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Maximum Mass of a Neutron Star*

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On the basis of Einstein's theory of relativity, the principle of causality, and Le Chatelier's principle, it is here established that the maximum mass of the equilibrium configuration of a neutron star cannot be larger than $3.2M_\odot$. The extremal principle given here applies as well when the equation of state of matter is unknown in a limited range of densities. The absolute maximum mass of a neutron star provides a decisive method of observationally distinguishing neutron stars from black holes.

The estimate of the range of the critical mass for a neutron star varies from 0.32 to $1.5M_\odot$. The greatest uncertainty comes from the equation of state at nuclear densities and above. In fact the knowledge of physical properties of neutron-star material at densities smaller than 10^{13} g/cm^3, essential to describe the properties of the crust of neutron stars[1] and perhaps the change in period of the pulsars,[2] is of no relevance for the determination of the maximum mass of a neutron star. The reason is that on increase of the central density the star becomes more and more compact and its crust becomes only a few tens of meters thick, or even less, depending on the models.[3] At nuclear densities and above, the equation of state is very poorly known because of the presence of strong interactions[4] between nucleons and threshold effects in the creation of resonances[5,6] because of unavailability of phase space.

In recent times it has become clear that the most powerful tool in determining the difference between neutron stars and black holes relies on the possible difference in mass of the two objects.[7] No possibility exists of differentiating them on the basis of electrodynamic properties.[8] Moreover, the recent discovery of x-ray sources in binary systems gives the possibility of determining the mass of a collapsed object with great accuracy.[9] We therefore have the clear need of establishing on solid ground the maximum mass of a neutron star. Instead of trying to analyze the details of nuclear interactions we follow here a different approach. We take that most extreme equation of state that produces the maximum critical mass compatible solely with these three conditions: (1) standard general-relativity equation of hydrostatic equilibrium, (2) Le Chatelier's principle, and (3) the principle of causality. While no suggestion is made that the resultant equation of state accurately represents the actual physical behavior of matter, it does illustrate a point of principle by yielding a maximum mass for the critical mass. It is not altogether new to approach the equation of state from the side of hydrostatic theory rather than from the side of the structure of matter. Gerlach[10] has shown that from a set of measurements on a sequence of stars at the end point of thermonuclear evolution one can, in principle, work back to deduce the equation of state, without any call on nuclear or elementary particle theory. We present here, first, an extremization technique for the case in which we assume that the equation of state is known everywhere apart from a finite range of pressure and densities

$$\rho_0 \leq \rho \leq \rho_1, \tag{1a}$$

$$p(\rho_0) \leq p \leq p(\rho_1); \tag{1b}$$

in order to avoid a "supraluminous" equation of state (velocity of sound greater then the speed of light, violation of causality; $c=1$ in our units), we demand that

$$dp/d\rho \leq 1. \tag{2}$$

Also, we require that pressure be a monotonically nondecreasing function of the density (Le Chatelier's principle, which implies no spontaneous collapse of matter locally, speed of sound real):

$$dp/d\rho \geq 0. \tag{3}$$

This last condition could, indeed, be relaxed; however, the proof is more straightforward assuming the validity of both conditions (2) and (3). We wish to choose the equation of state between ρ_0 and ρ_1, so as to extremize the mass of the neutron star. We integrate the standard general relativistic equations

$$dm(r)/dr = 4\pi\rho r^2 = H(\rho, r), \tag{4.1}$$

$$\frac{dp}{dr} = -\frac{(\rho+p)}{r(r-2m)}[4\pi r^2 p(r) + m(r)]$$

$$= G(\rho, p, m, r). \tag{4.2}$$

That is, the total mass of the star is given by

$$M = \int_0^R 4\pi\rho(r) r^2\, dr, \tag{5}$$

with R the radius of the star.

This problem may be formulated in terms of the standard calculus of variations with inequality constraints.[11] We give here an alternative formulation in terms of control theory,[12] which yields both necessary and sufficient conditions. We take ρ as the independent variable. Then Eq. (5) can also be written

$$M = M_1(\rho_c, \rho_1) + \int_{\rho_1}^{\rho_0} 4\pi r^2 \rho (dr/d\rho)\, d\rho + M_0(\rho_0, r_0, m_0). \tag{6}$$

Here with M_1 we have indicated the mass contained in the range of densities $\rho_1 \leq \rho \leq \rho_c$, which is clearly a constant, and with M_0 the mass contained in the range of densities $\rho \leq \rho_0$. Clearly M_0 is a function not only of ρ_0 but also of r_0 and m_0, which in turn are functions of the equation of state adopted in the range of densities $\rho_0 \leq \rho \leq \rho_1$.

Letting the primes denote derivatives with respect to ρ, we have

$$u = p', \tag{7.1}$$

$$u/G = r', \tag{7.2}$$

$$Hu/G = m', \tag{7.3}$$

where $u(\rho)$ is given by the known equation of state in the ranges $\rho_0 \geq \rho$ and $\rho \leq \rho_1$ and has to have

$$0 \leq u \leq 1 \text{ in the range } \rho_0 \leq \rho \leq \rho_1. \tag{7.4}$$

Equations (7) replace Eqs. (1)–(5). Here u becomes the so-called control variable.[12,13] Since we seek the maximum of M given by Eq. (6) with the constraints given by Eq. (1)–(3), we can introduce the generalized Hamiltonian[13]

$$H(\rho, p, m, r, y_1, y_2, y_3, u) = \{y_1 + y_2 H/G + y_3/G\}u + [M_0(\rho, r, m)]|_{\rho=\rho_0} + \mu[(p-a_1)]|_{\rho=\rho_0}, \tag{8}$$

where a_1 indicates the value of the pressure at $\rho = \rho_0$ and y_1, y_2, y_3 are a set of Lagrange multipliers which satisfy the Euler-Lagrange equations

$$y_1' = -\left(y_2 \frac{\partial}{\partial p}\frac{H}{G} + y_3 \frac{\partial}{\partial p}\frac{1}{G}\right)u, \tag{9.1}$$

$$y_2' = -\left(y_2 \frac{\partial}{\partial m}\frac{H}{G} + y_3 \frac{\partial}{\partial m}\frac{1}{G}\right)u, \tag{9.2}$$

$$y_3' = -\left(y_2 \frac{\partial}{\partial r}\frac{H}{G} + y_3 \frac{\partial}{\partial r}\frac{1}{G}\right)u. \tag{9.3}$$

Since u has to fulfill the inequality (7.4), we introduce the new variable

$$u = \sin^2\omega. \tag{10}$$

The corresponding Euler-Lagrange equation then gives

$$2\{y_1 + y_2 H/G + y_3/G\}\sin\omega\cos\omega = 0, \tag{11}$$

which allows the solutions $\omega = 0$ and $\omega = \pi/2$ or $u = 0$ and $u = 1$. It is clear that in the extremization technique of Eq. (6) we have to take into account not only the integral in the range of densities ρ_0

$\leq \rho \leq \rho_1$ but also the mass M_1; this contribution is automatically taken into account by the transversality conditions[13]

$$(y_1 + \partial M_0/\partial r)_{\rho=\rho_0} = 0, \tag{12.1}$$

$$(y_2 + \partial M_0/\partial m)_{\rho=\rho_0} = 0, \tag{12.2}$$

$$(y_3 + \partial M_0/\partial p + \mu)_{\rho=\rho_0} = 0, \tag{12.3}$$

which are the boundary conditions to be fulfilled in the integration of the system of Eq. (9). We can then conclude, in complete generality, that the desired extremum has to lie on the boundary of the allowed range of the control variable u, namely on path with $u=0$ and $u=1$ (see Fig. 1). To see which one of the paths maximizes the mass and at which point the "switch" of conditions from $u=0$ to $u=1$ has to be applied, it is necessary to proceed to a direct integration of Eqs. (7) and (9). We can then conclude that the path which maximizes the mass is given by path a in Fig. 1.[14]

If we now turn from this general problem to the special case of establishing an absolute upper limit to the mass of a neutron star, our variational principle applies much more directly and the problem greatly simplifies. The extremization of the Hamiltonian (8) together with the constraints (7), the differential equations (7), (9), and (11), and the transversality conditions simply tell us that the maximum mass is obtained for that equation of state which maximizes at *each* density the velocity of sound of the material. We know from general arguments that at densities below $\rho_0 = 4.6\times10^{14}$ g/cm^3 the equation of state of free degenerate neutrons, neglecting all nuclear interactions, maximizes the velocity of sound of neutron-star material,[3] in the sense that any realistic equation of state has smaller values of the sound velocity.[3] At densities larger than $\rho_0 = 4.6\times10^{14}$ g/cm^3 very little is known about the possible description of the interactions between nucleons at suprenuclear densities. Therefore we assume the equation of state with the highest conceivable velocity of sound, namely the one with velocity of sound equal to the speed of light. By direct integration of the equations of equilibrium for selected values of the central density (see Fig. 2) we can then conclude that no matter what the details of the equation of state at nuclear or supranuclear density, a neutron star can never have a mass larger than $3.2M_\odot$.

Finally, it is important to realize that although our arguments are presented here in the case of neutron stars with zero angular momentum they

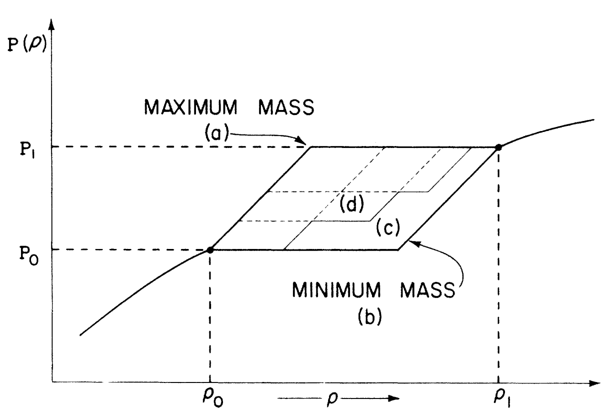

FIG. 1. "Allowed rhomboid" in the $p\,\rho$ plane. The equation of state is known for $\rho<\rho_0$ and $\rho>\rho_1$. In the range $\rho_0\leq\rho\leq\rho_1$ all equations of state compatible with the principle of causality and Le Chatelier's principle have to be contained inside the rhomboid. Path a (b) maximizes (minimizes) the mass of the neutron star; see Ref. 14.

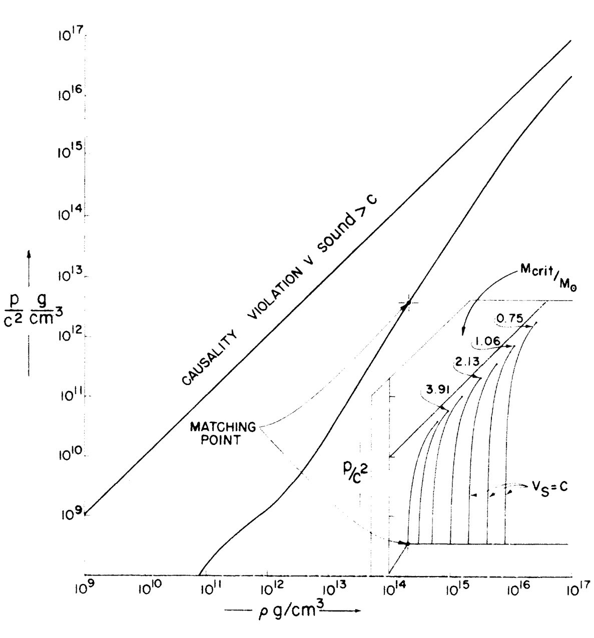

FIG. 2. Matching of the "allowed rhomboid" to the Harrison-Wheeler equation of state (Ref. 15) for a value of the density $\rho_0 = 2\times10^{14}$ g/cm^3 (lower right-hand side of the figure). The different paths followed correspond to an equation of state with $u=1$ (path a in Fig. 1) or to a combination of a path with $u=0$ and then $u=1$ (path b in Fig. 1). The difference in inclination of the lines with $u=1$ between Figs. 1 and 2 is due to the difference in scale: linear in Fig. 1 and logarithmic here. In these computations the integrations are carried out up to the value of critical central density at which the value of the critical mass is reached.

can indeed be applied to the case of pulsars and x-ray sources. The reason is that all the pulsars[16] and the pulsating binary x-ray sources[7] are rotating very slowly ($P \gtrsim 33$ msec), and in this region their masses can be affected by rotation only by a factor $\delta M/M < 0.2\omega^2/(M/R^3)$, where ω is the angular velocity, M the mass, and R the radius of the neutron star. Moreover, rapidly rotating neutron stars ($P \lesssim 0.1$ msec) would have a decay time $\tau = E_{\rm rot}/[-(dE/dt)_{\rm diss}]$ of approximately one week if dissipation is due to dipole magnetic rediation, or $\tau \lesssim 10$ min if dissipation is due to emission of gravitational radiation.[17] Even in this very extreme case ($P \lesssim 0.1$ msec) the critical mass of a neutron star would be changed by a factor smaller than 1.5.[18]

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