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Christodoulou's nonlinear gravitational-wave memory: Evaluation in the quadrupole approximation

Alan G. Wiseman and Clifford M. Will

Department of Physics, McDonnell Center for the Space Sciences, Washington University, St. Louis, Missouri 63130

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Christodoulou has found a new nonlinear contribution to the net change in the wave form caused by the passage of a burst of gravity waves ("memory of the burst"). We argue that this effect is nothing but the gravitational wave form generated by the stress energy in the burst itself. We derive an explicit formula for this effect in terms of a retarded-time integral of products of time derivatives of wave-zone gravitational wave forms. The resulting effect corresponds in size to a correction 2.5 post-Newtonian orders $[O((Gm/rc^2)^{5/2})=O((v/c)^5)]$ beyond the quadrupole approximation, and is therefore negligible for all but the most relativistic of systems. For gravitational bremsstrahlung from two stars moving at 3000 $\mathrm{km\,s^{-1}}$, the effect is much less than 10^{-10} of the usual linear quadrupole wave form, while for a system of coalescing binary compact objects we estimate that the effect is of order 10^{-1} for two neutron stars.

I. INTRODUCTION

Christodoulou [1] has recently found a new nonlinear gravitational-wave effect in which the net change in the gravitational wave form caused by a passing burst has a contribution whose source is the energy in the gravitational waves themselves. Thorne [2] has interpreted this as a hitherto overlooked contribution to the net change $\Delta h_{ij}^{\rm TT}$ in the transverse-traceless part of the gravitational wave form, called gravitational-wave "memory."

It has been known for some time that gravitational-wave memory could be produced by the stress-energy-momentum of matter, for example, by two stars passing each other in a small-angle scattering event [3]. The main assumption is that before and after the event, the system can be described by a set of freely moving, gravitationally unbound bodies of mass M_A and velocity $\mathbf{v}_A$. The change in the gravitational wave form is then given by [2]

$$\Delta h_{ij}^{\rm TT}=\Delta\sum_{A=1}^{N}\frac{4M_A}{r(1-v_A^2)^{1/2}}\left[\frac{v_A^i v_A^j}{1-\mathbf{N}\cdot\mathbf{v}_A}\right]^{\rm TT}, \qquad (1)$$

where r is the distance from source to detector, $\mathbf{N}$ is a unit vector from the source to the detector, and the superscript TT denotes the transverse-traceless part (see Ref. [4] for discussion of gravitational-wave conventions; we use units in which $G=c=1$). The notation Δ denotes a difference between initial and final values, when the sources can be viewed as freely moving.

Thorne [2] pointed out that the overlooked contribution to Eq. (1) was that caused by the gravitons emitted in the burst, each graviton itself being viewed as a freely moving particle carrying stress energy and capable of producing gravitational waves. A similar effect had been previously noticed for bursts of neutrinos [5]. Thorne argued that a simple replacement of particle energy $M_A/(1-v_A^2)^{1/2}$ with an appropriate graviton energy distribution $dE/d\Omega$ and of velocity $\mathbf{v}_A$ with a $(c=1)$ unit vector $\boldsymbol{\xi}$ would give the correct contribution to the memory. The formula that resulted was

$$\Delta h_{ij}^{\rm TT}=\frac{4}{r}\int\frac{dE}{d\Omega'}\left[\frac{\xi_i'\xi_j'}{1-\mathbf{N}\cdot\boldsymbol{\xi}'}\right]^{\rm TT}d\Omega', \qquad (2)$$

where the integral is over a solid angle and $\boldsymbol{\xi}'$ is a unit vector directed from the source to $d\Omega'$. This has the same form as the result obtained by Christodoulou, using rigorous asymptotic techniques, except that Christodoulou's wave form was evaluated at infinity along an outgoing null geodesic, while Thorne's was evaluated at late time but at a fixed radius.

It is possible to give a heuristic derivation of this effect that elucidates the similarities and differences between the results of Christodoulou and Thorne. On a flat background spacetime (we ignore the asymptotic deviation of the true null cones from the flat cones), in the harmonic gauge, the relaxed Einstein equation can be written as the integral equation for the "trace-reversed" wave form $\bar{h}^{ij} = h^{ij} - h\delta^{ij}$:

$$\bar{h}^{ij}(t,\mathbf{x}) = 4\int \tau^{ij}(t',\mathbf{x}')|\mathbf{x}-\mathbf{x}'|^{-1} \times \delta(t'-t+|\mathbf{x}-\mathbf{x}'|)d^4x' , \quad (3)$$

where τ^{ij} contains the stress-energy tensor of matter and gravity, the latter in the form of terms of quadratic and higher order in $\bar{h}^{\mu\nu}$ and its derivatives. In order to focus on the contribution to τ^{ij} from the outgoing gravitational waves themselves, we subtract from τ^{ij} the matter stress energy and the contribution from the near-zone gravitational fields. This part of the source provides the usual quadrupole and post-quadrupole contributions to the wave form. The remainder, denoted $\tilde{t}^{ij}$, is thus nonzero only in the wave zone, where it can be approximated as a null radiation field with $\tilde{t}^{ij}(t',\mathbf{x}') = \tilde{t}^{00}\xi'^i\xi'^j[1+O(r^{-1})]$, where ξ' is an outgoing radial unit vector. With these assumptions, we are in effect evaluating the "bundle" term identified and studied by Blanchet and Damour [6]. Note that $r'^2\tilde{t}^{00} \equiv dL/d\Omega'$, the differential flux, which is purely a function of retarded time $u' \equiv t'-r'$ and of direction Ω'. Changing integration variables from d^4x' to $du'r'^2dr' \times d\Omega'$, integrating over r', and denoting the result $\tilde{h}_{ij}$, we find

$$\tilde{h}_{ij}(t,\mathbf{x}) = 4\int_{-\infty}^{u} du' \int \frac{dL(u',\Omega')}{d\Omega'} \frac{\xi'_i\xi'_j}{t-u'-\mathbf{x}\cdot\xi'} d\Omega' , \quad (4)$$

where $u = t-r$. We now assume that the radiation is in the form of a burst so that the range of u' in the integral is bounded, we consider a field point $(t,\mathbf{x})$ corresponding to a time after the burst has passed, and take the limit along an outgoing null ray ($r \to \infty$, u = const). The result is that $t-u' \to r$, and

$$\lim_{r\to\infty,\, u \text{ fixed}} r\tilde{h}_{ij} = 4\int_{-\infty}^{u} du' \int \frac{dL}{d\Omega'} \frac{\xi'_i\xi'_j}{1-\mathbf{N}\cdot\xi'} d\Omega' , \quad (5)$$

where $\mathbf{N} = \mathbf{x}/r$. Since we expect $\tilde{h}_{ij}$ before the burst to vanish, and since $\int dL/d\Omega' du' = dE/d\Omega'$, we obtain the nonlinear memory, Eq. (2), in the asymptotic limit used by Christodoulou. Thorne, however, considers a gravitational-wave detector at a fixed r. In such a case, one is only interested in values of u in Eq. (4) corresponding to the passage of the burst ($u \approx u'$), or for a time Δt following the burst that is much shorter than the light travel time of the burst from source to detector ($\Delta t \ll r \approx 10^3$–$10^7$ yr). Under these assumptions, $t-u' \approx \Delta t + r \approx r$, and Eq. (5) holds without the limit, and represents the contribution to the wave form during the burst caused by the nonlinear effect of the burst itself. With either interpretation, we denote the "nonlinear memory" determined from Eq. (5) by $\Delta\tilde{h}_{ij}$. Notice that for a detector at fixed r, the memory vanishes as $t \to \infty$.

The purpose of this paper is to evaluate Eq. (5) explicitly in terms of the far-zone gravitational wave forms of radiating systems, and to apply the result to the examples of gravitational bremsstrahlung and binary-star coalescence.

II. EVALUATION OF THE NONLINEAR MEMORY FOR GENERAL SYSTEMS

The differential flux $dL/d\Omega'$ is given in terms of the far-zone gravitational wave form h_{ij} by

$$dL/d\Omega' = (r^2/32\pi)\dot{h}_{ij}^{TT}\dot{h}_{ij}^{TT} \equiv (r^2/32\pi)\dot{h}_{lm}\dot{h}_{pq}(P'_{il}P'_{jm} - \tfrac{1}{2}P'_{ij}P'_{lm})(P'_{ip}P'_{jq} - \tfrac{1}{2}P'_{ij}P'_{pq}) , \quad (6)$$

where an overdot denotes a derivative with respect to retarded time, $P'_{ij} \equiv \delta_{ij} - \xi'_i\xi'_j$, $P'_{il}P'_{lj} = P'_{ij}$, $P'_{ii} = 2$ (summation over repeated spatial indices is assumed). We then write Eq. (5) in the transverse-traceless form

$$\Delta\tilde{h}_{ij}^{TT} = \frac{4}{r}\int_{-\infty}^{u} du' \left[\int \frac{dL}{d\Omega'} \frac{\xi'_i\xi'_j}{1-\mathbf{N}\cdot\xi'} d\Omega'\right]^{TT} , \quad (7)$$

where the final TT operation uses projection operators $P_{ij} = \delta_{ij} - N_iN_j$.

We now work in the quadrupole approximation for h_{ij}; in this case, h_{ij} itself does not contain any unit vectors ξ' (higher-order corrections to h_{ij} do, however). Comparing Eqs. (6) and (7), we see that we must evaluate three angular integrals of unit vectors, and take the TT part of the (ij) indices of the results. Recalling that $\delta_{ij}^{TT} = 0$ and that $(N_iB_j)^{TT} = 0$ for any B_j, we find

$$\left[\int (1-\mathbf{N}\cdot\xi')^{-1}\xi'_i\xi'_j d\Omega'\right]^{TT} = 0 , \quad (8a)$$

$$\left[\int (1-\mathbf{N}\cdot\xi')^{-1}\xi'_i\xi'_j\xi'_l\xi'_m d\Omega'\right]^{TT} = (2\pi/3)(\delta_{i(l}\delta_{m)j})^{TT} , \quad (8b)$$

$$\left[\int (1-\mathbf{N}\cdot\xi')^{-1}\xi'_i\xi'_j\xi'_l\xi'_m\xi'_p\xi'_q d\Omega'\right]^{TT} = (2\pi/3)(\delta_{(ij}\delta_{lm}\delta_{pq)})^{TT} + \pi(\delta_{(ij}\delta_{lm}N_pN_{q)})^{TT} . \quad (8c)$$

Combining Eqs. (6)–(8), we obtain

$$\Delta\tilde{h}_{ij}^{TT} = (4/r)A_{ij}^{TT} , \quad (9)$$

where

$$A_{ij} = \int (r^2/720)[\dot{h}_{ij}(17\dot{h} + \dot{h}_{lm}N^lN^m) - 26\dot{h}_{il}\dot{h}_{lj} + 2\dot{h}_{il}\dot{h}_{jm}N^lN^m]du' . \quad (10)$$

Consider a system consisting of two spinless bodies in an orbit confined to the x-y plane. In the quadrupole approximation, h_{ij} is given by $(2/r)(\ddot{I}_{ij}-\frac{1}{2}\delta_{ij}\ddot{I})$, where I_{ij} is the moment-of-inertia tensor of the source [4], and so h_{ij} has components only in the x-y plane. The observer direction $\mathbf{N}$ is described by polar angles θ and ϕ. For this case, we evaluate the "+" and "×" polarizations of the memory [for formulas, cf. for example, Eqs. (18) and (19) of Ref. [7]], and find explicitly

$$\Delta\tilde{h}_{+}=\frac{1}{120r}\int_{-\infty}^{u}[g_1(r\dot{h}_{xx})^2+g_2(r\dot{h}_{yy})^2+g_3(r\dot{h}_{xy})^2+g_4r^2\dot{h}_{xx}\dot{h}_{yy}+g_5r^2\dot{h}_{xx}\dot{h}_{xy}+g_6r^2\dot{h}_{yy}\dot{h}_{xy}]du', \tag{11a}$$

$$\Delta\tilde{h}_{\times}=\frac{1}{120r}\int_{-\infty}^{u}[g_7(r\dot{h}_{xx})^2+g_8(r\dot{h}_{yy})^2+g_9(r\dot{h}_{xy})^2+g_{10}r^2\dot{h}_{xx}\dot{h}_{yy}+g_{11}r^2\dot{h}_{xx}\dot{h}_{xy}+g_{12}r^2\dot{h}_{yy}\dot{h}_{xy}]du', \tag{11b}$$

where the angular functions are given by

$$\begin{aligned}
&g_{1(2)}=\tfrac{1}{8}[(A+2B)\sin^2\theta\mp4(2+A^2)\cos2\phi+A\sin^2\theta\cos4\phi],\\
&g_3=-\tfrac{1}{2}\sin^2\theta(3A-4B+A\cos4\phi),\quad g_4=\tfrac{1}{4}\sin^2\theta(7A-6B-A\cos4\phi),\\
&g_{5(6)}=-\tfrac{1}{2}[2(2+A^2)\sin2\phi\mp A\sin^2\theta\sin4\phi],\quad g_{7(8)}=\tfrac{1}{4}\cos\theta(\pm2B\sin2\phi-\sin^2\theta\sin4\phi),\\
&g_9=2g_{10}=\cos\theta\sin^2\theta\sin4\phi,\quad g_{11(12)}=\cos\theta(-B\cos2\phi\pm\sin^2\theta\cos4\phi),
\end{aligned} \tag{12}$$

where $A\equiv1+\cos^2\theta$, and $B\equiv5+\cos^2\theta$, and the upper (lower) sign goes with the first (second) subscript. Equations (11) and (12) can then be used to evaluate the nonlinear wave form for various orbiting systems.

III. APPLICATION TO GRAVITATIONAL BREMSSTRAHLUNG AND COALESCING BINARIES

Gravitational bremsstrahlung is the radiation emitted during the small-angle scattering of two masses which approach each other from large separations at a relative velocity v_∞, with impact parameter b (which corresponds approximately to the distance of closest approach), and with $v_\infty^2\gg m/b$, where m is the total mass. The latter inequality ensures that the gravitational deflection is small ($\Delta\theta\approx m/bv_\infty^2$). In the quadrupole approximation, for bodies initially moving in the y direction, the gravitational wave form is given by [7]

$$h^{ij}=(2\mu/r)[(\text{const})^{ij}+(m/b)(4\hat{\lambda}^{(i}\delta^{j)y}-2\cos\chi\hat{n}^i\hat{n}^j)], \tag{13}$$

where μ is the reduced mass, and where

$$\hat{\mathbf{n}}\equiv\mathbf{e}_x\cos\chi+\mathbf{e}_y\sin\chi, \tag{14a}$$

$$\hat{\boldsymbol{\lambda}}\equiv-\mathbf{e}_x\sin\chi+\mathbf{e}_y\cos\chi, \tag{14b}$$

$$\cos\chi=b(b^2+v_\infty^2u^2)^{-1/2},\quad \sin\chi=v_\infty u(b^2+v_\infty^2u^2)^{-1/2}. \tag{14c}$$

Calculating the change in h^{ij} between $u=+\infty$ and $u=-\infty$ (fixed r), and determining the TT polarizations, we obtain, for the "linear" memory,

$$\Delta h_{+}=-2(2\mu/r)(m/b)(1+\cos^2\theta)\sin2\phi, \tag{15a}$$

$$\Delta h_{\times}=-4(2\mu/r)(m/b)\cos\theta\cos2\phi. \tag{15b}$$

Now substituting Eqs. (13) and (14) into Eqs. (11) and integrating over retarded time from $-\infty$ to $+\infty$, we obtain an estimate of the "nonlinear" memory:

$$\begin{aligned}\Delta\tilde{h}_{+}=(2\mu/r)(m/b)^3\eta v_\infty(\pi/1920)[&192\cos2\phi+\sin^2\theta(1756-128\cos2\phi-50\cos4\phi)\\&-\sin^4\theta(102-32\cos2\phi-25\cos4\phi)],\end{aligned} \tag{16a}$$

$$\Delta\tilde{h}_{\times}=-(2\mu/r)(m/b)^3\eta v_\infty(\pi/960)\cos\theta[96\sin2\phi-\sin^2\theta(16\sin2\phi+25\sin4\phi)], \tag{16b}$$

where $\eta=\mu/m$.

Notice that the nonlinear memory is smaller than the linear memory by $(m/b)^2v_\infty$. In a post-Newtonian expansion of the gravitational wave form in powers of $m/b<v_\infty^2$, this would correspond to $\frac{5}{2}$ orders beyond the quadrupole approximation, which is the order at which gravitational-radiation reactions effects must explicitly show up in the far-zone wave form [8]. This makes considerable sense, since the nonlinear memory is a direct consequence of the radiated gravitational energy [6]. Since $m/b\ll v_\infty^2<1$, the nonlinear effect in bremsstrahlung is much smaller than v_∞^5 times the linear effect. For bodies moving at 3000 km s^{-1}, $v_\infty^5\approx10^{-10}$. Thus this effect is completely swamped by the linear effect for any astrophysical sources of bremsstrahlung.

For a coalescing binary system of two compact objects such as neutron stars or black holes, the nonlinear memory can be calculated in principle by integrating Eqs. (11), first over the time during which the two-body orbit decays by gravitational radiation damping, then over the highly dynamical period when the two bodies coalesce, and finally over the period when the final body emits quasinormal-mode radiation. However, a complete wave form for this process is not in hand to date, although wave forms in each regime have been studied separately [9–11]. In order to obtain a crude estimate of the size of the nonlinear memory for this process, we restrict attention to the precoalescence contribution. We use the quadrupole approximation for the h_{ij} that appear in Eq. (11) together with an orbit evolved to late times near coalescence using equations of motion valid through post$^{5/2}$-Newtonian order (thus including the orbital effects of radiation reaction). Then $\bar{h}^{ij}$ is given by

$$r\bar{h}^{ij} = 4\mu(V^iV^j - mX^iX^j/R^3),$$
$$r\dot{\bar{h}}^{ij} = -4\mu m(4V^{(i}X^{j)} - 3X^iX^j\dot{R}/R)R^{-3}, \tag{17}$$

where **V**, **X**, and R are the relative velocity, separation vector, and distance between the two bodies, respectively. Time derivatives of **V** have been evaluated using the Newtonian equations of motion. For the time evolution of these variables we substitute analytical and numerical solutions for decaying orbits studied by Lincoln and Will [9a]. We then integrate from a suitably early time, when the bodies are well separated, and the gravitational wave form is small, up to a time determined either by the onset of hydrodynamics for neutron stars, or by the failure of the post$^{5/2}$-Newtonian approximation for black holes. From $u=-\infty$ to a time corresponding to a relative separation of around $40m$, integration of an analytic approximation to the orbit [Eqs. (3.6) and (3.12) of Ref. [9a]] yields $\Delta\tilde{h}_+ \approx \frac{1}{96}(\mu/r)(m/R)\sin^2\theta(17+\cos^2\theta)$. Surprisingly, $\Delta\tilde{h}_\times$ is zero in this approximation. For $\Delta\tilde{h}_+$, the effect is as large as 20% of the quadrupole wave form $O(\mu/r)(m/R)$ because the effect has built up over a radiation-reaction time scale. Subsequent evolution of $\Delta\tilde{h}_+$ determined numerically is plotted in Fig. 1 as the ratio of $\Delta\tilde{h}_+$ to the peak amplitude of the quadrupole wave form (including postquadrupole corrections). The ratio does not increase further, and in fact decreases at late times as the quadrupole contribution increases more rapidly. Whereas the quadrupole wave form oscillates at increasing frequencies $O((m/R^3)^{1/2})$, the nonlinear wave form varies smoothly, representing a slow drift. For a gravitational-wave detector, the important quantity is the *change* in the wave form from the time the signal is acquired, thus the initial values represented in Fig. 1 are not very meaningful.

Notice that a relative separation of $7m$ corresponds to 30 km for two neutron stars, or a separation when the stars are touching, and hydrodynamical effects take over. Numerical-hydrodynamics evolution of neutron-star coalescences show wave forms and luminosity that decrease very rapidly as the hydrodynamic phase proceeds [10b], and hence the nonlinear memory evaluated at $7m$, $\Delta\tilde{h}_+ \approx 0.03\mu/r$ (about $\frac{1}{10}$ of the quadrupole wave form at that separation) probably represents a realistic upper limit. This memory will persist after the burst has past. For black holes, the post-Newtonian approximation is breaking down by separations of $5m$, and it is difficult to make reliable estimates (in harmonic coordinates, the event horizon of an isolated black hole is at m).

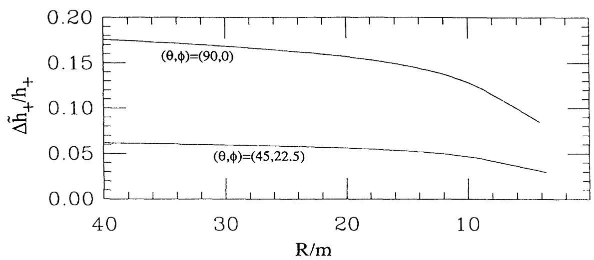

FIG. 1. Ratio of nonlinear memory $\Delta\tilde{h}_+$ to amplitude of quadrupole h_+ for binary coalescence as a function of separation, for two observation directions.

IV. CONCLUSIONS

We have argued that the Christodoulou nonlinear memory is a consequence of gravitational radiation generated by the outgoing gravitational radiation itself, and have derived an explicit formula that can be used to calculate this effect approximately for a wide variety of dynamical sources. For gravitational bremsstrahlung, the effect is utterly negligible, while for coalescing binaries, the effect builds up to a significant fraction of order $\frac{1}{10}$ of the usual quadrupole wave form. However, because it is a slowly varying drift, detecting it will pose different challenges from detecting the rapidly oscillating wave form. Furthermore, in highly relativistic systems such as coalescing neutron stars and black holes, higher-order (postquadrupole) corrections to the normal wave form will be of comparable size to or larger than the Christodoulou effect, further complicating its detectability. For example, slow drifts similar to the Christodoulou drift can be seen already in first-post-quadrupole corrections to the wave form, caused by interference between different harmonics [see Fig. 14(c) of Ref. [9a]], and in "hereditary" terms resulting from backscattering of the radiation from the curved-spacetime background [6].

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