

while the spinor $\Phi_{ABC'D'}$ is encoded in the following nine real scalars,[2]

$$\begin{aligned}
&\Phi_{00} := -\Phi_{ABC'D'}o^A o^B o^{C'} o^{D'}\,, \quad \Phi_{01} := -\Phi_{ABC'D'}o^A o^B o^{C'} \iota^{D'}\,, \quad \Phi_{02} := -\Phi_{ABC'D'}o^A o^B \iota^{C'} \iota^{D'}\,,\\
&\Phi_{10} := -\Phi_{ABC'D'}o^A \iota^B o^{C'} o^{D'}\,, \quad \Phi_{11} := -\Phi_{ABC'D'}o^A \iota^B o^{C'} \iota^{D'}\,, \quad \Phi_{12} := -\Phi_{ABC'D'}o^A \iota^B \iota^{C'} \iota^{D'}\,,\\
&\Phi_{20} := -\Phi_{ABC'D'}\iota^A \iota^B o^{C'} o^{D'}\,, \quad \Phi_{21} := -\Phi_{ABC'D'}\iota^A \iota^B o^{C'} \iota^{D'}\,, \quad \Phi_{22} := -\Phi_{ABC'D'}\iota^A \iota^B \iota^{C'} \iota^{D'}\,,
\end{aligned} \tag{3.62}$$

which together with Λ give the ten (real) independent components of the Ricci tensor. The spin coefficients (3.58), Weyl scalars (3.61) and Ricci scalars (3.62) are very natural quantities from the spinor perspective. From the tensor approach they instead appear as rather arbitrary combinations, as you are encouraged to check in the following exercise.

Exercise 3.8. Use the spinorial versions of the null frame, Weyl and Ricci tensors to show that the spin coefficients can be written as

$$\begin{aligned}
&\kappa = m^a l^b \nabla_b l_a\,, \quad \rho = m^a \bar{m}^b \nabla_b l_a\,, \quad \sigma = m^a m^b \nabla_b l_a\,, \quad \tau = m^a n^b \nabla_b l_a\,,\\
&\beta = \frac{1}{2}\left(n^a m^b \nabla_b l_a + m^a m^b \nabla_b \bar{m}_a\right)\,, \quad \epsilon = \frac{1}{2}\left(n^a l^b \nabla_b l_a - \bar{m}^a l^b \nabla_b m_a\right)\,,
\end{aligned} \tag{3.63}$$

the Weyl scalars as

$$\begin{aligned}
&\Psi_0 = -C_{abcd} l^a m^b l^c m^d\,, \quad \Psi_1 = -C_{abcd} l^a m^b l^c n^d\,, \quad \Psi_2 = -C_{abcd} l^a m^b \bar{m}^c n^d\,,\\
&\Psi_3 = -C_{abcd} l^a n^b \bar{m}^c n^d\,, \quad \Psi_4 = -C_{abcd} \bar{m}^a n^b \bar{m}^c n^d\,,
\end{aligned} \tag{3.64}$$

and the Ricci scalars as

$$\begin{aligned}
&\Phi_{00} = -\frac{1}{2}R_{ab} l^a l^b\,, \quad \Phi_{01} = -\frac{1}{2}R_{ab} l^a m^b\,, \quad \Phi_{02} = -\frac{1}{2}R_{ab} m^a m^b\,,\\
&\Phi_{10} = -\frac{1}{2}R_{ab} l^a \bar{m}^b\,, \quad \Phi_{11} = -\left(\frac{1}{2}R_{ab} l^a n^b - 3\Lambda\right)\,, \quad \Phi_{12} = -\frac{1}{2}R_{ab} m^a n^b\,,\\
&\Phi_{20} = -\frac{1}{2}R_{ab} \bar{m}^a \bar{m}^b\,, \quad \Phi_{21} = -\frac{1}{2}R_{ab} \bar{m}^a n^b\,, \quad \Phi_{22} = -\frac{1}{2}R_{ab} n^a n^b\,.
\end{aligned} \tag{3.65}$$

3.5.1 GHP Weights and Derivatives

A very useful organising principle in managing GHP quantities is to classify them based on how they transform under a dyad scaling,

$$\{o^A, \iota^A\} \mapsto \{\lambda o^A, \lambda^{-1}\iota^A\}\,, \tag{3.66}$$

[2] There seems to be a very unfortunate chain of typos in the literature at this point, involving (and implying inconsistencies among) [1,3,5]. It involves a sign when writing $\Phi_{00}, \Phi_{01}, \ldots$ in terms of dyad contractions with $\Phi_{ABC'D'}$. Here we chose that sign in such a way that when $\Phi_{00}, \Phi_{01}, \ldots$ are written in terms of contractions of the Ricci tensor and the null frame, they coincide with the corresponding expressions in the original paper by Newman and Penrose [1] (who use our sign conventions for the Riemann). We recall that the sign in the definition of Λ has also been chosen according to that criterion.

which preserves the orthonormal condition $o_A \iota^A = 1$ as long as the complex function λ is non-vanishing. We say that a quantity η has *GHP weight* (p,q), and write $\eta \mathrel{\mathring{=}} (p,q)$, if under (3.66) it transforms as

$$\eta \mapsto \lambda^p \bar{\lambda}^q \eta \,. \tag{3.67}$$

For example, under (3.66) the null frame is mapped into

$$\{l^a, n^a, m^a, \bar{m}^a\} \mapsto \{\lambda\bar{\lambda} l^a, \lambda^{-1}\bar{\lambda}^{-1} n^a, \lambda\bar{\lambda}^{-1} m^a, \lambda^{-1}\bar{\lambda}\bar{m}^a\} \,, \tag{3.68}$$

so their GHP weights are

$$l^a \mathrel{\mathring{=}} (1,1)\,, \quad n^a \mathrel{\mathring{=}} (-1,-1)\,, \quad m^a \mathrel{\mathring{=}} (1,-1)\,, \quad \bar{m}^a \mathrel{\mathring{=}} (-1,1)\,. \tag{3.69}$$

It can also be verified that

$$\begin{aligned} &\kappa \mathrel{\mathring{=}} (3,1)\,, \quad \sigma \mathrel{\mathring{=}} (3,-1)\,, \quad \rho \mathrel{\mathring{=}} (1,1)\,, \quad \tau \mathrel{\mathring{=}} (1,-1)\,, \\ &\Psi_i \mathrel{\mathring{=}} (4-2i, 0) \quad (i = 0, ..., 4)\,, \\ &\Phi_{s,r} \mathrel{\mathring{=}} (2-2s, 2-2r) \quad (s, r = 0, 1, 2)\,. \end{aligned} \tag{3.70}$$

The GHP weights of the complex conjugate, primed and starred versions of the quantities above follow by noticing that such operations change the weights as

$$\eta \mathrel{\mathring{=}} (p,q) \quad \mapsto \quad \bar{\eta} \mathrel{\mathring{=}} (q,p)\,, \quad \eta' \mathrel{\mathring{=}} (-p,-q)\,, \quad \eta^* \mathrel{\mathring{=}} (p,-q)\,. \tag{3.71}$$

The spin coefficients ϵ and β do not have definite GHP weights, since they do not transform as (3.67) under (3.66), and similarly for ϵ' and β'. We say they are not propertly weighted. However, they can be used to introduce derivative operators that act within properly weighted quantities (again assuming $\eta \mathrel{\mathring{=}} (p,q)$),

$$\begin{aligned} &þ\eta \coloneqq (l^a \nabla_a - p\epsilon - q\bar{\epsilon})\,\eta\,, \quad þ'\eta \coloneqq (n^a \nabla_a + p\epsilon' + q\bar{\epsilon}')\,\eta\,, \\ &ð\eta \coloneqq \left(m^a \nabla_a - p\beta + q\bar{\beta}'\right)\eta\,, \quad ð'\eta \coloneqq \left(\bar{m}^a \nabla_a + p\beta' - q\bar{\beta}\right)\eta\,. \end{aligned} \tag{3.72}$$

The operators þ and ð are called *thorn* and *eth*, and are well defined derivatives as shown in the next exercise.

Exercise 3.9. Obtain the transformation laws of ϵ and β under (3.66) and show that if $\eta \mathrel{\mathring{=}} (p,q)$ is a properly weighted quantity, then their thorn and eth derivatives are also properly weighted quantities with weights

$$þ\eta \mathrel{\mathring{=}} (p+1, q+1)\,, \quad þ'\eta \mathrel{\mathring{=}} (p-1, q-1)\,, \quad ð\eta \mathrel{\mathring{=}} (p+1, q-1)\,, \quad ð'\eta \mathrel{\mathring{=}} (p-1, q+1)\,. \tag{3.73}$$

Furthermore, show that

$$(þ\eta)' = þ'\eta'\,, \quad (þ'\eta)' = þ\eta'\,, \quad (ð\eta)' = ð'\eta'\,, \quad (ð'\eta)' = ð\eta'\,, \tag{3.74}$$

and

$$\overline{\text{þ}\eta} = \text{þ}\bar{\eta}\,, \quad \overline{\text{þ}'\eta} = \text{þ}'\bar{\eta}\,, \quad \overline{\eth\eta} = \eth'\bar{\eta}\,, \quad \overline{\eth'\eta} = \eth\bar{\eta}\,. \tag{3.75}$$

so that $\bar{\text{þ}} = \text{þ}, \bar{\text{þ}'} = \text{þ}'$ and $\bar{\eth} = \eth', \bar{\eth'} = \eth$.

Remark 3.3. There is a more systematic way of introducing the GHP derivatives in the language of principal line bundles. Although this approach will not be taken here, it clarifies the geometric origin of þ and ð, so we review briefly this idea in Appendix A.

3.5.2 GHP Structure Equations

The GHP quantities introduced above are not independent, since they are subject to the usual geometric structure equations: the commutation relations associated to the null frame, the relation between the curvature and (derivatives of) the spin coefficients, and the Bianchi identities. Equipped with the operators defined above, it is possible to write the geometric structure equations in terms of properly weighted quantities. Applying the definitions, one finds that the GHP frame derivatives are

$$\begin{aligned} \text{þ}l_a &= -\bar{\kappa}m_a - \kappa\bar{m}_a, \quad \text{þ}m_a = -\bar{\tau}'l_a - \kappa n_a, \\ \text{þ}'l_a &= -\bar{\tau}m_a - \tau\bar{m}_a, \quad \text{þ}'m_a = -\bar{\kappa}'l_a - \tau n_a, \\ \eth l_a &= -\bar{\rho}m_a - \sigma\bar{m}_a, \quad \eth m_a = -\bar{\sigma}'l_a - \sigma n_a, \\ \eth' l_a &= -\bar{\sigma}m_a - \rho\bar{m}_a, \quad \eth' m_a = -\bar{\rho}'l_a - \rho n_a. \end{aligned} \tag{3.76}$$

together with their primed and conjugate versions. The structure equations involving the curvature scalars will be given in vacuum, for simplicity. In the rest of this section it will be assumed that $\Phi_{ABC'D'} = 0$ and $\Lambda = 0$, and the equations we write are specific to that case. First, from a suitable use of Proposition 3.3 applied to the spinor dyad, and after projecting on the dyad itself, one finds

$$\begin{aligned} \eth\rho - \eth'\sigma &= (\rho - \bar{\rho})\tau + (\bar{\rho}' - \rho')\kappa - \Psi_1, \\ \text{þ}\rho - \eth'\kappa &= \rho^2 + \sigma\bar{\sigma} - \bar{\kappa}\tau - \kappa\tau', \\ \text{þ}\sigma - \eth\kappa &= (\rho + \bar{\rho})\sigma - (\tau + \bar{\tau}')\kappa + \Psi_0, \\ \text{þ}\rho' - \eth\tau' &= \rho'\bar{\rho} + \sigma\sigma' - \tau'\bar{\tau}' - \kappa\kappa' - \Psi_2, \end{aligned} \tag{3.77}$$

which, together with their primed and starred versions, are equivalent to the vacuum Einstein equations ($\Phi_{r,s} = \Lambda = 0$). Next, projecting on the dyad the spinor version of the Bianchi identity (3.43), one finds

$$\begin{aligned} (\text{þ} - 4\rho)\Psi_1 - (\eth' - \tau')\Psi_0 &= -3\kappa\Psi_2, \\ (\text{þ} - 3\rho)\Psi_2 - (\eth' - 2\tau')\Psi_1 &= \sigma'\Psi_0 - 2\kappa\Psi_3, \end{aligned} \tag{3.78}$$

together with their primed and starred versions. Finally, one can also verify that the commutator between GHP derivatives acting on a scalar with weights $\eta \overset{\circ}{=} (p,q)$ is given by

$$\begin{aligned} [\text{þ}, \text{þ}']\eta &= \Big[(\bar\tau - \tau')\eth + (\tau - \bar\tau')\eth' - p(\kappa\kappa' - \tau\tau' + \Psi_2) - q(\bar\kappa\bar\kappa' - \bar\tau\bar\tau' + \bar\Psi_2)\Big]\,\eta, \\ [\text{þ}, \eth]\eta &= \left[-\bar\tau'\text{þ} - \kappa\text{þ}' + \bar\rho\eth + \sigma\eth' - p(\rho'\kappa - \tau'\sigma + \Psi_1) - q(\bar\sigma'\bar\kappa - \bar\rho\bar\tau')\right]\eta. \end{aligned} \tag{3.79}$$

Although obtaining the equations above by projecting the spinor equations on the dyad is unproblematic, the computations are rather long and tedious. Fortunately, this can be done automatically with the help of software for symbolic tensor manipulation, as explained in the `mathematica` notebook that comes with these lecture notes. This makes use of a number of `xAct` packages, specially `SpinFrames`, designed to manipulate spinor and GHP equations.

Remark 3.4. Before the GHP formalism was conceived in [10], Newman and Penrose [1] had put forward another approach to transforming spinor equations into scalar ones, the so-called *Newman–Penrose* (NP) formalism. However, this follows immediately from the GHP formalism, simply by disregarding the weight structure of the various quantities, as well as the priming and starring operations. Thus, primed scalars are given their own name, e.g. $\epsilon' = -\gamma$, and instead of þ and ð one uses the directional derivative operators

$$D \coloneqq l^a\nabla_a\,, \quad \Delta \coloneqq n^a\nabla_a\,, \quad \delta \coloneqq m^a\nabla_a\,, \quad \bar\delta \coloneqq \bar m^a\nabla_a\,. \tag{3.80}$$

The complete relation between the original NP and the subsequent GHP formalisms is spelled out in [10]. In these notes we will always employ the GHP approach, since that allows one to reduce the form and number of equations significantly by benefiting from the weight structure of GHP scalars.

To conclude this section we will revisit three well-known results, Petrov's classification (discussed in Section 3.3), the Golberg–Sachs and peeling theorems, and show that they are expressed in a very natural (i.e. simple) form when formulated in terms of GHP quantities.

3.5.3 Petrov's Classification in GHP Language

Choose an orthonormal dyad in such a way that o^A is aligned with a principal spinor, $o^A \sim \kappa^{(1)A}$, or equivalently that $o^A o^{A'}$ is aligned with the corresponding PND, $o^A o^{A'} \sim k^{(1)AA'}$. Then using the canonical decomposition of Ψ_{ABCD} (see (3.35)) one has

$$\Psi_0 = \Psi_{ABCD}o^Ao^Bo^Co^D = \kappa^{(1)}_{(A}\kappa^{(2)}_B\kappa^{(3)}_C\kappa^{(4)}_{D)}o^Ao^Bo^Co^D \sim \kappa^{(1)}_{(A}\kappa^{(2)}_B\kappa^{(3)}_C\kappa^{(4)}_{D)}\kappa^{(1)A}\kappa^{(1)B}\kappa^{(1)C}\kappa^{(1)D} = 0\,, \tag{3.81}$$

since $\kappa^{(1)A}\kappa^{(1)}_A = 0$. This is true for any spacetime which, in general, will be of Petrov type I. Now assume the spacetime has a repeated PND (type II), with principal spinor $\kappa^{(1)A}$. Choosing $o^A \sim \kappa^{(1)A}$, besides finding $\Psi_0 = 0$ as shown above, one also has

$$\Psi_1 = \Psi_{ABCD}o^Ao^Bo^C\iota^D \sim \kappa^{(1)}_{(A}\kappa^{(1)}_B\kappa^{(2)}_C\kappa^{(3)}_{D)}\kappa^{(1)A}\kappa^{(1)B}\kappa^{(1)C}\iota^D = 0\,. \tag{3.82}$$

By a similar computation, in the case of a type D spacetime choosing o^A and ι^A along each PND entails $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$. Finally, in spacetimes of types III and N, picking o^A along the repeated PND implies that all Weyl scalars vanish except for Ψ_3 in the type III case, and Ψ_4 in the type N one.

In sum, we have found that if the spinor dyads are properly aligned with the repeated PNDs (if existent), then the Weyl scalars of the various Petrov types satisfy

$$\begin{aligned} \textit{Type I :} \quad & \Psi_0 = 0 \\ \textit{Type II :} \quad & \Psi_0 = \Psi_1 = 0 \\ \textit{Type D or II-II:} \quad & \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0 \\ \textit{Type III :} \quad & \Psi_0 = \Psi_1 = \Psi_2 = \Psi_4 = 0 \\ \textit{Type N or IV :} \quad & \Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0 \end{aligned} \tag{3.83}$$

The take home message is that, as anticipated in Section 3.3, algebraically special spacetimes have simple forms in terms of their GHP quantities. As an example of this, later on we will discuss the GHP description the Kerr black hole, an instance of type D spacetime.

3.5.4 The Peeling Theorem

The peeling theorem establishes that, under certain general assumptions that are expected to hold for asymptotically flat solutions, the behaviour of the Weyl scalars in a suitable choice of frame and coordinates is

$$\Psi_i = O\left(r^{-5+i}\right) , \quad i = 0, ..., 4 , \tag{3.84}$$

where it is assumed that infinity lies at $r \to \infty$ (the proof will not be discussed here, but can be found in [1]). This result can be interpreted as follows. Far away from sources and strong gravitational fields, the dominant Weyl scalar is Ψ_4 and the spacetime is approximately type N. As one moves backwards from the asymptotic region, Ψ_3 becomes important, making the spacetime effectively type III. It occurs similarly for the remaining Weyl scalars, in such a way that the PNDs "peel off" one by one from the type N bundle (where all PNDs point in the same direction, see Figure 1) as one falls backwards from infinity.

3.5.5 The Goldberg–Sachs Theorem

The Goldberg–Sachs theorem [15] establishes that *in a vacuum spacetime (potentially with a cosmological constant) that is not conformally flat, a null vector field is a repeated PND if and only if it is geodesic and shearfree*. In terms of GHP quantities, we notice that (from the first of (3.76)) l^a is geodesic only if $\kappa = 0$. It can also be shown that σ measures the failure of l^a to be shearfree (see [4] and [1]). Then, in terms of

GHP quantities the Goldberg–Sachs theorem is simply expressed as

$$\textit{If } R_{ab} = 0, \textit{ then:} \quad \kappa = \sigma = 0 \Leftrightarrow \Psi_0 = \Psi_1 = 0\,. \tag{3.85}$$

The conciseness thus achieved in the GHP language will be crucial in the applications to gravitational radiation discussed below .

4 Application to Black Holes and Gravitational Waves

In this section we use the spinorial tools developed above to approach some problems in black hole and gravitational wave physics. For simplicity, we will focus on vacuum spacetimes. We start in Section 4.1 by characterising vacuum type D solutions in terms of their GHP quantities, and discuss Kerr's spacetime as an important example. Next, in Section 4.2 we derive non-perturbative wave equations describing the propagation of curvature in vacuum, and in Section 4.3 we use the results in Sections 4.1 and 4.2 to study gravitational radiation. In particular, we provide a simple derivation of Teukolsky's equations, by starting from the curvature wave equations in Section 4.2 and linearising them on a vacuum type D space. We conclude by discussing their implications to gravitational radiation in Kerr's spacetime.

4.1 An Important Class: Vacuum Type D Solutions

The class of type D solutions plays a very important role in gravitational physics. It contains some of the most important solutions, such as the Kerr–Newman family describing charged and rotating black holes in the Einstein–Maxwell theory. Although these are very nontrivial solutions given by seemingly complicated spacetime geometries, the fact that they possess two distinct repeated PNDs allows one to describe them in relatively simple terms, using GHP variables. In particular, restricting to vacuum solutions and choosing l^a and n^a along the repeated PNDs (a choice sometimes referred to as a *principal frame*), the Golberg–Sachs theorem (3.85) applied to both geodesic families implies

$$\kappa = \sigma = \kappa' = \sigma' = 0\,, \quad \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0\,, \tag{4.1}$$

where we have used that $\Psi_0' = \Psi_4$ and $\Psi_1' = \Psi_3$. The Kerr family is an example of spacetime satisfying these conditions, as we discuss next.

4.1.1 An important example: the Kerr Black Hole

The Kerr spacetime was found by specifically looking after vacuum type D solutions (as can be noticed from the paper's title [16]). Its metric in Boyer–Lindquist coordinates looks rather complicated (we use

$x = \cos\theta$),

$$\begin{aligned} ds^2 = {} & \frac{\Delta - a^2(1-x^2)}{\Sigma}dt^2 + 4aMr\frac{1-x^2}{\Sigma}dtd\phi - \frac{\Sigma}{\Delta}dr^2 - \frac{\Sigma}{1-x^2}dx^2 \\ & - \left(\frac{(r^2+a^2)^2 - \Delta a^2(1-x^2)}{\Sigma}\right)(1-x^2)d\phi^2\,, \end{aligned} \tag{4.2}$$

where M and $J = aM$ are the ADM mass and angular momentum, and

$$\Delta = r^2 - 2Mr + a^2, \quad \Sigma = r^2 + a^2x^2\,. \tag{4.3}$$

However, we know it must have a simple description in terms of GHP quantities. Using the `mathematica` notebook that comes with these notes, you are encouraged to check that the following frame

$$\begin{aligned} l^a &= \frac{a^2+r^2}{\Delta}\partial_t + \partial_r + \frac{a}{\Delta}\partial_\phi\,, \\ n^a &= \frac{a^2+r^2}{2\Sigma}\partial_t - \frac{\Delta}{2\Sigma}\partial_r + \frac{a}{2\Sigma}\partial_\phi\,, \\ m^a &= i\frac{\sqrt{(1-x^2)/2}}{r+iax}\left(a\partial_t + \frac{1}{1-x^2}\partial_\phi + i\partial_x\right)\,, \end{aligned} \tag{4.4}$$

is a null frame where l^a and n^a are geodesic (with l^a affinely-parametrised). The non-vanishing, properly-weighted spin coefficients are

$$\rho = -\frac{1}{r-iax}\,, \quad \tau = -\frac{ia\sqrt{(1-x^2)/2}}{\Sigma}\,, \quad \rho' = -\frac{\Delta(r)}{2\Sigma}\rho\,, \quad \tau' = \frac{r+iax}{r-iax}\tau\,, \tag{4.5}$$

while the non-properly weighted ones are

$$\beta = \frac{x/2\sqrt{2}}{(r+iax)\sqrt{1-x^2}}\,, \quad \epsilon' = \frac{2\Delta(r) - (r-iax)\Delta'(r)}{4(r-iax)\Sigma}\,, \quad \beta' = \frac{x(r+iax) - 2ia}{2\sqrt{2}\sqrt{1-x^2}(r-iax)^2}\,. \tag{4.6}$$

In particular, we notice that $\kappa = \sigma = \kappa' = \sigma' = 0$ so, by the Golberg–Sachs theorem, the only non-trivial Weyl scalar must be Ψ_2. We get

$$\Psi_2 = -\frac{M}{(r-iax)^3}\,, \tag{4.7}$$

while the rest indeed vanish. Later on we will use the structure of vacuum type D solutions to describe the propagation of gravitational fluctuations about such spacetimes, with an emphasis on Kerr's space. However, we will first derive general, nonperturbative wave equations describing curvature propagation in any vacuum spacetime.

4.2 Curvature Wave Equations in Vacuum

Above we have shown that the Weyl spinor in vacuum satisfies the wave equation

$$\Box\Psi_{ABCD} - 6\Psi^{EQ}{}_{(AB}\Psi_{CD)EQ} = 0\,. \tag{4.8}$$

This provides a very natural way of deriving wave equations for the Weyl scalars. Since this is a fully symmetric spinorial equation, it has five complex independent components (see Proposition 2.2), which can be obtained by simply computing its five independent full projections with the dyad. This gives an equation for each of the five Weyl scalars. Although conceptually this is very simple, translating the dyad projections into GHP quantities is computationally expensive if attempted by hand. In the `mathematica` notebook that comes with this notes, we use the `SpinFrames` subpackage of `xAct` to do this operations efficiently. Here we simply report the resulting equations for Ψ_0 and Ψ_4 with a brief description of the steps followed, while a detailed explanation can be found in the associated notebook (en passant, we correct some typos in the literature):

- *Equation for* Ψ_0*:* Project (4.8) with $o^A o^B o^C o^D$, commute GHP operators so that non-primed ones act first, eliminate first derivatives of Ψ_1 by using Bianchi identities, and finally eliminate first derivatives of ρ and τ by using Ricci-rotation equations (3.77). The resulting equation is

$$\begin{aligned}&\left[\text{þ}'\text{þ} - \eth'\eth - \bar{\rho}'\text{þ} - 5\rho\text{þ}' + \bar{\tau}\eth + 5\tau\eth' + 4\sigma\sigma' - 4\kappa\kappa' - 10\Psi_2\right]\Psi_0\\ &+\left[4\text{þ}'\kappa - 4\eth'\sigma - 4(\bar{\rho}' - 2\rho')\kappa + 4(\bar{\tau} - 2\tau')\sigma + 10\Psi_1\right]\Psi_1\\ &+\left[-4\sigma\text{þ} + 4\kappa\eth - 12\kappa\tau + 12\rho\sigma\right]\Psi_2 = 0.\end{aligned} \tag{4.9}$$

- *Equation for* Ψ_4*:* From the knowledge of (4.9), an equation for Ψ_4 can be obtained just by priming equation (4.9) and commuting GHP operators so that non-primed ones act first. Alternatively, one can emulate the steps that lead to (4.9): project (4.8) with $\iota^A \iota^B \iota^C \iota^D$, commute GHP operators so that non-primed ones act first, eliminate first derivatives of Ψ_3 by using Bianchi identities, and finally eliminate first derivatives of ρ' and τ' by using Ricci-rotation equations (3.77). From either approach, one gets to the same equation for Ψ_4,[3]

$$\begin{aligned}&[\text{þ}'\text{þ} - \eth'\eth - (4\rho' + \bar{\rho}')\text{þ} - \rho\text{þ}' + (4\tau' + \bar{\tau})\eth + \tau\eth' + 4\rho\rho' - 4\tau\tau' - 2\Psi_2]\Psi_4\\ &+[4\text{þ}\kappa' - 4\eth\sigma' - 4(\bar{\rho} - 2\rho)\kappa' + 4(\bar{\tau}' - 2\tau)\sigma' + 10\Psi_3]\Psi_3\\ &+[-4\sigma'\text{þ}' + 4\kappa'\eth' - 12\kappa'\tau' + 12\rho'\sigma']\Psi_2 = 0.\end{aligned} \tag{4.10}$$

Proceeding this way it is also possible to obtain equations for $\Psi_{1,2,3}$, but we shall not discuss this here since we will not need those equations later on (a discussion of those equations can be found in [17]). To the best of our knowledge, the equations for Ψ_0 and Ψ_4 as presented here were first obtained by Stewart and Walker [12], by a judicious manipulation of GHP equations. Here instead we proceeded in an arguably more natural way, based on a single projection of (4.8) and subsequent elimination of some variables.

[3] In Stewart and Walker's paper [12], instead of $\bar{\tau}'$ there is a $\bar{\tau}$, missing a prime. This typo propagated also to [17].

We stress that these equations hold exactly on any vacuum spacetime. In addition, as we shall see next, they are written in such a way that their linearisation on certain algebraically special backgrounds is immediate, a fact that will prove very useful when considering gravitational waves.

4.3 Gravitational Radiation and Teukolsky's Equations

Understanding gravitational radiation is one of the main challenges in theoretical physics. From the perspective of astrophysics, this allows us to interpret the signals registered in gravitational wave detectors and, to date, this is one of the most promising windows into the strong-field structure of gravity. However, this is a very hard problem and approaching it requires some simplifying assumptions. Assuming gravitational wave sources of comparable masses and small separation distances requires employing numerical methods in constructing solutions. That would be the case of a comparable-mass binary black hole merger. However, numerical methods are limited by computational power and often fail in resolving large separation of scales, e.g. if one of the black holes of a binary is much smaller than the other, or capturing the features of gravitational waves at very late times after a black hole merger. In those cases, one models gravitational radiation as small fluctuations off a stationary background.

A paradigmatic example is the relaxation of a black hole towards equilibrium in the last stages after a black hole merger. The remnant black hole converges towards a final, quiescent state by emitting gravitational waves that oscillate at the hole's characteristic frequencies, the so-called quasinormal modes (QNMs). Observing this relaxation process, known as the black hole's "ringdown", can yield unprecedented tests about the nature of gravity if theoretical predictions for the QNMs are available. These include tests about black hole uniqueness and the high-energy completion of general relativity (you can consult [18] for a review on QNMs). Another important case is that of extreme-mass-ratio inspirals (EMRIs), which are black hole binaries where one of the holes is much smaller than the other. The smaller hole spends thousands of orbits in the deep, strong-field region of the larger hole before plunging and disappearing past its event horizon. The gravitational waves emitted during those orbits encode invaluable information about the structure of extreme gravitational fields, a fact that makes EMRIs one of the most important science cases of the future space-based detector LISA [19].

The theory developed in the previous sections is crucial in solving problems such as black hole relaxation or EMRIs described above. An instrumental result for that are the so-called *Teukolsky equations* [11], which describe the generation and propagation of linear gravitational fluctuations on vacuum type D spacetimes (in fact, the equations also exist in the presence of a cosmological constant, so they have found applications in the contexts of cosmology and holography, too).

The original derivation by Teukolsky is based on a judicious combination of some linearised NP equations. Here we follow an alternative approach, where most of the work has been done already by obtaining the nonperturbative curvature wave equations in Section 4.2 above.[4] Consider a linear, but otherwise fully general gravitational fluctuation of a type D background. We will use dots to denote linear fluctuations of GHP quantities and, in a slight abuse of notation, if no dot is written it will be assumed that the variable takes its background value (e.g. Ψ_0 and $\dot{\Psi}_0$ denote the backgorund value of Ψ_0 and its linear fluctuation, respectively). From the Goldberg–Sachs theorem we know that the background null frame can be chosen so that

$$\kappa = \sigma = \kappa' = \sigma' = 0\,, \quad \text{and} \quad \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0\,. \tag{4.11}$$

Taking into account the vanishing of these background quantities, the linearisation of (4.9) yields simply

$$\begin{aligned} &\left[\text{þ}'\text{þ} - \eth'\eth - \bar{\rho}'\text{þ} - 5\rho\text{þ}' + \bar{\tau}\eth + 5\tau\eth' - 10\Psi_2\right]\dot{\Psi}_0 \\ &+ \left[-4\dot{\sigma}\text{þ} + 4\dot{\kappa}\eth - 12\dot{\kappa}\tau + 12\rho\dot{\sigma}\right]\Psi_2 = 0\,. \end{aligned} \tag{4.12}$$

However, from the Bianchi identities in GHP form (3.78) the thorn and eth derivatives of Ψ_2 on a vacuum type-D space read

$$\text{þ}\Psi_2 = 3\rho\Psi_2\,, \quad \eth\Psi_2 = 3\tau\Psi_2\,, \tag{4.13}$$

so the last line in (4.12) vanishes, and one is left with a decoupled equation for $\dot{\Psi}_0$,

$$\left[\text{þ}'\text{þ} - \eth'\eth - \bar{\rho}'\text{þ} - 5\rho\text{þ}' + \bar{\tau}\eth + 5\tau\eth' - 10\Psi_2\right]\dot{\Psi}_0 = 0\,, \quad \textit{(Teukolsky equation for } \dot{\Psi}_0\textit{)}\,. \tag{4.14}$$

Applying a similar argument to (4.10) one also obtains a decoupled equation for $\dot{\Psi}_4$,[5]

$$[\text{þ}'\text{þ} - \eth'\eth - (4\rho' + \bar{\rho}')\text{þ} - \rho\text{þ}' + (4\tau' + \bar{\tau})\eth + \tau\eth' + 4\rho\rho' - 4\tau\tau' - 2\Psi_2]\dot{\Psi}_4 = 0\,, \quad \textit{(Teukolsky equation for } \dot{\Psi}_4\textit{)}\,. \tag{4.15}$$

Although we have derived these equations in vacuum for (fluctuations of) the gravitational Weyl spinor, in a very similar way it is possible to obtain analogous equations for lower-spin fields, such as the electromagnetic field or neutrinos. It is also possible to account for linear sources, where instead of setting the fluctuations of the Ricci scalars to zero one trades them by the components of the energy-momentum tensor of the linear source (e.g. the energy-momentum tensor of a point particle). If these equations are, furthermore, specialised to the Kerr background with the null frame (4.4), they admit separable solutions $\dot{\psi} = R(r)S(x)e^{-i(\omega t - m\phi)}$, where $\dot{\psi}$ denotes generically the variable describing the field's fluctuation (e.g.

[4] To the best of our knowledge, an alternative derivation along these lines was first considered by Ryan [20] and shortly after by Stewart and Walker [12], that we follow here. We also note that an approach to perturbations from curvature wave equations is likewise efficient and clarifying in other contexts too [21].

[5] Since our assumptions (4.11) are left invariant under priming, one can also obtain the equation for $\dot{\Psi}_4$ just by priming (4.14).

$\dot{\Psi}_0$), ω is its frequency and m its angular momentum along the hole's rotational axis. This yields two decoupled ODEs, one for $R(r)$ and one for $S(x)$.[6] The equation for $S(x)$ is a smooth deformation of the harmonic equation of spin-weighted spherical harmonics [22], while the equation for $R(r)$ is

$$\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{dR}{dr}\right)+\left(\frac{K^2-2is(r-M)K}{\Delta}+4is\omega r-\lambda\right)R=0\,, \tag{4.16}$$

where s is the spin of the fluctuation (for gravity $s=\pm 2$), λ a separation constant and $K^2=(r^2+a^2)\omega - am$. This equation can be generalised to account for a linear source, which amounts to including a term on the right-hand-side that is homogeneous in the linear energy-momentum tensor.

Besides having a very clear geometric origin (the wave equation of the Weyl spinor), these equations describe the behaviour of physically meaningful quantities. In particular, it can be shown that $\dot{\Psi}_{0,4}$ are fully gauge-invariant and control the fluxes of incoming and outgoing radiation at infinity. Restricting to monochromatic waves, one has

$$\frac{d^2E^{(\text{out})}}{dtd\Omega}=\lim_{r\to\infty}\frac{r^2}{4\pi\omega^2}|\dot{\Psi}_4|^2\,,\qquad \frac{d^2E^{(\text{in})}}{dtd\Omega}=\lim_{r\to\infty}\frac{r^2}{64\pi\omega^2}|\dot{\Psi}_0|^2\,, \tag{4.17}$$

where $d\Omega$ is the solid angle element [11]. In addition, some work that followed that of Teukolsky [23–27] established that solutions to (4.14) and (4.15) contain the information of the entire metric fluctuation, and are enough to reconstruct it in a particular gauge (the spinor description of this is explicitly given in [27]).

The fact that an equation as simple as (4.16) governs the fluctuations of rotating black holes has been instrumental in obtaining many of the predictions known to date about the behaviour of gravitational waves in the strong field regime. This includes (but is certainly not restricted to) high-precision computations of black hole QNMs in general relativity, and also accurate analysis of the gravitational radiation emitted by several astrophysically promising systems, such as EMRIs. This is, however, only an example of the use of spinor techniques in gravitational physics, and they are crucial in many other open challenges.

[6] However, we notice they share a common separation constant. In some cases, such as computing QNMs, this effectively couples both equations, that need to be solved simultaneously.

A Line Bundle Approach to the GHP Formalism

Here we follow the introduction of [28], although we also recommend the original discussions in [10, 29]. Assume two null directions l^a and n^a satisfying $l^a n_a = 1$ have been chosen globally across spacetime. The set of null frames aligned with l^a and n^a, together with the (right) action of the group $\mathbb{C}_\times$ (complex numbers without the origin)

$$(\ell_a, n_a, m_a; \lambda) \mapsto (\lambda\bar{\lambda}\ell_a, \lambda^{-1}\bar{\lambda}^{-1}n_a, \lambda\bar{\lambda}^{-1}m_a) \quad \forall\lambda \in \mathbb{C}_\times\,, \tag{A.1}$$

form a principal line bundle, consisting in a reduction of the null frame bundle. The associated connection 1-form is

$$\omega_a = -\epsilon'\ell_a + \epsilon n_a + \beta' m_a - \beta\bar{m}_a = \frac{1}{2}\left(n^b\nabla_a\ell_b + m^b\nabla_a\bar{m}_b\right)\,, \tag{A.2}$$

which transforms correctly under the action of $\mathbb{C}_\times$,

$$\omega_a \mapsto \omega_a + \lambda^{-1}\nabla_a\lambda\,. \tag{A.3}$$

The quantities $\eta \mathrel{\mathring{=}} (p, q)$ live in the vector bundle $E_{p,q}$ associated to the representation of $\mathbb{C}_\times$ with weights p, q. The covariant derivative on $E_{p,q}$ is

$$\Theta_a = \nabla_a - p\omega_a - q\bar{\omega}_a\,, \tag{A.4}$$

where we write ∇_a instead of ∂_a to include also sections of $E_{p,q}$ that take values on tensors (e.g. $R_{abcd}\ell^c m^d$). We notice that, by construction, Θ_a preserves GHP weights so in that sense $\Theta_a \mathrel{\mathring{=}} (0, 0)$. Then, þ and ð are introduced in a natural way simply as the directional derivatives associated to Θ_a,

$$þ = \ell^a\Theta_a\,, \quad þ' = n^a\Theta_a\,, \quad ð = m^a\Theta_a\,, \quad ð' = \bar{m}^a\Theta_a\,, \tag{A.5}$$

and it becomes manifest that they are well defined derivatives and that $þ \mathrel{\mathring{=}} (1, 1), þ' \mathrel{\mathring{=}} (-1, -1), ð \mathrel{\mathring{=}} (1, -1), ð' \mathrel{\mathring{=}} (-1, 1)$. Finally, the properties (3.74) follow directly from $(\Theta_a\eta)' = \Theta_a\eta'$.

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