GOLDEN OLDIE

Republication of: The dynamics of general relativity

**Richard Arnowitt · Stanley Deser ·
Charles W. Misner**

Abstract   This article—summarizing the authors' then novel formulation of General Relativity—appeared as Chap. 7, pp. 227–264, in *Gravitation*: *an introduction to current research*, L. Witten, ed. (Wiley, New York, 1962), now long out of print. Intentionally unretouched, this republication as Golden Oldie is intended to provide contemporary accessibility to the flavor of the original ideas. Some typographical corrections have been made: footnote and page numbering have changed–but not section nor equation numbering, etc. Current institutional affiliations are encoded in: arnowitt@physics.tamu.edu, deser@brandeis.edu, misner@umd.edu.

R. Arnowitt (✉)
Department of Physics, Texas A&M University, 4242 TAMU,
College Station, TX 77843-4242, USA
e-mail: arnowitt@physics.tamu.edu

S. Deser (✉)
Mailstop 057, Brandeis University, 415 South Street,
Waltham, MA 02454, USA
e-mail: deser@brandeis.edu

Lauritsen Laboratory, California Institute of Technology,
Pasadena, CA 91125, USA

C. W. Misner (✉)
Gravitation Theory Group, Department of Physics, Room 4205D,
University of Maryland, College Park, MD 20742-4111, USA
e-mail: misner@umd.edu

R. Arnowitt,[i] *S. Deser,*[ii] *and C. W. Misner*[iii]

The Dynamics
of General Relativity

1. Introduction[iv]

The general coordinate invariance underlying the theory of relativity creates basic problems in the analysis of the dynamics of the gravitational field. Usually, specification of the field amplitudes and their first time derivatives initially is appropriate to determine the time development of a field viewed as a dynamical entity. For general relativity, however, the metric field $g_{\mu\nu}$ may be modified at any later time simply by carrying out a general coordinate transformation. Such an operation does not involve any observable changes in the physics, since it merely corresponds to a relabeling under which the theory is invariant. Thus it is necessary that the metric field be separated into the parts carrying the true dynamical information and those parts characterizing the coordinate system. In this respect, the general theory is analogous to electromagnetic theory. In particular, the coordinate invariance plays a role similar to the gauge invariance of the Maxwell field. In the latter case, this gauge invariance also produces difficulties in separating out the independent dynamical modes, although the linearity here does simplify the analysis. In both cases, the effect

of invariance properties (both Lorentz and "gauge" invariance) is to introduce redundant variables in the original formulation of the theory to insure that the correct transformation properties are maintained. It is this clash with the smaller number of variables needed to describe the dynamics (*i.e.*, the number of *independent* Cauchy data) that creates the difficulties in the analysis. In Lorentz covariant field theories, general techniques (Schwinger, 1951, 1953) (valid both in the quantum and classical domains) have been developed to enable one to disentangle the dynamical from the gauge variables. We will see here that, while general relativity possesses certain unique aspects not found in other theories, these same methods may be applied. Two important advantages are obtained by proceeding in this fashion. First, the physics of Lorentz covariant field theory is well understood. Consequently, techniques which relate back to this area of knowledge will help one to comprehend better the physics of general relativity. Second, insofar as quantization of the theory is concerned, a formulation closely associated with general quantization techniques which include consistency criteria (*i.e.*, the Schwinger action principle) will be more appropriate for this highly non-linear theory. Direct correspondence principle quantization (which is suitable for linear theories without constraints) may well prove inadequate here.

A precise determination of the independent dynamical modes of the gravitational field is arrived at when the theory has been cast into canonical form and consequently involves the minimal number of variables specifying the state of the system. At this level, one will have all the relevant information about the field's behavior in familiar form. The canonical formalism, involving only the minimal set of variables (which will turn out to be four), is also essential to the quantization program, since it yields directly simple Poisson bracket (P.B.) relations among these conjugate, unconstrained, variables. Two essential aspects of canonical form are: (1) that the field equations are of first order in the time derivatives; and (2) that time has been singled out so that the theory has been recast into 3+1 dimensional form. These two features are characteristic of Hamilton (or P.B.) equations of motion, in contrast to the Lagrange equations. The first requirement may be achieved in general relativity, since its Lagrangian may be written in a form linear in the time derivatives (which is called the Palatini form). The type of variable fulfilling the second requirement is dictated by the desire for canonical form, and will be seen also to possess a natural geometrical interpretation.

The use of the Palatini Lagrangian and of 3+1 dimensional notation does not, of course, impair the general covariance of the theory under arbitrary coordinate transformations. In possessing this covariance, general relativity is precisely analogous to the parameterized form of mechanics in which the Hamiltonian and the time are introduced as a conjugate pair of variables of a new degree of freedom. When in parameterized form, a theory is invariant under an arbitrary re-parameterization, just as general relativity is invariant under an arbitrary change of coordinates. The action of general relativity will thus be seen to be in "already parameterized" form. The well-known relations between the usual canonical form and the parameter description will thus provide the key for deriving the desired canonical form for the gravitational field. We will therefore begin, in Section 2, with a brief review of parameterized particle mechanics. In Section 3, the Lagrangian of general relativity will be cast into Palatini and 3+1 dimensional form, and the geometrical significance of the variables will be discussed. We will see then that relativity has a form identical to parameterized mechanics. Section 4 completes the analysis, to obtain the canonical variables and their relations as well as the P.B. equations of motion.

Once canonical form is reached, the physical interpretation of quantities involved follows directly as in other branches of physics. Thus, the canonical variables themselves represent the independent excitations of the field (and hence provide the basis for defining gravitational radi-

ation in a coordinate-independent way). Further, the numerical value of the Hamiltonian for a particular state of the system provides the primary definition of total energy (a definition which amounts to comparing the asymptotic form of the spatial part of the metric with that of the exterior Schwarzschild solution). Similarly, the total momentum is defined from the generator of *spatial* translations. The energy and momentum are invariant under coordinate transformations not involving Lorentz rotations at spatial infinity, and behave as a four-vector under the latter. It is also possible to set up the analysis of gravitational radiation in a fashion closely analogous to electrodynamics by introducing a suitable definition of the wave zone. In this region, gravitational waves propagate as free radiation, independent of the strong field interior sources. The waves obey ordinary (flat-space) wave equations and consequently satisfy superposition. The Poynting vector may also be defined invariantly in the wave zone. In contrast, the Newtonian-like parts of the metric cannot be determined within the wave zone; they depend strongly on the interior non-linearities. These points are discussed in Section 5.

When the analysis is extended to include coupling of other systems to the gravitational field (Section 6), the above definition of energy may be used to discuss self-energy questions. In this way, the static gravitation and electromagnetic self-masses of point particle systems will be treated rigorously in Section 7. Here the canonical formalism is essential in order that one may recognize a pure particle (no wave) state. The vanishing of the canonical variables guarantees that there are no independent field excitations contributing to the energy. The total clothed mass of a classical electron turns out to be finite, independent of its bare mass and completely determined by its charge. Further, a "neutral" particle (one coupled only to the gravitational field) has a zero clothed mass, showing that the mass of a particle arises entirely from its interactions with other fields. The physical origin of these finite results is discussed at the beginning of Section 7 in terms of equivalence principle considerations. The self-stress $\mathfrak{T}^{ij}$ of the electron vanishes, showing that the particle is stable, its repulsive electrostatic self-forces being precisely cancelled by gravitational attraction without any *ad hoc* compensation being required. Thus, a completely consistent classical point charge exists when gravitation is included. These rigorous results are in contrast to the higher and higher infinities that would arise in a perturbation analysis of the same problem.

Whether gravitational effects will maintain the finiteness of self-energies in quantum theory (and if so, whether the effective cutoff will be appropriate to produce reasonable values) is at present an open question. In the final section (8), some speculative remarks are made on this problem. Since a complete set of P.B.'s has been obtained classically in Section 4, it is formally possible to quantize by the usual prescription of relating them to quantum commutators. However, the non-linear nature of the theory may necessitate a more subtle transition to the quantum domain. Section 8 also discusses some of these questions.

2. Classical Dynamics Background

2.1. *Action principle for Hamilton's equations.* As was mentioned in Section 1, general relativity is a theory in "already parameterized form." We begin, therefore, with a brief analysis of the relevant properties of the parameter formalism (see also Lanczos, 1949). For simplicity, we deal with a system of a finite number M of degrees of freedom. Its action may be written as

$$I = \int_{t_2}^{t_1} dt\, L = \int_{t_2}^{t_1} dt\, \left(\sum_{i=1}^{M} p_i \dot{q}_i - H(p, q) \right) \tag{2.1}$$

where $\dot{q} \equiv dq/dt$ and the Lagrangian has been expressed in a form linear in the time derivatives.

(This will be referred to as the first-order form since independent variation of p_i and q_i gives rise to the first-order equations of motion.) The maximal information obtainable from the action arises when not only p_i and q_i are varied independently, but t is also varied and endpoint variations are allowed. Postulating that the total δI is a function only of the endpoints $[\delta I = G(t_1) - G(t_2)]$ leads to: (1) the usual Hamilton equations of motion for p_i and q_i; (2) conservation of energy $(dH/dt = 0)$; and (3) the generating function

$$G(t) = \sum_i p_i \, \delta q_i - H \, \delta t \tag{2.2}$$

Here $\delta q_i = \delta_0 q_i + \dot{q}_i \, \delta t$ where $\delta_0 q_i$ denotes the independent ("intrinsic") variation of q_i. The generating function can easily be seen to be the conventional generator of canonical transformations. Thus $G_q = \sum_i p_i \, \delta q_i$ generates changes $q_i \to q_i + \delta q_i$, $p_i \to p_i$ while $G_t = -H \, \delta t$ generates the translation in time. That is, for G_q one has $[q_j, G_q] = \sum_i [q_j, p_i] \delta q_i = \delta q_j$, where $[A, B]$ means the Poisson bracket (P.B.), and for G_t one has $[q_i, G_t] = -[q_i, H] \delta t = -\dot{q}_i \, \delta t$ by the P.B. form of the equations of motion. The above elementary discussion may be inverted to show that, for the action of (2.1), if every variable occurring in H is also found in the $p\dot{q}$ term, then the theory is in canonical form and p_i and q_i obey the conventional P.B. relations. This is the classical equivalent of the Schwinger action principle (Schwinger, 1951, 1953).

2.2. *The action in parameterized form.* The motion of the system (2.1) is described in terms of one independent variable t (the "coordinate"). The action may be cast, as is well known, into parameterized form, in which the time is regarded as a function q_{M+1} of an arbitrary parameter τ:

$$I = \int_{\tau_2}^{\tau_1} d\tau \, L_\tau \equiv \int_{\tau_2}^{\tau_1} d\tau \left[\sum_{i=1}^{M+1} p_i q_i' \right] .$$

Here, $q' \equiv dq/d\tau$, and the constraint equation $p_{M+1} + H(p, q) = 0$ holds. One may equally well replace this constraint by an additional term in the action:

$$I = \int_{\tau_2}^{\tau_1} d\tau \left[\sum_{i=1}^{M+1} p_i q_i' - N R \right] \tag{2.3}$$

where $N(\tau)$ is a Lagrange multiplier. Its variation yields the constraint equation $R(p_{M+1}, p, q) = 0$, which may be any equation with the solution (occurring as a simple root) $p_{M+1} = -H$. The theory as cast into form (2.3) is now generally covariant with respect to arbitrary coordinate transformations $\bar{\tau} = \bar{\tau}(\tau)$, bearing in mind that N transforms as $dq/d\tau$. The price of achieving this general covariance has been not only the introduction of the $(M+1)$st degree of freedom, but, more important, the loss of canonical form, due to the appearance of the Lagrange multiplier N in the "Hamiltonian," $H' \equiv N R$. (N occurs in H' but not in $\sum_{i=1}^{M+1} p_i q_i'$.) A further striking feature which is due to the general covariance of this formulation is that the "Hamiltonian" H' vanishes by virtue of the constraint equation. This is not surprising, since the motion of any particular variable $F(p, q)$ with respect to τ is arbitrary, *i.e.*, F' may be given any value by suitable recalibration $\tau \to \bar{\tau}$.

2.3. *Reduction of parameterized action to Hamiltonian form—intrinsic coordinates.* As we shall see, the Lagrangian of general relativity may be written in precisely the form of (2.3). We will, therefore, be faced with the problem of reducing an action of the type (2.3) to canonical form

(2.1). The general procedure consists essentially in reversing the steps that led to (2.3). If one simply inserts the solution, $p_{M+1} = -H$, of the constraint equation into (2.3), one obtains

$$I = \int d\tau \left[\sum_{i=1}^{M} p_i q_i' - H(p, q) q_{M+1}' \right] .$$ (2.4)

All reference to the arbitrary parameter τ disappears when I is rewritten as

$$I = \int dq_{M+1} \left[\sum_{i=1}^{M} p_i (dq_i/dq_{M+1}) - H \right]$$ (2.5)

which is identical to (2.1)with the notational change $q_{M+1} \to t$. Equation (2.5) exhibits the role of the variable q_{M+1} as an "intrinsic coordinate." By this is meant the following. The equation of motion for q_{M+1} is $q_{M+1}' = N(\partial R/\partial q_{M+1})$ from (2.3). Also, none of the dynamical equations determine N as a function of τ. Thus N and hence q_{M+1}, are left arbitrary by the dynamics (though, of course, a choice of q_{M+1} as a function of τ fixes N). One is therefore free to choose $q_{M+1}(\tau)$ to be any desired function and use this function as the new independent variable (parameter): $q_i = q_i(q_{M+1})$, $p_i = p_i(q_{M+1})$, $i = I \dots M$. The action of (2.5), and hence the relations between q_i, p_i, and q_{M+1} are now independent of τ. They are manifestly invariant under the general "coordinate transformation" $\bar{\tau} = \bar{\tau}(\tau)$ (for the simple reason that τ itself no longer appears). The choice of q_{M+1} as the independent variable thus yields a manifestly τ-invariant formulation and gives an "intrinsic" specification of the dynamics. This is in contrast to the original one in which the trajectories of $q_1 \dots q_{M+1}$ are given in terms of some arbitrary variable τ (which is extraneous to the system).

In practice, we shall arrive at the intrinsic form (2.5) from (2.4) in an alternate way. Since the relation between q_{M+1} and τ is undetermined, we are free to specify it explicitly, *i.e.*, impose a "coordinate condition." If, in particular, this relation is chosen to be $q_{M+1} = \tau$ (a condition which also determines N), the action (2.4) then reduces (2.5) with the notational change $q_{M+1} \to \tau$; the non-vanishing Hamiltonian only arises as a result of this process. [Of course, other coordinate conditions might have been chosen. These would correspond to using a variable other than q_{M+1} as the intrinsic coordinate in the previous discussion.]

This simple analysis has shown that the way to reduce a parameterized action to canonical form is to insert the solution of the constraint equations and to impose coordinate conditions. Further, the imposition of coordinate conditions is equivalent to the introduction of intrinsic coordinates.

In field theory it will prove more informative to carry out this analysis in the generator. We exhibit here the procedure in the particle case: The generator associated with the action of (2.3) is

$$G = \sum_{i=1}^{M+1} p_i \, \delta q_i - N R \, \delta \tau$$ (2.6)

Upon inserting constraints, the generator reduces to

$$G = \sum_{i=1}^{M} p_i \, \delta q_i - H \, \delta q_{M+1} .$$ (2.7)

Imposing the coordinate condition $q_{M+1} = t$ then yields (2.2) From *this* form, one can immediately recognize the M pairs of canonical variables and the non-vanishing Hamiltonian of the theory.

One can, of course, perform the above analysis for a parameterized field theory as well. Here the coordinates appear as four new field variables $q^{M+\mu} = x^\mu(\tau^\alpha)$, and there are four extra momenta $p_{M+\mu}(\tau^\alpha)$ conjugate to them. Four constraint equations are required to relate these momenta to the Hamiltonian density and the field momentum density, and correspondingly, there are four Lagrange multipliers $N_\mu(\tau^\alpha)$ for a field. An example in which the scalar meson field is parameterized may be found in III.

3. First-Order Form of the Gravitational Field

3.1. *The Einstein action in first-order (Palatini) form.* The usual action integral for general relativity[1]

$$I = \int d^4x \, \mathcal{L} = \int d^4x \, \sqrt{-g} \, R \tag{3.1}$$

yields the Einstein field equations when one considers variations in the metric (*e.g.*, $g_{\mu\nu}$ or the density $\mathfrak{g}^{\mu\nu} = \sqrt{-g} \, g^{\mu\nu}$). These Lagrange equations of motion are then second-order differential equations. It is our aim to obtain a canonical form for these equations, that is, to put them in the form $\dot{q} = \partial H/\partial p$, $\dot{p} = -\partial H/\partial q$. As a preliminary step, we will restate the Lagrangian so that the equations of motion have two of the properties of canonical equations: (1) they are first-order equations; and (2) they are solved explicitly for the time derivatives. The second property will be obtained by a $3 + 1$ dimensional breakup of the original four-dimensional quantities, as will be discussed below. The first property is insured by using a Lagrangian linear in first derivatives. In relativity, this is called the Palatini Lagrangian, and consists in regarding the Christoffel symbols $\Gamma_\mu{}^\alpha{}_\nu$ as independent quantities in the variational principle (see, for example, Schrödinger, 1950). Thus, one may rewrite (3.1) as

$$I = \int d^4x \, \mathfrak{g}^{\mu\nu} R_{\mu\nu}(\Gamma) \tag{3.2}$$

where

$$R_{\mu\nu}(\Gamma) \equiv \Gamma_\mu{}^\alpha{}_{\nu,\alpha} - \Gamma_\mu{}^\alpha{}_{\alpha,\nu} + \Gamma_\mu{}^\alpha{}_\nu \Gamma_\alpha{}^\beta{}_\beta - \Gamma_\mu{}^\alpha{}_\beta \Gamma_\nu{}^\beta{}_\alpha \, . \tag{3.3}$$

Note that these covariant components $R_{\mu\nu}$ of the Ricci tensor do not involve the metric but only the affinity $\Gamma_\mu{}^\alpha{}_\nu$. Thus, by varying $g^{\mu\nu}$, one obtains directly the Einstein field equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2} \, g_{\mu\nu} \, R = 0 \, . \tag{3.4a}$$

These equations no longer express the full content of the theory, since the relation between the now independent quantities $\Gamma_\mu{}^\alpha{}_\nu$ and $g_{\mu\nu}$ is still required. This is obtained as a field equation by varying $\Gamma_\mu{}^\alpha{}_\nu$. One then finds

$$\mathfrak{g}^{\mu\nu}{}_{;\alpha} \equiv \mathfrak{g}^{\mu\nu}{}_{,\alpha} + \mathfrak{g}^{\mu\beta}\Gamma_\beta{}^\nu{}_\alpha + \mathfrak{g}^{\nu\beta}\Gamma_\beta{}^\mu{}_\alpha - \mathfrak{g}^{\mu\nu}\Gamma_\beta{}^\beta{}_\alpha = 0 \tag{3.4b}$$

which, as is well known, can be solved for $\Gamma_\mu{}^\alpha{}_\nu$ to give the usual relation $\Gamma_\mu{}^\alpha{}_\nu = \{^{\ \alpha}_{\mu\ \nu}\} \equiv \tfrac{1}{2} g^{\alpha\beta}(g_{\mu\beta,\nu} + g_{\nu\beta,\mu} - g_{\mu\nu,\beta})$.

The Palatini formulation of general relativity has a direct analog in Maxwell theory, the affinity corresponding to the field strength $F_{\mu\nu}$ and the metric to the vector potential A_μ. Here the Lagrangian is

$$\mathcal{L} = A_{\mu,\nu}F^{\mu\nu} + \tfrac{1}{4} \, F_{\mu\nu}F^{\mu\nu} \tag{3.5}$$

---

[1] We use units such that $16\pi\gamma c^{-4} = 1 = c$, where γ is the Newtonian gravitational constant; electric charge is in rationalized units. Latin indices run from 1 to 3, Greek from 0 to 3, and $x^0 = t$. Derivatives are denoted by a comma or the symbol ∂_μ.

with A_μ and $F_{\mu\nu}$ to be independently varied. The field equations then become

$$F^{\mu\nu}{}_{,\nu} = 0 \tag{3.6a}$$

and

$$A_{\nu,\mu} - A_{\mu,\nu} = F_{\mu\nu} \tag{3.6b}$$

which correspond to (3.4a) and (3.4b).

The next step in achieving canonical form is to single out time derivatives by introducing three-dimensional notation. For the Maxwell field, we thus define

$$E^i \equiv F^{0i} . \tag{3.7a}$$

Since the canonical form requires equations of first order in time derivatives (but not in space derivatives), we may use the equation $F_{ij} = A_{j,i} - A_{i,j}$ to eliminate F_{ij}. In terms of the abbreviation

$$B^i \equiv \tfrac{1}{2} \epsilon^{ijk}(A_{k,j} - A_{j,k}) \tag{3.7b}$$

the Lagrangian reads

$$\mathcal{L} = -E^i \partial_0 A_i - \tfrac{1}{2}\left(B^i B^i + E^i E^i\right) - A_0 E^i{}_{,i} . \tag{3.8}$$

At this stage, the Maxwell equations are obtained by varying $\mathcal{L}$ with respect to the independent quantities E^i, A_i, and A_0.

3-2. *Three-plus-one dimensional decomposition of the Einstein field.* The three-dimensional quantities appropriate for the Einstein field are (as will be discussed in detail later)

$$g_{ij} \equiv {}^4g_{ij} , \qquad N \equiv (-{}^4g^{00})^{-1/2} , \qquad N_i \equiv {}^4g_{0i} \tag{3.9a}$$

$$\pi^{ij} \equiv \sqrt{-{}^4g} \left({}^4\Gamma_p{}^0{}_q - g_{pq}\,{}^4\Gamma_r{}^0{}_s g^{rs}\right)g^{ip}g^{jq} . \tag{3.9b}$$

Here and subsequently we mark every four-dimensional quantity with the prefix ^{4}, so that all unmarked quantities are understood as three-dimensional. In particular, g^{ij} in (3.9b) is the reciprocal matrix to g_{ij} . The full metric ${}^4g_{\mu\nu}$ and ${}^4g^{\mu\nu}$ may, with (3.9a), be written

$$^4g_{00} = -(N^2 - N_i N^i) \tag{3.10}$$

where $N^i = g^{ij}N_j$, and

$$^4g^{0i} = N^i/N^2 , \qquad {}^4g^{00} = -1/N^2 \tag{3.11a}$$

$$^4g^{ij} = g^{ij} - (N^i N^j/N^2) . \tag{3.11b}$$

One further useful relation is

$$\sqrt{-{}^4g} = N\sqrt{g} . \tag{3.12}$$

In terms of the basic quantities of (3.9), the Lagrangian of general relativity becomes

$$\mathcal{L} = \sqrt{-{}^4g}\,{}^4R = -g_{ij}\,\partial_t \pi^{ij} - NR^0 - N_i R^i$$
$$-2(\pi^{ij}N_j - \tfrac{1}{2}\pi N^i + N^{|i}\sqrt{g})_{,i} \tag{3.13}$$

where

$$R^0 \equiv -\sqrt{g}\,[{}^3R + g^{-1}(\tfrac{1}{2}\,\pi^2 - \pi^{ij}\pi_{ij})] \tag{3.14a}$$

$$R^i \equiv -2\pi^{ij}{}_{|j} \, . \tag{3.14b}$$

The quantity ^3R is the curvature scalar formed from the spatial metric g_{ij}, $_|$ indicates the covariant derivative using this metric, and spatial indices are raised and lowered using g^{ij} and g_{ij}. (Similarly, $\pi \equiv \pi^i{}_i$.) As in the electromagnetic example, we have allowed second-order space derivatives to appear by eliminating such quantities as $\Gamma_i{}^k{}_j$ in terms of $g_{ij,k}$.

One may verify directly that the first-order Lagrangian (3.13) correctly gives rise to the Einstein equations. One obtains

$$\partial_t g_{ij} = 2N g^{-1/2}(\pi_{ij} - \tfrac{1}{2}\,g_{ij}\pi) + N_{i|j} + N_{j|i} \tag{3.15a}$$

$$\begin{aligned}
\partial_t \pi^{ij} = & - N\sqrt{g}\,(^3R_{ij} - \tfrac{1}{2}\,g^{ij}\,{}^3R) + \tfrac{1}{2}\,N g^{-1/2} g^{ij}(\pi^{mn}\pi_{mn} - \tfrac{1}{2}\pi^2) \\
& - 2N g^{-1/2}(\pi^{im}\pi_m{}^j - \tfrac{1}{2}\pi\pi^{ij}) + \sqrt{g}\,(N^{|ij} - g^{ij}\,N^{|m}{}_{|m}) \\
& + (\pi^{ij}N^m)_{|m} - N^i{}_{|m}\pi^{mj} - N^j{}_{|m}\pi^{mi}
\end{aligned} \tag{3.15b}$$

$$R^\mu(g_{ij}, \pi^{ij}) = 0 \, . \tag{3.15c}$$

Equation (3.15a), which results from varying π^{ij}, would be viewed as the defining equation for π^{ij} in a second-order formalism. Variation of N and N_i yields equations (3.15c), which are the $^4G^0{}_\mu \equiv {}^4R^0{}_\mu - \tfrac{1}{2}\delta^0_\mu\,{}^4R = 0$ equations, while equations (3.15b) are linear combinations of these equations and the remaining six Einstein equations ($^4G_{ij} = 0$).

3.3. *Geometrical interpretation of dynamical variables.* Before proceeding with the reduction to canonical form, it is enlightening to examine, from a geometrical point of view, our specific choices (3.9) of three-dimensional variables. Geometrically, their form is governed by the requirement that the basic variables be three-covariant under all coordinate transformations which leave the t=const surfaces unchanged. Any quantities which have this property can be defined entirely within the surface (this is clearly appropriate for the 3+1 dimensional breakup). One fundamental four-dimensional object which is clearly also three-dimensional is a curve $x^\mu(\lambda)$ which lies entirely within the 3-surface, *i.e.*, $x^0(\lambda) = $ const. The vector $v^\mu \equiv dx^\mu/d\lambda$ tangent to this curve is therefore also three-dimensional. The restriction that the curve lie in the surface t=const is then $v^0 = 0$, and conversely *any* vector V^μ, with $V^0 = 0$ is tangent to some curve in the surface. Three such independent vectors are $V^\mu_{(i)} = \delta^\mu_i$. Given any covariant tensor $A_{\mu...\nu}$, its projection onto the surface is then $V^\mu_{(i)} \ldots V^\nu_{(j)} A_{\mu...\nu} = A_{i...j}$. Thus, the *covariant* spatial components of any four-tensor form a three-tensor which depends only on the surface[2] (in contrast to the contravariant spatial components which are scalar products with gradients rather than tangents, and hence depend also on the choice of spatial coordinates in the immediate neighborhood of the surface). This accounts for the choice of g_{ij}, rather than $^4g^{ij}$. In contrast, N and N_i do not have the desired invariance and, in fact, by choosing coordinates such that the $x^i = $ const lines are normal to the surface, one obtains $N_i = 0$. (If x^0 is arranged to measure proper time along these lines, one has also $N = 1$.) By the same argument, one can see that A_i and F_{ij} are appropriate three-dimensional quantities in a general relativistic discussion of the Maxwell field.

The quantity which plays the role of a momentum is more difficult to define within the surface, since it refers to motion in time leading out of the original $t = $ const surface. Such a quantity

---

[2]Of course, any three-dimensional operations (such as contraction of indices with the three-metric) on a three-tensor yield quantities defined on the three-surface.

is, however, provided by the second fundamental form K_{ij} (see, for example, Eisenhart, 1949), which gives the radii of curvature of the $t = \text{const}$ surface as measured in the surrounding four-space. These "extrinsic curvatures" describe how the normals to the surface converge or diverge, and hence determine the geometry of a parallel surface at an infinitesimally later time. Since K_{ij} describes a geometrical property of the $t = \text{const}$ surface, as imbedded in four-space, it again does not depend on the choice of coordinates away from the surface. This may also be seen from a standard definition, $K_{ij} = -n_{(i;j)}$, which expresses K_{ij} as the covariant spatial part of the tensor $n_{(\mu;\nu)}$ (the four-dimensional covariant derivative of the unit normal, $n_\mu = -N\delta_\mu^0$ to the surface).[3] For convenience in ultimately reaching canonical form, we have chosen, instead of K_{ij}, the closely related variable $\pi^{ij} = -\sqrt{g}\left(K^{ij} - g^{ij}K\right)$. Thus, the geometrical analysis defines g_{ij} and g_{ij} as suitable quantities, unaffected by the choice of coordinates later in time, while N and N_i describe how the coordinate system will be continued off the $t = \text{const}$ surface.

3.4. *Initial value problem and dynamical structure of field equations.* Returning to the field equations (3.15), we may now analyze them from the point of view of the initial value problem. If one specifies g_{ij}, π^{ij} and N, N_i initially, it is clear that the equations uniquely determine g_{ij} and π^{ij} at a later time, while N and N_i remain undetermined then. Since the latter merely express the continuation of the coordinates, the intrinsic (coordinate-independent) geometry of space-time is determined uniquely by an initial choice of g_{ij} and π^{ij}. This choice is restricted, however, by the four constraint equations (3.15c) which relate these twelve variables at the initial time. Subject to these compatibility conditions, then, the (g_{ij}, π^{ij}) constitute a complete set of Cauchy data for the theory.

The maintenance in time of these constraints is guaranteed by the Bianchi identities $({}^4G_\mu{}^\nu{}_{;\nu} \equiv 0)$. Hence

$$ {}^4G_\mu{}^0{}_{,0} = -\,{}^4G_\mu{}^i{}_{;i} - {}^4G_\mu{}^\nu\,{}^4\Gamma_\nu{}^0{}_0 + {}^4G_\nu{}^0\,{}^4\Gamma_\mu{}^\nu{}_0\,. \tag{3.16} $$

Thus, by virtue of the dynamic equations ${}^4G_{ij} = 0$ (and consequently of their spatial derivatives) at $t = 0$, the constraints ${}^4G_\mu{}^0 = 0$ hold at all times if they hold initially.

In electrodynamics, the constraint equation corresponding to (3.15c) is obtained by varying A_0 in (3.8), and is $F^{0i}{}_{,i} = \boldsymbol{\nabla} \cdot \boldsymbol{E} = 0$. The identity ensuring its maintenance in time is just $F^{\mu\nu}{}_{,\mu\nu} \equiv 0$, which may be rewritten as $(F^{0i}{}_{,i})_{,0} = -(F^{i\nu}{}_{,\nu})_{,i}$. The right-hand side then vanishes by the dynamic equations $F^{i\nu}{}_{,\nu} = 0$.

While the twelve variables (g_{ij}, π^{ij}) constitute a complete set of Cauchy data, they do not provide a minimal set (which the canonical formalism will eventually give to be two pairs, corresponding to two degrees of freedom). We may now count the number of minimal variables. Of the twelve g_{ij}, π^{ij}, we may eliminate four by using the constraint equations (3.15c). There will correspondingly be four "Bianchi" identities among the twelve equations of motion (3.15a) and (3.15b). As we have seen, N and N_i determine the continuation of the coordinate system without affecting the intrinsic geometry (*i.e.*, the physics of the field). For every choice of N and N_i as functions of the remaining eight Cauchy data (which represents a choice of coordinate frame), there will result four equations stating that the time derivatives of four of the remaining eight (g, π) variables vanish. [More precisely, a choice of coordinate frame is made by specifying four functions q^μ of (g_{ij}, π^{ij}) as the coordinates x^μ. The equations for $\partial_t q^\mu$ $(= \delta_0^\mu)$ then determine N and N_i.] Thus, after these coordinate conditions are imposed, we are left with four dynamic equations of the

---

[3] Thus one has $K_{ij} = -n_{(i;j)} = -n_{(i,j)} + n_\mu\Gamma_i{}^\mu{}_j = N\Gamma_i{}^0{}_j$.

form $\partial_t u_a = f_a(u)$ $(a = 1, 2, 3, 4)$. These equations govern the motion of a system of two degrees of freedom. This is to be expected, since the linearized gravitational field is a massless spin two field, and the self-interaction of the full theory should not alter such kinematical features as the number of degrees of freedom.

3.5. *The gravitational field as an already parameterized system.* To conclude this section, we point out the characteristic properties of the Einstein Lagrangian of (3.13). We reproduce it here with a divergence[4] and a total time derivative[5] discarded:

$$\mathcal{L} = \pi^{ij} \, \partial_t g_{ij} - N R^0 - N_i R^i \,, \qquad R^\mu = R^\mu(g_{ij}, \, \pi^{ij}) \,. \tag{3.17}$$

Equation (3.17) is thus precisely in the form of a parameterized theory's Lagrangian as in (2.3). This form just expresses the invariance of the theory with respect to transformations of the four coordinates x^μ and hence the x^μ are parameters in exactly the same sense that τ was in the particle case. That the N and N_i are truly Lagrange multipliers follows from the fact that they do not appear in the pq' (*i.e.*, $\pi^{ij} \, \partial_t g_{ij}$) part of $\mathcal{L}$. Their variation yields the four constraint equations R^μ, $= 0$. The "Hamiltonian" $\mathcal{H}' \equiv N R^0 + N_i R^i$ vanishes due to the constraints. The true non-vanishing Hamiltonian of the theory will arise only after the constraint variables have been eliminated and coordinate conditions chosen. The analysis leading to the canonical form is carried out in the next section.

4. Canonical Form for General Relativity

4.1. *Analysis of generating functions.* We are now in a position to cast the general theory into canonical form. The geometrical considerations of Section 3 were useful in obtaining the Lagrangian in the form (3.17), which, in the light of Section 2, we recognized as the Lagrangian of a parameterized field theory corresponding to (2.3).

The reduction of (3.17) to the canonical form analogous to (2.1) requires an identification of the four extra momenta to be eliminated by the constraint equations (3.15c). To this end we consider the generator arising from (3.17):

$$G = \int d^3 x \, [\pi^{ij} \, \delta g_{ij} + T^0{}_\mu{}' \, \delta x^\mu] \,. \tag{4.1}$$

The $T^0{}_\mu{}' \, \delta x^\mu$ term comes from the independent coordinate variations. However, $T^0{}_\mu{}'$ vanishes as a consequence of the constraint equations.[6] For example, $T^0{}_0{}' = -N R^0 - N_i R^i = 0$. When the constraints are inserted, G reduces to

$$G = \int d^3 x \, \pi^{ij} \, \delta g_{ij} \tag{4.2}$$

[corresponding to (2.7)] where four of the twelve (g_{ij}, π^{ij}) are understood to have been expressed in terms of the rest by solving $R^\mu = 0$ for them. This elimination exhausts the content of the

---

[4]A divergence in a theory with constraints cannot necessarily be discarded, since such a term may cease to be a divergence upon elimination of constraints. In IIIA, it is shown that the divergences neglected in this work are indeed to be discarded.

[5]The addition of a total time derivative in a Lagrangian does not, of course, alter the equations of motion and corresponds to a canonical transformation in the generator. The properties of canonical transformations in general relativity are discussed in IVA.

[6]More precisely $T^0{}_i{}'$ reduces to an irrelevant divergence.

constraint equations. Finally, as in the particle case, coordinate conditions (now four in number) must be chosen and this information inserted into (4.2), leaving one now with only four dynamical variables (π^A, ϕ_A). If, in fact, the generator at this stage has the form

$$G = \int d^3x \left[\sum_{A=1}^{2} \pi^A \, \delta\phi_A + \mathfrak{T}^0_{\ \mu}(\pi^A, \, \phi_A)\delta x^\mu \right] \tag{4.3}$$

then the theory is clearly in *canonical* form with π^A and ϕ_A as canonical variables and $\int \mathfrak{T}^0_{\ \mu} \, \delta x^\mu$ the generator of translations δx^μ. In (4.3), $\mathfrak{T}^0_{\ \mu}[\pi^A, \, \phi_A]$ has arisen in the elimination of the extra momenta $p_{M+\mu}$, by solving the constraint equations, and the x^μ, now represent the four variables chosen as coordinates $q_{M+\mu}$.

For the Maxwell field, the generator corresponding to (4.2) is, by (3.8),

$$G = \int d^3x[-E^i \, \delta A_i - \tfrac{1}{2}(E^i E^i + B^i B^i)\delta t + (\mathbf{E}\times\mathbf{B}) \cdot \delta\mathbf{r}]$$

with the constraint $E^i_{\ ,i} = 0$ understood to have been eliminated. The solution of the constraint equation is, of course, well known in terms of the orthogonal decomposition $E^i = E^{iT} + E^{iL}$ ($\nabla\cdot\mathbf{E}^T \equiv 0 \equiv \nabla\times\mathbf{E}^L$). The longitudinal component E^{iL} of the electric field vanishes, and therefore the canonical form is reached when one inserts this information into G:

$$G = \int d^3x[-E^{iT} \, \delta A_i{}^T - \tfrac{1}{2}(E^{iT} E^{iT} + B^i B^i)\delta_t + (\mathbf{E}^T\times\mathbf{B}) \cdot \delta\mathbf{r}] \ .$$

Note that $A_i{}^L$ has automatically disappeared from the kinetic term (by orthogonality), and never existed in $\mathbf{B} \equiv \nabla\times\mathbf{A} = \nabla\times\mathbf{A}^T$. Thus, the two canonically conjugate pairs of Maxwell field variables are $-E^{iT}$ and $A_i{}^T$.

4.2. *Analysis of constraint equations in linearized theory-orthogonal decomposition of the metric.* In order to achieve the form (4.3) for relativity, it is useful to be guided by the linearized theory. Here one must treat the constraint equations to second order since our general formalism shows that the Hamiltonian arises from them. Through quadratic terms, equations (3.15c) may be written in the form

$$g_{ij,ij} - g_{ii,jj} = \mathcal{P}_2^0[g_{ij}, \pi^{ij}] \tag{4.4a}$$

$$-2\pi^{ij}_{\ ,j} = \ \mathcal{P}_2^i[g_{ij}, \pi^{ij}] \tag{4.4b}$$

where $\mathcal{P}_2^0$ and $\mathcal{P}_2^i$ are purely quadratic functions of g_{ij} and π^{ij}. These equations determine one component of g_{ij} and three components of π^{ij} in terms of the rest. The content of equations (4.4) can be seen more easily if one makes the following linear orthogonal decomposition on g_{ij} and π^{ij}. For any symmetric array $f_{ij} = f_{ji}$ one has

$$f_{ij} = f_{ij}{}^{TT} + f_{ij}{}^{T} + (f_{i,j} + f_{j,i}) \tag{4.5}$$

where each of the quantities on the right-hand side can be expressed uniquely as a linear functional of f_{ij}. The quantities $f_{ij}{}^{TT}$ are the two transverse traceless components of $f_{ij}(f_{ij}{}^{TT}_{\ ,j} \equiv 0, \ f_{ii}{}^{TT} \equiv 0)$. The trace of the transverse part of f_{ij}, *i.e.*, f^T, uniquely defines $f_{ij}{}^T$ according to

$$f_{ij}{}^T \equiv \tfrac{1}{2}[\delta_{ij}f^T - (1/\nabla^2)f^T_{\ ,ij}] \tag{4.6}$$

(and clearly, $f_{ij}{}^T{}_{,j} = 0$, $f_{ii}{}^T = f^T$. The operator $1/\nabla^2$ is the inverse of the flat space Laplacian, with appropriate boundary conditions. The longitudinal parts of f_{ij} reside in the remaining part, $f_{i,j} + f_{j,i}$. Decomposing f_i into its transverse and longitudinal (curl-less) parts, one has $f_i = f_i{}^T + \frac{1}{2} f^L{}_{,i}$ ($f_i{}^T{}_{,i} \equiv 0$). The remainder then becomes $f_i{}^T{}_{,j} + f_j{}^T{}_{,i} + f^L{}_{,ij}$. One may express f_i, f^T and $f_{ij}{}^{TT}$ in terms of f_{ij} by

$$f_i = (1/\nabla^2)[f_{ij,j} - \tfrac{1}{2}(1/\nabla^2)f_{kj,kji}] \tag{4.7a}$$

$$f^T = f_{ii} - (1/\nabla^2)f_{ij,ij} \tag{4.7b}$$

$$f_{ij}{}^{TT} = f_{ij} - f_{ij}{}^T[f_{mn}] - \{f_{i,j}[f_{mn}] + f_{j,i}[f_{mn}]\}. \tag{4.7c}$$

The foregoing orthogonal decomposition for a symmetric tensor is just the extension of the usual decomposition of a vector into longitudinal and transverse parts employed in electromagnetic theory.

Returning to (4.4), one has

$$-\nabla^2 g^T = \mathcal{P}_2{}^0 \tag{4.8a}$$

$$-2\nabla^2(\pi^{iT} + \pi^L{}_{,i}) = \mathcal{P}_2{}^i . \tag{4.8b}$$

To first order, one sees that g^T and π^i vanish.[7] These structures begin, therefore, at second order where $g^T = -(1/\nabla^2)\mathcal{H}_{\text{lin}}$ and $-2(\pi^{iT}+\pi^L{}_{,i}) = (1/\nabla^2)\mathfrak{T}_{\text{lin}}^{0i}$. Here $\mathcal{H}_{\text{lin}}$, and $\mathfrak{T}_{\text{lin}}^{0i}$ are obtained from $\mathcal{P}_2^0$ and $\mathcal{P}_2^i$ by setting g^T and π^i equal to zero there. They are just the linearized theory's Hamiltonian and field momentum densities.[8] The analysis has thus shown that the constraint equations can be solved for g^T and π^i (in terms of the remaining variables) as the four extra momenta. To see explicitly that $\mathcal{H}_{\text{lin}}$, and $\mathfrak{T}_{\text{lin}}^{0i}$ generate the appropriate time and space translations, one must return to the generator. Inserting the orthogonal decomposition (4.5) for both g_{ij} and π^{ij} into (4.2), one obtains

$$G = \int d^3x\, [\pi^{ijTT}\, \delta g_{ij}{}^{TT} + \pi^{ijT}\, \delta g_{ij}{}^T + 2(\pi^i{}_{,j} + \pi^j{}_{,i})\delta g_{i,j}] . \tag{4.9}$$

The cross-terms in (4.9) have vanished due to the orthogonality of the decomposition (e.g., $\int d^3x\, \pi^{ijTT}\delta g_{i,j} = -\int d^3x\, \pi^{ijTT}{}_{,j}\, \delta g_i = 0$). We have also used here the fact that taking the variation of a quantity does not alter its transverse or longitudinal character in such a linear breakup and that the derivatives commute with the variation. Equation (4.9) may be brought to the desired form by further integration by parts and addition of a total variation:[9]

$$G = \int d^3x\, \{\pi^{ijTT}\, \delta g_i{}^{TT} - (-\nabla^2 g^T)\delta[-(1/2\nabla^2)\pi^T] + [-2\nabla^2(\pi^{iT} + \pi^L{}_{,i})]\}\delta g_i \tag{4.10}$$

4.3. *Imposition of coordinate conditions.* Equation (4.10) is now in the form of (2.7). The final step in reduction to canonical form is to impose coordinate conditions. The structure of (4.10)suggests that one choose as coordinate conditions

$$t = -(1/2\nabla^2)\pi^T \tag{4.11a}$$

---

[7]We use boundary conditions such that g^T and π^i vanish asymptotically. Note also that π^{iT} and π^L are independently determined by (4.8b).

[8]More precisely, $\mathcal{H}_{\text{lin}}$ differs from the linearized Hamiltonian density by a divergence which vanishes when coordinate conditions (4.11) are imposed.

[9]Addition of a total variation to a generator corresponds to the addition of a total time derivative in the Lagrangian, as stated earlier.

$$x^i = g_i . \tag{4.11b}$$

Alternately, these coordinate conditions can be written in more conventional form[10] by eliminating π^T and g_i via (4.7):

$$\pi^{ii}{}_{,jj} - \pi^{ij}{}_{,ij} = 0 \tag{4.11c}$$

$$g_{ij,j} = 0 . \tag{4.11d}$$

One can see that these coordinate conditions are acceptable by looking at those of the field equations that involve $\partial_t g_i$ and $\partial_t \pi^T$ The linear part of equation (3.15a) gives, as the equations for the longitudinal part of g_{ij},

$$\partial_t(g_{i,j} + g_{j,i}) = N_{i,j} + N_{j,i} . \tag{4.12}$$

The Lagrange multipliers $N_i \equiv g_{0i}$ are functions determined only when coordinate conditions are imposed and must vanish at infinity where space is flat. Inserting (4.11b) into (4.12) gives, consistent with the boundary conditions, $N_i = 0$ everywhere. Similarly, from (3.15b), one has

$$\partial_t[-(1/2\nabla^2)\pi^T] = N . \tag{4.13}$$

Condition (4.11a) implies $N \equiv (-g_{00})^{-1/2} = 1$, again consistent with the required asymptotic limit.

Alternately, one can see that equations (4.11) are physically appropriate coordinate conditions by a direct comparison with the known results of linearized theory as discussed in I. Thus, as mentioned above, $\mathcal{H}_{\text{lin}} = -\nabla^2 g^T$ and $\mathfrak{T}^{0i}_{\text{lin}} = -2\nabla^2(\pi^{iT} + \pi^L{}_{,i})$ are the linearized theory's Hamiltonian and momentum densities and so their coefficients in the generator (4.10) must be δt and δx^i respectively, in order that the form (4.3), also obtained in I, be reproduced.

4.4. *Hamiltonian form and independent variables for linearized theory.* Since the generator is now

$$G = \int d^3x \, [\pi^{ijTT} \delta g_{ij}{}^{TT} - \mathcal{H}_{\text{lin}}(\pi^{ijTT}, \, g_{ij}{}^{TT}) \delta t + \mathfrak{T}^{0i}_{\text{lin}}(\pi^{ijTT}, g_{ij}{}^{TT}) \, \delta x^i] \tag{4.14}$$

the linearized theory has been put into canonical form, with $g_{ij}{}^{TT}$ and π^{ijTT} as the two canonically conjugate pairs of variables.

We will now see the usefulness of the linearized theory in suggesting the choice of canonical variables for the full theory. Since the identification is made from the bilinear part of the Lagrangian $\pi^{ij} \partial_t g_{ij}$, which is the same as for the linearized theory, the greater complexity of the full theory, *i.e.*, its self-interaction, is to be found only in the non-linearity of the constraint equations. Even in the constraint equations, the linearized theory will guide us in choosing g^T and π^i as the four extra momenta to be solved for.

4.5. *Hamiltonian form and independent variables for the full theory.* The full theory can now easily be put into canonical form. The generator of (4.10) is, of course, also correct for the full theory since it comes from the bilinear part of the Lagrangian. The constraint equations (3.15c) now read (in a coordinate system to be specified shortly)

$$-\nabla^2 g^T = \mathcal{P}^0[g_{ij}{}^{TT}, \, \pi^{ijTT}; \, g^T, \, \pi^i; \, g_i, \, \pi^T] \tag{4.15a}$$

$$-2\nabla^2(\pi^{iT} + \pi^L{}_{,i}) = \mathcal{P}^i[g_{ij}{}^{TT}, \, \pi^{ijTT}; \, g^T, \, \pi^i; \, g_i, \, \pi^T] \tag{4.15b}$$

---

[10]The differential form (4.11c,d) of the coordinate conditions may be integrated to yield (4.11a,b) either by imposing appropriate boundary conditions or by a procedure given in Appendix B of III.

where $\mathcal{P}^\mu$ are non-linear functions of g_{ij} and π^{ij}. One can again solve these (coupled) equations (at least by a perturbation-iteration expansion) for g^T and π^i. Thus, one can again choose $-\nabla^2 g^T$ and $-2\nabla^2(\pi^{iT} + \pi^L{}_{,i})$ as the four extra momenta to be eliminated. We denote the solutions of equations

$$-\nabla^2 g^T = -\mathfrak{T}^0{}_0[g_{ij}{}^{TT}, \pi^{ijTT}, g_i, \pi^T] \tag{4.16a}$$

$$-2\nabla^2(\pi^{iT} + \pi^L{}_{,i}) = -\mathfrak{T}^0{}_i[g_{ij}{}^{TT}, \pi^{ijTT}, g_i, \pi^T] . \tag{4.16b}$$

These equations are the counterpart of $p_{M+1} = -H$ in the particle case.

As we have seen, the four constraint equations are maintained in time as a consequence of the other field equations. Hence, after inserting equations (4.16) into (3.15a,b), one finds that four of these twelve (those for $\partial_t g^T$ and $\partial_t \pi^i$) are "Bianchi" identities, leaving eight independent equations in the twelve variables $g_{ij}{}^{TT}$, π^{ijTT}, g_i, π^T and N, and N_i. These equations are linear in the time derivatives of the first eight variables.

We now impose the coordinate conditions (4.11) which determine π^T and g_i. The $\partial_t g_i$ and $\partial_t \pi^T$ equations become determining equations for N and N_i [the full theory's analog of (4.12) and (4.13)]. N and N_i are no longer 1 and 0 respectively, but now become specific functionals of $g_{ij}{}^{TT}$ and π^{ijTT}, which could (in principle) be calculated explicitly.[11] In the last four equations, then, N and N_i may, in principle, be eliminated, leaving a system of four equations involving only $g_{ij}{}^{TT}$ and π^{ijTT}, and linear in their time derivatives. We will now see that this reduced system is in Hamiltonian form.

The generator (4.10) reduces to canonical form [with coordinate, conditions (4.11) imposed and constraints (4.16) inserted]:

$$G = \int d^3x \, [\pi^{ijTT} \, \delta g_{ij}{}^{TT} + \mathfrak{T}^0{}_0 \, \delta t + \mathfrak{T}^0{}_i \, \delta x^i] \tag{4.17a}$$

while the Lagrangian now becomes

$$\mathcal{L} = \pi^{ijTT} \, \partial_t g_{ij}{}^{TT} + \mathfrak{T}^0{}_0 . \tag{4.17b}$$

It can be shown that the solutions $\mathfrak{T}^0{}_\mu$ of the constraint equations do not depend explicitly on the coordinates x^μ of (4.11) (see III). This is not unexpected, since the variables g_{ij} and π^{ij} appearing on the right-hand side of (4.15) do not depend explicitly on the coordinates in this frame. (Thus, only $g_{i,j} = x^i{}_{,j} = \delta^i{}_j$ and $\pi^T = -2\nabla^2 t = 0$ appear in g_{ij} and π^{ij}.)

4.6. *Fundamental Poisson brackets and P.B. equations of motion for general relativity.* With the generator now in canonical form, we can immediately write down the fundamental equal time P.B. relations for $g_{ij}{}^{TT}$ and π^{ijTT}. These are[12]

$$[g_{ij}{}^{TT}(\mathbf{x}), \pi^{mnTT}(\mathbf{x}')] = \delta^{mn}{}_{ij}(\mathbf{x} - \mathbf{x}') \tag{4.18a}$$

$$[g_{ij}{}^{TT}(\mathbf{x}), g_{mn}{}^{TT}(\mathbf{x}')] = 0 = [\pi^{ijTT}(\mathbf{x}), \pi^{mnTT}(\mathbf{x}')] . \tag{4.18b}$$

The $\delta^{mn}{}_{ij}(\mathbf{x})$ in (4.18a) is a conventional Dirac δ-function modified in such a way that the transverse-traceless nature of the variables on the left-hand side is not violated. Note that the definition of this modified δ-function does not depend on the metric; it is symmetric, transverse, and traceless on each pair of indices:

---

[11] An example of such an explicit determination of N and N_i for coordinate conditions (4.22) is given in (7.7) and (7.9).

[12] The variables being used as the coordinates x^μ (4.11a,b) of course have vanishing P.B. with *all* variables.

$$\delta^{mn}{}_{ij} = \delta^{nm}{}_{ij} = \delta^{mn}{}_{ji} = \delta^{ij}{}_{mn} \tag{4.19a}$$

$$\delta^{mm}{}_{ij} = 0 = \delta^{mn}{}_{ii} \tag{4.19b}$$

$$\delta^{mn}{}_{ij,j} = 0 . \tag{4.19c}$$

From the form of equations (4.17), one also has the P.B. equations of motion:

$$\partial_t g_{ij}{}^{TT} = [g_{ij}{}^{TT}, H] = \delta H / \delta \pi^{ijTT} \tag{4.20a}$$

$$\partial_t \pi^{ijTT} = [\pi^{ijTT}, H] = -\delta H / \delta g_{ij}{}^{TT} \tag{4.20b}$$

where $H \equiv -\int d^3x \, \mathfrak{T}^0{}_0$ is the Hamiltonian. The last equalities in equations (4.20) follow from equations (4.18). That (4.18) and (4.20) are consistent with the Lagrangian equations obtained by varying (4.17b) is now immediate.[13] Corresponding to the time translation equations (4.20), one also has for spatial displacements

$$\partial_k g_{ij}{}^{TT} = [P_k, g_{ij}{}^{TT}] = -\delta P_k / \delta \pi^{ijTT} \tag{4.21a}$$

$$\partial_k \pi^{ijTT} = [P_k, \pi^{ijTT}] = \delta P_k / \delta g_{ij}{}^{TT} \tag{4.21b}$$

where $P_k \equiv +\int \mathfrak{T}^0{}_k \, d^3x$ is the total momentum operator. With the canonical momentum $P_k{}^c \equiv \int d^3x \, \mathfrak{T}^0{}_k{}^c \equiv -\int d^3x \, \pi^{mnTT} g_{mn}{}^{TT}{}_{,k}$, equations (4.21) are obvious. The quantities P_k and $P_k{}^c$ actually coincide, since their respective integrands differ by a divergence of canonical variables, as shown in IIIA.

4.7. *Alternate canonical form arising from different coordinate conditions.* The canonical form of equations (4.17) is, of course, not unique. First one can make the usual type of canonical transformations among the canonical variables $g_{ij}{}^{TT}$ and π^{ijTT}, leaving the coordinate conditions fixed. A more general class is available, however: the generator (4.2) can also be reduced to canonical form with coordinate conditions other than (4.11). That the reduction can always be carried out is shown in Appendix A of III. An example which will be of use later is given by the coordinate conditions

$$x^i = g_i - (1/4\nabla^2) g^T{}_{,i} \tag{4.22a}$$

$$t = -(1/2\nabla^2)(\pi^T + \nabla^2 \pi^L) \tag{4.22b}$$

or, in differential form,

$$g_{ij,jkk} - \tfrac{1}{4} g_{kj,kji} - \tfrac{1}{4} g_{jj,kki} = 0 \tag{4.22c}$$

$$\pi^{ii} = 0 . \tag{4.22d}$$

In this frame, the canonical variables are $g_{ij}{}^{TT}$ and π^{ijTT} (where the ^{TT} symbol now refers to the coordinate system (4.22) and hence these canonical variables are *different* from the set previously considered). The $\mathfrak{T}^0{}_\mu$ in this frame are the solutions of the constraint equations (4.16) for $\nabla^2 g^T$ and $-2\nabla^2(\pi^{iT} + \pi^L{}_{,i})$, which will be *different* functions of the new canonical variables, since again ^{T} and ^{L} are referred to (4.22) rather than (4.11). This frame is of interest since, for $g_{ij}{}^{TT} = 0$, the metric g_{ij} reduces to the isotropic form, i.e., $g_{ij} = (1 + \tfrac{1}{2} g^T)\delta_{ij}$. We will discuss, in Section 5, a preferred class of physically equivalent frames which includes the two given here.

---

[13]The primary consistency criterion on the canonical reduction is that equations (4.20) be identical to the Einstein equations in the coordinate frame (4.11); this is established in IIIA.

It is perhaps worth noting that, with the coordinates of (4.22), the orthogonal decomposition of g_{ij} takes on the simple form

$$g_{ij} = g_{ij}{}^{TT} + \delta_{ij}(1 + \tfrac{1}{2}\, g^T\, [g_{mn}{}^{TT},\, \pi^{mnTT}])\,. \tag{4.23}$$

Thus, the dynamical aspects of the theory are, as expected, to be found in the deviation of the metric from its flat space value. By tracing (4.23), one obtains a local relation giving the canonical variable $g_{ij}{}^{TT}$ at a point in this coordinate system in terms of g_{ij} at that point: $g_{ij}{}^{TT} = g_{ij} - \tfrac{1}{3}\,\delta_{ij}g_{kk}$. The g^T term, as we have seen, is directly given by the Hamiltonian density, and, as will be shown in the next section, carries the asymptotic (Newtonian) m/r term in the metric.

7-5. Energy and Radiation

5.1. *Expressions for the energy and momentum P_μ of the gravitational field.* The canonical formalism developed in the previous section has brought out the formal features of general relativity which have their counterpart in usual Lorentz covariant field theories. As a consequence, the physical interpretation of the gravitational field may be carried out in terms of the same quantities that characterize other fields, *e.g.*, energy, momentum, radiation flux (Poynting vector).

The energy E of the gravitational field is just the numerical value of the Hamiltonian for a particular solution of the field equations. In obtaining this numerical value, the form of the Hamiltonian as a function of the canonical variables is irrelevant, and one may make use of the equation, $\mathfrak{T}^0{}_0 = \nabla^2 g^T$ to express E as a surface integral. One has then[14]

$$P^0 \equiv E = -\int d^3\mathbf{r}\, \nabla^2 g^T = \; = \; -\oint dS_i\, g^T{}_{,i}$$
$$= \; \oint dS_i\, (g_{ij,j} - g_{jj,i}) \tag{5.1}$$

where dS_i is the two-dimensional surface element at spatial infinity. Similarly, the total field momentum P_i may be written as

$$P^i = -2 \oint (\pi^i{}_{,j} + \pi^j{}_{,i})dS_j = -2 \oint \pi^{ij}\, dS_j\,. \tag{5.2}$$

In (5.1) and (5.2) we have assumed, as throughout, that the coordinate system is asymptotically rectangular. The basic requirement for an energy to be at all defined is, of course, that space-time become flat at spatial infinity (so that the integrals in (5.1,2) are well defined). The rectangularity requirement may then be conveniently imposed to avoid unnecessarily complicated expressions.

5.2. *Coordinate invariance of energy-momentum expression; Lorentz four-vector nature of P_μ.* The usefulness of formulas (5.1) and (5.2) is enhanced by the fact that, in evaluating the surface integrals at infinity, it is not necessary to employ there the original canonical coordinate frame. At spatial infinity, where the metric approaches the Lorentz value, coordinate transformations, which preserve this boundary condition, must approach the identity transformation (excluding, for the moment, Lorentz transformations at infinity). For the transformation $\bar{x}^\mu = x^\mu - \xi^\mu$ then, one has

---

[14]It should be emphasized that, while the *energy* and *momentum* are indeed divergences, the integrands in the *Hamiltonian* and in the space translation generators $\mathfrak{T}^0{}_\mu[g^{TT}, \pi^{TT}]$ are *not* divergences when expressed as functions of the canonical variables.

$\xi^{\mu}{}_{,\nu} \to 0$ (since only $\partial \bar{x}^{\mu}/\partial x^{\nu}$ occurs in the transformation laws of $g_{\mu\nu}$ and $\Gamma_{\rho}{}^{\alpha}{}_{\sigma}$). Let us first keep only the linear terms in the transformation:

$$\bar{g}_{ij} = g_{ij} + \xi^{i}{}_{,j} + \xi^{j}{}_{,i} \tag{5.3a}$$

$$\bar{\pi}^{ij} = \pi^{ij} - \left(\delta_{ij}\xi^{0}{}_{,ll} - \xi^{0}{}_{,ij}\right). \tag{5.3b}$$

For the entire formalism, it is necessary that $g_{\mu\nu} - \eta_{\mu\nu}$ and $\Gamma_{\rho}{}^{\alpha}{}_{\sigma}$ fall off at least as $\sim 1/r$ at spatial infinity. Hence, $\xi^{\mu}{}_{,\nu}$, must have a similar behavior. As can be seen from (4.5), these transformations affect only g_i and π^T leaving $g^T{}_{ij}$, $, \pi^i{}_{,j}$, and the canonical variables invariant to $O(1/r)$. However, in (5.1) and (5.2), one needs to know $g^T{}_{,i}$ and $\pi^i{}_{,j}$, to $O(1/r^2)$, so that one must also investigate the quadratic terms in the transformation law. These present a problem only if a derivative acting on $\xi^{\mu}{}_{,\nu}$ does not alter its order, *i.e.*, if one has a "coordinate wave" $\xi^{\mu}{}_{,\nu} \sim e^{ipx}/r$ whose derivatives are again $\sim 1/r$. A detailed investigation (IVC) shows that expressions (5.1) and (5.2) for P^{μ} *are* invariant under *all* transformations preserving $\bar{g}_{\mu\nu} - \eta_{\mu\nu} \sim 1/r$, provided one averages the surface integrals over the oscillatory terms. The prescription of averaging is justified, since the energy and momentum can be defined directly from the leading $\sim 1/r$ terms of g^T and π^i, in the coordinate frame (4.11). [For instance, from (4.16a), one sees that $g^T = (1/\nabla^2)\mathfrak{T}^0{}_0 \sim E/4\pi r$.] However, as shown in IVC, these $1/r$ terms are in fact coordinate invariant and the averaged P^{μ} integrals agree with them. Thus, by either prescription, our formula for P_{μ} is unaffected by changes of coordinates not involving Lorentz transformations at spatial infinity.

The definition of energy and momentum, as given by the canonical formalism, may be shown to coincide with that arising from the stress energy pseudotensor of Landau and Lifshitz. In this definition, the energy and momentum can also be given in surface integral form. (Here, too, one must average over any oscillatory terms to obtain a coordinate-independent result.) The linear terms in this pseudotensor are then just the surface integrals in (5.1) and (5.2), while the higher powers are negligible at infinity (using the averaging). This agreement also enables us to note directly that P^{μ} behaves as a Lorentz vector under rigid Lorentz transformations of the coordinate system, since this property is manifest for the pseudotensor expression. That the energy of a system should be defined from the asymptotic metric alone follows from the equivalence principle, the total gravitational mass being measurable by the acceleration of a test particle at infinity.

Since P^{μ} is constant, due to the fact that $\mathfrak{T}^0{}_{\mu}$ does not depend on x^{μ}, explicitly, it can be evaluated at any given time. For this purpose, one should need only those initial Cauchy data required to specify the state of the system uniquely. Prior to the imposition of coordinate conditions, these are g_{ij} and π^{ij} (and not, for example, $g_{0\mu}$) as we have seen in Section 3. The last members of (5.1) and (5.2) are so expressed. When any (asymptotically rectangular) coordinate conditions are imposed, one needs only the two pairs of canonical variables of that frame to specify the state of the system and, of course, to calculate P^{μ} (using the formula $P_{\mu} = \int d^3r\, \mathfrak{T}^0{}_{\mu}$).

5.3. *Conditions for equality of Hamiltonian and energy—"Heisenberg representation."* As was discussed in the previous section, there are an infinite number of coordinate conditions that may be invoked to put the theory in canonical form. The question therefore arises as to whether the above physical results are affected by the coordinate conditions chosen. While, for example, we have seen that the P_{μ} of (5.1) and (5.2) is invariant under change of frame, this does not prove that different canonical forms, arising from different frames, will yield the same integral for P_{μ}. Indeed, not every canonical form of the theory has the same value of the Hamiltonian, as one is free to perform Hamilton–Jacobi (H.-J.), or partial H.-J., transformations. One must therefore.

take as primary, frames and canonical variables which are in the "Heisenberg representation." More precisely, we require that in the expression giving the full metric $g_{\mu\nu}$ at any time in terms of the canonical variables at that time, there be no explicit coordinate dependence. The preferred role is given to the metric in this requirement since basic measurements (rods and clocks) refer to $g_{\mu\nu}$ directly. Similarly, in standard particle mechanics, the directly measured quantities (which appear in the original Lagrangian) are the basic "Heisenberg" variables, and one requires that any other "Heisenberg" set be related to them in a time-independent fashion. More generally, even if the original Lagrangian (which always defines the Heisenberg representation) is not in canonical form, the same requirement is to be imposed. For relativity, the basic variables in the Einstein Lagrangian are $g_{\mu\nu}$. (Note that the question of whether a given canonical form is Heisenberg thus does not require a comparison with some particular canonical form, and also that each of the two forms in Section 4 is indeed Heisenberg.) With this criterion, it is clear that the Hamiltonian densities will also have no explicit coordinate dependence, since the canonical equations can be obtained by substituting $g_{\mu\nu}[g^{\text{can}}, \pi^{\text{can}}]$ into the original field equations. This step produces no explicit coordinate dependence in the Heisenberg representation. It can be shown (see IVA) that all Heisenberg forms are linked by coordinate transformations of the type $\bar{x}^{\mu} = x^{\mu} + f^{\mu}[g^{c}(x), \pi^{c}(x)]$ where the f^{μ} do not depend on x^{μ} explicitly. Thus the relation between Heisenberg frames is determined by the intrinsic geometry and does not contain functions of the coordinates, which are independent of the physics of the state. As a result of this form of the transformations, one finds that P_{μ} is numerically the same in any Heisenberg frame for a given physical situation. Thus (5.1) gives correctly the numerical value of the Hamiltonian of any Heisenberg frame and could have been derived in any Heisenberg canonical form.

5.4. *"Waves" as excitations of canonical variables.* Excitations of the canonical modes of a field provide a definition of what one calls waves. This is, of course, the same definition as that given in electrodynamics. In electrodynamics, the gauge invariance partly obscures the fact that only the transverse parts $\mathbf{A}^{T}$ and $\mathbf{E}^{T}$ of the vector potential and electric field need be examined to recognize waves. These variables, as we saw in Section 4, are just the canonical ones for the Maxwell field. Correspondingly, in general relativity we may utilize the non-vanishing of the canonical variables as the criterion of the existence of waves. As in electrodynamics, when sources (among which are included the non-linear self-interactions) are present, the radiation and induction effects can be meaningfully separated only in the "wave zone"; but also as in electrodynamics, the wave concept can be employed consistently nearer the sources where it is of heuristic value, although the definition is somewhat arbitrary.

An example of an application of the wave criteria is furnished by the time-symmetric case ($\pi^{ij} = 0$ initially). In the frame of (4.22), it was seen that $\pi_{ij}{}^{TT} = 0$ if, and only if, the metric is isotropic, $g_{ij} = (1 + \frac{1}{2} g^{T})\delta_{ij}$. One need not explicitly transform to this frame to verify whether the canonical variables vanish in it; it is sufficient to examine the three-tensor (see, for example, Eisenhart, 1949, Section 28):

$$R_{ijk} \equiv R_{ij|k} - R_{ik|j} + \tfrac{1}{4}(g_{ik}R_{,j} - g_{ij}R_{,k}) \qquad (5.4)$$

which vanishes if, and only if, $g_{ij}{}^{TT}$ vanishes in the frame[15] of (4.22). A class of time-symmetric

---

[15]In terms of the canonical formalism, the time-symmetric situation $\pi^{ij} = 0$ can be viewed as the requirement that excitations in the canonical momentum variable π^{ijTT} vanish. This follows from the fact that if $\pi^{ijTT} = 0$, then $\pi^{i} = 0$ by the constraint equations (3.15c); consequently, $\pi^{T} = 0$ in the frame of (4.22). Thus, in this frame, $\pi^{ijTT} = 0$ implies $\pi^{ij} = 0$.

waves is furnished by the initial conditions recently discussed by Brill (1959).

5.5. *Radiation and definition of wave-zone.* The concept of radiation embodies the idea of field excitations escaping to infinity and propagating independently of their sources. In linear field theories, the region in which this occurs (the wave zone) is simply one which is many wavelengths away from the source. In general relativity, for strong fields, nonlinear effects are important, and act as effective sources, so that merely going beyond the matter sources is not, in general, adequate to define free radiation. To have a wave zone situation, one must set up further conditions such that the non-linear self-interaction terms do not influence the propagation of the waves. Such further conditions are also necessary in quantum electrodynamics due to vacuum polarization. The latter brings in effective non-linear contributions to the Maxwell action, so that in the presence of an arbitrary external electromagnetic field, the self-coupling can produce distortion of the "waves" (Delbrück scattering); again, scattering of two photons can occur, even in the absence of external fields, thus affecting superposition, that is, free propagation. The usual definitions of radiation would thus fail in such situations and, in fact, the "wave zone" is then defined only when the self-coupling is negligible. One must therefore require that in the wave zone, the various field amplitudes be small when the theory is non-linear. On the other hand, the process of going to the asymptotic wave zone region is not an *a priori* equivalent to taking the linearized approximation. In general relativity, we have previously seen that $(g^T, \pi^i) \sim P^\mu/r$ asymptotically, where P^μ is clearly determined by the interior region. Thus, components of the metric other than the wave modes survive asymptotically, and in fact may be comparable in size to the wave modes.

To define the wave zone, we consider a general situation in which the gravitational canonical modes behave as fe^{ikx}/r asymptotically in some region (with wave numbers k up to some maximum $k_{\max}$); beyond some point (the wave front) they are assumed to vanish very rapidly. (This last assumption is needed to make the energy of the waves finite.) The first wave zone requirement is the familiar one that $kr \gg 1$; this implies that gradients and time derivatives acting on the canonical modes are also $\sim 1/r$. Second, we demand that for all components of the metric, $|g_{\mu\nu} - \eta_{\mu\nu}| \sim a/r \ll 1$. This criterion can always be met (for radiation escaping to infinity) by taking r large enough, and waiting for the wave to reach this distance. This condition implies that, not only is the wave disturbance weak ($f/r \ll 1$), but also the Newtonian-like parts of the field (*e.g.*, g^T) are small ($P^\mu/r \ll 1$). It insures that quadratic (and higher) terms in $g_{\mu\nu} - \eta_{\mu\nu}$, are negligible compared to the linear ones. Finally, we impose the condition that $|\partial g/\partial(kx)|^2 \ll |g - \eta|$. In terms of $k_{\min}$, the minimum frequency (or wave number) to be considered (with, of course, $k_{\min}r \gg 1$), one has $k_{\min}r \gg k_{\max}(a/r)^{1/2}$. Again, for large enough r, this inequality can always be achieved. [For fixed r, it sets a lower bound on frequencies which may be treated in the wave zone.] The purpose of this requirement is to make the non-linear structures containing gradients also small compared to the leading linear terms. The last two conditions together are arranged to guarantee the absence of all self-interactions in the wave zone.

5.6. *General structure of gravitational radiation; superposition, coordinate-independence.* With the above definition of the wave zone, one finds (see IVB) that the field equations (3.15a,b) reduce to

$$\partial_t g_{ij} = 2(\pi^{ij} - \tfrac{1}{2}\delta_{ij}\pi^{ll}) + N_{i,j} + N_{j,i} + O(1/r^2) \tag{5.5a}$$

$$\begin{aligned}
\partial_t \pi^{ij} = &\tfrac{1}{2}(g_{ij,kk} + g_{kk,ij} - g_{ik,kj} - g_{jk,ki}) \\
&- \tfrac{1}{2}\delta_{ij}(g_{kk,ll} - g_{kl,kl}) - (\delta_{ij}N_{,kk} - N_{,ij}) \\
&+ O(1/r^2)
\end{aligned} \tag{5.5b}$$

where $O(1/r^2)$ is much less than the leading terms. Using a result of IVB that the orthogonal parts of $\sim 1/r$ and $\sim 1/r^2$ structures are also at most $\sim 1/r$ and $\sim 1/r^2$ respectively, one finds that

$$\partial_t g_{ij}{}^{TT} = 2\pi^{ijTT} \tag{5.6a}$$

$$\partial_t \pi^{ijTT} = \tfrac{1}{2}\nabla^2 g_{ij}{}^{TT} \tag{5.6b}$$

to the order $1/r$. Thus the *rigorous* dynamical modes obey the linearized theory's equations in the wave zone. Equations (5.6) are, of course, source-free, so that the radiation propagates independently of its origin, and without any self-interaction or dependence on the Newtonian-like parts g^T and π^i of the field. The flat-space wave equation with constant coefficients, $\Box^2_{\text{flat}} g_{ij}{}^{TT} = 0$, represented by (5.6), indicates the absence of curvature effects on the radiation. Further, there are no coordinate effects left, since the coordinate-dependent components g_i, π^T, and $g_{0\mu}$, have disappeared.

We have seen previously that $g_{ij}{}^{TT}$ and π^{ijTT} are invariant to the order $1/r$ under coordinate transformations leaving the metric asymptotically flat and not involving Lorentz transformations. Thus, in a fixed Lorentz frame, g^{TT} and π^{TT} represent a coordinate-independent description of the radiation. The detailed analysis of IVB shows how any coordinate waves which may be present can be identified unambiguously.

The above derivation may be carried through equally well in the presence of coupling to matter (*e.g.*, the Maxwell field), provided of course, the matter system is bounded within the interior region (and has finite energy content). Under these circumstances, any gravitational waves in the wave zone are still independent of the matter sources. If, however, electromagnetic radiation is also propagating in the wave zone with small enough amplitude (and $\sim 1/r$), the gravitational radiation is still free, being unaffected to $\sim 1/r$ by the electromagnetic waves, since the latter couple quadratically ($\sim 1/r^2$) in the field equations. Hence, in the common wave zone, the systems are correctly independent.

5.7. *Measurables in the wave zone: the Poynting vector, radiation amplitudes. Relation to curvature tensor.* Since the dynamics of the rigorous canonical modes is identical, in the wave zone, to that given by linearized theory, any radiation arising from strong interior fields can be precisely simulated by a corresponding solution of the linearized theory with sources. Such an analog metric can be made everywhere weak, so that linearized theory is everywhere valid, and one may arrange the sources so that in the wave zone, the correct g^T and π^i result. An absorber of gravitational radiation in the wave zone clearly cannot distinguish between the true and analog situations. One is therefore free to use the Poynting vector of linearized theory to measure the energy flux. From the symmetric stress-tensor of I, one finds straightforwardly that

$$T_{\text{lin}}^{i0} = \pi^{lmTT}(2g_{li,m}^{TT} - g_{lm,i}^{TT}) \tag{5.7}$$

in the coordinate frame of (4.11). However, it is easy to see that the general form of T_{lin}^{i0}, is coordinate-invariant through terms of $0(1/r^2)$, as is the right-hand side of (5.7). Hence (5.7) provides an invariant formula for the wave-zone Poynting vector. We remark that, as in electrodynamics, the physical part of the Poynting vector is obtained by averaging over oscillatory terms. This averaging is of use in proving that T_{lin}^{i0} is coordinate-independent, since it also eliminates "coordinate wave" effects. It may also be noted that, in (5.7), no terms involving g^T, π^i have survived to $O(1/r^2)$, showing that the Newtonian-like parts do not affect the energy flux of radiation. Similarly, in electrodynamics $\mathbf{E}^L$ does not contribute to $\mathbf{E}\times\mathbf{B}$ since $\mathbf{E}^L \sim 1/r^2$.

The canonical variables $g_{ij}{}^{TT}$ and π^{ijTT} are, in general, non-local functions of the metric and its first derivatives. However, it is possible to obtain their $1/r$ terms by measurements entirely within the wave zone. From the condition $kr \gg 1$, it is possible to isolate sufficiently accurately the k Fourier component of g_{ij} and $\partial_t g_{ij}$, and to obtain $g_{ij}{}^{TT}(k)$, $\pi^{ijTT}(k)$ algebraically from these. Equivalently, one may measure the k-component of the curvature tensor $R_\alpha{}^\mu{}_{\nu\beta}$ to obtain these physical quantities since it may be shown that in the wave zone

$$g_{ij}{}^{TT} = (2/\mathbf{k}^2)R_i{}^m{}_{mj} \tag{5.8a}$$

$$\pi_{ijTT} = -(ik_\ell \mathbf{k}^2)R_i{}^0{}_{lj} \ . \tag{5.8b}$$

Such measurements find a parallel in Maxwell theory, where $\mathbf{E}^T$, $\mathbf{A}^T$ are found from $\mathbf{E}$ and $\mathbf{B}$ by

$$\mathbf{E}^T = \mathbf{E} - \mathbf{k}[(\mathbf{k} \cdot \mathbf{E})/\mathbf{k}^2] \tag{5.9a}$$

$$\mathbf{A}^T = i(\mathbf{k} \times \mathbf{B})/\mathbf{k}^2 \ . \tag{5.9b}$$

In fact, electromagnetic wave measurements are commonly of this type.

It should be noted that just as the definition of electromagnetic radiation is not restricted to outgoing radiation (but may include mixtures of outgoing and incoming waves), this also is true of the above definition of gravitational radiation. For the purely outgoing case, the $1/r$ part of the curvature tensor is of type II—Null in Petrov classification (Petrov, 1954, and Pirani, 1957). We have seen in this section that the leading ($\sim 1/r$) physical terms in the field past the wave front are g^T and π^i, and that these yield invariantly the energy-momentum vector of the system. In the wave zone itself, the dynamical modes specifying the radiation, as well as the Poynting vector, could be obtained invariantly. Actually, a good deal of detailed information about the strong fields in interior regions is obtainable by purely asymptotic ("S-matrix") measurements,[16] so that a large portion, at least, of the description of a system is manifestly frame-independent, and can be gotten without the need for introducing apparatus into strong field regions and hence having to analyze its behavior there.

6. Extension to Coupling with Other Systems

6.1. *Lagrangian for coupled Einstein-Maxwell-charged particle system.* Up to now we have considered the free gravitational field, sketching in the process the main physical features which have emerged there. The extension to coupling enables us to verify that these characteristics remain unchanged and provides a basis for the self-energy calculations of the next section. The analysis is completely analogous to the discussion of the free field and so we limit ourselves to a brief outline for the case of coupling to electrodynamics with point charges (more details may be found in V and VA). The addition to the Einstein Lagrangian is

$$\begin{aligned}
\mathcal{L}_M = {} & \tfrac{1}{2}\left(A_{\mu,\nu} - A_{\nu,\mu}\right)\mathfrak{F}^{\mu\nu} + \tfrac{1}{4}\left(-^4g\right)^{1/2}\mathfrak{F}^{\mu\nu}\mathfrak{F}^{\alpha\beta}\,{}^4\mathfrak{g}_{\mu\alpha}\,{}^4\mathfrak{g}_{\nu\beta} \\
& + \int ds\, e(dx^\mu(s)/ds)A_\mu(x)\,\delta^4(x - x(s)) \\
& + \int ds\, \{\pi_\mu(dx^\mu/ds) - \tfrac{1}{2}\lambda'(s)(\pi_\mu\pi_\nu\,{}^4g^{\mu\nu} + m_0{}^2)\}\delta^4(x - x(s)) \ .
\end{aligned} \tag{6.1}$$

---

[16]J. Plebanski (private communication) has shown how properties of orbits of test particles may be examined by purely asymptotic measurements (using geodesic projections) even if the orbits remain entirely within the strong field interior region.

We are here employing a first-order formalism for the particle as well as for the Maxwell field. The Maxwell part of $\mathcal{L}_M$ is the straightforward covariant generalization of (3.5), with the field strength $\mathfrak{F}^{\mu\nu}$ now a tensor density. Thus, A_μ and $\mathfrak{F}^{\mu\nu}$ are varied separately, as are $x^\mu(s)$, the mechanical momentum $\pi_\mu(s)$, and the Lagrange multiplier $\lambda'(s)$. The particle Lagrangian is in parameterized form (with arbitrary parameter[17] s), as is required for manifest covariance. The δ^4-function is a scalar density, invariantly defined in a metric-independent way by $\int \delta^4(x)d^4x = 1$; the δ^3-function is similarly defined in three-space, i.e., $\int \delta^3(\mathbf{r})d^3r = 1$. Introducing the notation $\mathcal{E}^i \equiv \mathfrak{F}^{0i}$ and $\mathcal{B}^i \equiv \frac{1}{2}\epsilon^{ijk}(A_{k,j} - A_{j,k})$ (where $\epsilon^{ijk} = \epsilon_{ijk} = 0, \pm 1$ is totally antisymmetric) and using the gravitational variables of Section 2, one obtains

$$
\begin{aligned}
\mathcal{L}_M = {} & A_i\,\delta_0\mathcal{E}^i + \delta^3(\mathbf{r} - \mathbf{r}(t))[(\pi_i(t) + eA_i)(dx^i/dt) + \pi^0] \\
& + A_0[e\,\delta^3(\mathbf{r} - \mathbf{r}(t)) - \mathcal{E}^i{}_{,i}] \\
& - \tfrac{1}{2}\lambda[g^{ij}\pi_i\pi_j - N^{-2}(\pi_0 - N^i\pi_i)^2 + m_0{}^2]\delta^3(\mathbf{r} - \mathbf{r}(t)) \\
& - \tfrac{1}{2}Ng^{-1/2}g_{ij}[\mathcal{E}^i\mathcal{E}^j + \mathcal{B}^i\mathcal{B}^j] + N^i[\epsilon_{ijk}\mathcal{E}^j\mathcal{B}^k] \,.
\end{aligned}
\tag{6.2}
$$

We have also performed the s integration which led to the replacement of λ' by $\lambda \equiv [\lambda'(s)(ds/dx^0(s)]_{x^0(s=t)}$. It is convenient next to the eliminate the non-gravitational constraints obtained from varying A_0 and λ. These are:

$$
\mathcal{E}^i{}_{,i} = e\,\delta^3(\mathbf{r} - \mathbf{r}(t))
\tag{6.3a}
$$

$$
\pi^\mu\pi_\mu + m_0{}^2 = 0
\tag{6.4a}
$$

whose solutions are

$$
\mathcal{E}^{iL} = -\nabla(e/4\pi|\mathbf{r} - \mathbf{r}(t)|)
\tag{6.3b}
$$

$$
\pi_0 = N^i\pi_i - N(g^{ij}\pi_i\pi_j + m_0{}^2)^{1/2} \,.
\tag{6.4b}
$$

Here, we have again used the orthogonal breakup of $\mathcal{E}^i$ in the same flat-space sense as in Section 4. Therefore, $\mathcal{E}^i$ is to be regarded subsequently as an abbreviation for $\mathcal{E}^{iT} + \mathcal{E}^{iL}$, with $\mathcal{E}^{iL}$ expressed from (6.3b).

6.2. *Canonical reduction of coupled system.* At this stage, the matter Langrangian is in the reduced form

$$
\begin{aligned}
\mathcal{L}_M = {} & (-\mathcal{E}^{iT})\,\partial_0 A_i{}^T + [p_i\,dx^i/dt]\delta^3(\mathbf{r} - \mathbf{r}(t)) \\
& - \tfrac{1}{2}Ng^{-1/2}g_{ij}[\mathcal{E}^i\mathcal{E}^j + \mathcal{B}^i\mathcal{B}^j] \\
& - N[g^{ij}(p_i - eA_i{}^T)(p_j - eA_j{}^T) + m_0{}^2]^{1/2}\,\delta^3(\mathbf{r} - \mathbf{r}(t)) \\
& + N^i[\epsilon_{ijk}\mathcal{E}^j\mathcal{B}^k + (p_i - eA_i{}^T)\delta^3(\mathbf{r} - \mathbf{r}(t))]
\end{aligned}
\tag{6.5}
$$

where $p_i \equiv \pi_i + eA_i{}^T$. The first two terms of $\mathcal{L}_M$ are in the standard $\Sigma p\dot{q}$ form and thus $(A_i{}^T, -\mathcal{E}^{iT})$ are the canonical variables for the electro-magnetic field, while $p_i(t)$, $x^i(t)$ are those of the particle. Note that the variables A_i, $\mathcal{E}^i$, p_i or x^i, which arose naturally in obtaining canonical form, are also the appropriate variables from the geometrical (Cauchy problem) considerations of Section 3. Thus, A_i, p_i, π_i are covariant spatial components of the corresponding four-vectors, while $x^i(t)$ locates the particle within the three-surface. The three-vector density character of $\mathcal{E}^i$ is established

---

[17]Equation (6.1) is covariant against any reparameterization $\bar{s} = \bar{s}(s)$ with λ' transforming as a "vector": $\bar{\lambda}' = \lambda'(ds/d\bar{s})$ (just as N does in Section 2).

from its relation to the covariant spatial components of the four-dimensional dual $\mathcal{F}_{\mu\nu}$ of the field strength:

$$\mathcal{E}^i = \tfrac{1}{2} \epsilon^{ijk}\, \mathcal{F}_{jk} \equiv \tfrac{1}{2} \epsilon^{ijk}(\tfrac{1}{2}\, \epsilon_{jk\mu\nu}\mathfrak{F}^{\mu\nu})$$

The total Lagrangian, which is the sum of $\mathcal{L}_M$ and the gravitational part (3.13), now has precisely the parameterized form of (2.3), as a consequence of the general covariance of each part. The coefficients of N and N^i now yield the extended constraint equations

$$gR + \tfrac{1}{2}\pi^2 - \pi^{ij}\pi_{ij} = \sqrt{g}\,\delta^3(\mathbf{r}-\mathbf{r}(t))[g^{ij}(p_i - eA_i^{\,T})(p_j - eA_j^{\,T}) + m_0^{\,2})]^{1/2} + \tfrac{1}{2}g_{ij}[\mathcal{E}^i\mathcal{E}^j + \mathcal{B}^i\mathcal{B}^j] \quad (6.6a)$$

$$-2\pi_{ij}{}^{|j} = \delta^3(\mathbf{r}-\mathbf{r}(t))(p_i - eA_i^{\,T}) + \epsilon_{ijk}\mathcal{E}^j\mathcal{B}^k . \quad (6.6b)$$

Equations (6.6) can again be solved for

$$\nabla^2 g^T = \mathfrak{T}^0{}_0$$

and

$$-2\nabla^2(\pi^{iT} + \pi^i{}_{,i}) = \mathfrak{T}^0{}_i .$$

When the coordinate conditions (4.11) are again imposed, the theory is in canonical form with the *same* gravitational canonical variables. One sees most clearly now how the gravitational canonical variables characterize independent gravitational field excitations. All the other components of the metric, *e.g.*, g^T and N, N^i depend on the non-gravitational variables as well. In contrast, only $g_{ij}{}^{TT}$ and π^{ijTT} can be specified initially, irrespective of the excitations of the matter system.

The quantity $-\mathfrak{T}^0{}_0$ now represents the Hamiltonian density of the total system[18] and depends only on the canonical variables of all three parts. Further, the expressions (5.1) and (5.2) now give the energy-momentum of the total system. The fact that the total energy of the interacting system is obtainable from purely metric quantities is analogous to the expression in electrodynamics of the total charge in terms of the integral of the longitudinal electric field. It is worth noting that the choice of coordinate conditions is unaffected by coupling. Finally, we record the equations of motion of the gravitational field including coupling. The extension of the constraint equations (3.15c) has been given by (6.6) above; the $\partial_t g_{ij}$ equations (3.15a) are unchanged, since they are of the type $m\dot{x} = p$ (namely, they are defining equations for Christoffel symbols). However, one must now add a term $\tfrac{1}{2}N\mathfrak{T}_M{}^{ij}$ to the right-hand side of (3.15b) for $\partial_t\pi^{ij}$, where $\mathfrak{T}_M{}^{ij}$ is the symmetric spatial stress-energy tensor density of the matter:

$$\begin{aligned}
\mathfrak{T}_M{}^{ij} &\equiv g^{ij}g^{jk}\sqrt{g}\,{}^4\mathfrak{T}_{Mlk} \\
&= \delta^3(\mathbf{r}-\mathbf{r}(t))(p^i - eA^{iT})(p^j - eA^{jT})[(\mathbf{p}-e\mathbf{A}^T)^2 + m_0^{\,2}]^{-1/2} \\
&\quad + g^{-1/2}[\tfrac{1}{2}g^{ij}(\mathcal{E}^m\mathcal{E}_m + \mathcal{B}^m\mathcal{B}_m) - \mathcal{E}^i\mathcal{E}^j - \mathcal{B}^i\mathcal{B}^j] . \quad (6.7)
\end{aligned}$$

7-7. Static Self-Energies

7.1. *Physical basis for finiteness of self-energy when gravitation is included.* The canonical formalism developed for coupled systems in the previous section allows one to define pure particle states by the vanishing of the canonical variables referring to field excitations. For particles at rest, therefore, the energies of such states are just the total rest and interaction energies, which include, of course, their self-energies.

---

[18]In the flat space limit, this Hamiltonian density is correctly the usual one for electrodynamics.

The aim of classical point electron theory, since its inception, has been to obtain a finite, model-independent electromagnetic self-energy, and if possible, to dispense with mechanical mass altogether. Thus the total mass of the particle would arise from its coupling to the field. Such a program, however, was not feasible, since the self-energy diverged linearly, with no realistic compensation possible. In terms of renormalization theory, this implied an infinite "bare" mechanical mass. Since gravitational interaction energy is negative on the Newtonian level, it may be expected to provide compensation. Indeed a simple argument yields a limit on the self-energy due to just such a compensation. Consider a *bare* mass m_0 distributed in a sphere of radius ϵ. In the Newtonian limit, the total energy (*i.e.*, the clothed mass) m is given by $m = m_0 - \frac{1}{2}\gamma m_0^2/\epsilon$. For sufficiently small ϵ, m could become zero and then negative. In general relativity, the principle of equivalence states that it is the total energy that interacts gravitationally and not just the bare mass. Thus, as the interaction energy grows more negative, were a point reached where the total energy vanished, there could be no further interaction energy. Consequently, there can be no negative total energy, in contrast to the negative infinite self-energy of Newtonian theory. General relativity effectively replaces m_0 by m in the interaction term: $m = m_0 - \frac{1}{2}\gamma m^2/\epsilon$. Solving for m yields $m = \gamma^{-1}[-\epsilon + \epsilon^2 + 2\gamma m_0\epsilon)^{1/2}]$, which shows that $m \to 0$ as $\epsilon \to 0$.

More interesting is the fact that the gravitational interactions produce a natural cutoff for the Coulomb self-energy of a point charge. Here the self-mass resides in the Coulomb field $\frac{1}{2}\int (e/4\pi r^2)^2 \, d^3\mathbf{r}$. By the general argument above on gravitational compensation, one expects that the Coulomb energy near the origin (which is, in fact, "denser" than the neutral particle's δ-function distribution) will have a very strong gravitational self-interaction, resulting in a vanishing *total* contribution to the self-mass from the region near the origin. Thus, the integral effectively extends down only to some radius a, yielding $m_{\text{EM}} = \frac{1}{2}\int_a^\infty (e/4\pi r^2)^2 d^3r = (e^2/4\pi)/2a$. We can determine this effective flat-space cutoff a by the same equivalence principle argument. Without the gravitational contribution, the mass is $m_0 + \frac{1}{2}(e^2/4\pi\epsilon)$, so the total clothed mass is determined by the equation $m = m_0 + \frac{1}{2}e^2/4\pi\epsilon - \frac{1}{2}\gamma m^2/\epsilon$. This yields

$$m = \gamma^{-1}\{-\epsilon + [\epsilon^2 + 2m_0\epsilon\gamma + (e^2/4\pi)\gamma]^{1/2}\} \, . \tag{7.1}$$

We will see below that this formula is a rigorous consequence of the field equations. In the limit $\epsilon \to 0$, we have $m = (e^2/4\pi)^{1/2}\gamma^{-1/2}$. The bare mechanical mass m_0 again does not contribute to the clothed mass. Our result is then equivalent to a cutoff $a = \frac{1}{2}(e^2/4\pi)^{1/2}\gamma^{1/2}$ on the flat-space Coulomb integral.

7.2. *Calculation of charged and neutral particle self-energies from initial data.* We begin the self-energy calculation, then, by obtaining a solution of the field equations for a pure one-particle state, *i.e.*, a state containing no independent excitations of the gravitational or electromagnetic fields in the rest frame. This requires that $g_{ij}^{TT} = \pi^{ijTT} = A^{iT} = \mathcal{E}^{iT} = p_i = 0$ on the $t = $ const surface where the energy is being computed. With the coordinate conditions of (4.22), we are therefore dealing with the time-symmetric situation $\pi^{ij} = 0$. According to (4.23), then, the metric is isotropic; it is convenient to write it as $g_{ij} = \chi^4(r)\,\delta_{ij}$. From Section 7-5, the energy is the coefficient of $1/(32\pi r)$ in the asymptotic form of $\chi(r)$. The field equation determining g^T and hence χ is (6.6a). One has[19]

$$\sqrt{g}\ ^3R = -8\chi\nabla^2\chi = m_0\,\delta^3(\mathbf{r}) + \tfrac{1}{2}\,\chi^{-2}\mathcal{E}^{iL}\mathcal{E}^{iL} \tag{7.2a}$$

[19]Professor L.N. Cooper has pointed out to us that (7.2a) with $e = 0$ $[-8\chi\nabla^2\chi = \rho_0(r)$, where ρ_0 is any bare mass distribution] can be obtained by simple equivalence principle arguments starting from Newtonian theory. The Poisson equation $\nabla^2\phi = 4\pi\gamma\rho_0 = \frac{1}{4}\rho_0$ for the gravitational potential ϕ, must be corrected to include the particle's

and the electric field $\mathbf{E}^L$ by (6.3b) is

$$\mathcal{E}^{iL} = (-e/4\pi r)_{,i} \, . \tag{7.3}$$

A formal solution of (7.2a) may be found by setting $\chi^2 = \psi^2 - \phi^2$ (Misner and Wheeler, 1957). One finds

$$-8\chi\nabla^2\chi = 8(\phi\nabla^2\phi - \psi\nabla^2\psi) + 8(\psi^2 - \phi^2)^{-1}(\phi\nabla\psi - \psi\nabla\phi)^2$$
$$= m_0\,\delta^3(\mathbf{r}) + \tfrac{1}{2}\,(\psi^2 - \phi^2)^{-1}(\mathbf{E}^L)^2 \, . \tag{7.2b}$$

If one then makes the assumption

$$\mathbf{E}^L = 4(\phi\nabla\psi - \psi\nabla\phi) \tag{7.4}$$

with $\phi = e/16\pi r$ and $(1 + m/32\pi r) = \psi$, (7.4) correctly reproduces (7.3). Equation (7.2b) then determines the total mass m to be the $\epsilon \to 0$ limit of (7.1):

$$m = \lim_{\epsilon \to 0} 16\pi\{-\epsilon + [\epsilon^2 + (e/8\pi)^2 + m_0\epsilon/8\pi]^{1/2}\} \, . \tag{7.5}$$

In (7.5) the parameter ϵ has been introduced by setting $\delta^3(\mathbf{r})/r$ equal to $\delta^3(\mathbf{r})/\epsilon$, and is thus essentially the "radius" of the δ-function. [This interpretation of the δ-function corresponds to viewing $\delta^3(\mathbf{r})$ as the limit of a shell distribution $\delta(r - \epsilon)/4\pi r^2$ of radius ϵ. In V, it is shown that the results of this section are, in fact, independent of the model chosen for $\delta^3(\mathbf{r})$ in the point limit.] The result (7.5) is that $m = 2|e|$ and hence the total mass is finite and independent of the bare mechanical mass. The gravitationally renormalized electrostatic self-energy is now finite. The analysis also points out that mass only arises if a particle has nongravitational interaction with a field of finite range.[20] For example, an electrically neutral particle coupled to a Yukawa field would acquire a mass by virtue of *this* coupling.

A solution may also be obtained for the case of two particles of like charge. The results are consistent with those obtained above for one particle (see V). The energy is just $2(|e_1| + |e_2|)$, that is, just the sum of the individual masses, independent of both the mechanical masses and the interparticle separation r_{12}. The absence of an interaction term can be traced to the cancellation of the Newtonian attraction of the masses with the Coulomb repulsion of the like charges.[21] This also checks with the result expected at large separation for particles whose mass and charge are related by $m_{1,2} = (e^2_{1,2}/4\pi\gamma)^{1/2}$. For opposite charges, such a cancellation should not occur. However, we have not been able to obtain a rigorous solution for a pure particle state of two opposite charges.

All the usual Newtonian theory results may be obtained from (7.5) or from the corresponding equation [(3.8) of V] for two bodies in the appropriate limit. This "dilute" limit consists in regarding

gravitational self-energy $\tfrac{1}{2}\rho\phi$ as part of the source, *i.e.*, $\nabla^2\phi = \tfrac{1}{4}\rho \equiv \tfrac{1}{4}(\rho_0 + \tfrac{1}{2}\rho\phi)$. Eliminating ρ, one obtains $\nabla^2\phi = \tfrac{1}{4}\rho_0(1 - \tfrac{1}{2}\phi)^{-1}$. In terms of $\chi = 1 - \tfrac{1}{2}\phi$, this is just (7.2a) in the neutral case. It is interesting that this argument yields the rigorous field equations in the frame (4.22) for this situation. For the point electric charge, the same argument with ρ_0 replaced by $\rho_0 + \tfrac{1}{2}\rho_e\phi_e$ (ϕ_e being the electrostatic potential) leads to an equation for ϕ which, while different from (7.2a), does yield the correct total energy.

[20]It is interesting to note that, even though $m = 0$ for the neutral particle, this does not imply that space is everywhere flat. The metric is indeed flat for $r > \epsilon$, but rises steeply in the interior. For example, $\int_0^\infty d^3r\sqrt{^3g}\,{}^3R = m_0$, which shows that space is curved at the origin.

[21]This cancellation implies that the solution should be static (since there is no initial potential between the particles). This has indeed been shown to be the case for $m_{1,2} = 2|e_{1,2}|$ by Papapetrou (1947).

m_0 and e small compared to ϵ before passing to $\epsilon = 0$, thus preventing the strong (non-Newtonian) gravitational interactions. Alternately, the "dilute" limit represents a perturbation expansion of the theory in powers of the coupling constant γ. From (7.1) one easily sees that

$$E \sim m_0 + (e^2/4\pi - \gamma m_0{}^2)/2\epsilon + O(1/\epsilon^2) . \tag{7.6}$$

[For two bodies, the Newtonian and Coulomb interaction terms $(e_1 e_2/4\pi - \gamma (m_0)_1 (m_0)_2)/r_{12}$ also appear.] Since m_0 and e are unrelated, there is no cancellation in the self-energy term of (7.6). In fact, the perturbation expansion consists of an infinite series of more and more divergent terms. The inapplicability of such an expansion is shown by the finite rigorous answer $m = 2|e|$. If one attempted to use standard perturbation renormalization techniques, one would indeed find that the theory was unrenormalizable (see V).

7.3. *Determination of full metric due to a charge. Stability of the particle. Vanishing of self-stress.* We have been able to find the self energy purely from the initial value (constraint) equations. The full solution of the problem consists also in the specification of the N, $N_i (g_{0\mu})$ for our frame as well as the time development of the system (fuller details are in VA). The equations determining N and N_i are obtained by taking the time derivatives of the coordinate conditions (4.22c,d). These are basically linear combinations of the right-hand sides of (3.15a,b) [including, of course, the coupling term from (6.7)]. Fortunately, the initial values and the coordinate conditions for our problem reduce these equations to quite simple form. At the initial time, (3.15a) becomes

$$\partial_0 g_{ij} = N_{i|j} + N_{j|i} . \tag{7.7}$$

Taking the time derivative of (4.22c), one obtains a homogeneous equation for N_i which has the solution

$$N_i(\mathbf{r}, 0) = 0 . \tag{7.8}$$

The relevant part of (3.15b) is now just the $\partial_0 \pi^{ii}$ equation. The coordinate condition (4.22d), the isotropic form of g_{ij}, and the constraint equation (7.2) simplify this equation to

$$\partial_m(\chi^2 \, \partial_m N) = \tfrac{1}{4} N\{\chi^{-2} \mathcal{E}^{iL} \mathcal{E}^{iL} + m_0 \, \delta^3(\mathbf{r})\} . \tag{7.9}$$

For the point particle, the solution for N is just (see VA):

$$N = (1 + |e|/8\pi r)^{-1} = \chi^2 . \tag{7.10}$$

Note that the total solution for $g_{\mu\nu}$, is everywhere *non-singular* (except, of course, at the particle). In VA, it was shown that this coordinate system led to a *completely* non-singular metric $g_{\mu\nu}$ initially, for a spread-out distribution of charge and mass with arbitrary m_0 and ϵ. (This is in contrast to the standard Reissner-Nordstrom metric in isotropic coordinates for which N always has a singularity for small enough ϵ.)

Further, the point charge is a stable object, since the equations (7.8) and (7.10) for $g_{\mu\nu}$ initially, lead to a static solution ($\partial_t g_{\mu\nu} = \partial_t \pi^{ij} = 0$). The cancellation of the (repulsive) electrostatic self-forces by the gravitational ones is made clear by examining the components of $\mathfrak{T}^{ij}$, the total system's spatial stress density. The conservation requirement on the total stress tensor $\mathfrak{T}^{\mu\nu}$ implies that $\mathfrak{T}^{ij}{}_{,j} = -\mathfrak{T}^{0i}{}_{,0} = 0$ for the static case, where $\mathfrak{T}^{ij}$ are the total system's spatial stresses. Thus, in the notation of the orthogonal decomposition, $\mathfrak{T}^{ij}$ reduces to $\mathfrak{T}^{ij} = \mathfrak{T}^{ijTT} + \mathfrak{T}^{ijT}$, since it is transverse. Spherical symmetry means that $\mathfrak{T}^{ijTT}$, vanishes, since no preferred transverse direction

can be distinguished. Hence, $\mathfrak{T}^{ij}$ has at most one independent component which may be taken as $\mathfrak{T}^{ii}$. For our static point solutions $\mathfrak{T}^{ii}$ (as calculated from either the Landau-Lifshitz or Einstein pseudotensors) vanishes everywhere for arbitrary m_0 and e. In the rest frame, then, $\mathfrak{T}^{\mu\nu} = \rho\, \delta_0^\mu\, \delta_0^\nu$ where $\int d^3r\, \rho = m$, the clothed mass. In a moving system, therefore, $\mathfrak{T}^{\mu\nu} = \rho(dx^\mu/d\tau)(dx^\nu/d\tau)$ (τ is Lorentz proper time). The vanishing of $\mathfrak{T}^{ij}$ in the rest frame, then, is necessary for the structure of the total stress tensor to be that of a renormalized mass m. (This requirement is stronger than the usual one that $P_\mu = \int d^3r\, \mathfrak{T}^0{}_\mu$ transform like the energy-momentum of a particle.[22]) Thus the point charge is a completely stable object, without any *ad hoc* pressure (cohesive) terms required, and its mass is completely determined by its field interactions.

7.4. *Comparison with standard Reissner-Nordstrom solutions.* It is interesting to compare our results with the standard discussion of Schwarzschild (neutral) and Riessner-Nordstrom (charged) solutions. In these approaches, no bare matter parameters were ever introduced, so that the self-energy problem could not be formulated. The source term used in the standard treatment is the Maxwell stress tensor, together with a matter source term of a perfect fluid with four-velocity U^μ:

$$T^{\mu\nu} = (\rho_{00} + p_0)U^\mu U^\nu + p_0 g^{\mu\nu} . \tag{7.11}$$

The scalar "proper rest-mass density" ρ_{00} is, as can be seen from this equation, the mass density in the locally inertial rest-frame, while p_0 is the pressure in this frame (see, for example, Møller, 1952). In our treatment, on the other hand, the matter was introduced dynamically rather than as an externally prescribed source. This was accomplished by including a Lagrangian for the particle. Comparing (7.11) with the source terms in our equations,[23] one finds that $\rho_{00} = m_0 \chi^{-6}\, \delta^3(\mathbf{r})$, which shows that ρ_{00} includes "clothing" effects through the χ^{-6}-factor. The pressure terms in the usual source tensor (7.11) are introduced expressly in order to make the distribution stable. Since a pressure term summarizes the presence of other forces (not being treated dynamically), these forces would contribute to the clothing of the original mechanical mass and hence the parameter m_0 would now represent this original mechanical mass *plus* the clothing due to the forces giving rise to the pressure term. In our analysis, where no phenomenological pressure terms are invoked, one cannot at the outset ask that a solution be static. In fact, extended initial distributions of matter and charge are not, in general, stable, although the point limit does have this property.

8. Outlook

8.1. *Discussion of quantization.* In this survey, we have seen that general relativity may be viewed as an ordinary classical field, once the meaning of the coordinate invariance has been analyzed. It was then possible to treat the dynamics of the gravitational field according to techniques common to classical field theory. In this way, many physical properties could be directly understood from their counterparts in, say, electrodynamics. For example, the canonical formalism provided a unique definition of energy and gravitational radiation. The gravitational field, of course, also has aspects not found elsewhere. In particular, its sources are the total energies of *all* other fields. The attractive, static interaction, part of the total energy provided the possibility of a compensating effect on the flat-space self-energies of other fields. Thus, a stable classical point electron of finite

---

[22]The standard static Reissner-Nordstrom solution has $\mathfrak{T}^{ii} \neq 0$ but $\int d^3r\, \mathfrak{T}^{ii} = 0$. Thus it obeys the weaker condition on P_μ but its total stress tensor is not that of a particle. This is due to the presence of the *phenomenological* pressure terms which were needed there to stabilize the particle.

[23]In VA, the comparison with (7.11) was carried out in terms of a "dust" model of the particle, that is, with a Lagrangian describing the dynamics of a continuous distribution of matter in gravitational interaction, and without phenomenological pressure forces.

mass exists when gravitation is included.

A realistic elementary particle theory must, of course, be formulated in quantum terms to determine whether the quantum self-mass is also finite. A full quantum treatment includes the quantization of the dynamical modes of the gravitational field, and not only of the effects arising due to a q-number source. Since a complete set of Poisson bracket relations among the classical canonical variables has been obtained, a correspondence principle quantization may be performed immediately by transcribing the P.B.'s into commutators. Such a quantization may indeed be valid when appropriate care of the ordering of the canonical operators in the non-linear parts of the Hamiltonian is taken. However, certain consistency criteria proper to quantum theory and extending beyond this possible ambiguity must be examined. The reduced system (involving only the canonical formalism and its two degrees of freedom) does not represent the full statement of general relativity: one also has equations to determine $g_{0\mu}$, for example, which are not part of this canonical theory as such. Furthermore, one encounters these variables as soon as one makes a Lorentz transformation from the initial frame,[24] an operation that must be allowed for any sensible quantum theory. Since the equations defining $g_{0\mu}$ are now quantum ones, one must establish the consistency of the ordering of the Hamiltonian and the $g_{0\mu}$ equations in one Lorentz frame with those in another frame. Finally, we saw classically that there was an infinite number of *a priori* equivalent sets of simple canonical variables with conserved Hamiltonians, each of which could, of course, be used as a basis for such a quantum scheme; quantum mechanically, however, the relation between these sets of variables need no longer be one of a unitary transformation due to the operator character of the variables. Hence these are no longer *a priori* equivalent starting points for quantization. The classical canonical transformations among these sets will cause the coordinates associated with one set to be functions of both the canonical variables and coordinates of the other. This would lead to the phenomenon, in the quantum theory, of the coordinates of one set being q-numbers when expressed in terms of the variables of the other set,[25] as was discussed in II and IVA.

In view of the many ambiguities which could arise in an attempt to quantize consistently at this level, it would seem more fruitful to return to the Lagrangian in four-dimensional form, *i.e.*, $\mathfrak{g}^{\mu\nu} R_{\mu\nu}[\Gamma^a{}_{\lambda\rho}]$ and try to repeat our reduction to canonical form within the framework of quantum theory. There, one can use the manifest Lorentz covariance of the original Lagrangian as an aid in proving the Lorentz covariance of the canonical quantum form that should arise. Further, the ordering ambiguities are drastically reduced: since at most cubic terms enter in this Lagrangian, one can show easily that there is a simple three-parameter family of available Hermitian quantum Lagrangians, all of which are generally covariant. These different orderings of the Lagrangian differ from each other only by double commutators, *i.e.*, by effects of order $\hbar^2$. The basic requirement of consistency between the Lagrange and Heisenberg equations of motion should single out one of these forms, since commutators of Bose fields do not contribute to Lagrange equations but presumably affect the Heisenberg ones. In an investigation based on the four-dimensional form of the quantum Lagrangian, one ultimately expects to arrive at canonical forms very similar to those obtained here.

---

[24]In electrodynamics one similarly has that the gauge functions (*i.e.*, the scalar potential and longitudinal part of the vector potential) mix with the dynamical variables (*i.e.*, the transverse parts of the vector potential) when a Lorentz transformation is made.

[25]A similar situation could arise in particle mechanics with a transformation $\bar{t} = t + f(p, q)$. Here one would not so transform, since a preferred time coordinate t has been decided upon; in gravitation, however, no such single preferred coordinate frame exists in the interior, at least classically. Whether the quantum theory may force a preferred frame through the consistency requirements is not known.

Thus the classical results should represent an excellent guide in formulating the quantum theory.[26]

8.2. *Speculations on quantum self-energy problem.* Leaving aside these technical questions on the rigor of gravitational quantization, however, one may speculate as to the quantum effects on the self-energy problem. The numerical value obtained classically for the mass m of a point particle with electronic charge, $m = e/(4\pi\gamma)^{1/2} \simeq 10^{18} m_e$, is much too large. Of course, one would not expect classical theory to give correct numerical values for masses. Any realistic discussion of self-masses must be based on quantum theory. However, if the effective flat-space cutoff ($a \sim [e/(4\pi)^{1/2}]\gamma^{1/2} \sim 10^{-34}$ cm) obtained in this paper were to hold also in quantum theory,[27] the numerical values for the mass and charge would be quite different. For example, using such a cutoff in Landau's estimate (Landau, 1956) for the renormalized charge, one finds $e_r{}^2/4\pi \approx 10^{-2}$ (essentially independent of the bare charge) as noticed by Landau. Thus, Landau obtains

$$
\begin{aligned}
e_r{}^2 &= e^2[1 + (2/3\pi)\nu(e^2/4\pi)\ln\{(\hbar/mc)/a\}]^{-1} \\
&\approx [(\nu/12\pi^2)\ln\{(\hbar/mc)/a\}]^{-1}
\end{aligned}
\tag{8.1}
$$

where $\nu \approx 10$ is essentially the number of charged fields. Equation (8.1) was obtained by summing the dominant terms in each self-energy diagram. With an effective cutoff of physical origin, the usual objections to making an estimate of what otherwise is a divergent series need no longer hold. The methods of Landau also yield a formula for the renormalized mass in terms of the bare mass and the charge.[28] In the classical theory, we saw that the bare mass did not enter. To what extent this is maintained in the quantum theory (*e.g.*, to what extent the bare mass distribution remains effectively a δ-function) is not clear. It should also be emphasized that we have assumed in the above discussion the simplest possibility, that the gravitational effects on the quantum theory can be summarized in terms of a cutoff of the type considered by Landau.

Finally, the treatment given here has not touched on several major topics of interest in the classical theory. For example, the problem of motion (Einstein–Infeld–Hoffman analysis),has not been examined from the present viewpoint. Here, it is hoped that the canonical methods will shed further light on the manner in which the coupling of particles with the various components of the metric determines their motion. A particular issue where these techniques might be relevant is the question of radiation by the moving masses. Some further topics involve spaces with non-flat boundary conditions, which arise in cosmology, as well as spaces where the topology is not Euclidian. It remains to be seen whether an approach of this type can be applied to such situations.

---

[26]It has been shown in I that, if the theory can be quantized at all, it obeys Bose statistics (as intuitively expected).

[27]An effective quantum gravitational cutoff might, on dimensional grounds, be $\sim (\gamma\hbar c^{-3})^{1/2} \sim 10^{-33}$ cm. This differs from the classical a merely by the factor $a^{1/2} \equiv (e^2/4\pi\hbar c)^{1/2}$ and would not affect the discussion in the text.

[28]This formula, $m = m_0(e^2/e_r{}^2)^{9/4\nu}$, does not make clear what relation between m, m_0, and e_r might be expected, since the estimate of (8.1) fails to determine e with sufficient accuracy.

REFERENCES

Arnowitt, R., and S. Deser, 1959, *Phys. Rev.*, **113**, 745 (I).

Arnowitt, R., S. Deser, and C.W. Misner:

    1959, *Phys. Rev.*, **116**, 1322 (II).

    1960, *Phys. Rev.*, **117**, 1595 (III).

    1960, *Nuov. Cim.*, **15**, 487.

    1960, *Phys. Rev. Lett.*, **4**, 375.

    1960, *Phys. Rev.*, **118**, 1100 (IV).

    1960, *J. Math. Phys.*, **1**, 434 (IIIA).

    1960, *Phys. Rev.*, **120**, 313 (V).

    1960, *Phys. Rev.*, **120**, 321 (VA).

    1960, *Ann. Phys.*, **11**, 116 (VB).

    1961, *Nuov. Cim.*, **19**, 668 (IVA).

    1961, *Phys. Rev.*, **121**, 1556 (IVB).

    1961, *Phys. Rev.*, **122**, 997 (IVC).

Brill, D., 1959, *Ann. Phys.*, **7**, 466.

Eisenhart, L.P., 1949, *Riemannian Geometry*, Princeton University Press, Princeton.

Lanczos, C., 1949, *The Variational Principles of Mechanics*, Toronto University Press.

Landau, L.D., 1955, in *Niels Bohr and the Development of Modern Physics*,
    Pergamon Press, London.

Misner, C.W., and J.A. Wheeler, 1957, *Ann. Phys.*, **2**, 592.

Møller, C., 1952, *The Theory of Relativity*, Oxford University Press.

Papapetrou, A., 1947, *Proc. Roy. Irish Acad.*, **51A**, 191.

Petrov, A.Z., 1954, *Sci. Not. Kazan State Univ.*, **114**, 55.

Pirani, F.A.E., 1957, *Phys. Rev.*, **105**, 1089.

Schrödinger, E., 1950, *Space-Time Structure*, Cambridge University Press.

Schwinger, J., 1951, *Phys. Rev.*, **82**, 914.

——— 1953, *Phys. Rev.*, **91**, 713.