ON SPHERICALLY SYMMETRICAL ACCRETION

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(Received 1951 October 3)

Summary

The special accretion problem is investigated in which the motion is steady and spherically symmetrical, the gas being at rest at infinity. The pressure is taken to be proportional to a power of the density. It is found that the accretion rate is proportional to the square of the mass of the star and to the density of the gas at infinity, and varies inversely with the cube of the velocity of sound in the gas at infinity. The factor of proportionality is not determined by the steady-state equations, though it is confined within certain limits. Arguments are given suggesting that the case physically most likely to occur is that with the maximum rate of accretion.

1. The importance of the accretion of interstellar gas by stars has been recognized since the work of Hoyle and Lyttleton (**1, 2, 3**). In their work the problem, later investigated in detail by Bondi and Hoyle (**4**), was that in which the rate of accretion was limited principally by the relative motion of the star and the gas cloud, the effects of pressure being considered negligible in comparison with the dynamical effects. The result derived in these papers was that

$$dM/dt = 2\pi\alpha(GM)^2 V^{-3}\rho_\infty, \qquad (1)$$

where M is the mass of the star, dM/dt is the rate of accretion, ρ_∞ the density of the gas cloud far from the star, V is the relative velocity of the star and the distant (undisturbed) parts of the cloud, G is the constant of gravitation, and α is a numerical constant which was first estimated to be equal to 2. Later work (**4**) showed that the steady-state equations did not determine α, although it seemed likely that it should always be between 1 and 2. It was also shown that if the star entered a cloud of uniform density with a plane boundary, α settled down to a value near 1·25.

In all this work pressure effects were neglected, the argument being that any heat generated would be radiated away rapidly, so that the temperature of the gas was always very low. Considerable mathematical simplification is introduced by this assumption, and it was shown that it was likely to be satisfied in most cases of astrophysical interest (**3**). The mathematical difficulties of the more general problem, in which both dynamical and pressure effects are considered, seem insuperable at present. However, the extreme case of negligible dynamical effects is again far simpler, and will be discussed in this paper. It may reasonably be expected that the case discussed here together with the case discussed previously bracket the complete problem.

2. The problem to be discussed may be defined as follows:

A star of mass M is at rest in an infinite cloud of gas, which at infinity is also at rest and of uniform density ρ_∞ and pressure p_∞. The motion of the gas is spherically symmetrical and steady, the increase in mass of the star being ignored so that the field of force is unchanging. The pressure p and density ρ are related everywhere by

$$p/p_\infty = (\rho/\rho_\infty)^\gamma, \tag{2}$$

where γ is a constant satisfying $1 \leqslant \gamma \leqslant \frac{5}{3}$.

With a suitable choice of γ, equation (2) is equivalent to the physical condition that no heat is radiated or conducted away. Hence the solution should provide the most complete contrast possible with the problem previously investigated. The equations governing the problem are easily set up. If we take r to be the radial coordinate and v the *inward* velocity of the gas, the equation of continuity is

$$4\pi r^2 \rho v = \text{constant} = A \text{ (say)}, \tag{3}$$

where A is the accretion rate.

Bernoulli's equation is

$$\frac{v^2}{2} + \int_{p_\infty}^{p} \frac{dp}{\rho} - \frac{GM}{r} = \text{constant } (=0). \tag{4}$$

The constant is readily seen to vanish by virtue of the boundary conditions at infinity. Combining (2) and (4) we have

$$\frac{v^2}{2} + \frac{\gamma}{\gamma - 1}\frac{p_\infty}{\rho_\infty}\left[\left(\frac{\rho}{\rho_\infty}\right)^{\gamma-1} - 1\right] = \frac{GM}{r}. \tag{5}$$

Equations (3) and (5) are two equations for the two variables v and ρ in terms of r, the distance from the centre of the star.

The equations may be made non-dimensional by the appropriate use of the velocity of sound in the gas at infinity, which as usual we denote by c. By the well-known formula

$$c^2 = \gamma p_\infty/\rho_\infty. \tag{6}$$

Let us introduce non-dimensional variables, x, y, z, to replace r, v, ρ, respectively, as follows:

$$\begin{aligned} r &= xGM/c^2, \\ v &= yc, \\ \rho &= z\rho_\infty. \end{aligned} \tag{7}$$

Then (3) and (5) take the non-dimensional form

$$x^2 y z = \lambda, \tag{8}$$

$$\tfrac{1}{2}y^2 + (z^{\gamma-1} - 1)/(\gamma - 1) = 1/x, \tag{9}$$

where λ is given by

$$A = 4\pi\lambda(GM)^2 c^{-3}\rho_\infty. \tag{10}$$

Accordingly λ is the non-dimensional parameter determining the accretion rate. It plays the same role as α in equation (1). It will also be observed that the relative velocity V of equation (1) has been replaced by c in (10).

3. The explicit solution of equations (8) and (9) for general γ is possible not in terms of the variables y and z but only if an auxiliary variable depending only on $y^2/z^{\gamma-1}$ is introduced. It is particularly interesting that mathematical requirements lead to the introduction of this variable, since

$$u = yz^{-(\gamma-1)/2} \tag{11}$$

has immediate physical significance as the ratio of the local bulk velocity v of the gas to the local velocity of sound $(\gamma p/\rho)^{1/2}$. Substituting (11) into (8) and solving for y and z we have

$$y = u^{2/(\gamma-1)}(\lambda/x^2)^{(\gamma-1)/(\gamma+1)}, \tag{12}$$

$$z = (\lambda/x^2 u)^{2/(\gamma-1)}. \tag{13}$$

Then (9) becomes

$$\tfrac{1}{2}u^{4/(\gamma+1)}\left(\frac{\lambda}{x^2}\right)^{2(\gamma-1)/(\gamma+1)} + \frac{1}{\gamma-1}\left(\frac{\lambda}{x^2 u}\right)^{2(\gamma-1)/(\gamma+1)} = \frac{1}{x} + \frac{1}{\gamma-1}. \tag{14}$$

Rearranging the terms and multiplying by $(x^2/\lambda)^{2(\gamma-1)/(\gamma+1)}$ we find that (14) takes the form

$$f(u) = \lambda^{-2(\gamma-1)/(\gamma+1)} g(x), \tag{15}$$

where

$$f(u) = \tfrac{1}{2}u^{4/(\gamma+1)} + \frac{1}{\gamma-1}u^{-2(\gamma-1)/(\gamma+1)} = u^{4/(\gamma+1)}\left(\frac{1}{2} + \frac{1}{\gamma-1}\frac{1}{u^2}\right), \tag{16}$$

$$g(x) = x^{4(\gamma-1)/(\gamma+1)}\left[\frac{1}{x} + \frac{1}{\gamma-1}\right] = \frac{x^{4(\gamma-1)/(\gamma+1)}}{\gamma-1} + x^{-(5-3\gamma)/(\gamma+1)}. \tag{17}$$

A study of the functions f and g serves to determine u as a function of λ and x. The variables y and z are then readily found by (12) and (13).

4. We shall first assume that $1 < \gamma < \frac{5}{3}$. The two limiting cases $\gamma = 1$, $\gamma = \frac{5}{3}$ will be examined later. With this assumption both f and g are each the sum of a positive and negative power of the respective variables and hence each of them has a minimum. The minimum of $f(u)$ occurs for $u = u_m = 1$ and is of value

$$f = \tfrac{1}{2}(\gamma+1)/(\gamma-1) = f_m \text{ (say)}.$$

The minimum of $g(x)$ occurs for $x = x_m = \frac{1}{4}(5-3\gamma)$ and is of value

$$g = \frac{1}{4}\frac{\gamma+1}{\gamma-1}[\tfrac{1}{4}(5-3\gamma)]^{-(5-3\gamma)/(\gamma+1)} = g_m \text{ (say)}.$$

In our problem x varies between infinity and the value corresponding to the surface of the star. This last value is very small indeed. As an example, if the star is taken to be the Sun, and $c = 1$ km/s corresponding to a gas temperature at infinity of nearly 3 000 deg. K, the surface value of x is only 5×10^{-6}. Even for a red giant the surface value of x would be less than 10^{-2} unless the temperature of the gas at infinity were quite improbably high (more than, say, 5×10^4 deg. K). Accordingly x will attain the value x_m in the physically significant interval. Hence, somewhere in that interval, the right-hand side of (15) will reach a value as low as $\lambda^{-2(\gamma-1)/(\gamma+1)} g_m$. But the lowest value f can reach is f_m. Hence λ cannot exceed λ_c, where

$$\lambda_c = \left(\frac{g_m}{f_m}\right)^{(\gamma+1)/2(\gamma-1)} = \left(\frac{1}{2}\right)^{(\gamma+1)/2(\gamma-1)}\left(\frac{5-3\gamma}{4}\right)^{-(5-3\gamma)/2(\gamma-1)}. \tag{18}$$

Our first result is therefore that the accretion rate A cannot exceed the value

$$4\pi\lambda_c(GM)^2 c^{-3}\rho_\infty. \tag{19}$$

Table I gives the value of λ_c for a few values of γ.

TABLE I

γ	1	1·2	$1·4 = \frac{7}{5}$	1·5	$\frac{5}{3}$
λ_c	$\frac{1}{4}e^{3/2} \doteqdot 1·12\ldots$	$\frac{1}{4}(0·7)^{-3·5} \doteqdot 0·872\ldots$	0·625	0·500	0·250

5. In order to obtain a more detailed picture it is necessary to discuss $f(u)$ and $g(x)$ more fully, and to take account of the boundary conditions of the problem. Since, at infinity, v vanishes but p and ρ tend to finite limits, it follows that u tends to zero there.

It is now easy to consider the problem graphically. Fig. 1 shows f and g drawn for the typical case $\gamma=\frac{7}{5}$. The resulting variation of u as a function of x is shown in Fig. 2 for the cases (i) $\lambda=\frac{1}{4}\lambda_c$, (ii) $\lambda=\lambda_c$ and (iii) $\lambda=4\lambda_c$. It will be seen that the boundary condition at infinity implies that u is very small near infinity (points beyond A on the graphs). For $\lambda<\lambda_c$, as x diminishes, u rises gradually to a maximum (B) and diminishes to zero (C) as x tends to zero. The closer λ approaches λ_c from below, the sharper the maximum B. No part of the curve A′B′C′ (on which u is very large both at infinity and near $x=0$) is of physical significance, since u given by this curve does not satisfy the boundary condition at infinity. No jump from the curve ABC to A′B′C′ is possible, since this would imply an infinite acceleration. Along ABC the variable u is always less than unity, so that the motion is subsonic. Along A′B′C′ the value of u exceeds unity.

The case $\lambda=\lambda_c$ is quite different. For in this the curves have contact at B = B′. Coming from A, the physically significant curve can continue either to C or to C′. In the first alternative, the curve is the limiting form of the curves for $\lambda<\lambda_c$. The curve has a discontinuous tangent at B and there is hence a finite jump in the acceleration. Although this is perhaps physically not very plausible, there does not seem to be any argument disallowing this motion altogether. It may be significant that at B the value of u is unity, so that the bulk velocity equals the velocity of sound.

The curve ABC′ is perfectly smooth and monotonic. For $x>x_B$ the motion is subsonic, while for $x<x_B$ the motion is supersonic. The system is in a state quite different from any state possible for $\lambda<\lambda_c$.

If λ exceeds λ_c then the pattern of the curves changes as indicated, and no solution is possible.

We see hence that there are two quite different types of motion. Type I exists for $\lambda\leqslant\lambda_c$. The motion is everywhere subsonic (except at $x=x_m$ if $\lambda=\lambda_c$), and u has a single maximum which is less than or equal to unity. The bulk velocity v has a maximum if $\gamma<\frac{3}{2}$ but not if $\gamma\geqslant\frac{3}{2}$. (This follows from a simple consideration of y' in terms of u.) For $\gamma<\frac{3}{2}$ the velocity v tends to zero as r tends to zero, for $\gamma=\frac{3}{2}$ it tends to a limit (equal to $\frac{1}{4}c\lambda$), and for $\gamma>\frac{3}{2}$ it tends to infinity. The density is always a monotonic function of the radius. Type II exists only if $\lambda=\lambda_c$. In this case u, v, ρ are all monotonic functions of the radius.

The special case $\lambda=0$ may be briefly referred to here. In this case the gas is at rest, forming a tenuous continuation of the star. Since $y=0$ we have by (9)

$$z^{\gamma-1}=1+(\gamma-1)/x. \tag{20}$$

Figs. 3, 4 and 5 show y and z as functions of x for $\gamma=1$, $\frac{7}{5}$ and $\frac{5}{3}$ respectively. Three values of λ are taken in each case, namely $\lambda=\lambda_c$, $\lambda=\frac{1}{4}\lambda_c$ and the case $\lambda=0$ just referred to. It will be seen from the figures that z does not depend very critically on λ, varying only slightly between the extreme cases $\lambda=0$ and $\lambda=\lambda_c$, especially in the case of the higher γ values.

6. It remains to discuss the two limiting cases $\gamma=1$ and $\gamma=\frac{5}{3}$ respectively.

If $\gamma=1$ equation (9) becomes

$$\tfrac{1}{2}y^2+\ln z=1/x. \tag{9$'$}$$

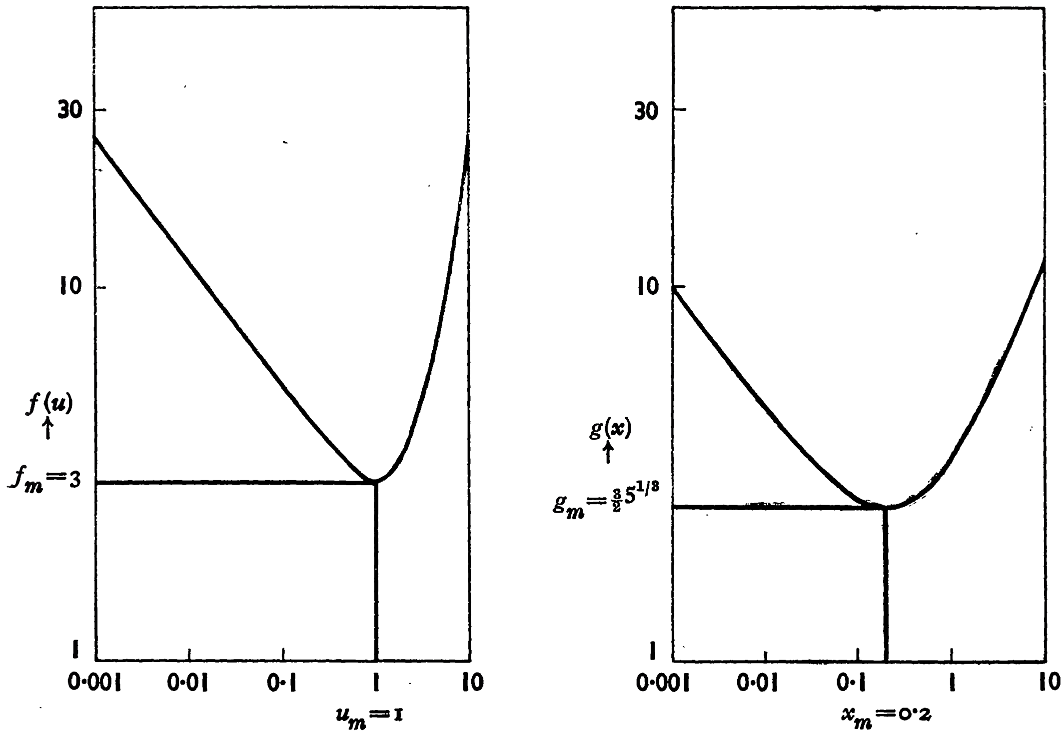

FIG. 1.—*f(u) and g(x) for* $\gamma=\frac{7}{5}$.

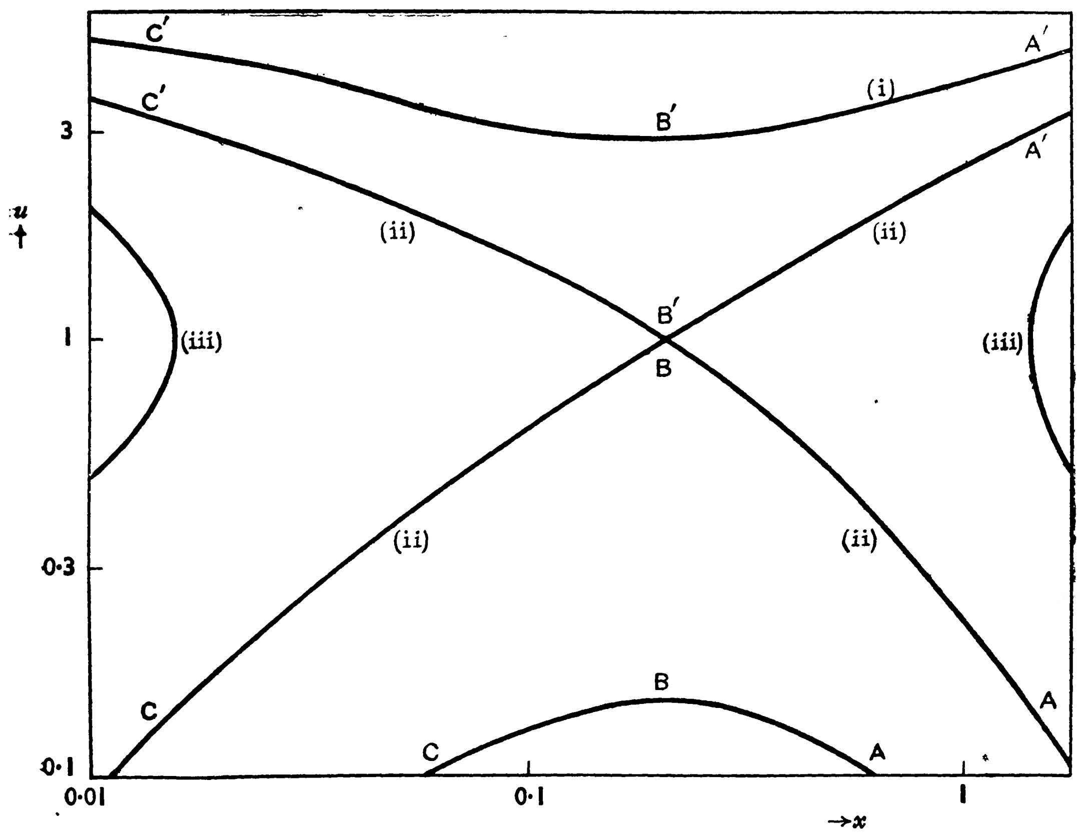

FIG. 2.—*u as function of x for* $\gamma=\frac{7}{5}$.

(i) $\lambda=\frac{1}{4}\lambda_c$;
(ii) $\lambda=\lambda_c$;
(iii) $\lambda=4\lambda_c$.

It is easy to eliminate z between (8) and (9′) since, by (11), $y=u$ in the present case. Accordingly

$$\tfrac{1}{2}y^2-\ln y=-\ln\lambda+(1/x+2\ln x). \qquad (14')$$

The minimum of the left-hand side occurs for $y=1$ and equals $\frac{1}{2}$, while the minimum of the bracket on the right-hand side occurs for $x=\frac{1}{2}$ and equals $2-2\ln 2$. Accordingly

$$\lambda_c=\tfrac{1}{4}e^{3/2}=1\cdot 120\ldots, \quad \ln\lambda_c=0\cdot 1138\ldots. \qquad (18')$$

This is also the limit of expression (18) as $\gamma\to 1$, so that λ_c is continuous at $\gamma=1$. Fig. 4 represents this case.

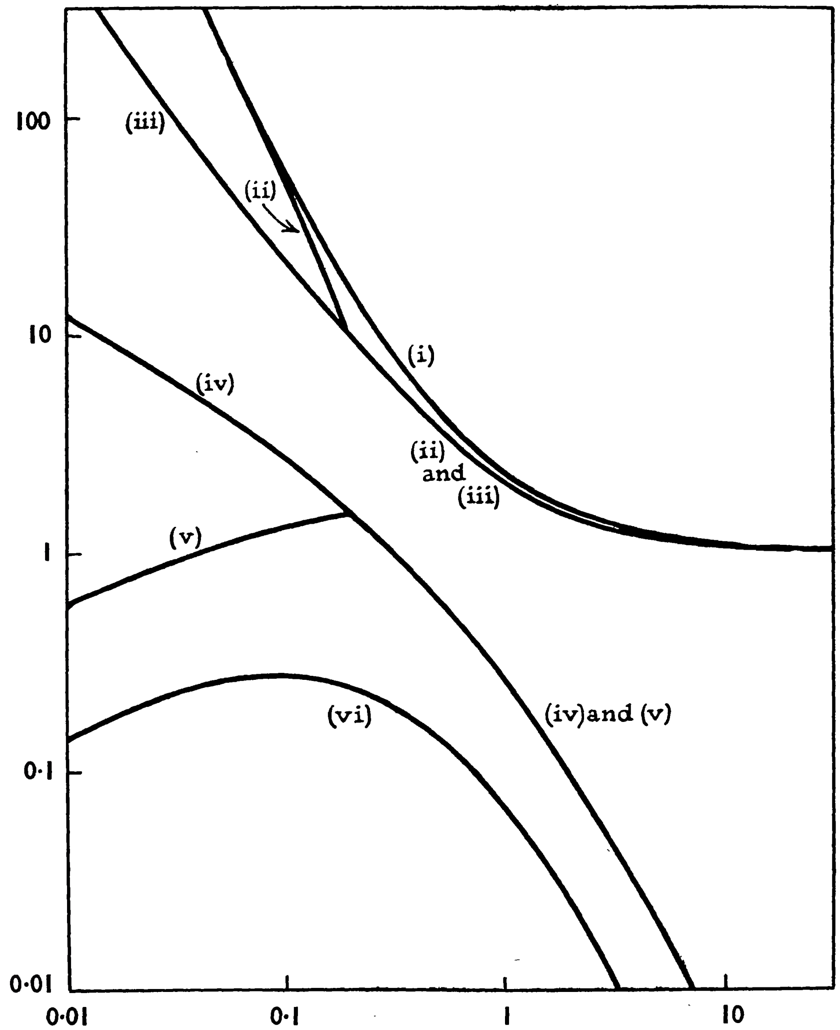

FIG. 3.—*y and z as functions of x for $\gamma=\frac{7}{5}$.*
(i) z ($\lambda=0$);
(ii) z ($\lambda=\lambda_c$, Type I);
(iii) z ($\lambda=\lambda_c$, Type II);
(iv) y ($\lambda=\lambda_c$, Type II);
(v) y ($\lambda=\lambda_c$, Type I);
(vi) y ($\lambda=\frac{1}{4}\lambda_c$).

The case $\gamma=\frac{5}{3}$ is an even simpler extension of the ordinary case. The only real change is that x_m (which decreases monotonically as γ increases) now equals zero. Accordingly there is now no difference between Type I and Type II motions for $\lambda=\lambda_c=\frac{1}{4}$ (Fig. 5).

It can be seen that u, y and z, considered as functions of x and γ, are continuous in $1 \leqslant \gamma \leqslant \frac{5}{3}$.

7. The final question that must be considered is what determines the value of λ in any actual case. It has been seen that the steady-state equations possess a solution whenever $0 \leqslant \lambda \leqslant \lambda_c$. The particular value of λ actually occurring must therefore be determined by other considerations. This is analogous to the

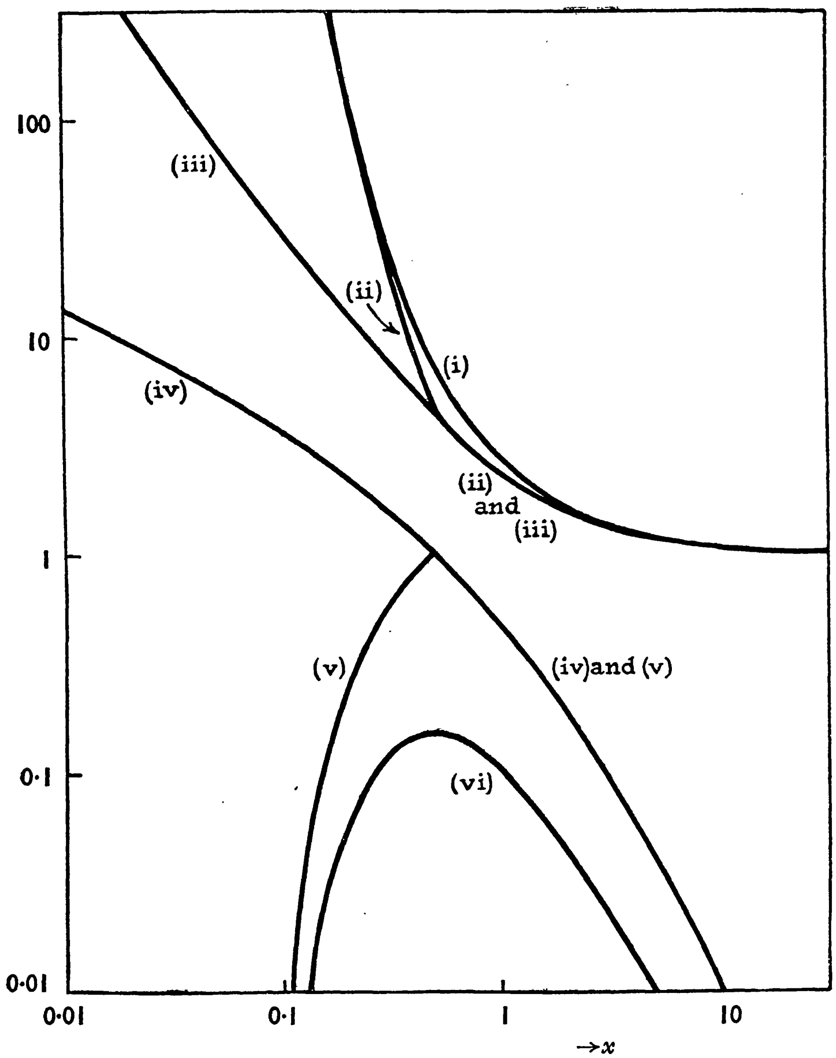

FIG. 4.—y and z *as functions of* x *for* $\gamma = 1$.
(i) z ($\lambda = 0$);
(ii) z ($\lambda = \lambda_c$, Type I);
(iii) z ($\lambda = \lambda_c$, Type II);
(iv) y ($\lambda = \lambda_c$, Type II);
(v) y ($\lambda = \lambda_c$, Type I);
(vi) y ($\lambda = \frac{1}{4}\lambda_c$).

velocity-limited case of accretion where α is not determined by the steady-state equations (4). The method used there to determine a specific value of α was to consider the case in which the star entered a cloud of gas with a plane boundary. The use of such a model would lead to very great mathematical difficulties in the present case, since the conditions of time-independence and spherical symmetry would have to be dropped simultaneously.

The boundary conditions at the surface of the star do not seem to be of help in the problem, since the star will swallow up any material falling into it without imposing any real conditions on the velocity, density, or pressure of the incoming materials.

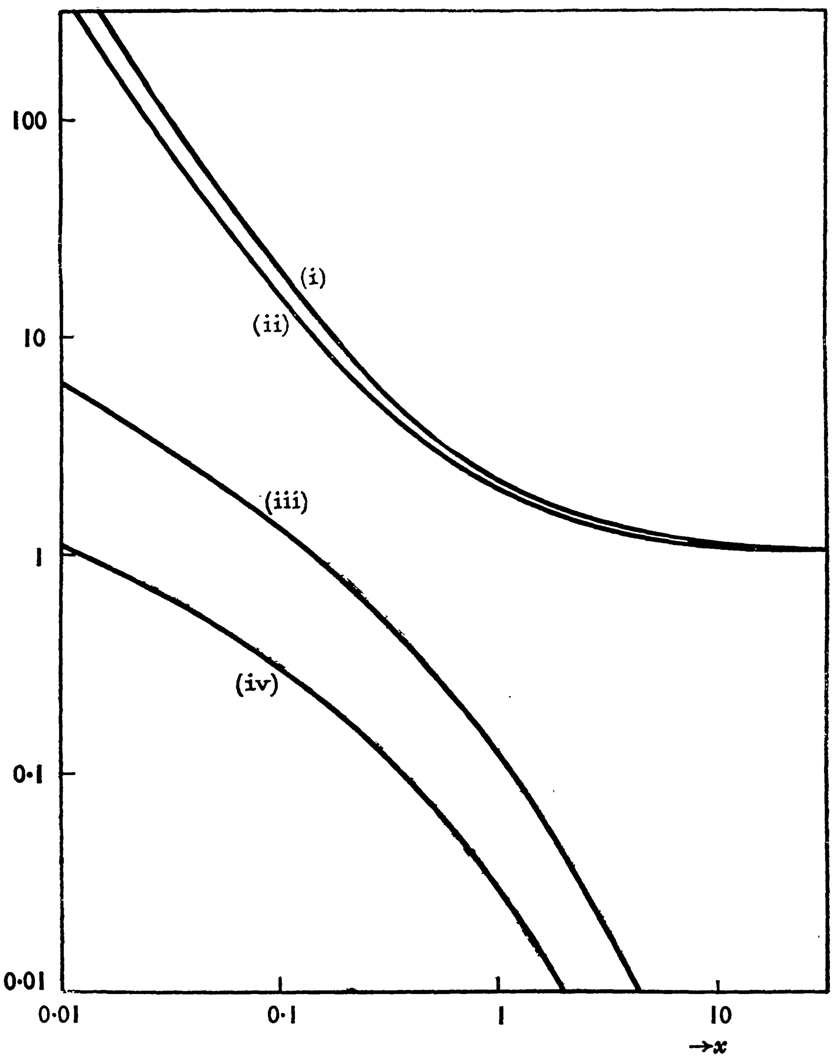

FIG. 5.—*y and z as functions of x for* $\gamma=\frac{5}{3}$.
(i) z ($\lambda=0$);
(ii) z ($\lambda=\lambda_c$);
(iii) y ($\lambda=\lambda_c$);
(iv) y ($\lambda=\frac{1}{2}\lambda_c$).

There remains the possibility of investigating the stability of the system with respect to small disturbances. Even if only spherically symmetrical perturbations are admitted, a partial differential equation of considerable complexity results. It is easily seen from it that disturbances are in part propagated with the velocity of sound relatively to the material, but the nature of the part that remains behind is not easily found. However, this may be a possible method of approach to the problem.

There is yet another possibility of investigating the stability of the system, and that is by comparing the energy of the system in its various states. The state with the lowest energy would then be expected to be the only stable state. Owing to the fact that our system is not isolated (the star itself not being considered part of the system) the validity of this approach is not quite assured, and similarly doubts may arise owing to the infinite extent of the system. Nevertheless, since the comparison of the energy of the system in every spherical shell leads to the same result, it seems very likely that the method gives the correct answer.

The energy of the gas per unit mass is constant by virtue of our assumptions (*cf.* equation (4)). Accordingly a comparison of the densities is all that is involved. Consider now z as function of λ for fixed x. Then it may be seen from equations (13), (15) and (16) that for u less than unity z decreases as λ increases. Accordingly the energy of the system in every spherical shell is lower for the Type I state with $\lambda=\lambda_c$ than for any other Type I state. Comparing now, for $\lambda=\lambda_c$, the Type I and the Type II states, it is immediately seen that, for $x \geqslant x_m$, the densities are the same, but that for $x<x_m$ the density (and hence the energy) is lower in the Type II state.

Accordingly the system has, in the sense described, the lowest energy in the Type II state, and we may expect to find a natural system in this state with $\lambda=\lambda_c$. If $\gamma=\frac{5}{3}$, the difference between the Type II state and the Type I state with $\lambda=\lambda_c$ disappears, and we would expect to find the system in this joint state. The result that the Type II state is the one most likely to be realized is very satisfactory, since the behaviour of all the functions is most uniform and smooth in this state. The result is also in agreement with the intuitive idea that, since there is nothing to stop the process of accretion, it takes place at the greatest possible rate, i.e. with $\lambda=\lambda_c$.

8. The two cases of accretion that have been examined so far may be called velocity-limited and temperature-limited respectively. The intermediate range of cases presents far greater difficulties. However, it may be possible to conjecture what the result is in the following way.

In the velocity-limited case of accretion the accretion rate A is given (**4**) by

$$A \doteqdot 2{\cdot}5\pi(GM)^2V^{-3}\rho_\infty, \tag{21}$$

while for, say, $\gamma=\frac{3}{2}$ in the temperature-limited case the result is

$$A=2\pi(GM)^2c^{-3}\rho_\infty. \tag{22}$$

If we therefore write down the formula

$$A \doteqdot 2\pi(GM)^2(V^2+c^2)^{-3/2}\rho_\infty,$$

it seems likely that it represents the order of magnitude of the accretion rate in the intermediate case, in which a star of mass M moves with relative velocity V in a uniform cloud of gas, in which the undisturbed density and the velocity of sound have the values ρ_∞ and c respectively. This formula, in agreement with intuitive ideas, suggests that if c exceeds V, temperature (pressure) imposes the chief limitation on the rate of accretion; whereas if V exceeds c, dynamical limitations are of greater importance.

The limitations due to pressure have probably been somewhat overestimated in this work. For if the cloud is able to radiate away some of the heat of compression then the adiabatic law will not apply, the pressure near the star will be diminished, and the accretion rate somewhat increased. How large this effect will be depends on the composition of the cloud. If there is a high proportion of constituents (such as hydrogen molecules (**2**)) that easily radiate at moderate temperatures, then the effect will be appreciable. In this case the effective value of γ will be closer to unity than to the standard value for the gas in question. If γ equals unity the process is isothermal.

The work of the present paper, together with previous work, is likely to give a fair estimate of the order of magnitude of accretion in all cases of physical interest. Further progress in this field will probably require the consideration of non-steady states.

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1951 *October* 2.

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