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Gravitational radiation from a particle in circular orbit around a black hole. II. Numerical results for the nonrotating case

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One promising source of gravitational waves for future ground-based interferometric detectors is the last several minutes of inspiral of a compact binary. Observations of the gravitational radiation from such a source can be used to obtain astrophysically interesting information, such as the masses of the binary components and the distance to the binary. Accurate theoretical models of the waveform are needed to construct the matched filters that will be used to extract the information. We investigate the applicability of post-Newtonian methods for this purpose. We consider the particular case of a compact object (e.g., either a neutron star or a stellar mass black hole) in a circular orbit about a much more massive Schwarzschild black hole. In this limit, the gravitational radiation luminosity can be calculated by integrating the Teukolsky equation. Numerical integration is used to obtain accurate estimates of the luminosity dE/dt as a function of the orbital radius r_0. These estimates are fitted to a post-Newtonian expansion of the form $dE/dt = (dE/dt)_N \sum_k a_k x^k$, where $(dE/dt)_N$ is the standard quadrupole-formula expression and $x \equiv (M/r_0)^{1/2}$. From our fits we obtain values for the expansion coefficients a_k up through order x^6. While our results are in excellent agreement with low-order post-Newtonian calculations, we find that the post-Newtonian expansion converges slowly. Corrections beyond x^6 may be needed to achieve the desired accuracy for the construction of the template waveforms.

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I. INTRODUCTION

Detection of gravitational waves using the planned kilometer-size laser interferometer detectors [Laser Interferometer Gravitational Wave Observatory (LIGO) [1], VIRGO [2] and presumably others] will generally rely on matched filter techniques [3] to extract the signal from the noise in which it will be embedded [4]. Detector sensitivity will be restricted to the range 10–1000 Hz, with maximum sensitivity at $\sim$ 100 Hz [1]. For the case of gravitational radiation originating from inspiraling neutron-star or black-hole binary systems, the gravitational-wave frequency sweeps through 10 Hz approximately 1 to 10 min before the final merger. For the matched filter technique to be successful, and for reliable determinations of the source parameters to be possible, it is necessary to construct theoretical template waveforms that are accurate to within $\sim$ 1 rad of phase over most of the 10^3–10^4 cycles emitted in these last few minutes [5, 6].

It seems unlikely that straightforward numerical integration of the Einstein equations will be able to produce accurate waveforms in the near future. In the absence of such a direct numerical calculation of the waveform, a post-Newtonian expansion is one possible approximate procedure for finding accurate waveforms for coalescing compact binaries. Several groups [7–9] have been calculating post-Newtonian approximations, including wave generation and radiation reaction, to the gravitational waveform from inspiraling binaries: Lincoln and Will [7], for example, have recently calculated such waveforms in a post-Newtonian expansion where the orbital elements are described accurately through order $(v/c)^4$, but that only includes the lowest-order contribution to the radiation reaction (i.e., that given by the quadrupole formula). But these estimates are not yet accurate enough for our purposes: neglect of the next-order term in the radiation reaction introduces errors in the inspiral rate of order $(v/c)^2 \sim 1\%$ and these errors result in an accumulated phase error of 1 rad in a few hundred cycles.

We address the following question: to what order must one carry such a post-Newtonian expansion in order to calculate the waveforms sufficiently accurately? Our tools and our understanding of the Einstein equations are

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not sufficient to obtain the answer to this question in full generality; however, we can answer it in the special case where $\mu/M \ll 1$ (where μ is the mass of the smaller body and M is the mass of the larger one). In this limit the less massive body acts as a small perturbation upon the stationary spacetime geometry of the more massive one. Thus, using the linear perturbation theory pioneered by Regge and Wheeler [10], Zerilli [11], and Teukolsky [12], we can obtain the waveform and luminosity to all orders in v/c (and first order in μ/M). Detweiler [13] was the first to compute the waveform and gravitational radiation luminosity in this way.

For our purposes it is necessary to determine only the luminosity dE/dt as a function of r_0. From this, we can obtain the binary inspiral rate $dr_0/dt = -(dr_0/dE)(dE/dt)$ and consequently the rate of change of the gravitational wave frequency $df/dt = (df/dr_0)(dr_0/dt)$ with fractional error of order μ/M (here r_0 denotes the orbital radius of the particle in Schwarzschild coordinates, and in dr_0/dE, E is the particle's orbital energy).

We examine the convergence of the post-Newtonian expansion for dE/dt using high-precision numerical calculations. At large orbital radii the luminosity approaches

$$\left(\frac{dE}{dt}\right)_N \equiv \frac{32}{5}\frac{\mu^2 M^3}{r_0^5}, \tag{1.1}$$

which follows directly from the quadrupole formula [14]. At finite r_0, dE/dt may be expressed as this "Newtonian" result times a power series in $x \equiv (M/r_0)^{1/2}$. The coefficients may be obtained by fitting that expansion to our numerical results for dE/dt. In fact, the two lowest-order corrections are already known exactly:

$$\frac{dE}{dt} = \left(\frac{dE}{dt}\right)_N \left(1 - \frac{1247}{336}x^2 + 4\pi x^3 + \cdots\right). \tag{1.2}$$

The coefficient of x^2 in this expansion was computed by Wagoner and Will [15] and by Gal'tsov [16]. Recently one of us (Poisson [17], paper I) derived analytically the 4π coefficient after our numerical fits had revealed it with high accuracy. An alternate derivation of this coefficient, valid for arbitrary mass ratios, was subsequently given by Wiseman [18].

Our numerical fits provide higher-order terms in this series, as well. We thus obtain a high-order post-Newtonian expansion for the power radiated. By truncating this expansion at nth order, we obtain an estimate of the inspiral rate which can be compared to the high-precision numerical answers.

Paper I [17] reviews the perturbation formalism and finds solutions using approximate, analytical methods. In Sec. II of this paper we briefly review the Teukolsky perturbation formalism and describe how we solve for the perturbation. We describe our numerical methods and estimate the accuracy of our results in Sec. III. In Sec. IV we model our numerical results for dE/dt in terms of a post-Newtonian expansion, and determine the values of the expansion coefficients. Section V examines the implications of our results for the construction of template waveforms, and Sec. VI offers some concluding observations.

II. SUMMARY OF THE FORMALISM

We model the binary system as a point particle of mass μ following a circular orbit around a nonrotating black hole of mass M, with $\mu \ll M$. The gravitational radiation is described as a perturbation of the Schwarzschild geometry driven by the point particle's stress-energy. The perturbation formalism as we have used it is presented in full detail in paper I [17] (cf. Detweiler [13]); we shall only briefly summarize it here.

In the Newman-Penrose formalism [19], the gravitational perturbations associated with outgoing radiation are described by the complex Weyl scalar Ψ_4. The perturbation equation for Ψ_4 is separable in space and time. (See [10] for an analysis of the Schwarzschild case [12], for the general case, and [20] for a comprehensive account of this problem.) The time dependence of the perturbation is $e^{-i\omega t}$ and the angular dependence is described by spin-weighted spherical harmonics with spin-weight $s = -2$ (so that $l \geq 2$ and $|m| \leq l$, where l and m are the usual spherical harmonic degree and order). The radial dependence $R_{\omega\ell m}(r)$ is governed by the inhomogeneous Teukolsky equation [12].

The source term of the inhomogeneous Teukolsky equation depends only on the stress-energy tensor of the point particle and has support only at the particle's radial location r_0. There are boundary conditions at both the horizon and at spatial infinity: the radiation must be strictly "down-going" (i.e., flowing into the black hole only) at the horizon, and out-going at spatial infinity. In the limit $\mu \ll M$ the test particle orbit does not evolve and thus the orbit is a source of radiation only at harmonics of the Keplerian orbital frequency Ω:

$$\omega = m\Omega, \tag{2.1}$$

where $\Omega = (M/r_0^3)^{1/2}$.

Like Detweiler [13], we solve the inhomogeneous Teukolsky equation by means of a Green's function constructed from the two linearly independent solutions, $R^H_{\omega\ell m}(r)$ and $R^\infty_{\omega\ell m}(r)$, of the homogeneous Teukolsky equation. The function $R^H_{\omega\ell m}(r)$ satisfies the ingoing-wave boundary condition at the event horizon but is a superposition of ingoing and outgoing waves at infinity. Correspondingly, $R^\infty_{\omega\ell m}(r)$ satisfies the outgoing-radiation boundary condition at infinity, but is a superposition of "down-going" and "up-going" radiation at the horizon.

The Green function analysis implies that the asymptotic behavior of $R^\infty_{\omega\ell m}$ determines the corresponding behavior of $R_{\omega\ell m}$. We thus have that

$$R_{\omega\ell m}(r \to \infty) = \mu Z_{\ell m} r^3 e^{i\omega r^*}, \tag{2.2}$$

where $Z_{\ell m}$ is a constant and

$$r^* = r + 2M\ln(r/2M - 1) \tag{2.3}$$

is the usual tortoise radial coordinate. Note that $Z_{\ell m}$ is independent of μ. The factor $e^{i\omega r^*}$ in Eq. (2.2) corresponds to purely outgoing radiation at infinity. For such solutions $R_{\omega\ell m}$, the associated Weyl scalar falls off like $e^{i\omega r^*}/r$ at infinity.

Physically interesting information may be obtained from $R_{\omega\ell m}(r \to \infty)$. In particular, the luminosity is given by

$$\frac{dE}{dt} = \mu^2 \sum_{\ell=2}^{\infty} \sum_{m=1}^{\ell} \frac{|Z_{\ell m}|^2}{2\pi\omega^2}. \tag{2.4}$$

Note that the sum has been restricted to positive spherical harmonic order m. This restriction is possible because the contributions from negative m are the same as those from positive m, and there is no contribution from $m = 0$.

The number $Z_{\ell m}$ encapsulates all the information about the source and the wave propagation from the near zone to infinity. It depends on (i) r_0 directly, (ii) $R^H_{\omega\ell m}(r)$ and its first derivative evaluated at $r = r_0$, and (iii) the constant $B^{\rm in}_{\omega\ell m}$, defined by

$$R^H_{\omega\ell m}(r \to \infty) = r^{-1} B^{\rm in}_{\omega\ell m} e^{-i\omega r^*} + r^3 B^{\rm out}_{\omega\ell m} e^{i\omega r^*}. \tag{2.5}$$

The direct dependence of $Z_{\ell m}$ on r_0 may be written down explicitly [17]; the evaluation of the other (wave-propagation) elements requires the numerical integration of the homogeneous Teukolsky equation.

It is clear from the asymptotic form of $R^H_{\omega\ell m}$ [cf. Eq. (2.5)] that determination of $B^{\rm in}_{\omega\ell m}$ by direct integration of the Teukolsky equation is difficult [21–24]. Since we are interested only in perturbations of Schwarzschild spacetime, we can instead integrate the Regge-Wheeler equation:

$$\left[f^2 \frac{d^2}{dr^2} + \frac{2M}{r^2} f \frac{d}{dr} + \omega^2 - V(r)\right] X^H_{\omega\ell m}(r) = 0, \tag{2.6}$$

with the potential

$$V(r) = f\left[\ell(\ell+1)/r^2 - 6M/r^3\right] \tag{2.7}$$

where $f = 1 - 2M/r$. The solution $X^H_{\omega\ell m}$ is chosen to satisfy the down-going radiation boundary conditions at the black-hole horizon:

$$X^H_{\omega\ell m}(r \to 2M) \sim e^{-i\omega r^*}. \tag{2.8}$$

Its asymptotic behavior at large r is then

$$X^H_{\omega\ell m}(r \to \infty) \sim A^{\rm in}_{\omega\ell m} P(\omega r) e^{-i\omega r^*} + A^{\rm out}_{\omega\ell m} \bar{P}(\omega r) e^{i\omega r^*}, \tag{2.9}$$

where the overbar denotes complex conjugation, and $P(\omega r)$ is a power series in $1/\omega r$ whose coefficients can be determined by substituting Eq. (2.9) into Eq. (2.6).

We can recover $R^H_{\omega\ell m}$ at any r from $X^H_{\omega\ell m}$ and its first derivative at the same r by using an algebraic transformation due to Chandrasekhar [25]. In this way, we find that $B^{\rm in}_{\omega\ell m}$ is determined by $A^{\rm in}_{\omega\ell m}$.

III. NUMERICAL INTEGRATION

The only major numerical task is to integrate the Regge-Wheeler equation (2.6), determine $X^H_{\omega\ell m}(r)$ and its first derivative at $r = r_0$, and find $A^{\rm in}_{\omega\ell m}$.

We integrated the Regge-Wheeler equation for each $5 \geq \ell \geq 2$, $\ell \geq m > 0$, and $\omega = m\Omega$. If r is chosen as the independent variable, the Regge-Wheeler equation is singular at the black-hole horizon where the boundary conditions are imposed. Alternatively, if the tortoise coordinate r^* [cf. Eq. (2.3)] is chosen as the independent variable, the singularity is "pushed off" to $r^* \to -\infty$. We nevertheless chose to adopt r as the independent variable, hence avoiding the numerical inversion of $r^*(r)$; we found that the singular character of the Regge-Wheeler equation is not a serious impediment. The integration was started at $r = r_i = 2M(1+\varepsilon)$ for small ε ($10^{-4} \gtrsim \varepsilon \gtrsim 10^{-10}$). Equation (2.8) was used to infer the boundary conditions $X^H_{\omega\ell m}(r_i)$ and $X^{H\prime}_{\omega\ell m}(r_i)$ (where a prime denotes differentiation with respect to r). Using the algorithm of Bulirsch and Stoer [26], we then integrated outward from $r = r_i$: the integrator spent a long time in the singular region near $r = 2M$ and then

TABLE I. Numerical values for the gravitational power, including multipoles up through $\ell = 5$, for several values of orbital radius. Caution: This table is printed with a large number of significant places. This is not intended to indicate the accuracy of our calculations. In fact, these numbers are accurate to between six and ten significant places. Many places are offered solely for the purpose of precise reproduction and testing of our results.

r_0/M	$(M/\mu)^2 dE/dt$
6	$9.334792562807677 \times 10^{-4}$
8	$1.957426618374151 \times 10^{-4}$
10	$6.147308500354202 \times 10^{-5}$
12	$2.428366476150825 \times 10^{-5}$
24	$7.545091436738473 \times 10^{-7}$
36	$1.0045251750454977 \times 10^{-7}$
48	$2.4040127560388038 \times 10^{-8}$
70	$3.681880101348405 \times 10^{-9}$
80	$1.8945354109627725 \times 10^{-9}$
90	$1.0541121060132224 \times 10^{-9}$
100	$6.238202648137723 \times 10^{-10}$
120	$2.51576748851826 \times 10^{-10}$
140	$1.1670297381469564 \times 10^{-10}$
150	$8.274457551383351 \times 10^{-11}$
300	$2.6073383971227096 \times 10^{-12}$
400	$6.201576437236146 \times 10^{-13}$
500	$2.0350485987533726 \times 10^{-13}$
600	$8.186438165393261 \times 10^{-14}$
700	$3.7902840858413365 \times 10^{-14}$
800	$1.945130362386846 \times 10^{-14}$
900	$1.0798725934420318 \times 10^{-14}$
1000	$6.378752721731783 \times 10^{-15}$
2000	$1.9965668413422893 \times 10^{-16}$
3000	$2.630686436939572 \times 10^{-17}$
4000	$6.244509398335936 \times 10^{-18}$
6000	$8.225583027365644 \times 10^{-19}$
8000	$1.952253057505779 \times 10^{-19}$
12000	$2.571245492447089 \times 10^{-20}$
16000	$6.10213764672353 \times 10^{-21}$
24000	$8.036335624329943 \times 10^{-22}$
32000	$1.9071315991777994 \times 10^{-22}$

TABLE II. Decomposition of the gravitational power into multipole contributions, for $r_0/M = 10$. See caution in Table I.

ℓ	m	$(M/\mu)^2 dE/dt$
2	2	$5.368795477495964 \times 10^{-5}$
2	1	$1.9316094014509972 \times 10^{-7}$
3	3	$6.426082556370699 \times 10^{-6}$
3	2	$4.79591646035441 \times 10^{-8}$
3	1	$5.714898908903078 \times 10^{-8}$
4	4	$9.53960064291412 \times 10^{-7}$
4	3	$8.778757573084952 \times 10^{-9}$
4	2	$5.262245330003326 \times 10^{-10}$
4	1	$1.457585631445564 \times 10^{-13}$
5	5	$1.5241547683239266 \times 10^{-7}$
5	4	$1.4921162878190968 \times 10^{-9}$
5	3	$1.8291012379756077 \times 10^{-10}$
5	2	$3.8193532390971175 \times 10^{-13}$
5	1	$2.367637176972572 \times 10^{-16}$
	Sum $= 6.147308500354202 \times 10^{-5}$	

proceeded swiftly.

We first integrated from $r = r_i$ to $r = r_0$, and obtained the values of $X^{H}_{\omega\ell m}(r_0)$ and $X^{H\prime}_{\omega\ell m}(r_0)$. We then resumed the integration, proceeding until r reached a value of order $1/\omega$. Equation (2.9) and its derivative were then used to approximate $A^{\rm in}_{\omega\ell m}$. For each subsequent integration step in r, the corresponding approximate value of $A^{\rm in}_{\omega\ell m}$ was estimated; this procedure yields a sequence $A^{\rm in}_{\omega\ell m}(r)$ of numbers which converges to $A^{\rm in}_{\omega\ell m}$ in a way governed by Eq. (2.9). Since the values of $A^{\rm in}_{\omega\ell m}(r)$ approach $A^{\rm in}_{\omega\ell m}$ as a power series in $1/\omega r$ (with the constant term being the correct answer), $A^{\rm in}_{\omega\ell m}$ is obtained rapidly and to high accuracy by using repeated Richardson extrapolation [27]. The Richardson extrapolator successively doubled the target radius for integration, extrapolating the sequence $A^{\rm in}_{\omega\ell m}(r)$ to obtain an estimate for $r = \infty$. The Richardson limit was assumed to be reached when the next step made a relative change in the answer of less than one part in 10^6. The integrator was then stopped, and we obtained the value of $A^{\rm in}_{\omega\ell m}$. In general the residual relative error incurred by cutting off the extrapolation here is much smaller than 10^{-6}; it is probably around 10^{-9}. The remaining operations were purely algebraic. For the given values of ℓ and m the number $Z_{\ell m}$ was evaluated, which lead to the power radiated by the mode (ℓ, m). We repeated this calculation for a wide range of orbital radii; for each r_0 the calculations were performed for $2 \le \ell \le 5$ and $1 \le m \le \ell$. Complete tables of all our numerical results are available [28]; to facilitate the reader's ability to check our results, we have also included a small subset of our numerical data in the tables. Table I lists the summed contributions of the modes $l \le 5$ for a range of r_0. Tables II to IV list all the individual mode contributions for three different values of r_0.

The accuracy of our results is difficult to estimate. All computations were performed in IEEE double-precision arithmetic [29]. The numerical integration was done with a relative local truncation error no greater than one part in 10^{10} and the number of steps of integration was not large; so, there was not much opportunity for the global error to grow. The Richardson limit we obtained was invariant (in the first six places) to changes of the initial radius as we varied ε from 10^{-4} to 10^{-10} (we used $\varepsilon = 10^{-6}$): thus the accuracy of our answers is not particularly affected by the singular character of the Regge-Wheeler equation near $r = 2M$.

Finally, and most importantly, after minor debugging, the solution process behaved as expected: the Richardson extrapolator did act as if the extrapolated quantity was in fact the first term of a power series, chopping off successive terms and converging faster than any polynomial. This gives us great confidence in the answers. We can be sure that we have at least 6 significant digits in the answers, and internal evidence from the structure of the residuals in our fits gives us reason to believe that our answers have about nine significant places (see Fig. 2).

TABLE III. Decomposition of the gravitational power into multipole contributions, for $r_0/M = 150$. See caution in Table I.

ℓ	m	$(M/\mu)^2 dE/dt$
2	2	$8.199125786292143 \times 10^{-11}$
2	1	$1.5533943517118724 \times 10^{-14}$
3	3	$7.293520702420003 \times 10^{-13}$
3	2	$2.9082089243401785 \times 10^{-16}$
3	1	$6.74697151150845 \times 10^{-17}$
4	4	$7.971664741718715 \times 10^{-15}$
4	3	$3.9142213792047955 \times 10^{-18}$
4	2	$4.505010440900235 \times 10^{-18}$
4	1	$8.521671202482892 \times 10^{-23}$
5	5	$9.309999512986334 \times 10^{-17}$
5	4	$4.8428899552929203 \times 10^{-20}$
5	3	$1.1404657697295971 \times 10^{-19}$
5	2	$1.5786392264188516 \times 10^{-23}$
5	1	$1.8449226677094525 \times 10^{-25}$
	Sum $= 8.274457551383351 \times 10^{-11}$	

TABLE IV. Decomposition of the gravitational power into multipole contributions, for $r_0/M = 1000$. See caution in Table I.

ℓ	m	$(M/\mu)^2 dE/dt$
2	2	$6.369945055198321 \times 10^{-15}$
2	1	$1.7759713704759728 \times 10^{-19}$
3	3	$8.614435276459479 \times 10^{-18}$
3	2	$5.059610388808615 \times 10^{-22}$
3	1	$7.895853658131901 \times 10^{-22}$
4	4	$1.430413566682988 \times 10^{-20}$
4	3	$1.034676636333062 \times 10^{-24}$
4	2	$8.001814004174917 \times 10^{-24}$
4	1	$2.254175411299116 \times 10^{-29}$
5	5	$2.5368671341710977 \times 10^{-23}$
5	4	$1.943844777526346 \times 10^{-27}$
5	3	$3.0755337304511715 \times 10^{-26}$
5	2	$6.332662426296351 \times 10^{-31}$
5	1	$4.965606233081542 \times 10^{-32}$
	Sum $= 6.378752721731783 \times 10^{-15}$	

IV. POST-NEWTONIAN EXPANSION

In this section we fit our numerical results for dE/dt to a post-Newtonian expansion of the form

$$\frac{dE}{dt} = \left(\frac{dE}{dt}\right)_N \sum_{k=0} a_k x^k, \tag{4.1}$$

where $x = (M/r_0)^{1/2}$ and $(dE/dt)_N$ is the quadrupole approximation luminosity [cf., Eq. (1.1)]. The first four terms are already known [15–17]:

$$a_0 = 1, \quad a_1 = 0, \quad a_2 = -\tfrac{1247}{336}, \quad a_3 = 4\pi. \tag{4.2}$$

By fitting Eq. (4.1) to our numerical results (we use an equal-weight least-squares fitting criterion, implemented with the method of singular-value decomposition [30]), we are able to estimate a few of the higher-order coefficients.

Equation (2.4) shows that the gravitational power has contributions from arbitrarily high-order terms in the multipole expansion. In this paper, we have included multipoles only up through $\ell = 5$, and this will induce errors in the values of the post-Newtonian coefficients. However, Paper I shows that a multipole of order ℓ only contributes to the total power as a correction of order $(M/r_0)^{\ell-2}$. By computing terms up through $\ell = 5$ we therefore incur a relative error of order $(M/r_0)^4 = x^8$. Consequently, our numbers allow, in principle, the accurate determination of the post-Newtonian coefficients up through order x^7. This is not achieved in practice, because the higher-order coefficients are increasingly sensitive to numerical error.

To illustrate this point, we tested how accurately we could recover, by least-squares fitting, the coefficients shown in (4.2). Typically, our fits yield the theoretically derived coefficients with the following relative errors: $\Delta a_0/a_0 \sim 10^{-9}$, $\Delta a_1 \sim 10^{-6}$, $\Delta a_2/a_2 \sim 10^{-5}$, and $\Delta a_3/a_3 \sim 10^{-3}$.

Figure 1 shows the residual obtained by subtracting the theoretically derived terms from the numerical results. The residual, Δ, is defined by

$$\Delta = \frac{dE/dt - (dE/dt)_N \left\{1 - \frac{1247}{336}x^2 + 4\pi x^3\right\}}{(dE/dt)_N\, x^4}. \tag{4.3}$$

The coefficient a_4 is Δ at $x = 0$ and the coefficient a_5 is the slope of Δ at $x = 0$. A least-squares best fit of a cubic to those points for which $r_0/M > 100$ gives $\Delta \simeq -4.8924 - 37.981x + 135.15x^2 + 19.714x^3$. From this we conclude that

$$a_4 \simeq -4.89, \quad a_5 \simeq -38, \quad a_6 \simeq +135. \tag{4.4}$$

The natural question is now: how accurate are the estimations of a_4, a_5, and a_6? This is difficult to ascertain. The quality of the fits is measured in terms of the standard deviation σ, essentially the rms deviation of each data point from the fitted curve. A small value of σ therefore corresponds to a good fit. But a good fit by no means implies an accurate determination of the post-Newtonian coefficients: it is possible to change slightly the value of a_4 without affecting the quality of the fit, provided that a_5 and a_6 are modified accordingly. Thus a small uncertainty in a_4 is translated into large variations in the higher-order coefficients. However, this much is clear: the value of a_4 may not be changed by a large amount without affecting significantly the quality of the fits. Thus we are confident that our estimation of a_4 is fairly accurate; we believe, but cannot prove, that it is correct to within 2%. It is much harder to evaluate our accuracy on a_5 and a_6; as a rough indication, we would suggest that a_5 is precise to within 10% and that a_6 is precise to 50%.

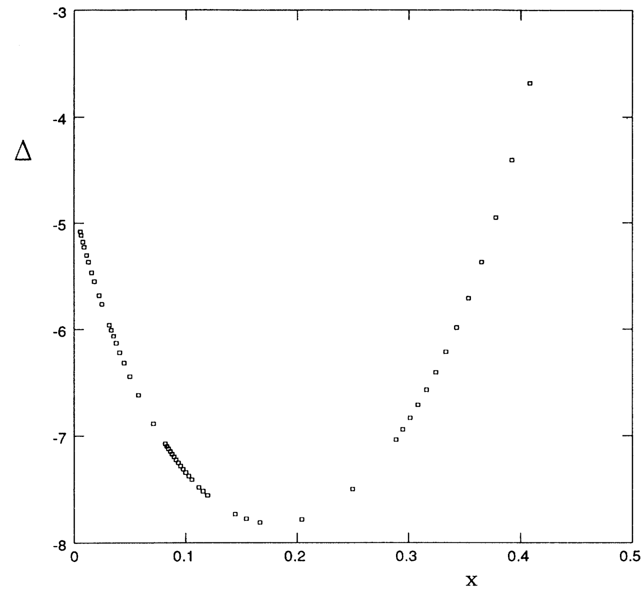

FIG. 1. The residual Δ plotted against x. The intercept is a measure of a_4 and the slope near the axis is a measure of a_5. Although the curve appears symmetrical, it is not well approximated by a parabola.

V. IMPLICATIONS FOR THE CONSTRUCTION OF TEMPLATE WAVEFORMS

We can now test the feasibility of using the post-Newtonian approximation to calculate template waveforms for the last few thousand orbits of a compact binary.

First we derive a rough criterion indicating how accurately dE/dt must be known to ensure that the phase of the template waveform matches the true waveform to within one radian. Consider two template waveforms corresponding to the same physical situation (same two masses, same distance from observer, etc.) — one constructed using an approximate form for $\dot{E}(r_0)$, the second constructed using the correct function $\dot{E}(r_0)$ (a dot denotes a time derivative). Let the two waveforms have the same phase $\Phi = 0$ and same frequency $f = f_0$ at some initial time $t = 0$. The relative difference in energy loss $\Delta\dot{E}$ implies a difference in the rate of change of the wave frequency, with $\Delta\dot{f}/\dot{f} \sim \Delta\dot{E}/\dot{E}$. Hence the relative phase error between the templates, $\Delta\Phi$, will (initially) accumulate quadratically in time: $\Delta\Phi \sim (\Delta\dot{f}_0)t^2$. The entire inspiral lasts a time of order $t \sim f_0/\dot{f}_0$, and the total phase accumulated in that time is $\Phi \sim f_0 t$. Combining these relations, we find that $\Delta\Phi$, at the end of the inspiral, is roughly given by

$$\Delta\Phi/\Phi \sim \Delta\dot{E}/\dot{E}, \tag{5.1}$$

where the right side refers to the relative error in $\dot{E}$ when the frequency sweeps past f_0.

For the planned ground-based detectors, the relevant value of f_0 is 10 Hz (below which the signal is expected to be undetectable due to inadequate isolation from seismic noise). This frequency corresponds to (i) $r_0/M \simeq 50$ and $\Phi \simeq 4 \times 10^3$ in the case of two 10 $M_\odot$ black holes, (ii) to $r_0/M \simeq 70$ and $\Phi \simeq 2 \times 10^4$ for a $10M_\odot$ black hole and a $1.4M_\odot$ neutron star, and (iii) to $r_0/M \simeq 175$ and $\Phi \simeq 10^5$ for two $1.4M_\odot$ neutron stars. (Here we assume that the expansions for non-negligible mass μ/M will be subject to the same convergence problems as for negligible μ/M.)

How accurately can template wave forms be calculated for these three cases when dE/dt is improved from the standard quadrupole formula by including post-Newtonian corrections up through order x^3 (the order through which the expansion coefficients are currently known analytically)? Figure 2 and the criterion described in Eq. (5.1) show that a post-Newtonian approximation through order x^3 leads to a cumulative phase error of ~ 10 radians for case (i), ~ 20 radians for case (ii), and ~ 20 radians for case (iii).

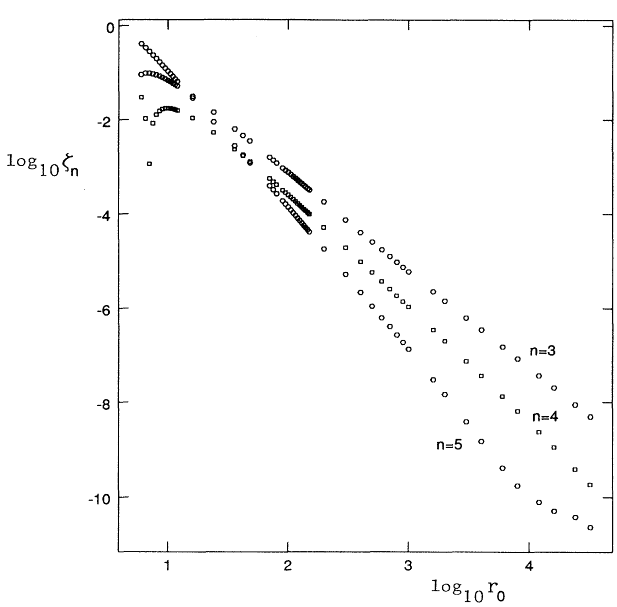

FIG. 2. Plots of $\log_{10}\zeta_n$ ($n = 3,\ldots,5$) as functions of $\log_{10} r_0$, where $(dE/dt) \times \zeta_n \equiv dE/dt - (dE/dt)_N \sum_{k=0}^{n} a_k x^k$. The values of a_k are listed in Eqs. (4.2) and (4.4). Each curve represents, as a function of orbital radius, the relative error between the numerical values of dE/dt and a post-Newtonian expansion truncated to order x^n. Notice that the asymptotic behavior of the residuals for each approximation appears to be a power law, except when the residuals get below 10^{-10}. Here we are probably seeing the appearance of the numerical errors in our calculations.

Figure 2 further shows that by carrying the expansion to order x^5 the cumulative phase accuracy would improve by a factor of approximately 2 to 6 — a rather modest improvement considering the effort required to extend the post-Newtonian calculations to this order. Clearly the slow convergence of the post-Newtonian expansion is due to the large values of the expansion coefficients at order x^5 and x^6.

The proposed Laser Gravitational-Wave Observatory in Space (LAGOS) would be most sensitive to gravitational radiation in the bandwidth 10^{-3} Hz $\lesssim f \lesssim 10^{-2}$ Hz [31]. This corresponds to the frequency of the radiation from compact stellar mass objects orbiting at or near the last stable circular orbit about small supermassive black holes ($4\times10^5\,M_\odot \lesssim M \lesssim 4\times10^6\,M_\odot$ for Schwarzschild). Black holes like these may lurk in the centers of active galactic nuclei [32, 33] and normal galaxies [34–37] (including our own [38, 39]). In this case $\mu/M \lesssim 10^{-6}$ and the approximation $\mu/M \ll 1$ is almost exact.

Depending on the mass of the black hole and the compact object, the radiation can spend anywhere from hundreds of years (for a $1M_\odot$ compact object and a $4\times10^5 M_\odot$ black hole) to seconds (for a 4×10^6) in the LAGOS bandwidth. Observation times, on the other hand, will be no longer than 1/3 to 1 year. Consequently, the number of cycles that can be observed by LAGOS varies from a very few to on order 3×10^4.

Since the orbital radius is so close to the last stable circular orbit for sources that would be observed by LAGOS, $x = (M/r_0)^{1/2} \simeq 2/5$ is large and many terms would be needed in a post-Newtonian expansion in order to find an adequate approximation for dE/dt. It would be nonsensical to proceed in this way, however: assuming circular orbits, calculations of the kind described here [but involving $\max(\ell) \gtrsim 20$] provide the exact dE/dt and consequently the exact df/dt for the evolution of the orbits. With df/dt and the exact wave forms (which have not been calculated here, but can be found from the $Z_{\ell m}$; *cf.* Detweiler [13]), numerical wave forms appropriate for the constructions of matched filters for LAGOS may be found. Elliptical capture orbits (which may be more important for LAGOS than for LIGO [41]) may also be evolved (both in eccentricity and radius) by solving the Teukolsky equation (only now the radiation is not restricted to harmonics of the orbital frequency).

VI. CONCLUSIONS

We have solved numerically the Teukolsky equation to find the gravitational-wave luminosity of a test mass in a circular orbit about a Schwarzschild black hole. Our results are in excellent agreement with low-order post-Newtonian calculations and allow us to find approximate values for higher-order post-Newtonian expansion coefficients. We find that, for the purpose of computing template wave forms for the planned interferometric detectors, the expansion converges rather slowly.

We have explored the possibility that, by fitting a somewhat different function to our data, the convergence of the post-Newtonian expansion could be improved. For example, paper I shows that each (ℓ, m) contribution to dE/dt possesses a simple pole at $r_0 = 3M$. We verified the existence of this pole in our numerical data. One might suspect that by fitting a power series in x to $(1-3M/r_0)^{-1}(dE/dt)/(dE/dt)_N$, we could obtain better convergence. Unfortunately, removing the pole does not help. We also tried a number of higher-order Padé approximations, without obtaining more rapid convergence.

We feel that the particular problem studied here provides a useful testbed for exploring more fundamental questions concerning the convergence of approximation formalisms designed to include radiation reaction. To our knowledge there exists no general algorithm that allows one to solve radiation reaction problems to arbitrary order in a post-Newtonian expansion (as distinguished from, say, a post-Minkowski expansion [42]). In this sense, the theory is much less well developed than, say, perturbative quantum electrodynamics. Nor is it understood whether an infinite post-Newtonian expansion (assuming one could be generated) would converge or merely be asymptotic to an appropriate solution of the field equations.

Such questions acquire precise formulations in the context of a test mass in a circular orbit around a spherical body. To wit, consider the function $P(x) = (dE/dt)/(dE/dt)_N$. A post-Newtonian expansion of the gravitational-wave luminosity essentially generates the derivatives of $P(x)$ at $x = 0$. We know that $P(x)$ is at least C^3 at $x = 0$, but is it C^∞ at $x = 0$? A preliminary calculation by Ori [43] suggests that the sixth derivative of $P(x)$ fails to exist at $x = 0$, due to the presence of a $x^6 \ln x$ term in the post-Newtonian expansion. If this is true, can one transform to a slightly different variable, in terms of which the function is C^∞? A detailed understanding of this rather simple problem might help clarify the general status of post-Newtonian formalisms that include radiation reaction.

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