Relativistic Tidal Love Numbers:

Tests of Strong-Field Gravity

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Resumo

Os números de Love de maré (NLM) relacionam a estrutura multipolar de um corpo com os campos de maré que o perturbam. Em relatividade geral (RG), curiosamente, os NLM de buracos negros (BNs) são precisamente zero. Este resultado intrigante motiva-nos a analisar esta propriedade sob cenários mais gerais.

Nesta tese estudamos o caso de "buracos de minhoca" mostrando que, mesmo em configurações ultracompactas, os NLM são não-nulos. Interessamo-nos, também, por soluções de BNs em teorias para além da RG no vácuo. Os nossos resultados neste âmbito mostram que a propriedade "zero-Love" é extensiva a BNs descritos em gravidade de Brans-Dicke e de Einstein-Maxwell. Ainda nesta prespectiva estudamos os NLM de BNs em gravidade de Chern-Simons e, interessantemente, descobrimos que estes são não-nulos. Este resultado providencia a primeira prova de que os NLM de BNs podem ser não triviais em gravidade modificada. Argumentamos que este resultado pode ser usado como um método inovador para testar a gravidade em regime forte e para identificação de possíveis objectos exóticos compactos através de futuras medições de ondas gravitacionais.

A nossa metodologia começa com a perturbação das soluções de BN usando um formalismo de perturbações lineares. Posteriormente, impomos a gauge de Regge-Wheeler e resolvemos as equações de campo para a teoria usando condições fronteiras apropriadas. Das soluções resultantes identificamos os momentos multipolares induzidos e os campos de maré que nos permitem calcular os NLM.

Alguns destes resultados e mais pesquisa sobre NLM de objectos exóticos compactos e BNs em gravidade modificada podem ser encontrados em [1].

Palavras-chave: Números de Love de maré, buracos de minhoca, buracos negros, relatividade geral, gravidade modificada.

Abstract

The tidal Love numbers (TLNs) relate the induced multipolar structure of a body with the perturbing tidal environment. Interestingly, in general relativity (GR), TLNs of black holes (BHs) are precisely zero. This intriguing result motivate us to analyze this property under more general scenarios.

In this thesis we study the case of wormholes finding that, even in ultracompact configurations, their TLNs are non-zero. We are also interested in some BH solutions in gravity theories beyond vacuum GR. In this context our results show that the "zero-Love" property is extensive to BHs described in Brans-Dicke and Einstein-Maxwell gravity. Still in this perspective, we study the TLNs of non-rotating BHs in Chern-Simons gravity and, interestingly, we find that they are non-zero. This result provides the first evidence that TLNs of BHs can be non-trivial in modified gravity. We argue that this result may be used as a novel method to test strong-field gravity and to identify possible exotic compact objects with future precise gravitational wave measurements.

Our methodology starts with the perturbation of our BH solutions using a linear perturbation formalism. We then impose the Regge-Wheeler gauge and solve the perturbed field equations of the theory using appropriate boundary conditions. From the resulting solutions, we identify the induced multipole moments and the tidal fields which allow us to compute the TLNs.

Some of these results and further research on TLNs of exotic compact objects and BHs in modified gravity theories can be found in Ref. [1].

Keywords: Tidal Love numbers, wormholes, black holes, general relativity, modified gravity.

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Nomenclature

Abreviations

TLN Tidal Love numbers.

GR General Relativity.

BH Black hole.

ECO Exotic compact object.

LIGO Laser Interferometer Gravitational-Wave Observatory.

NS Neutron star.

GW Gravitational wave.

EM Electromagnetic.

STF Symmetric and tracefree.

Symbols

k_l Newtonian tidal Love number.

k_l^B Axial-type gravitational tidal Love number.

k_l^E Polar-type gravitational tidal Love number.

$\mathcal{K}_l^B$ Axial-type electromagnetic tidal Love number.

$\mathcal{K}_l^E$ Polar-type electromagnetic tidal Love number.

1 Introduction

1.1 Motivation

In order to fully understand the motivation behind this thesis one needs to understand the importance of studying tidal effects. First of all, tidal deformation occurs whenever a body of finite size is deformed due to its gravitational interaction with other bodies. The gravitational interaction between the Earth, Moon and Sun is a well-known example of a system where tidal effects are important, with implications in different fields of study such as geophysics, astrophysics and engineering.

The theory of tidal deformation had its origin in the studies about Earth's oceanic tides. This research allowed the accurate prediction of Earth's oceanic tides providing essential information to everyone who depends on oceans and seas for their livelihood. Several types of civil engineering projects like bridges and harbors depend on the height of the oceanic tides. The planning and development of such projects requires the study and prediction of these tides. Tidal effects are felt not only on liquids, like oceans and rivers, but also in solids and, for that reason, highly sensitive ground experiments, like those at the Large Hadron Collider and the Laser Interferometer Gravitational-Wave Observatory (LIGO) can be affected by planetary tides [4, 5].

In astrophysics, the study of tidal effects is essential in a broad range of astronomical scales, from small objects like stars (for Newtonian and relativistic stars [6, 7]) to large celestial bodies such as galaxies (e.g. the formation of tidal tails [8]). Another well-known example of the importance of tidal effects is the process of tidal capture, where two unbound celestial bodies can become bound due to tidal effects [9]. These tidal processes can be particularly strong and important in the regime that characterizes compact objects where tidal effects can give rise to several interesting and extreme phenomena (e.g. tidal disruptions).

Binaries of neutron stars (NSs) and black holes (BHs) are promising sources of gravitational waves (GWs) [10, 11] as confirmed by the detection of GWs from BH mergers by the LIGO Collaboration [12, 13]. A compact binary system (i.e. a binary system composed by compact objects) loses energy and angular momentum by emission of GWs which leads to a decrease in its orbital radius and a increase in its frequency. Initially, the separation between the two objects is sufficiently large for the objects to be considered as point masses, however, at some point in the inspiral, the orbital separation becomes sufficiently small that tidal effects become relevant. The influence of the NS's internal structure in the GW signal emitted by the inspiral is characterized by one parameter: the tidal Love numbers (TLNs). In Newtonian gravity, the Love numbers measure the deformability of an object immersed in an external tidal field [2, 14]. The relationship between the TLNs and their effect on the GW signal was studied in recent years and prospects of measuring the TLNs of NSs by the the current generation of ground based GW detectors were formulated [3, 7, 15]. Since TLNs encode information about the internal structure of the deformed body, the measurement of GWs can provide a window to probe the interior of these compact objects. NSs are one of the most compact objects in the universe, with densities much larger than the nuclear density. One of the most pressing problems in modern astrophysics is to understand and model the internal structure of these objects. The measurement of TLNs through GW detection may provide direct and model-independent constraints for the equation of state of these objects, one of the most intriguing and active research topics in modern astrophysics.

Motivated by these works, a relativistic theory of Love numbers was developed with success. Interestingly, it was found that the TLNs of BHs are precisely zero, at least within the framework of general relativity (GR). These results were obtained for non-rotating BHs for weak tidal fields [16, 17] and generalized for strong tidal fields [18]. Further studies [19–22] extended this result for slowly rotating BHs to second order in the spin for axisymmetric spacetimes [22] and first order in the spin for general spacetimes [20]. These results lead to the conjecture that the TLNs of rotating BHs are zero to any order in the spin parameter, at least in the axisymmetric case.

The exact vanishing of the TLNs of BHs poses the intriguing problem of "natureness" in GR which has been a recent discussion topic [23]. This BH property can be tested with future precise GW data (which encodes the TLNs of the objects), and possible deviations can indicate that we are in presence of new physics.

In this thesis, we argue that deviations of this "zero-Love" property can occur in modified gravity. It is known that, in some modified gravity theories, unperturbed BHs may not be described by the Kerr-class of metrics. Here, we also show that some BHs in these theories can have different multipolar responses to perturbing tidal fields and, therefore, they may exhibit non-zero TLNs. We will also discuss this property for exotic compact objects (ECOs), in particular for wormholes (cf. Refs. [24, 25] for a review). More details on the TLNs of other ECOs are present in Ref. [1].

The research developed in this thesis and related works [1] attempts to contribute to a better comprehension of the "zero-Love" property of BHs. In the dawn of the GW astronomy age, we may expect to use future GW data to measure precisely the TLNs of BHs and with it to test gravity, GR and beyond.

1.2 Exotic Compact Objects

The current state-of-art of stellar structure evolution suggests that matter, even in extreme forms, cannot support the self-gravity present in massive compact objects and, naturally, these ultracompact objects tend to collapse to BH states. However, if we consider other types of objects, composed by other forms of matter, and that rely on different supporting mechanisms, we can construct objects that are almost as compact as BHs and do not possess an event-horizon. These are the so-called ECOs or BH mimickers. Arguably, the most well-study ECOs are boson-stars, self-gravitating configurations of massive scalar fields (cf. Refs. [26, 27] for some review on the topic); gravastars, objects whose interior is a de Sitter spacetime and its internal structure is characterized by an exotic equation-of-state (cf. Refs. [28, 29]); wormholes, objects composed by exotic matter that connect two separate universes (or spacetime regions).

In this thesis we are concerned in more detail with the TLNs of wormholes. The concept of wormhole first appeared in the context of Reissner-Nordström or Kerr spacetimes, where it played the role of objects of the quantum foam that connected different regions of the spacetime [30], however, these wormholes could not be crossed back and forth. The desire to conceive a wormhole that, throught it, allowed the passage from one region of the spacetime to another, led to a first research about traversable wormholes [31]. Thereforward, the research on wormholes has grown substantially, culminating on the work [24] and more recently in [25].

The study of wormholes and other ECOs is of extreme interest to modern astrophysics and gravitation physics. From an observational point of view, it would be interesting to conduct searches for this types of objects in order to understand if they are real astrophysical objects or just hypothetical, exotic solutions of the Einstein's equations. One of the most promising methods to search for this BH mimickers is through GW signals [32] since the presence of the objects surface is more challenging to test in the electromagnetic (EM) spectrum. Furthermore, the oscillation modes of BHs are very well-known [33] and they can be distinguished from the oscillation modes of ECOs that leave a clear imprint in the GW signal due to the presence of a surface [34]. Note, however, that this approach must be taken with care since it is possible that ECOs and BHs may exhibit very similar waveforms until late stages of the ringdown signal [35].

1.3 Modified Gravity

Perhaps one of the most simple extensions of GR is the Einstein-Maxwell gravity which allows the existence of BHs with electric charge. Charged BHs have been disregarded as serious candidates for astrophysical BHs due to several processes that tend to neutralize rapidly their charge (e.g. quantum discharge effect [36], electron-positron pair production [37–39] and charge neutralization by the environment like astrophysical plasmas). However, recent works motivated the possibility of charged BHs to be serious astrophysical candidates by interaction with minicharged dark matter [40]. A viable candidate for cold dark matter are dark fermions, a model of particle which possess a fractional electric charge or that are charged under a hidden $U(1)$ symmetry [41–46]. These fermions possess a charge much smaller than the electron charge and, therefore, their coupling with the EM sector is suppressed. In these minicharged dark matter theories, the charge-neutralization arguments mentioned above can be circumvented and consequently, charged BHs can be realistic astrophysical BHs [40].

There are both theoretical and observational evidence that GR should be modified at large and/or low energy scales [47]. From a purely theoretical perspective one can state that modern physics is based on the pillars of Einstein's theory of GR and quantum mechanics. There seems to exist a mismatch between these two theories since GR is a purely classical theory and not renormalizable in the usual quantum field theory sense. This mismatch poses a great obstacle for the development of a quantum theory of gravity, one of the most interesting problems in theoretical physics. Meanwhile, it was shown that adding quadratic curvature terms to Einstein-Hilbert action will make the theory renormalizable [48]. Furthermore, quantum mechanics and GR are valid in two different scales and it is interesting to understand the physics in the limit where the two scales meet.

From an observational point of view, there is evidence that motivate studies on modified gravity theories. Modern cosmology measurements provide evidence for the existence of dark matter and dark energy, and a non-zero cosmological constant [49–51]. This interpretation of the results leads to different conceptual issues like the cosmological constant problem and the coincidence problem, where the former refers to the low measured value of the cosmological constant and the later to the remarkable coincidence between the dark matter density and the present matter density [52, 53]. Arguments as the fact that GR does not present a dynamical cosmological constant solution [52] and that ultraviolet corrections to GR would leak-down to cosmological scales showing up as low-energy corrections, motivate investigations on modified gravity theories. These theories should differ from GR at both high and/or low energy scales, while agreeing with GR at intermediate scales where this theory is extremely well-tested [54]. Furthermore, the recent detection of GWs from BH mergers provides the first steps for the direct testing of strong-field gravity. The results obtained from GW measurements are in accordance with the results predicted by Einstein's theory of GR and therefore impose more constrains on how modifications to this theory should appear in the strong-field regime [55].

Whitin this final part, we will concentrate on strong-field modifications of gravity [47]. In order to study the gravity in this limit, it is natural to investigate the behavior of NSs and BHs, two types of extreme compact objects related with high-curvature regions of the spacetime. Although the Kerr-Newman spacetime metric [56] is a solution of several different modified gravity theories [57, 58], generally these theories present different dynamics and GW emissions [59–61].

One of the most natural ways to modify gravity consists in including scalar degrees of freedom in the gravitational sector of the theory through a nonminimal coupling. This type of theory is named scalar-tensor gravity and is motivated by different possible fundamental theories and cosmologic scenarios. Another class of modified theories of gravity that satisfy the requirements and the motivations described above are the quadratic theories of gravity [57, 62]. In these theories the Einstein-Hilbert action is considered the first term of a possibly infinite expansion containing all curvature invariants. These modifications to gravity are supported by low-energy effective string theory and loop quantum gravity, two candidates for a quantum description of gravity [63–65]. Despite the ability of making the theory renormalizable, these quadratic terms come with the cost of introducing higher-derivative terms in the field equations. In general, these higher-order field equations are prone to the Ostrogradski instability and the appearance of ghosts [66]. In order to avoid these higher-order derivatives in the field equations, the quadratic curvature terms must appear in a specific combination corresponding to the Gauss-Bonnet invariant or the theory must be considered as an effective action, valid up to second order correction in the curvature, obtained by the truncation of a more general theory. The motivation for considering the effective field theory approach is not only driven by the removal of the higher-order derivatives of the field equations, but also because it arises naturally in some low-energy expansion of string theory.

Considering this, throughout this work we will focus on one particular case of quadratic gravity: the Chern-Simons gravity [67, 68]. This gravity theory is qualitatively different from the remaining quadratic theories since this action predicts corrections to GR in the presence of parity-odd sources, such as rotating objects. Further details about this theory will be presented in Chapter 5 of this thesis.

The arguments discussed above, together with the prospect of measuring TLNs from GW signals, motivate our research on TLNs in modified gravity theories as methods of testing strong-field gravity. The research developed in this Master thesis, added with future works and comparison with data obtained from GW observations might lead to new constraints in these theories and new insights on how GR may be modified in the strong field regime.

1.4 Newtonian Theory of Tidal Love Numbers

The theory of Love numbers emerges naturally from the theory of tidal deformation. We take the case of a finite size object immersed in a tidal environment, which can be composed by one or more bodies and can be simpler or extremely complex. We simplify this problem by containing all these gravitational effects in one external potential V. The external potential V can be naturally expanded in a Taylor's series around the body's center-of-mass [2],

$$V = \sum_{l=0}^{\infty} \frac{1}{l!} \partial_L V x^L, \tag{1.1}$$

where L is multi-index containing l individual indices. A summation over a repeated multi-index is equivalent to a summation over all the individual indices. In this case the potential is differentiated l times with respect to x^j and evaluated at the body's center-of-mass. A detailed explanation of this notation is present on Appendix A. Noticing that $l = 0$ is a constant term, we can remove it from the expansion since it will not contribute for the equations of motion. The $l = 1$ term can also be removed due to the fact that it is related with the body's dipole moment which vanishes by virtue of the definition of center-of-mass (see chapters 1 and 2 of Ref. [2]). Eq. (1.1) can be rewritten as

$$V = -\sum_{l=2}^{\infty} \frac{1}{l(l-1)} \mathcal{E}_L x^L, \tag{1.2}$$

where we defined $\mathcal{E}_L$ as

$$\mathcal{E}_L := -\frac{1}{(l-2)!} \partial_L V. \tag{1.3}$$

The quantity $\mathcal{E}_L$ is called gravitational tidal moment and characterizes the external tidal environment. The fact that the external potential satisfies Laplace's equation implies that the tensors $\mathcal{E}_L$ are symmetric and tracefree (STF).

We will restrict this problem to the case of static tides by imposing that $\mathcal{E}_L$ does not depend on time. In general, tidal moments are time-dependent, but we will assume that this dependence is sufficiently small that the tidal field is never able to the take the body out of hydrostatic equilibrium.

In absence of the external perturbations, the body is in an initial hydrostatic equilibrium with an unperturbed internal structure. However, due to the gravitational interaction with the external environment, this hydrostatic equilibrium will be disrupted, resulting in modifications of its internal and multipolar structure. In the framework of linear perturbation theory, the equations of motion yield a proportionality relation between the external tidal field moment $\mathcal{E}_L$ and the induced multipole mass moment $I_{\langle L\rangle}$,

$$I_{\langle L\rangle} = -\frac{2k_l(l-2)!}{G(2l-1)!!}R^{2l+1}\mathcal{E}_L, \tag{1.4}$$

where the proportionality constant k_l is defined as the TLN of the object. The introduction of the factor R^{2l+1}/G guarantees that the TLN is adimensional, and the remaining numerical factors are introduced by convenience in order to be in accordance with the standard notation in the literature [2, 17].

The general expression for the body's gravitational potential U in terms of its multipole mass moments is,

$$U = G\sum_{l=0}^{\infty}\frac{(2l-1)!!}{l!}I^{\langle L\rangle}\frac{n_{\langle L\rangle}}{r^{l+1}}, \tag{1.5}$$

where $n_{\langle L\rangle}$ is defined as,

$$x_L = n_{\langle L\rangle}r^l\,. \tag{1.6}$$

Noticing that the dipole mass term vanishes we can rewrite Eq. (1.5) as,

$$U = \frac{GM}{r} - \sum_{l=2}^{\infty}\frac{(2l-1)!!}{l!}I^{\langle L\rangle}\frac{n_{\langle L\rangle}}{r^{l+1}}. \tag{1.7}$$

Substituting Eq. (1.4) in the potential (1.5) and adding the external potential V (1.2) we obtain the complete gravitational potential,

$$\Phi = U + V, \tag{1.8}$$

around a tidally deformed body [2, 17, 69],

$$\Phi = \frac{GM}{r} - \sum_{l=2}^{\infty}\frac{1}{l(l-1)}\left[1 + 2k_l\left(\frac{R}{r}\right)^{2l+1}\right]\mathcal{E}_L x^L, \tag{1.9}$$

where the first term is the body's undisturbed potential, the first part of sum are terms that grow with r^L correspondent to the contribution of the external tidal field and the second part are terms that decay with $r^{-(L+1)}$ that correspond to the body's induced multipolar structure.

1.5 Relativistic Theory of Tidal Love Numbers

The prospect of measuring TLNs from the GW signals of compact object binaries [7] led to the development of a fully relativistic theory of TLNs [16, 17, 69]. In this thesis we will develop the study of TLNs of fully relativistic objects such as BHs and ECOs which requires the use of this theory and, for those reasons, the main features of the relativistic theory of TLNs will be introduced here.

To generalize the Newtonian definition of Love numbers to a relativistic definition we need to replace the Newtonian potential formulation of the gravitational interaction by a geometrical description of the tidal interaction. Several works were made regarding the tidal deformation of bodies in GR. A relativistic definition of multipole moments was developed by Geroch and Hansen [70, 71], where they used a complex mathematical formulation to describe the changes in the asymptotic behavior of the spacetime in terms of two quantities, the mass multipole moments $\mathfrak{M}$ and current multipole moments $\mathfrak{J}$. This formalism has the useful property of not depending on any particular choice of coordinate system and therefore it is an extremely powerful description for proving general theorems. Another definition of relativistic multipole moments for stationary, asymptoticaly flat spacetimes was given by Thorne in 1980 where the body's multipole moments can be extracted and defined from the asymptotic spacetime metric [72]. This method comes as a natural extension of the procedure to read the mass and angular momentum of an object from the asymptotic limit of the spacetime metric [73]. Thorne's definition of multipole moments requires the use of a specific coordinate system, the asymptotically Cartesian and mass centered (ACMC) coordinates. In this system of coordinates the metric becomes asymptotically Minkownskian sufficiently fast and the center of coordinates lies at the center-of-mass of the source. Further details on this definition can be found on references [72, 74] and in the appendix of Ref. [75]. Gursel [76] showed the equivalence between the multipole moments defined by Thorne [72] and the multipole moments defined by Geroch and Hansen [70, 71]. In recent years most of the works followed the Geroch-Hansen normalization of multipole moments and for this reason we shall take this normalization during the course of this work.

So far, these works focused on the body's multipole moments and in the description of the spacetime metric in terms of these moments. When considering tidal deformation due to external gravitational sources we need to account for the direct contibution of these external gravitational effects in the spacetime description and, in some sense, find a relativistic generalization of Eqs. (1.2)–(1.3). Following the multipolar approach of Ref. [72], some studies about the multipole expansion of the external universe were developed [74, 77]. Regarding the external universe decomposition we follow Ref. [17] and define the STF polar and axial[1] tidal multipole moments of order l as $\mathcal{E}_{a_1\ldots a_l} \equiv [(l-2)!]^{-1}\langle C_{0a_10a_2;a_3\ldots a_l}\rangle$ and $\mathcal{B}_{a_1\ldots a_l} \equiv [\frac{2}{3}(l+1)(l-2)!]^{-1}\langle \epsilon_{a_1bc}C^{bc}_{a_20;a_3\ldots a_l}\rangle$, where C_{abcd} is the Weyl tensor, a semicolon denotes a covariant derivative, ϵ_{abc} is the permutation symbol, the angular brackets denote symmetrization of the indices a_i and all traces are removed. The polar (axial) moments $\mathcal{E}_{a_1\ldots a_l}$ ($\mathcal{B}_{a_1\ldots a_l}$) can be decomposed in a basis of even (odd) parity spherical harmonics.

We denote by $\mathcal{E}_{lm}$ and $\mathcal{B}_{lm}$ the amplitudes of the polar and axial components of the external tidal field with harmonic indices (l, m), where m is the azimuthal number ($|m| \leq l$). The structure of the external tidal field is entirely encoded in the coefficients $\mathcal{E}_{lm}$ and $\mathcal{B}_{lm}$ (cf. Ref. [17] for details).

As a result of the external perturbation, the mass and current multipole moments (M_l and S_l, respectively) of the compact object will be deformed. In linear perturbation theory, these deformations are proportional to the applied tidal field. In the nonrotating case, mass (current) multipoles have even (odd) parity, and therefore they only depend on polar (axial) components of the tidal field. When the deformed

[1] It is slightly more common to use the distinction electric/magnetic components rather than polar/axial. Since we shall discuss also EM fields, we prefer to use the former distinction.

object is rotating, this symmetry is broken due to the introduction of spin-tidal couplings. In this case there exists a series of selection rules that allow to define a wider class of "rotational" TLNs [20–22, 78]. Hence, we can define the TLNs as [3, 17]

$$\begin{aligned} k_l^E &:= -\frac{1}{2}\frac{l(l-1)}{M^{2l+1}}\sqrt{\frac{4\pi}{2l+1}}\frac{M_l}{\mathcal{E}_{l0}}, \\ k_l^B &:= -\frac{3}{2}\frac{l(l-1)}{(l+1)M^{2l+1}}\sqrt{\frac{4\pi}{2l+1}}\frac{S_l}{\mathcal{B}_{l0}}, \end{aligned} \tag{1.10}$$

where k_l^E is called polar-type TLN, k_l^B is called axial-type TLN, and M is the mass of the object. The factor M^{2l+1} was introduced to make the above TLNs dimensionless. It is customary to normalize the TLNs by introducing powers of the object's radius R rather than powers of its mass M. Here, we adopted the latter nonstandard choice to be in accordance with Ref. [1]. We argue that this definition is more appropriate to our discussion since the radius of some ECOs is not a well defined quantity. The remaining normalization factors were chosen such that our definition of TLNs agrees with the standard definition of Hinderer, Binnington and Poisson (HBP) by

$$k_{l\,\mathrm{ours}}^{E,B} = \left(\frac{R}{M}\right)^{2l+1} k_{l\,\mathrm{HBP}}^{E,B}\,. \tag{1.11}$$

1.6 Electromagnetic and Scalar Love Numbers

In this work we will introduce the study of TLNs of BHs in modified theories of gravity and also the TLNs of ECOs. These types of objects typically require the presence of extra fields which are (non)minimally coupled to the metric tensor. Here we shall consider representative example of both scalar and vector fields and, correspondingly, we shall also consider external scalar and EM fields.

We decompose an external EM field in its electric and magnetic components ($\mathfrak{E}_{lm}$ and $\mathfrak{B}_{lm}$, respectively), which can induce an electric and magnetic multipole moment (Q_l and J_l, respectively) on a charged body. Similarly to the gravitational case, we can define an analogous of the gravitational TLNs as

$$\begin{aligned} \mathcal{K}_l^E &:= -\frac{1}{2}\frac{l(l-1)}{M^{2l+1}}\sqrt{\frac{4\pi}{2l+1}}\frac{Q_l}{\mathfrak{E}_{l0}}, \\ \mathcal{K}_l^B &:= -\frac{3}{2}\frac{(l-1)l}{(l+1)M^{2l+1}}\sqrt{\frac{4\pi}{2l+1}}\frac{J_l}{\mathfrak{B}_{l0}}, \end{aligned} \tag{1.12}$$

where $\mathfrak{E}_l$ and $\mathfrak{B}_l$ are the amplitudes of the azimuthal component of the applied EM field with polar and axial parity, respectively. We restrict this analysis to the case of nonspinning objects, where the spacetime is spherically symmetric and, without loss of generality, we can define the TLNs in the axisymmetric ($m = 0$) case. Clearly, this property does not hold when the object is spinning [19, 20, 22]. We define $\mathcal{K}_l^E$ as the electric TLN and $\mathcal{K}_l^B$ as the magnetic TLN (not to be confused with the gravitational one in Eq. (1.10)). Note that the electric and magnetic TLNs are simply proportional to the electric and magnetic susceptibilities [79].

Finally, an applied scalar field $\mathcal{E}_{lm}^S$ can induce a scalar multipole moment Φ_l associated to a scalar

TLN,

$$k_l^S := -\frac{1}{2}\frac{l(l-1)}{M^{2l+1}}\sqrt{\frac{4\pi}{2l+1}}\frac{\Phi_l}{\mathcal{E}_l^S}\,, \tag{1.13}$$

where $\mathcal{E}_l^S$ is the amplitude of the azimuthal component of the external scalar field with harmonic index l.

Similarly to the case for gravitational tidal fields, we will assume that the external scalar and EM fields are small enough so we can study their effects on the deformations of the objects through linear perturbation theory. Since the background is spherically symmetric, perturbations with different parity and different harmonic index l decouple.

1.7 State of the Art

In 2005, the process of tidal deformation of a BH by an orbiting body of much smaller mass was studied [80]. The authors used a linear perturbation approach to describe slight perturbations to the Schwarzschild metric caused by the external tidal field. The gauge freedom was fixed by choosing the Regge-Wheeler gauge and, after the calculations, they concluded that for a static tidal field there are no induced quadrupole moment on the BH. They argued that the vanishing quadrupole moment of the BH could be caused by an inappropriate gauge choice and discussed that the definition of induced multipole moment of a BH is ambiguous. These arguments were analyzed in following researches and it was shown that they can be contradicted.

Later, the problem of tidal deformation of a NS with the purpose of computing its TLN k_2 was addressed [3, 7]. In their work, the authors calculated the external spacetime metric (perturbed Schwarzschild metric using Regge-Wheeer gauge) and matched it with the interior spacetime metric (calculated using a polytropic pressure-density relation). The star's quadrupole moment and the static external quadrupolar tidal field were obtained using the asymptotic behavior of the spacetime metric and, from this procedure, the TLN can be calculated as a function of the star's parameters.

After these works, a generalization of the Newtonian theory of Love numbers to a precise relativistic theory was developed [17]. This relativistic definition lead to the introduction of two TLNs, an electric-type TLN related to the even-sector of the perturbations which can be related to the Newtonian TLN, and a magnetic-type TLN[2] related to the odd-sector of the perturbations which is a purely relativistic effect. Applying perturbations to the Schwarzschild metric written in Eddington-Finkelstein coordinates using light-cone gauge conditions, they reached the conclusion that both TLNs must be zero for a BH. In their work they calculate the TLNs for stars using a polytropic equation of state to describe the stellar interior. Their work also showed that the TLNs are gauge invariant, contradicting the argument of Ref. [80]. They argued also that there should be no ambiguity problem in the definition of induced multipole moment on the BH.

Simultaneously to Ref. [17] another similar work was conducted [16]. In this work, the authors studed the problem of the tidal deformation of bodies in the framework of GR and presented precise definitions for electric- and magnetic-type TLNs and, in addition, they defined shape Love numbers. Their results

[2]We remark that, in our notation, the electric-type (magnetic-type) TLN of Ref. [17] corresponds to our polar-type (axial-type) TLN.

lead to the conclusion that the induced quadrupole moment of a BH is zero, however the authors state that this does not imply that the correct value of the TLN of a BH is zero and, in order to reach this conclusion, one should study the diverging diagrams that enter the computation of interacting point masses at the 5-loop (or 5PN) level. However, in Ref. [17] this argument is debated and the authors argue that it has no relevant importance to the conclusions. This relativistic formulation of TLNs of Ref. [17] was later complemented with the relativistic definition of surficial Love numbers (related with the shape Love numbers) [69]. In this work the authors revisit the concept of surficial (shape) Love number of Ref. [16] and define it in terms of the deformed curvature of the body's surface. With this approach the authors develop a fully relativistic theory of surficial Love numbers which can be implement for material bodies and for black holes. The authors also derive a compactness-dependent relation between the polar-type and the surficial Love numbers of a perfect fluid body which agrees in the Newtonian limit with the relation between the Newtonian tidal and shape Love numbers.

With this relativistic formulation developed with success, further studies were developed to generalize this theory to rotating compact objects. A first approach to study the TLNs of rotating compact objects was attempted simultaneously by two independent works [20, 21]. Both of these studies were concerned about the description of the exterior geometry of a spinning compact object immersed in a tidal environment. Ref. [21] considers the external geometry expanded to the second order in the spin parameter while Ref. [20] focus on a linear order expansion. However, the first work restricts the analysis to axisymmetric spacetimes, while the second generalizes the study to non-axisymmetric spacetimes. When the deformed object is a BH, these studies shown that the TLNs are precisely zero and, therefore, both of these works are complementary. These results lead to the conjecture that, at least in the axisymmetric case, the TLNs of a rotating BH are zero to any order in the spin parameter.

1.8 Thesis Outline

Regarding the organization of the thesis it is as follows. In Chapter 1, we introduced the purpose of this thesis by defining the relevance of tidal deformation and Love numbers. We provided an introduction to the theory of TLNs and distinguished between the Newtonian theory, based on a potential formulation of gravity, and the relativistic theory, based on a geometric formulation of gravity. Furthermore, we introduced the new concepts of electric, magnetic and scalar TLNs that will be necessary for this thesis. The remaining of this work is divided into three main chapters. In Chapter 2 we will continue the introduction to the study of TLNs by applying the Newtonian theory to study the case of a fluid sphere immersed on a tidal field. After this chapter we will focus on fully relativistic objects and, for this reason, in Chapter 3, we explain how to identify and calculate TLNs in relativistic theories of gravity. In Chapter 4 we will advance to the relativistic theory of TLNs in the framework of GR and Einstein-Maxwell gravity. In Sec. 4.2, we will present the calculation of the TLNs of an uncharged BH, showing the "zero-Love property" of BHs and, in Sec. 4.3, this calculation will be extended to the case where the deformed object is a charged BH. Still in the framework of GR we will study, in Sec. 4.4, the TLNs of wormholes. Finally, in Chapter 5 we will present the calculations and study of tidal deformation in modified gravity theories, where, in Sec. 5.1, we will study the TLNs of a BH in scalar-tensor gravity (exemplifying for Brans-Dicke gravity) and, in Sec. 5.2, we will introduce the study of TLNs in quadratic theories of gravity where the TLNs of a BH in Chern-Simons gravity will be calculated. The conclusions of this dissertation are discussed in Chapter 6.

Love Numbers in Newtonian Gravity

2

Before attempting to calculate the TLNs of fully relativistic objects, it is interesting to understand how they are calculated in the context of the Newtonian description of gravity. In this section we complement the definitions of Newtonian TLNs (cf. Sec. 1.4 of Chapter 1) with an example on how to calculate the TLNs of a fluid star following Ref. [2].

2.1 Love Numbers of a Fluid Star

For this purpose, we consider the case of a non-rotating[1], unperturbed fluid star of radius $r = R$ characterized by its density $\rho(r)$ and by its pressure $p(r)$ which is immersed on a tidal environment that induces deformations on the internal structure of the body. The system described can be represented by Fig. 2.1.

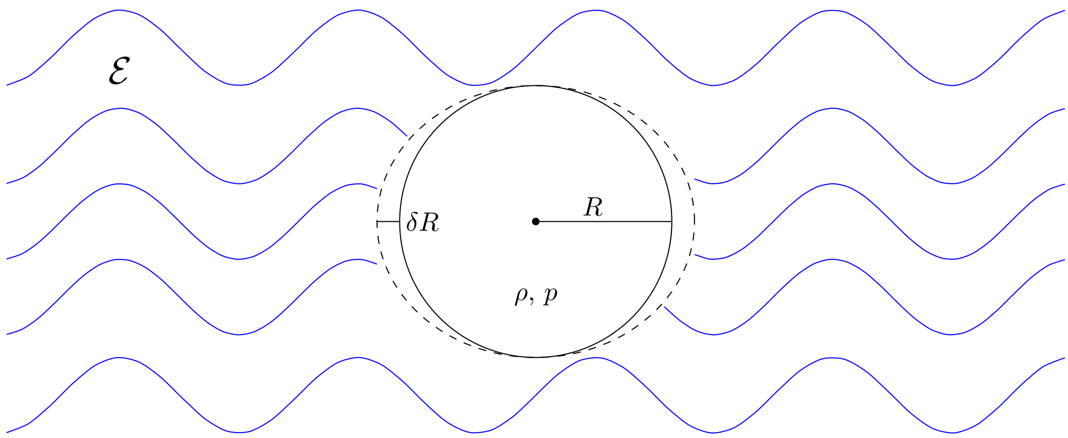

Figure 2.1: Pictorial description of a spherical fluid star of radius R characterized by density ρ and pressure p, immersed in an external tidal environment characterized by the tidal field moment $\mathcal{E}$. The external perturbations break the initial hydrostatic equilibrium and deform the planet by inducing a radial displacement δR and changing its multipolar structure.

When considering our fluid sphere system there are only two forces that we must take into account:

[1]The assumption that the fluid is non-rotating can be relaxed as it is done in Ref. [2]. In order to account for the body's rotation one needs to be careful with the choice of coordinate frame, being more convenient to work in the rotating frame. However, in this frame one needs to account for the resulting (non-inertial) centrifugal force. The remaining calculations are identical to the ones described in this chapter.

the gravitational force, related with the gravitational potential, and the force related to the pressure which prevents the star's collapse.

When there are no external interactions the star is in an initial hydrostatic equilibrium stage characterized by the annulment of the gravitational force and the internal pressure of the body,

$$\frac{dp}{dr} - \rho\frac{dU}{dr} = 0\,, \tag{2.1}$$

and the unperturbed gravitational potential is governed by Poisson's equation,

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dU}{dr}\right) = -4\pi G\rho\,. \tag{2.2}$$

In order to fully characterize the system and the unperturbed state of the body we need one remaining equation, the equation of state. This thermodynamic equation relates a set of state variables that describe a body under certain physical conditions. One of the most simple and common equations of state is the polytropic equation which describes a body whose internal pressure is proportional to a power of its density [81, 82],

$$p = K\rho^{\Gamma}\,, \tag{2.3}$$

where K and Γ are two constants. One can define the polytropic index n by the relation

$$\Gamma := 1 + 1/n\,. \tag{2.4}$$

For this problem, we will consider that our central object can be described by a polytropic model and use Eq. (2.3) to complete the description of our system.

The body is now perturbed by the introduction of an external potential V. The equation of motion for this perturbed system is described by the Euler equation[2] [83],

$$\frac{du^i}{dt} = \partial_i\Phi - \frac{\partial_i\overline{p}}{\overline{\rho}}, \tag{2.5}$$

where d/dt represents the convective time derivative associated with the motion of fluid elements, u^i denotes the i component of the velocity of the fluid element at position $\boldsymbol{x}$ and time t, $\Phi = \overline{U} + V$ is the total gravitational potential and $\overline{X}$ denotes a perturbed quantity of the unperturbed quantity X. The perturbations of a fluid quantity X can be described in terms its Lagrangian or Eulerian descriptions. Here, we explain the main differences between these two perspectives. An Eulerian perturbation δX corresponds to a macroscopic perspective, where the value of a perturbed quantity is compared with its unperturbed value at the same point in space and time,

$$\delta X := \overline{X}(t, \boldsymbol{x}) - X(t, \boldsymbol{x}). \tag{2.6}$$

[2] In fluid mechanics the Euler equation is usually expressed with the total time derivative replaced in terms of its components $d/dt := \partial/\partial t + \boldsymbol{v}\cdot\nabla$. However, for this calculation it is convenient to write Euler equation as Eq. (2.5) since the unperturbed body will be considered to be in equilibrium such that $d/dt = \partial/\partial t$.

A Lagrangian perturbation ΔX is related with a microscopic view of the fluid, where the value of a perturbed quantity in a fluid element is compared with its value at an initial point $\boldsymbol{x}$ that is displaced by a vector $\boldsymbol{\xi}$ to its final configuration,

$$\Delta X := \overline{X}(t, \boldsymbol{x} + \boldsymbol{\xi}) - X(t, \boldsymbol{x}). \tag{2.7}$$

Expanding the first term in the right-hand side of the previous equation in Taylor series we obtain a relation between the Lagrangian and the Eulerian representation of the perturbations,

$$\Delta X = \overline{X}(t, \boldsymbol{x}) + \xi^i \partial_i X - X(t, \boldsymbol{x}) = \delta X + \xi^i \partial_i X\,. \tag{2.8}$$

With the previous relations in mind and, using the conservation of mass of a fluid element during the displacement, we get

$$\delta\rho = -\partial_j \left(\rho \xi^j\right), \qquad \Delta\rho = -\rho \partial_j \xi^j. \tag{2.9}$$

By virtue of the equation of state Eq. (2.3), the density and pressure perturbations are related by

$$\frac{\Delta p}{p} = \Gamma \frac{\Delta\rho}{\rho}\,. \tag{2.10}$$

Substituting in the previous equation the expression (2.9) for the Lagragian density perturbation $\Delta\rho$, and making use of the relation between the two representations of the perturbation described in Eq. (2.8), the Eulerian description of the pressure perturbation can be written as

$$\delta p = -\Gamma p \partial_i \xi^i - \xi^i \partial_i p\,. \tag{2.11}$$

If we specifically consider $\boldsymbol{\xi}$ to be the displacement of a fluid element between two surfaces with constant density ρ in the unperturbed and perturbed configurations, such that $\Delta\rho \equiv 0$, Eqs. (2.9) and (2.10) yield

$$\delta\rho = -\xi^j \partial_j \rho\,, \tag{2.12}$$

$$\partial_j \xi^j = 0\,. \tag{2.13}$$

Furthermore, from Eq. (2.10), it is ensured that the Lagrangian pressure perturbation is zero, $\Delta p = 0$, and Eq. (2.11) yields

$$\delta p = -\xi^i \partial_i p\,. \tag{2.14}$$

We shall also consider that the fluid element suffers a displacement that leads the body from one equilibrium stage, in the initial configuration, to another equilibrium stage, in the final state,

$$\frac{d\xi^i}{dt} = 0\,. \tag{2.15}$$

Using this relation, and perturbing the quantities in Eq. (2.5) using a linear perturbation approach, we can write the Euler equation in terms as,

$$\frac{\delta\rho}{\rho^2}\partial_j p - \frac{1}{\rho}\partial_j \delta p + \partial_j \phi = 0\,, \tag{2.16}$$

where $\phi := \delta U + V$ and Eq. (2.1) was used to eliminate the unperturbed terms.

The perturbed quantities can be written in a spherical harmonic decomposition,

$$\xi^r = \sum_{lm} = r\xi_{lm}(r)Y_{lm}(\theta,\varphi)\,, \tag{2.17}$$

$$\delta\rho = \sum_{lm} \rho_{lm}(r)Y_{lm}(\theta,\varphi)\,, \tag{2.18}$$

$$\delta p = \sum_{lm} p_{lm}(r)Y_{lm}(\theta,\varphi)\,, \tag{2.19}$$

$$\delta U = \sum_{lm} U_{lm}(r)Y_{lm}(\theta,\varphi)\,, \tag{2.20}$$

$$V = \sum_{lm} V_{lm}(r)Y_{lm}(\theta,\varphi)\,, \tag{2.21}$$

where, by virtue of Eqs. (2.12) and (2.14), we presented only the radial part of ξ since the unperturbed density ρ and pressure p depend only on the radial coordinate and, therefore, only this component will be important. Using the former decomposition in Eqs. (2.12) and (2.14) we obtain

$$\rho_{lm} = -r\rho'\xi_{lm}\,, \tag{2.22}$$

$$p_{lm} = -rp'\xi_{lm} = \frac{\rho G m}{r}\xi_{lm}\,, \tag{2.23}$$

where the prime denotes a differentiation with respect to r.

Poisson's equation for the body's potential implies the relation

$$r^2U''_{lm} + 2rU'_{lm} - l(l+1)U_{lm} = -4\pi G r^2 \rho_{lm}\,, \tag{2.24}$$

in the interior of the body $r < R$ where $\rho_{lm} \neq 0$. In the outer region $\rho_{lm} = 0$ and the potential is

$$U^{\mathrm{ext}}_{lm}(r) = \frac{4\pi G}{2l+1}\frac{I_{lm}}{r^{l+1}}\,, \tag{2.25}$$

where I_{lm} are the body's multipole moments.

The functions V_{lm} can be found by requiring that the external potential must satisfy the Laplace's equation,

$$\nabla^2 V = 0\,, \tag{2.26}$$

and, by substituting Eq. (2.21) in the previous expression, we get

$$r^2V''_{lm} + 2rV'_{lm} - l(l+1)V_{lm} = 0\,. \tag{2.27}$$

The external potential must be regular at $r = 0$ and, with this condition, the only possible solution for Eq. (2.27) is

$$V_{lm}(r) = \frac{4\pi}{2l+1} d_{lm} r^l \,, \tag{2.28}$$

where the coefficients d_{lm} are defined as the moments of the external potential.

Substituting Eqs. (2.17)–(2.21) in Eq. (2.16), we obtain

$$p'_{lm} = \frac{Gm}{r^2}\rho_{lm} + \rho\left(U'_{lm} + V'_{lm}\right) \,, \tag{2.29}$$

$$p_{lm} = \rho(U_{lm} + V_{lm}) \,, \tag{2.30}$$

from the radial and angular components, respectively. Differentiating Eq. (2.30) and by substituting it in Eq. (2.29) it is straightforward to obtain

$$\frac{Gm}{r^2}\rho_{lm} = -\rho'(U_{lm} + V_{lm}) \,, \tag{2.31}$$

which by means of Eqs. (2.22)–(2.23) yields the final form of Euler's equation in terms of the harmonic functions,

$$\frac{Gm}{r}\xi_{lm} = U_{lm} + V_{lm} \,. \tag{2.32}$$

The system of equations described by Eqs. (2.22), (2.23) and (2.32) implies that the perturbed functions ρ_{lm}, p_{lm} and U_{lm} can be specified entirely by the function ξ_{lm}, for a given potential V_{lm} (or potential moment d_{lm}). The next immediate step is to find a relation between ξ_{lm} and V_{lm}, which will give information about the deformed structure of the star and allow the computation of the tidal Love numbers. Thus, we need to obtain a relation between the body's multipole moments I_{lm} and the tidal moment d_{lm}. Using Eq. (2.32) and its derivative evaluated at the body's surface $r = R$ we obtain the relations

$$\frac{GM}{R}\xi_{lm}(R) = \frac{4\pi}{2l+1}\left(\frac{GI_{lm}}{R^{l+1}} + d_{lm}R^l\right) \,, \tag{2.33}$$

$$\frac{GM}{R}\left[R\xi'_{lm}(R) - \xi_{lm}(R)\right] = \frac{4\pi}{2l+1}\left[-(l+1)\frac{GI_{lm}}{R^{l+1}} + l d_{lm}R^l\right] \,, \tag{2.34}$$

where we substituted V_{lm} by Eq. (2.28) and U_{lm} by Eq. (2.25). This system of equations can be transformed into two separate equations for the tidal moment d_{lm} and the body's multipole moment I_{lm},

$$d_{lm} = \frac{GM}{4\pi R^{l+1}}\left[R\xi'_{lm}(R) + l\xi_{lm}(R)\right] \,, \tag{2.35}$$

$$I_{lm} = -\frac{M}{4\pi}R^l\left[R\xi'_{lm}(R) - (l+1)\xi_{lm}(R)\right] \,. \tag{2.36}$$

It is clear from Eqs. (2.35) and (2.36) that one can write a linear relation between d_{lm} and I_{lm},

$$I_{lm} = \frac{2}{G}k_l R^{2l+1} d_{lm} \,, \tag{2.37}$$

where we define the quantity

Table 2.1: Tidal Love numbers (TLNs) of a fluid star with unperturbed density ρ and pressure p, characterized by a polytropic equation of state, $p = K\rho^\Gamma$. Details on the calculation of the TLNs are given in the main text. The TLNs are present explicitly for the $l = 2$ to $l = 7$ multipoles, and for selected values of the polytropic index given by the relation $\Gamma = 1 + 1/n$. We verify that, as n increases, the TLN becomes smaller. For any fixed polytropic index, we observe that, as the multipolar order grows, the TLN decreases. These results agree with Ref. [2].

l	k_l	$n = 1/2$	$n = 2/3$	$n = 1$	$n = 3/2$	$n = 2$	$n = 3$	$n = 4$
2	k_2	4.49×10^{-1}	3.76×10^{-1}	2.60×10^{-1}	1.43×10^{-1}	7.39×10^{-2}	1.44×10^{-2}	1.20×10^{-3}
3	k_3	2.03×10^{-1}	1.65×10^{-1}	1.06×10^{-1}	5.29×10^{-2}	2.44×10^{-2}	3.70×10^{-3}	2.24×10^{-4}
4	k_4	1.25×10^{-1}	9.85×10^{-2}	6.02×10^{-2}	2.74×10^{-2}	1.15×10^{-2}	1.41×10^{-3}	6.60×10^{-5}
5	k_5	8.76×10^{-2}	6.74×10^{-2}	3.93×10^{-2}	1.66×10^{-2}	6.42×10^{-3}	6.56×10^{-4}	2.49×10^{-5}
6	k_6	6.60×10^{-2}	4.98×10^{-2}	2.78×10^{-2}	1.10×10^{-2}	3.97×10^{-3}	3.47×10^{-4}	1.10×10^{-5}
7	k_7	5.22×10^{-2}	3.87×10^{-2}	2.08×10^{-2}	7.75×10^{-3}	2.63×10^{-3}	2.01×10^{-4}	5.50×10^{-5}

$$k_l := \frac{l + 1 - \eta_l(R)}{2\,[l + \eta_l(R)]}\,, \tag{2.38}$$

and η_l is the Radau's function defined as

$$\eta_l := \frac{r\xi'_{lm}}{\xi_{lm}}\,. \tag{2.39}$$

We now recall Eq. (1.2) and try to find a relation between the symmetric-trace-free decomposition of the potential and the harmonic decomposition in Eq. (2.21). Using Eq. (1.6) to relate x^L with $n^{\langle L\rangle}$ and using the decomposition of an STF tensor in spherical harmonics present in Eq. (A.6), together with the expression for V_{lm} in Eq. (2.28), it is straightforward to obtain a relation between the potential moment d_{lm} and the tidal moment $\mathcal{E}_L$,

$$d_{lm} = -\frac{(l-2)!}{(2l-1)!!}\mathcal{E}_L \mathcal{Y}^{\langle L\rangle}_{lm}\,. \tag{2.40}$$

The same procedure can be done for the mass multipole moments. The conversion between the multipole moments in the two notations is given by

$$I_{lm} = \mathcal{Y}^{\langle L\rangle}_{lm} I_{\langle L\rangle}\,. \tag{2.41}$$

A direct substitution of Eqs. (2.40) and (2.41) in Eq. (2.37) implies that the function k_l corresponds to the tidal Love number defined in Eq. (1.4).

The TLNs of a fluid star are completely determined by Eq. (2.38) as it was shown in Ref. [2]. In order to calculate the Radau's function we make use of Eq. (2.24), where we substitute the functions V_{lm} and U_{lm} by Eq. (2.27) and (2.32), respectively. This way, we obtain the Clairaut's equation,

$$r^2\xi''_{lm} + 6\mathcal{D}(r)(r\xi'_{lm} + \xi_{lm}) - l(l+1)\xi_{lm} = 0\,, \tag{2.42}$$

where the function $\mathcal{D}(r)$ contains the details of the unperturbed configuration and it is defined by

$$\mathcal{D}(r) := \frac{4\pi\rho(r)r^3}{3m(r)}. \tag{2.43}$$

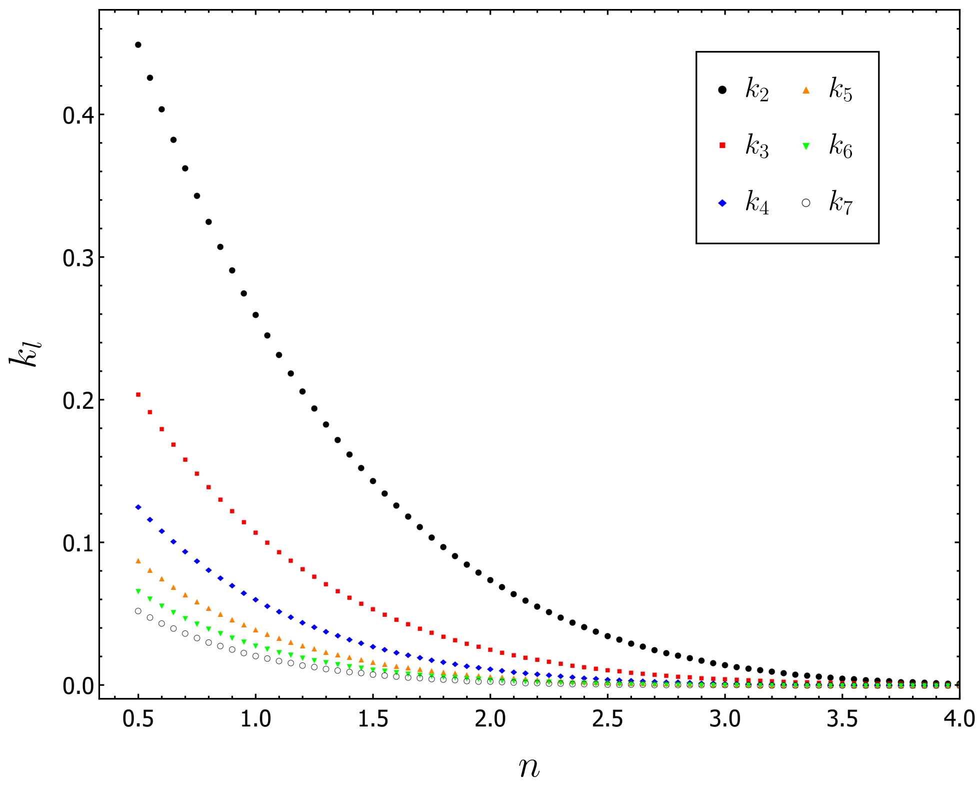

Figure 2.2: Plot of the $l = 2$ and $l = 3$, Newtonian tidal Love numbers (TLNs) for a fluid star with unperturbed density ρ and pressure p, characterized by a polytropical equation of state, $p = K\rho^\Gamma$. The TLNs are evaluated for different polytropic indexes n defined by the relation $\Gamma = 1 + 1/n$. We observe that the TLNs are positive and decrease when the polytropic index grows. For a fixed polytropic index the TLNs becomes smaller with the increase of the multipolar order.

Substituting Eq. (2.39) in Eq. (2.42) we obtain the Radau's equation,

$$r\eta_l' + \eta_l(\eta_l - 1) + 6\mathcal{D}(r)(\eta_l + 1) - l(l + 1) = 0\,, \tag{2.44}$$

which can be solved in order to determine the Radau's function for a given equation of state. We solve numerically this equation for the $l = 2$ to $l = 7$ multipoles and, using Eq. (2.38), we compute the TLNs of a polytropic fluid star. The TLNs for selected polytropic indices and multipolar orders are present in Table 2.1 and in Fig. 2.2.

2.2 Love Numbers of a Homogeneous Fluid Star

The previous calculations required that both the potential and its first derivative were continuous across the star's surface, however, that is no longer the case if we consider an unperturbed fluid star with homogeneous density,

$$\rho(r) = \rho_0\Theta(R - r), \tag{2.45}$$

where $\Theta(R - r)$ is the Heaviside function.

Substituting Eq. (2.45) in Eq. (2.22) we verify that,

$$\rho_{lm} = r\rho_0\xi_{lm}\delta(R - r), \tag{2.46}$$

where we used $d\Theta(x)/dx = \delta(x)$, being $\delta(x)$ the Dirac delta function.

Analyzing Eq. (2.46) one can conclude that the density perturbations of an homogeneous fluid affects only its boundary $r = R$.

Appropriate boundary conditions can be found by studying the potential and its derivative at the surface of the fluid. We have seen that the density perturbation ρ_{lm} vanished in the interior of the body and, therefore, Eq. (2.24) can be integrated in this region. Noticing that the potential inside the body must be regular at $r = 0$, we conclude that the only valid solution is

$$U_{lm}^{\text{int}} = \frac{4\pi G}{2l+1} A_{lm} r^l \,, \tag{2.47}$$

where c_{lm} is an integration constant. Since the external solution continues to be described by Eq. (2.25) we can use the fact that the potential is continuous across the star's surface,

$$U_{lm}(R+\epsilon) - U_{lm}(R-\epsilon) = 0 \,, \tag{2.48}$$

to find the integration constants c_{lm} as functions of the body's multipole moments,

$$A_{lm} = \frac{I_{lm}}{R^{(2l+1)}} \tag{2.49}$$

As previously mentioned, the derivative of the body's potential is discontinuous across the border $r = R$ and depends on the perturbed function ξ_{lm},

$$U'_{lm}(R+\epsilon) - U'_{lm}(R-\epsilon) = -\frac{3GM}{R^2}\xi_{lm}(R). \tag{2.50}$$

Substituting in Eq. (2.50) the expressions for the external and internal potential, described by Eqs. (2.25) and (2.47), and making use of the relation Eq. (2.49) we obtain a final relation between the body's multipole moments and the perturbed function ξ_{lm}.

$$I_{lm} = \frac{3}{4\pi} M R^l \xi_{lm}(R). \tag{2.51}$$

To compute the TLN we need to relate the function ξ_{lm} with d_{lm}. This relation is obtained by substituting in Eq. (2.32), the expressions for U_{lm} and V_{lm} given by Eqs. (2.25) and (2.28),

$$d_{lm} = \frac{2(l-1)GI_{lm}}{3R^{2l+1}}. \tag{2.52}$$

Thus, we obtain again a linear relation between the applied potential moment and the multipole moments. Comparing with Eq. (2.37) we verify that the TLN k_l for a homogeneous fluid star is,

$$k_l = \frac{3}{4(l-1)} \,, \tag{2.53}$$

in accordance to Ref. [2].

3 Tidally Deformed Relativistic Object

In the previous chapter we illustrated the Newtonian theory of TLNs using as an example the tidal deformation of a fluid spherical body. However, motivated by the possibility of measuring TLNs with GW detection, the study of Love numbers of fully relativistic objects became an active research topic in recent years. In order to study TLNs of BHs (in GR and modified gravity) and ECOs [1], it was necessary to develop a relativistic theory for Love numbers which was presented in Chap. 1. As previously mentioned, the TLNs can be precisely defined in terms of the body's multipole moments and the external tidal fields. We generalized this definition by introducing the concepts of EM TLNs and scalar TLNs which arise in some types of BHs studied in this work. These new TLNs are defined in terms of the EM and scalar multipole moments and tidal fields (1.12)–(1.13). The definitions of the TLNs (1.10)–(1.13) require that we are able to identify the multipole and tidal field moments from our problem. The purpose of this chapter is to define the multipole and tidal field moments in relativistic theories of gravity and demonstrate how to extract them from the metric and other relevant physical quantities.

3.1 Multipole Moments of a Relativistic Object

As mentioned in Chapter 1, Thorne developed a method that allowed the definition and extraction of the multipole moments from the spacetime metric [72]. This method requires that the spacetime can be covered by an ACMC system of coordinates. In this system of coordinates the spacetime metric becomes Minkowski at sufficiently large radii and the origin of the spatial coordinates lies at the center of mass of the body such that there is no dipole mass moment. The asymptotic spacetime metric can be written as an expansion in inverse powers of the radial coordinate and the expansion coefficients are defined as the body's multipole moments,

$$g_{00} = -1 + \frac{2M}{r} + \sum_{l\geq 2} \frac{1}{r^{l+1}} \left(\frac{2}{l!} M^{\langle a_1 \ldots a_l \rangle} n^{a_1 \ldots a_l} + (l' < l \,\text{harmonics}) \right), \tag{3.1}$$

$$g_{0\varphi} = -2 \sum_{l\geq 1} \frac{1}{r^{l+1}} \left(\frac{1}{l!} \epsilon^{jka_l} S^{\langle k a_1 \ldots a_{l-1} \rangle} n^{\langle a_1 \ldots a_l \rangle} + (l' < l \,\text{harmonics}) \right). \tag{3.2}$$

where $M^{\langle a_1 \dots a_l \rangle}$ and $S^{\langle a_1 \dots a_l \rangle}$ are the body's multipole mass and current moments respectively. The braces in the superscript denote a STF quantity (cf. Appendix A). We follow the procedure of Ref. [75] and choose the Geroch-Hasen normalization for the multipole moments in the asymptotic metric (3.1)–(3.2).

In this work we are interested in studying spherically symmetric spacetimes. In Appendix A, we demonstrate that, when the spacetime is symmetric with respect to an axis $\vec{k}$, the multipole moments can be decomposed as

$$M^{\langle a_1 \dots a_l \rangle} = (2l-1)!! M_l k^{\langle a_1 \dots a_l \rangle}\,, \qquad S^{\langle a_1 \dots a_l \rangle} = (2l-1)!! S_l k^{\langle a_1 \dots a_l \rangle}\,, \tag{3.3}$$

reducing to two scalar multipole moments M_l and S_l. Using these relations and the SFT properties described in Eqs. (A.4)–(A.5) we obtain

$$M^{\langle a_1 \dots a_l \rangle} n^{\langle a_1 \dots a_l \rangle} = l! M_l P_l (\cos\theta)\,, \tag{3.4}$$

$$\epsilon^{jka_l} S^{\langle k a_1 \dots a_{l-1} \rangle} n^{a_1 \dots a_l} = (l-1)! \epsilon^{ijk} n^i k^k S_l P_l' (\cos\theta)\,. \tag{3.5}$$

Substituting the multipole moment decompositions (3.4)–(3.5) in Eqs. (3.1)–(3.2) we get

$$g_{00} = -1 + \frac{2M}{r} + \sum_{l \geq 2} \frac{1}{r^{l+1}} \left(M_l P_l (\cos\theta) + (l' < l \,\text{harmonics}) \right), \tag{3.6}$$

$$g_{0\varphi} = -2 \sin^2\theta \sum_{l \geq 1} \frac{1}{r^l} \frac{S_l}{l} \left(P_l' (\cos\theta) + (l' < l \,\text{harmonics}) \right). \tag{3.7}$$

In order to obtain a final expression we rewrite the asymptotic spacetime metric (3.6)–(3.7) as

$$g_{00} = -1 + \frac{2M}{r} + \sum_{l \geq 2} \left(\frac{2}{r^{l+1}} \left[\sqrt{\frac{4\pi}{2l+1}} M_l Y^{l0} + (l' < l \,\text{pole}) \right] \right), \tag{3.8}$$

$$g_{0\varphi} = \frac{2J}{r} \sin^2\theta + \sum_{l \geq 2} \left(\frac{2}{r^l} \left[\sqrt{\frac{4\pi}{2l+1}} \frac{S_l}{l} S_\varphi^{l0} + (l' < l \,\text{pole}) \right] \right), \tag{3.9}$$

where we have used the conversion between spherical harmonics $Y^{lm}(\theta, \varphi)$ and the associated Legendre polynomials $P_{lm}(\cos\theta)$,

$$Y^{lm}(\theta, \varphi) = \sqrt{\frac{(2l+1)}{4\pi} \frac{(l-m)!}{(l+m)!}} P_{lm}(\cos\theta) e^{im\varphi}\,, \tag{3.10}$$

with $m = 0$.

3.2 Relativistic Tidal Field Moments

Definitions (3.1)–(3.2) allow us to compute and extract the multipole moments of a relativistic object, however, they do not account for the effects of the external universe that are essential in the study of tidal deformations. These effects were studied in Refs. [74, 77] and we shall follow their notation.

We consider that the external universe is characterized by a tidal environment which can be decomposed into two STF tensors: a polar tidal field $\mathcal{E}_L$ and in an axial tidal field $\mathcal{B}_L$. These tensors can be properly defined in terms of the Weyl tensor of the external universe,

$$\mathcal{E}_L \equiv [(l-2)!]^{-1} \left\langle C_{0a_1 0a_2;a_3\ldots a_l}\right\rangle \,, \qquad \mathcal{B}_L \equiv \left[\frac{2}{3}(l+1)(l-2)!\right]^{-1} \left\langle \epsilon_{a_1 bc} C^{bc}_{a_2 0;a_3\ldots a_l}\right\rangle \,, \tag{3.11}$$

where C_{abcd} is the Weyl tensor, a semicolon denotes a covariant derivative, ϵ_{abc} is the permutation symbol, the angular brackets denote symmetrization of the indices a_i and all traces are removed.

Following Ref. [17] we can use the tidal moments $\mathcal{E}_L$ and $\mathcal{B}_L$ to define the tidal potentials. In order to calculate the TLNs we will use the polar- and axial-type tidal potentials defined as

$$\mathcal{E}^{(l)} := \mathcal{E}_L n^L \,, \qquad \mathcal{B}^{(l)} := \mathcal{B}_L n^L \,. \tag{3.12}$$

These tidal moments can be decomposed in a spherical harmonic base with amplitudes $\mathcal{E}^{(l)}_m$ and $\mathcal{B}^{(l)}_m$.

$$\mathcal{E}^{(l)} = \sum_m \mathcal{E}^{(l)}_m Y^{lm} \,, \qquad \mathcal{B}^{(l)} = \sum_m \mathcal{B}^{(l)}_m Y^{lm} \,. \tag{3.13}$$

In the axysymmetric case we can take $m = 0$ and amplitudes of the tidal fields can be extracted from the tt- and $t\varphi$-components of the asymptotic spacetime metric [17, 22],

$$g_{00} \to -\frac{2}{l\,(l-1)}\mathcal{E}^{(l)}_0 Y^{l0} r^l \,, \qquad g_{0\varphi} \to \frac{2}{3l\,(l-1)}\mathcal{B}^{(l)}_0 S^{l0}_\varphi r^{l+1} \,. \tag{3.14}$$

3.3 Asymptotic Spacetime of a Deformed Body

Combining the body expansion (3.8)–(3.9) with the external spacetime contribution (3.14) we are able to write the multipole expansion of the total spacetime metric in the exterior of a tidally deformed body. This metric is valid in the asymptotic limit and provides a method to extract the multipole moments and tidal fields of any spacetime metric written as,

$$g_{00} = -1 + \frac{2M}{r} + \sum_{l\geq 2}\left(\frac{2}{r^{l+1}}\left[\sqrt{\frac{4\pi}{2l+1}} M_l Y^{l0} + (l' < l\,\text{pole})\right] - \frac{2}{l(l-1)} r^l \left[\mathcal{E}_l Y^{l0} + (l' < l\,\text{pole})\right]\right) \,, \tag{3.15}$$

$$g_{0\varphi} = \frac{2J}{r}\sin^2\theta + \sum_{l\geq 2}\left(\frac{2}{r^l}\left[\sqrt{\frac{4\pi}{2l+1}}\frac{S_l}{l} S^{l0}_\varphi + (l' < l\,\text{pole})\right] + \frac{2r^{l+1}}{3l\,(l-1)}\left[\mathcal{B}_l S^{l0}_\varphi + (l' < l\,\text{pole})\right]\right) \,, \tag{3.16}$$

This analysis must be extended to account for characteristic types of BHs that will naturally emerge in this work, specifically, BHs which are solutions of Einstein-Maxwell theory of gravity or modified theories of gravity like scalar-tensor and quadratic theories of gravity. In these theories, BHs are not only described by the spacetime metric, but also by an EM potential A_μ and a scalar field Φ, respectively. We will assume that our spacetime will suffer from EM and scalar perturbations caused by external EM and scalar tidal fields, respectively, in addition to the spacetime metric perturbations. These new external tidal fields will induce changes in the body's multipolar structure in such a way that its EM and scalar multipole moments will be modified. Thus, we are able to define and compute new types of TLNs as introduced in Eqs. (1.12)–(1.13).

We choose the normalization of the multipole and external tidal fields moments such that the asymptotic behavior of the EM potential and scalar field are

$$A_t = -\frac{Q}{r} + \sum_{l\geq 1}\left(\frac{2}{r^{l+1}}\left[\sqrt{\frac{4\pi}{2l+1}}Q_l Y^{l0} + (l' < l\,\mathrm{pole})\right] - \frac{2}{l(l-1)}r^l\left[\mathfrak{E}_l Y^{l0} + (l' < l\,\mathrm{pole})\right]\right), \quad (3.17)$$

$$A_\varphi = \sum_{l\geq 1}\left(\frac{2}{r^l}\left[\sqrt{\frac{4\pi}{2l+1}}\frac{J_l}{l}S^{l0}_\varphi + (l' < l\,\mathrm{pole})\right] + \frac{2r^{l+1}}{3l\,(l-1)}\left[\mathfrak{B}_l S^{l0}_\varphi + (l' < l\,\mathrm{pole})\right]\right), \quad (3.18)$$

$$\Phi = \Phi_0 + \sum_{l\geq 1}\left(\frac{2}{r^{l+1}}\left[\sqrt{\frac{4\pi}{2l+1}}\Phi_l Y^{l0} + (l' < l\,\mathrm{pole})\right] - \frac{2}{l(l-1)}r^l\left[\mathcal{E}^{\mathrm{S}}_l Y^{l0} + (l' < l\,\mathrm{pole})\right]\right). \quad (3.19)$$

An appropriate comparison between the solution of the field equations and the expansions (3.15)–(3.19) will provide a method to extract the relevant multipole moments and compute the TLNs.

3.4 Linear Spacetime Perturbations

In order to find the TLNs we need to calculate the expressions for the induced mass and current multipole moments as a function of the external tidal field. For this purpose we will perturb the spacetime metric as

$$g_{\mu\nu} = \overset{(0)}{g}_{\mu\nu} + h_{\mu\nu}\,, \quad (3.20)$$

where $\overset{(0)}{g}_{\mu\nu}$ is the background spacetime metric and $h_{\mu\nu}$ is a small perturbation to the spacetime metric that we can use to employ a linear perturbation theory approach. We will consider stationary spherically symmetric background metrics which are described by

$$\overset{(0)}{g}_{\mu\nu} = -e^\Gamma dt^2 + e^\Lambda dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\,. \quad (3.21)$$

We decompose $h_{\mu\nu}$ in spherical harmonics allowing us to separate the perturbation into even and odd parts,

$$h_{\mu\nu} = h^{\mathrm{even}}_{\mu\nu} + h^{\mathrm{odd}}_{\mu\nu}\,. \quad (3.22)$$

Choosing the Regge-Wheeler gauge [84] the most general form for $h_{\mu\nu}$ is

$$h^{\mathrm{even}}_{\mu\nu} = \begin{pmatrix} e^\Gamma H^{lm}_0(t,r)Y^{lm} & H^{lm}_1(t,r)Y^{lm} & 0 & 0 \\ H^{lm}_1(t,r)Y^{lm} & e^\Lambda H^{lm}_2(t,r)Y^{lm} & 0 & 0 \\ 0 & 0 & r^2 K^{lm}(t,r)Y^{lm} & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta K^{lm}(t,r)Y^{lm} \end{pmatrix}, \quad (3.23)$$

$$h^{\rm odd}_{\mu\nu} = \begin{pmatrix} 0 & 0 & h_0^{lm}(t,r)S_\theta^{lm} & h_0^{lm}(t,r)S_\varphi^{lm} \\ 0 & 0 & h_1^{lm}(t,r)S_\theta^{lm} & h_1^{lm}(t,r)S_\varphi^{lm} \\ h_0^{lm}(t,r)S_\theta^{lm} & h_1^{lm}(t,r)S_\theta^{lm} & 0 & 0 \\ h_0^{lm}(t,r)S_\varphi^{lm} & h_1^{lm}(t,r)S_\varphi^{lm} & 0 & 0 \end{pmatrix}, \tag{3.24}$$

with $\left(S_\theta^{lm}, S_\varphi^{lm}\right) \equiv \left(-Y^{lm}_{,\varphi}/\sin\theta,\, \sin\theta\, Y^{lm}_{,\theta}\right)$.

The EM potential and scalar field will also be perturbed as

$$A_\mu = A_\mu^{^{(0)}} + \delta A_\mu\,, \tag{3.25}$$

$$\Phi = \Phi^{^{(0)}} + \delta\Phi\,, \tag{3.26}$$

where $A^{^{(0)}}$ and $\Phi^{^{(0)}}$ are background quantities while δA_μ and $\delta\Phi$ are small perturbations.

We can separate the EM potential in even and odd parts as we have done for the metric perturbation,

$$\delta A_\mu = \delta A_\mu^{\rm even} + \delta A_\mu^{\rm odd}, \tag{3.27}$$

where we write the odd and even parity terms as in references [85, 86],

$$\delta A_\mu^{\rm even} = \left(\frac{u_1^{lm}(t,r)}{r} Y^{lm},\, \frac{u_2^{lm}(t,r)\, e^{-\Gamma}}{r} Y^{lm},\, \frac{u_3^{lm}(t,r)}{l(l+1)} Y_b^{lm}\right)\,, \tag{3.28}$$

$$\delta A_\mu^{\rm odd} = \left(0, 0, u_4^{lm}(t,r) S_b^{lm}\right)\,, \tag{3.29}$$

with $Y_b^{lm} \equiv \left(Y^{lm}_{,\theta}, Y^{lm}_{,\varphi}\right)$. Henceforward we shall drop the (lm) superscripts in the perturbation functions.

Regarding the scalar field perturbation, we write it in the usual spherical harmonic decomposition,

$$\delta\Phi = \delta\phi(t,r) Y^{lm}\,. \tag{3.30}$$

Solving the appropriate field equations for the theory in study will give us expressions for the metric functions in (3.23)–(3.24), for the EM functions in (3.28)–(3.29) and for the scalar fields.

We use a linear perturbation theory approach by considering that all perturbed quantities are sufficiently small, $h_{\mu\nu} \ll 1$, $\delta A_\mu \ll 1$ and $\delta\Phi \ll 1$, and for that reason, we shall neglect second and higher orders terms in perturbed quantities.

Love Numbers in General Relativity

4

4.1 Introduction

As previously mentioned, this work is concerned with Love numbers of fully relativistic objects and, for that reason, their relativistic formulation presented in Chapter 3 will be applied here to objects described in Einstein's theory of GR. This theory is characterized by the action (see e.g. Ref. [73])

$$S_{\rm GR} = \frac{c^4}{16\pi G}\int d^4x\sqrt{-g}\,R + S_{\rm matt}\,, \tag{4.1}$$

where c is the speed of light, G is the gravitational constant, R is the Ricci scalar and $S_{\rm matt}$ is the action for any matter fields that appear in the theory. For the rest of this thesis we consider geometrized units $G = c = 1$.

Varying the action (4.1) with respect to the spacetime metric leads to Einstein's equations,

$$G_{\mu\nu} = 8\pi T_{\mu\nu}\,, \tag{4.2}$$

where $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ is the stress-energy tensor,

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm matt}}{\delta g^{\mu\nu}}\,. \tag{4.3}$$

If the matter Lagrangian $L_{\rm matt}$ is composed by some matter field ψ (scalar, vector or tensor), such that $L_{\rm matt} = L_{\rm matt}(g^{\mu\nu}, \psi)$, governing equations for the field can be obtained by varying the action with respect to it,

$$\frac{\delta S_{\rm matt}}{\delta\psi} = 0\,. \tag{4.4}$$

Considering $S_{\rm matt}$ to be composed only by EM radiation, such that it can be written as

$$S_{\rm matt} = -\frac{1}{16\pi}\int d^4x\sqrt{-g}F_{\mu\nu}F^{\mu\nu}\,, \tag{4.5}$$

we can write the resulting stress-energy tensor as

$$T_{\alpha\beta} = \frac{1}{4\pi}\left(g^{\mu\gamma}F_{\alpha\mu}F_{\beta_\gamma} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}g_{\alpha\beta}\right)\,, \tag{4.6}$$

where $F_{\mu\nu}$ is the Maxwell tensor related with the 4-potential A_μ by

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}\,. \tag{4.7}$$

In this scenario, Eqs. (4.4) will be obtained by varying action (4.1) with respect to the 4-potential A^μ. From here, we get Maxwell's equations in curved spacetime,

$$\nabla_\nu F^{\mu\nu} = 0\,. \tag{4.8}$$

Eqs. (4.2) and (4.8) compose the set of field equations that we need to solve in order to compute the TLNs in the framework of GR.

This chapter is divided into two parts. In the first one, correspondent to Sec. 4.2, we review the effects of external perturbations on the spacetime around a non-rotating, uncharged BH [16, 17]. In the second part, which corresponds to Sec. 4.3, we generalize previous works in the literature and extend the analysis to perturbations of a non-rotating charged BH.

We will study two different types of perturbations: gravitational and EM. The former will be directly related with the gravitational TLNs while the latter will lead to the introduction of a new class of Love numbers, the EM TLNs, analogues of the gravitational TLNs for the EM field and defined by expression (1.12).

For an uncharged BH these perturbations are decoupled and we can solve Einstein's equations to calculate the gravitational TLNs and Maxwell's equations to calculate the EM TLNs. However, in a charged BH, the gravitational and EM perturbations are coupled and we must solve the coupled system of Einstein and Maxwell's equations to determine both TLNs.

4.2 Love Numbers of a Non-Rotating, Uncharged Black Hole

In this section we will focus on the calculation of the TLNs of a non-rotating, uncharged BH. Several works present in the literature concluded that the TLNs of this BH are zero [16, 17] and we shall confirm these results here.

We consider spherically symmetric BHs described by the line element (3.21). Uncharged BHs have a zero vector potential and therefore zero background Maxwell and stress-energy tensors,

$$\overset{(0)}{A_\mu} = 0\,, \qquad \overset{(0)}{F_{\mu\nu}} = 0\,, \qquad \overset{(0)}{T_{\mu\nu}} = 0\,, \tag{4.9}$$

where $\overset{(0)}{X}$ denotes the background value of the quantity X.

The solution of field equations (4.2) and (4.8) describing this type of BH is the Schwarzschild metric [87],

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,, \tag{4.10}$$

where the quantity M can be identified as the BH's mass.

However, spacetime metric (4.10) only represents the unperturbed (background) state of the BH and, to fully describe a BH immersed on an external tidal environment, we need to apply perturbations to the spacetime.

The presence of a tidal environment induces changes in the geometry of the spacetime and, for this reason, we write the perturbed metric $g_{\mu\nu}$ according to Eq. (3.20) where the unperturbed metric $\overset{(0)}{g}_{\mu\nu}$ is given by (4.10) and the perturbations are described by Eqs. (3.22)–(3.24). The EM field is perturbed according to Eq. (3.25), where the perturbations are described by Eqs. (3.27)–(3.29). By direct comparison of Eqs. (3.21) and (4.10), the metric functions used in these perturbations are $e^{\Gamma} = e^{-\Lambda} = 1 - 2M/r$.

The stress-energy tensor (4.6) is quadratic in the EM potential, which is a purely perturbed quantity for an uncharged BH. Thus, under the linear perturbation theory assumption, we conclude that stress-energy tensor is zero,

$$T_{\mu\nu} = 0\,. \tag{4.11}$$

For the same reason, the covariant derivative in Maxwell's equations (4.8) will contain only unperturbed terms of the spacetime metric $g_{\mu\nu}$. These facts imply that the metric and EM perturbations are decoupled and that we can solve separately Einstein and Maxwell's equations to obtain expressions for the metric and EM perturbations respectively.

4.2.1 Stationary Perturbations of Uncharged Black Holes

We will consider that the time variations in our tidal system are sufficiently small such that we can work under a regime of non-dynamical tides and assume stationary perturbations (i.e. the perturbed functions are time independent). Using this assumption, we check that the tr-component of Einstein's equations yields $H_1 = 0$ and, similarly for the axial sector, we obtain that $h_1 = 0$. The components of Einstein's equations (4.2) are not all linear independent and can be reduced to two second-order differential equations, one governing the polar perturbations and another governing the axial. From the $\theta\theta$-component proportional to Y_θ we obtain $H_2 = H_0$. From the tt-, rr- and $\theta\theta$-components we obtain expressions for K and its first two derivatives. Substituting these in the $r\theta$-component of (4.2) we obtain a second-order differential equations for H_0,

$$r^2(r-2M)^2 H_0'' + 2r(r-2M)(r-M)H_0' - \left(l(l+1)r^2 - 2l(l+1)Mr + 4M^2\right) H_0 = 0. \tag{4.12}$$

Regarding the axial sector of the theory, the $t\varphi$-component of Einstein's equations automatically yield a second-order differential equation for h_0,

$$h_0'' + \frac{h_0(4M - l(l+1)r)}{r^2(r-2M)} = 0. \tag{4.13}$$

We can now solve independently Maxwell's equations (4.8) using the stationary perturbations assumption such that the coefficients in EM perturbation described by Eqs. (3.28)–(3.29) are independent of time. The t-component of Maxwell's equation yields a second-order differential equation for the polar function u_1,

$$u_1'' - \frac{l(l+1)u_1}{r(r-2M)} = 0\,, \tag{4.14}$$

and the φ-component of (4.8) provides the differential equation for the axial function u_4,

$$u_4'' + \frac{2Mu_4' - l(l+1)u_4}{r(r-2M)} = 0\,. \tag{4.15}$$

We must now solve Eqs. (4.12)–(4.13) to obtain expressions for the perturbing coefficients and identify the gravitational polar- and axial-type TLNs. Analogously, solving Eqs. (4.12)–(4.13) will allow us to identify the new EM TLNs.

4.2.2 Gravitational and Electromagnetic Love Numbers

Fortunately, Eqs. (4.12)–(4.15) can be solved analytically, however, each of these equations will yield two integration constants that we need to fix with appropriate boundary conditions. When considering a material body (e.g. a NS) there is another set of equations describing its internal structure (e.g a polytropic equation). In those cases, the boundary conditions are obtained by matching the inner and outer solutions at the body's surface [3]. Our case is considerably simpler since BHs are not material bodies and therefore, as a boundary condition, we need to only to impose that the perturbations are regular across the horizon, $r_h = 2M$. For $l = 2$ the solutions regular at the horizon are

$$\begin{aligned}
H_0 &= -r^2\mathcal{E}_2 + 2Mr\mathcal{E}_2 \equiv -\mathcal{E}_2 r^2 f\,,\\
h_0 &= \frac{r^3\mathcal{B}_2}{3} - \frac{2}{3}Mr^2\mathcal{B}_2 \equiv \frac{1}{3}\mathcal{B}_2 r^3 f\,,\\
u_1 &= -r^3\mathfrak{E}_2 + 3Mr^2\mathfrak{E}_2 - 2M^2 r\mathfrak{E}_2\,,\\
u_4 &= -2r^3\mathfrak{B}_2 + 3Mr^2\mathfrak{B}_2\,,
\end{aligned} \tag{4.16}$$

where $f = 1-2M/r$ and the integration constants were normalized such that the leading order, asymptotically divergent coefficients of Eqs. (4.16) are proportional to the external fields described in Eqs. (3.15)–(3.18). The solutions in (4.16) do not contain asymptotically decaying terms and we can immediately conclude that the presence of the tidal environment does not induce any multipolar response on the body. Thus, by comparing with expansions (3.15)–(3.18), the four body's multipole moments M_2, S_2, Q_2 and J_2 are zero, and, according to Eqs. (1.10)–(1.12), so are their respective TLNs. Since this procedure can be generalized to higher values of l we conclude that,

$$k_l^E = 0\,, \qquad k_l^B = 0\,, \qquad \mathcal{K}_l^E = 0\,, \qquad \mathcal{K}_l^B = 0\,. \tag{4.17}$$

Throughout this thesis we will find differential equations that cannot be solved analytically. Here, we present another method to study the solutions of Eqs. (4.12)–(4.15) without the need to solve them explicitly. In more complex examples we will use this method to calculate the Love numbers. The solutions of Eqs. (4.12)–(4.15) can be written schematically as

$$H_0 = H_0^{\rm div} + H_0^{\rm dec}\,, \qquad h_0 = h_0^{\rm div} + h_0^{\rm dec}\,, \qquad u_1 = u_1^{\rm div} + u_1^{\rm dec}\,, \qquad u_4 = u_4^{\rm div} + u_4^{\rm dec}\,, \tag{4.18}$$

where the first term on the right-hand side of each expression is regular at the horizon r_h and divergent at large distances, $r \gg M$, whereas the second term of the right-hand side decay at large distances but diverge at the horizon. More specifically, when $r \to \infty$,

$$\begin{aligned} H_0^{\rm div} &\sim -\frac{2}{l(l-1)} r^l \sum_{i=0}^{\infty} \frac{a_H^{(i)}}{r^i}\,, \\ h_0^{\rm div} &\sim \frac{2}{3l(l-1)} r^{l+1} \sum_{i=0}^{\infty} \frac{a_h^{(i)}}{r^i}\,, \\ u_1^{\rm div} &\sim -\frac{2}{l(l-1)} r^{l+1} \sum_{i=0}^{\infty} \frac{a_{u1}^{(i)}}{r^i} \\ u_4^{\rm div} &\sim -\frac{2(l+1)}{3(l-1)} r^{l+1} \sum_{i=0}^{\infty} \frac{a_{u4}^{(i)}}{r^i}\,, \end{aligned} \tag{4.19}$$

$$\begin{aligned} H_0^{\rm dec} &\sim \sqrt{\frac{16\pi}{2l+1}} r^{-(l+1)} \sum_{i=0}^{\infty} \frac{b_H^{(i)}}{r^i}\,, \\ h_0^{\rm dec} &\sim \frac{1}{l}\sqrt{\frac{16\pi}{2l+1}} r^{-l} \sum_{i=0}^{\infty} \frac{b_h^{(i)}}{r^i}\,, \\ u_1^{\rm dec} &\sim \sqrt{\frac{16\pi}{2l+1}} r^{-l} \sum_{i=0}^{\infty} \frac{a_{u4}^{(i)}}{r^i}\,, \\ u_4^{\rm dec} &\sim -(l+1)\sqrt{\frac{16\pi}{2l+1}} r^{-l} \sum_{i=0}^{\infty} \frac{a_{u4}^{(i)}}{r^i}\,, \end{aligned} \tag{4.20}$$

where the prefactors are included for future convenience. Note that all subleading coefficients are related to the dominant ones and can be computed by solving the asymptotic expansion of the differential equations iteratively.

It is interest to note that solutions (4.18) can be found in closed form and, requiring regularity at the horizon, we can look only at the divergent solutions. Focusing on $l = 2$ perturbations, the diverging series can be written as

$$H_0^{\rm div} = -a_H^{(0)} r^2 + 2a_H^{(0)} M r \equiv -a_H^{(0)} r^2 f\,, \tag{4.21}$$

$$h_0^{\rm div} = \frac{1}{3} a_h^{(0)} r^3 - \frac{2}{3} a_h^{(0)} M r^2 \equiv \frac{1}{3} a_h^{(0)} r^3 f\,, \tag{4.22}$$

$$u_1^{\rm div} = -a_{u1}^{(0)} r^3 + 3M a_{u1}^{(0)} r^2 - 2M^2 a_{u1}^{(0)} r\,, \tag{4.23}$$

$$u_4^{\rm div} = -2a_{u4}^{(0)} r^3 + 3a_{u4}^{(0)} M r^2\,, \tag{4.24}$$

and by comparison with Eqs. (3.15)–(3.18) we identify $a_H^{(0)} = \mathcal{E}_2$, $a_h^{(0)} = \mathcal{B}_2$, $a_{u1}^{(0)} = \mathfrak{E}_2$ and $a_{u4}^{(0)} = \mathfrak{B}_2$. Since the subdominant terms in Eqs. (4.21)–(4.24) do not mix with the respective decaying series (4.20), we can clearly identify $b_H^{(0)} = M_l$, $b_h^{(0)} = S_l$, $b_{u1}^{(0)} = Q_l$ and $b_{u4}^{(0)} = J_l$. Therefore, we conclude that there are no induced multipole moments and, by Eqs. (1.10)–(1.12), the gravitational and EM TLNs of a uncharged BH are zero,

$$k_2^E = 0\,, \qquad k_2^B = 0\,, \qquad \mathcal{K}_2^E = 0\,, \qquad \mathcal{K}_2^B = 0\,. \tag{4.25}$$

Although the calculations were presented explicitly for $l = 2$ perturbations, the same results can be obtained for the Love number for any multipole order with $l > 2$, leading to the conclusion that the gravitational and EM TLNs are zero for a non-rotating, uncharged BH,

$$k_l^E = 0\,, \qquad k_l^B = 0\,, \qquad \mathcal{K}_l^{E)} = 0\,, \qquad \mathcal{K}_l^B = 0\,. \tag{4.26}$$

4.3 Love Numbers of a Non-Rotating, Charged Black Hole

We now generalize the calculations of the previous section to the case non-rotating, charged BHs with the purpose of calculating their TLNs. The unperturbed state of this BHs is described by the Reissner-Nordström line element [88, 89],

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right), \tag{4.27}$$

where M and Q can be identified as the BH's mass and charge, respectively.

This spacetime is also described by a non-vanishing EM potential,

$$A_\mu^{(0)} = (-Q/r, 0, 0, 0)\,, \tag{4.28}$$

which, by means of Eqs. (4.6)–(4.7), leads to non-vanishing background Maxwell and stress-energy tensors. These zeroth order terms in the Maxwell and stress-energy tensors are the main difference from the previous case and, because of them, a coupling between the EM and metric perturbations will appear in Einstein and Maxwell's equations (4.2) and (4.8). Similarly to the previous case, we will introduce the effects of the tidal environment by perturbing the spacetime metric and vector potential according to Eqs. (3.20) and (3.25). The expressions for the perturbation functions of the metric and EM fields are governed by Eqs. (3.23)–(3.24) and (3.28)–(3.29) with $e^\Gamma = e^{-\Lambda} = 1 - 2M/r + Q^2/r^2$.

As mentioned above, the presence of background EM fields will introduce a coupling between gravitational and EM perturbations. This facts implies that for a charged BH we can no longer treat the gravitational perturbations separately from the EM perturbations as in Sec. 4.2, but instead we must solve the coupled system of differential equations.

4.3.1 Stationary Perturbations of Charged Black Holes

Similarly to Sec. 4.2 we analyze stationary perturbations, in such a way that all the perturbation coefficients are time independent, and reduce Eqs. (4.2) and (4.8) to a simpler system of equations. Since the field equations preserve the parity of the system we can treat separately the polar and axial functions.

The polar sector of Eqs. (4.2) and (4.8) can be reduced to two coupled differential equations for H_0 and u_1. From the tr-component of Eq. (4.2) we can find $H_1 = 0$ and H_2 can also be found from the part of the $\theta\theta$-component proportional to Y_θ. The tt-, rr-, and $\theta\theta$- components can be used to substitute K and its derivatives in the remaining equations. With the appropriate substitutions the $r\theta$-component of Einstein's equations and t-component of Maxwell's equations provide a system of two coupled differential equations for the metric perturbation H_0 and for the EM perturbation u_1.

Similarly, the axial sector of Eqs. (4.2) and (4.8) is already in the form of two coupled equations for h_0 and u_4. Furthermore, from the $r\phi$-component of Eq. (4.2) we find that $h_1 = 0$.

The reduced systems of equations can be written schematically as,

$$\mathcal{D}_1^{(2)} H_0 + \frac{4Q}{r^3 - 2Mr^2 + Q^2 r}\mathcal{D}_1^{(1)} u_1 = 0\,, \tag{4.29}$$

$$\mathcal{D}_2^{(2)} u_1 + \frac{Q}{r}\mathcal{D}_2^{(1)} H_0 = 0\,, \tag{4.30}$$

$$\mathcal{D}_3^{(2)} h_0 - \frac{4Q}{l(l+1)r^2} u_4' = 0\,, \tag{4.31}$$

$$\mathcal{D}_4^{(2)} u_4 - \frac{l(l+1)Q}{r(r-2M)+Q^2}\mathcal{D}_4^{(1)} h_0 = 0\,, \tag{4.32}$$

where we defined the operators,

$$\begin{aligned}\mathcal{D}_1^{(2)} :=& \frac{d^2}{dr^2} - \frac{2(M-r)}{r(r-2M)+Q^2}\frac{d}{dr} - \\ & - \frac{Q^2 r\left(\left(l^2+l-2\right)r - 4M\right) + r^2\left(-2l(l+1)Mr + l(l+1)r^2 + 4M^2\right) + 2Q^4}{r^2\left(r(r-2M)+Q^2\right)^2}\,,\end{aligned} \tag{4.33}$$

$$\mathcal{D}_1^{(1)} := \frac{d}{dr} + \frac{\left(Q^2 - r^2\right)}{r\left(r(r-2M)+Q^2\right)}\,, \tag{4.34}$$

$$\mathcal{D}_2^{(2)} := \frac{d^2}{dr^2} + \frac{4Q^2 - l(l+1)r^2}{r^2\left(r(r-2M)+Q^2\right)}\frac{d}{dr}\,, \tag{4.35}$$

$$\mathcal{D}_2^{(1)} := \frac{d}{dr} + \frac{2\left(Mr - Q^2\right)}{r\left(r(r-2M)+Q^2\right)}\,, \tag{4.36}$$

$$\mathcal{D}_3^{(2)} := \frac{d^2}{dr^2} - \frac{r(l(l+1)r - 4M) + 2Q^2}{r^2\left(r(r-2M)+Q^2\right)}\,, \tag{4.37}$$

$$\mathcal{D}_4^{(2)} := \frac{d^2}{dr^2} + \frac{2\left(Mr - Q^2\right)}{r\left(r(r-2M)+Q^2\right)}\frac{d}{dr} - \frac{l(l+1)}{r(r-2M)+Q^2}\,, \tag{4.38}$$

$$\mathcal{D}_4^{(1)} := \frac{d}{dr} - \frac{2}{r}\,. \tag{4.39}$$

In order to compute the Love numbers of a charged BH we do not need to find an analytic solution of Eqs. (4.29)–(4.32). We can follow a similar procedure to the one taken in the previous section and use expansion techniques to extract and identify the Love numbers.

4.3.2 Gravitational and Electromagnetic Love Numbers

One can now solve Eqs. (4.29)–(4.32) to obtain expressions for the perturbed functions. These solutions can be written schematically as Eqs. (4.18). As mentioned the first term on the right-hand side of each expression is regular at the horizon $r_h = M + \sqrt{M^2 - Q^2}$ and divergent at large distances, $r \gg M$, whereas the second term of the right-hand side decay at large distances but diverge at the horizon. More specifically, when $r \to \infty$, these expressions are described by Eqs. (4.19)–(4.20). Note that all subleading coefficients are related to the dominant ones and can be computed by solving the asymptotic expansion of the differential equations iteratively.

Remarkably, these solutions can be found in a closed form. Imposing regularity at the event-horizon we can look only to the diverging solutions. Focusing on $l=2$ perturbations we can write,

$$H_0^{\rm div} = -a_H^{(0)} r^2 + r(2a_H^{(0)}M - 2a_{u1}^{(0)}Q) - a_H^{(0)}Q^2 + 4a_{u1}^{(0)}MQ - \frac{2a_{u1}^{(0)}Q^3}{r}\,, \tag{4.40}$$

$$u_1^{\rm div} = -a_{u1}^{(0)} r^3 + r^2\left(3a_{u1}^{(0)}M - \frac{1}{2}a_H^{(0)}Q\right) + r\left(a_H^{(0)}MQ - 2a_{u1}^{(0)}M^2 - 2a_{u1}^{(0)}Q^2\right) - \tag{4.41}$$
$$- \frac{1}{2}a_H^{(0)}Q^3 + 3a_{u1}^{(0)}MQ^2 - \frac{a_{u1}^{(0)}Q^4}{r}\,,$$

$$h_0^{\rm div} = \frac{1}{3}a_h^{(0)} r^3 + r^2\left(a_{u4}^{(0)}Q - \frac{2}{3}a_h^{(0)}M\right) + r\left(\frac{1}{3}a_h^{(0)}Q^2 - 2a_{u4}^{(0)}MQ\right) + \frac{2}{3}a_{u4}^{(0)}Q^3 + \frac{2a_{u4}^{(0)}MQ^3}{3r} - \frac{a_{u4}^{(0)}Q^5}{3r^2}\,, \tag{4.42}$$

$$u_4^{\rm div} = -2a_{u4}^{(0)} r^3 + r^2\left(3a_{u4}^{(0)}M - \frac{1}{2}a_h^{(0)}Q\right) + \frac{1}{2}a_h^{(0)}Q^3 - 3a_{u4}^{(0)}MQ^2 + \frac{2a_{u4}^{(0)}Q^4}{r}\,. \tag{4.43}$$

By direct comparison with Eqs. (3.15)–(3.18), we can identify $a_H^{(0)} = \mathcal{E}_2$, $a_h^{(0)} = \mathcal{B}_2$, $a_{u1}^{(0)} = \mathfrak{E}_2$ and $a_{u4}^{(0)} = \mathfrak{B}_2$. Unfortunately, these general solutions yield subdominant terms that decay at large distances. This mixing between asymptotically diverging and asymptotically decaying terms is problematic to analysis of Love number for several reasons. Mainly, this "mixing problem" poses the question whether these terms correspond to some subleading tidal field contribution or to the body response still needs to be analyzed. Furthermore, if these terms correspond to the body's response, it could indicate that the body is developing a lower order multipole than the one of the applied tidal field. In this thesis we did not develop this analysis and postponed it for future work. Fortunately, a physically relevant case that makes the analysis much more treatable is a system with vanishing EM tidal fields. The tidal deformation of a charged BH by an external uncharged compact object is an example of a physical situation that satisfies our assumption.

With the assumption that all EM fields are zero,

$$\mathfrak{E}_l = 0\,, \qquad \mathfrak{B}_l = 0\,, \tag{4.44}$$

the solutions (4.40)–(4.43) take the form:

$$H_0^{\rm div} = -\mathcal{E}_2 r^2 + 2\mathcal{E}_2 Mr - \mathcal{E}_2 Q^2 \equiv -\mathcal{E}_2 r^2 f\,, \tag{4.45}$$

$$u_1^{\rm div} = -\frac{1}{2}\mathcal{E}_2 Q r^2 + r\mathcal{E}_2 MQ - \frac{1}{2}\mathcal{E}_2 Q^3 \equiv -\frac{1}{2}\mathcal{E}_2 Q r^2 f\,, \tag{4.46}$$

$$h_0^{\rm div} = \frac{1}{3}\mathcal{B}_2 r^3 - \frac{2}{3}\mathcal{B}_2 M r^2 + \frac{1}{3}\mathcal{B}_2 Q^2 r \equiv \frac{1}{3}\mathcal{B}_2 r^3 f\,, \tag{4.47}$$

$$u_4^{\rm div} = -\frac{1}{2}\mathcal{B}_2 Q r^2 + \frac{1}{2}\mathcal{B}_2 Q^3\,, \tag{4.48}$$

where $f = 1 - 2M/r + Q^2/r^2$.

Note that, with the assumption of vanishing EM tidal fields, the EM Love numbers as defined in Eq. (1.12) have no meaning. Thus, we will restrict our analysis to the gravitational Love numbers which allow us to focus only on Eqs. (4.45) and (4.47).

An immediate look over Eqs. (4.45) and (4.47) confirms that there are no decaying terms. The absence of this terms allow us to to compare these expressions with Eqs. (3.15)–(3.16) and extract the body's response without any type of problems. This comparison leads to the conclusion that, in absence of external EM fields, there are no induced mass and current quadrupoles and therefore, by means of Eq. (1.10), the polar-type and axial-type Love numbers of charged BHs are zero,

$$k_2^E = 0\,, \qquad k_2^B = 0\,. \tag{4.49}$$

This results indicate that the multipolar structure of charged BH is not affected when immersed in a purely gravitational tidal environment. A charged BH will maintain its vanishing mass, current and EM quadrupole structure when acted upon by gravitational tidal fields and all the relevant Love numbers are zero. Although the calculations above were specified for $l = 2$, this procedure can be generalized to higher order multipoles and obtain the same conclusions:

$$k_l^E = 0\,, \qquad k_l^B = 0\,. \tag{4.50}$$

4.4 Love Numbers of a Wormhole

A natural extension of the procedure developed in the previous sections is the calculation of tidal Love numbers of wormholes. The Schwarzschild solution written in the usual coordinates (4.10) is valid in the range $r \in \,]2M, +\infty[$, however, this solution can be extended using appropriate coordinate transformations to describe the maximal extension of the spacetime. The maximal analytical solution describes the existence of a wormhole spacetime composed by two qualitatively identical universes connected by a bridge [25, 31]. One appropriate and intuitive method to construct wormhole solutions [24] consists in taking two copies of the ordinary Schwarzschild solution (4.10) and remove from them the four-dimensional regions described by

$$\Omega_{1,2} \equiv \{r_{1,2} \le r_0 | r_0 > 2M\}\ . \tag{4.51}$$

With this procedure we obtain two geodesic incomplete manifolds that are bounded by the the timelike hypersurfaces

$$\partial\Omega_{1,2} \equiv \{r_{1,2} = r_0 | r_0 > 2M\}\ . \tag{4.52}$$

The two copies are now glued together by identifying these two boundaries, $\partial\Omega_1 = \partial\Omega_2$, such that the resulting spacetime is geodesically complete and possess two distinct regions connected by a wormhole with a throat at $\partial\Omega$. Since the wormhole spacetime is composed by two Schwarzschild spacetimes, the stress-energy tensor vanishes everywhere except on the throat of the wormhole. The patching at the throat requires a thin-shell of matter with surface density and surface pressure

$$\sigma = -\frac{1}{2\pi r_0}\sqrt{1-\frac{2M}{r_0}}\,, \qquad p = \frac{1}{4\pi r_0}\frac{1-M/r_0}{\sqrt{1-2M/r_0}} \tag{4.53}$$

We use the radial tortoise coordinate r_* to cover the two patches of the spacetime. The tortoise coordinate can be related with the Schwarzschild radial coordinate r by,

$$\frac{dr}{dr_*} = \pm\left(1 - \frac{2M}{r}\right) . \tag{4.54}$$

Furthermore, we can assume without loss of generality that the tortoise coordinate at the throat is zero $r_*(r_0) = 0$, such that the domain of one universe corresponds to $r_* > 0$ whereas the other domain corresponds to $r_* < 0$.

4.4.1 Perturbations and Boundary Conditions

Here we discuss the wormhole's perturbation formalism, and since the two regions of the wormhole are described by two Schwarzschild metrics, we will use the same formalism developed for a non-rotating uncharged BH, where the metric is perturbed according to Eq. (3.20) and the even and odd sector perturbations can be described as Eqs. (3.23)-(3.24). The only remaining issue are boundary conditions at the wormhole throat which require a delicate handling. The boundary conditions are imposed to us by Darmois-Israel junction conditions [34, 90] and their application is easier if the thin-shell's worldtube coincides with a fixed coordinate sphere at a constant radius, however, this choice is incompatible with choosing the Regge-Wheeler gauge in both interior and the exterior of the wormhole. Since we desire to combine the advantages of using the Regge-Wheeler formalism to simplify the field equations and the convenience of the matching conditions in a fixed sphere, we will carry out our matching in the following way. We first construct a coordinate system where the metric perturbations will no longer be Regge-Wheeler, but any mass on the shell will remain static. We carry our matching at the throat and obtain junction conditions that relate the interior and exterior metric perturbations and also equations of motion for matter on the shell. With an appropriate gauge transformation we map the shell to a fix location and write the full metric in this new coordinate system. As final step, we match the components of the new metric along the shell and apply the Darmois-Israel junction conditions to the extrinsic curvature of the metric. In the dynamical case, the junction conditions read

$$[[h_0]] = [[h_1]] = 0 \,, \tag{4.55}$$

for axial-type perturbations. However, since we are considering stationary perturbations, h_1 is identically zero and we get a second order differential equation for h_0. In practice we impose that h_0 and its derivative with respect to r_* be smooth. Here, the symbol "[[...]]" gives the "jump" in a given quantity across the spherical shell (or $r = r_0$), i.e. $[[X]] \equiv X(r_{0_+}) - X(r_{0_-})$. For polar perturbations, we find that

$$[[K]] = 0 \qquad \text{and} \qquad [[dK/dr_*]] = -8\pi\sqrt{f(r_0)}\delta\Sigma, \tag{4.56}$$

where $\delta\Sigma$ is the perturbation to the surface energy density. Although once could solve for a generic equation of state, we will assume that the throat material is stiff, and that $\delta\Sigma \sim 0$.

4.4.2 Polar-type Love numbers of a Wormhole

In Sec. 4.2 we verified that Einstein's equation for stationary polar-type perturbations of the Schwarzschild metric lead to the differential equation Eq. (4.12),

$$r^2(r-2M)^2H_0''+2r(r-2M)(r-M)H_0'-\left(l(l+1)r^2-2l(l+1)Mr+4M^2\right)H_0=0, \tag{4.57}$$

where K and K' can be related with H_0 and its first derivative by

$$\begin{aligned} K &= \frac{2Mr(2M-r)H_0'+H_0\left(2\left(l^2+l-4\right)Mr-\left(l^2+l-2\right)r^2+4M^2\right)}{\left(l^2+l-2\right)r(2M-r)}\,, \\ K' &= \frac{r(2M-r)H_0'-2MH_0}{r(2M-r)}\,. \end{aligned} \tag{4.58}$$

Equation (4.12) can be solved on both sides of the throat to obtain,

$$H_{\text{left}}=C_1P_l^2\left(\frac{r}{M}-1\right)+C_2Q_l^2\left(\frac{r}{M}-1\right)\,, \tag{4.59}$$

$$H_{\text{right}}=C_3P_l^2\left(\frac{r}{M}-1\right)+C_4Q_l^2\left(\frac{r}{M}-1\right)\,, \tag{4.60}$$

where P_l^m and Q_l^m are the associated Legendre polynomials of first and second kind, respectively. As discussed in Sec. 4.2 the terms proportional to C_1 and C_3 are asymptotically divergent and can therefore be identified with the tidal fields, whereas the terms proportional to C_2 and C_4 decay at large distances with r^{-3} and are related with the body's response.

We now must use boundary conditions to match these two solutions. On the "other side" of the wormhole, we consider that there are no external tidal fields, and we require asymptotic flatness in that domain, $H_{\text{left}}(r\to\infty)\to 0$. In order to integrate the solution across our universe we use the boundary conditions (4.56). Note that, by virtue of relation (4.54), the continuity of the derivative with respect to the tortoise coordinate corresponds to a discontinuity in the Schwarzschild radial coordinate r. The asymptotic flateness in the "other universe" and the conditions (4.56) allow us to fix three of the integration constants as functions of the applied tidal field. Unfortunately the full expression of H_{right} is to cumbersome to present in the main part of this thesis, however it can be found in the appended notebook [91].

From this solution we can extract the Love number using the first expression in Eq. (1.10). In Fig. 4.1 we present the polar-type Love numbers as a function of r_0. It is clear that the Love numbers are non-zero for an arbitrary throat's radius $r_0>2M$. It is interesting to see that the Love numbers of wormholes have negative sign, contrasting with is known for neutron-stars [3, 16, 17]. Furthermore, we observe that the Love number tend to zero as the radius of the throat approaches the Schwarzschild radius, $r_0\to 2M$. It is interesting to analyze the BH limit of the solution, i.e, the limit where the r_0 is close to the Schwarzschild radius. We first expand the solution in powers of $\xi:=r_0/(2M)-1$ and then take the asymptotic limit of the resulting solution with a second expansion in powers of $1/r$. At an appropriate order in the expansion parameters the solution can be written as

$$H_{\rm right}(r) = -r^2\mathcal{E}_2 + 2Mr\mathcal{E}_2 - \frac{8M^5\mathcal{E}_2}{r^3(15\log(\xi)+40)} + \mathcal{O}(\xi, r^{-4})\,. \tag{4.61}$$

Comparing this solution with Eq. (3.15) we verify that, at leading order in ξ, the induced mass quadrupole is

$$M_2 = -\frac{2M^5\mathcal{E}_2}{\sqrt{5\pi}(3\log(\xi)+8)}\,, \tag{4.62}$$

and using the first expression in Eqs. (1.10) we can calculate the polar-type Love number of an wormhole,

$$k_E^{(2)} = \frac{4}{15\log(\xi)+40}\,. \tag{4.63}$$

The previous procedure can be generalized to higher order multipoles, for example, the next leading order Love number is

$$k_E^{(3)} = \frac{8}{210\log(\xi)+735}\,. \tag{4.64}$$

4.4.3 Axial-type Love numbers of a Wormhole

The axial perturbations are governed by the differential equation (4.13),

$$h_0'' + \frac{h_0(4M - l(l+1)r)}{r^2(r-2M)} = 0. \tag{4.65}$$

which can be solved independently on both sides of the throat. For $l = 2$ this differential equation leads to

$$h_0^{\rm left} = C_1 r^2(r-2M) + \frac{C_2\left(2M\left(2M^3+2M^2r+3Mr^2-3r^3\right) - 3r^3(r-2M)\log(1-2M/r)\right)}{24M^5 r}\,, \tag{4.66}$$

$$h_0^{\rm right} = C_3 r^2(r-2M) + \frac{C_4\left(2M\left(2M^3+2M^2r+3Mr^2-3r^3\right) - 3r^3(r-2M)\log(1-2M/r)\right)}{24M^5 r}\,, \tag{4.67}$$

Using the same notation as employed for the polar perturbations, we will consider that the left-side of the throat does not contain tidal fields and therefore the perturbed solution must be asymptotically flat, $h_0^{\rm left}(r\to\infty)\to 0$. The remaining boundary conditions are obtained by specifying the matching of the two regions of the spacetime at the throat

$$\left[h_0(r_*)\right]_{r_*=0} = 0\,, \qquad \left[\frac{dh_0(r_*)}{dr_*}\right]_{r_*=0} = 0\,. \tag{4.68}$$

Imposing the previous boundary conditions we can describe the complete spacetime in terms of an single free constant that can be related with the external tidal field. The full solution is to cumbersome to present here and can be found in Ref. [91]. Similarly to case of polar-type tidal Love numbers we find that the axial-type Love numbers for a wormhole are negative for arbitrary values of the throat's radius and tend to zero when in the BH limit. The behavior of the Love number for $l = 2$ is presented

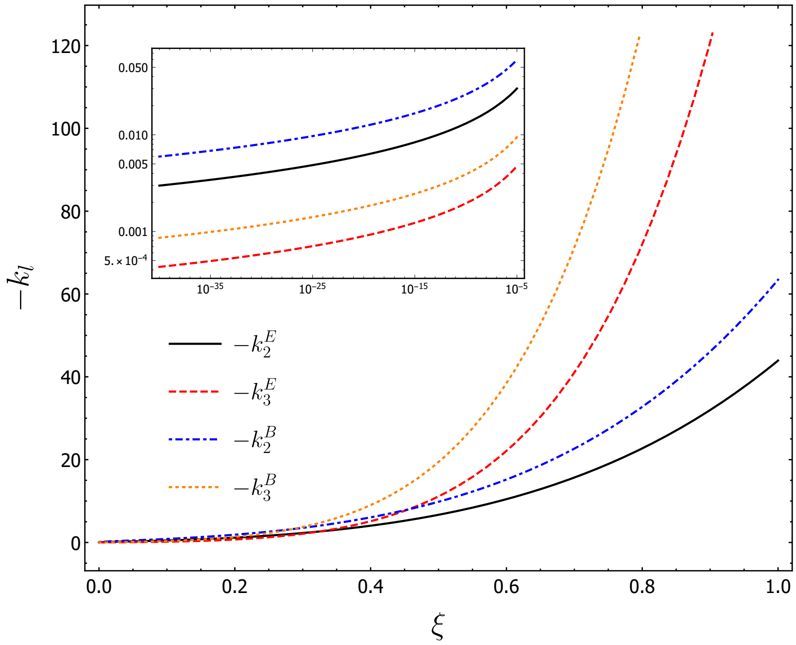

Figure 4.1: Plot of the $l = 2$ and $l = 3$, axial- and polar-type tidal Love numbers (TLNs) for a stiff wormhole constructed by patching two Schwarzschild spacetimes at $r = r_0 > 2M$, where r_0 is the throat's radius. The plot is described as a function of the adimensional parameter $\xi := (r - 2M)/(2M)$. We verify that the TLNs are in general non-zero and grow in magnitude with the throat's radius. Furthermore, the wormhole's Love numbers are negative, contrasting with the neutron star case [3]. We note that in the limit where the radius of the thorat approaches the Schwarzschild radius $r = 2M$ the TLN tends to zero as expected. In detail we present the BH limit of the solution. We verify that even when $r_0 \sim 2M$ the TLNs can be significanly different.

in Fig. 4.1. Similarly to the case for polar-type Love numbers is interesting to analyze the BH limit of the solution. Expanding the solution in powers of ξ and taking the asymptotic limit we found that h_0 behaves according to

$$h_0^{\text{right}}(r) = \frac{r^3\mathcal{B}_2}{3} - \frac{2}{3}r^2 M\mathcal{B}_2 - \frac{16M^5\mathcal{B}_2}{r^2(60\log(\xi)+155)} + +\mathcal{O}(\xi, r^{-3})\,. \tag{4.69}$$

Comparing the previous solution with the expansion (3.16) we extract the induced current quadrupole moment on the wormhole,

$$S_2 = -\frac{16M^5\mathcal{B}_2}{\sqrt{5\pi}(12\log(\xi)+31)}\,. \tag{4.70}$$

Making use of the second expression on Eqs. (1.10) we conclude that the axial-type Love number behaves as

$$k_B^{(2)} = \frac{32}{60\log(\xi)+155}\,. \tag{4.71}$$

This procedure can also be employed for higher multipolar orders. For $l = 3$ we obtain,

$$k_B^{(3)} = \frac{32}{420\log(\xi)+1463}\,. \tag{4.72}$$

4.4.4 Black hole limit

The logarithmic dependence of the polar and axial Love numbers is particularly interesting for discussion on quantum corrections at the horizon. This dependence implies that, even when the throat's radius is very close to the Schwarzschild radius (i.e differences at Plank length scale), the Love numbers can be relatively larger as shown in the inset plot of Fig. 4.1.

It is also relevant to determine the magnitude of the Love numbers in this BH limit, i.e, when $r_0 - 2M \sim l_{\text{Plank}} \sim 1.616 \times 10^{-33}$cm. For a mass range of $M \in [1, 100]M_\odot$, the $l = 2$ and $l = 3$ polar- and axial-type Love numbers are

$$\begin{aligned} k_2^E &= -3 \times 10^{-3}\,, \qquad & k_2^B &= -6 \times 10^{-3}\,, \\ k_3^E &= -4 \times 10^{-4}\,, \qquad & k_3^B &= -9 \times 10^{-4}\,. \end{aligned} \tag{4.73}$$

These results indicate that with sufficient precision in the detection of the Love numbers of GW signals, i.e $\mathcal{O}(10^{-4})$, the detection of GW may provide a valid experimental test of the existence of these objects. Furthermore, it seems that this logarithmic dependence is characteristic of ultracompact objects [1] which may indicate that putative deviations of the "no Love" property of BHs may be relatively large, even when the object is almost as compact as a BH. Another interesting idea that arises is the fact that possible quantum corrections at the BH horizon may be governed by a similar behavior. Thus, the analysis of the Love numbers can provide a valid method to test these corrections.

5 Love Numbers of Black Holes in Modified Gravity

The previous chapter was dedicated to Einstein's theory of GR where we studied the TLNs of non-rotating, uncharged BHs verifying their "zero-Love" rule and then proved that charged BHs can also exhibit this property. Following that, we extended this analysis to wormholes where we verified that they have negative TLNs and that, in the BH limit, these numbers can be relatively large due to their logarithmic behavior. We will now extend the studies of TLNs to modified theories of gravity. Throughout this chapter, we are going to restrict ourselves to two specific classes of modified gravity: scalar-tensor theories and quadratic theories.

5.1 Love Numbers in Scalar-Tensor gravity

Scalar-tensor gravity is one of the most natural and direct extensions to GR, in which one or more scalar fields are included in the gravitational sector of the action, through a nonminimal coupling. The motivation to study scalar-tensor theories comes from the fact that this types of theories have important applications in astrophysics. Furthermore, possible fundamental theories such as string theory [63], Kaluza-Klein-like theories [92] and braneworld scenarios [93, 94] yield scalar fields that are nonminimally coupled to gravity.

The most general scalar-tensor action that is at most quadratic in the field derivatives can be written using the Bergmann-Wagoner formulation [95, 96] as

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[\Phi R - \frac{\omega\left(\Phi\right)}{\Phi}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - U(\Phi)\right] + S_{\rm matt}\,, \tag{5.1}$$

where Φ is a scalar field that is nonminimally coupled to gravity, $\omega(\Phi)$ is a coupling function that depends on the scalar field, $U(\Phi)$ is an arbitrary scalar field potential that depends also on Φ. $S_{\rm matt}$ is the usual matter action that depends on certain matter fields.

Action (5.1) is usually referred as the Jordan-frame action, and it is interesting to note that it can be written in a different form by performing a scalar field redefinition $\psi = \psi(\Phi)$ and a conformal transformation of the spacetime metric $g_{\mu\nu} \to g^\star_{\mu\nu} := A^{-2}(\psi)g_{\mu\nu}$. In particular, using the conformal factor $A(\psi) = 1/\sqrt{\Phi(\psi)}$ the action (5.1) transforms into

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g^\star} \left[R^\star - 2g^{\star\mu\nu}\partial_\mu\psi\partial_\nu\psi - V(\psi)\right] + S_{\rm matt} \,, \tag{5.2}$$

which is referred to as the Einstein-frame action. The quantities $g^\star$ and $R^\star$ are the metric determinant and the Ricci tensor of the new conformal metric $g^\star_{\mu\nu}$ and new potential $V(\psi)$ is defined as $V(\psi) \equiv \psi U(\Phi(\psi))$.

Writing the action in the form of Eq. (5.2) has the advantage that the gravitational sector of the action becomes minimally coupled to gravity, however, it comes with the cost of a nonminimal coupling in the matter sector.

Since actions (5.1) and (5.2) are simply different representations of the same physical theory, we can choose to use the action that simplifies the calculations for our problem.

5.1.1 Love Numbers in Brans-Dicke Gravity

Introduction

As an example of scalar-tensor theory, we will study perturbations to Brans-Dicke gravity [97]. Brans-Dicke gravity can be obtained from action (5.1) when the function $\omega(\Phi)$ is constant $\omega(\Phi) = \omega_{\rm BD}$ and there is no scalar field potential $U(\Phi) = 0$. Thus, this theory can be described by the following action,

$$S = \frac{1}{16\pi} \int dx^4 \sqrt{-g} \left(\Phi R - \frac{\omega_{\rm BD}}{\Phi}\partial_\mu\Phi\partial^\mu\Phi\right) + S_{\rm matt} \,, \tag{5.3}$$

where R is the Ricci scalar, $\omega_{\rm BD}$ is a dimensionless coupling constant and $S_{\rm matt}$ is the matter's action.

The first term in the integral is the usual Lagrangian density for GR with a non-minimal coupling between the Ricci tensor and the scalar field, and the second term is the Lagrangian density of a scalar field Φ, where the scalar field in the denominator is included to adimensionalize the constant $\omega_{\rm BD}$.

To obtain Brans-Dicke's field equations, we vary the action with respect to the scalar field Φ and the contravariant metric $g^{\mu\nu}$. Working in the Jordan's frame, we have

$$G_{\mu\nu} = \frac{8\pi}{\Phi}T_{\mu\nu} + \frac{\omega_{\rm BD}}{\Phi^2}\left(\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}g_{\mu\nu}\partial_\lambda\Phi\partial^\lambda\Phi\right) + \frac{1}{\Phi}\left(\nabla_\mu\nabla_\nu\Phi - g_{\mu\nu}\Box\Phi\right) \,, \tag{5.4}$$

$$\Box\Phi = \frac{1}{3 + 2\omega_{\rm BD}} 8\pi T \,, \tag{5.5}$$

where $T_{\mu\nu}$ is the stress-energy tensor defined by Eq. (4.3) and T is its trace.

Once more we will focus on the vacuum solutions such that $T_{\mu\nu} = 0$, $T = 0$ and Eqs. (5.4)–(5.5) simplify to,

$$G_{\mu\nu} = \frac{\omega_{\rm BD}}{\Phi^2}\left(\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}g_{\mu\nu}\partial_\lambda\Phi\partial^\lambda\Phi\right) + \frac{1}{\Phi}\nabla_\mu\nabla_\nu\Phi \,, \tag{5.6}$$

$$\Box\Phi = 0 \,. \tag{5.7}$$

Background Metric

We focus on static, spherically symmetric solutions of vacuum field equations (5.6)–(5.7). Ref. [98] shown that solutions with variable Φ will lead to solutions which do not possess an event-horizon and, therefore, cannot describe a unperturbed BH. Assuming a constant scalar field, $\Phi = 1$, we verify that Eqs. (5.6)–(5.7) reduce to the usual Einstein's equations and, by virtue of Birkhoff's theorem, the only possible solution describing a static non-rotating BH is the Schwarzschild line element (4.10).

An identical conclusion is reached in Ref. [99], where by geometrical arguments it is proven that a stationary spacetime describing a BH is a solution of Brans-Dicke's field equations if and only if it is a solution of Einstein's field equations. Recently this was generalized to more general theories and scenarios [100]

Therefore, a stationary, non-rotating BH in Brans-Dicke gravity is characterized by

$$\overset{(0)}{g}_{\mu\nu} dx^\mu dx^\nu = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,, \tag{5.8}$$

$$\overset{(0)}{\Phi} = 1. \tag{5.9}$$

Perturbations to Brans-Dicke equations of motion

We will apply perturbations to the spacetime metric and to the scalar field according to Eqs. (3.20) and (3.26). The perturbation of the spacetime metric is decomposed in even and odd parity sectors according to Eqs. (3.24)–(3.23), where the functions in these perturbations are $e^\Gamma = e^{-\Lambda} = 1 - 2M/r$, and the scalar perturbation is decomposed according to Eq. (3.30). The computations become much more simple by taking into account that derivatives in Eqs. (5.6)–(5.7) can only act on $\delta\Phi$ since the background scalar field is constant.

In Brans-Dicke gravity, $\delta\Phi$ transforms as a scalar under rotations and therefore the right-hand side of Eq. (5.6) vanishes for odd-parity perturbations. Consequently, this parity sector of the perturbed Brans-Dicke's field equations reduce to the GR case. Recalling the results in Chapter 4, this conclusion implies that the axial-type gravitational TLNs of a non-rotating BH in Brans-Dicke gravity are zero,

$$k_l^B = 0\,. \tag{5.10}$$

We will now consider the case of even-parity perturbations such that the $h_{\mu\nu}$ takes the form of Eq. (3.23) and $\delta\Phi$ is written according to Eq. (3.30). We will also restrict to static perturbations such that the functions in Eqs. (3.23) and (3.30) are time-independent.

We focus on first on Eq. (5.7) which provides us with a differential equation for $\delta\Phi$. This equation can be written as

$$\delta\Phi'' - \frac{2(r-M)\delta\Phi' - l(l+1)\delta\Phi}{r(2M-r)} = 0\,, \tag{5.11}$$

which yields the solution

$$\delta\Phi = C_1 P_l\left(\frac{r}{M} - 1\right) + C_2 Q_l\left(\frac{r}{M} - 1\right)\,, \tag{5.12}$$

where P_l and Q_l are the first and second Legendre polynomials respectively and C_1 and C_2 are integration constants that are fixed through boundary conditions.

It is possible to verify that the term proportional to C_2 is divergent at the event-horizon, $r_h = 2M$, and therefore we must take $C_2 = 0$. The term proportional to C_1 is regular at the event-horizon and diverges at large distances with r^l. Comparing with Eq. (3.19), we conclude that C_1 is proportional to the scalar tidal field, $C1 \propto \mathcal{E}_l^S$. Thus, the solution for the scalar field perturbation with correct boundary conditions is

$$\delta\Phi = C_1 P_l \left(\frac{r}{M} - 1\right) . \tag{5.13}$$

The Legendre polynomial in Eq. (5.13) does not contain decaying terms which implies that there are no scalar multipole moments and, by Eq. (1.13) the scalar Love number is zero,

$$k_l^S = 0 . \tag{5.14}$$

The particular case when there is no scalar tidal field (i.e. $C_1 = 0$ in Eq. (5.12)) is trivial since there are no scalar field perturbations, $\delta\Phi = 0$. In this scenario, the perturbed field equations for Brans-Dicke reduce to the equations studied in Sec. 4.2 for an uncharged in GR. This argument leads to the conclusion that polar-type TLNs of a non-rotating BH in Brans-Dicke gravity immersed on a purely gravitational tidal field are zero,

$$k_l^E = 0 . \tag{5.15}$$

It is more interesting to analyze tidal environments that may include scalar tidal fields $\mathcal{E}_l^S \neq 0$. From an independent angular component in the $\theta\theta$-component of Eq. (5.6) we get

$$H_2 = H_0 - 2\delta\Phi , \tag{5.16}$$

and, using the $tt-$ rr- and the remaining $\theta\theta$-component of Eq. (5.6), we find expressions for K and its first two derivatives. Substituting the resulting expressions in the $r\theta$-component of Eq. (5.6) we obtain a coupled second-order equation,

$$\mathcal{D}_H^{(2)} H + \mathcal{D}_\Phi^{(2)} \delta\Phi = 0 , \tag{5.17}$$

where we defined the operators

$$\mathcal{D}_H^{(2)} = \frac{d^2}{dr^2} + \frac{2M - 2r}{2Mr - r^2}\frac{d}{dr} + \frac{2l(l+1)Mr - l(l+1)r^2 - 4M^2}{r^2(r-2M)^2} , \tag{5.18}$$

$$\mathcal{D}_\Phi^{(2)} = \frac{d^2}{dr^2} + \frac{2M - 2r}{2Mr - r^2}\frac{d}{dr} + \frac{2l(l+1)Mr - l(l+1)r^2 + 4M^2}{r^2(r-2M)^2} . \tag{5.19}$$

Substituting Eq. (5.13) in Eq. (5.17) to eliminate $\delta\Phi$, we obtain an inhomogeneous differential equation for H_0. Specifying for $l = 2$ we obtain

$$H_0'' + \frac{2(r-M)}{r(r-2M)}H_0' - \frac{2\left(2M^2 - 6Mr + 3r^2\right)}{r^2(r-2M)^2}H_0 - \frac{4M^2\left(2M^2 - 6Mr + 3r^2\right)}{3r^2(r-2M)^2}\mathcal{E}_2^{\mathrm{S}} = 0 , \tag{5.20}$$

where we used the relation $C_1 \equiv -2/3M^2\mathcal{E}_2^{\rm S}$. Imposing regularity at the horizon we find

$$H_0 = -r^2\mathcal{E}_2 + 2Mr\mathcal{E}_2 - \frac{2}{3}M^2\mathcal{E}_2^{\rm S}\,. \tag{5.21}$$

Comparing Eq. (5.21) with expansion (3.15) we see that there are conclude that there is no mass quadrupole moment induced on a Brans-Dicke BH and consequently the polar-type gravitational TLN is zero,

$$k_2^E = 0\,. \tag{5.22}$$

Despite the differential equation (5.20) and subsequent calculations were presented for $l = 2$, this procedure can be generalized for higher values of l.

We now recall the main results of this section. Here, we analyzed the tidal deformation of a non-rotating BH described in scalar-tensor gravity, in particular Brans-Dicke gravity. We described a BH immersed in a general tidal environment that can be composed by gravitational and scalar tides. We solved the perturbed Brans-Dicke equations and found that the BH does not develop any multipolar response to the external perturbations. Thus, this results led to the conclusion that the TLNs (polar, axial,and scalar) are zero. Therefore, we verified that the "zero-Love" property of BHs is still valid in Brans-Dicke gravity and conclude that the detection of TLNs will not provide constrains to this theory.

5.2 Love Number in Quadratic Theories of Gravity

One of the most interesting problems in theoretical physics is to accommodate GR in the framework of quantum field theories and develop a quantum theory of gravity. However, the fact that GR is not a renormalizable theory in the usual quantum field theory sense poses a large obstacle to this problem. One proposed solution to solve it is to consider modifications to GR (c.f. Ref. [47] for a review). Specifically, this problem can be circumvented if we consider the Einstein-Hilbert action as the first term in a more complex action that may be composed of infinite terms containing all possible curvature invariants. It was shown that adding quadratic curvature terms to the action will make it renormalizable [48] and, at this order, the only independent curvature invariants are:

$$R^2, \qquad R^2_{\mu\nu} \equiv R_{\mu\nu}R^{\mu\nu}, \qquad R^2_{\mu\nu\rho\sigma} \equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, \qquad {}^*RR := \frac{1}{2}\epsilon^{cdef}R^a_{\ bef}R^b_{\ acd}. \tag{5.23}$$

An action composed simply by the sum of this terms do not contribute to any modifications to the field equations in four spacetime dimensions since their integrals account only for boundary terms which go to zero. Thus, scalar degrees of freedom are introduced in the theory by coupling the higher-order curvature terms with dynamical fields. With this addition, we will obtain differences from GR at the field equations level. The most general action that includes the quadratic order curvature terms coupled to one scalar field is [57, 62],

$$
\begin{aligned}
S = \frac{1}{16\pi}\int \sqrt{-g}\, d^4x \big[& R - 2\nabla_\mu\Phi\nabla^\mu\Phi - V(\Phi) + \\
& + f_1(\Phi)\, R^2 + f_2(\Phi)\, R^2_{\mu\nu} + f_3(\Phi)\, R^2_{\mu\nu\rho\sigma} + f_4(\Phi)\, {}^*RR\big] + \\
& + S_{\rm matt}\left[\Psi, \gamma(\Phi)\, g_{\mu\nu}\right] ,
\end{aligned}
\tag{5.24}
$$

where $V(\Phi)$ is the scalar field self-potential and $f_i(\phi)$ $(i = 1, ..., 4)$ are functions that couple the scalar field with the higher-order curvature terms. The matter section of the action is composed by the matter fields Ψ and we included a universal nonminimal coupling with the spacetime metric.

Applying dimensional analysis on action (5.24), we verify that the coupling functions have the dimensions of length squared (i.e. inverse of curvature). This functions introduce a new fundamental length scale in these theories and one of the main objectives in current and future researches is to determine the magnitude of this new length scale.

Field equations generated from action (5.24) will generally be of higher-order which leads to the appearance of ghost degrees of freedom and to the Ostrogradski instability [66]. In order to avoid these problems, the quadratic curvature terms must appear in the form of the Gauss-Bonnet scalar,

$$
R^2_{\rm GB} \equiv R^2 - 4R^2_{\mu\nu} + R^2_{\mu\nu\rho\sigma} \,, \tag{5.25}
$$

or instead, action (5.24) must be considered as the truncation, up to second order in curvature, of a more general theory. This corresponds to an effective field theory approach, and action (5.24) takes the roll of an effective action.

In this chapter, we are going to focus on Chern-Simons gravity, a particular quadratic extension of GR. The procedure developed here can be, in principle, applied for other quadratic theories of gravity such as EdGB, however the analysis for this theory was postponed for future work.

5.2.1 Love Numbers of Black Holes in Chern-Simons gravity

Introduction

One quadratic theory of gravity that has received much attention in recent years as a possible extension to GR is Chern-Simons gravity (c.f. [68] and references therein). This theory is a 4-dimensional modification to GR that can be characterized in terms of its action

$$
S = S_{\rm EH} + S_{\rm CS} + S_\Phi + S_{\rm matt}, \tag{5.26}
$$

where $S_{\rm EH}$ is the Einstein-Hilbert action, $S_{\rm CS}$ is the Chern-Simons term, S_Φ is the scalar field contribution and $S_{\rm matt}$ contains any other matter fields. This terms are defined as,

$$S_{\rm EH} = \frac{1}{16\pi}\int d^4x\sqrt{-g}R, \tag{5.27}$$

$$S_{\rm CS} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\frac{1}{4}f(\Phi)^*RR, \tag{5.28}$$

$$S_\Phi = -\frac{1}{16\pi}\int d^4x\sqrt{-g}\frac{\beta(\Phi)}{2}\left[\nabla_a\Phi\nabla^a\Phi + V(\Phi)\right], \tag{5.29}$$

$$S_{\rm matt} = \int d^4x\sqrt{-g}\mathcal{L}_{\rm matt}, \tag{5.30}$$

where Φ is a scalar field characteristic of the theory named Chern-Simons coupling field, and $\mathcal{L}_{\rm matt}$ is a matter Lagrangian density which does not depend on Φ. The function $\beta(\Phi)$ and the function $f(\Phi)$ are two dimensional coupling functions of the theory and $^*RR$ is the Pontryagin scalar defined in (5.23).

Here we present the two coupling functions $f(\Phi)$ and $\beta(\Phi)$ to include the different notations in the literature, however, since one can always redefine the scalar field in order to fix one of them, we will take $\beta(\Phi) = 1$. Note that, by taking $f_1 = f_2 = f_3 = 0$ and with an appropriate redefinition of the scalar field and the coupling functions in action (5.24), we obtain Chern-Simons gravity.

In this work we will neglect the Chern-Simons coupling field potential, $V(\Phi) = 0$, and any other matter fields of the theory, $\mathcal{L}_{\rm matt} = 0$. Furthermore, we will apply an effective field theory approach such that the coupling function $f(\Phi)$ can be expanded in a power series,

$$f(\Phi) = \eta + \alpha_{\rm CS}\Phi + \mathcal{O}(\Phi^2)\,. \tag{5.31}$$

Since the Pontryagin scalar is a topological invariant, the constant term $f(\Phi) = \eta$ does not produce any modifications from GR. Using this fact, we can restrict to the case when $\eta = 0$ and, neglecting second order terms in the $\alpha_{\rm CS}$, we can write the coupling function as $f(\Phi) = \alpha_{\rm CS}\Phi$. Using this approach, the action for the Chern-Simons gravity takes the form of

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[R - \frac{1}{2}g^{ab}\nabla_a\Phi\nabla_b\Phi + \frac{\alpha_{\rm CS}}{4}\Phi\,^*RR\right]. \tag{5.32}$$

Varying action (5.32) with respect to the scalar field and to the spacetime metric, we can derive the equations of motion of Chern-Simons gravity,

$$R_{ab} = -\alpha_{\rm CS}C_{ab} + \frac{1}{2}\Phi_{,a}\Phi_{,b}, \tag{5.33}$$

$$\Box\Phi = -\frac{\alpha_{\rm CS}}{4}{}^*RR, \tag{5.34}$$

where,

$$C_{ab} := \Phi_c\epsilon^{cde(a}\nabla_e R^{b)}_d + \Phi_{;dc}{}^*R^{dabc}. \tag{5.35}$$

Background Metric

In order to determine the TLNs of a BH in Chern-Simons gravity we need first to describe the unperturbed state of this BH. We will study the case of a spherically symmetric stationary solution of Chern-Simons field equations (5.33)–(5.34).

This background metric can be described by the line element (3.21) and substituting it in (5.33)–(5.34) we can find expressions for the metric functions.

For spherically symmetric line elements, the Pontryagin scalar vanishes,

$$^{*}RR = 0\,, \tag{5.36}$$

and, if the coupling field is spherically symmetric, $\Phi = \Phi(t, r)$, we can check that

$$C_{ab} = 0\,. \tag{5.37}$$

Neglecting second order contributions of the scalar field we check that the field equation (5.33) together with Eq. (5.37) reduce to

$$R_{ab} = 0\,, \tag{5.38}$$

which yields the well-known Schwarzschild metric as solution. Using Eq. (5.36) we can substitute it in Eq. (5.34) such that it takes the form $\Box\Phi = 0$. Imposing regular boundary conditions at the horizon $r_h = 2M$ and at infinity, we conclude that the coupling scalar field vanishes. Thus, the unperturbed spacetime describing a non-rotating BH in Chern-Simons gravity is characterized by the line element and coupling field [59, 61, 101]

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,, \tag{5.39}$$

$$\Phi = 0\,. \tag{5.40}$$

Perturbations to Chern-Simons spacetime

Despite the fact that the spacetime description of a non-rotating BH in Chern-Simons is identical to GR, in general,this theory predicts a different linear response and therefore different GW emission [59, 61]. Based on this argument, we expect that TLNs can have non-trivial solutions in this gravity theory.

Following the procedure of the previous sections, we perturb this spacetime following a linear perturbation theory approach. The spacetime metric is disturbed from its unperturbed state by a small perturbation $h_{\mu\nu}$ described by Eqs. (3.23)–(3.24) with $e^{\Gamma} = e^{-\Lambda} = 1 - 2M/r$. Similarly to Sec. 5.1, the scalar field is perturbed according to Eq. (3.26), where $\Phi^{(0)} = 0$.

As in the previous cases, we can analyze the effects of polar and axial perturbations separately. Since the scalar field transforms as a pseudoscalar in Chern-Simons gravity, the axial perturbations are coupled with scalar perturbations, while polar perturbations are decoupled.

To simplify the problem we consider only stationary perturbations imposing that the perturbed functions in Eqs. (3.23), (3.24), (3.30), (3.28) and (3.29) do not depend on time.

The field equations for Chern-Simons modified gravity yield a system of nine differential equations. One can check that these equations are not all linearly independent and, in particular, they can be reduced to a system of two coupled second-order differential equations for the scalar and axial-type perturbations and another differential equation governing the polar perturbations.

In the following sections we use the perturbed spacetime explained in this section to compute the TLNs for a BH in Chern-Simons modified gravity.

Polar-type Love Numbers in Chern-Simons Gravity

We can immediately deduce that the gravitational polar-type TLNs in Chern-Simons gravity is trivial. As we have seen, the background spacetime is Schwarzschild and the polar and scalar perturbations are decoupled. It can be concluded that polar perturbations of a Schwarzschild BH in Chern-Simons are identical to GR [61, 102]. Therefore, we conclude that polar-type TLNs in modified Chern-Simons gravity are governed by an identical system as the one in GR. Using the same procedure as employed in Sec. 4.2, this system of equations can be reduced to Eq. (4.13) which leads to the conclusion that

$$k_l^E = 0, \tag{5.41}$$

showing that polar-type TLNs for a BH in Chern-Simons modified gravity are zero. The fact that polar-type TLNs vanish in Chern-Simons gravity implies that constrains to this theory via GW detection will be extremely difficult since the polar-type TLNs are the dominant corrections to the inspiral waveform [7, 103–105].

Axial-type Love Numbers in Chern-Simons Gravity

Contrasting with polar-type TLNs, we expect that the axial-type TLNs of a BH in Chern-Simons gravity can be non-trivial. Due to the existing coupling between the scalar and polar-type functions, the equations of motion governing these functions are much more complex to study.

We can check that the $t\varphi$-, $r\varphi$- and $\theta\varphi$-components of Eq. (5.33), together with the scalar field equation (5.34), yield four differential equations that govern the axial-type and scalar perturbations. From the $r\varphi$-component of Eq. (5.33), we conclude directly that $h_1 = 0$ which satisfies the $\theta\varphi$-component. The $t\varphi$-component of Eq. (5.33) and Eq. (5.34) form a set of two coupled differential equations for the metric function h_0 and the scalar perturbation $\delta\phi(r)$. These equations can be written schematically as

$$\mathcal{D}_1^{(2)} h_0 - \frac{6\alpha_{\rm CS} M}{r^3} \mathcal{D}_1^{(1)} \delta\phi = 0\,, \tag{5.42}$$

$$\mathcal{D}_2^{(2)} \delta\phi + \frac{6\alpha_{\rm CS} l(l+1) M}{r^4(2M-r)} \mathcal{D}_2^{(1)} h_0 = 0\,, \tag{5.43}$$

where we define the differential operators

$$\mathcal{D}_1^{(2)} := \frac{d^2}{dr^2} + \frac{l(l+1)r - 4M}{r^2(2M-r)}\,, \tag{5.44}$$

$$\mathcal{D}_1^{(1)} := \frac{d}{dr} - \frac{1}{r}\,, \tag{5.45}$$

$$\mathcal{D}_2^{(2)} := \frac{d^2}{dr^2} + \frac{2M-2r}{2Mr-r^2}\frac{d}{dr} + \frac{l^2+l}{2Mr-r^2}\,, \tag{5.46}$$

$$\mathcal{D}_2^{(1)} := \frac{d}{dr} - \frac{2}{r}\,. \tag{5.47}$$

In order to find the TLNs we need to solve the coupled system of Eqs. (5.42)–(5.43). This system can be solved numerically for a generic coupling $\zeta_{\rm CS} := \alpha_{\rm CS}/M^2$ or perturbatively when $\zeta_{\rm CS} \ll 1$. We start by discussing the numerical method to compute the TLNs.

The perturbed functions can be written as

$$h_0 = h_0^{\rm div} + h_0^{\rm dec}\,, \tag{5.48}$$

$$\delta\phi = \delta\phi^{\rm div} + \delta\phi^{\rm dec}\,, \tag{5.49}$$

where the first terms on the right-hand side of the equations are divergent at large distances and the second terms at the right-hand side decay at large distances. In the asymptotic limit these terms can be written as an expansion series,

$$h_0^{\rm div} \sim \frac{2}{3l(l-1)} r^{l+1} \sum_{i=0}^{\infty} \frac{a_h^{(i)}}{r^i}\,, \tag{5.50}$$

$$\delta\phi^{\rm div} \sim -\frac{2}{l(l-1)} r^{l} \sum_{i=0}^{\infty} \frac{a_h^{(i)}}{r^i}\,, \tag{5.51}$$

$$h_0^{\rm dec} \sim \frac{2}{l}\sqrt{\frac{4\pi}{2l+1}}\, r^{-l} \sum_{i=0}^{\infty} \frac{b_h^{(i)}}{r^i}\,, \tag{5.52}$$

$$\delta\phi^{\rm dec} \sim 2\sqrt{\frac{4\pi}{2l+1}}\, r^{-l} r^{-(l+1)} \sum_{i=0}^{\infty} \frac{b_h^{(i)}}{r^i}\,, \tag{5.53}$$

where the subdominant coefficients in the series can be related with the dominant coefficients by virtue of the field equations (5.42)–(5.43). The factors in front of the diverging series were introduced in order to identify the dominant coefficients $a_h^{(0)}$ and $a_\phi^{(0)}$ as the amplitudes of the axial-type gravitational field and the amplitude of the external scalar field, respectively.

In general, the diverging series for h_0 will contain terms proportional to r^{-l} and similarly, $\delta\phi^{\rm div}$ will contain terms proportional to $r^{-(l+1)}$ such that the diverging and decaying series are mixed. This mixing will introduce difficulties in the extraction of the Love number due to certain ambiguities that emerge in the definition of the multipole moments.

In order to circumvent this problem we focus on the simpler case and impose that there is no scalar tidal field,

$$\mathcal{E}^S = 0\,. \tag{5.54}$$

This simplification allow us to write the diverging series in a closed form, in particular, for $l = 2$, we can write the divergent series for h_0 as,

$$h_0^{\mathrm{div}} = \frac{1}{3}\mathcal{B}_2 r^3 - \frac{2}{3} r^2 \mathcal{B}_2 M \equiv \frac{\mathcal{B}_2}{3} r^3 f\,, \tag{5.55}$$

with $f \equiv 1 - 2M/r$. This expansion does not contain terms that decay with $1/r^2$ or faster and therefore we identify $b_h^{(0)} \equiv S_2$. Interestingly, this is the same expression that we found for GR.

This imposition implies that we will not consider perturbations to the spacetime caused by a scalar tidal field and therefore it makes no sense to compute the scalar Love number as defined in expression (1.13).

We now integrate numerically Eqs. (5.42)–(5.43) imposing regular boundary conditions at the event horizon and match the resulting solution with the asymptotic behavior described by Eqs. (5.50)–(5.53) to identify the current quadrupole moment and axial-type tidal field. Using the second expression of (1.10) we are able to compute the axial-type TLNs for a BH in Chern-Simons gravity. In contrast with the results obtain for GR we conclude that, for finite values of the coupling parameter, the current quadrupole moment does not vanish and we obtain non-zero TLNs.

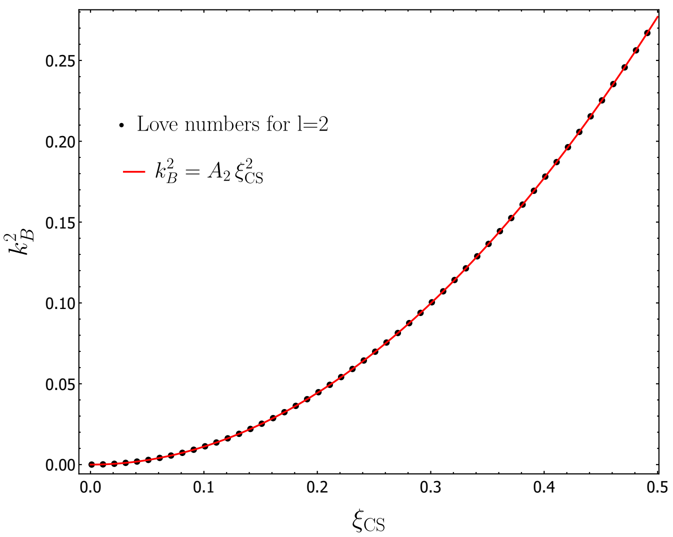

Figure 5.1: Axial-type tidal Love numbers (TLNs) of a BH in Chern-Simons gravity for $l = 2$ perturbations calculated for different values of the adimensional coupling constant $\xi_{\mathrm{CS}} := \alpha_{\mathrm{CS}}/M^2$. The dots correspond to the values obtained directly from a numerical integration and the line corresponds to the fitted funtion. The fit yields the parameter $A_{\mathrm{CS}} = 1.11$. We verify that the axial-type TLNs of a BH in Chern-Simons gravity are well-fitted by a quadratic expression of the coupling constant.

With the purpose to determine the behavior of the TLN, we plotted the $l = 2$ axial-type gravitational TLNs for different values of the coupling parameter ζ_{CS}. We fitted a power law function of ζ_{CS} to the numerical values and found that the TLN grows quadratically with the coupling parameter α_{CS},

$$k_2^B = 1.11 \zeta_{\mathrm{CS}}^2\,, \tag{5.56}$$

where the fit's accuracy is guaranteed by a relative error $\epsilon_k \lesssim 0.1\%$ defined by

$$\epsilon_k := \left| \frac{k_{\mathrm{fit}} - k_{\mathrm{num}}}{k_{\mathrm{fit}}} \right|\,, \tag{5.57}$$

where $k_{\rm fit}$ is the value of k_2^B using Eq. (5.56) and $k_{\rm num}$ is the value calculated numerically. In Fig. 5.1 present the results for the axial-type gravitational TLN using the numerical method.

This analysis leads to the conclusion that the gravitational axial-type TLNs of a nonrotating BH in modified Chern-Simons gravity are non-zero and behave according to Eq. (5.56).

The same results can be obtained by an analytical method developed in Ref. [1]. This approach consists in solving Eqs. (5.42)–(5.43) perturbatively by expanding the metric and scalar perturbations as,

$$h_{\mu\nu} = h^{(0)}_{\mu\nu} + \zeta_{\rm GB}^2 h^{(2)}_{\mu\nu} + \mathcal{O}(\zeta_{\rm GB}^3)\,, \tag{5.58}$$

$$\delta\Phi = \zeta_{\rm GB}\delta\phi^{(1)} + \mathcal{O}(\zeta_{\rm GB}^2)\,. \tag{5.59}$$

The advantage of employing this perturbative approach is that the equations decouple from each other. To $\mathcal{O}(\zeta_{\rm CS}^1, \epsilon^1)$, the scalar field equation reads

$$\mathcal{D}_S^{(2)}\delta\phi^{(1)} = \frac{12\mathcal{B}_2 M}{r^2(r-2M)}\,, \tag{5.60}$$

where

$$\mathcal{D}_S^{(2)} := \frac{d^2}{dr^2} - \frac{2(M-r)}{r^2-2Mr}\frac{d}{dr} - \frac{6}{r^2-2Mr}. \tag{5.61}$$

The solution which is regular at $r = 2M$ reads

$$\delta\phi^{(1)} = \frac{1}{6}M^2\Big(-18\mathcal{B}_2(3(y-2)y+2)\,\text{polylog}\left[2, 1-\frac{y}{2}\right] + 3\mathcal{B}_2\left(36y + \pi^2(-3(y-2)y-2) - 54\right) + $$
$$+9\mathcal{B}_2\log\left(\frac{2}{y}\right)\left((-3(y-2)y-2)\log\left(\frac{2}{y}\right) - 12(y-1)\right) - 2\mathcal{E}_2^S(3(y-2)y+2)\Big) \tag{5.62}$$

where $\mathcal{E}_2^S$ is the amplitude of the (quadrupolar) scalar tidal field as defined in Eq. (3.19) and $y := r/M$. In Chern-Simons gravity, compact objects can possess scalar charge and would naturally produce a scalar tidal field, however, here, we shall focus on the simpler case where there is no scalar tidal field, $\mathcal{E}_2^S = 0$, due to difficulties described previously.

Using the previous solution, the $\mathcal{O}(\zeta_{\rm CS}^2, \epsilon^1)$ equation for the axial perturbation is an inhomogeneous differential equation that reads

$$\mathcal{D}_A^{(2)} h_0^{(2)} = \mathcal{S}_A^{(2)}\,, \tag{5.63}$$

with

$$\mathcal{D}_A^{(2)} := \frac{d^2}{dr^2} + \frac{6r-4M}{r^2(2M-r)}\,, \tag{5.64}$$

$$\mathcal{S}_A^{(2)}(r) := \frac{3\mathcal{B}_2 M}{(y-2)y^4}\Big(-6(y-2)\left(3y^2-2\right)\text{polylog}\left[2, 1-\frac{y}{2}\right] + (y-2)\Big(\pi^2\left(2-3y^2\right) + \tag{5.65}$$
$$+3\log\left(\frac{2}{y}\right)\left(\left(2-3y^2\right)\log\left(\frac{2}{y}\right) + 6(y-2)y - 8\right) + 36y + 18\Big) + 6y(3(y-2)y+2)\log\left(\frac{y}{2}\right)\Big)\,,$$

This inhomogeneous differential equation can be solved using the usual Green's function method. The solution with the appropriate boundary conditions takes the form of

$$h_0^{(2)}(r) = \frac{\Psi_+(r)}{W}\int_{2M}^{r} dr' \mathcal{S}_A^{(2)}(r')\Psi_-(r') + \frac{\Psi_-(r)}{W}\int_r^{\infty} dr' \mathcal{S}_A^{(2)}(r')\Psi_+(r')\,, \tag{5.66}$$

where the two linearly independent solutions of the homogeneous problem read

$$\Psi_-(r) = C_1 r^2 (r-2M)\,, \tag{5.67}$$

$$\Psi_+(r) = \frac{C_2}{24M^5 r}\left(2M\left(2M^3+2M^2r+3Mr^2-3r^3\right)+3r^3(2M-r)\log\left(1-\frac{2M}{r}\right)\right)\,, \tag{5.68}$$

and

$$W \equiv \Psi_+'(r)\Psi_-(r) - \Psi_-'(r)\Psi_+(r) = C_1 C_2\,, \tag{5.69}$$

is the Wronskian.

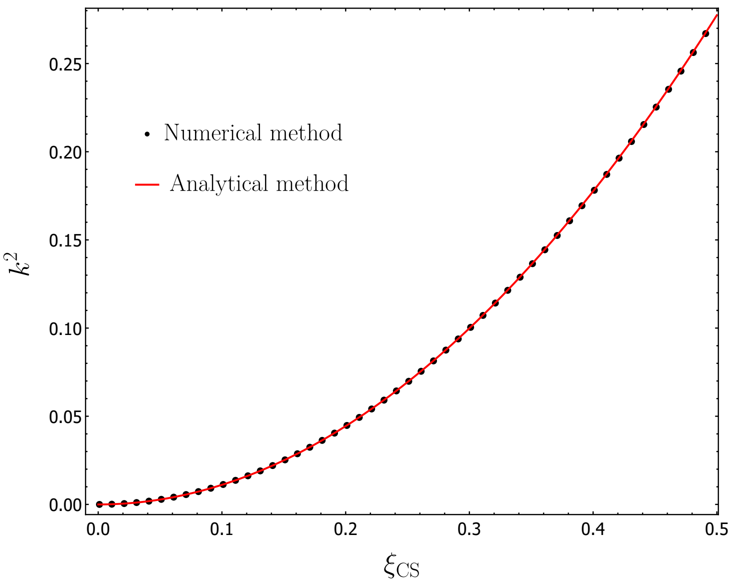

Figure 5.2: Axial TLNs of a BH in Chern-Simons gravity for $l = 2$ perturbations calculated for different values of the coupling constant $\xi_{\rm CS} := \alpha_{\rm CS}/M^2$. The dots correspond to the values obtained directly from a numerical integration and the line corresponds to analytical result in Eq. (5.72). The agreement between the numerical and perturbative methods validates our results.

We notice that, in the absence of scalar tidal field, the source term, Ψ_- and Ψ_+ behave as

$$\begin{aligned}\mathcal{S}_A^{(2)} &\sim \mathcal{B}_2/r^5\,,\\ \Psi_-(r) &\sim C_1 r^3\,,\\ \Psi_+(r) &\sim -\frac{C_2}{5r^2}\end{aligned} \tag{5.70}$$

at large distances, the first integral in Eq. (5.66) converges, whereas the second integral does not contribute to the current quadrupole S_2. Fortunately, it is possible to compute these integrals in closed form and obtain the asymptotic behavior of $h_0^{(2)}(r)$:

$$h_0^{(2)}(r) \to -\frac{9\left(\mathcal{B}_2 M^5(8\zeta(3)-9)\right)}{5r^2} + \mathcal{O}(M^3/r^3)\,, \tag{5.71}$$

where ζ is the Riemann's ζ function. By comparing the above result with Eq. (3.16) and using Eq. (1.10), it is straighforward to obtain

$$k_2^B = \frac{9(8\zeta(3)-9)}{5}\zeta_{\rm CS}^2 \approx 1.11\zeta_{\rm CS}^2\,, \tag{5.72}$$

which is in perfect agreement with the numerical result obtained previously. This leads to the conclusion that, in the Chern-Simons gravity, the axial-type TLN is nonzero and proportional to $\zeta_{\rm CS}^2$, as expected. The comparison between the numerical and perturbative methods is represented in Fig. 5.2

Therefore, by proving that the axial-type TLNs of BHs in Chern-Simons gravity are non-zero, we provided an example that TLNs of BHs can break the "zero-Love" rule in modified gravity theories, motivating the importance to study them. This result was calculated for the simpler case where the scalar external field vanishes, however, the more more general case where the system is allowed to have scalar tides remains open. Furthermore, this thesis also study the dependence of the Love numbers with the coupling constant of the theory, proving that the Love numbers grow quadratically with $\alpha_{\rm CS}$ according to Eq. (5.56). In principle, this dependence would motivate the possibility of constraining the dynamical Chern-Simons gravity with independent GW measurements of the axial-type TLN, however, since this term is subdominant in the GW signal, its detectability is much more challenging.

6 Conclusions

In this thesis we reviewed the relativistic theory of TLNs developed in the last few years and we extended it to account for more general astrophysical scenarios. The existing definition of gravitational TLNs was complemented with the introduction of two other classes of Love numbers: the EM and scalar TLNs. With these new sets of Love numbers the former relativistic definition can now be applied to scenarios in which the gravitational sector is coupled with EM and/or scalar sectors of the theory, for example ECOs and modified gravity.

In the first part of this work we studied the Newtonian theory of TLNs, where we applied it to the case of a fluid star following Ref. [2]. This calculation was performed for stars with a polytropic equation of state and for stars with homogeneous density. For the first case, we used numerical integration techniques to solve the final differential equation, while in the second case a complete analytical solution was found. Furthermore, we found that in both cases the TLNs are all positive.

Thereforward, we started the discussion of relativistic TLNs with the case of non-rotating, uncharged BHs within GR. As expected from the literature [16, 17] we found that the TLNs are zero. This "zero-Love" property of BHs poses an intriguing result and motivates the search for possible transgressions of this property [23]. Our purpose in this thesis is to check whether this property still applies to modified theories of gravity or if in these cases the BHs have non-vanishing TLNs.

One of the simplest modifications to GR arises in the Einstein-Maxwell theory of gravity. In light of this theory, we studied perturbations of uncharged and charged BHs. In the first case the gravitational sector is decoupled from the EM sector of the theory and, applying an almost identical treatment as the one used for GR, we concluded that all the TLNs (gravitational and EM) are zero. The second case is more complex due to the mixing of the two sectors of the theory and because of that must be handled with care. Considering this, for the perturbations of charged BHs we studied purely gravitational tidal environments (i.e. no EM tidal fields) and concluded that the gravitational TLNs are zero. Thus, our work showed that charged BHs, at least in this simpler case, exhibit the "zero-Love" property found for uncharged BHs. For more general tidal environments, intriguing questions about the TLN's definition and the tidal field contributions arise, making this an interesting topic to analyze in a future work.

Another important part of this research was regarding TLNs in modified gravity theories where the action is coupled with scalar degrees of freedom, with focus on the cases of scalar-tensor gravity and quadratic gravity. As an example of scalar-tensor gravity we studied the TLNs of a BH in Brans-Dicke gravity. We found that TLNs (gravitational and scalar) are zero and therefore BHs in this theory continue to present the "zero-Love" property. Within quadratic gravity, we studied in specific the Chern-Simons family of solutions. In Chern-Simons gravity the polar sector of the perturbation equations is identical to the one in GR and we concluded that the polar-type TLNs are zero. However, the axial case is more intriguing due to the coupling with the scalar perturbations. We found that, in absence of scalar tidal fields, the axial-type TLNs are non-zero presenting the first example of a violation of the "zero-Love" rule. In the more general case, in which we allow the existence of scalar tidal fields, we verify that there seems to exist a non-trivial mixing between subdominant terms of tidal fields and terms proportional to the body's response. This problem is qualitatively similar to the one found for charged BHs and poses the same problems of TLN's definition and interpretation.

The extended relativistic theory of TLNs present here can be applied several examples ECOs. In this thesis, we dedicated our research to the TLNs of wormholes. Perhaps not so suprisingly we found that the they are non-zero, even in the limit where the wormhole is almost as compact as a BH. This procedure can be applied to other types of ECOs, for example gravastars and boson stars [1]. Another interesting result is the fact the wormhole's TLNs are always negative, contrasting with the well-known TLNs of less exotic objects (e.g. Newtonian fluid stars and NSs).

The fact that BHs in some modified gravity theories have non-zero TLNs may provide new methods to test gravity and the "zero-Love" property of BHs. This work can also be applied for future search of ECOs using GW signals. As shown in this thesis and in Ref. [1], the measurement of TLNs can be a valid method of ECOs' identification. In GW analysis, an ECO signal is typically distinguished from a BH signal through the final ringdown modes (where the presence of the object's surface is more clear). However, here, we show that ECOs, even in ultracompact configurations, have a different tidal response to the one of BHs, which may indicate the possibility of testing the nature of the GW sources through the analysis of the late stage inspiral.

We verified also that, in more complex gravity theories and tidal environments, the multipole analysis is non-trivial. In this situation we cannot distinguish with clarity the tidal field contributions from the body's multipolar response. Perhaps more concerning is the fact that, in these scenarios, there are lower order multipoles (i.e. more dominant at large distances) that seem to appear. It is necessary in future works to focus on this problem since it poses an obstacle in defining unambiguously the TLNs.

There are several possible extensions for this work once the previous problem is solved. The first and most pressing one is to analyze the more general case, where the system is allowed to have other types of tidal fields. The analysis of the TLNs of a BH in EdGB gravity is a topic started with this thesis and will be discussed in future works [106]. To better relate with realistic astrophysical scenarios, it would be interesting to study the TLNs of rotating BHs in modified gravity. Furthermore, since NSs binaries are promising sources of GWs, the calculation of TLNs of NSs in modified gravity theories is important for future GW research and could be an interesting research topic.

In this work we restricted ourselves to the case of static tides, however, at late stages of the inspiral, this approximation is no longer valid and we should assume a dynamical tidal regime. Recently, it was shown that the presence of dynamic tides can have stronger influence on the TLN when compared with static tides [107]. It is therefore necessary to develop this theory of dynamic tides to the cases already considered.

The same problems studied here for TLNs can be studied for surficial Love numbers, a set of parameters that characterize the induced curvature change on the body's surface due to external tidal fields [69]. In Newtonian gravity and GR, a useful mathematical relation exists between the tidal and surficial Love numbers and it would be interesting to study this relation for BHs in modified gravity and ECOs.

Finally, future GW measurements will provide us important data that encodes information about the internal structure of these GW sources. These data should be analyzed in order to compare the theoretical results obtained for the Love numbers, in this and upcoming researches, with the experimental results obtained from the detectors. Future data treatment is essential in the investigation of new physics around BHs, in the search for ECOs and to constrain and test gravity in the strong-field regime.

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A Notation

During the course of this work we will mention some tensors that contain several indices and, for that reason, it is useful to introduce the multi-index notation. In this notation, a tensor A with l indices is written as,

$$A^L \equiv A^{a_1 \ldots a_l}. \tag{A.1}$$

Naturally, in this abbreviate notation, a product $A_L B^L$ contains an implicit summation over all the repeated indices.

We shall represent the radial vector with length r by $\boldsymbol{x}$ and the unit radial vector by $\boldsymbol{n}$. It sometimes more useful to decompose quantities in a base that involves tensorial combinations of $\boldsymbol{n}$ instead of the usual spherical harmonic decomposition. Each of this quantities constructed from $\boldsymbol{n}$ as the property of being symmetric under the exchange of any two indices and also tracefree in each pair of indices. Tensors with this construction are called symmetric and tracefree and will be represented with angular brackets enveloping its indices.

We shall also abbreviate the product of l radial vectors by,

$$n_L \equiv n_{i_1 \ldots i_l}\,, \qquad x_L \equiv r^l n_L \equiv x_{i_1 \ldots i_l}\,. \tag{A.2}$$

We remark some useful properties for the following deductions: The product of an arbitrary tensor A_L with an STF tensor $B^{\langle L \rangle}$ as the property

$$A_L B^{\langle L \rangle} = A_{\langle L \rangle} B^{\langle L \rangle}\,, \tag{A.3}$$

and the product of two unity STF vectors satisfy the following relations,

$$n_{\langle L \rangle} k^{\langle L \rangle} = \frac{l!}{(2l-1)!!} P_l(\boldsymbol{n} \cdot \boldsymbol{k})\,. \tag{A.4}$$

$$n_{\langle L \rangle} k^{\langle jL \rangle} = \frac{l!}{(2l+1)!!} \left[\frac{dP_{l+1}}{d(\boldsymbol{n} \cdot \boldsymbol{k})} k^j - \frac{dP_l}{d(\boldsymbol{n} \cdot \boldsymbol{k})} n^j \right]\,. \tag{A.5}$$

It is convenient to have tools to change from an STF base to a spherical harmonic decomposition.

This can be done by noticing that the STF tensors $n^{\langle L\rangle}$ may be decomposed in a spherical harmonic base as,

$$n^{\langle L\rangle} := \frac{4\pi l!}{2l+1!!}\sum_{m=-l}^{l}\mathcal{Y}_{lm}^{\langle L\rangle}Y_{lm}(\theta\varphi)\,, \tag{A.6}$$

where $\mathcal{Y}_{lm}^{\langle L\rangle}$ is a constant STF tensor that satisfies the $\mathcal{Y}_{l,-m}^{\langle L\rangle} = (-1)^m\mathcal{Y}_{lm}^{*\langle L\rangle}$. The decomposition of spherical harmonics in an STF basis (i.e the inverse relation of Eq. (A.6)) is,

$$Y_{lm}(\theta,\varphi) = \mathcal{Y}_{lm}^{*\langle L\rangle}n_{\langle L\rangle}\,, \tag{A.7}$$

The relation between the multipole moments written in STF base and in spherical harmonic decomposition is given by

$$I_{lm} = \mathcal{Y}_{lm}^{\langle L\rangle}I_{\langle L\rangle}\,, \tag{A.8}$$

$$I^{\langle L\rangle} = \frac{4\pi l!}{(2l+1)!!}\sum_{m=-l}^{l}\mathcal{Y}_{lm}^{*\langle L\rangle}I_{lm}\,. \tag{A.9}$$

When a body possess symmetry with respect to an axis $\vec{k}$, the properties of STF tensors imply that the multipole moment $I^{\langle L\rangle}$ is proportional to $k^{\langle i_1\cdots}k^{i_l\rangle}$,

$$I^{\langle L\rangle} = \alpha_l k^{\langle i_1\cdots}k^{i_l\rangle}\,, \tag{A.10}$$

where α_l is a proportionality constant that must be determined. Substituting $I^{\langle L\rangle}$ by Eq. (A.9), and aligning the z direction of our system with the $\vec{k}$ such that the only non-vanishing moments correspond to $m=0$, we obtain,

$$\frac{4\pi l!}{(2l+1)!!}\mathcal{Y}_{l0}^{*\langle L\rangle}I_{l0} = \alpha_l k^{\langle i_1\cdots}k^{i_l\rangle}\,. \tag{A.11}$$

We can now multiply on both sides by $n_{\langle L\rangle}$, use property (A.7) on the left-hand side and property (A.4) on the right-hand side to write Eq. (A.11) as

$$\frac{4\pi l!}{(2l+1)!!}Y_{l0}I_{l0} = \alpha_l\frac{l!}{(2l-1)!!}P_l(\cos\theta)\,. \tag{A.12}$$

Recalling the relation between the spherical harmonics and the associated Legendre polynomial,

$$Y^{lm}(\theta,\varphi) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}P_{lm}(\cos\theta)e^{im\varphi}\,, \tag{A.13}$$

and substituting in Eq. (A.12), we get

$$\alpha_l = \sqrt{\frac{4\pi}{2l+1}}I_{l0}\,. \tag{A.14}$$

Substituting the previous expression in Eq. (A.10) we obtain the decomposition of the multipole moment in the STF notation

$$I^{\langle L\rangle} = M_l k^{\langle i_1\cdots}k^{i_l\rangle}\,. \tag{A.15}$$

where we defined the quantity

$$M_l \equiv \sqrt{\frac{4\pi}{2l+1}} I_{l0} \,. \tag{A.16}$$

The multipole moments in the Geroch-Hansen normalization are related with the multipole moments $I^{\langle L \rangle}$ as,

$$M^{\langle L \rangle} = (2l-1)!! I^{\langle L \rangle} \,, \tag{A.17}$$

and therefore, combining Eq. (A.17) with relation (A.15) we obtain the Eqs. (3.4)–(3.5).