Tidal Deformability of Gravitational Atoms

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Declaration

I declare that this document is an original work of my own authorship and that it fulfills all the requirements of the Code of Conduct and Good Practices of the Universidade de Lisboa.

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Resumo

Na última década, a deteção de ondas gravitacionais (OG) provou ser uma ferramenta útil para estudar objetos astrofísicos, nomeadamente sistemas com um campo gravítico forte como os buracos negros (BNs). Espera-se que esta revele utilidade experimental para os modelos de extensão do Modelo Padrão como a natureza de partículas ultraleves e Matéria Escura.

Se existirem campos bosonicos ultraleves em torno de BNs, o processo de superradiância, que é um mecanismo de amplificação de ondas, pode levar a uma instabilidade que causa o seu crescimento exponencial em amplitude e extrai massa e momento angular do BN até que se forme um condensado (ou nuvem) que roda com o BN. Esta nuvem depois emite sinais de OGs que podem ser analisados e transmitir conhecimento em relação a nova física.

Nesta tese focamo-nos na deformabilidade de maré destes BNs ”vestidos” com um campo escalar através do cálculo de alguns dos seus números de Love de maré (NLM). Estes coeficientes quantificam a resposta induzida na estrutura multipolar de um objeto devido à sua interação gravítica com outro corpo maciço. Fazemos isto nas aproximações de limite Newtoniano, onde se assume que o campo gravítico é fraco e os objetos têm velocidades baixas, e de pequeno-acoplamento (entre o campo escalar e o BN).

Visto que foi provado que os BNs têm NLM nulos, qualquer deteção de OGs cuja assinatura implique NLM não-nulos pode indicar a existência de partículas ultraleves.

Palavras-chave: Buracos negros, Superradiância, Campos bosonicos ultraleves, Números de Love de maré.

Abstract

In the last decade, gravitational wave (GW) detection has proved to be a useful tool to study astrophysical objects, namely systems with a strong gravitational field like black holes (BHs). It is expected that it will shed some light into models which extend the Standard Model such as the nature of ultralight particles and Dark Matter.

If there exist ultralight bosonic fields surrounding BHs, the process of superradiance, which is a mechanism of wave amplification, may lead to an instability causing their exponential growth in amplitude and extracting mass and angular momentum from the BH until a co-rotating condensate (or cloud) is formed. This cloud then emits GW signals which can be analysed and provide insight into new physics.

In this thesis we focus on the tidal deformability of these BHs ”dressed” with a scalar field through the computation of some of their tidal Love numbers (TLNs). These are coefficients which quantify the induced response in the multipolar structure of an object from its gravitational interaction with another massive body. We do this in the Newtonian limit, where one assumes a weak gravitational field and slow-motion for the objects, and small-coupling (between the scalar field and the BH) approximations.

Since it has been proved that BHs have zero TLNs, any GW detection whose signature leads to non-vanishing TLNs may indicate the existence of ultralight particles.

Keywords: Black holes, Superradiance, Ultralight bosonic fields, Tidal Love numbers.

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Glossary

ACMC	Asymptotically Cartesian and mass centered
BH	Black hole
GW	Gravitational wave
STF	Symmetric and tracefree
TLN	Tidal Love number

Chapter 1

Introduction

In 1915, Einstein published his theory of General Relativity [1] which describes how the four-dimensional spacetime which we perceive is affected by matter through the field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{1.1}$$

where the left-hand side pertains to the geometry of spacetime through the metric tensor $g_{\mu\nu}$ (which then gives the Ricci tensor $R_{\mu\nu}$ and Ricci scalar R) and the right-hand side pertains to the matter involved in the problem through the energy-momentum tensor $T_{\mu\nu}$ (G is the universal gravitational constant and c is the speed of light). Its description is best understood under Wheeler's quote "Spacetime tells matter how to move; matter tells spacetime how to curve." [2].

One possible solution of these equations in vacuum (that is, when $T_{\mu\nu} = 0$) is the Kerr metric which describes the spacetime geometry around a spinning black hole (BH), or in other words, a Kerr BH. This object has some interesting properties like frame-dragging and the existence of an ergoregion. The former refers to the fact that any observer [1] with zero angular momentum at a finite distance is forced to co-rotate with the BH. The latter is defined as the region between the event horizon and an infinite-redshift surface (i.e. a surface such that any light ray emitted from it will be infinitely redshifted at infinity) called the ergosurface, located at a radial distance $r_{\text{ergo}} = M + \sqrt{M^2 - a^2 \cos^2\theta}$ where M is the mass of the BH, $a = J/(Mc)$ is the BH's spin parameter (proportional to its angular momentum J) and θ is the polar angle coordinate. The ergosurface is a static limit, which means that no static observer is allowed inside the ergoregion.

The combination of these two facts means that all observers inside the ergoregion are forced to rotate with the BH and, as Penrose showed in a famous thought experiment called the Penrose process, allows for the extraction of energy from it [3]. Although the Penrose process applies to particles, it was later discovered [4, 5] that there is a similar effect for the case of waves in what is now called rotational superradiance, whereby they become amplified.

As we shall explain in Chapter 2, this plays a very important role in the study of BHs surrounded by bosonic (that is, with integer spin) fields since an instability might be triggered which, through superradi-

[1] Having timelike four-velocity.

ance, leads to the formation of a cloud (composed by the mass of the field) around the BH, co-rotating with it. This effect has been theoretically shown to occur both for scalar (spin 0) fields [6, 7], vector (spin 1) fields [8–11] and even massive tensor (spin 2) fields [12–14]. Interestingly, in the non-relativistic limit, the eigenfunctions of this system are solutions to a Schrödinger-like equation and so it can be called a ”gravitational atom” [15].

The fact that the process then reaches a quasi-stationary state would seem to be in contradiction to the ”no-hair” theorems [16–19] which state that all Kerr BHs are only described by two quantities (mass and spin). For this reason, these systems are sometimes called ”hairy” or ”dressed” BHs. However, most of these configurations either eventually decay into a Kerr BH over very long timescales (as occurs for real fields, see e.g. Ref. [7]), or are outside the scope of the assumptions made in the non-hair theorems (as can happen for complex fields, see e.g Ref. [20]).

1.1 Motivation

1.1.1 Ultralight Particles

More than one century later, experimental discoveries are still being made which show just how useful and accurate General Relativity is in describing the universe, in the astrophysical context. One of these examples is the recent detection of gravitational waves (GWs) by the LIGO/Virgo collaboration [21–25] which confirms that spacetime may have periodic deformations, i.e. waves, which propagate at the speed of light and can contain information about the emitting system, just like electromagnetic waves. In particular, binaries of Neutron Stars and BHs make for interesting sources.

As it turns out, a bosonic cloud around a BH dissipates in time through the emission of GWs [7, 26, 27] and this provides an essential opportunity for detecting new particles [15, 28–30]. Let $m = \mu\hbar$ (where $\hbar$ is the reduced Planck constant) be the mass of the field and M the mass of the BH. The superradiant instabilities which form the cloud only have a relevant (i.e. sufficiently short) time scale when the gravitational coupling verifies $\mu M \lesssim 1$ [31]. This implies that we should expect the particles corresponding to the bosonic field to be ultralight (for astrophysically relevant time scales). Since all elementary bosons of the Standard Model are either massless or very massive, it is expected that new, unknown particles can be detected from this system. The range of masses one could potentially probe goes from 10^{-20} eV to 10^{-10} eV encompassing models from Quantum Chromodynamics axions [32–34] (at the upper end) to Dark Matter candidates [35–37] (at the lower end).

1.1.2 Tidal Love Numbers

We now make a small detour into introducing tidal effects. The closest consequence of tidal deformations to our daily lives is the sea tides. They are caused by inhomogeneities in the gravitational attraction of the Moon and Sun on the Earth. The quadrupolar shape of the tidal deformation leads to the twice-per-day phenomenon of low and high tides. Other examples are close binary-star systems, where the tidal deformation changes each star's gravitational potential, consequently leading to observable perturbations in the orbital motion of the system [38] and the ”tails” visible in galaxies [39].

The gravitational tidal Love numbers (TLNs) are quantities which provide insight into how an object is being tidally deformed by another. They are named after the British geophysicist A.E.H. Love (1863-1940), who introduced them in the 20th century [40] and are also known in the astronomical and celestial mechanics literature as ”apsidal constants”. In the case of General Relativity, it has been proved that the TLNs of isolated BHs are zero [41–44] whilst this is not the case for BHs surrounded by matter fields [27, 45].

As was seen for the case of Neutron Stars [46–48], GWs carry a signature of the TLNs of the emitting objects, hence non-vanishing TLNs may provide a way to confirm the detection of the ultralight particles we mentioned earlier. This is the main motivation for the work done in this thesis.

In Chapter 4, in order to study the tidal deformability of a gravitational atom, we consider a Kerr BH surrounded by a scalar cloud in the Newtonian limit. Hence, the BH is approximated by a point particle and the polar-type TLNs which will be defined in Chapter 3 are calculated, providing the analog of the Newtonian TLNs. These results are meant to be a stepping stone into a fully relativistic analysis to be done in the future.

1.2 Thesis Outline

The thesis is organized with two introductory chapters leading up to the final chapter where new results are presented. In Chapter 2 superradiance is the main topic, providing the explanation into how a system of a BH surrounded by a scalar cloud can be formed through instabilities. Section 2.1 serves both the purpose of giving an historical introduction into the topic and to reference some recent works of research. We will try to mention mainly the most famous or relevant results (for a more complete treatment, we refer the reader to [31]).

In Chapter 3 the concept of TLNs is presented in Newtonian gravity and General Relativity. This chapter sets the stage into how the calculations of the following chapter will be made. For more details regarding tidal deformations in Newtonian theory, we refer the reader to the excellent reference [38].

Finally, in Chapter 4 we present the main results of this thesis which are the computation of the TLNs of a BH surrounded by a scalar cloud in the Newtonian limit. We do this for two configurations of the background scalar field.

In regard to the unit systems, Section 2.2 is written with Planck units, Section 2.3 and Chapter 3 with geometrized units, Sections 4.1 and 4.2 use nonrelativistic units and finally, in Sections 4.3 through 4.5 we revert back to Planck units.

Chapter 2

Black Hole Superradiance

2.1 An Historical Overview of Superradiance

The origin of the term ”superradiance” goes back to 1954 when Dicke discussed coherent emission of radiation by a quantum gas [49]. He called this a ”super-radiant” gas. However, it is interesting to note that, before this time, a number of different works had already discussed phenomena which were later perceived to fall under the category of superradiance [31, 50]. Namely the Klein paradox [51] (due to the relation between pair production and superradiance [31]), the Vavilov-Cherenkov effect (discovered experimentally in 1934 and explained theoretically in 1937 [52]) and the anomalous Doppler effect [53]. In all of them there is some form of radiation enhancement or amplification (i.e. superradiance) given a certain inequality[1].

The Klein paradox consists of a beam of particles (originally only fermions were considered but works involving bosons later appeared, as in Ref. [31]) being scattered in a high enough potential barrier. We know today that it contained the first indication that superradiance is not possible for fermionic fields.

In the context of BH physics, the first works discussing superradiance (not counting the Penrose process, from 1969, since it was not discovered in the context of superradiance, as we have already discussed in the Introduction) only came in 1971 when Zel'dovich studied rotational superradiance [4, 5]. He showed that electromagnetic waves scattering off rotating absorbing surfaces may be amplified under certain conditions, one of them being

$$\omega < m\Omega \tag{2.1}$$

where ω is the frequency of the monochromatic wave, m the azimuthal number with respect to the rotation axis and Ω the angular velocity of the body.

Most surprisingly, spontaneous pair production of particles by a rotating body (when considering quantum effects) was predicted, eventually leading to Hawking's result on BH evaporation [54].

In the three following years, the possibility that scattered waves could extract rotational energy from BHs was used by Teukolsky and Press to do a quantitative analysis of ”superradiant scattering” and

[1] The Klein paradox is a case of superradiant amplification of current in a quantum mechanical context and the inequality involves the energy of the particles while the other two effects are classical examples of spontaneous superradiance (only one charged particle is considered) and the inequalities are written for the particle's velocity.

the possibilities of ”BH bombs” (assuming the BH is confined) and ”floating orbits” were discussed [55]. Along with [56], these works pioneer the use of the term ”superradiance” in the context of BHs. We note that in 1973, a proof also arose for the absence of superradiance for massless fermions [57] which was later generalized to the massive case [58, 59].

In the period of 1976-1980, the instability of a Kerr BH under massive scalar perturbations is discovered [60] and then formalized [61, 62]. We will explore the latter in the next section.

The next 20 years do not present noteworthy developments related to superradiance but are important mainly for the development of BH perturbation theory and the AdS/CFT duality conjecture, which will both be applied to this topic. Then, in the 2000s, there was a revival regarding studies of the superradiant instability starting with [63], followed by numerical calculations [64, 65] and leading to the prediction of the existence of ultralight fields around spinning BHs [15]. It was found that measurements of the spin and mass of BHs, along with GW observations, could be used to prove or disprove this hypothesis [15, 28–30] (for a more complete list of references on this subject see [31]). The possibility of ”hairy” (meaning, in this case, with matter fields around them) BHs has been studied extensively in recent decades and is still an active research topic.

In 2014 and 2015, there was a big leap forward when superradiance was shown to occur at the full nonlinear level [66] and when the simulated adiabatic evolution of the superradiant instability of the Kerr spacetime, in the presence of an accretion disk and GW emission, was found to lead to the growth of a scalar cloud and a subsequent depletion through the emission of GWs [6, 7]. A full nonlinear evolution of the superradiant instability within Numerical Relativity then followed in 2017 for the case of massive vector fields [9]. It found the growth of a vector cloud with the extraction of mass from the BH. In the following year, this cloud was shown to emit GWs (just as in the scalar case) [26], once again at the full nonlinear level.

The final noteworthy development, which was also a motivation for this thesis, was the recent study of the behaviour of these superradiant clouds under the influence of another body [67–70], namely with respect to tidal deformations [27, 71].

2.2 Superradiant Instabilities due to Massive Scalar Fields

For a more concrete understanding of this topic, we will now present the most studied case of superradiant instability, which is the propagation of massive scalar fields on a fixed Kerr geometry [62, 65]. Here we follow Ref. [62], but first we need to discuss some approximations. Throughout this section we use Planck units $G = c = \hbar = 1$.

Let us consider a complex scalar field Ψ with mass μ. The equation of motion resulting from the action

$$S = \int d^4x\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\nabla_\mu\Psi^*\nabla_\nu\Psi - \frac{\mu^2}{2}\Psi^*\Psi\right) \tag{2.2}$$

is the Klein-Gordon equation

$$\Box\Psi = \mu^2\Psi. \tag{2.3}$$

Since the energy-momentum tensor is given by a variation of this action with respect to the metric tensor, it will depend quadratically on the field. In order to use perturbation theory to analyse this system, we may neglect backreaction effects on the metric (because the fluctuations of order ϵ in Ψ induce fluctuations of order ϵ^2 in $T_{\mu\nu}$ and therefore, the Einstein field equations (1.1) remain unchanged). This assures us that the spacetime geometry remains fixed.

On the other hand, since we are interested in exploring condition (2.1) and it was shown in Refs. [62, 72] that in the low-frequency ($\omega M \ll 1$) and small-coupling ($\mu M \ll 1$) limits the differential equations for this system may be solved analytically, we will consider them to be satisfied. The latter limit is equivalent to $\lambda_C \gg r_g$ where $\lambda_C = 1/\mu$ is the (reduced) Compton wavelength of the particle associated to this field and $r_g = M$ is the gravitational radius of the BH. This is justified, not only due to simplifications in the calculations but also if one is considering that the fields are ultralight, such that the gravitational coupling with the BH is very small.

We may now proceed with the actual calculations. Our geometry is that of the Kerr spacetime (in Boyer-Lindquist coordinates)

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4aMr}{\Sigma}\sin^2\theta dt d\varphi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{2Ma^2 r}{\Sigma}\sin^2\theta\right)\sin^2\theta d\varphi^2 \tag{2.4}$$

where M is identified with the BH mass, a is related to the BH's angular momentum J by $J = aM$ and the auxiliary quantities are defined as $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$. Defining the radial coordinates $r_\pm = M \pm \sqrt{M^2 - a^2}$ (where the positive sign corresponds to the event horizon), one may also write $\Delta = (r - r_+)(r - r_-)$. Note that, in order for the quantity inside the square roots to be non-negative, $a \leq M$ must be satisfied.

In [73], it was shown that equation (2.3) is separable by the ansatz $\Psi = e^{-i\omega t}e^{im\varphi}R(r)S(\theta)$. Calculating the covariant derivatives from the metric given by (2.4), we may write the Klein-Gordon equation as a partial differential equation

$$\frac{\partial}{\partial r}\left(\Delta\frac{\partial\Psi}{\partial r}\right) - \frac{a^2}{\Delta}\frac{\partial^2\Psi}{\partial\varphi^2} - \frac{4Mra}{\Delta}\frac{\partial^2\Psi}{\partial\varphi\partial t} - \frac{(r^2+a^2)^2}{\Delta}\frac{\partial^2\Psi}{\partial t^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\Psi}{\partial\varphi^2} + a^2\sin^2\theta\frac{\partial^2\Psi}{\partial t^2} - \mu^2 r^2\Psi - \mu^2 a^2\cos^2\theta\Psi = 0 \tag{2.5}$$

and substitute the previous ansatz to obtain angular and radial differential equations

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS}{d\theta}\right) + \left[a^2(\omega^2 - \mu^2)\cos^2\theta - \frac{m^2}{\sin^2\theta} + \lambda\right]S = 0 \tag{2.6}$$

$$\Delta\frac{d}{dr}\left(\Delta\frac{dR}{dr}\right) + [\omega^2(r^2+a^2)^2 - 4aMrm\omega + a^2m^2 - \Delta(\mu^2r^2 + a^2\omega^2 + \lambda)]R = 0 \tag{2.7}$$

where λ is a separation constant. Note that (2.6) may be seen as an eigenvalue equation for λ. The eigenfunctions are known [74, 75] to be the spheroidal wave functions[2] $S_{\ell m}(\theta)$ where ℓ and m are

[2]The angular prolate spheroidal wave functions satisfy a differential equation which is equivalent to the differential equation for the angular oblate spheroidal wave functions under the transformations $c \to \pm ic$, using the notation in [74]. Therefore, also

integers and $|m| \leq \ell$ and $\ell \geq 0$. Although an analytic expression for the eigenvalues for all values of ℓ and m is not known, we may use the series expansion in powers of $a^2(\omega^2 - \mu^2)$ [74, 75], i.e., $\lambda = \ell(\ell+1)+\mathcal{O}(a^2(\omega^2-\mu^2))$. From our assumptions, we may write $\omega a \leq \omega M \ll 1$ and $\mu a \leq \mu M \ll 1$ and hence[3], $\lambda \simeq \ell(\ell+1)$. On the other hand, neglecting $a^2(\omega^2-\mu^2)$ in (2.6) results in the differential equation for the spherical harmonics, which means that in this limit, the spheroidal wave functions become the spherical harmonics.

After solving equation (2.7) with boundary conditions corresponding to exponentially decaying waves at infinity and ingoing waves at the event horizon (for details, we refer the reader to Appendix A), the following expression for the frequencies is obtained:

$$\begin{aligned}\omega \simeq \mu - \frac{\mu}{2}\left(\frac{M\mu}{n+\ell+1}\right)^2 + i\mu(\mu M)^{4\ell+4}\left(\frac{am}{M} - 2\mu r_+\right)\frac{2^{4\ell+1}(2\ell+1+n)!}{(n+\ell+1)^{2\ell+4}n!}\left[\frac{\ell!}{(2\ell)!(2\ell+1)!}\right]^2 \times \\ \times \prod_{j=1}^{\ell}\left[j^2\left(1-\frac{a^2}{M^2}\right)+\left(\frac{am}{M}-2\mu r_+\right)^2\right]\end{aligned} \tag{2.8}$$

where n is a non-negative integer (for a more thorough understanding of this quantity, see Appendix A). These frequencies describe the characteristic oscillation modes of a BH under an external perturbation and are called quasibound states, since they decay at infinity but, in general, are not pure bound states because they are complex. Remembering that the time dependence of the scalar field is $e^{-i\omega t} = e^{-i\omega_R t}e^{\omega_I t}$, then one sees that $\omega_I < 0$ corresponds to a damped, stable perturbation with damping time $\tau = -1/\omega_I$ and $\omega_I > 0$ corresponds to an unstable perturbation, i.e. an instability, where the amplitude of the field grows exponentially in a time scale given by $\tau_{\text{inst}} = 1/\omega_I$. Looking at (2.8), we see that the sign of this quantity is determined by the sign of $am/M - 2\mu r_+$. One finds that only modes with $m > 0$ are unstable (and this automatically requires that $\ell \geq 1$ because of the relation $|m| \leq \ell$). There are two variables which can tell us when an instability is occurring: ω_R or a.

First, by writing

$$\frac{am}{M} - 2\mu r_+ = 2r_+\left(\frac{am}{2Mr_+} - \mu\right) \tag{2.9}$$

and recognizing that $a/(2Mr_+) = \Omega_H$ is the angular velocity of a Kerr BH measured by a zero angular momentum observer (that is, an observer with a timelike four-velocity which falls into the BH with zero angular momentum) at the event horizon and that $\mu \sim \omega_R$ when $\mu M \ll 1$, we conclude that there are superradiant instabilities when $\omega_R < m\Omega_H$, which was precisely the Zel'dovich condition (2.1). Secondly, by writing $am/M - 2\mu r_+ = (a - 2\mu M r_+/m)m/M$, we see that instabilities are verified for $a > 2\mu M r_+/m$.

Furthermore, one may also analyse how strong the instabilities are by looking at the expression for the growth time τ_{inst}. This depends on the coupling μM, the dimensionless spin a/M and on the mode numbers (n, ℓ, m). The strongest instability occurs for $\ell = m = 1$, $n = 0$ and highly-spinning BHs [31, 65].

in that notation, one may choose the eigenfunctions of (2.6) to be either prolate as $S(\pm ic, \cos\theta)$ or oblate as $S(c, \cos\theta)$ with $c^2 = a^2(\omega^2 - \mu^2)$.

[3]Actually, this approximation is not entirely accurate. It was shown in [8] that it led [62] to a wrong result by a factor of 2. As one can see in Appendix A, the right procedure is to consider $\lambda = \ell'(\ell'+1)$ with $\ell' = \ell + \epsilon$ and then take the limit $\epsilon \to 0$.

2.3 Scalar Clouds

In this section we use geometrized units $G = c = 1$.

As we've shown in the previous section, the presence of a scalar field around a spinning BH may trigger an exponential growth of the field through instability processes. What are the consequences of this phenomenon?

Changing the radial coordinate to the tortoise coordinate $dr_* = (r^2+a^2)dr/\Delta$, one can show that (2.7) reduces to a Schrödinger-like equation with an effective potential [73]. An analysis of its shape shows that the superradiant amplification takes place inside the ergoregion and a well is present between the centrifugal barrier created by the event horizon and the ergoregion. Furthermore, from the exponential factor $e^{-\sqrt{\mu^2-\omega^2}r}$ in (A.5), a typical decay radius $r_d \sim 1/(\mu^2 M)$ is expected for the field, which means most of it is confined in the volume bounded by this radius.

The continuous process of amplification and reflection leads to the formation of a scalar "cloud" [7, 76] around the spinning BH through extraction of mass and angular momentum by the field. As this process is developing, the BH's spin will continually decrease until $a \sim 2\mu M r_+/m$ and the superradiance condition is saturated. Then, a quasi-stationary state is reached and the cloud is slowly dissipated through GW emission [7, 27]. In the limit of small-coupling $\mu M \ll 1$ (which we are considering), the time scale of the emission of GWs is much larger than the time scale of the superradiant instability such that the two processes may be considered separately [27].

Using (A.23) as $k \simeq M\mu^2/(n+\ell+1)$ and the relation between the confluent hypergeometric U and the generalized Laguerre polynomials we mentioned in Appendix A, the normalized radial eigenfunction (A.5) of the field may be written as

$$R_{n\ell}(r) = \left[\frac{n!}{2(n+\ell+1)(2\ell+n+1)!}\right]^{1/2} \left(\frac{2M\mu^2}{n+\ell+1}\right)^{3/2+\ell} r^\ell \exp\left(-\frac{M\mu^2}{n+\ell+1}r\right) L_n^{(2\ell+1)}\left(\frac{2M\mu^2}{n+\ell+1}r\right). \tag{2.10}$$

From this function, one finds that the cloud "peaks" at $r_c \sim (n+\ell+1)^2/(\mu^2 M)$ [28] far away from the event horizon, meaning that it is a good approximation to neglect the BH's spin. This allows us to use spherical harmonics instead of spheroidal harmonics $S_{\ell m}(\theta)e^{im\varphi} \simeq Y_{\ell m}(\theta,\varphi)$. Thus, within this approximation, the wave function which gives the time variation and spatial distribution of the cloud is

$$\Psi_{n\ell m}(t,r,\theta,\varphi) = e^{-i\omega_{n\ell m}t} R_{n\ell}(r) Y_{\ell m}(\theta,\varphi), \tag{2.11}$$

where $\omega_{n\ell m}$ are the eigenfrequencies (2.8). Note that, in general, the condensate is non-axisymmetric and oscillates, for example for the most unstable mode $\ell = m = 1$, according to (note that Re denotes the real part[4])

$$\text{Re}\Psi = \text{Re}(e^{-i\omega_{n\ell m}t} R_{n1} Y_{11}) \propto R_{n1}(r)\cos(\varphi - \omega_R t)\sin\theta. \tag{2.12}$$

Hence, from this expression one can see that after the superradiant process ends, the nonspherical monochromatic cloud will emit GWs with frequency $\omega_R \sim 2\mu$, typically with a wavelength smaller than

[4]Here we take the real part because there is only GW emission for real fields. In fact, the energy-momentum tensor of a complex scalar field has no time-dependence in the quasi-stationary regime and therefore does not radiate [20].

r_c ruling out the use of the quadrupole formula [7, 28]. In spite of this obstacle, the emission can be computed using BH perturbation theory [6, 30].

Therefore, under the approximation $\mu M \ll 1$, it was shown in Ref. [7] that for the mode $\ell = m = 1$, the energy and angular momentum fluxes carried away in the gravitational radiation is given by

$$\dot{E}_{\text{GW}} = \frac{484 + 9\pi^2}{23040} \left(\frac{M_c}{M}\right)^2 (M\mu)^{14}; \tag{2.13}$$

$$\dot{J}_{\text{GW}} = \frac{\dot{E}_{\text{GW}}}{\omega_R}, \tag{2.14}$$

where M_c is the mass of the cloud. An analysis containing more than one mode can also be found in [77].

Finally, according to the no-hair theorems [16–19], the cloud should dissipate and the system should give rise to a Kerr BH with a lower spin and no hair (if one considers that only a single mode with $\ell = m$ has formed during superradiance). The entire process we've described in this section may be viewed in Figure 2.1.

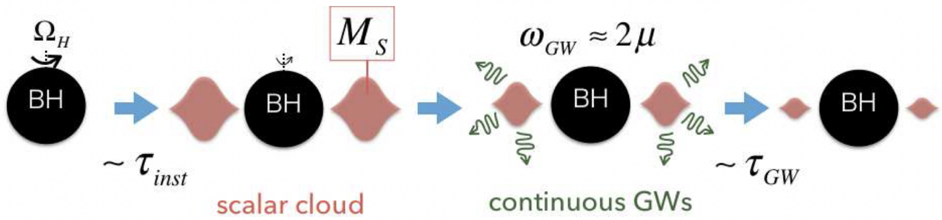

Figure 2.1: Sketch of the stages provoked by superradiant instabilities due to the fluctuations of a massive ultralight scalar field around a BH with angular velocity Ω_H at the event horizon: when the frequencies of the quasibound states (2.8) satisfy $\omega_R < m\Omega_H$, an exponential amplification of the field is triggered with a time scale τ_{inst}. As it is confined and forced to co-rotate with the BH, it successively extracts energy and angular momentum from it until a condensate (or "cloud") with mass M_{S} is formed. Then, a stationary state is reached which leads to the emission of GWs with frequency $\omega_{\text{GW}} \sim 2\mu$ in a time scale τ_{GW}. For example, $\tau_{\text{inst}} \sim 0.07$ yr [76] for a BH of mass $M_i \sim 10M_\odot$ before the superradiant process and spin $a \sim M_i$ and $\tau_{\text{GW}} \sim 6 \times 10^4$ yr [76] for a BH of mass $M_f \sim 10M_\odot$ after the superradiant process and spin $a \sim M_f$. Eventually, the cloud dissipates due to GW emission. Taken from [76].

Chapter 3

Tidal Love Numbers

Throughout this chapter we use geometrized units $G = c = 1$.

3.1 Tidal Deformations in Newtonian Gravity

3.1.1 Multipole Expansion of the Gravitational Potential

In the theory of Newtonian gravity, the fundamental equation governing the gravitational field is Poisson's equation

$$\nabla^2 U = 4\pi\rho \tag{3.1}$$

where $U(t, \mathbf{x})$ is the Newtonian potential and $\rho(t, \mathbf{x})$ is the density of the distribution of matter which is "creating" the gravitational field. The variables t and $\mathbf{x}$ represent time and the position vector (measured in an inertial frame of reference), respectively. The formal solution to this equation is

$$U(t, \mathbf{x}) = -\int \frac{\rho(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\, dx' \tag{3.2}$$

where the continuous integration variable $\mathbf{x}'$ describes the position within the volume of matter, measured with respect to a certain reference frame. This can be shown, for instance, by applying the operator ∇^2 to both sides of (3.2) and using the identity $\nabla^2(|\mathbf{x} - \mathbf{x}'|^{-1}) = -4\pi\delta(\mathbf{x} - \mathbf{x}')$ (derivable from the properties of the Dirac delta) to recover (3.1).

In astrophysics, one is often interested in bodies with some approximation to spherical symmetry, as in the case of planets or stars. One cannot assume complete spherical symmetry because the effects of rotation, interaction with other bodies and internal stresses often lead to some deviations. Therefore, we will use the method of multipole expansions, which is a useful and powerful tool in describing the gravitational field of these objects. This method consists of writing the angular part of the quantities involved in the system (in this case, the potential and density) as an infinite sum over multipoles (indexed by an integer l) and, if the deviations from spherical symmetry are small, the contributions from higher multipole moments are progressively smaller, which means one usually needs only a finite and small number of multipoles to accurately describe the system.

The main analytical tool we will use for this method is the spherical-harmonic functions $Y_{lm}(\theta, \varphi)$, labelled by an integer l which ranges from 0 to ∞ and another integer m, which ranges from $-l$ to l for each value of l. They also depend on the polar angle θ and the azimuthal angle φ which are spherical polar coordinates. Several useful identities regarding these functions are available on Appendix B.

Having this context in mind, one assumes that U and ρ are "well-behaved" functions so that we are allowed to write the following expansions[1]:

$$\begin{aligned}
\rho(t, r, \theta, \varphi) &= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \rho_{lm}(t, r) Y_{lm}(\theta, \varphi) \\
U(t, r, \theta, \varphi) &= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} U_{lm}(t, r) Y_{lm}(\theta, \varphi)
\end{aligned} \tag{3.3}$$

in which the expansion coefficients are given by

$$\begin{aligned}
\rho_{lm}(t, r) &= \int \rho(t, r, \theta, \varphi) Y^*_{lm}(\theta, \varphi)\, d\Omega \\
U_{lm}(t, r) &= \int U(t, r, \theta, \varphi) Y^*_{lm}(\theta, \varphi)\, d\Omega.
\end{aligned} \tag{3.4}$$

Writing the differential operator ∇^2 in spherical coordinates and using equations (3.3), (B.1) and (B.4) in (3.1), we are left with a differential equation in the radial coordinate for U_{lm} which can be solved by using the method of Green's function (we omit the details and refer the interested reader to [38]). This radial solution is given by

$$U_{lm}(t, r) = -\frac{4\pi}{2l+1} \left[r^l \int_r^{\infty} \frac{\rho_{lm}(t, r')}{r'^{l+1}} r'^2 dr' + \frac{1}{r^{l+1}} \int_0^r r'^l \rho_{lm}(t, r') r'^2 dr' \right] \tag{3.5}$$

and one then substitutes back in (3.3) to get the full solution for the potential:

$$U(t, r, \theta, \varphi) = -\sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \left[r^l \int_r^{R} \frac{\rho_{lm}(t, r')}{r'^{l+1}} r'^2 dr' Y_{lm}(\theta, \varphi) + \frac{Y_{lm}(\theta, \varphi)}{r^{l+1}} \int_0^r r'^l \rho_{lm}(t, r') r'^2 dr' \right]. \tag{3.6}$$

A few comments need to be made regarding this result. First, we truncated the first integral to an arbitrary radius R that surrounds the matter distribution. This is justified by the fact that the density ρ_{lm} is only non-vanishing in the interior of the body. On the exterior of the body, one should use

$$U_{\text{ext}}(t, r, \theta, \varphi) = -\sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{Y_{lm}(\theta, \varphi)}{r^{l+1}} \int_0^R r'^l \rho_{lm}(t, r') r'^2 dr', \tag{3.7}$$

since the mentioned integral vanishes. Lastly, we note that (3.6) could also have been obtained from (3.2) by decomposing $|\mathbf{x} - \mathbf{x}'|^{-1}$ in the spherical-harmonic basis and using the properties of the Legendre

[1] Here we use the fact that the spherical harmonics form a complete set of orthonormal functions - see Appendix B - and we are using spherical coordinates.

polynomials (once again, we refer the reader to [38] for further details).

We now define the multipole moments of the mass distribution:

$$I_{lm}(t) \equiv \int_0^R r'^l \rho_{lm}(t, r') r'^2 dr' = \int r^l \rho(t, \mathbf{x}) Y^*_{lm}(\theta, \varphi)\, d^3x. \tag{3.8}$$

In the second step we used equation (3.4). Note also that the domain of integration is taken over the volume occupied by the matter (hence the renaming of the integration variable). These quantities allow us to write the exterior potential as

$$U_{\text{ext}}(t, \mathbf{x}) = -\sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} I_{lm}(t) \frac{Y_{lm}(\theta, \varphi)}{r^{l+1}} \tag{3.9}$$

and will be important when discussing tidal deformations. The moment corresponding to $l = m = 0$ is known as the monopole moment, the moments corresponding to $l = 1$ are known as the dipole moments, the ones corresponding to $l = 2$ are called quadrupole, and so on. In the case of spherical symmetry, only I_{00} is non-vanishing and in the case of axial symmetry about the z axis, only the moments with $m = 0$ are non-vanishing, with U being independent from the azimuthal angle. Let us compute some moments for future convenience.

Defining the mass of a body as $M \equiv \int \rho(t, \mathbf{x}) d^3x$ and using (B.2) gives

$$I_{00} = \int \rho\, Y^*_{00}\, d^3x = \frac{1}{\sqrt{4\pi}} \int \rho\, d^3x = \frac{M}{\sqrt{4\pi}}. \tag{3.10}$$

Furthermore, defining the center-of-mass of this body (which we label by A) as $\mathbf{r}_A \equiv \int \rho(t, \mathbf{x}) \mathbf{x} d^3x/M$ and choosing a coordinate system whose origin coincides with the center-of-mass, $\int \rho(t, \mathbf{x}) \mathbf{x} d^3x = 0$ which implies, using (B.2) again,

$$I_{10} = \int \rho\, r Y^*_{10}(\theta)\, d^3x = \sqrt{\frac{3}{4\pi}} \int \rho\, r \cos\theta\, d^3x = \sqrt{\frac{3}{4\pi}} \int \rho\, z\, d^3x = 0 \tag{3.11}$$

$$I_{1\pm1} = \int \rho\, r Y^*_{1\pm1}(\theta, \varphi)\, d^3x = \mp\sqrt{\frac{3}{8\pi}} \int \rho\, r \sin\theta\, e^{\mp i\varphi}\, d^3x = \mp\sqrt{\frac{3}{8\pi}} \int \rho\, (x \mp iy)\, d^3x = 0 \tag{3.12}$$

from the definition of spherical coordinates. Substituting these four mass multipole moments in (3.9), we get

$$U_{\text{ext}}(t, \mathbf{x}) = -\frac{M}{r} - \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{I_{lm}(t)}{r^{l+1}} Y_{lm}(\theta, \varphi). \tag{3.13}$$

The monopole term (first term on the right-hand side) is the exterior potential due to a spherical body which proves, as we've mentioned before, that for bodies with spherical symmetry only I_{00} is non-zero.

3.1.2 Decomposition of the Exterior Potential in Symmetric Tracefree Tensors

We now introduce an alternative way of decomposing the gravitational potential which consists of tensorial combinations of the unit vector $\mathbf{n} \equiv \mathbf{x}/r$, instead of spherical harmonics. This will allow us to express the final results of Section 3.1 in a different way to the one we've been using so far and which are commonly used in the General Relativity literature.

These quantities are symmetric and tracefree (both properties being verified on all pairs of indices) and are hence known as symmetric tracefree tensors (or STF tensors). It turns out that they generate an irreducible representation of the rotation group and therefore, there is a one-to-one correspondence between them and the spherical-harmonic functions [78].

In order to do this, we must introduce the so-called multi-index notation to which the reader may find an explanation in Appendix B, as well as the required identities for our calculations. We also use the Einstein summation convention from now on.

Let us focus on the exterior part of U. Considering a point $\mathbf{x}$ outside the matter distribution such that $|\mathbf{x}| > |\mathbf{x}'|$, one may write the Taylor expansion:

$$\begin{aligned} \frac{1}{|\mathbf{x}-\mathbf{x}'|} &= \frac{1}{r} - x'^j \frac{\partial}{\partial x^j}\left(\frac{1}{r}\right) + \frac{1}{2!} x'^j x'^k \frac{\partial^2}{\partial x^j \partial x^k}\left(\frac{1}{r}\right) - \cdots \\ &= \frac{1}{r} - x'^j \frac{\partial}{\partial x^j}\left(\frac{1}{r}\right) + \frac{1}{2!} x'^{jk} \partial_{jk}\left(\frac{1}{r}\right) - \cdots \\ &= \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} x'^L \partial_L \left(\frac{1}{r}\right) \end{aligned} \tag{3.14}$$

where we used $r = |\mathbf{x}|$ and the notation $\partial_i \equiv \partial/\partial x^i$. Substituting in (3.2) gives

$$U_{\text{ext}}(t,\mathbf{x}) = -\sum_{l=0}^{\infty} \frac{(-1)^l}{l!} I^{\langle L\rangle} \partial_{\langle L\rangle}\left(\frac{1}{r}\right) \tag{3.15}$$

with the definition

$$I^{\langle L\rangle}(t) \equiv \int \rho(t,\mathbf{x}') x'^{\langle L\rangle} d^3x' \tag{3.16}$$

for the STF multipole moments, by analogy with (3.8). Note that we used the fact that $\partial_L(r^{-1})$ is a STF tensor and by equation (B.14), $x'^L \partial_{\langle L\rangle}(r^{-1}) = x'^{\langle L\rangle}\partial_{\langle L\rangle}(r^{-1})$. In fact, ∂_L is symmetric because the partial derivatives commute and $\partial_L(r^{-1})$ is tracefree because any pair of indices in $\partial_{j_1 j_2 \cdots j_l}$ which is contracted gives a laplacian operator (which commutes with ∂_j for any j) and $\nabla^2(r^{-1}) = -4\pi\delta(\mathbf{x})$ vanishes in the exterior region.

Finally, applying equation (B.15) to (3.15) results in

$$U_{\text{ext}}(t,\mathbf{x}) = -\sum_{l=0}^{\infty} \frac{(2l-1)!!}{l!} I^{\langle L\rangle} \frac{n_{\langle L\rangle}}{r^{l+1}} = -\frac{M}{r} - \sum_{l=2}^{\infty} \frac{(2l-1)!!}{l!} \frac{I^{\langle L\rangle}}{r^{l+1}} n_{\langle L\rangle}, \tag{3.17}$$

which are the equivalent of (3.9) and (3.13), respectively, in a STF tensor basis. Note that, when $l = 0$, the property $n!! = (n+1)!/(n+1)!!$ of the double factorial should be used for $(2l-1)!!$.

3.1.3 Tidal Environment

Since we've already discussed the gravitational field created by a single (spherical or nearly spherical) isolated body of mass M, the next step in the Newtonian theory of gravity is to consider this body to be subject to the effect of other massive bodies. Therefore, we move on to a system composed by a finite number of bodies, each one modelled by a perfect fluid, all in orbital motion around each other and surrounded by vacuum. They are considered isolated, in the sense that no matter is ejected from, nor accreted by, each body but they are all subject to the gravitational attraction of the others.

The configuration is characterized by two length scales: the typical size R of each body and the typical separation r between bodies. The fact that they are isolated means that $R \ll r$, which has the consequence that the orbital time scale τ_{orb} given by Kepler's third law is much greater than the time scale of hydrodynamical processes taking place within each body τ_{int}, i.e. $\tau_{\text{int}} \ll \tau_{\text{orb}}$ (for details regarding the estimations of these time scales, see [38]). Hence, the external, inter-body dynamics may be decoupled from the internal, intra-body dynamics.

Suppose we want to analyse the effect (tidal or otherwise) of the system on any particular body A. Then, we want to distinguish the contributions to the gravitational potential due to this body, U_A, and due to all the other bodies, V. The total gravitational potential is then given by

$$U = U_A + V, \tag{3.18}$$

where both components are solutions of the Poisson equation (3.1), given the respective mass densities of each body in the system (here B denotes all the bodies which are not A):

$$U_A(t, \mathbf{x}) = -\int_A \frac{\rho(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x', \quad V(t, \mathbf{x}) = -\sum_{B \neq A} \int_B \frac{\rho(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x'. \tag{3.19}$$

We call U_A the internal potential and V the external potential.

Let $\mathbf{r}_A$ be the center-of-mass of body A and $\mathbf{r}_B$ the center-of-mass of body B, measured in an inertial frame. Then, because of our previous assumption that each body is isolated, which can be expressed as $R_A \ll |\mathbf{r}_A - \mathbf{r}_B|$, where R_A denotes the characteristic size of body A, the fact that the external potential varies over the larger scale $|\mathbf{r}_A - \mathbf{r}_B|$ when compared to the variable $\mathbf{x}$, which varies over R_A, allows us to write the Taylor expansion at $\mathbf{r}_A$:

$$V(t, \mathbf{x}) = \sum_{l=0}^{\infty} \frac{1}{l!} (x - r_A)^L \partial_L V(t, \mathbf{r}_A) \tag{3.20}$$

where $r_A = |\mathbf{r}_A|$ and each derivative is evaluated at $\mathbf{x} = \mathbf{r}_A$. If we fix our reference frame so that the origin coincides with this center-of-mass (meaning $\mathbf{r}_A \equiv \mathbf{0}$), then this equation becomes

$$\begin{aligned} V(t, \mathbf{x}) &= \sum_{l=0}^{\infty} \frac{1}{l!} x^L \partial_L V(t, \mathbf{0}) \\ &= V(t, \mathbf{0}) + x^i \partial_i V(t, \mathbf{0}) + \sum_{l=2}^{\infty} \frac{1}{l!} \mathcal{E}_L(t) x^L \end{aligned} \tag{3.21}$$

with the definition

$$\mathcal{E}_L(t) \equiv \partial_L V(t, \mathbf{0}). \tag{3.22}$$

These quantities are named tidal moments and serve to specify the external tidal environment to which body A is subjected. Since Laplace's equation $\nabla^2 V = 0$ is satisfied within the volume occupied by body A (because the density of each body B is zero in this region), by the same arguments we used for $\partial_L(r^{-1})$ after equation (3.16), one concludes $\partial_L V$ is STF, which implies that $\mathcal{E}_L$ is also a STF tensor.

Besides Poisson's equation, A also obeys some differential equations of fluid dynamics, which we will not discuss here (for details see [38]) and in both these equations and Poisson's equation, all terms involving V are either the gradient of V or the laplacian operator of V. Therefore, the first term of the series in (3.21), corresponding to $l = 0$, may be discarded for being spatially constant and playing no part in the dynamics of the system. From Euler's equation (one of the previously mentioned equations from fluid dynamics), one finds that there is a fictitious force originating from the effective potential $V - \mathbf{a}_A \cdot \mathbf{x}$ where $\mathbf{a}_A$ is the acceleration of body A, due to the fact that the center-of-mass frame is not inertial. The term corresponding to $l = 1$ in (3.21) is precisely cancelled by this extra term and we get

$$V - \mathbf{a}_A \cdot \mathbf{x} = \sum_{l=2}^{\infty} \frac{1}{l!} \left[\mathcal{E}_L(t) x^L - \frac{I_A^{\langle L \rangle}(t)}{M} \mathcal{E}_{iL}(t) x^i \right], \tag{3.23}$$

where $I_A^{\langle L \rangle}$ are the STF multipole moments of body A and M its mass. When the body is spherical, or nearly spherical (as we've been assuming so far), the second term may be neglected and one obtains the final approximation to the effective external potential of the system, which we label by "tidal" due to the fact that it is a series in the tidal moments:

$$U_{\text{tidal}} \equiv V - \mathbf{a}_A \cdot \mathbf{x} \simeq \sum_{l=2}^{\infty} \frac{1}{l!} \mathcal{E}_L(t) x^L. \tag{3.24}$$

We make the observation that, since our working hypotheses led to the relation $\tau_{\text{orb}} \gg \tau_{\text{int}}$, the time dependence of the tidal moments is actually too slow to take the body out of (hydrostatic) equilibrium, which means we may neglect it and, henceforth, we will work with static tides and write $\mathcal{E}_L$.

3.1.4 Love Numbers in Newtonian Gravity

We now have all the necessary tools at our disposal to introduce the TLNs, which quantify the response of a body due to the surrounding tidal environment. We will show the results both using the spherical-harmonic decomposition and the STF decomposition.

Depending on the problem at hand, body A may be subject not only to the external potential U_{tidal}, but to an effective potential which includes other contributions. However, we will only consider the former case[2]. The presence of U_{tidal} creates a perturbation in the fluid equations which describe the body and, consequently, affects all variables (density, pressure, velocity, etc.). As a consequence, the potential U_A changes to $U_A + \delta U_T$, with δU_T a small perturbation.

Having this in mind, we assume that δU_T contains all the induced multipoles on body A from the external potential U_{tidal} and thus, A starts out being spherically symmetric (in the absence of U_{tidal}): $U_A = -M/r$. This means that all terms with $l \geq 2$ in the series (3.6) belong to δU_T. In particular, outside

[2]One example of an additional contribution would be the centrifugal potential from a rotating reference frame. In this case we assume that the body is nonrotating, such that one can neglect these terms.

the body:

$$\delta U_T = -\sum_{l=2}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{I_{lm}(t)}{r^{l+1}}Y_{lm}(\theta,\varphi) \tag{3.25}$$

$$= -\sum_{l=2}^{\infty}\frac{(2l-1)!!}{l!}\frac{I^{\langle L\rangle}}{r^{l+1}}n_{\langle L\rangle} \tag{3.26}$$

from (3.13) and (3.17).

The total gravitational potential is $U = U_A + \delta U_T + U_{\text{tidal}}$. Since the external part verifies $\nabla^2 U_{\text{tidal}} = 0$, its solution in a multipole expansion may be written as

$$U_{\text{tidal}} = \sum_{l=2}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}d_{lm}r^l Y_{lm}(\theta,\varphi) \tag{3.27}$$

by imposing regularity at the origin. The numerical factor was chosen for convenience and d_{lm} are the moments of the driving potential (analogously to I_{lm}), to be determined at each specific case.

The effect of the tidal environment on the body's multipole moments is measured through the gravitational TLNs k_l. Therefore, they may be defined by the relation of proportionality $I_{lm} \equiv -2k_l R^{2l+1}d_{lm}$ (we dropped the label "A" from the radius in accordance with the literature. Note that, since the time dependence of the driving potential is neglected, the time dependence in I_{lm} is also dropped. This, along with the spherical symmetry of U_A, is also the reason why k_l has no dependence on m) and one obtains

$$\begin{aligned} U = U_A + \delta U_T + U_{\text{tidal}} &= -\frac{M}{r} + \sum_{l=2}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\left(d_{lm}r^l - \frac{I_{lm}}{r^{l+1}}\right)Y_{lm}(\theta,\varphi) \\ &= -\frac{M}{r} + \sum_{l=2}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\left[1 + 2k_l\left(\frac{R}{r}\right)^{2l+1}\right]d_{lm}r^l Y_{lm}(\theta,\varphi) \end{aligned} \tag{3.28}$$

in the body's exterior, when using the spherical-harmonic basis.

For the decomposition in STF tensors, we need to write the defining relation of the TLNs in the new formalism. First, inserting (B.12) and (B.16) in (3.24) and comparing with (3.27) we get

$$d_{lm} = \frac{1}{(2l-1)!!}\mathscr{Y}^{\langle L\rangle}_{lm}\mathcal{E}_L. \tag{3.29}$$

Then, multiplying both sides of the equation by $Y_{lm}(\theta,\varphi)$ and summing in m with the help of (B.17) results in

$$\mathcal{E}_L = \frac{4\pi l!}{2l+1}\sum_{m=-l}^{l}d_{lm}\mathscr{Y}^{*\langle L\rangle}_{lm}. \tag{3.30}$$

Finally, substituting $I_{lm} = -2k_l R^{2l+1}d_{lm}$ and (3.30) in (B.19) gives

$$I_{\langle L\rangle} = -\frac{2k_l}{(2l-1)!!}R^{2l+1}\mathcal{E}_L, \tag{3.31}$$

which is the proportionality relation we were looking for. The decomposition of the total potential from equations (3.26) and (3.24) using (B.14), (B.12) and (3.31) thus becomes

$$\begin{aligned} U = U_A + \delta U_T + U_{\text{tidal}} &= -\frac{M}{r} - \sum_{l=2}^{\infty} \frac{(2l-1)!!}{l!} \frac{I_{\langle L \rangle}}{r^{l+1}} n^{\langle L \rangle} + \sum_{l=2}^{\infty} \frac{1}{l!} \mathcal{E}_L x^L \\ &= -\frac{M}{r} + \sum_{l=2}^{\infty} \frac{1}{l!} \left[\mathcal{E}_L - \frac{I_{\langle L \rangle}}{r^{2l+1}} (2l-1)!! \right] x^L \\ &= -\frac{M}{r} + \sum_{l=2}^{\infty} \frac{1}{l!} \left[1 + 2k_l \left(\frac{R}{r} \right)^{2l+1} \right] \mathcal{E}_L x^L \end{aligned} \tag{3.32}$$

which is the equivalent of (3.28) in the STF formalism.

3.2 Relativistic Tidal Love Numbers

The generalization of the Newtonian theory of tidal interactions we've discussed so far to a relativistic context consists, as expected, in the use of the spacetime metric tensor instead of a potential scalar function. The metric, which describes the gravitational field of a given system, may also be expanded in multipole moments which are then perturbed by external tidal fields inducing a response.

3.2.1 Multipole Expansion of the Metric

Although the study of TLNs in General Relativity is fairly recent, the theory of multipolar expansions of the metric goes back to the 1970s, when Geroch and Hansen [79, 80] used a mathematical formulation to model the asymptotic behavior of spacetime in terms of the mass multipole moments $\mathfrak{M}$ and the current (or angular momentum) multipole moments $\mathfrak{J}$. The results from these works have the advantage of being coordinate independent. Then, in 1980, Thorne [78] developed an alternative theory where a body's multipole moments may be determined from the stationary and asymptotically flat spacetime metric. Unlike the Geroch-Hansen formalism, Thorne uses a particular coordinate system (asymptotically Cartesian and mass centered (ACMC) coordinates) but it turns out that both formalisms are equivalent, as was shown by Gürsel in [81].

Let us consider a spacetime described by the ACMC coordinate system. In this case, the spacetime metric tends to a Minkowski metric at sufficiently large radii and the origin of the spatial coordinates coincides with the center-of-mass of the source. We saw in the previous section that the latter choice implied the vanishing of the dipole mass moment and the same happens in this case. The asymptotic (covariant) spacetime metric is then given by [82]

$$g_{00} = -1 + \frac{2M}{r} + \sum_{l=2}^{\infty} \frac{1}{r^{l+1}} \left[\frac{2}{l!} M^{\langle L \rangle} n^{\langle L \rangle} + (l' < l \text{ harmonics}) \right], \tag{3.33}$$

$$g_{0j} = -2 \sum_{l=1}^{\infty} \frac{1}{r^{l+1}} \left[\frac{1}{l!} \epsilon^{jka_l} S^{\langle k a_1 \cdots a_{l-1} \rangle} n^{\langle a_1 \cdots a_l \rangle} + (l' < l \text{ harmonics}) + (l \text{ harmonics with parity } (-1)^l) \right] \tag{3.34}$$

where M is the mass of the object, L is a multi-index notation for l indices, angular brackets represent a STF tensor (see Appendix B) and ϵ^{ijk} is the Levi-Civita symbol. $M^{\langle L\rangle}$ and $S^{\langle L\rangle}$ are the body's mass and current multipole moments, respectively. Since most recent works follow the Geroch-Hansen normalization, we will do the same here. This means that, in the weak-field limit (when the mass multipole moments should be equal to the Newtonian ones), the relativistic mass multipole moments relate to the Newtonian ones (3.16) as $M^{\langle L\rangle} \to (2l-1)!!I^{\langle L\rangle}$ whilst in Thorne's paper one has $M^{\langle L\rangle} \to I^{\langle L\rangle}$ [82].

When the spacetime is axially symmetric, one can show (for details see [82] and [83]) that components (3.33) and $g_{0\varphi}$ may be written as

$$g_{00} = -1 + \frac{2M}{r} + \sum_{l=2}^{\infty} \frac{2}{r^{l+1}} \left[\sqrt{\frac{4\pi}{2l+1}} M_{l0} Y_{l0}(\theta) + (l' < l \text{ pole}) \right], \tag{3.35}$$

$$g_{0\varphi} = \frac{2J}{r} \sin^2\theta + \sum_{l=2}^{\infty} \frac{2}{r^{l}} \left[\sqrt{\frac{4\pi}{2l+1}} \frac{S_{l0}}{l} S^{l0}_{\varphi}(\theta) + (l' < l \text{ pole}) \right] \tag{3.36}$$

where J is the angular momentum of the source, S^{l0}_{φ} is the φ component of the vector spherical harmonics S^{lm}_{a} (see Appendix B) and M_{l0} and S_{l0} are the scalar mass and current multipole moments (in other words, the equivalent of $M^{\langle L\rangle}$ and $S^{\langle L\rangle}$ in the spherical-harmonic basis. Since we are considering axial symmetry, the only non-vanishing moments are the ones with $m = 0$). If the symmetry is about the direction aligned with the unit vector $\mathbf{k}$, then[3]

$$M^{\langle L\rangle} = (2l-1)!!M_{l0}k^{\langle L\rangle}, \;\; S^{\langle L\rangle} = (2l-1)!!S_{l0}k^{\langle L\rangle}, \tag{3.37}$$

where $k^{\langle L\rangle}$ is the STF tensor constructed from $\mathbf{k}$. Equations (3.35) and (3.36) are the relativistic generalization of (3.13).

3.2.2 Tidal Environment

The first works describing the external universe to which a massive body may be subject to were [84, 85]. However, we will base our notation in [42].

This environment is characterized by the STF tidal polar and axial[4] multipole moments $\mathcal{E}_L$ and $\mathcal{B}_L$, respectively. These are defined by the expressions

$$\mathcal{E}_{a_1\cdots a_l} \equiv \frac{\langle C_{0a_1 0a_2;a_3\cdots a_l}\rangle}{(l-2)!}, \qquad \mathcal{B}_{a_1\cdots a_l} \equiv \frac{\langle \epsilon_{a_1bc} C^{bc}_{\;\;a_2 0;a_3\cdots a_l}\rangle}{\frac{2}{3}(l+1)(l-2)!} \tag{3.38}$$

where C_{abcd} is the Weyl tensor of the external universe and a semicolon denotes a covariant derivative. Just like in Newtonian theory, we assume the tidal moments have a sufficiently slow variation in time and therefore the time dependence is neglected.

They owe their name due to the fact that their decomposition in spherical harmonics is made in a polar or axial basis (see Appendix B). Given that there are no scalar axial spherical harmonics, there can be no scalar tidal potential $\mathcal{B}_l$ corresponding to $\mathcal{B}_L$. This implies one must define the vector potential

[3]This is only true in the Geroch-Hansen normalization. Other normalizations give different numerical factors. For more details about this, we refer the reader to [82].

[4]One may also use the denomination electric (instead of polar) and magnetic (instead of axial).

$\mathcal{B}_a^{(l)}$ where $a = \theta, \varphi$, to encode the components of $\mathcal{B}_L$ in this basis. This is not the case for $\mathcal{E}_L$, which has the corresponding scalar tidal potential $\mathcal{E}^{(l)}$. Having all this in mind, the tidal potentials of the external universe in the spherical-harmonic basis are given by:

$$\mathcal{E}^{(l)} = \sum_{m=-l}^{l} \mathcal{E}_{lm} Y_{lm}(\theta, \varphi), \qquad \mathcal{B}_a^{(l)} = \frac{1}{l} \sum_{m=-l}^{l} \mathcal{B}_{lm} S_a^{lm}(\theta, \varphi). \tag{3.39}$$

Considering that the source is nonrotating and axisymmetric (so that the polar and axial perturbations decouple and $m = 0$), it was shown in [42, 86] that the tidal perturbations to the asymptotic spacetime metric components (3.35), which we will call h_{00} and (3.36), which we will call $h_{0\varphi}$ may be written as

$$h_{00} = -\frac{2}{l(l-1)} \mathcal{E}_{l0} Y_{l0}(\theta) r^l, \qquad h_{0\varphi} = \frac{2}{3l(l-1)} \mathcal{B}_{l0} S_\varphi^{l0}(\theta) r^{l+1}. \tag{3.40}$$

3.2.3 Love Numbers in General Relativity

Gathering our results so far, the full asymptotic spacetime metric components 00 and 0φ sourced by a nonrotating, axisymmetric massive body immersed in a tidal environment, in ACMC coordinates (in the exterior of the body), may be written as

$$g_{00} = -1 + \frac{2M}{r} + \sum_{l=2}^{\infty} \left\{ \frac{2}{r^{l+1}} \left[\sqrt{\frac{4\pi}{2l+1}} M_{l0} Y_{l0}(\theta) + (l' < l \text{ pole}) \right] - \frac{2}{l(l-1)} r^l [\mathcal{E}_{l0} Y_{l0}(\theta) + (l' < l \text{ pole})] \right\}, \tag{3.41}$$

$$g_{0\varphi} = \frac{2J}{r} \sin^2\theta + \sum_{l=2}^{\infty} \left\{ \frac{2}{r^l} \left[\sqrt{\frac{4\pi}{2l+1}} \frac{S_{l0}}{l} S_\varphi^{l0}(\theta) + (l' < l \text{ pole}) \right] + \frac{2r^{l+1}}{3l(l-1)} [\mathcal{B}_{l0} S_\varphi^{l0}(\theta) + (l' < l \text{ pole})] \right\}. \tag{3.42}$$

In linear perturbation theory, the induced response of the mass and current multipole moments will be proportional to the applied tidal field, defining the relativistic TLNs. Since we are considering a nonrotating body, the parity of the multipole moments is conserved after the deformation, such that the mass multipole moments (which are polar) are related to the polar sector of the tidal field and the current multipole moments (which are axial) are related to the axial sector of the tidal field. When rotation is taken into account, additional Love numbers can be defined due to spin-tidal couplings [87–89] but in our case, one defines the following [42, 46]:

$$k_l^P \equiv -\frac{1}{2} \frac{l(l-1)}{M^{2l+1}} \sqrt{\frac{4\pi}{2l+1}} \frac{M_{l0}}{\mathcal{E}_{l0}}, \tag{3.43}$$

$$k_l^A \equiv -\frac{3}{2} \frac{l(l-1)}{(l+1) M^{2l+1}} \sqrt{\frac{4\pi}{2l+1}} \frac{S_{l0}}{\mathcal{B}_{l0}}. \tag{3.44}$$

k_l^P are the gravitational polar-type TLNs and k_l^A are the gravitational axial-type TLNs. In accordance with [83], the factor M^{2l+1} where M is the mass of the object was introduced to make them dimensionless. This definition differs from that of the works of Hinderer, Binnington and Poisson, which use the radius R instead. Since there are some objects in General Relativity for which this is not a well-defined quantity (for example the bosonic clouds formed by superradiance we discussed in the previous chapter), it is often more useful to use the former approach. In order to obtain the TLNs of the latter convention, one need only multiply equations (3.43) and (3.44) by $(M/R)^{2l+1}$ [83].

Substituting in the metric components gives

$$g_{00} = -1 + \frac{2M}{r} + \sum_{l=2}^{\infty} \left\{ -\frac{2}{l(l-1)} r^l \left[1 + 2k_l^P \left(\frac{M}{r} \right)^{2l+1} \right] \mathcal{E}_{l0} Y_{l0}(\theta) + \cdots \right\}; \tag{3.45}$$

$$g_{0\varphi} = \frac{2J}{r} \sin^2\theta + \sum_{l=2}^{\infty} \left\{ \frac{2}{3l(l-1)} r^{l+1} \left[1 - \frac{2(l+1)}{l} k_l^A \left(\frac{M}{r} \right)^{2l+1} \right] \mathcal{B}_{l0} S_{\varphi}^{l0}(\theta) + \cdots \right\} \tag{3.46}$$

where dots represent all the terms originating from the $l' < l$ poles (in equation (3.41) for (3.45) and in (3.42) for (3.46)). These are the relativistic generalization of (3.28). In fact, it was shown in [42] that in the nonrelativistic limit[5], one recovers the Newtonian potential and thus, the polar-type TLN becomes[6] the Newtonian gravitational TLN k_ℓ.

It is interesting to note that the TLNs of isolated (or naked) BHs were shown to be zero, both in the Schwarzschild case [41, 42] and, more recently, in the Kerr case [43, 44]. However, this may not be the case for dressed BHs [27, 45], i.e. those surrounded by matter fields, as we will see in the next chapter.

In the context of modified theories of gravity and Exotic Compact Objects, there are typically extra fields (non)minimally coupled to the metric tensor. If they are subject to external scalar (or vector) perturbations, one can also define so-called scalar (or vector) TLNs [83]. However, we will not need these coefficients in this thesis.

[5]There are some conventional differences between that work and this one. Since all Newtonian potentials in [42] have opposite sign from ours, their nonrelativistic limit $g_{00} = -1 + 2U$ must be $g_{00} = -1 - 2U$ in our case. To compare the Newtonian limit (1.8) with (3.31) of this thesis, one needs to change the sign of the first term (due to a typo in that paper) and divide the second term by $(l-2)!$ because of the difference in the definition of the Newtonian STF tidal moment $\mathcal{E}_L$.

[6]In our case, the polar-type TLN is only proportional to the Newtonian TLN because of different normalizations in Sections 3.1 and 3.2. Also note that the Newtonian TLNs from Section 3.2 are written in terms of radius (following the Hinderer, Binnington and Poisson convention) whilst the relativistic TLNs of Section 3.1 are written in terms of mass. We made these choices so that the current section agrees with [27, 83].

Chapter 4

Tidal Love Numbers of Gravitational Atoms

In this chapter we compute the gravitational TLNs of a BH dressed with a scalar condensate like the one discussed in Section 2.3 (i.e. a gravitational atom), in the Newtonian limit. That means we ”read” the polar-type TLN (3.43) from the Newtonian limit of (3.45): $g_{00} = -1 - 2U$ (when using geometrized units) where U is the total Newtonian potential of the system, as in (3.18). The proof for the components of the metric in this scenario may be found in any textbook on General Relativity (see, for example, Section §18.4 of Ref. [90]).

As we mentioned in footnote 6, these TLNs will not be exactly equal to the Newtonian TLNs k_l but only proportional.

Throughout this chapter, greek letters take the values $0, 1, 2, 3$ and Latin letters take the values $1, 2, 3$. The symbol ∂_a denotes the partial derivative $\partial/\partial x^a$. Since we are interested in taking the Newtonian limit, factors of G and c are reinserted in Sections 4.1 and 4.2, so as to keep track of the order of magnitude of the different terms in the equations.

4.1 Model Description

Our system is very similar to the one in [27]. We start by considering a complex scalar field Φ with mass $m = \mu\hbar$ propagating on the background spacetime of a spinning BH of mass M and that a superradiant instability led to the formation of a scalar cloud, after a time scale τ_{inst}. The system is in a quasi-stationary state, as mentioned in Section 2.3, and the cloud is fully formed.

In order to take the Newtonian limit of this system, we must assume that the gravitational field is weak and that means considering the small-coupling limit

$$\alpha \equiv \frac{GM\mu}{c} \ll 1 \tag{4.1}$$

like in Section 2.2. As we saw in Section 2.3, the size of the cloud is of the order r_g/α^2 where r_g is the gravitational radius of a BH (defined from the Schwarzschild radius) $r_g \equiv GM/c^2$, which means that in this limit $r_c \gg r_g$ so that the cloud is very far away from the BH.

We then assume that there is some external body orbiting the system with angular frequency Ω_{orb} and apply the framework of Section 3.1.3, implying that body A, which is the BH, is very far from body B which is the perturbing body. B is unspecified so that the results may be applied to any tidal environment (point particles, stars, BHs, etc.). Working in the center-of-mass frame of body A with the z axis aligned with the orbital angular momentum vector, the dynamics reduces to a one-body problem and the relation between the orbital radius r_{orb} and the orbital frequency is given by Kepler's law $\Omega^2_{\text{orb}} = GM/r^3_{\text{orb}}$. In order to consider the frequency dependence of the problem perturbatively, we will assume that the orbital motion is sufficiently slow such that $\Omega_{\text{orb}} \ll \mu\alpha^2 c^2$. Kepler's law then gives $r_{\text{orb}} \gg r_c$, which means the external body is very far from the cloud. The two assumptions regarding the length scales mean that the region in which we intend to solve the equations of motion is

$$r_c \ll r \ll r_{\text{orb}}. \tag{4.2}$$

In order to take the BH potential into account in the Newtonian limit, we will model it using a point particle approximation, which obviously implies spherical symmetry in its unperturbed configuration. Remembering what was said in Section 3.1.4, this implies that all mass multipoles with $l \geq 2$ are originated in response to the tidal field and can be included in the perturbed potential. We neglect any effects of backreaction or accretion on the spacetime metric due to the presence of Φ, given that those effects are only important at higher perturbative orders than the ones we intend to consider here (see the comment regarding the field fluctuations in Section 2.2).

4.2 Field Equations

The two equations which govern the dynamics of the problem in the relativistic case are the Einstein equations (1.1) and the Klein-Gordon equation $\Box\Phi = \mu^2 c^2 \Phi$ and we need to write them in the Newtonian limit. At this point in the analysis, we take coordinates $x^\mu \equiv (ct, x, y, z)$.

There are two main approximations for this procedure. One is the use of a linearized theory of gravity where the spacetime metric is considered to be slightly deviated from the Minkowski metric $(\eta_{\mu\nu}) \equiv \text{diag}(-1, +1, +1, +1)$, i.e. $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$, and the other is the slow-motion approximation where time derivatives are neglected when compared to spatial derivatives due to the relation $\partial_0 \sim \varepsilon \partial_i$, with $\varepsilon = \mathcal{O}(|h_{\mu\nu}|)$.

Let U be the Newtonian potential (3.18) where $|U| \ll c^2$. The metric tensor components are then given by

$$g_{00} = -1 - \frac{2U}{c^2} + \mathcal{O}\left(\frac{1}{c^4}\right), \qquad g_{ij} = \left(1 - \frac{2U}{c^2}\right)\delta_{ij} + \mathcal{O}\left(\frac{1}{c^4}\right), \qquad g_{0i} = \mathcal{O}\left(\frac{1}{c^3}\right). \tag{4.3}$$

On the other hand, the total energy-momentum tensor of the system consists of the energy-momentum tensor of the BH, given by the expression for a point particle at rest at the origin: $T^{\text{BH}}_{\mu\nu} = \rho c^2 \delta^0_\mu \delta^0_\nu$ with

$\rho = M\delta(\mathbf{x})$, plus the energy-momentum tensor of the complex scalar field:

$$T^{\mathsf{S}}_{\mu\nu} = \frac{\hbar^2}{2}(\partial_\mu\Phi^*\partial_\nu\Phi + \partial_\mu\Phi\partial_\nu\Phi^*) - \frac{\hbar^2}{2}g_{\mu\nu}(\partial_\alpha\Phi^*\partial^\alpha\Phi + \mu^2c^2\Phi^*\Phi). \tag{4.4}$$

Note that only component 00 of the energy-momentum tensor of the BH is non-vanishing, which means this is the component of Einstein's equations we should study. In order to factor out the high-frequency oscillations of the field [91], we introduce another scalar field Ψ through the relation

$$\Phi \equiv \frac{1}{\sqrt{\mu\hbar}}e^{-i\mu c^2 t}\Psi. \tag{4.5}$$

Using equations (4.3) to compute the Ricci tensor and $\Box\Phi$ (for details see Appendix C), we get

$$\nabla^2 U = 4\pi G\rho + 4\pi G m|\Psi|^2; \tag{4.6}$$

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + mU\Psi. \tag{4.7}$$

Note that equation (4.6) is Poisson's equation (3.1) with mass density $\rho + m|\Psi|^2$ ($m|\Psi|^2$ is precisely the mass density of the auxiliary scalar field which is what was expected) and equation (4.7) is Schrödinger's equation of nonrelativistic quantum mechanics with potential $V_{\mathsf{Sch.}} = mU$. Thus, the Newtonian limit of the Einstein-Klein-Gordon system is the Schrödinger-Poisson system.

4.3 Linearized Perturbations

Henceforth, we use Planck units $G = c = \hbar = 1$, which allows us to use results from Section 2.2.

We now introduce the external body mentioned in Section 4.1 which perturbs the dressed BH. The order of the perturbations will be encoded in two small parameters ϵ and ϵ_p. The former quantifies the amplitude of the field (or cloud) Ψ and the latter quantifies the strength of the response to the tidal field. Our ansatz for the perturbation expansion of the system is

$$\Psi = \epsilon\psi + \epsilon_p\delta\Psi, \qquad U = U_A + \epsilon^2\delta U + \epsilon_p\delta U_T. \tag{4.8}$$

The term $\epsilon^2\delta U$ measures the response of the potential due to the presence of the field (it is of order ϵ^2 because the energy-momentum tensor is quadratic in Ψ) which, in turn, affects the geometry of the problem through Einstein's field equations, $\epsilon_p\delta U_T$ is the equivalent of δU_T in Chapter 3 which means it has the induced multipoles on the potential U_A and $\epsilon_p\delta\Psi$ gives the cloud's response to the tidal perturbation. The Schrödinger-Poisson system should provide the correct relations between the effects at play by mixing powers of ϵ and ϵ_p, which will allow us to study the full response of the dressed BH.

Inserting (4.8) into (4.6) and (4.7) results in:

$$\nabla^2 U_A + \epsilon^2\nabla^2\delta U + \epsilon_p\nabla^2\delta U_T = 4\pi\rho + 4\pi\mu(\epsilon^2|\psi|^2 + \epsilon\epsilon_p\psi^*\delta\Psi + \epsilon\epsilon_p\psi\delta\Psi^* + \epsilon_p^2\delta\Psi^*\delta\Psi); \tag{4.9}$$

$$i\epsilon\frac{\partial\psi}{\partial t}+i\epsilon_p\frac{\partial\delta\Psi}{\partial t}=-\frac{1}{2\mu}(\epsilon\nabla^2\psi+\epsilon_p\nabla^2\delta\Psi)+\mu(\epsilon\psi U_A+\epsilon_p U_A\delta\Psi+\epsilon^3\delta U\psi$$
$$+\epsilon^2\epsilon_p\delta U\delta\Psi+\epsilon_p\epsilon\delta U_T\psi+\epsilon_p^2\delta U_T\delta\Psi). \quad (4.10)$$

Equating the same powers of the perturbation coefficients ϵ and ϵ_p on both sides, one obtains up to first order in the perturbation functions (terms like $\delta U\delta\Psi$ are dropped):

$$\nabla^2 U_A = 4\pi\rho \quad (4.11)$$

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2\mu}\nabla^2\psi+\mu U_A\psi \quad (4.12)$$

$$\nabla^2\delta U = 4\pi\mu|\psi|^2 \quad (4.13)$$

$$\nabla^2\delta U_T = 4\pi\mu\epsilon(\psi^*\delta\Psi+\psi\delta\Psi^*) \quad (4.14)$$

$$i\frac{\partial\delta\Psi}{\partial t} = -\frac{1}{2\mu}\nabla^2\delta\Psi+\mu U_A\delta\Psi+\mu\epsilon\delta U_T\psi \quad (4.15)$$

Since we are assuming the BH is a point particle, it must be spherically symmetric which means, remembering what was said at the end of Section 3.1.1, that the solution of (4.11) is

$$U_A(r) = -\frac{M}{r}. \quad (4.16)$$

Substituting in (4.12) we obtain the usual Schrödinger equation with a Coulomb potential (in Planck units) $-\mu M/r$. Therefore, this equation is analogous to the non-relativistic limit of the hydrogen atom and has solutions, expanded in the basis of eigenstates[1], given by [31, 68]

$$\psi(t,r,\theta,\varphi)=\sum_{\ell_i=0}^{\infty}\sum_{m_i=-\ell_i}^{\ell_i}e^{-iE_{n\ell_i}t}R_{n\ell_i}(r)Y_{\ell_i m_i}(\theta,\varphi), \quad (4.17)$$

where

$$E_{n\ell_i}\equiv\omega_{n\ell_i}-\mu\simeq-\frac{\mu^3M^2}{2(n+\ell_i+1)^2} \quad (4.18)$$

because $\omega_{n\ell_i}$ are the real part of (2.8) (remember that there is no imaginary part because we've assumed there is no more instability at this stage) and $R_{n\ell_i}$ are the functions (2.10). In this case the subscripts 'i' are not indices but merely a label. We use this label in order to distinguish the cloud's angular momentum numbers from the angular momentum numbers of the gravitational and scalar perturbations that we discuss below.

Since we will perform all calculations using the confluent hypergeometric U instead of the generalized Laguerre polynomials, we write, also using (4.18) to ease the notation:

$$R_{n\ell_i}(r)=\frac{(-1)^n\left(8\mu|E_{n\ell_i}|\right)^{3/4}}{\sqrt{2n!(n+\ell_i+1)(n+2\ell_i+1)!}}\left(\sqrt{8\mu|E_{n\ell_i}|}r\right)^{\ell_i}e^{-\sqrt{8\mu|E_{n\ell_i}|}r/2}U\left(-n,2\ell_i+2,\sqrt{8\mu|E_{n\ell_i}|}r\right). \quad (4.19)$$

When writing (4.17), we considered that the frequency of the cloud is monochromatic (thereby leaving

[1] In order to write the angular dependence in the basis of spherical harmonics we used the fact that in the small-coupling limit, as we've explained in Section 4.1, the cloud is far away from the BH which means its spin may be neglected, implying $S_{\ell_i m_i}(\theta)e^{im_i\varphi}\simeq Y_{\ell_i m_i}(\theta,\varphi)$ as we mentioned in Section 2.3.

out an integral in $\omega_{n\ell_i}$). We will also analyse each mode (n, ℓ_i, m_i) separately, namely $(0,0,0)$ and $(0,1,1)$, so from now on the sum signs will be dropped in calculations. Note that equation (4.13) refers only to the perturbations of the metric, which are out of the scope of this thesis. Therefore, we will leave it unsolved.

In the remaining part of this chapter, the procedure will consist of solving (4.15) to obtain $\delta\Psi$ and then "feeding" that result into (4.14) to get δU_T. Then we can write the multipole expansion $g_{00} = -1 - 2U$ and determine the TLNs (it is not important that δU isn't known because it does not appear at order ϵ_p and so does not contribute to the result). As we've mentioned, this will be done for a scalar cloud in a quasi-stationary state: first at $\ell_i = m_i = 0$ and then at $\ell_i = m_1 = 1$, always considering the fundamental level $n = 0$.

4.4 Solving the Perturbation Equations

Let us choose the following ansatz for the perturbation functions, following Ref. [92]:

$$\begin{aligned}\delta\Psi(t,r,\theta,\varphi) = &\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\sum_{\ell_j=0}^{\infty}\sum_{m_j=-\ell_j}^{\ell_j} e^{-iE_{n\ell_i}t}\frac{1}{r}[\hat{Z}_1^{\ell_j m_j}(\omega,r)Y_{\ell_j m_j}(\theta,\varphi)e^{-i\omega t}\\ &+ (\hat{Z}_2^*)^{\ell_j m_j}(\omega,r)Y^*_{\ell_j m_j}(\theta,\varphi)e^{i\omega t}];\end{aligned} \tag{4.20}$$

$$\delta U_T(t,r,\theta,\varphi) = \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\sum_{l=2}^{\infty}\sum_{m=-l}^{l}[\hat{u}^{lm}(\omega,r)Y_{lm}(\theta,\varphi)e^{-i\omega t} + (\hat{u}^*)^{lm}(\omega,r)Y^*_{lm}(\theta,\varphi)e^{i\omega t}], \tag{4.21}$$

where hats denote functions in the frequency-domain. Henceforth, integrals and summation symbols are written without limits. Just like in equation (4.17), 'j' is a label and not an index. Note that the factor $e^{-iE_{n\ell_i}t}$ of $\delta\Psi$ is the same as that of ψ, because this allows us to use separation of variables to solve the field equations. The first sum in δU_T starts at $l = 2$ due to equations (3.25) and (3.26).

Substituting in (4.14) and (4.15) and inverting the appropriate Fourier transforms results in the four following equations:

$$\begin{aligned}\sum_{l,m}[\mathcal{D}\hat{u}^{lm}(\omega,r)]Y_{lm}(\theta,\varphi) = \frac{4\pi\mu\epsilon}{r}R_{n\ell_i}(r)\sum_{\ell_j,m_j}&[\hat{Z}_1^{\ell_j m_j}(\omega,r)Y_{\ell_j m_j}(\theta,\varphi)Y^*_{\ell_i m_i}(\theta,\varphi)\\ &+ \hat{Z}_2^{\ell_j m_j}(\omega,r)Y_{\ell_j m_j}(\theta,\varphi)Y_{\ell_i m_i}(\theta,\varphi)];\end{aligned} \tag{4.22}$$

$$\begin{aligned}\sum_{l,m}[\mathcal{D}(\hat{u}^*)^{lm}(\omega,r)]Y^*_{lm}(\theta,\varphi) = \frac{4\pi\mu\epsilon}{r}R_{n\ell_i}(r)\sum_{\ell_j,m_j}&[(\hat{Z}_2^*)^{\ell_j m_j}(\omega,r)Y^*_{\ell_j m_j}(\theta,\varphi)Y^*_{\ell_i m_i}(\theta,\varphi)\\ &+ (\hat{Z}_1^*)^{\ell_j m_j}(\omega,r)Y^*_{\ell_j m_j}(\theta,\varphi)Y_{\ell_i m_i}(\theta,\varphi)];\end{aligned} \tag{4.23}$$

$$\sum_{\ell_j,m_j}[\mathcal{D}_+\hat{Z}_1^{\ell_j m_j}(\omega,r)]Y_{\ell_j m_j}(\theta,\varphi) = 2\epsilon\mu^2 r R_{n\ell_i}(r)\sum_{l,m}\hat{u}^{lm}(\omega,r)Y_{lm}(\theta,\varphi)Y_{\ell_i m_i}(\theta,\varphi); \tag{4.24}$$

$$\sum_{\ell_j,m_j}[\mathcal{D}_-(\hat{Z}_2^*)^{\ell_j m_j}(\omega,r)]Y^*_{\ell_j m_j}(\theta,\varphi) = 2\epsilon\mu^2 r R_{n\ell_i}(r)\sum_{l,m}(\hat{u}^*)^{lm}(\omega,r)Y^*_{lm}(\theta,\varphi)Y_{\ell_i m_i}(\theta,\varphi), \tag{4.25}$$

with the differential operators defined by:

$$\mathcal{D} \equiv \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2}; \tag{4.26}$$

$$\mathcal{D}_{\pm} \equiv \frac{d^2}{dr^2} + 2\mu(E_{n\ell_i} \pm \omega) + \frac{2\mu^2 M}{r} - \frac{\ell_j(\ell_j+1)}{r^2}. \tag{4.27}$$

Now we multiply each equation by the complex conjugate of the spherical-harmonic function which is on its left-hand side and integrate in the 2-sphere, applying (B.4). This gives

$$\begin{aligned}\mathcal{D}\hat{u}^{lm}(\omega,r) = \frac{4\pi\mu\epsilon}{r}R_{n\ell_i}(r)\sum_{\ell_j,m_j}\hat{Z}_1^{\ell_j m_j}(\omega,r)\int Y_{\ell_j m_j}(\theta,\varphi)Y^*_{\ell_i m_i}(\theta,\varphi)Y^*_{lm}(\theta,\varphi)\,d\Omega \\ + \frac{4\pi\mu\epsilon}{r}R_{n\ell_i}(r)\sum_{\ell_j,m_j}\hat{Z}_2^{\ell_j m_j}(\omega,r)\int Y_{\ell_j m_j}(\theta,\varphi)Y_{\ell_i m_i}(\theta,\varphi)Y^*_{lm}(\theta,\varphi)\,d\Omega;\end{aligned} \tag{4.28}$$

$$\begin{aligned}\mathcal{D}(\hat{u}^*)^{lm}(\omega,r) = \frac{4\pi\mu\epsilon}{r}R_{n\ell_i}(r)\sum_{\ell_j,m_j}(\hat{Z}_2^*)^{\ell_j m_j}(\omega,r)\int Y^*_{\ell_j m_j}(\theta,\varphi)Y^*_{\ell_i m_i}(\theta,\varphi)Y_{lm}(\theta,\varphi)\,d\Omega \\ + \frac{4\pi\mu\epsilon}{r}R_{n\ell_i}(r)\sum_{\ell_j,m_j}(\hat{Z}_1^*)^{\ell_j m_j}(\omega,r)\int Y^*_{\ell_j m_j}(\theta,\varphi)Y_{\ell_i m_i}(\theta,\varphi)Y_{lm}(\theta,\varphi)\,d\Omega;\end{aligned} \tag{4.29}$$

$$\mathcal{D}_+\hat{Z}_1^{\ell_j m_j}(\omega,r) = 2\epsilon\mu^2 rR_{n\ell_i}(r)\sum_{l,m}\hat{u}^{lm}(\omega,r)\int Y_{lm}(\theta,\varphi)Y_{\ell_i m_i}(\theta,\varphi)Y^*_{\ell_j m_j}(\theta,\varphi)\,d\Omega; \tag{4.30}$$

$$\mathcal{D}_-(\hat{Z}_2^*)^{\ell_j m_j}(\omega,r) = 2\epsilon\mu^2 rR_{n\ell_i}(r)\sum_{l,m}(\hat{u}^*)^{lm}(\omega,r)\int Y^*_{lm}(\theta,\varphi)Y_{\ell_i m_i}(\theta,\varphi)Y_{\ell_j m_j}(\theta,\varphi)\,d\Omega. \tag{4.31}$$

This system of equations allows one to solve the problem for any desired set of modes (n,ℓ_i,m_i), (n,ℓ_j,m_j) and tidal deformation multipoles (l,m). However, we point out once again that $n=0$ is imposed so that, from now on, only that value will appear in the equations above.

4.4.1 Case $\ell_i = m_i = 0$

Let us start with a spherically symmetric $\ell_i = m_i = 0$ spatial distribution for the cloud. Given that $Y_{00} = 1/\sqrt{4\pi}$ is a constant, it may be taken out of all the integrals in (4.28)-(4.31). Then, using (B.4) we see that all of them give $\delta_{l\ell_j}\delta_{mm_j}/\sqrt{4\pi}$ so that the equations become

$$\mathcal{D}\hat{u}^{lm}(\omega,r) = \frac{2\sqrt{\pi}\mu\epsilon}{r}R_{00}(r)[\hat{Z}_1^{lm}(\omega,r) + \hat{Z}_2^{lm}(\omega,r)]; \tag{4.32}$$

$$\mathcal{D}(\hat{u}^*)^{lm}(\omega,r) = \frac{2\sqrt{\pi}\mu\epsilon}{r}R_{00}(r)[(\hat{Z}_1^*)^{lm}(\omega,r) + (\hat{Z}_2^*)^{lm}(\omega,r)]; \tag{4.33}$$

$$\mathcal{D}_+\hat{Z}_1^{\ell_j m_j}(\omega,r) = \frac{\epsilon\mu^2}{\sqrt{\pi}}rR_{00}(r)\hat{u}^{\ell_j m_j}(\omega,r); \tag{4.34}$$

$$\mathcal{D}_-(\hat{Z}_2^*)^{\ell_j m_j}(\omega,r) = \frac{\epsilon\mu^2}{\sqrt{\pi}}rR_{00}(r)(\hat{u}^*)^{\ell_j m_j}(\omega,r). \tag{4.35}$$

From equations (4.34) and (4.35), one can see that the particular solution for $\hat{Z}_1^{\ell_j m_j}$ and $(\hat{Z}_2^*)^{\ell_j m_j}$ will be of order[2] ϵ. Plugging in equations (4.32) and (4.33) one can see that the most general solution for $\hat{u}^{lm}$ should be of the form $\hat{u}^{lm} \sim \hat{u}^{lm}_{(0)} + \epsilon^2 \hat{u}^{lm}_{(2)}$. Hence we perform the following additional perturbation expansion:

$$\hat{Z}_1^{\ell_j m_j}(\omega, r) = \epsilon (\hat{Z}_1)^{\ell_j m_j}_{(1)}(\omega, r); \tag{4.36}$$

$$(\hat{Z}_2^*)^{\ell_j m_j}(\omega, r) = \epsilon (\hat{Z}_2^*)^{\ell_j m_j}_{(1)}(\omega, r); \tag{4.37}$$

$$\hat{u}^{lm}(\omega, r) = \hat{u}^{lm}_{(0)}(\omega, r) + \epsilon^2 \hat{u}^{lm}_{(2)}(\omega, r). \tag{4.38}$$

Substituting in equations (4.32)-(4.35) and matching powers of ϵ results in:

$$\frac{d^2 \hat{u}^{lm}_{(0)}}{dr^2} + \frac{2}{r}\frac{d\hat{u}^{lm}_{(0)}}{dr} - \frac{l(l+1)}{r^2}\hat{u}^{lm}_{(0)} = 0; \tag{4.39}$$

$$\frac{d^2 (\hat{u}^*)^{lm}_{(0)}}{dr^2} + \frac{2}{r}\frac{d(\hat{u}^*)^{lm}_{(0)}}{dr} - \frac{l(l+1)}{r^2}(\hat{u}^*)^{lm}_{(0)} = 0; \tag{4.40}$$

$$\frac{d^2 \hat{u}^{lm}_{(2)}}{dr^2} + \frac{2}{r}\frac{d\hat{u}^{lm}_{(2)}}{dr} - \frac{l(l+1)}{r^2}\hat{u}^{lm}_{(2)} = \frac{2\sqrt{\pi}\mu}{r} R_{00}[\hat{Z}_1^{lm} + \hat{Z}_2^{lm}]; \tag{4.41}$$

$$\frac{d^2 (\hat{u}^*)^{lm}_{(2)}}{dr^2} + \frac{2}{r}\frac{d(\hat{u}^*)^{lm}_{(2)}}{dr} - \frac{l(l+1)}{r^2}(\hat{u}^*)^{lm}_{(2)} = \frac{2\sqrt{\pi}\mu}{r} R_{00}[(\hat{Z}_1^*)^{lm} + (\hat{Z}_2^*)^{lm}]; \tag{4.42}$$

$$\frac{d^2 (\hat{Z}_1)^{\ell_j m_j}_{(1)}}{dr^2} - \left[-2\mu(E_{00} + \omega) - \frac{2\mu^2 M}{r} + \frac{\ell_j(\ell_j+1)}{r^2}\right](\hat{Z}_1)^{\ell_j m_j}_{(1)} = \frac{\mu^2}{\sqrt{\pi}} r R_{00} \hat{u}^{\ell_j m_j}_{(0)}; \tag{4.43}$$

$$\frac{d^2 (\hat{Z}_2^*)^{\ell_j m_j}_{(1)}}{dr^2} - \left[-2\mu(E_{00} - \omega) - \frac{2\mu^2 M}{r} + \frac{\ell_j(\ell_j+1)}{r^2}\right](\hat{Z}_2^*)^{\ell_j m_j}_{(1)} = \frac{\mu^2}{\sqrt{\pi}} r R_{00} (\hat{u}^*)^{\ell_j m_j}_{(0)}. \tag{4.44}$$

Equations (4.39) and (4.40) are the radial part of Laplace's equation or, in other words, the radial part of a homogeneous Poisson equation (3.1). Therefore, the solution is analogous to (3.5), i.e. renaming constants: $\hat{u}^{lm}_{(0)}(\omega, r) = A_{lm}(\omega) r^l + B_{lm}(\omega) r^{-l-1}$. Since it should describe the external tidal field, from equation (3.27) one can see that $B_{lm} = 0$ and it becomes $\hat{u}^{lm}_{(0)}(\omega, r) = A_{lm}(\omega) r^l$. We can now obtain some constraints on A_{lm} using what we know about the external potential.

We know from equation (3.19) that it (in this case $V \equiv \epsilon^2 \delta U + \epsilon_p \delta U_T$ but only $\epsilon_p \delta U_T$ contributes to the tidal response) is given by

$$V(t, \mathbf{x}) = -\int_B \frac{\rho(r, \mathbf{x})}{|\mathbf{x} - \mathbf{x}'|} d^3x', \tag{4.45}$$

because there is only one body B sourcing it. Furthermore, we know that [38]:

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y^*_{lm}(\theta', \varphi') Y_{lm}(\theta, \varphi), \tag{4.46}$$

where (r, θ, φ) are the spherical polar coordinates of $\mathbf{x}$, (r', θ', φ') are the spherical polar coordinates of $\mathbf{x}'$, $r_< \equiv \min(r, r')$ and $r_> \equiv \max(r, r')$. We wish to substitute in V, which means the summation starts at $l = 2$ because of equation (3.27). On the other hand, remembering that we chose the z axis aligned

[2] We can discard the solutions to the homogeneous version of equations (4.34) and (4.35) since we are only interested in perturbations to the scalar profile sourced by the tidal field.

with the orbital angular momentum (see Section 4.1) then the orbit happens on the plane $z = 0$ which means B has polar angle $\theta' = \pi/2$ at all times. We also assume that the orbit is circular such that the remaining coordinates of B are $r' = r_{\text{orb}}$ and $\varphi' = \Omega_{\text{orb}} t$. Having this in mind, all positions $\mathbf{x}'$ are constant in space and may be taken out of the integral (4.45). Finally, considering that the region we're working in is given by (4.2), $r < r'$ is verified and we get:

$$V(t,\mathbf{x}) = -M_B \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{1}{r_{\text{orb}}^{l+1}} r^l Y_{lm}^* \left(\frac{\pi}{2}, \Omega_{\text{orb}} t\right) Y_{lm}(\theta,\varphi) \tag{4.47}$$

$$= -\sum_{l=2}^{\infty} \sum_{m=-l}^{l} \frac{C_{lm}}{r_{\text{orb}}^{l+1}} r^l Y_{lm}(\theta,\varphi) e^{-im\Omega_{\text{orb}} t} \tag{4.48}$$

where we defined $M_B \equiv \int_B \rho(t,\mathbf{x}')\, d^3x'$ as the mass of body B and we used $Y_{lm}^*(\pi/2, \Omega_{\text{orb}} t) = C_{lm} e^{-im\Omega_{\text{orb}} t}$ with C_{lm} a constant which vanishes for $l + m$ due to the value of the associated Legendre polynomial $P_l^m(0)$ and satisfies $C_{l,-m} = (-1)^m C_{lm}$ so that the potential is a real function [93, 94]. Fourier-transforming V into the frequency-domain, one may write

$$\begin{aligned} \hat{V}(\omega, r) &= \int_{-\infty}^{+\infty} V(t,\mathbf{x}) e^{i\omega t}\, dt = -\sum_{l=2}^{\infty} \sum_{m=-l}^{l} \frac{C_{lm}}{r_{\text{orb}}^{l+1}} r^l Y_{lm}(\theta,\varphi) \int_{-\infty}^{+\infty} e^{i(\omega - m\Omega_{\text{orb}})t}\, dt \\ &= \sum_{l=2}^{\infty} \sum_{m=-l}^{l} c_{lm} r^l Y_{lm}(\theta,\varphi) \delta(\omega - m\Omega_{\text{orb}}), \end{aligned} \tag{4.49}$$

with c_{lm} a redefinition of the constants inside the summation symbol (having the same symmetries as C_{lm}). In this case, in order for $\hat{V}$ to be real, $c_{l,-m}\delta(-\omega + m\Omega_{\text{orb}}) = (-1)^m c_{lm}\delta(\omega - m\Omega_{\text{orb}})$ must be true. Comparing the radial coefficients of (4.49) with $\hat{u}_{(0)}^{lm}(\omega, r) = A_{lm}(\omega) r^l$, we get $A_{lm}(\omega) = c_{lm}\delta(\omega - m\Omega_{\text{orb}})$ thereby inheriting the symmetry $A_{l,-m}(-\omega) = (-1)^m A_{lm}(\omega)$.

Since (4.40) is the same equation as (4.39), it has the same solution: $(\hat{u}^*)_{(0)}^{lm} = \hat{u}_{(0)}^{lm}$; which has the consequences of $(\hat{Z}_1)_{(1)}^{\ell_j m_j}(\omega, r) = (-1)^{m_j} (\hat{Z}_2^*)_{(1)}^{\ell_j, -m_j}(-\omega, r)$ and $(\hat{Z}_1)_{(1)}^{\ell_j m_j}$, $(\hat{Z}_2^*)_{(1)}^{\ell_j m_j}$ being real. This, in turn, implies $\hat{u}_{(2)}^{lm} = (\hat{u}^*)_{(2)}^{lm}$ so that all functions are real.

Noting that we are dealing with a linear system of differential equations, all the remaining undetermined functions will have a factor $\delta(\omega - m\Omega_{\text{orb}})$. Therefore, we choose the following ansatz:

$$(\hat{Z}_1)_{(1)}^{\ell_j m_j}(\omega, r) = (\hat{Z}_1)_{(1),t}^{\ell_j m_j}(\omega, r)\delta(\omega - m_j\Omega_{\text{orb}}); \tag{4.50}$$

$$\hat{u}_{(2)}^{lm}(\omega, r) = \hat{u}_{(2),t}^{lm}(\omega, r)\delta(\omega - m\Omega_{\text{orb}}) \tag{4.51}$$

which results in the same equations as (4.41)-(4.44) but for the functions with a 't' label. We will use the method of Green's function[3] to solve (4.43) and (4.41) since (4.42) and (4.44) follow from the relations of the previous paragraph.

[3] For details on the procedure, see [95].

The solution of (4.43) with the appropriate boundary conditions is

$$(\hat{Z}_1)^{\ell_j m_j}_{(1)}(\omega,r) = \frac{(\hat{Z}_{1,+})^{\ell_j m_j}_{(1)}(\omega,r)}{W}\int_0^r (\hat{Z}_{1,-})^{\ell_j m_j}_{(1)}(\omega,r')S_Z(\omega,r')dr' + \frac{(\hat{Z}_{1,-})^{\ell_j m_j}_{(1)}(\omega,r)}{W}\int_r^\infty (\hat{Z}_{1,+})^{\ell_j m_j}_{(1)}(\omega,r')S_Z(\omega,r')dr' \tag{4.52}$$

where

$$S_Z(\omega,r) \equiv \frac{A_{\ell_j m_j}(\omega)}{\sqrt{\pi}}\mu^2 r^{\ell_j+1}R_{00}(r), \tag{4.53}$$

the solutions for the homogeneous equation are given by the Whittaker functions:

$$(\hat{Z}_{1,-})^{\ell_j m_j}_{(1)}(\omega,r) = h_1 M_{\frac{1}{\sqrt{1+\frac{\omega}{E_{00}}}},\ell_j+\frac{1}{2}}\left(\sqrt{-8\mu(E_{00}+\omega)}r\right); \tag{4.54}$$

$$(\hat{Z}_{1,+})^{\ell_j m_j}_{(1)}(\omega,r) = h_2 W_{\frac{1}{\sqrt{1+\frac{\omega}{E_{00}}}},\ell_j+\frac{1}{2}}\left(\sqrt{-8\mu(E_{00}+\omega)}r\right), \tag{4.55}$$

with h_1, h_2 constants and

$$W \equiv (\hat{Z}_{1,-})^{\ell_j m_j}_{(1)}\frac{d(\hat{Z}_{1,+})^{\ell_j m_j}_{(1)}}{dr} - (\hat{Z}_{1,+})^{\ell_j m_j}_{(1)}\frac{d(\hat{Z}_{1,-})^{\ell_j m_j}_{(1)}}{dr} = -h_1 h_2 \frac{\Gamma(2+2\ell_j)\sqrt{-8\mu(E_{00}+\omega)}}{\Gamma\left(\ell_j+1-\frac{1}{\sqrt{1+\frac{\omega}{E_{00}}}}\right)} \tag{4.56}$$

is the Wronskian [74]. We only need the solutions for $\ell_j \geq 2$ since these are the ones that source $\hat{u}^{lm}_{(2)}(\omega,r)$ with $l \geq 2$, which means there are no singularities in the denominator. Nevertheless, using (4.19), we can write the solutions for $\ell_j > 0$:

$$(\hat{Z}_1)^{\ell_j m_j}_{(1)}(\omega,r) = -\frac{\mu^2 A_{\ell_j m_j}(\omega)}{2\sqrt{2\pi}}\frac{\Gamma(\ell_j)}{\Gamma(2+2\ell_j)}\frac{r^{\ell_j+1/2}}{\sqrt{8\mu|E_{00}|}}e^{-\sqrt{8\mu|E_{00}|}r/2}\left[\sqrt{\pi}\frac{\Gamma(2+2\ell_j)}{\Gamma(2+\ell_j)} \times W_{1,\ell_j+\frac{1}{2}}\left(\sqrt{8\mu|E_{00}|}r\right)I_M(r) + \frac{1}{\sqrt{\pi}}M_{1,\ell_j+\frac{1}{2}}\left(\sqrt{8\mu|E_{00}|}r\right)I_W(r)\right] + f_{\ell_j}(r)A_{\ell_j m_j}(\omega)\frac{\omega}{E_{00}} + \mathcal{O}\left(\frac{\omega^2}{E_{00}}\right) \tag{4.57}$$

with

$$I_M(r) = \left[4+8\ell_j+(4\ell_j+2)\sqrt{8\mu|E_{00}|}r+(\sqrt{8\mu|E_{00}|}r)^2\right]I_{\ell_j+\frac{1}{2}}\left(\frac{\sqrt{8\mu|E_{00}|}r}{2}\right) - \left[2\sqrt{8\mu|E_{00}|}r+(\sqrt{8\mu|E_{00}|}r)^2\right]I_{\ell_j-\frac{1}{2}}\left(\frac{\sqrt{8\mu|E_{00}|}r}{2}\right); \tag{4.58}$$

$$I_W(r) = \left[4+8\ell_j+(4\ell_j+2)\sqrt{8\mu|E_{00}|}r+(\sqrt{8\mu|E_{00}|}r)^2\right]K_{\ell_j+\frac{1}{2}}\left(\frac{\sqrt{8\mu|E_{00}|}r}{2}\right) + \left[2\sqrt{8\mu|E_{00}|}r+(\sqrt{8\mu|E_{00}|}r)^2\right]K_{\ell_j-\frac{1}{2}}\left(\frac{\sqrt{8\mu|E_{00}|}r}{2}\right), \tag{4.59}$$

where $I_{\ell_j \pm 1/2}$ and $K_{\ell_j \pm 1/2}$ are Modified Bessel functions of the First Kind and Second Kind, respectively and f_{ℓ_j} is an undetermined function of r. Remember the assumption $\Omega_{\text{orb}} \ll \alpha^2\mu = \mu^3 M^2$ from Section 4.1. Since the solutions only have support at $\omega = m\Omega_{\text{orb}}$, we have $\omega \sim \Omega_{\text{orb}} \ll \mu^3 M^2 \sim E_{n\ell_i}$ from equation (4.18), which means we are justified in expanding in powers of ω/E_{00}.

Using the fact that $(\hat{Z}_2)^{\ell_j m_j}_{(1)}$ is real, one may write $(\hat{Z}_2)^{\ell_j m_j}_{(1)}(\omega, r) = (-1)^{m_j}(\hat{Z}_1)^{\ell_j, -m_j}_{(1)}(-\omega, r)$ so that

$$(\hat{Z}_1)^{lm}_{(1)}(\omega, r) + (\hat{Z}_2)^{lm}_{(1)}(\omega, r) = (\hat{Z}_1)^{lm}_{(1)}(\omega, r) + (-1)^m(\hat{Z}_1)^{l,-m}_{(1)}(-\omega, r) \tag{4.60}$$

implies that the terms of order ω/E_{00} cancel: $f_l(r)[A_{lm}(\omega) - (-1)^m A_{l,-m}(-\omega)]\omega/E_{00} = 0$ due to the properties of A_{lm}. The right-hand side of (4.41) then becomes

$$\begin{aligned} S_u(\omega, r) \equiv & -\mu^3 A_{lm}(\omega)(8\mu|E_{00}|)^{1/4}\frac{\Gamma(l)}{\Gamma(2+2l)} r^{l-1/2} e^{-\sqrt{8\mu|E_{00}|r}} \left[\sqrt{\pi}\frac{\Gamma(2+2l)}{\Gamma(2+l)}\right. \\ & \left. \times W_{1,l+\frac{1}{2}}\left(\sqrt{8\mu|E_{00}|r}\right) I_M(r)\Big|_{\ell_j = l} + \frac{1}{\sqrt{\pi}} M_{1,l+\frac{1}{2}}\left(\sqrt{8\mu|E_{00}|r}\right) I_W(r)\Big|_{\ell_j = l}\right] + \mathcal{O}\left(\frac{\omega^2}{E_{00}^2}\right). \end{aligned} \tag{4.61}$$

Using the same method to solve (4.41), we have:

$$\hat{u}^{lm}_{(2)}(\omega, r) = \hat{u}^{(2)}_{+}(\omega, r)\int_0^r \frac{\hat{u}^{(2)}_{-}(\omega, r')S_u(\omega, r')}{W(r')}dr' + \hat{u}^{(2)}_{-}(\omega, r)\int_r^\infty \frac{\hat{u}^{(2)}_{+}(\omega, r')S_u(\omega, r')}{W(r')}dr' \tag{4.62}$$

with the solutions of the homogeneous equation (again being the radial part of the Laplace equation) $\hat{u}^{(2)}_{-}(\omega, r) \equiv d_1(\omega) r^l$ and $\hat{u}^{(2)}_{+}(\omega, r) \equiv d_2(\omega) r^{-l-1}$ giving the Wronskian $W(r) = -d_1(\omega)d_2(\omega)(2l+1)/r^2$.

Since $r_{\text{orb}} \gg r$ (i.e. the external body is very far away) from equation (4.2), we look for the asymptotic solution (up to order ω^2/E_{00}^2):

$$\begin{aligned} \hat{u}^{lm}_{(2)}(\omega, r) \longrightarrow & -\frac{r^{-l-1}}{2l+1}\int_0^\infty r'^{l+2} S_u(\omega, r')dr' \\ & = A_{lm}(\omega)\frac{\mu^3(8\mu|E_{00}|)^{1/4}}{2l+1}\frac{\Gamma(l)}{\Gamma(2+2l)}\frac{\mathcal{I}}{r^{l+1}}, \end{aligned} \tag{4.63}$$

where

$$\begin{aligned} \mathcal{I} = \int_0^\infty & r'^{2l+3/2} e^{-\sqrt{8\mu|E_{00}|r'}} \left[\sqrt{\pi}\frac{\Gamma(2+2l)}{\Gamma(2+l)} W_{1,l+\frac{1}{2}}\left(\sqrt{8\mu|E_{00}|r'}\right) I_M(r')\Big|_{\ell_j = l}\right. \\ & \left. + \frac{1}{\sqrt{\pi}} M_{1,l+\frac{1}{2}}\left(\sqrt{8\mu|E_{00}|r'}\right) I_W(r')\Big|_{\ell_j = l}\right] dr'. \end{aligned} \tag{4.64}$$

Although we were not able to obtain the integral of the second function inside square brackets analytically, we checked numerically that it gave the same result as the integral of the first function inside square brackets, for some values of l. Therefore, the results that follow are written under the hypothesis that this is true for all $l \geq 2$.

Substituting $E_{00} = -\mu^3 M^2/2$ from (4.18), we obtained (again, asymptotically)

$$\begin{aligned}\hat{u}^{lm}_{(2)}(\omega, r) = \frac{A_{lm}(\omega)}{\mu^{4l+1}M^{2l+2}}\frac{2(l+2)\Gamma(4+4l)\Gamma(l)}{4^l\Gamma(3+3l)}\Big[{}_2F_1(l, 4+4l; 3+3l; -1) \\ - 2\,{}_2F_1(l+1, 4+4l; 3+3l; -1)\Big]\frac{1}{r^{l+1}},\end{aligned} \tag{4.65}$$

resulting in the full radial perturbation potential:

$$\begin{aligned}\hat{u}^{lm}(\omega, r) = A_{lm}(\omega)r^l + \frac{\epsilon^2}{r^{l+1}}\frac{A_{lm}(\omega)}{\mu^{4l+1}M^{2l+2}}\frac{2(l+2)\Gamma(4+4l)\Gamma(l)}{4^l\Gamma(3+3l)}\Big[{}_2F_1(l, 4+4l; 3+3l; -1) \\ - 2\,{}_2F_1(l+1, 4+4l; 3+3l; -1)\Big].\end{aligned} \tag{4.66}$$

4.4.2 Case $\ell_i = m_i = 1$

For the next calculation, we choose $\ell_i = m_i = 1$ in (4.17). We also choose $l = 2$ tidal perturbations only, since this is the most important contribution in the expansion and provides insight into how to solve the field equations for higher multipole orders. The procedure is entirely analogous to the previous case. Therefore, inserting these values in (4.28)-(4.31), using the auxiliary formula (B.8) with selection rules (B.9) and performing expansions (4.36)-(4.38) gives the linear system of differential equations:

$$\frac{d^2\hat{u}^{2m}_{(0)}}{dr^2} + \frac{2}{r}\frac{d\hat{u}^{2m}_{(0)}}{dr} - \frac{6}{r^2}\hat{u}^{2m}_{(0)} = 0; \tag{4.67}$$

$$\frac{d^2(\hat{u}^*)^{2m}_{(0)}}{dr^2} + \frac{2}{r}\frac{d(\hat{u}^*)^{2m}_{(0)}}{dr} - \frac{6}{r^2}(\hat{u}^*)^{2m}_{(0)} = 0; \tag{4.68}$$

$$\frac{d^2\hat{u}^{2m}_{(2)}}{dr^2} + \frac{2}{r}\frac{d\hat{u}^{2m}_{(2)}}{dr} - \frac{6}{r^2}\hat{u}^{2m}_{(2)} = S_u; \tag{4.69}$$

$$\frac{d^2(\hat{u}^*)^{2m}_{(2)}}{dr^2} + \frac{2}{r}\frac{d(\hat{u}^*)^{2m}_{(2)}}{dr} - \frac{6}{r^2}(\hat{u}^*)^{2m}_{(2)} = S_{u^*}; \tag{4.70}$$

$$\frac{d^2(\hat{Z}_1)^{\ell_j m_j}_{(1)}}{dr^2} - \left[-2\mu(E_{01}+\omega) - \frac{2\mu^2 M}{r} + \frac{\ell_j(\ell_j+1)}{r^2}\right](\hat{Z}_1)^{\ell_j m_j}_{(1)} = S_{Z_1}; \tag{4.71}$$

$$\frac{d^2(\hat{Z}^*_2)^{\ell_j m_j}_{(1)}}{dr^2} - \left[-2\mu(E_{01}-\omega) - \frac{2\mu^2 M}{r} + \frac{\ell_j(\ell_j+1)}{r^2}\right](\hat{Z}^*_2)^{\ell_j m_j}_{(1)} = S_{Z_2}, \tag{4.72}$$

with the definitions for the source terms:

$$\begin{aligned}S_u \equiv \frac{4\pi\mu}{r}R_{01}\Bigg[-\sqrt{\frac{(m-1)(m-2)}{40\pi}}(\hat{Z}_1)^{1,m+1}_{(1)} + \sqrt{\frac{3(m+3)(m+4)}{280\pi}}(\hat{Z}_1)^{3,m+1}_{(1)} \\ + \sqrt{\frac{(m+1)(m+2)}{40\pi}}(\hat{Z}_2)^{1,m-1}_{(1)} - \sqrt{\frac{3(m-3)(m-4)}{280\pi}}(\hat{Z}_2)^{3,m-1}_{(1)}\Bigg];\end{aligned} \tag{4.73}$$

$$\begin{aligned}S_{u^*} \equiv \frac{4\pi\mu}{r}R_{01}\Bigg[\sqrt{\frac{(m+1)(m+2)}{40\pi}}(\hat{Z}^*_2)^{1,m-1}_{(1)} - \sqrt{\frac{3(m-3)(m-4)}{280\pi}}(\hat{Z}^*_2)^{3,m-1}_{(1)} \\ - \sqrt{\frac{(m-1)(m-2)}{40\pi}}(\hat{Z}^*_1)^{1,m+1}_{(1)} + \sqrt{\frac{3(m+3)(m+4)}{280\pi}}(\hat{Z}^*_1)^{3,m+1}_{(1)}\Bigg];\end{aligned} \tag{4.74}$$

$$S_{Z_1} \equiv 2\mu^2 r R_{01}\left[-\sqrt{\frac{(m_j-2)(m_j-3)}{40\pi}}\hat{u}^{2,m_j-1}_{(0)}\delta_{1\ell_j}+\sqrt{\frac{3(m_j+2)(m_j+3)}{280\pi}}\hat{u}^{2,m_j-1}_{(0)}\delta_{3\ell_j}\right]; \quad (4.75)$$

$$S_{Z_2} \equiv 2\mu^2 r R_{01}\left[\sqrt{\frac{(m_j+2)(m_j+3)}{40\pi}}(\hat{u}^*)^{2,m_j+1}_{(0)}\delta_{1\ell_j}-\sqrt{\frac{3(m_j-2)(m_j-3)}{280\pi}}(\hat{u}^*)^{2,m_j+1}_{(0)}\delta_{3\ell_j}\right]. \quad (4.76)$$

Comparing (4.67)-(4.72) with (4.39)-(4.44) we see that exactly the same arguments may be used to show that all functions are real, $\hat{u}^{2m}_{(0)}(\omega,r)=A_m(\omega)r^2$ such that $A_m(\omega)=c_m\delta(\omega-m\Omega_{\text{orb}})$ with c_m a constant which vanishes when m is odd, $A_{-m}(-\omega)=(-1)^m A_m(\omega)$ and $(\hat{Z}_1)^{\ell_j m_j}_{(1)}(\omega,r)=(-1)^{m_j}(\hat{Z}^*_2)^{\ell_j,-m_j}_{(1)}(-\omega,r)$. Using these properties and the linearity of the system to write

$$(\hat{Z}_1)^{\ell_j m_j}_{(1)}(\omega,r)=(\hat{Z}_1)^{\ell_j m_j}_{(1),t}(\omega,r)\delta(\omega-(m_j-1)\Omega_{\text{orb}}); \quad (4.77)$$

$$\hat{u}^{2m}_{(2)}(\omega,r)=\hat{u}^{2m}_{(2),t}(\omega,r)\delta(\omega-m\Omega_{\text{orb}}), \quad (4.78)$$

we find that it reduces to the following two equations:

$$\frac{d^2(\hat{Z}_1)^{\ell_j m_j}_{(1),t}}{dr^2}-\left[-2\mu(E_{01}+\omega)-\frac{2\mu^2 M}{r}+\frac{\ell_j(\ell_j+1)}{r^2}\right](\hat{Z}_1)^{\ell_j m_j}_{(1),t}(\omega,r)=S^{\ell_j m_j}_{Z,t}(r); \quad (4.79)$$

$$\frac{d^2\hat{u}^{2m}_{(2),t}}{dr^2}+\frac{2}{r}\frac{d\hat{u}^{2m}_{(2),t}}{dr}-\frac{6}{r^2}\hat{u}^{2m}_{(2),t}(\omega,r)=S_{u,t}(\omega,r), \quad (4.80)$$

with the respective source terms

$$S^{\ell_j m_j}_{Z,t}(r)\equiv 2\mu^2 c_{m_j-1}r^3 R_{01}(r)\left[-\sqrt{\frac{(m_j-2)(m_j-3)}{40\pi}}\delta_{1\ell_j}+\sqrt{\frac{3(m_j+2)(m_j+3)}{280\pi}}\delta_{3\ell_j}\right]; \quad (4.81)$$

$$\begin{aligned}S_{u,t}(\omega,r)\equiv{}&\frac{4\pi\mu}{r}R_{01}(r)\Bigg[-\sqrt{\frac{(m-1)(m-2)}{40\pi}}(\hat{Z}_1)^{1,m+1}_{(1),t}(\omega,r)-(-1)^m\sqrt{\frac{(m+1)(m+2)}{40\pi}}\\&\times(\hat{Z}_1)^{1,-m+1}_{(1),t}(-\omega,r)+\sqrt{\frac{3(m+3)(m+4)}{280\pi}}(\hat{Z}_1)^{3,m+1}_{(1),t}(\omega,r)+(-1)^m\sqrt{\frac{3(-m+3)(-m+4)}{280\pi}}\\&\times(\hat{Z}_1)^{3,-m+1}_{(1),t}(-\omega,r)\Bigg].\end{aligned} \quad (4.82)$$

Clearly (4.79) only needs to be solved for $\ell_j=1$ and $\ell_j=3$ as well as $m_j=\pm m+1$, in order to solve (4.80). Proceeding like in the previous case, the solutions and Wronskian of the homogeneous version of (4.79):

$$(\hat{Z}_{1,-})^{\ell_j m_j}_{(1),t}(\omega,r)=h_1 M_{\frac{2}{\sqrt{1+\frac{\omega}{E_{01}}}},\ell_j+\frac{1}{2}}\left(\sqrt{-8\mu(E_{01}+\omega)}r\right); \quad (4.83)$$

$$(\hat{Z}_{1,+})^{\ell_j m_j}_{(1),t}(\omega,r)=h_2 W_{\frac{2}{\sqrt{1+\frac{\omega}{E_{01}}}},\ell_j+\frac{1}{2}}\left(\sqrt{-8\mu(E_{01}+\omega)}r\right); \quad (4.84)$$

$$W^{\ell_j}_Z=-h_1h_2\frac{\Gamma(2+2\ell_j)\sqrt{-8\mu(E_{01}+\omega)}}{\Gamma\left(\ell_j+1-\frac{2}{\sqrt{1+\frac{\omega}{E_{01}}}}\right)} \quad (4.85)$$

give the solutions with the intended boundary conditions:

$$\begin{aligned}(\hat{Z}_1)^{1\pm m+1}_{(1),t}(\omega,r) &= \frac{(\hat{Z}_{1,+})^{1,\pm m+1}_{(1),t}(\omega,r)}{W^1_Z}\int_0^r (\hat{Z}_{1,-})^{1,\pm m+1}_{(1),t}(\omega,r')S^{1,\pm m+1}_{Z,t}(\omega,r')dr' \\ &+ \frac{(\hat{Z}_{1,-})^{1,\pm m+1}_{(1),t}(\omega,r)}{W^1_Z}\int_r^\infty (\hat{Z}_{1,+})^{1,\pm m+1}_{(1),t}(\omega,r')S^{1,\pm m+1}_{Z,t}(\omega,r')dr' \\ &= 2\mu^2\frac{(8\mu|E_{01}|)^{5/4}}{\sqrt{24}}c_{\pm m}\sqrt{\frac{(\pm m-1)(\pm m-2)}{40\pi}}\frac{1}{\Gamma(4)\sqrt{8\mu|E_{01}|}}\left[\frac{720r^2e^{-\sqrt{8\mu|E_{01}|}r/2}}{(\sqrt{8\mu|E_{01}|})^3}\frac{E_{01}}{\omega}\right. \\ &\left.+\frac{2r^2e^{-\sqrt{8\mu|E_{01}|}r/2}}{(\sqrt{8\mu|E_{01}|})^3}\left(-480+9(\sqrt{8\mu|E_{01}|}r)^2+(\sqrt{8\mu|E_{01}|}r)^3\right)\right]+\mathcal{O}\left(\frac{\omega}{E_{01}}\right)\end{aligned} \tag{4.86}$$

and

$$\begin{aligned}(\hat{Z}_1)^{3\pm m+1}_{(1),t}(\omega,r) &= \frac{(\hat{Z}_{1,+})^{3,\pm m+1}_{(1),t}(\omega,r)}{W^3_Z}\int_0^r (\hat{Z}_{1,-})^{3,\pm m+1}_{(1),t}(\omega,r')S^{3,\pm m+1}_{Z,t}(\omega,r')dr' \\ &+ \frac{(\hat{Z}_{1,-})^{3,\pm m+1}_{(1),t}(\omega,r)}{W^3_Z}\int_r^\infty (\hat{Z}_{1,+})^{3,\pm m+1}_{(1),t}(\omega,r')S^{3,\pm m+1}_{Z,t}(\omega,r')dr' \\ &= -2\mu^2\frac{(8\mu|E_{01}|)^{5/4}}{\sqrt{24}}c_{\pm m}\sqrt{\frac{3(\pm m+3)(\pm m+4)}{280\pi}}\frac{1}{\Gamma(8)\sqrt{8\mu|E_{01}|}} \\ &\times\frac{1680e^{-\sqrt{8\mu|E_{01}|}r/2}}{\sqrt{8\mu|E_{01}|}}\left(4r^4+\sqrt{8\mu|E_{01}|}r^5\right)+\mathcal{O}\left(\frac{\omega}{E_{01}}\right).\end{aligned} \tag{4.87}$$

For these calculations, the following derivatives of the confluent hypergeometric functions were required [96]:

$$\frac{\partial}{\partial a}M(a,b,z)\Big|_{a=0} = \frac{z}{b}{}_2F_2(1,1;2,b+1;z), \qquad \frac{\partial}{\partial a}U(a,4,z)\Big|_{a=0} = \frac{2}{z^3}+\frac{3}{z^2}+\frac{3}{z}-\log(z). \tag{4.88}$$

where ${}_2F_2$ is a generalized hypergeometric function. Proceeding to solve (4.80) asymptotically, we have $\hat{u}^{2m}_+(\omega,r)\equiv d_2(\omega)r^{-3}$ and $\hat{u}^{2m}_-(\omega,r)\equiv d_1(\omega)r^2$ as the solutions of the homogeneous equation and the Wronskian $W_u(r)=-5d_1(\omega)d_2(\omega)/r^2$. Hence, up to order ω/E_{01},

$$\begin{aligned}\hat{u}^{2m}_{(2),t}(\omega,r) &= \hat{u}^{2m}_+(\omega,r)\int_0^r\frac{\hat{u}^{2m}_-(\omega,r')S_{u,t}(\omega,r')}{W(r')}dr'+\hat{u}^{(2)}_-(\omega,r)\int_r^\infty\frac{\hat{u}^{2m}_+(\omega,r')S_{u,t}(\omega,r')}{W(r')}dr' \\ &\longrightarrow -\frac{1}{5r^3}\int_0^\infty r'^4S_{u,t}(\omega,r')dr' \\ &= \frac{c_m}{r^3}\left(-\frac{864m}{\mu^9M^6}\frac{E_{01}}{\omega}+\frac{3840+672m^2}{\mu^9M^6}\right),\end{aligned}$$

resulting in (substituting $E_{01}=-\mu^3M^2/8$ from (4.18))

$$\hat{u}^{2m}(\omega,r) = A_m(\omega)r^2+\frac{\epsilon^2A_m(\omega)}{r^3}\left(\frac{108m}{\mu^6M^4}\frac{1}{\omega}+\frac{3840+672m^2}{\mu^9M^6}\right). \tag{4.89}$$

4.5 Final Results

As we've seen throughout Section 4.4, we've obtained a metric expansion which depends on the number m and therefore, we need to rewrite (3.41) so as to define m-dependent TLNs. All we need to do, according to Refs. [85, 97] and equation (3.39), is to recover the m dependence of the mass multipole moments and the tidal moments (as well as the spherical-harmonic functions of course):

$$\begin{aligned} g_{00} = -1 + \frac{2M}{r} + \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \Bigg\{ \frac{2}{r^{l+1}} \left[\sqrt{\frac{4\pi}{2l+1}} M_{lm} Y_{lm}(\theta,\varphi) + (l' < l \text{ pole}) \right] \\ - \frac{2}{l(l-1)} r^l [\mathcal{E}_{lm} Y_{lm}(\theta,\varphi) + (l' < l \text{ pole})] \Bigg\}. \end{aligned} \tag{4.90}$$

Redefining the polar-type gravitational TLNs as

$$k_{lm}^P \equiv -\frac{1}{2} \frac{l(l-1)}{M^{2l+1}} \sqrt{\frac{4\pi}{2l+1}} \frac{M_{lm}}{\mathcal{E}_{lm}}, \tag{4.91}$$

we get

$$g_{00} = -1 + \frac{2M}{r} + \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \left\{ -\frac{2}{l(l-1)} r^l \left[1 + 2k_{lm}^P \left(\frac{M}{r} \right)^{2l+1} \right] \mathcal{E}_{lm} Y_{lm}(\theta,\varphi) + \cdots \right\}, \tag{4.92}$$

where dots represent all the terms originating from the $l' < l$ poles.

4.5.1 Case $\ell_i = m_i = 0$

In order to split the dependency in m and ω, let us write $\hat{u}^{lm}(\omega, r) = A_{lm}(\omega) H_l(r)$ with the definition

$$H_l(r) \equiv r^l + \frac{\epsilon^2}{r^{l+1}} \frac{1}{\mu^{4l+1} M^{2l+2}} \frac{2(l+2)\Gamma(4+4l)\Gamma(l)}{4^l \Gamma(3+3l)} \Big[{}_2F_1(l, 4+4l; 3+3l; -1) - 2\, {}_2F_1(l+1, 4+4l; 3+3l; -1) \Big]. \tag{4.93}$$

Then, using (4.21), the fact that $\hat{u}^{lm}$ is real and property (B.3), the 00 component of the metric in the Newtonian limit is given by:

$$\begin{aligned} g_{00} &= -1 - 2U = -1 - 2(U_A + \epsilon_p \delta U_T + \epsilon^2 \delta U) \\ &= -1 + \frac{2M}{r} - 2\epsilon_p \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \left[\int \frac{d\omega}{2\pi} \hat{u}^{lm}(\omega, r) Y_{lm}(\theta,\varphi) e^{-i\omega t} + \int \frac{d\omega}{2\pi} (\hat{u}^*)^{lm}(\omega, r) Y_{lm}^*(\theta,\varphi) e^{i\omega t} \right] - 2\epsilon^2 \delta U \\ &= -1 + \frac{2M}{r} - \frac{\epsilon_p}{\pi} \sum_{l=2}^{\infty} \sum_{m=-l}^{l} H_l(r) c_{lm} Y_{lm}(\theta,\varphi) e^{-im\Omega_{\text{orb}} t} - 2\epsilon^2 \delta U \\ &= -1 + \frac{2M}{r} - \frac{\epsilon_p}{\pi} \sum_{l=2}^{\infty} \sum_{m=-l}^{l} r^l \Bigg\{ + \frac{\epsilon^2}{r^{2l+1}} \frac{1}{\mu^{4l+1} M^{2l+2}} \frac{2(l+2)\Gamma(4+4l)\Gamma(l)}{4^l \Gamma(3+3l)} \Big[{}_2F_1(l, 4+4l; 3+3l; -1) \\ &\quad - 2\, {}_2F_1(l+1, 4+4l; 3+3l; -1) \Big] \Bigg\} c_{lm} Y_{lm}(\theta,\varphi) e^{-im\Omega_{\text{orb}} t} - 2\epsilon^2 \delta U. \end{aligned} \tag{4.94}$$

Note that there is no sum when $l+m$ is odd due to the vanishing of c_{lm}. Thus, we can only define a tidal response when l and m have the same parity (i.e. $l+m$ is even). Comparing with (4.92), we find that for each l,

$$k_l^P = \frac{\epsilon^2}{\mu^{4l+1}M^{4l+3}} \frac{(l+2)\Gamma(4+4l)\Gamma(l)}{4^l\Gamma(3+3l)} \Big[{}_2F_1(l,4+4l;3+3l;-1) - 2\,{}_2F_1(l+1,4+4l;3+3l;-1)\Big] \quad (4.95)$$

for those values of m having the same parity as l.

It is interesting to write this result in terms of parameters related to the scalar cloud. We will do this using the cloud's mass M_c and radius r_c. In the small-coupling limit we have, up to order ϵ^2 in perturbations, that the energy-density of the scalar field is given by $\rho_\text{S} = -(T_0^0)^\text{S} \simeq \mu\epsilon^2|\psi|^2$ and therefore:

$$M_c = \int \rho_\text{S}\, d^3x = \mu\epsilon^2 \int |\psi|^2 d^3x \simeq \mu\epsilon^2 \quad (4.96)$$

by inserting (4.17) with $n = \ell_i = m_i = 0$. On the other hand, $r_c \sim (n+\ell_i+1)^2/(\mu^2 M) = 1/(\mu^2 M)$ from Section 2.3. Hence:

$$\begin{aligned} k_l^P &= \frac{(l+2)\Gamma(4+4l)\Gamma(l)}{4^l\Gamma(3+3l)} \Big[{}_2F_1(l,4+4l;3+3l;-1) - 2\,{}_2F_1(l+1,4+4l;3+3l;-1)\Big] \frac{1}{(\mu M)^{4l+2}} \left(\frac{M_c}{M}\right); \\ &= \frac{(l+2)\Gamma(4+4l)\Gamma(l)}{4^l\Gamma(3+3l)} \Big[{}_2F_1(l,4+4l;3+3l;-1) - 2\,{}_2F_1(l+1,4+4l;3+3l;-1)\Big] \frac{M_c r_c^{2l+1}}{M^{2l+2}}, \end{aligned} \quad (4.97)$$

when l and m have the same parity. The scaling we find with the radius of the cloud is in perfect agreement with the Newtonian TLNs of other matter systems, see e.g. Refs. [45, 83].

4.5.2 Case $\ell_i = m_i = 1$

Using equation (4.89) and the same identities as in the metric expansion of the previous case gives

$$\begin{aligned} g_{00} &= -1 - 2U = -1 - 2(U_0 + \epsilon^2\delta U + \epsilon_p \delta U_T) = -1 + \frac{2M}{r} - 2\epsilon_p \sum_{m=-2}^{2} \Bigg[\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \hat{u}^{2m}(\omega,r) Y_{2m}(\theta,\varphi) e^{-i\omega t} \\ &+ \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} (\hat{u}^*)^{2m}(\omega,r) Y_{2m}^*(\theta,\varphi) e^{i\omega t} \Bigg] - 2\epsilon^2 \delta U \\ &= -1 + \frac{2M}{r} - \frac{2\epsilon_p}{\pi} \sum_{\substack{m=-2 \\ m\neq 0}}^{2} r^2 \left[1 + \frac{\epsilon^2}{r^5} \left(\frac{108}{\mu^6 M^4 \Omega_\text{orb}} + \frac{3840 + 672m^2}{\mu^9 M^6} + \mathcal{O}\left(\frac{\Omega_\text{orb}}{\mu^3 M^2}\right)\right)\right] \\ &\times c_m Y_{2m}(\theta,\varphi) e^{-im\Omega_\text{orb} t} - \frac{2\epsilon_p}{\pi} r^2 \left(1 + \frac{\epsilon^2}{r^5}\frac{3840}{\mu^9 M^6}\right) c_0 Y_{20}(\theta,\varphi) - 2\epsilon^2\delta U, \end{aligned} \quad (4.98)$$

with $c_m = 0$ for odd values of m. Comparing with (4.92), one obtains

$$k_{20}^P = \frac{1920\epsilon^2}{\mu^9 M^{11}}; \qquad k_{2,\pm 2}^P = \frac{54\epsilon^2}{\mu^6 M^9 \Omega_\text{orb}} + \frac{3264\epsilon^2}{\mu^9 M^{11}}, \quad (4.99)$$

where the result for $m = \pm 2$ is valid up to order $\Omega_{\text{orb}}/(\mu^3 M^2)$.

As it turns out, the mass of the cloud in this case is the same as the previous case (4.96), up to order ϵ^2. However, the size gives $r_c \sim (n + \ell_i + 1)^2/(\mu^2 M) = 4/(\mu^2 M)$, therefore

$$k^P_{2,\pm 2} = \left(\frac{54\mu^3 M^2}{\Omega_{\text{orb}}} + 3264\right)\frac{1}{(\mu M)^{10}}\left(\frac{M_c}{M}\right) = \left(\frac{27}{512}\frac{\mu^3 M^2}{\Omega_{\text{orb}}} + \frac{51}{16}\right)\frac{M_c r_c^5}{M^6}; \tag{4.100}$$

$$k^P_{20} = \frac{1920}{(\mu M)^{10}}\left(\frac{M_c}{M}\right) = \frac{15}{8}\frac{M_c r_c^5}{M^6}. \tag{4.101}$$

Chapter 5

Conclusions

Throughout this thesis, we described how a bosonic cloud may be formed around a BH due to superradiant instabilities and we tried to provide a unified discussion of tidal Love numbers including Newtonian gravity and General Relativity. The first part introduced the system in which Chapter 4 would focus and the second presented the framework which allowed us to make the calculations of the TLNs. As we saw in equation (4.12), in the Newtonian limit, the background field is described, at lowest order, by a Schrödinger equation analogous to the one of the Hydrogen atom with a Coulomb interaction between the BH and the field, which justifies the name ”gravitational atom”. The BH was assumed to be a point particle.

We successfully computed the polar-type TLNs in the Newtonian limit by solving the field equations of the Schrödinger-Poisson system (4.6), (4.7) which we showed in Appendix C to be derived from the Newtonian limit of the Einstein field equations (1.1) and the Klein-Gordon equation $\Box\Phi = \mu^2 c^2 \Phi$. We assumed the external body perturbating the system describes circular orbits.

The cases where the background field describing the cloud is in the $\ell_i = m_i = 0$ and $\ell_i = m_i = 1$ modes were considered. In the first case, a result for the TLNs in all multipole orders l of the system in equation (4.97) was obtained and in the second, we arrived at equations (4.100) and (4.101) for the TLNs describing the quadrupolar response. In the former case, the TLNs are not defined when l and m have the same parity and in the latter case, the TLNs are not defined for odd m (that is, $m = \pm 1$).

For the first case, an important note is in order regarding the integral (4.64). Although the integral of the term with the Whittaker function $W_{1,l+\frac{1}{2}}$ can be easily computed, the integral of the term involving $M_{1,l+\frac{1}{2}}$ was not analytically obtained. We verified for some values of l that the two gave the same result and we assumed that to be the case for all other values of l, although an exact proof is missing. However, integral (4.64) can be computed for specific values of l and the TLNs exactly obtained for those values. We verified that equation (4.97) was correct for $l = 2$ and $l = 3$. Equation (4.97) is valid up to order $\Omega_{\text{orb}}^2/(\mu^3 M^2)^2$.

In the second case, we surprisingly found that the TLNs with $m \neq 0$ have a term of order $\mu^3 M^2/\Omega_{\text{orb}}$ which diverges when $\Omega_{\text{orb}} \to 0$. This is thought to originate from the vanishing of the Wronskian (4.85) when $\ell_j = 1$. Equation (4.100) is valid up to order $\Omega_{\text{orb}}/\mu^3 M^2$ and equation (4.101) is valid at all orders since the higher-order terms are proportional to m, which vanishes in this case. Note that this procedure can be easily generalized to higher multipoles l simply from using the selection rules (B.9) in the system (4.28)-(4.31).

The reason why the term of order $\Omega_{\text{orb}}/\mu^3 M^2$ was not computed is the difficulty that was encountered in computing derivatives of the confluent hypergeometric function $U(a, b, z)$ with respect to the first parameter. We were able to obtain expression (4.88) when $a = 0$ but not for other values of a. Unfortunately, no useful mathematical literature was found which was useful in solving this problem. It was also the reason why we were not able to solve the perturbation equations for excited states $n \neq 0$ of the background field. Nevertheless, these difficulties could of course be circumvented by solving the equations numerically for given values of the parameters but our focus was to find analytical formulas for the TLNs, which we hope will act as a benchmark point for future numerical works.

One of the motivations for this work was Ref. [27] where the scalar and axial vector TLNs[1] of a BH surrounded by a scalar cloud were computed in a relativistic framework. For the TLNs of the quadrupolar mode $l = 2$, we found a dependence of α^{-10} in the coupling constant different from their result of α^{-8}. However, our results for the Newtonian gravitational TLNs agree in the dependence of the radius of the cloud with Ref. [45] where TLNs of BHs surrounded by matter shells were determined. The dependence on the radius we find is also in agreement with the one in Ref. [83], where TLNs of boson stars were considered, which further supports our claim that the results of [27] are not an accurate toy model for the Newtonian gravitational TLNs we computed. Note that [27] says that their dependence on α agrees with the dimensional analysis made in [68], but we believe that this analysis is also incomplete given that the variables which multiply the term $(0.1/\alpha)^8$ (see equation (4.11) in Ref. [68]) may depend on α and therefore one is left unclarified as to what the dependence on the coupling constant actually is. Since the TLNs affect the "gravitational waveforms" of a binary system, the detectability analysis made in [27] should be revisited.

5.1 Future Work

This is the first work in which Newtonian TLNs for a BH surrounded by a scalar cloud were computed. The procedure developed in Section 4.4 seems to be easily applied for higher l multipoles of the tidal field. One could then also see the effect that these higher modes have on the induced multipoles of lower l modes. Another possible generalization of these calculations are the states with $n \neq 0$, but in order to solve the field equations, one has to tackle the mathematical difficulty of computing derivatives of confluent hypergeometric functions with respect to the first argument.

It is our plan to compute the gravitational TLNs of a BH surrounded by a bosonic cloud in a fully relativistic context, where one can then take the Newtonian limit and compare with the results obtained here.

Finally, a possible interesting scenario where one could apply the procedure of Chapter 4 is when the BH is surrounded by a massive vector (or Proca) field.

[1] Undefined in this thesis. For details, see [83].

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Appendix A

Radial Functions and Quasibound States

Here we present the necessary steps to go from equation (2.7) to (2.8). We will use the matching asymptotics method, which consists of solving the differential equation in the regions of large r and small r and then matching the coefficients of the solutions. Note that, since the eigenvalues of the angular equation are not exactly integers (because of the power series we wrote in Section 2.2 or, in other words, because the BH is spinning), we will take into account $\lambda = \ell'(\ell'+1)$ with $\ell' = \ell + \epsilon$ (where ϵ is a small parameter) from the beginning.

Let us first consider $r \gg M$. Remembering that we are in the small-coupling and low-frequency regime and that $M \geq a$, then we may choose to keep only terms of order r^2/M^2 or higher:

$$\omega^2(r^2+a^2)^2 - 4aMrm\omega + a^2m^2 - \Delta[\mu^2r^2 + a^2\omega^2 + \ell'(\ell'+1)] \simeq (\omega^2-\mu^2)r^4 + 2M\mu^2r^3 - \ell'(\ell'+1)r^2. \tag{A.1}$$

On the other hand, using $\Delta \simeq r^2$, we may verify that

$$\Delta\frac{d}{dr}\left(\Delta\frac{dR}{dr}\right) \simeq r^3\frac{d^2}{dr^2}(rR). \tag{A.2}$$

Therefore, (2.7) becomes, in this limit,

$$\frac{d^2}{dr^2}(rR) + \left[\omega^2 - \mu^2 + \frac{2M\mu^2}{r} - \frac{\ell'(\ell'+1)}{r^2}\right]rR = 0 \tag{A.3}$$

up to order M^2/r^2.

Defining the auxiliary variables $k^2 \equiv \mu^2 - \omega^2$, $\nu \equiv M\mu^2/k$ and $x \equiv 2kr$ and substituting above gives

$$\frac{d^2}{dx^2}(xR) + \left[-\frac{1}{4} + \frac{\nu}{x} - \frac{\ell'(\ell'+1)}{x^2}\right]xR = 0 \tag{A.4}$$

which is the Whittaker equation [74] and therefore, its solution is a linear combination of the Whittaker functions $M_{\nu,\ell'+\frac{1}{2}}(x)$ and $W_{\nu,\ell'+\frac{1}{2}}(x)$. However, we want to impose a decaying wave at infinity (or, in other words, we want the radial wave function to be normalizable) which implies that the correct solution for (A.4) is $xR \propto W_{\nu,\ell'+\frac{1}{2}}(x) = x^{\ell'+1}e^{-x/2}U(\ell'+1-\nu, 2\ell'+2, x)$ where U is a confluent hypergeometric function (also called Tricomi's function) and, consequently, the solution to (2.7) when $r \gg M$ is

$$R(r) = C_1(2kr)^{\ell'}e^{-kr}U(\ell'+1-\nu, 2\ell'+2, 2kr) \tag{A.5}$$

with C_1 a constant.

We now turn to the limit $r \ll \max(\ell/\omega, \ell/\mu)$ of (2.7). If one, once again, defines new variables $z = (r-r_+)/(r_+-r_-)$ and $P = (am - 2Mr_+\omega)/(r_+-r_-)$, then $\Delta = z(z+1)(r_+-r_-)^2$ and (2.7) becomes [98]

$$\begin{aligned} z(z+1)\frac{d}{dz}\left[z(z+1)\frac{dR}{dz}\right] + \Bigg\{P^2 + \left[\frac{8M\omega r_+}{r_+-r_-}\left(\omega r_+ - \frac{am}{2r_+}\right) - \ell'(\ell'+1) - r_+^2\mu^2 - a^2\omega^2\right]z \\ + [a^2\omega^2 - \ell'(\ell'+1) + 2\mu^2a^2 - 3\mu^2r_+^2 + 6r_+^2\omega^2]z^2 + 2(r_+-r_-)[2r_+(\omega^2-\mu^2) + M\mu^2]z^3 \\ + (r_+-r_-)^2(\omega^2-\mu^2)z^4\Bigg\}R = 0. \end{aligned} \tag{A.6}$$

Since $r_\pm = \mathcal{O}(M)$, neglecting all terms of order $(\mu M)^2$ or $(\omega M)^2$ results in

$$z(z+1)\frac{d}{dz}\left[z(z+1)\frac{dR}{dz}\right] + [P^2 - \ell'(\ell'+1)z(z+1)]R = 0. \tag{A.7}$$

The general solution is a linear combination of the associated Legendre polynomials $P_{\ell'}^{\pm 2iP}(1+2z)$ and $Q_{\ell'}^{\pm 2iP}(1+2z)$ (with the same sign in each case. Both signs are possible because of the symmetries of these functions and because the differential equation depends on P^2 so that the defining sign of P is irrelevant). We choose the negative signs as convention. Near the event horizon, $z \sim 0$ and the solutions become $P_{\ell'}^{-2iP}(1+2z) \sim z^{iP}$ and $Q_{\ell'}^{-2iP}(1+2z) \sim z^{-iP}$. We wish to impose only ingoing waves at the event horizon, which means the solution should be proportional to e^{-ikr_*} (the reason for this is that equation (2.7) reduces to a Schrödinger-type equation [72, 73] in terms of r_* whose solutions are given by $e^{\pm ikr_*}$) where r_* is the tortoise coordinate defined by [99]

$$r_* = \int \frac{r^2+a^2}{\Delta}dr = r + \frac{2Mr_+}{r_+-r_-}\ln\left(\frac{r-r_+}{2M}\right) - \frac{2Mr_-}{r_+-r_-}\ln\left(\frac{r-r_-}{2M}\right) \tag{A.8}$$

up to an integration constant, and $k \equiv \omega - am/(2Mr_+)$. Noting that $P = -2Mr_+k/(r_+-r_-)$, one can show, using (A.8), that $z^{\pm iP} \sim e^{\mp ikr_*}$ near the event horizon. Hence, the appropriate functions are $P_{\ell'}^{-2iP}(1+2z)$ and from the relation [74]

$$P_\beta^\alpha(x) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{1+x}{1-x}\right)^{\alpha/2} {}_2F_1\left(-\beta, \beta+1; 1-\alpha; \frac{1-x}{2}\right), \tag{A.9}$$

where ${}_2F_1$ is a hypergeometric function, one obtains

$$
\begin{aligned}
R(z) &= C_2\left(\frac{z}{z+1}\right)^{iP} {}_2F_1(-\ell',\ell'+1;1+2iP;-z)\\
\Rightarrow R(r) &= C_2\left(\frac{r-r_+}{r-r_-}\right)^{iP} {}_2F_1\left(-\ell',\ell'+1;1+2iP;-\frac{r-r_+}{r_+-r_-}\right)
\end{aligned}
\tag{A.10}
$$

with a constant C_2.

Before comparing the two radial solutions, we need to make an observation regarding the eigenvalues of equation (A.3). Notice that this equation is similar to the one for an electron in a hydrogen atom so that the solution should also be similar. In fact, since [74] $L_n^{(\alpha)}(x) \propto U(-n,\alpha+1,x)$, by comparing the first argument of U in (A.5) with the subscript index of the generalized Laguerre polynomial in the radial wave function corresponding to an electron in a hydrogen atom, one concludes that ν is the equivalent of the principal quantum number n. Therefore, it must also be quantized according to the relation $\nu = \ell' + 1 + n$ where n is non-negative (this n is not the principal quantum number of the hydrogen atom, we just chose this letter in convention with the literature) and gives the number of nodes for each solution. However, in the case of a BH, the boundary condition for small values of the radial distance (i.e. at the event horizon) is different from the boundary condition of the radial wave function for the electron in the hydrogen atom, given that in the former case there are ingoing waves and in the latter case there is regularity. Given this boundary condition for the case of the BH, we expect the field amplitude to either decay or grow in time due to energy conservation. This means that the eigenfrequencies ω should be complex (as we will explain below) and, consequently, the respective mode ν should also be complex. Hence, we write $\nu - \ell' - 1 = n + \delta\nu$ with $\delta\nu$ a small complex number and (A.5) becomes $R(r) = C_1(2kr)^{\ell'} e^{-kr} U(-n-\delta\nu, 2\ell'+2, 2kr)$.

Let us proceed with the matching of the two solutions (A.5) and (A.10). Given that, as we've mentioned previously, $\ell' = \ell + \epsilon$ is not an integer, we are allowed to use the relation [74]

$$
U(-n-\delta\nu,2\ell'+2,2kr) = \frac{\Gamma(-2\ell'-1)}{\Gamma(-2\ell'-1-n-\delta\nu)} M(-n-\delta\nu,2\ell'+2,2kr) \tag{A.11}
$$

$$
+\frac{\Gamma(1+2\ell')}{\Gamma(-n-\delta\nu)}(2kr)^{-2\ell'-1} M(-2\ell'-1-n-\delta\nu,-2\ell',2kr). \tag{A.12}
$$

Since $M(a,b,z)\sim 1$ and $e^{-z}\sim 1$ for $z \ll 1$, then (A.5) is given by

$$
R(r) \sim C_1 \frac{\Gamma(-2\ell'-1)}{\Gamma(-2\ell'-1-n-\delta\nu)}(2kr)^{\ell'} + C_1\frac{\Gamma(1+2\ell')}{\Gamma(-n-\delta\nu)}(2kr)^{-\ell'-1} \tag{A.13}
$$

when $kr \ll 1$. On the other hand, performing the transformation [74]

$$
\begin{aligned}
{}_2F_1(-\ell',\ell'+1;1+2iP;-z) &= \frac{\Gamma(1+2iP)\Gamma(2\ell'+1)}{\Gamma(\ell'+1)\Gamma(\ell'+1+2iP)} z^{\ell'} {}_2F_1\left(-\ell',-\ell'-2iP;-2\ell';-\frac{1}{z}\right)\\
&+\frac{\Gamma(1+2iP)\Gamma(-2\ell'-1)}{\Gamma(-\ell')\Gamma(-\ell'+2iP)} z^{-\ell'-1} {}_2F_1\left(\ell'+1,\ell'+1-2iP;2\ell'+2;-\frac{1}{z}\right),
\end{aligned}
\tag{A.14}
$$

we may write

$$R(r) \sim C_2 \left(\frac{r}{r-r_-}\right)^{iP} \left[\frac{\Gamma(1+2iP)\Gamma(2\ell'+1)}{\Gamma(\ell'+1)\Gamma(\ell'+1+2iP)} \left(\frac{r}{r_+-r_-}\right)^{\ell'} + \frac{\Gamma(1+2iP)\Gamma(-2\ell'-1)}{\Gamma(-\ell')\Gamma(-\ell'+2iP)} \left(\frac{r}{r_+-r_-}\right)^{-\ell'-1}\right] \tag{A.15}$$

as the asymptotic expression of (A.10) when $r/M \gg \max(P,\ell)$ (or $z \gg 1$ - that's why we used ${}_2F_1(a,b;c;-1/z) \sim 1$). Note that our matching region is $M\max(P,\ell) \ll r \ll 1/k$. Equating the coefficients of $r^{\ell'}$ and $r^{-\ell'-1}$ for both solutions in this region we get:

$$C_1 \frac{\Gamma(-2\ell'-1)}{\Gamma(-2\ell'-1-n-\delta\nu)}(2k)^{\ell'} = C_2 \left(\frac{r}{r-r_-}\right)^{iP} \frac{\Gamma(1+2iP)\Gamma(2\ell'+1)}{\Gamma(\ell'+1)\Gamma(\ell'+1+2iP)} \frac{1}{(r_+-r_-)^{\ell'}} \tag{A.16}$$

$$C_1 \frac{\Gamma(1+2\ell')}{\Gamma(-n-\delta\nu)}(2k)^{-\ell'-1} = C_2 \left(\frac{r}{r-r_-}\right)^{iP} \frac{\Gamma(1+2iP)\Gamma(-2\ell'-1)}{\Gamma(-\ell')\Gamma(-\ell'+2iP)} \frac{1}{(r_+-r_-)^{-\ell'-1}}. \tag{A.17}$$

After solving the system, one finds the following relation:

$$\frac{\Gamma(\ell'+1+2iP)}{\Gamma(-\ell'+2iP)}(2k)^{2\ell'+1} = \frac{\Gamma(-2\ell'-1-n-\delta\nu)}{\Gamma(-2\ell'-1)\Gamma(-n-\delta\nu)} \frac{\Gamma(-\ell')}{\Gamma(-2\ell'-1)} \frac{\Gamma(2\ell'+1)^2}{\Gamma(\ell'+1)}(r_+-r_-)^{-2\ell'-1}. \tag{A.18}$$

We will now compute the ratios of the Gamma functions involving negative quantities, having also in mind that $\delta\nu \ll 1$. We should remember that, since we are using the perturbing quantity ϵ, the arguments are well-defined and therefore, we may use Euler's reflection formula $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ and $\Gamma(1+z) = z\Gamma(z)$. By taking $\epsilon \to 0$, we get:

$$\begin{aligned}
&\lim_{\epsilon\to 0} \frac{\Gamma(-2\ell'-1-n-\delta\nu)}{\Gamma(-2\ell'-1)\Gamma(-n-\delta\nu)} = \lim_{\epsilon\to 0} \frac{1}{\Gamma(-2\ell'-1)}[\Gamma(-2\ell'-1-n)+\mathcal{O}(\delta\nu)][(-1)^{n+1}n!\delta\nu+\mathcal{O}(\delta\nu^2)] \\
&= \lim_{\epsilon\to 0} \frac{\Gamma(-2\ell'-1-n)}{\Gamma(-2\ell'-1)}(-1)^{n+1}n!\delta\nu+\mathcal{O}(\delta\nu^2) \simeq \lim_{\epsilon\to 0} \frac{\Gamma(-2\ell-1-n-2\epsilon)}{\Gamma(-2\ell-1-2\epsilon)}(-1)^{n+1}n!\delta\nu \\
&= \lim_{\epsilon\to 0} \frac{(-2\ell-1-2\epsilon)\cdots(-2\epsilon)\Gamma(1-2\epsilon)}{(-2\ell-1-n-2\epsilon)\cdots(-2\epsilon)\Gamma(1-2\epsilon)}(-1)^{n+1}n!\delta\nu = \frac{(-1)^{2\ell+1}}{(-1)^{2\ell+1+n}} \frac{(2\ell+1)!}{(2\ell+1+n)!}(-1)^{n+1}n!\delta\nu \\
&= -\frac{(2\ell+1)!}{(2\ell+1+n)!}n!\delta\nu;
\end{aligned} \tag{A.19}$$

$$\lim_{\epsilon\to 0} \frac{\Gamma(-\ell')}{\Gamma(-2\ell'-1)} = \lim_{\epsilon\to 0} \frac{\Gamma(-\ell-\epsilon)}{\Gamma(-2\ell-1-2\epsilon)} = \lim_{\epsilon\to 0} \frac{(-2\ell-1-2\epsilon)\cdots(-2\epsilon)\Gamma(1-\epsilon)}{(-\ell-\epsilon)\cdots(-\epsilon)\Gamma(1-2\epsilon)} = 2(-1)^{\ell+1}\frac{(2\ell+1)!}{\ell!}. \tag{A.20}$$

For the ratio on the left-hand side of (A.18), identity $|\Gamma(1+m+bi)|^2 = \prod_{k=1}^{m}(k^2+b^2)\pi b/\sinh(\pi b)$ (when $m \in \mathbb{N}$) is needed which may be proved using the multiplication theorem for the Gamma function. Making use of it, one obtains

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\frac{\Gamma(\ell'+1+2iP)}{\Gamma(-\ell'+2iP)}=\frac{\sin[\pi(\ell+1-2iP)]}{\pi}\Gamma(\ell+1+2iP)\Gamma(\ell+1-2iP)\\
&=\frac{\sin[\pi(\ell+1-2iP)]}{\pi}|\Gamma(\ell+1+2iP)|^2=-\frac{\cos[\pi(\ell+1)]\sin(2iP)}{\pi}\frac{2\pi P}{\sinh(2\pi P)}\prod_{j=1}^{\ell}(j^2+4P^2)\\
&=(-1)^\ell 2iP\prod_{j=1}^{\ell}(j^2+4P^2).
\end{aligned}
\tag{A.21}
$$

Inserting (A.19)-(A.21) in (A.18) gives

$$
\delta\nu = iP[2k(r_+-r_-)]^{2\ell+1}\frac{(2\ell+1+n)!}{n!}\left[\frac{\ell!}{(2\ell)!(2\ell+1)!}\right]^2\prod_{j=1}^{\ell}(j^2+4P^2). \tag{A.22}
$$

Finally, we just need to use the quantization relation $\nu = n+\ell+1+\delta\nu$ (we replaced ℓ' by ℓ because we no longer need ϵ) with $\nu\equiv M\mu^2/k$ to obtain the real and imaginary part of the frequencies $\omega=\omega_R+i\omega_I$. Note that, when ω is real, we may neglect $\delta\nu$ and write

$$
\begin{aligned}
\frac{M\mu^2}{k}=n+\ell+1 &\Rightarrow \mu^2-\omega_R^2=\mu^2\left(\frac{M\mu}{n+\ell+1}\right)^2\\
&\Rightarrow \omega_R\simeq\mu-\frac{\mu}{2}\left(\frac{M\mu}{n+\ell+1}\right)^2
\end{aligned}
\tag{A.23}
$$

which means that, when $M\mu\ll 1$, then $\omega_R\sim\mu$. However, if one allows ω to be complex with $M\omega_I\ll 1$, one gets (using the previous equation):

$$
\begin{aligned}
\frac{M\mu^2}{k}=n+\ell+1+\delta\nu &\Rightarrow \mu^2-\omega^2\simeq\left(\frac{M\mu^2}{n+\ell+1}\right)^2\left(1-\frac{2\delta\nu}{n+\ell+1}\right)\\
&\Leftrightarrow i\omega_I\simeq\frac{\delta\nu}{M}\left(\frac{M\mu}{n+\ell+1}\right)^3.
\end{aligned}
\tag{A.24}
$$

Inserting (A.22) with $k\simeq M\mu^2/(n+\ell+1)$ and $P\simeq(am-2Mr_+\mu)/(r_+-r_-)$ results in

$$
\begin{aligned}
\omega_I\simeq\mu(\mu M)^{4\ell+4}&\left(\frac{am}{M}-2\mu r_+\right)\frac{2^{4\ell+1}(2\ell+1+n)!}{(n+\ell+1)^{2\ell+4}n!}\left[\frac{\ell!}{(2\ell)!(2\ell+1)!}\right]^2\times\\
\times\prod_{j=1}^{\ell}&\left[j^2\left(1-\frac{a^2}{M^2}\right)+\left(\frac{am}{M}-2\mu r_+\right)^2\right].
\end{aligned}
\tag{A.25}
$$

Since the scalar field has a time dependence $e^{-i\omega t}=e^{-i\omega_R t}e^{\omega_I t}$, when $\omega_I>0$ its amplitude grows exponentially in time (one of the two hypotheses we had suggested) with a time scale of $\tau=1/\omega_I$ and there is, in fact, an instability. The frequencies ω represent the quasibound states of a Kerr BH under scalar perturbations, as we explain in Chapter 2.

Appendix B

Properties of the Spherical Harmonics and Symmetric Trace-Free Tensors

B.1 Spherical Harmonics

They are solutions of the differential (or eigenvalue) equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_{\ell m}}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 Y_{\ell m}}{\partial\varphi^2}=-\ell(\ell+1)Y_{\ell m} \tag{B.1}$$

and may be written explicitly in relation to the associated Legendre polynomials as

$$Y_{\ell m}(\theta,\varphi)=\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}P_\ell^m(\cos\theta)e^{im\varphi}. \tag{B.2}$$

For $m<0$, the following relation is useful:

$$Y_{\ell,-m}(\theta,\varphi)=(-1)^m Y^*_{\ell m}(\theta,\varphi). \tag{B.3}$$

We use the normalization

$$\int Y_{\ell m}(\theta,\varphi)Y^*_{\ell' m'}(\theta,\varphi)\,d\Omega=\delta_{\ell\ell'}\delta_{mm'} \tag{B.4}$$

where $d\Omega\equiv\sin\theta d\theta d\varphi$ is an element of solid angle and the integral is calculated on the entire 2-sphere (any surface $r=$ constant), from $\theta=0$ to $\theta=\pi$ and from $\varphi=0$ to $\varphi=2\pi$.

They form a complete set of orthonormal functions [38], meaning that any square-integrable function on the 2-sphere $f(\theta,\varphi)$ may be expanded as

$$f(\theta,\varphi)=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}f_{\ell m}Y_{\ell m}(\theta,\varphi) \tag{B.5}$$

with coefficients given by

$$f_{\ell m} = \int f(\theta, \varphi) Y^*_{\ell m}(\theta, \varphi)\, d\Omega. \tag{B.6}$$

Integrals involving three spherical harmonics may be computed using Wigner 3-j symbols:

$$\begin{aligned}\int Y_{\ell_1 m_1}(\theta,\varphi) Y_{\ell_2 m_2}(\theta,\varphi) Y_{\ell_3 m_3}(\theta,\varphi)\, d\Omega = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix} \\ \times \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix}.\end{aligned} \tag{B.7}$$

In the case of this work, it happens that at least one of the functions is a complex conjugate, so the following formula might be used, because of the symmetry (B.3):

$$\begin{aligned}\int Y^*_{\ell_1 m_1}(\theta,\varphi) Y_{\ell_2 m_2}(\theta,\varphi) Y_{\ell_3 m_3}(\theta,\varphi)\, d\Omega = (-1)^{m_1} \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}} \\ \times \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -m_1 & m_2 & m_3 \end{pmatrix}.\end{aligned} \tag{B.8}$$

Note that this integral is only non-vanishing when the following selection rules are satisfied:

$$-m_1 + m_2 + m_3 = 0, \qquad |\ell_1 - \ell_2| \le \ell_3 \le \ell_1 + \ell_2, \qquad \ell_1 + \ell_2 + \ell_3 = 2p, \quad \text{for } p \in \mathbb{Z}. \tag{B.9}$$

There are also vector spherical harmonics $\{Y_a^{\ell m}, S_a^{\ell m}\}$ which form a complete basis for vectors in the 2-sphere [100]. These are defined as

$$Y_a^{\ell m}(\theta,\varphi) \equiv \left(\frac{\partial Y_{\ell m}}{\partial \theta}, \frac{\partial Y_{\ell m}}{\partial \varphi} \right), \tag{B.10}$$

$$S_a^{\ell m}(\theta,\varphi) \equiv \left(-\frac{1}{\sin\theta} \frac{\partial Y_{\ell m}}{\partial \varphi}, \sin\theta \frac{\partial Y_{\ell m}}{\partial \theta} \right) \tag{B.11}$$

where $a = \theta, \varphi$. They are characterized by their transformation under parity inversion ($\theta \to \pi - \theta, \varphi \to \varphi + \pi$). The vector spherical harmonics $Y_a^{\ell m}$ are called polar (or even or electric) since they transform like the scalar spherical harmonics, $Y_a^{\ell m} \to (-1)^\ell Y_a^{\ell m}$, and the vector spherical harmonics $S_a^{\ell m}$ are called axial (or odd or magnetic) because they transform like $S_a^{\ell m} \to (-1)^{\ell+1} S_a^{\ell m}$.

B.2 Symmetric Trace-Free Tensors

In the multi-index notation, when the same variable is repeated with different indices, such as $x^j x^k x^n$, it is written as x^{jkn} (the same goes for lower indices). Furthermore, an uppercase index L stands for ℓ individual indices ($x^{i_1 i_2 \cdots i_\ell}$ may be written as x^L and the same for lower indices). Using the Einstein summation convention in this multi-index notation means that in a quantity like $x^L y_L$, every individual index is being summed. This section is written adopting this convention.

The position vector is written in Cartesian coordinates as $\mathbf{x} = (x, y, z)$ and its distance from the origin is given by the variable $r = \sqrt{x^2+y^2+z^2}$. From these two quantities one can define the unit radial vector as $\mathbf{n} \equiv \mathbf{x}/r$ or, in multi-index notation,

$$n_L \equiv x_L/r^\ell. \tag{B.12}$$

Tensorial combinations of this vector which are symmetric and tracefree are called symmetric tracefree (STF) tensors and provide an alternative to the spherical-harmonic decomposition. STF tensors have their indices between angular brackets as in $A^{\langle L\rangle}$. The general formula for STF products of components of the unit radial vector is [38]

$$n^{\langle j_1 j_2 \cdots j_\ell\rangle} = \sum_{p=0}^{[\ell/2]} (-1)^p \frac{\ell!(2\ell-2p-1)!!}{(\ell-2p)!(2\ell-1)!!(2p)!!} \delta^{(j_1 j_2}\delta^{j_3 j_4}\ldots\delta^{j_{2p-1}j_{2p}} n^{j_{2p+1}} n^{j_{2p+2}} \ldots n^{j_\ell)}, \tag{B.13}$$

where angular brackets denote the symmetric part of a tensor and $[\ell/2]$ is the largest integer not larger than $\ell/2$: equal to $\ell/2$ when ℓ is even and to $(\ell-1)/2$ when ℓ is odd. Note that the sum on the right-hand side consists of the product $n^{j_1 j_2 \cdots j_\ell}$ (which is not tracefree) plus a combination of products of Kronecker deltas with products of components of n. Since the quantity on the left-hand side of this equation forms the basis of STF tensors and all tensors have an STF part and a non-STF part (one just needs to symmetrize its indices and remove all traces and then separate this part out), then the difference between the STF part and the non-STF part of all tensors is a sum of terms involving Kronecker deltas. In other words, any tensor A^L may be written as $A^{\langle L\rangle}$ plus combinations of Kronecker deltas. This means that, when an arbitrary tensor A^L is multiplied by an STF tensor $B_{\langle L\rangle}$, these extra terms all vanish because the Kronecker deltas contract pairs of a traceless tensor and only the STF part of A^L remains.

We've just proved that (B.13) implies

$$A^L B_{\langle L\rangle} = A^{\langle L\rangle} B_{\langle L\rangle}. \tag{B.14}$$

It can be shown by induction [38] that

$$\partial_{\langle L\rangle}\left(\frac{1}{r}\right) = (-1)^\ell (2\ell-1)!! \frac{n_{\langle L\rangle}}{r^{\ell+1}}. \tag{B.15}$$

Since $n^{\langle L\rangle}$ satisfies the same eigenvalue equation as the spherical harmonics, the correspondence between the two may be written as

$$n^{\langle L\rangle} = \frac{4\pi\ell!}{(2\ell+1)!!} \sum_{m=-\ell}^{\ell} \mathscr{Y}^{\langle L\rangle}_{\ell m} Y_{\ell m}(\theta,\varphi) \tag{B.16}$$

where $\mathscr{Y}^{\langle L\rangle}_{\ell m}$ is a constant STF tensor which satisfies $\mathscr{Y}^{\langle L\rangle}_{\ell,-m} = (-1)^m \mathscr{Y}^{*\langle L\rangle}_{\ell m}$. The inverse relation is

$$Y_{\ell m}(\theta,\varphi) = \mathscr{Y}^{*\langle L\rangle}_{\ell m} n_{\langle L\rangle}. \tag{B.17}$$

Analogously, the conversion between the two bases for the multipole moments may be written as

$$I_{\ell m} = \mathscr{Y}^{\langle L\rangle}_{\ell m} I_{\langle L\rangle}; \tag{B.18}$$

$$I^{\langle L\rangle} = \frac{4\pi\ell!}{(2\ell+1)!!}\sum_{m=-\ell}^{\ell} \mathscr{Y}^{*\langle L\rangle}_{\ell m} I_{\ell m}. \tag{B.19}$$

Appendix C

Newtonian Limit of the Einstein and Klein-Gordon equations

In this appendix, we present the reader with the steps necessary to take the Newtonian limit of the Einstein field equations (1.1) and Klein-Gordon equation

$$\Box\Phi = \mu^2 c^2 \Phi \tag{C.1}$$

where Φ is the scalar field defined in Chapter 4 and the spacetime metric has the usual components in the Newtonian limit (4.3), assuming there is a gravitational potential U sourced by the system, such that $|U| \ll c^2$.

The Newtonian limit allows us to use linearized gravity $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$ and $\eta_{\mu\nu}$ the Minkowski metric defined in Chapter 4 as well as the slow-motion approximation $\partial_0 \sim \varepsilon\partial_i$ with $\varepsilon = \mathcal{O}(|h_{\mu\nu}|)$, defined for coordinates $x^\mu \equiv (ct, x, y, z)$.

From equations (4.3) one obtains

$$h_{00} = -\frac{2U}{c^2}; \tag{C.2}$$

$$h_{ij} = -\frac{2U}{c^2}\delta_{ij}; \tag{C.3}$$

$$h_{0i} = 0. \tag{C.4}$$

Note that, by convention, indices are raised and lowered using the Minkowski metric and so $h^{00} = h_{00}$, $h^{0i} = -h_{0i}$, $h^{ij} = h_{ij}$ and $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \mathcal{O}(|h_{\mu\nu}|^2)$. From here we get the Christoffel symbols and Riemann tensor:

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2}\eta^{\mu\alpha}(\partial_\lambda h_{\nu\alpha} + \partial_\nu h_{\lambda\alpha} - \partial_\alpha h_{\nu\lambda}) + \mathcal{O}(|h_{\mu\nu}|^2); \tag{C.5}$$

$$R^\mu{}_{\nu\lambda\sigma} = \partial_\lambda \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\lambda} + \mathcal{O}(|h_{\mu\nu}|^2), \tag{C.6}$$

the components of the Ricci tensor:

$$R_{00} \simeq \frac{1}{c^2}\nabla^2 U + \mathcal{O}\left(\frac{1}{c^4}\right); \tag{C.7}$$

$$R_{ij} \simeq \delta_{ij}\frac{1}{c^2}\nabla^2 U + \mathcal{O}\left(\frac{1}{c^4}\right); \tag{C.8}$$

$$R_{0i} \simeq \mathcal{O}\left(\frac{1}{c^3}\right), \tag{C.9}$$

and finally the Ricci scalar:

$$R \simeq \frac{2}{c^2}\nabla^2 U + \mathcal{O}\left(\frac{1}{c^3}\right). \tag{C.10}$$

Consequently, we obtain the left-hand side of Einstein's equations for the component 00:

$$R_{00} - \frac{1}{2}g_{00}R \simeq \frac{1}{c^2}\nabla^2 U + \frac{1}{2}\left(1+\frac{2U}{c^2}\right)\frac{2}{c^2}\nabla^2 U \simeq \frac{2}{c^2}\nabla^2 U. \tag{C.11}$$

For the right-hand side we need to obtain component 00 of the energy-momentum tensor of the system $T_{\mu\nu} = T^{\mathsf{BH}}_{\mu\nu} + T^{\mathsf{S}}_{\mu\nu}$, using the definitions of Chapter 4: $T^{\mathsf{BH}}_{\mu\nu} = \rho c^2 \delta^0_\mu \delta^0_\nu$ and (4.4). Defining the auxiliary scalar field Ψ with (4.5) and imposing[1]

$$\mathcal{O}\left(\frac{|\partial_t \Psi|}{c^2|\Psi|}\right) = \mathcal{O}\left(\frac{U}{c^2}\right); \tag{C.12}$$

$$\mathcal{O}\left(\frac{|\partial_i \Psi|}{c|\Psi|}\right) = \mathcal{O}\left(\frac{\sqrt{U}}{c}\right), \tag{C.13}$$

we get

$$T^{\mathsf{S}}_{00} \simeq \mu\hbar c^2|\Psi|^2. \tag{C.14}$$

Hence, (1.1) gives

$$\frac{2}{c^2}\nabla^2 U = \frac{8\pi G}{c^4}(\rho c^2 + \mu\hbar c^2|\Psi|^2) \Rightarrow \nabla^2 U = 4\pi G\rho + 4\pi G m|\Psi|^2, \tag{C.15}$$

which is equation (4.6).

Now we focus on Klein-Gordon's equation (C.1). The d'Alembertian of Φ, using (4.5), is

$$\begin{aligned} \Box\Phi &= \frac{1}{\sqrt{-g}}\partial_\mu(g^{\mu\nu}\sqrt{-g}\partial_\nu\Phi) \\ &\simeq \frac{e^{-i\mu c^2 t}}{\sqrt{\mu\hbar}}\left(\mu^2 c^2\Psi + 2i\mu\frac{\partial\Psi}{\partial t} - 2\mu^2 U\Psi + \nabla^2\Psi\right). \end{aligned} \tag{C.16}$$

[1]The motivation for this assumption can be found in Ref. [101]: Appendix A, footnote 1.

Substituting in (C.1), we find

$$\begin{aligned}&\frac{e^{-i\mu c^2 t}}{\sqrt{\mu\hbar}}\left(\mu^2c^2\Psi+2i\mu\frac{\partial\Psi}{\partial t}-2\mu^2U\Psi+\nabla^2\Psi\right)=\mu^2c^2\frac{e^{-i\mu c^2 t}}{\sqrt{\mu\hbar}}\Psi\\&\Rightarrow i\hbar\frac{\partial\Psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\Psi+mU\Psi.\end{aligned}\tag{C.17}$$

which is (4.7).

Then, in the Newtonian limit of a scalar field (with factored-out high-frequency oscillations) propagating in a gravitational field, one can write

$$\begin{cases}R_{00}-\frac{1}{2}g_{00}R=\frac{8\pi G}{c^4}T_{00}\\ \Box\Phi=\mu^2c^2\Phi\end{cases}\Rightarrow\begin{cases}\nabla^2U=4\pi G\rho+4\pi Gm|\Psi|^2\\ i\hbar\frac{\partial\Psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\Psi+mU\Psi\end{cases},\tag{C.18}$$

so that the Einstein-Klein-Gordon system becomes the Schrödinger-Poisson system of differential equations.