

Tables of 16-digits Quazi Normal Frequencies (QNFs) of the Quasi Normal Modes (QNMs) for different boundary conditions in Schwarzschild metric

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I. 16-DIGITS PRECISION QNFS OF QNMS: SOME GENERAL REMARKS

Being still an elusive physical problem, the enough-precision-detection of QNMs is the only undisputable way to confirm or reject rigorously the existence of event horizon of the observed Extremely Compact Objects (ECOs), i.e., to confirm or reject rigorously their Black Hole (BH) nature which remains thus far hypothetical.

There is no problem to produce theoretical values of QNFs with very high precision using the exact solutions of the Regge-Wheeler and Zerilli equations together with proper high-precision (up to 512 digits) numerical methods, and ignoring the fact that this is a quite time-consuming procedure.

The difficulty is to find, with a good precision, some starting point for calculation of each specific QNF. If we already have in our disposal high-precision tables for corresponding QNFs, we are looking for, the further detailed study of the problem becomes much easier and solvable in a relatively short time.

Indeed, having enough high-precision points with increasing overtone number $n = 0, \dots, n_{max}$ and angular index $l = 2, \dots, l_{max}$ we can obtain with a good-enough precision starting points for next $n = n_{max} + 1$ and/or $l = l_{max} + 1$, using a simple numerical extrapolation.

Below, we present such useful tables for basic boundary problems in Schwarzschild metric, choosing $n_{max} = 24$ for Black-Hole-Sommerfeld Boundary Conditions (BHSBC), $n_{max} = 21$ for Dirichlet-Sommerfeld Boundary Conditions (DSBC) and Neumann-Sommerfeld Boundary Conditions (NSBC), and $l_{max} = 7$ in all cases.

The above maximal overtone numbers $n_{max} = 24$ and $n_{max} = 21$ in these tables are chosen from additional physical reason: to have a ratio $\tau[0, l]/\tau[n_{max}, l] \sim 50 \div 100$ for all included in the tables damping times $\tau[n, l] = 1/\Im(\omega_{n,l})$. Taking into account the sensitivity of the existing GW detectors and near future ones, these limitations seems to be enough for all practical purposes now and in foreseeing future.

The 16-digits-precision QNFs permit us to study in detail nontrivial properties of QNMs and corresponding new phenomena.

For some additional detail of the used methods, as well as for additional references see the recent paper arXiv:1912.13432.

II. 16-DIGITS PRECISION BHSBC QNMS IN SCHWARZSCHILD METRIC

In Table I we present for the first time the lowest 150 in number Schwarzschild BHSBC QNFs with 16-digits precision. These frequencies were obtained for value $r_\infty = 50$ of "the actual infinity" and are stable under variations of $r_\infty \geq 40 \div 100$.

According to the well-known isospectral theorem, see S. Chandrasekhar, The Mathematical Theory of Black Holes, Oxford University Press, New York (1983), the BHSBC QNFs for the Regge-Wheeler and Zerilli equations are the same, nevertheless the QNMs are different¹.

We checked numerically this theorem using our methods and confirmed it with much grater precision than 16-digits one. (We actually obtained values of RWBH and ZBH QNFs with not less than 3E-28 precision.)

We also confirmed with higher precision the original result of the paper Plamen P. Fiziev, D. Staicova, PRD **84**, 127502 (2011) about the frequency $\omega_{8,2}$ of the ninth mode $QNM_{8,2}$ of BHSBC QNMs.

Indeed, it is *not* algebraically special mode, as often incorrectly stated in the literature before this paper, as a result of not-enough-precision calculations and/or incorrect understanding of the role of algebraically special solutions of Regge-Wheeler and Zerilli equations.

¹ One must make difference between the QNFs $\omega_{n,l}$ and the very solutions $QNM_{n,l}$ of the corresponding equations.

It was shown already in the paper S. Chandrasekhar, Proc. R. Soc. London, Ser. A **392** 1 (1984) that the algebraically special solution with pure imaginary frequencies $\omega_{as}(l) = i 4 \binom{l+2}{4}$ do not describe QNMs since algebraically special solutions satisfy completely different boundary conditions.

For example, in the previously published tables of accurate BHSBC QNFs (see, E. Berti, <https://pages.jh.edu/~eberti2/ringdown>), the obtained value for QNF of $n = 8, l = 2$ mode (with declared precision $0.9154E-10$) is

$$\omega_{8,2} = 0.4615178773933189E-15 + i 3.999999999999607,$$

and indeed can be interpreted as numerically identical with the frequency of the algebraically special mode $\omega_{as}(2) = 0 + i 4$. However, this does not mean that the very solutions for corresponding QNMs coincide.

The other values of BHSBC QNFs in the Berti tables (up to the declared there precision, which is lower than our one) are in a good correspondence with our BHSBC QNFs in Table 1.

The same incorrect interpretation as for the frequency $\omega_{8,2}$ of the $QNM_{8,2}$ mode can be found in the literature about BHSBC QNMs also for higher values of l . See for example the paper K.D. Kokkotas, B.G. Schmidt, Living Rev. Rel. 2:2(1999) about the frequency $\omega_{40,3}$ of the $QNM_{40,3}$ mode. Indeed, $\omega_{40,3}$ is very closed to the frequency $\omega_{as}(3)$ but this does not mean that the $QNM_{40,3}$ mode coincides with corresponding algebraically special solution.

Our result for $\omega_{8,2}$ (see in Table I) with precision $3E-21$ is

$$\omega_{8,2} = 0.0306490095213001613678 + i 3.99682368371788648697$$

It is curious that up to the low precision $1.2E-3$, its imaginary part coincides with the value $\omega_{8,2} = 0 + i 0.9980$, obtained already in the paper E.W. Leaver, Proc. Roy. Soc. London A402, 285 (1985).

Our result for $\omega_{8,2}$ is also close to the low-precision one of the paper P.T. Leung, A. Maassen van den Brink, K.W. Mak, and K. Young, Class. Quantum Grav. 20, L217 (2003): $\omega_{8,2} = 0.03248 + i (0.003436)$, called there "unconventional damped mode".

We intend to consider in a separate paper the surprising closeness of the pure imaginary algebraically special frequencies $\omega_{as}(l)$ and the QNFs $\omega_{n(l),l}$ of QNMs with arbitrary $l = 2, 3, 4, 5, 6, 7, \dots$ and corresponding

$$n(l) = 2|\omega_{as}(l)| = 8, 40, 120, 280, 560, 1008, \dots$$

III. 16-DIGITS PRECISION RWDSBC AND RWNSBC QNMS

In Table II we present for the first time the lowest $126 = \#\{n = 0, \dots, 20\} \times \#\{l = 2, \dots, 7\}$ in number RWDSBC QNMs with 16-digits precision.

In Table III we present for the first time the lowest $126 = \#\{n = 0, \dots, 20\} \times \#\{l = 2, \dots, 7\}$ in number RWNSBC QNMs with 16-digits precision.

IV. 16-DIGITS PRECISION ZDSBC AND ZNSBC QNMS

In Table IV we present for the first time the lowest $126 = \#\{n = 0, \dots, 20\} \times \#\{l = 2, \dots, 7\}$ in number ZDSBC QNMs with 16-digits precision.

In Table V we present for the first time the lowest $126 = \#\{n = 0, \dots, 20\} \times \#\{l = 2, \dots, 7\}$ in number ZNSBC QNMs with 16-digits precision.

TABLE I: The first 150 = $\# \{n = 0, \dots, 24\} \times \# \{l = 2, \dots, 7\}$ BHSBC QNMs:

	l=2		l=3		l=4	
n=0	0.7473433688360837+ i	0.1779246313778714	1.1988865768749801+ i	0.1854060958898952	1.6183567550644783+ i	0.1883279219778465
n=1	0.6934219937583269+ i	0.5478297505824696	1.1652876060665989+ i	0.5625962268700881	1.5932630640690050+ i	0.5686686988096814
n=2	0.6021069092247328+ i	0.9565539664461436	1.1033698015569026+ i	0.9581855019339245	1.5454190652134186+ i	0.9598163502423262
n=3	0.5030099243711812+ i	1.4102964048669907	1.0239238221166748+ i	1.3806741919384782	1.4796734600110771+ i	1.3678486380357641
n=4	0.4150291596261312+ i	1.8936897817327031	0.9403480116303105+ i	1.8312987850101932	1.4030310185033328+ i	1.7964779435183281
n=5	0.3385988061860873+ i	2.3912161082716936	0.8627729572843077+ i	2.3043027242818109	1.3231449987139982+ i	2.2459535070258131
n=6	0.2665046804903760+ i	2.8958212523240763	0.7953190483515142+ i	2.7918244854451831	1.2462177493318392+ i	2.7133725366864099
n=7	0.1856446673404019+ i	3.4076823444122711	0.7379845517794569+ i	3.2876890567135324	1.1758176500595310+ i	3.1943413612269196
n=8	0.0306490095213002+ i	3.9968236837178865	0.6892366371902328+ i	3.7880656083859256	1.1131495329460165+ i	3.6846352672861472
n=9	0.1265270102512120+ i	4.6052895303170810	0.6473662632026929+ i	4.2907978995346410	1.0579947959000437+ i	4.1809824581259490
n=10	0.1531069257719723+ i	5.1216532347630114	0.6109218038621915+ i	4.7947091006032262	1.0095255767342033+ i	4.6811230891156460
n=11	0.1651962892832580+ i	5.6308847452159756	0.5787681902376187+ i	5.2991592113172892	0.9667709343913342+ i	5.1835833007719587
n=12	0.1714558441266348+ i	6.1373894878623803	0.5500387921677459+ i	5.8037991423876351	0.9288218550634754+ i	5.6874259463168030
n=13	0.1747885247471990+ i	6.6424604768651185	0.5240723189739770+ i	6.3084389558273000	0.8948977608982729+ i	6.1920603143412764
n=14	0.1764778815211750+ i	7.1466413966158451	0.5003594309230369+ i	6.8129771848332704	0.8643527300062831+ i	6.6971145417397564
n=15	0.1771807742403373+ i	7.6502104162908118	0.4785024409917175+ i	7.3173623260239959	0.8366600846335703+ i	7.2023544306413287
n=16	0.1772658049187587+ i	8.1533291773508901	0.4581860051784482+ i	7.8215714414015455	0.8113919289726468+ i	7.7076327365706982
n=17	0.1769534131502148+ i	8.6561004191053506	0.4391559775484251+ i	8.3255980380260667	0.7881998748755235+ i	8.2128575545591868
n=18	0.1763816962479508+ i	9.1585939137992661	0.4212040775412284+ i	8.8294450940412808	0.7667988067959268+ i	8.7179724914928534
n=19	0.1756402482612920+ i	9.6608594981426344	0.4041566354510808+ i	9.3331210067385230	0.7469538394498572+ i	9.2229441087748872
n=20	0.1747887765200059+ i	10.1629341990172931	0.3878661894016806+ i	9.8366372391145961	0.7284700567811314+ i	9.7277538809450225
n=21	0.1738679005724426+ i	10.6648464011229381	0.3722050738885733+ i	10.3400069788646298	0.7111845032023364+ i	10.2323929890660138
n=22	0.1729056889963511+ i	11.1666184221354119	0.3570603915553292+ i	10.8432444201415975	0.6949599397189766+ i	10.7368589150428142
n=23	0.1719217573281911+ i	11.6682681781352330	0.3423299270665694+ i	11.3463644471213075	0.6796799629350830+ i	11.2411531942345555
n=24	0.1709299134350313+ i	12.1698103028379546	0.3279186686484419+ i	11.8493825989012928	0.6652451710499623+ i	11.7452799219613456
	l=5		l=6		l=7	
n=0	2.0245906242707010+ i	0.1897410321632190	2.4240196413042610+ i	0.1905316916841641	2.8194702412186451+ i	0.1910192585520944
n=1	2.0044420557811167+ i	0.5716347635445261	2.4071479487775075+ i	0.5732998502274939	2.8049418760593583+ i	0.5743281309876688
n=2	1.9653915216168816+ i	0.9606569120281502	2.3741473492043160+ i	0.9611289855616705	2.7763636910157773+ i	0.9614184183414450
n=3	1.9100080122354146+ i	1.3611138172992129	2.3265401237021192+ i	1.3571819619650303	2.7347118512439621+ i	1.3546981306199594
n=4	1.8421636877374112+ i	1.7763951847768269	2.2666482029451821+ i	1.7642055422633084	2.6814787909603095+ i	1.7563617153701450
n=5	1.7666715213950490+ i	2.2083649279349598	2.1975066219316506+ i	2.1842300595983707	2.6186560432430898+ i	2.1682362026611846
n=6	1.6884963336414257+ i	2.6569939653077173	2.1225941060005232+ i	2.6182799265691312	2.5486421553191528+ i	2.5916062105005454
n=7	1.6118305655987273+ i	3.1205617144260267	2.0453958551820870+ i	3.0662407198672669	2.4740562236909457+ i	3.0270703886208021
n=8	1.5395121680296760+ i	3.5963615514239028	1.9689365987368918+ i	3.5269848508678991	2.3974808170896682+ i	3.4744922328386394
n=9	1.4729934146474527+ i	4.0814941444598210	1.8954638294546218+ i	3.9987184108686691	2.3212017250789719+ i	3.9330803098419225
n=10	1.4126801439483277+ i	4.5733970571184854	1.8263712008198453+ i	4.4793954393279353	2.2470217350485556+ i	4.4015774798139347
n=11	1.3583498740275840+ i	5.0700473187496527	1.7623182795628087+ i	4.9670493546925296	2.1761909204393772+ i	4.8784941268206512
n=12	1.3094791281350258+ i	5.5699510445906858	1.7034373034688331+ i	5.4599797225635947	2.1094438962726289+ i	5.3623161534763733
n=13	1.2654461489096388+ i	6.0720461185108614	1.6495355140456049+ i	5.9568131723433930	2.0471011000755165+ i	5.8516478994747377
n=14	1.2256371820383541+ i	6.5755923569524504	1.6002502549311354+ i	6.4564877725047323	1.9891882304478193+ i	6.3452852565652626
n=15	1.1894934383857590+ i	7.0800787164169986	1.5551511355765966+ i	6.9582042513564733	1.9355434907996731+ i	6.8422353413707754
n=16	1.1565262189889366+ i	7.5851535661007834	1.5138000011727982+ i	7.4613705121292669	1.8859000718455170+ i	7.3417040984366100
n=17	1.1263167465640948+ i	8.0905751338562295	1.4757823709010362+ i	7.9655517265165396	1.8399431834921383+ i	7.8430691615493460
n=18	1.0985096025841975+ i	8.5961772141930140	1.4407214672564861+ i	8.4704299054819166	1.7973462849918727+ i	8.3458489514138734
n=19	1.0728041881283564+ i	9.1018456995538867	1.4082824279725180+ i	8.9757728905359241	1.7577923004995223+ i	8.84967371110527821
n=20	1.0489462295075869+ i	9.6075025573229702	1.3781714015446217+ i	9.4814112681133297	1.7209848420162324+ i	9.3542607781602325
n=21	1.0267201421636726+ i	10.1130948628336542	1.3501322550035787+ i	9.9872214512858056	1.6866531776935555+ i	9.8593945443920134
n=22	1.0059424901926284+ i	10.6185872537810967	1.3239423960445606+ i	10.4931133722649300	1.6545534962419534+ i	10.3649107022510643
n=23	0.9864565227429729+ i	11.1239567044716488	1.2994084908211257+ i	10.9990215488596190	1.6244681115231680+ i	10.8706840839893076
n=24	0.9681276686586011+ i	11.6291888809852162	1.2763624520360460+ i	11.5048985935182353	1.5962036225045152+ i	11.3766193694713331

TABLE II: The first 126 = $\#\{n = 0, \dots, 20\} \times \#\{l = 2, \dots, 7\}$ RWDSBC QNMs:

	l=2	l=3	l=4
n=0	0.8858481605601723+ i 0.2803070595842718	1.3488840244761040+ i 0.2591039332271598	1.7722168572962863+ i 0.2283795480363940
n=1	0.9108831913717330+ i 0.9837041458219459	1.4193590595241325+ i 0.9100630451824658	1.8806973871044682+ i 0.8451879554755047
n=2	0.9197913784892056+ i 1.8779064367389418	1.4166061972606540+ i 1.7035217124291293	1.9062179699463342+ i 1.5770402575221327
n=3	1.0005293385811563+ i 2.8499314270779648	1.4237133194815165+ i 2.6139090548849290	1.8997381348727247+ i 2.4124505875279082
n=4	1.1285936232756674+ i 3.8329537814318643	1.4806413111408711+ i 3.5830371292934343	1.9091525795677651+ i 3.3310040729968596
n=5	1.2801575243190196+ i 4.8140294842602536	1.5797533100643314+ i 4.5695401886054306	1.9564189781663906+ i 4.2969296916440887
n=6	1.4443063609372282+ i 5.7918773471737308	1.7059137706624443+ i 5.5582428943276831	2.0393415338424283+ i 5.2824566882994539
n=7	1.6158891231923628+ i 6.7669695245540181	1.8490322579493902+ i 6.5449698704268108	2.1487000225631866+ i 6.2737741029056250
n=8	1.7922192670616052+ i 7.7399057739489553	2.0032104180445407+ i 7.5289055590279606	2.2765295712564975+ i 7.2653186625221883
n=9	1.9717667006004426+ i 8.7111705120229858	2.1649516086451649+ i 8.5101718368878048	2.4173167949621220+ i 8.2551158447305858
n=10	2.1535968305375671+ i 9.6811261955697252	2.3320899894317446+ i 9.4891120112221113	2.5674117716065686+ i 9.2426104165193660
n=11	2.3371063146609214+ i 10.6500413082878702	2.5032208978585354+ i 10.4660816723054101	2.7243805426493804+ i 10.2277859481481332
n=12	2.5218879673226051+ i 11.6181164480758916	2.6773948170166186+ i 11.4413929568694870	2.8865619508580866+ i 11.2108163099377099
n=13	2.7076568373475112+ i 12.5855036483965721	2.8539465125198144+ i 12.4153058774229361	3.0527914277521374+ i 12.1919279639717022
n=14	2.8942074188282038+ i 13.5523199391466249	3.0323954335593127+ i 13.3880333210056689	3.2222308841048522+ i 13.1713465745936857
n=15	3.0813876928029390+ i 14.5186567809357060	3.2123852253899889+ i 14.3597490071355411	3.3942624745592693+ i 14.1492779074923757
n=16	3.2690827432649701+ i 15.4845866725882448	3.3936456071075482+ i 15.3305950855197327	3.5684207397949316+ i 15.1259027252516818
n=17	3.4572040606261888+ i 16.4501678369933263	3.5759675401011583+ i 16.3006885168729322	3.7443481781104252+ i 16.1013772419238193
n=18	3.6456823517120191+ i 17.4154475925422327	3.7591865750651682+ i 17.2701261833765979	3.9217654410197387+ i 17.0758355765800588
n=19	3.8344625824885273+ i 18.3804648149419905	3.9431713918722494+ i 18.2389889070643348	4.1004508757457631+ i 18.0493927104436513
n=20	4.0235004830905580+ i 19.3452517608096452	4.1278157302447635+ i 19.2073445919869939	4.2802261826595500+ i 19.0221473438733943
	l=5	l=6	l=7
n=0	2.1763096024232943+ i 0.19851944384118787	2.5693763019567680+ i 0.1711821119599594	2.9553137283803623+ i 0.1465402797518588
n=1	2.3145425985015263+ i 0.78720696074576065	2.7319155489168702+ i 0.7346817262545633	3.1385297512989524+ i 0.6864907392951928
n=2	2.3699591281747231+ i 1.47961907436099551	2.8134439982807468+ i 1.3987250808379484	3.2424664254297809+ i 1.3282022857895404
n=3	2.3772425542922310+ i 2.25805063537620471	2.8413658209051287+ i 2.1367030683833196	3.2911538674055685+ i 2.0363707341880375
n=4	2.3718204950773562+ i 3.11729308357360930	2.8411409543591590+ i 2.9457679378622542	3.3043529001752981+ i 2.8065993556551560
n=5	2.3832193957059353+ i 4.04043057497796981	2.8377454733461425+ i 3.8201794612223904	3.3012352805763682+ i 3.6371399576431863
n=6	2.4251164149689777+ i 5.00331271091445262	2.8506070221303816+ i 4.7462010142770161	3.2999039304678894+ i 4.5219753588996675
n=7	2.4973413795565475+ i 5.98614581370507402	2.8889990868358749+ i 5.7064663359926853	3.3138458367247960+ i 5.4499673267631882
n=8	2.5942266520316194+ i 6.97718025746911723	2.9535473557208434+ i 6.6862114238149693	3.3497390649729504+ i 6.4080427053531320
n=9	2.7097888204337434+ i 7.97052857677842860	3.0407177175867926+ i 7.6756142869173382	3.4085098159171870+ i 7.3847928851428371
n=10	2.8392767852125167+ i 8.96349858183739071	3.1460873138971354+ i 8.6689976182701137	3.4879330062044270+ i 8.3720091889024405
n=11	2.9792233381931746+ i 9.95495820914926073	3.2657227416714755+ i 9.6633276882802031	3.5847483149954673+ i 9.3644459427529831
n=12	3.1271489901780293+ i 10.9445063938412994	3.3965010971293692+ i 10.6570692242390846	3.6957630271504392+ i 10.3589784141300582
n=13	3.2812729555886803+ i 11.9320764850757755	3.5360426656890329+ i 11.6494945066482543	3.8182504048511571+ i 11.3538273363121019
n=14	3.4402984592347644+ i 12.9177488799380762	3.6825536335248002+ i 12.6403013619140549	3.9500110132403699+ i 12.3480188837077085
n=15	3.6032649411334673+ i 13.9016624898520631	3.8346780715483955+ i 13.6294083071066900	4.0893110847359948+ i 13.3410483488826710
n=16	3.7694482831305015+ i 14.8839730946043914	3.9913818236658462+ i 14.6168454957714161	4.2347927251425962+ i 14.3326804063768179
n=17	3.9382933107946285+ i 15.8648342049537026	4.1518669852957983+ i 15.6026966425806833	4.3853903859258054+ i 15.3228326430951616
n=18	4.1093676167120529+ i 16.8443888340664280	4.3155102708174922+ i 16.5870681159899504	4.5402627608402952+ i 16.3115079736626161
n=19	4.2823294972023199+ i 17.8227665591076262	4.4818187714052914+ i 17.5700726243491771	4.6987401245333170+ i 17.2987554126911748
n=20	4.4569053284632116+ i 18.8000831073143113	4.6503980518899168+ i 18.5518208080477235	4.8602844917641081+ i 18.2846472957936322

TABLE III: The first $126 = \#\{n = 0, \dots, 20\} \times \#\{l = 2, \dots, 7\}$ RWNSBC QNMs:

	l=2	l=3	l=4
n=0	0.7678388276269696+ i 0.0543539183831632	1.1982372804997224+ i 0.0442275933669605	1.5917421259164475+ i 0.0322720013817410
n=1	0.9053097945617747+ i 0.6121330154983818	1.387669555244613+ i 0.5718295027298831	1.8260451728959502+ i 0.5240523159698668
n=2	0.9077186590280061+ i 1.4158894747588353	1.4197179185588348+ i 1.2935465591025799	1.8968673457418071+ i 1.2016186729515744
n=3	0.9525484403714988+ i 2.3612329385265134	1.4149042868827628+ i 2.1494849354481493	1.9026537503002828+ i 1.9850752847701031
n=4	1.0608112129337773+ i 3.3416524949538371	1.4462665689834934+ i 3.0953479140005325	1.9003815013387846+ i 2.8650132825724987
n=5	1.2024595609922865+ i 4.3240037485120297	1.5262474599311641+ i 4.0758424477680794	1.9281230907976160+ i 3.8109332920183862
n=6	1.3611452569513406+ i 5.3033906716092627	1.6403913981232034+ i 5.0641892018336937	1.9942369187064830+ i 4.7887897614950174
n=7	1.5294251818829130+ i 6.2797590673472908	1.7759174853026351+ i 6.0520418162303973	2.0914642444594095+ i 5.7780943118130058
n=8	1.7036092482740163+ i 7.2536928385083080	1.9250810566730681+ i 7.0373491631494970	2.2108410554012108+ i 6.7698291211790587
n=9	1.8816831858578879+ i 8.2257345846404379	2.0833524699842565+ i 8.0198915182715599	2.3456676374145301+ i 7.7605770948165152
n=10	2.0624572615710209+ i 9.1963022209911480	2.2479901028085542+ i 8.9999363178308114	2.4914502709054335+ i 8.7492175746303932
n=11	2.2451835831988399+ i 10.1657064042225866	2.4172560079514066+ i 9.9778413110147041	2.6452118596016518+ i 9.7355203349895014
n=12	2.4293681053994685+ i 11.1341782522046710	2.5899989372556424+ i 10.9539410709684898	2.8049456832527091+ i 10.7195850948817331
n=13	2.6146711054600707+ i 12.1018917511459634	2.7654262626437311+ i 11.9285204179766117	2.9692638541268283+ i 11.7016196253702565
n=14	2.8008511303111234+ i 13.0689798520263928	2.9429738768451806+ i 12.9018142620957678	3.1371804476640308+ i 12.6818523859718884
n=15	2.9877317612941933+ i 14.0355457158351238	3.1222287700413164+ i 13.8740145454326104	3.3079772011422474+ i 13.6604994852203583
n=16	3.1751810394337161+ i 15.0016705668827473	3.3028811167337730+ i 14.8452781074268154	3.4811187302929264+ i 14.6377538551800190
n=17	3.3630982534487950+ i 15.9674192307911307	3.4846935684420227+ i 15.8157336517629174	3.6561977061196444+ i 15.6137834374240767
n=18	3.5514051850269475+ i 16.9328440925838197	3.6674809533452193+ i 16.7854874453237037	3.8328985198110611+ i 16.5887328364641696
n=19	3.7400401475559660+ i 17.8979879670293078	3.8510964823904694+ i 17.7546278376674127	4.0109726241901192+ i 17.5627260962342062
n=20	3.9289538291779038+ i 18.8628862105469281	4.0354221456484282+ i 18.7232288045623855	4.1902214259767015+ i 18.5358696258923239
	l=5	l=6	l=7
n=0	1.9687920192478368+ i 0.0223752400530696	2.3364112902931818+ i 0.0148431773213412	2.6976855944162281+ i 0.0094138897074712
n=1	2.2419912021985824+ i 0.4776212382494015	2.6452223174127553+ i 0.4339010943481894	3.0405616026635262+ i 0.3929701765375724
n=2	2.3450721431930982+ i 1.1257041643832886	2.7745428585117697+ i 1.0595120377346801	3.1914976783189949+ i 0.9999671359681350
n=3	2.3754235687803971+ i 1.8609359147170935	2.8296882033191998+ i 1.7612835568518243	3.2689117880333678+ i 1.6767331184129036
n=4	2.3734418140954756+ i 2.6802202735079174	2.8420011010768722+ i 2.5345766920099992	3.2992791618157145+ i 2.4158212929197307
n=5	2.3741904932747261+ i 3.5735779715278570	2.8380784368095026+ i 3.3769864160362046	3.3028913516303492+ i 3.2162004650597113
n=6	2.4003525601446501+ i 4.5190505410699071	2.8413507630090678+ i 4.2788320781459002	3.2991285515764802+ i 4.0745895267136759
n=7	2.4579639381985351+ i 5.4935693253461435	2.8665874065542252+ i 5.2237393693114374	3.3044186004548612+ i 4.9822621069304365
n=8	2.5432786086634067+ i 6.4813671493459308	2.9183615042791175+ i 6.1950523936187364	3.3290196500038897+ i 5.9266213200703861
n=9	2.6501445624424743+ i 7.4739495876242358	2.9947503730289575+ i 7.1804136987510170	3.3765193692610560+ i 6.8950808064980434
n=10	2.7731453923880186+ i 8.4672655846847801	3.0915303713640131+ i 8.1722282835556229	3.4459881477095201+ i 7.8777608499397656
n=11	2.9082022970776381+ i 9.4595294094629005	3.2044485371537574+ i 9.1662929333202867	3.5345070639253600+ i 8.8680062154308920
n=12	3.0523804054157351+ i 10.4500348586812407	3.3299741615257288+ i 10.1604218222959781	3.6387753887726653+ i 9.8617262353641282
n=13	3.2035796919644231+ i 11.4385759846922646	3.4653738530080864+ i 11.1535389840227611	3.7558150232473491+ i 10.8565415416721980
n=14	3.3602823467120196+ i 12.4251724041957528	3.6085802672185871+ i 12.1451590854714948	3.8831670880640756+ i 11.8511223832710772
n=15	3.5213738308790836+ i 13.4099394845177326	3.7580343621130008+ i 13.1351057795522549	4.0188756461823888+ i 12.8447574796031486
n=16	3.6860212950281151+ i 14.3930270134512043	3.9125528833273088+ i 14.1233614246007789	4.1614057050718868+ i 13.8370930197914370
n=17	3.8535915254604096+ i 15.3745905548473377	4.0712283000097745+ i 15.1099870710960938	4.3095546421559091+ i 14.8279793922360149
n=18	4.0235951544973163+ i 16.3547785878249766	4.2333560929954137+ i 16.0950798269220018	4.4623761359375530+ i 15.8173821021604344
n=19	4.1956482177674118+ i 17.3337273143151617	4.3983825191362267+ i 17.0787502178733268	4.6191200554964254+ i 16.8053300775220065
n=20	4.3694452477219285+ i 18.3115591694787852	4.5658670509878331+ i 18.0611103386318915	4.7791865812567571+ i 17.7918856449040425

TABLE IV: The first $126 = \#\{n = 0, \dots, 20\} \times \#\{l = 2, \dots, 7\}$ ZDSBC QNMs:

	l=2	l=3	l=4
n=0	0.8704059934878189+ i 0.3137651879321848	1.3471437024665020+ i 0.2681314865367667	1.7721875092045215+ i 0.2318524597728932
n=1	0.8363217394978740+ i 1.0336109434514693	1.4068533694813524+ i 0.9238463071095937	1.8772125870544346+ i 0.8507868527575280
n=2	0.7614274750915682+ i 1.9568451571639639	1.3903507892419267+ i 1.7205413035078289	1.8985320112244668+ i 1.5838367772434898
n=3	0.7363811864137559+ i 2.9963319126826728	1.3821702286574486+ i 2.6354012181648454	1.8872266696613616+ i 2.4204298992420882
n=4	0.8092586457403744+ i 4.1830835257354618	1.4237843057289288+ i 3.6101418598099607	1.8915497970213017+ i 3.3405049106695365
n=5	1.0731625854724573+ i 5.2563105466460450	1.5076896232799124+ i 4.6027752891874590	1.9337583832097216+ i 4.3082161733390177
n=6	1.3020562201026084+ i 6.2556612279093258	1.6185343580403225+ i 5.5979787436044266	2.0117126693593251+ i 5.2956402891583382
n=7	1.5142097711003153+ i 7.2390463140358618	1.7460693197392322+ i 6.5916211708689170	2.1161604683609188+ i 6.2888964756824392
n=8	1.7186722869431389+ i 8.2159424795974557	1.8842818702747185+ i 7.5829864073463742	2.2391063858569109+ i 7.2824017366015176
n=9	1.9190230257205624+ i 9.1893347466394420	2.0295768180811593+ i 8.5723283988326890	2.3750185712605350+ i 8.2741783209188208
n=10	2.1169313510263303+ i 10.1605226204210831	2.1796938743879879+ i 9.5601605702527780	2.5202361886224820+ i 9.2636722131076642
n=11	2.3132753474532123+ i 11.1301888759301441	2.3331361105560645+ i 10.5470611274501345	2.6723182564730572+ i 10.2508692676359993
n=12	2.5085591916721166+ i 12.0987385988877001	2.4888696540204919+ i 11.5336393785946360	2.8295985682413240+ i 11.2359458618362237
n=13	2.7030922763685133+ i 13.0664331062629130	2.6461698019555446+ i 12.5205591564659074	2.9909085384816402+ i 12.2191310195015730
n=14	2.8970746461766857+ i 14.0334514728221812	2.8045602529901498+ i 13.5085890422271310	3.1554066226563682+ i 13.2006530209133992
n=15	3.0906412173515257+ i 14.9999221049641387	2.9638365079234062+ i 14.4986680737683544	3.3224718354189183+ i 14.1807203356737904
n=16	3.2838862046784926+ i 15.9659403666522909	3.124025407093258+ i 15.4919639977654104	3.4916357567205070+ i 15.1595165538376462
n=17	3.4768773518499281+ i 16.9315790886072842	3.2865830668656822+ i 16.4898356516890469	3.6625380239751166+ i 16.1372008721101638
n=18	3.6696645972554736+ i 17.8968951727813459	3.4531216170201803+ i 17.4934312339471973	3.8348964788218972+ i 17.1139105756885848
n=19	3.8622855529739144+ i 18.8619339245551479	3.6274139326588714+ i 18.5024528502591719	4.0084866782100590+ i 18.0897640220680326
n=20	4.0547690823588210+ i 19.8267319969143175	3.8130264351597616+ i 19.5134324267083961	4.1831275309649929+ i 19.0648635246308391
	l=5	l=6	l=7
n=0	2.1765518042091272+ i 0.2001366763110427	2.5696314377733037+ i 0.1720314807979103	2.9555331322038756+ i 0.1470245163529821
n=1	2.3133292032044649+ i 0.7899488373042343	2.7314465299784803+ i 0.7361957745307304	3.1383453197870440+ i 0.6874005951262354
n=2	2.3670453269921495+ i 1.4829744712975059	2.8121550018278449+ i 1.4005987296004171	3.2418366338057314+ i 1.3293400524307910
n=3	2.3723938247754530+ i 2.2618954311052926	2.8391627932283321+ i 2.1388407013362187	3.2900378404102267+ i 2.0376718527132177
n=4	2.3648543129490202+ i 3.1216823496964206	2.8379302669142177+ i 2.9481553538788170	3.3027041082169211+ i 2.8080401936962396
n=5	2.3740668530082783+ i 4.0454855389241135	2.8334626814546989+ i 3.8228530369418780	3.2990125904340870+ i 3.6387245982515860
n=6	2.4137913941063031+ i 5.0091254959487038	2.8452293849830413+ i 4.7492121046215786	3.2970812126294213+ i 4.5237247170182036
n=7	2.4838792927797979+ i 5.9927613813108062	2.8825325613090031+ i 5.7098542475290821	3.3104142433664987+ i 5.4519059768080192
n=8	2.5786564154136819+ i 6.9846161964334995	2.9460071898940450+ i 6.6899978810980212	3.3457014103490136+ i 6.4101897525039769
n=9	2.6921293370164956+ i 7.9787914052382642	3.0321180326155737+ i 7.6798091682250386	3.4038733414946227+ i 7.3871599186533226
n=10	2.8195397224341598+ i 8.9725911470647226	3.1364385578820163+ i 8.6736049072114015	3.4827050284543158+ i 8.3746021902371108
n=11	2.9574158361171070+ i 9.9648824042207104	3.2550321345760635+ i 9.6683488712223857	3.5789347320158776+ i 9.3672676710290642
n=12	3.1032753942193563+ i 10.9552640491324684	3.3847735855160220+ i 10.6625048500664690	3.6893682201044155+ i 10.3620300238134045
n=13	3.2553358264839125+ i 11.9436696240217612	3.5232816924904915+ i 11.6553448306214296	3.8112775534344962+ i 11.3571092550822010
n=14	3.4122991562893823+ i 12.9301797686046945	3.6687616530324329+ i 12.6465665915965647	3.9424624353336276+ i 12.3515312395238338
n=15	3.5732039662813954+ i 13.9149336269793009	3.8198568756087864+ i 13.6360886864128457	4.0811884994497878+ i 13.3447911644570849
n=16	3.7373254907302420+ i 14.8980871898939116	3.9755327466516681+ i 14.6239413264205863	4.2260974363949525+ i 14.3366536825798388
n=17	3.9041080403755938+ i 15.8797941597111264	4.1349910344869138+ i 15.6102082831558692	4.3761234060296475+ i 15.3270363916127053
n=18	4.0731187801332949+ i 16.8601977272294165	4.2976082111411745+ i 16.5949959743651359	4.5304248931241354+ i 16.3159422273334060
n=19	4.2440156379158890+ i 17.8394276383398987	4.4628911819439449+ i 17.5784171493473187	4.6883320189693277+ i 17.3034202265520883
n=20	4.4165246631146465+ i 18.8175997840732277	4.6304453645963651+ i 18.5605824822531753	4.8493066829641754+ i 18.2895427446680643

TABLE V: The first $126 = \#\{n = 0, \dots, 20\} \times \#\{l = 2, \dots, 7\}$ ZNSBC QNMs:

	l=2	l=3	l=4
n=0	0.7791962435245995+ i 0.0684569968611839	1.2030921735161418+ i 0.0476722606344778	1.5942694406926746+ i 0.0334420676633112
n=1	0.8651732024304556+ i 0.6547111477546830	1.3813751407454793+ i 0.5839642214140248	1.8245468782581011+ i 0.5289776698927590
n=2	0.7939800314111605+ i 1.4769848991091885	1.4007941020402245+ i 1.3089053321083694	1.8914313642160924+ i 1.2078841123731487
n=3	0.7450367136071302+ i 2.4663000420812256	1.3811223470047639+ i 2.1685288184093693	1.8926441707386225+ i 1.9924308541085384
n=4	0.7368763037125811+ i 3.5659481900236920	1.3970643406410382+ i 3.1195381559269225	1.8853471747314408+ i 2.8737045040874281
n=5	0.9459004530034103+ i 4.7394129588425609	1.4617972798198575+ i 4.1059525084673399	1.9079870287520876+ i 3.8213001439290038
n=6	1.1907791955862268+ i 5.7592576880725459	1.5606990421618645+ i 5.1006200393151151	1.9690862409466749+ i 4.8010130419042788
n=7	1.4095026576115563+ i 6.7485065397140807	1.6807895448044570+ i 6.0951733187253812	2.0613769062895282+ i 5.7922411573846105
n=8	1.6171422668864292+ i 7.7280697275522168	1.8141904923250151+ i 7.0876402778038514	2.1758586826160518+ i 6.7859275186911416
n=9	1.8192522516064561+ i 8.7029836109328362	1.9562674568558392+ i 8.07791776184859708	2.3058072291264224+ i 7.7786461030805417
n=10	2.0182311317539034+ i 9.6751593931995807	2.1041827791191051+ i 9.0664227774036868	2.4467144985945422+ i 8.7692760704826909
n=11	2.2152729564728217+ i 10.6455212899001777	2.2561048275543787+ i 10.0537077674225896	2.5955946549423799+ i 9.7575892036855162
n=12	2.4110360878882267+ i 11.6145884357226981	2.4107927252619201+ i 11.0403638274873558	2.7504350569864462+ i 10.7436877033929570
n=13	2.6059121802692997+ i 12.5826831667455854	2.5673787773287617+ i 12.0270215512326190	2.9098433406424183+ i 11.7277819071443215
n=14	2.8001483158860513+ i 13.5500203012800649	2.7252643367176120+ i 13.0143890826962470	3.0728298741783905+ i 12.7101028767895160
n=15	2.9939078400247844+ i 14.5167506621164931	2.8840996912666036+ i 14.0033112257145441	3.2386731128094261+ i 13.6908693870096261
n=16	3.18730219474397569+ i 15.4829844341027295	3.0438579183763983+ i 14.9948364260241426	3.4068346308198255+ i 14.6702771384315891
n=17	3.3804131923652849+ i 16.4488046658942377	3.2050506270427135+ i 15.9902458769438769	3.5769041938184234+ i 15.6484969787224544
n=18	3.5732965245109822+ i 17.4142755466581834	3.3691449551662558+ i 16.9908841190206778	3.7485633610831137+ i 16.6256765866248104
n=19	3.7659961422909443+ i 18.3794477259510589	3.5390442807955163+ i 17.9973950934974700	3.9215607876808370+ i 17.6019432762294851
n=20	3.9585448989878083+ i 19.3443618686621784	3.7186853690100039+ i 19.0080538732959299	4.0956950910044268+ i 18.5774069495717789
	l=5	l=6	l=7
n=0	1.9702910930940089+ i 0.0228409464481625	2.3373842150721172+ i 0.0150441254206242	2.6983593675163178+ i 0.0095037780913750
n=1	2.2415974917130034+ i 0.4800352547045645	2.6451427695867747+ i 0.4352348416806698	3.0405782162494863+ i 0.3937697824768232
n=2	2.3430659862559728+ i 1.1288052581660507	2.7736882132955697+ i 1.0612416620383219	3.1911010051729202+ i 1.0010154136057298
n=3	2.3715863311497277+ i 1.8645420373224484	2.8279635225929215+ i 1.7632997049978360	3.2680497457326741+ i 1.6779615725250273
n=4	2.3675593798254051+ i 2.6843227875776478	2.8393102321342340+ i 2.5368378739824161	3.2979059882739084+ i 2.4171944540129029
n=5	2.3661381188494701+ i 3.5782836427175507	2.8343405445366607+ i 3.3795101715735235	3.3009622972953822+ i 3.2177114925866304
n=6	2.3901125166349546+ i 4.5244743028553478	2.8365232637900624+ i 4.2816676019987017	3.2966095235427866+ i 4.0762531493137451
n=7	2.4455677554616573+ i 5.4997785171718718	2.8606649736226643+ i 5.2269343158954965	3.3012926745215622+ i 4.9841028774755229
n=8	2.5287604319380268+ i 6.4883904365100853	2.9113570226714127+ i 6.1986370462977446	3.3252849399965484+ i 5.9286618703429710
n=9	2.6335284077998717+ i 7.4817975725195388	2.9866794117929382+ i 7.1844030201088688	3.3721817826431323+ i 6.8973364439995984
n=10	2.7544463233641412+ i 8.4759423347537011	3.0824053849201738+ i 8.1766286131293083	3.4410553772502458+ i 7.8802400576353351
n=11	2.8874295338159426+ i 9.4690370483945624	3.1942783185566045+ i 9.1711066993535637	3.5289858403398445+ i 8.8707131069801879
n=12	3.0295395770126306+ i 10.4603751460211089	3.3187647255446920+ i 10.1656498985324724	3.6326708596447138+ i 9.8646626113346971
n=13	3.1786741853564672+ i 11.4497508012212645	3.4531293412523773+ i 11.1591817060973089	3.7491309463527671+ i 10.8597081092836176
n=14	3.3333140833020000+ i 12.4371838712657892	3.5953035930672925+ i 12.1512166532172084	3.8759061881244092+ i 11.8545193772595648
n=15	3.4923437180985581+ i 13.4227899710402966	3.7437276262650425+ i 13.1415784028099962	4.0110399231480414+ i 12.8483849532008650
n=16	3.6549294965645078+ i 14.4067191149018355	3.8972176347202478+ i 14.1302493675405952	4.1529966575043335+ i 13.8409509731740731
n=17	3.8204376288685475+ i 15.3891270705152255	4.0548657003459046+ i 15.1172906587330605	4.3005734194129834+ i 14.8320678253173348
n=18	3.9883782791276468+ i 16.3701625031184460	4.2159670223570224+ i 16.1027994391538019	4.4528236399460187+ i 15.8217010335183464
n=19	4.1583670868295961+ i 17.3499617887978584	4.3799676454389169+ i 17.0868862807148680	4.6089970089035570+ i 16.8098795488207997
n=20	4.3300982380466979+ i 18.3286475296554859	4.5464268768879146+ i 18.0696633160487452	4.7684935735909160+ i 17.7966657195740543