Laws of motion and precession for black holes and other bodies

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Laws of motion and precession are derived for a Kerr black hole or any other body which is far from all other sources of gravity ("isolated body") and has multipole moments that change slowly with time. Previous work by D'Eath and others has shown that to high accuracy the body moves along a geodesic of the surrounding spacetime geometry, and Fermi-Walker transports its angular-momentum vector. This paper derives the largest corrections to the geodesic law of motion and Fermi-Walker law of transport. These corrections are due to coupling of the body's angular momentum and quadrupole moment to the Riemann curvature of the surrounding spacetime. The resulting laws of motion and precession are identical to those that have been derived previously, by many researchers, for test bodies with negligible self-gravity. However, the derivation given here is valid for any isolated body, regardless of the strength of its self-gravity. These laws of motion and precession can be converted into equations of motion and precession by combining them with an approximate solution to the Einstein field equations for the surrounding spacetime. As an example, the conversion is carried out for two gravitationally bound systems of bodies with sizes much less than their separations. The resulting equations of motion and precession are derived accurately through post$^{1.5}$-Newtonian order. For the special case of two Kerr black holes orbiting each other, these equations of motion and precession (which include couplings of the holes' spins and quadrupole moments to spacetime curvature) reduce to equations previously derived by D'Eath. The precession due to coupling of a black hole's quadrupole moment to surrounding curvature may be large enough, if the hole lives at the center of a very dense star cluster, for observational detection by its effects on extragalactic radio jets. Unless the hole rotates very slowly, this quadrupole-induced precession is far larger than the spin-down of the hole by tidal distortion ("horizon viscosity"). When the hole is in orbit around a massive companion, the quadrupole-induced precession is far smaller than geodetic precession.

I. INTRODUCTION AND OVERVIEW

A. The problem of motion in general relativity

The problem of how a body moves through spacetime has long been of central importance in general relativity, both as an issue of principle and as a foundation for observational predictions. The equivalence principle guarantees that test bodies of vanishingly small mass move along geodesics of spacetime, and that if they have vanishingly small spin angular momentum, they Fermi-Walker transport it with themselves as they move. A body whose mass, angular momentum, and other multipole moments are finite but small on scales set by the surrounding spacetime will suffer corrections to this geodesic motion and Fermi-Walker spin transport, and in realistic situations those corrections can be observationally important. For example, they are responsible for the "general precession" of the Earth's spin axis [precession of the equinoxes; see, e.g., Exercise 16.4 of Misner, Thorne, and Wheeler, hereafter referred to as MTW (Ref. 1)]. As another example, the corrections can cause a precession of the jets emerging along the spin axis of a supermassive black hole at the center of a very dense star cluster (Sec. V of this paper).

While the history of the problem of motion in general relativity is a long one (for overviews see Refs. 2—4), the problem of the motion of a body with strong internal gravity (black hole or neutron star) has been investigated only relatively recently: D'Eath[5] and Kates[3] have established that for a strongly gravitating body, as for a test particle, if the mass and spin are vanishingly small, then the motion and transport are geodesic and Fermi-Walker (see also Manasse,[6] and Demianski and Grishchuk[7]). However, the corrections to geodesic motion and Fermi-Walker transport seem not to have been studied, with three exceptions: D'Eath[8] has studied the largest corrections for the special case of a binary black-hole system; Damour[9] has done the same for the motion, but not precession, of the bodies in any compact binary system; and Dixon[10] (see also Ehlers and Rudolph[11]) has developed an elegant treatment of all the corrections for bodies with nonsingular, matter interiors (not black holes), but a treatment in which the center-of-mass world line and the multipole moments are not tied in any as yet known way to observations that an external observer can make.

The purpose of this paper is to derive the leading corrections to the laws of geodesic motion and Fermi-Walker transport for any strongly gravitating body, in a form that can make contact with external observations (Secs. II and III), and to then convert those laws into ex-

plicit equations of motion and precession for the special case of a system of several bodies with sizes small compared to their separations and relative speeds small compared to the speed of light (Sec. IV). By "equations of motion and precession" we mean differential equations for the world line and angular momentum of the body, complete and ready for solution. By "laws of motion and precession" we mean general expressions for the rate of change of the body's momentum and angular momentum in terms of a coupling of the body's multipole moments to the curvature of the external universe. The distinction between laws of motion and equations of motion was introduced by Havas and Goldberg[12] and will be discussed in further detail below.

B. Approximation methods used in this paper

From the outset it must be clear that, in dealing with the problem of motion in a form that makes contact with external observations, there can be no discussion without approximation. Already implicit in the description of the problem is the idea that one can separate spacetime into a part which represents the body and a part which represents the spacetime of the external universe. This can be done approximately if the size of the body is small compared to the characteristic length scales of the surrounding spacetime ("isolated body"). One would expect, in the same approximation, to be able to define the mass, momentum, and angular momentum of the body and to derive laws of motion for their rates of change. On the other hand, for a general spacetime with no large difference of body length scales and external length scales, there is no possible separation into body plus external universe, no body energy, momentum, or angular momentum, and no equations of motion. All one can do is solve the Einstein equations for the fully coupled system of body plus universe.

The definitions of total mass, momentum, and angular momentum of a system that resides alone in an asymptotically flat spacetime have received considerable attention[13] because they are simple and accessible to rigorous methods. There is no evidence, however, that the real universe is asymptotically flat and no possibility of making observations at infinity even if it were. It is frequently the case, however, that portions of the spacetime encountered in nature are characterized by two different length scales—the "short" length scales of a "body" and the much longer length scale of the curvature of the "external universe." The problem of interest in this case is the *approximate* definition of the mass, momentum, and angular momentum of the body, and the evaluation of *approximate* expressions for their rates of change. This problem will be considered in this paper.

The external gravitational field of a moving and precessing body may be characterized by multipole moments[14] whose magnitudes depend on two length scales

$$M \equiv (\text{mass of body}) , \quad L \equiv (\text{size of body}) \gtrsim M ; \tag{1.1}$$

specifically, the magnitudes of the l-pole moments are $\lesssim ML^l$. These multipole moments can vary with time due to rotation, precession, or internal changes of structure such as starquakes; but we insist that they vary slowly,

$$T \equiv (\text{time scale for changes of moments}) \gg L . \tag{1.2a}$$

For a Kerr black hole with mass M and dimensionless rotation parameter $\chi \equiv (\text{angular momentum})/M^2 \leq 1$, L is approximately M, and T is infinite (except for very slow precession effects). For a neutron star with a nonaxisymmetric density distribution, rotating with angular velocity Ω, $L \simeq (\text{radius of star})$ and $T \simeq 1/\Omega$.

Each body studied in this paper will be presumed *"isolated"* in the following sense: It lives in a possibly complex external universe, but its immediate vicinity will be presumed devoid of matter and nongravitational fields. The external universe near the body will have a vacuum Riemann curvature tensor characterized by three length scales $\mathscr{R}$, $\mathscr{L}$, and $\mathscr{T}$,

$$\begin{aligned} \mathscr{R} &= (\text{radius of curvature}) , \\ \mathscr{L} &= (\text{inhomogeneity scale}) , \\ \mathscr{T} &= (\text{time scale for changes of curvature}) ; \end{aligned} \tag{1.3}$$

and the body is presumed *"isolated"* in the sense that all these scales are large compared to the body's size L,

$$\mathscr{R} \gg L , \quad \mathscr{L} \gg L , \quad \mathscr{T} \gg L . \tag{1.2b}$$

The three length scales $\mathscr{R}$, $\mathscr{L}$, and $\mathscr{T}$ are defined more precisely in terms of the components of the Riemann curvature tensor of the external universe in the neighborhood of the isolated body's world tube. In the local, asymptotic rest frame of the body, with nearly locally Lorentz coordinates, the Riemann tensor can be split into "electric" and "magnetic" parts (Ref. 15 and the Appendix of this paper):

$$\mathscr{E}_{jk} = R_{j0k0} , \quad \mathscr{B}_{jk} = \tfrac{1}{2}\epsilon_{jpq}R^{pq}{}_{k0} , \tag{1.4a}$$

where latin indices denote spatial components, the index 0 denotes a temporal component, and ϵ_{jpq} is the three-dimensional, flat-space, spatial Levi-Cività tensor. (Throughout we shall use the notation and sign conventions of MTW, including setting $G=c=1$.) In terms of these two parts of the external curvature, the external length scales are defined by

$$\begin{aligned} &\mathscr{E}_{jk} \sim \mathscr{B}_{jk} \sim \frac{1}{\mathscr{R}^2} , \quad \mathscr{E}_{jk,i} \sim \mathscr{B}_{jk,i} \sim \frac{\mathscr{E}_{jk}}{\mathscr{L}} , \\ &\mathscr{E}_{jk,0} \sim \mathscr{B}_{jk,0} \sim \frac{\mathscr{E}_{jk}}{\mathscr{T}} , \end{aligned} \tag{1.4b}$$

where commas denote derivatives. For example, if the body of interest is in a binary system with another ("external") body of mass M_E, then $\mathscr{L}$ is the separation between the two bodies, $\mathscr{R} \simeq (\mathscr{L}^3/M_E)^{1/2}$ is the radius of curvature of spacetime at distance $\mathscr{L}$ from the other body; and $\mathscr{T}$ is of order $1/2\pi$ times the orbital period, $\mathscr{T} \simeq [\mathscr{L}^3/(M+M_E)]^{1/2}$. Typically, as in this case, $\mathscr{L}$ is the shortest of the external length scales, $\mathscr{L} \lesssim \mathscr{R}$ and $\mathscr{L} \lesssim \mathscr{T}$, and $\mathscr{B}_{jk}$ is much smaller than $\mathscr{E}_{jk}$ (though we treat it formally as of the same magnitude). The "isola-

tion" requirements $\mathscr{R} \gg L$, $\mathscr{L} \gg L$, and $\mathscr{T} \gg L$ imply that, if significantly strong gravitational waves are impinging on the body from the external universe, their reduced wavelength $\lambda\!\!\!^{-} \equiv \lambda/2\pi \simeq \mathscr{L} \simeq \mathscr{T}$ must be large compared to the size of the body.

For an isolated body one can split spacetime up into three regions: The "body's neighborhood," which is a world tube surrounding it and extending, as measured in the body's local asymptotic rest frame, out to some radius $r_I \gg L$; a "buffer region" extending from radius r_I to some larger radius $r_0 \ll \mathscr{L} \lesssim (\mathscr{R}$ and $\mathscr{T})$; and the "external universe" located outside radius r_0. In the body's neighborhood its own gravitational effects dominate, but in the buffer region and the external universe the gravitational effects of other bodies are important. As Weyl[16] and Einstein and Grommer[17] realized, the vacuum Einstein equations in the buffer region determine the motion and precession of the body. Modern variants of their analyses involve separate mathematical treatments of the external universe and the body's neighborhood, and a matching of those treatments in the buffer region ("method of matched asymptotic expansions," introduced into this subject by Manasse and Wheeler[6] and expounded, e.g., in Refs. 3–5 and 18, Sec. 20.6 of MTW,[1] and Sec. II of this paper).

In our variant of the matching (Secs. II and III) we will expand the spacetime metric as a function of radius r in the buffer region in several power series: One power series characterizes the gravitational effects of the body and thus is an expansion ("body expansion") in powers of M/r, L/r, and r/T (strength of gravity, distance from source, and distance to wave zone), with expansion coefficients that are the body's multipole moments and their time derivatives. Another power series characterizes the gravitational effects of the external spacetime and thus is an expansion ("external-universe expansion") in powers of $r/\mathscr{R}$, $r/\mathscr{L}$, and $r/\mathscr{T}$, with expansion coefficients that are the external Riemann tensor and its derivatives. A third power series characterizes the gravitational interactions between the body and the external spacetime and thus is an expansion ("interaction expansion") in powers of M/r, L/r, and r/T simultaneously with $r/\mathscr{R}$, $r/\mathscr{L}$, and $r/\mathscr{T}$. The inner and outer edges of the buffer region r_I and r_0 are chosen such that all of the dimensionless expansion parameters M/r, L/r, r/T, $r/\mathscr{R}$, $r/\mathscr{L}$, and $r/\mathscr{T}$ are small throughout the buffer region. The interaction expansion can be derived from the body expansion and external-universe expansion by iterating the Einstein field equation. From certain portions of the interaction expansion, which are embodied in certain "conservation-law" surface integrals, we shall infer the laws of motion and precession of the body.

Analogous matching techniques have been used previously to study special cases of the motion and precession of compact bodies (Refs. 3–9; see Sec. I A above). However, in this paper we shall aspire to a level of generality not previously attempted.

C. Summary of the main results of this paper

In this paper we shall derive the largest deviations from geodesic motion and Fermi-Walker transport for an "isolated" black hole ($\mathscr{R} \gg L$, $\mathscr{L} \gg L$, $\mathscr{T} \gg L$), or any other isolated body with slowly changing moments ($T \gg L$), moving through an arbitrary external spacetime. The force and torque that produce the deviations from geodesic motion and Fermi-Walker transport will arise from couplings of the body's multipole moments to the external spacetime curvature and to the curvature's spatial gradients. The external curvature will be characterized by the values of its "electric" and "magnetic" parts $\mathscr{E}_{jk}$ and $\mathscr{B}_{jk}$ at the body's location (or, more precisely, in the buffer region around the body). Note that if the external spacetime is nearly Newtonian, its curvature can be expressed as

$$\mathscr{E}_{ij} \cong \Phi_{,ij} \ , \quad \mathscr{B}_{ij} \cong -\tfrac{1}{2} H_{(i,j)} \ , \tag{1.5}$$

where Φ is the Newtonian potential, H_i is the deDonder-gauge "gravitomagnetic field" $H_i = \epsilon_i{}^{jk} g_{0k,j}$, and the parentheses denote symmetrization. The relevant multipole moments of the body will be its intrinsic angular momentum $\mathscr{S}_j$, its mass quadrupole moment $\mathscr{I}_{jk}$, and its current quadrupole moment $\mathscr{S}_{jk}$. As discussed, e.g., in Ref. 14, these moments are symmetric, trace-free, spatial tensors that characterize the body's external gravitational field. These moments are defined in the body's "local, asymptotic rest frame," which is a coordinate system in the buffer region that is as nearly globally inertial and Lorentz as possible and in which the body is momentarily at rest. Our conventions for defining the quadrupole moments will be those of Thorne,[14] which for nearly Newtonian bodies in Cartesian coordinates reduce to

$$\mathscr{I}^{jk} = \left[\int \rho x^j x^k d^3x \right]^{\rm STF} ,$$
$$\mathscr{S}^{jk} = \left[\int x^j \epsilon^k{}_{pq} x^p \rho v^q d^3x \right]^{\rm STF} . \tag{1.6}$$

Here ρ is density, v^q is velocity, and the superscript STF means "symmetrize and make trace-free." For a Kerr black hole with spin axis in the direction $\vec{s}$ (with $\vec{s}$ a unit spatial vector in the hole's asymptotic, local rest frame), the angular momentum and quadrupole moments are (Sec. XI D of Ref. 14)

$$\vec{\mathscr{S}} = M^2 \chi \vec{s} \ , \quad \mathscr{I}_{jk} = M^3 \chi^2 (\tfrac{1}{3} \delta_{jk} - s_j s_k) \ , \quad \mathscr{S}_{jk} = 0 \ . \tag{1.7}$$

Here $\chi \leq 1$ is the hole's dimensionless rotation parameter (usually denoted a/M).

The mass M, momentum P_i, and angular momentum $\mathscr{S}_i$ of the body can be inferred from the body's metric in the buffer region. This can be done precisely only up to the uncertainties introduced by the presence of the external universe. We shall show in Sec. III F that those uncertainties have magnitudes

$$(\text{uncertainty in } M) \sim \frac{ML^2}{\mathscr{R}^2} \ , \tag{1.8a}$$

$$(\text{uncertainty in } P^i) \sim \frac{ML^2}{\mathscr{R}^2} \ , \tag{1.8b}$$

$$(\text{uncertainty in } \mathscr{S}^i) \sim \frac{M^3 L}{\mathscr{R}^2} \ . \tag{1.8c}$$

One way to think about these uncertainties is this: Different physicists, motivated by particular applications or mathematical convenience, might try to define in different precise manners the mass, momentum, and angular momentum of a body in an external universe. If these different definitions are all to agree with the standard definitions in the limit as the external universe becomes flat, then they will differ from each other by amounts of order (1.8).

Equations for the rates of change of the momentum and angular momentum, when expressed in terms of the multipole moments of the body and the curvature of the external universe, are the body's laws of motion and precession. The derivation of these laws can be given at several levels of rigor. At the lowest level there is the following simple argument.

The rates of change of momentum and angular momentum of a body could be evaluated by integrating expressions for the momentum flux and angular momentum flux over a closed two surface in the buffer region. What would be used in these integrations is the external universe's buffer-zone gravitational field, which is fully characterized by its curvature, and the body's buffer-zone gravitational field, which is fully characterized by its multipole moments.[14] Now, only the values of the body's multipole moments are important for the gravitational field, not their source. In particular these values carry no information as to whether the source is compact or diffuse. Thus, expressed in terms of the multipole moments and external curvature, the laws of motion and precession for a strongly relativistic body must be the same as for a nearly Newtonian one with negligible self-gravity. These weak-gravity laws of motion and precession are well known and extensively studied,[19–21] though their usual derivations are not via the above surface-integral route.

At a higher level of rigor, one can derive the laws of motion and precession by carrying out in detail the surface-integral calculation that is only contemplated in the above argument. We shall do this in Sec. III.

Our surface-integral calculation is surely not endowed with the ultimate of rigor. We believe that it would be of interest to give a discussion of the laws of motion and precession with rigor more nearly like that of current studies of asymptotically flat spacetimes,[13] and at several points in this paper we will try to indicate routes by which this might be achieved.

Our surface-integral derivation of the laws of motion and precession as presented in Sec. III is made conceptually complicated by its use of expansions in six independent dimensionless parameters: M/r, L/r, r/T, $r/\mathscr{R}$, $r/\mathscr{L}$, and $r/\mathscr{T}$. To build up confidence in the conceptual foundations of our method, we precede these computations of Sec. III by an analysis restricted to a Kerr black hole (Sec. II). In that analysis $L \simeq M$, $T=\infty$, and we formally regard $\mathscr{T} \sim \mathscr{L} \sim \mathscr{R}$, so there are only two dimensionless expansion parameters: M/r and $r/\mathscr{R}$. This simplification permits us to describe in a simple manner the matched-asymptotic-expansion conceptual basis for our computations.

Not surprisingly, in view of the "simple argument" above, the results of our calculations in Secs. II and III are the same as for a body with negligible or weak self-gravity:[19–21] As measured in its local asymptotic rest frame, the body's mass is conserved,

$$\frac{dM}{dt} \ll \frac{ML}{\mathscr{R}^2} ; \tag{1.9a}$$

its linear momentum (which is momentarily zero) changes at the rate

$$\frac{dP^j}{dt} = -\mathscr{S}^a \mathscr{B}_a{}^j ; \tag{1.9b}$$

and its angular momentum changes at the rate

$$\frac{d\mathscr{S}^j}{dt} = -\epsilon^j{}_{ab} \mathscr{I}^a{}_c \mathscr{E}^{cb} - \tfrac{4}{3} \epsilon^j{}_{ab} \mathscr{S}^a{}_c \mathscr{B}^{cb} . \tag{1.9c}$$

We shall call Eq. (1.9b) the "law of motion" and Eq. (1.9c) the "law of precession" for the body as it moves through the external universe. For a Kerr black hole, with moments given by Eqs. (1.7), these laws of motion and precession reduce to

$$\frac{dP^j}{dt} = -M^2 \chi s^a \mathscr{B}_a{}^j , \tag{1.9b'}$$

$$\frac{d\mathscr{S}^j}{dt} = \epsilon^j{}_{ab} \Omega_T^a \mathscr{S}^b , \quad \Omega_T^a \equiv -\mathscr{E}^a{}_c M \chi s^c . \tag{1.9c'}$$

The quantity $\vec{\Omega}_T$ is called the "angular velocity of torqued precession."

Note the magnitudes of the time changes (1.9):

$$\frac{dM}{dt} \ll \frac{ML}{\mathscr{R}^2} , \quad \frac{dP^j}{dt} \sim \frac{ML}{\mathscr{R}^2} , \quad \frac{d\mathscr{S}^j}{dt} \sim \frac{ML^2}{\mathscr{R}^2} . \tag{1.10}$$

Comparison with Eqs. (1.8b) and (1.8c) shows that the change in P^j need only evolve for a time $\Delta t \sim 100L$ and the change in $\mathscr{S}^j$ for a time $\Delta t \sim 100M^2/L$ to become much larger than the uncertainties in the definitions of P^j and $\mathscr{S}^j$. In this sense a meaning can be given to the laws of motion and precession even in the face of uncertainties in the quantities whose rates of change they describe.

In the law of precession (1.9c), the first term (coupling of mass quadrupole to electric-type curvature) will almost always produce the dominant torque, and always will do so for a black hole [cf. Eqs. (1.7) and (1.9c')], but occasionally the second term (coupling of current quadrupole to magnetic-type curvature) may be comparably strong.

For an isolated Kerr black hole, as for an isolated, rigidly rotating, perfect-fluid body, the spin axis and the angular-momentum direction always coincide. Consequently, at the level of precision of Eq. (1.9c') for $d\mathscr{S}^j/dt$, in which the distortion of the black hole by the external universe is unimportant, the angular velocity of precession $\vec{\Omega}_T$ of $\mathscr{S}^j$ is also the angular velocity of precession of the spin axis. Thus, a black hole does not nutate at this leading order of accuracy. It would be interesting to know whether nutation occurs at higher orders.

The laws of motion and precession [Eqs. (1.9b) and (1.9c)] can be rewritten in frame-independent geometric

language by regarding the body's multipole moments and the external curvatures as four-tensors orthogonal to the body's four-momentum:

$$P^\alpha{}_{|\beta}u^\beta = -\mathscr{B}^{\alpha\beta}\mathscr{S}_\beta \,, \tag{1.11a}$$

$$\mathscr{S}_{\alpha|\beta}u^\beta = -\epsilon_{\mu\alpha\beta\gamma}\mathscr{I}^{\beta\delta}\mathscr{E}_\delta{}^\gamma u^\mu - \tfrac{4}{3}\epsilon_{\mu\alpha\beta\gamma}\mathscr{S}^{\beta\delta}\mathscr{B}_\delta{}^\gamma u^\mu \,. \tag{1.11b}$$

Here all quantities are vectors and tensors in the external spacetime: P^α is the body's four-momentum; $u^\alpha \equiv P^\alpha/M$ is its four-velocity; the vertical bar denotes a covariant derivative with respect to the metric of the external spacetime; $\epsilon_{\mu\alpha\beta\gamma}$ is the Levi-Cività tensor; and $\mathscr{S}_\alpha$, $\mathscr{I}_{\alpha\beta}$, $\mathscr{S}_{\alpha\beta}$, $\mathscr{E}_{\alpha\beta}$, and $\mathscr{B}_{\alpha\beta}$ are the quantities $\mathscr{S}_j$, $\mathscr{I}_{jk}$, $\mathscr{S}_{jk}$, and $\mathscr{B}_{jk}$ viewed as four-tensors orthogonal to u^α, which reside on the body's world line in the external spacetime.

If one wishes to integrate these equations to determine the detailed motion and precession of the body, one must first augment them by two things: (i) "constitutive relations" for the body which determine $\mathscr{I}_{\alpha\beta}$ and $\mathscr{S}_{\alpha\beta}$ in terms of P^α and $\mathscr{S}^\alpha$ [for a Kerr black hole the constitutive relations are Eqs. (1.7), and for a rigidly but slowly rotating body they are discussed by Thorne and Gürsel[22]]; and (ii) a description of the external universe which is sufficiently detailed to permit location of the body on a specific world line and to permit computation of the curvature $\mathscr{E}_{\alpha\beta}$, $\mathscr{B}_{\alpha\beta}$ along that world line. When so augmented, the "laws of motion" (1.11) become concrete "differential equations of motion" for the body's world line. In some cases the external universe will be insensitive to gravitational "back-action" forces from the body, and as a result the computation of $\mathscr{E}_{\alpha\beta}$ and $\mathscr{B}_{\alpha\beta}$ will be straightforward; an example is an external universe consisting of a single very massive object ($M_E \gg M$) about which the body of interest orbits (see Secs. IV and V). As we shall see at the end of Sec. IV B, in this test-body case the effects of coupling to external curvature are of the same negligible magnitude as the effects of back action. In other cases the body's gravity may strongly influence the external universe, thus requiring that the evolution of the external universe and the motion and precession of the body be solved as a coupled problem by self-consistent methods. Section IV will present an example of this: the motions and precessions of a system of several or many bodies all of comparable mass, but with sizes L small compared to their separations $\mathscr{L}$, and velocities small compared to the velocity of light, computed to "post$^{1.5}$-Newtonian order" [fractional corrections $\sim(M/\mathscr{L})^{3/2}$ beyond Newtonian]. Not surprisingly, although the bodies may be black holes or neutron stars, the equations of motion and precession for the system have the standard form that has been derived previously for bodies with weak internal gravity.[19–21] When specialized to a black-hole binary system these equations of motion and precession reduce to those of D'Eath.[8]

The forces and torques in the laws of motion (1.9) are just the leading terms in a power-series expansion in internal length scales (M and L) over external length scales ($\mathscr{R}$, $\mathscr{L}$, and $\mathscr{T}$), and also in M/T and L/T. The higher-order forces and torques are all small compared to those of Eqs. (1.9) for the case of a black hole or a rotationally distorted star interacting with a generic external gravitational field. There are situations, however, when the higher-order contributions to the laws of motion become important. We shall give a few examples.

For a body that is surrounded by purely electric-type gravity ($\mathscr{B}_{ab}=0$) or a body that is rotating very slowly ($\mathscr{S}_a \cong 0$) and is distorted by internal stresses ($\mathscr{I}_{ab}\neq 0$), the force (1.9b), which is of order $ML/\mathscr{R}^2$ in our expansion, may be strongly suppressed compared to the force

$$\frac{dP^j}{dt} = -\tfrac{1}{2}\mathscr{E}^j{}_{ab}\mathscr{I}^{ab} \sim \frac{ML^2}{\mathscr{R}^2\mathscr{L}} \tag{1.12}$$

(where $\mathscr{E}_{jab}$ is defined below). We will not derive this force in this paper, but, as Zhang[23] shows, it can be derived by a higher-order iteration of the techniques of Sec. III or, alternatively, by Newtonian considerations. [The "simple argument" following Eqs. (1.8) guarantees that this force will have precisely the same form as in Newtonian theory even if the body is strongly relativistic.] This force arises from a coupling of the body's mass quadrupole moment $\mathscr{I}_{ab}$ to the "electric-type octupole moment" $\mathscr{E}_{jab}$ of the external Riemann curvature. That octupole moment is equal to the fully symmetrized, "electric" part of the gradient of the external Riemann tensor [Eq. (A9b) of the Appendix]

$$\mathscr{E}_{ijk} = (R_{i0j0|k})^S \equiv \tfrac{1}{3}(R_{i0j0|k} + R_{j0k0|i} + R_{k0i0|j}) \,, \tag{1.13a}$$

and in Newtonian theory it is given by

$$\mathscr{E}_{ijk} = \frac{\partial^3\Phi}{\partial x^i \partial x^j \partial x^k} \,. \tag{1.13b}$$

At the level of accuracy of our calculations, $O(ML/\mathscr{R}^2)$, the body's total mass-energy M is conserved. However, as we shall show in Sec. III E [Eq. (3.14a)], if we were to carry our calculations to higher accuracy for a body with $T \ll \mathscr{R}^2/M$ or $\mathscr{T} \ll \mathscr{R}^2/M$, we would find that the largest nonzero contributions to dM/dt have the forms and magnitudes

$$\begin{aligned}\frac{dM}{dt} &= \mu_1 \mathscr{E}_{jk}\frac{d\mathscr{I}^{jk}}{dt} + \mu_2 \mathscr{I}^{jk}\frac{d\mathscr{E}_{jk}}{dt} + \mu_3 \mathscr{B}_{jk}\frac{d\mathscr{S}^{jk}}{dt} \\ &\quad + \mu_4 \mathscr{S}^{jk}\frac{d\mathscr{B}_{jk}}{dt} \\ &\sim \frac{ML^2}{\mathscr{R}^2 T} + \frac{ML^2}{\mathscr{R}^2\mathscr{T}} \,,\end{aligned} \tag{1.14}$$

where the μ_j are constants of order unity. If the body's moments $\mathscr{I}_{jk}$ and $\mathscr{S}_{jk}$ are constant while $\mathscr{E}_{jk}$ and $\mathscr{B}_{jk}$ oscillate on a time scale $\mathscr{T}$, the mass M will oscillate with an amplitude of order the uncertainties (1.8a) in its own definition, $\Delta M \sim ML^2/\mathscr{R}^2$. The only way the changes (1.14) can grow large enough to exceed the uncertainties $ML^2/\mathscr{R}^2$ in the definition of M is if the body's moments and the external curvature oscillate in approximate resonance with each other, i.e., if $\mathscr{T} \sim T$. For example, the body's mass quadrupole moment $\mathscr{I}_{jk}$ might oscillate sinusoidally due to its rotation,[22] and the curvature $\mathscr{E}_{jk}$ might oscillate at roughly the same frequency due to a passing gravitational wave (this is the case for a "mechanical heterodyne detector" of gravitational waves[24]). Since Eq. (1.14) is physically significant only when $\mathscr{T} \sim T$ and

when allowed to build up for a time $\Delta t \gg T \sim \mathcal{T}$, only its average, $\langle dM/dt \rangle$, over times $\Delta t \gg T \sim \mathcal{T}$ has significance. Zhang[23] has computed explicitly the coefficients $\mu_1 - \mu_2$ and $\mu_3 - \mu_4$ appearing in that time average and has shown that they are the same as one would compute for a weakly gravitating body using Newtonian theory (for the $\mathcal{I}^{jk}$ terms) and linearized theory (for both the $\mathcal{I}^{jk}$ terms and the $\mathcal{S}^{jk}$ terms):

$$\left\langle \frac{dM}{dt} \right\rangle = -\frac{1}{2}\left\langle \mathcal{E}_{jk} \frac{d\mathcal{I}^{jk}}{dt} \right\rangle - \frac{2}{3}\left\langle \mathcal{B}_{jk} \frac{d\mathcal{S}^{jk}}{dt} \right\rangle . \tag{1.15}$$

If the body's quadrupole and higher-order moments are varying due to pulsation or nonaxisymmetric rotation, those variations will produce gravitational waves, and radiation reaction will induce time changes of the body's mass, momentum, and angular momentum [Eqs. (4.16′), (4.20′), and (4.23′) of Ref. 14, with signs reversed]. Those changes can be derived by the same technique of surface integrals in the buffer region as we use here to derive the laws of motion and precession (1.9) (Sec. 3.3.2 of Ref. 25). However, since this paper deals with coupling to external curvature, we shall ignore radiation reaction throughout it.

The remainder of this paper contains our two derivations of the laws of motion and precession (1.9) (Sec. II for a Kerr black hole, Sec. III for a general isolated body), our conversion of those laws of motion and precession into equations of motion and precession for a several-body system (Sec. IV), and some concluding remarks about astrophysical applications (Sec. V).

II. DERIVATION OF LAWS OF MOTION AND PRECESSION FOR A KERR BLACK HOLE

The work of D'Eath,[5,8] Kates,[3,18] and Damour[4] has shown that the method of matched asymptotic expansions is a powerful tool for deriving laws of motion and equations of motion for strongly relativistic systems such as black holes. In this section we shall review this method and use it to derive the corrections to geodesic motion and Fermi-Walker spin transport for a rotating black hole.

A. The spacetime of a black hole in an external universe

In a frame where it is momentarily at rest an isolated Kerr black hole may be characterized by a single dimensional parameter, its mass M, and by its dimensionless spin $\chi = a/M$ and its spin direction s^j. A spacetime which could be described as a black hole moving in an external universe should be characterized by two sets of length scales: First, by the scale of the black hole M, later to be identified with its mass; and second, by length scales $\mathcal{R}$=(radius of curvature), $\mathcal{L}$=(spatial inhomogeneity scale), $\mathcal{T}$=(time scale for changes of curvature) characterizing the external universe. To simplify the discussion, we shall regard $\mathcal{R}$, $\mathcal{L}$, and $\mathcal{T}$ as formally having the same magnitude, so for asymptotic-expansion purposes there is only one external length scale, $\mathcal{R} \equiv \min(\mathcal{R}, \mathcal{L}, \mathcal{T})$.

Can we identify a mass, momentum, and angular momentum for the black hole in such a spacetime? Mass, momentum, and angular momentum are quantities which can be *precisely* associated only with a system that resides alone in an asymptotically flat spacetime, and this is not the case for the black hole under discussion. When M is much smaller than $\mathcal{R}$, however, we can identify a region which is simultaneously far inside the external matter in the sense of having a scale much smaller than $\mathcal{R}$ and far outside the black hole in the sense of being at distances from it much larger than its scale M. This is the buffer region described in the Introduction, and it plays the role of an *approximate* asymptotically flat region for the body. In it we expect to be able to define *approximate* notions of mass, momentum, and angular momentum which become more and more precise as M becomes small compared to $\mathcal{R}$. We shall now show how to do this. For economy we shall continue to denote the spacetime's black hole parameters by M, χ, and s^j, understanding that though they are precisely defined, they represent mass, angular momentum, and spin direction only in the approximate sense described above.

Consider the family of solutions to Einstein's equation, generated by making M smaller and smaller and keeping $\mathcal{R}$ fixed. As M becomes much smaller than $\mathcal{R}$ we can identify a region much smaller in scale than $\mathcal{R}$ in which the spacetime metric can be expanded as

$$g = g^{[0]} + \mathcal{R}^{-1} g^{[1]} + \mathcal{R}^{-2} g^{[2]} + \cdots , \tag{2.1}$$

where $g^{[0]}$ is the Kerr metric and the succeeding terms represent the corrections due to the external matter. As we shall see below, the $\mathcal{R}^{-1} g^{[1]}$ term vanishes. Numbers for the mass M, momentum P^i, and angular momentum $\mathcal{S}^i$ of the black hole can be extracted from $g^{[0]}$ by fitting it with a Kerr geometry having these parameters. This procedure does not yield a precise identification of a mass, momentum, and angular momentum for the hole because the expansion (2.1) is not unique: One can move pieces out of the $\mathcal{R}^{-2} g^{[2]}$ part of the metric and into $g^{[0]}$, thereby changing the M, P^i, and $\mathcal{S}^i$ attributed to the hole. This implies that there are uncertainties of magnitude $\Delta M \sim M^3/\mathcal{R}^2$, $\Delta P^i \sim M^3/\mathcal{R}^2$, and $\Delta \mathcal{S}^i \sim M^4/\mathcal{R}^2$ in the definitions of the body's mass, momentum, and angular momentum (Sec. II E below).

The procedure for extracting the mass, momentum, and angular momentum of the hole from the metric can be made more definite by considering in the buffer region the surface integrals of a pseudotensor "potential" whose values asymptotically give these quantities for an asymptotically flat spacetime:

$$M(r) = (16\pi)^{-1} \oint H^{0\alpha 0j}{}_{,\alpha} d^2 S_j , \tag{2.2a}$$

$$P^i(r) = (16\pi)^{-1} \oint H^{i\alpha 0j}{}_{,\alpha} d^2 S_j , \tag{2.2b}$$

$$\mathcal{S}^i(r) = (16\pi)^{-1} \oint \epsilon^i{}_{jk} (x^j H^{k\alpha 0l}{}_{,\alpha} + H^{jl0k}) d^2 S_l \tag{2.2c}$$

[cf. Eqs. (20.6) and (20.9) of MTW]. The surface integrals are over any closed two-surface in the buffer region. For definiteness we have taken these to be surfaces of constant $r = [(x^1)^2 + (x^2)^2 + (x^3)^2]^{1/2}$ in the approximately Lorentz coordinate system which can be introduced in the hole's local asymptotic rest frame. In Eqs. (2.2) $\epsilon^i{}_{jk}$ is the three-dimensional, flat-space Levi-Cività symbol [no factor of $(-g)^{1/2}$; integrations performed as though in

asymptotically flat spacetime], d^2S_j is the flat-space surface element, and $H^{\alpha\beta\gamma\delta}$ is the pseudotensor "potential." Expression (2.2a) normally gives the time component P^0 of the four-momentum, but since the integration is performed in the body's local asymptotic rest frame where P^i is zero (aside from tiny effects of nongeodesic motion), P^0 reduces to the body's mass M.

Were we to evaluate Eqs. (2.2) using the Kerr metric we would obtain well-defined functions of r, $M^{[0]}(r)$, $P^{[0]i}(r)$, and $\mathcal{S}^{[0]i}(r)$, which asymptotically would become the mass M, momentum P^i, and angular momentum $\mathcal{S}^i$ of the Kerr geometry. However, M, P^i, and $\mathcal{S}^i$ could equally well be read off from the behavior of these functions for smaller r.

Consider the integrals (2.2) for the family of spacetimes parametrized by a decreasing M and fixed $\mathcal{R}$. If the range of r at which the integrals are evaluated shrinks to zero at a slower rate than M (say, as M^ϵ, $0<\epsilon<1$), then the surface is increasingly far from the black hole on its scale but increasingly small on the external-universe scale. The integrals in Eq. (2.2) become increasingly well defined in the sense of an asymptotic expansion in M. To the extent that these integrations, carried out with the full metric, coincide in this range with $M^{[0]}(r)$, $P^{[0]i}(r)$, and $\mathcal{S}^{[0]i}(r)$, one can extract a mass M, momentum P^i, and angular momentum $\mathcal{S}^i$ for the black hole. To the extent that the corrections to $g^{[0]}$ in Eq. (2.1) make the integrals noncoincident with the Kerr forms there will be an imprecision in the definitions of these quantities. For further details see Sec. II E below.

B. Laws of motion and precession from matched asymptotic expansion

The laws of motion and precession for the black hole in the external universe follow from expressions for the rates of change of its mass, momentum, and angular momentum. The rates of change of the momentum and angular momentum of the black hole are contained in the terms $g^{[1]}$, $g^{[2]}$, . . . , in (2.1). They could be extracted directly from a calculation of these terms (see Sec. III for further discussion) but equivalently and more conveniently are calculated from the expressions for the rates of change of the surface integrals (2.2). These can be expressed as surface integrals of a symmetric energy-momentum pseudotensor in the standard way,

$$\dot{M}(r)=-\oint t^{0j}d^2S_j\ , \tag{2.3a}$$

$$\dot{P}^i(r)=-\oint t^{ij}d^2S_j\ , \tag{2.3b}$$

$$\dot{\mathcal{S}}^i(r)=-\oint \epsilon^i{}_{jk}x^j t^{kl}d^2S_l\ . \tag{2.3c}$$

Here t^{ij} is the pseudotensor; the dots over M, P^i, and $\mathcal{S}^i$ denote time derivatives, $\dot{M}\equiv dM/dt$; and the surface integrals again are over the closed two-surface of constant r in the buffer region. These surface integrals are derived as mathematical identities satisfied by the $M(r)$, $P^i(r)$, $\mathcal{S}^i(r)$ of Eqs. (2.2) in Sec. 20.5 of MTW and in Sec. 96 of Landau and Lifshitz (LL),[26] but those derivations use Gauss's theorem in a form valid only if the body's interior has Euclidean topology. Although one might fear that this topological constraint invalidates the derivation when the body is a black hole, it does not do so. As discussed on page 42 of Ref. 25, the derivation and the final formulas (2.3) are valid for black holes as well as for normal bodies. When performing concrete calculations in this paper we shall use for $t^{\mu\nu}$ the Landau-Lifshitz pseudotensor

$$t^{\mu\nu}=(-g)t^{\mu\nu}_{\rm LL}=\text{ [Eq. (20.22) of MTW]}$$
$$=\text{ [Eq. (96.9) of LL] }. \tag{2.3d}$$

If in the family of spacetimes characterized by a decreasing M and fixed $\mathcal{R}$, the surface over which the integrations (2.3) are performed shrinks as described above for the integrals (2.2) (i.e., $r\propto M^\epsilon$, $0<\epsilon<1$), then these integrals become increasingly precisely defined. By evaluating the dominant term of an expansion of these integrals in powers of M and expressing the result in terms of the parameters characterizing the black hole and the local curvature of the external universe, we acquire the laws of motion and precession of the hole.

It is not difficult to guess from general considerations the form that the laws of motion and precession will take. The ingredients from which the right-hand sides of integrals like (2.2) and (2.3) are composed are the parameters M, χ, and s^i of the Kerr black hole; the flat, spatial Levi-Cività tensor ϵ_{ijk}; and the metric $g^{(0)}$ of the external spacetime [which we shall discuss in Eq. (2.5) below]. The results must not depend on the choice of coordinates for $g^{(0)}$ and thus can depend on it only through its Riemann tensor $\mathscr{E}_{ij}$ and $\mathscr{B}_{ij}$ in the vacuum, buffer region. The results must be dimensionally correct and transform correctly under time reversal (for which $M\to+M$, $P^i\to-P^i$, $\mathcal{S}^i\to-\mathcal{S}^i$, $s_i\to-s_i$, $\mathscr{E}_{ij}\to+\mathscr{E}_{ij}$, $\mathscr{B}_{ij}\to-\mathscr{B}_{ij}$, and $\epsilon_{ijk}\to+\epsilon_{ijk}$), under spatial reflections (for which, viewed "actively" rather than "passively," $M\to+M$, $P^i\to+P^i$, $\mathcal{S}^i\to-\mathcal{S}^i$, $s^i\to-s^i$, $\mathscr{E}_{ij}\to+\mathscr{E}_{ij}$, $\mathscr{B}_{ij}\to-\mathscr{B}_{ij}$, and $\epsilon_{ijk}\to-\epsilon_{ijk}$) and under spatial rotations (which requires correct balancing of indices). The results could also depend on how the value of r at which the integrals (2.2) and (2.3) are evaluated shrinks to zero. We have taken r to go as M^ϵ, $0<\epsilon<1$. For the laws of motion at leading order in M to make sense, they must be independent of ϵ and this will indeed be shown to be the case. We can therefore guess the results by using the limiting value $\epsilon=1$. The above requirements fix the form of the leading-order expressions for $\dot{M}$, $\dot{P}^i$, and $\dot{\mathcal{S}}^i$ as M is decreased and χ and s^i remain fixed:

$$\dot{M}=\mu(\chi)M^3\mathscr{E}_{ij}s^is^j\ , \tag{2.4a}$$

$$\dot{P}^i=\alpha(\chi)M^2\mathscr{B}^i{}_js^j\ , \tag{2.4b}$$

$$\dot{\mathcal{S}}^i=\beta(\chi)M^3\epsilon^i{}_{jk}\mathscr{E}^k{}_ls^js^l\ . \tag{2.4c}$$

Here μ, α, and β are dimensionless functions of χ which are independent of the details of the external universe because there is no way for them to depend on those details. They are properties of Kerr black holes. To complete a derivation of the laws of motion and precession it remains only to compute these coefficients. To ensure their meaning we must also show that M, P^i, and $\mathcal{S}^i$ are defined to a precision which is higher order in M than the leading

terms in Eq. (2.4). We shall compute α and β below, but we shall not compute μ for two reasons: (i) μ is not important for the hole's motion or precession, and (ii) because Eq. (2.4a) involves $\mathscr{E}_{ij}$ it entails for its precise evaluation higher-order calculations than (2.4b) and (2.4c).[27] We shall investigate questions of precision in Sec. II E below.

To derive the laws of motion and precession (2.4b) and (2.4c) (and thence the functions α and β) from the surface integrals (2.3) requires knowing the pseudotensor t^{ij} to order $M^3/\mathscr{R}^2$ — and this, in turn, requires knowing the first three terms $g^{[0]}$, $g^{[1]}$, and $g^{[2]}$ in the expansion (2.1) of the distorted Kerr metric. Fortunately, we shall not need the full details of those terms. All we shall need is a few properties of them, which can be established from a discussion of how they might be computed in principle. This we now give.

One cannot simply solve Einstein's equation for the successive terms in the expansion (2.1): The solution is determined in part by the boundary conditions at infinity and the expansion (2.1) is not valid there. To deal with this a second expansion is introduced: an expansion of the spacetime metric in powers of the mass of the black hole keeping fixed χ, s_i, and the parameters of the external universe. We write

$$g=g^{(0)}+Mg^{(1)}+M^2g^{(2)}+\cdots . \tag{2.5}$$

Roughly speaking, the metric $g^{(0)}$ describes the external spacetime in the absence of the black hole, and the succeeding terms describe the corrections due to the black hole's presence. A key defining characteristic of $g^{(0)}$ is that it is smooth at the black hole's location. However, just as there is ambiguity in the choice of the unperturbed Kerr metric [Kerr-type terms of order $\mathscr{R}^{-2}g^{[2]}$ can be moved into the $g^{[0]}$ of Eq. (2.1) thereby changing the M, P^i, and $\mathscr{S}^i$ attributed to the hole], so also there is ambiguity in the choice of the external metric $g^{(0)}$: Terms of order $Mg^{(1)}+M^2g^{(2)}+\cdots$, which vary smoothly with r as $r\to 0$, can be moved into $g^{(0)}$, thereby changing the external metric and its curvatures $\mathscr{E}_{ij}$, $\mathscr{B}_{ij}$ by small fractional amounts. When one converts the laws of motion into equations of motion for a specific situation, this freedom to make small changes in $g^{(0)}$ can be a powerful aid in optimizing the accuracy of the resulting equations; see Sec. IV for an example.

The expansion (2.5) may well not be valid arbitrarily close to the black hole where the curvatures it produces become strong. However, it will be valid outside a distance from the black hole of order its mass, and in particular at infinity.

The assumption of an isolated black hole, specifically the requirement $M/\mathscr{R}\ll 1$, implies that the expansions (2.1) and (2.5) have a common region of validity which includes the buffer region. To solve Einstein's equation for a black hole in an external universe, one can take a family of solutions of the form (2.1) satisfying correct boundary conditions at the black hole and a family of solutions of the form (2.5) satisfying correct boundary conditions at infinity, and match the two families in the region of common validity. The assumption that this procedure of matched asymptotic expansions can be carried out will be enough to determine the laws of motion.

Imagine that one has obtained by the method of matched asymptotic expansions a family of black-hole-in-external-universe solutions parametrized by M. As already described, the laws of motion are obtained by evaluating the surface integrals in Eqs. (2.3b) and (2.3c) at a radius $r\propto M^{\epsilon}$, $0<\epsilon<1$ as M shrinks. To study the integrands of these expressions at these small values of r, we expand the spacetime metric simultaneously in M and $\mathscr{R}^{-1}$ in the buffer region where both expansions (2.1) and (2.5) are valid. This will also be an expansion in powers of r since M, $\mathscr{R}$, and r are the only dimensional quantities. Writing out only the powers of the dimensional quantities involved, omitting the coefficients, and denoting the Minkowski metric by η, the expansion has the following structure:

$$\begin{array}{lccccccccc}
g= & & \eta & \& & \dfrac{M}{r} & \& & \dfrac{M^2}{r^2} & \& & \dfrac{M^3}{r^3} & \& & \cdots & g^{[0]} \\
& \& & 0 & \& & \dfrac{M}{\mathscr{R}} & \& & \dfrac{M^2}{r\mathscr{R}} & \& & \dfrac{M^3}{r^2\mathscr{R}} & \& & \cdots & \mathscr{R}^{-1}g^{[1]} \\
& \& & \dfrac{r^2}{\mathscr{R}^2} & \& & \dfrac{Mr}{\mathscr{R}^2} & \& & \dfrac{M^2}{\mathscr{R}^2} & \& & \dfrac{M^3}{r\mathscr{R}^2} & \& & \cdots & \mathscr{R}^{-2}g^{[2]} \\
& \& & \dfrac{r^3}{\mathscr{R}^3} & \& & \dfrac{Mr^2}{\mathscr{R}^3} & \& & \dfrac{M^2r}{\mathscr{R}^3} & \& & \dfrac{M^3}{\mathscr{R}^3} & \& & \cdots & \mathscr{R}^{-3}g^{[3]} \\
& \& & \cdots & & & & & & & & & & & \\
& & g^{(0)} & & & Mg^{(1)} & & & M^2g^{(2)} & & & M^3g^{(3)} & . &
\end{array} \tag{2.6}$$

Here, following the notation of Penrose,[28] "&" is to be read "and a term of the form. . . ." In principle any given term could be multiplied by a logarithmic r dependence,[29] but this does not happen in practice at the orders we shall encounter. Successive rows in this tableau correspond to $g^{[0]}$, $g^{[1]}$, $g^{[2]}$, etc. while successive columns correspond to $g^{(0)}$, $g^{(1)}$, $g^{(2)}$, etc. The first row is thus the expansion of the Kerr geometry in powers of M while the first

column is the expansion of the external spacetime in inverse powers of $\mathscr{R}$. The assumption of the existence of matched asymptotic expansions dictates the powers of M and $\mathscr{R}$ which occur in Eq. (2.6). For example, there are no terms of the form $M\mathscr{R}/r^2$ because these do not occur in (2.1) and none of the form $r^2/M\mathscr{R}$ because these do not occur in (2.5). The powers of r follow from dimensional analysis once the powers of M and $\mathscr{R}$ are specified. Two assumptions about coordinates have been used in Eq. (2.6). The first arises because the Kerr metric becomes asymptotically flat at large r and the metric of the external universe without black hole becomes locally flat at small r. These limits coincide in the Minkowski term η of Eq. (2.6). Consistency requires that both the Kerr metric $g^{[0]}$ and the external universe metric $g^{(0)}$ be expressed in coordinates for which the metric functions coincide in these limits. We have chosen these to be rectangular Lorentz coordinates (t,x^i). This is the *only* restriction imposed on the coordinates by the method of matched asymptotic expansions so that there is considerable gauge freedom left. We in particular are free to use any coordinates to express the Kerr geometry as long as it is asymptotically Lorentz and are free to use any coordinates for the metric of the external universe provided it is Lorentz at small r. We shall exploit the first possibility in what follows but more immediately we use the second to *eliminate any term in the metric $g^{(0)}$ of the external universe which is linear in r.* Physically this corresponds to choosing the coordinates to be nonrotating and nonaccelerating relative to the local inertial frames of the external universe. (See the Appendix and Sec. III for further discussion.)

The absence of terms of order $\mathscr{R}^{-1}$ in the metric $g^{(0)}$ of the external universe means that all terms of this order vanish, i.e., $\mathscr{R}^{-1}g^{[1]}=0$. To see this imagine iterating Einstein's equation to calculate the elements of the tableau in (2.6) starting from the known expansions of the Kerr metric $g^{[0]}$ and external-universe metric $g^{(0)}$. A given order in M and $\mathscr{R}$ of the expansion of Einstein's equation resulting from the metric expansion (2.6) will be a linear equation for the unknown metric perturbation of that order driven by a nonlinear combination of known lower-order terms. (See Sec. III for a more explicit discussion.) If there are no terms of order $\mathscr{R}^{-1}$ in the external-universe metric, there are no driving terms for any metric perturbation of this order. Since the boundary conditions at small and large r also do not require $\mathscr{R}^{-1}$ terms, they vanish identically. Thus, $\mathscr{R}^{-1}g^{[1]}$ and the second row of (2.6) are zero.

The tableau (2.6) permits us to understand more clearly the origin of the uncertainties in the mass, momentum, and angular momentum of our black hole: The $M^3/r\mathscr{R}^2$ term in the tableau will contain a contribution of the form $\mathscr{E}_{ij}\mathscr{I}^{ij}/r\sim M^3\mathscr{E}_{ij}s^is^j/r$ to g_{00}. We are free to move this contribution out of the $M^3/r\mathscr{R}^2$ term of the tableau and into the M/r term, if we wish, thereby changing the mass we attribute to the hole by an amount $\Delta M\sim M^3/\mathscr{R}^2$ [Eq. (1.8a)]. Such a move does not change at all the sum of all the terms in the tableau. Similar rearrangements of other terms will produce changes of magnitudes (1.8b) and (1.8c) in the P^i and $\mathscr{I}^i$ we attribute to the hole. And by absorbing the $Mr^2/\mathscr{R}^3$ term into $r^2/\mathscr{R}^3$ we can change $g^{(0)}$ by a fractional amount of order $M/\mathscr{R}$, thereby helping to optimize the accuracy of the equations of motion that follow from our laws of motion (Sec. IV).

We shall now identify the terms in the tableau (2.6) which contribute to the surface integrals for $\dot{P}^i$ and $\dot{\mathscr{I}}^i$ [Eqs. (2.3)] as M shrinks to zero and the radius of the integration surface shrinks as M^ϵ, $0<\epsilon<1$. For definiteness consider the Landau-Lifshitz pseudotensor $t^{ij}=(-g)t^{ij}_{\rm LL}$, displayed in Eq. (20.22) of MTW. $(-g)t^{ij}_{\rm LL}$ depends only on the metric and its first derivatives and is quadratic in these first derivatives. From this we see that, for small M, products of coefficients in (2.6) which together vary as r^0 contribute to $\dot{P}^i$ and as r^{-1} to $\dot{\mathscr{I}}^i$. The combinations of interest are those of lowest order in M as M shrinks to zero.

C. Derivation of the law of motion, $\dot{P}^i$

A little study of the tableau (2.6) convinces one of the following: The only terms in t^{ij} which contribute to the surface integral for $\dot{P}^i$ evaluated at a radius $r\sim M^\epsilon$ for small M are those constructed from products of first derivatives of the part of the Kerr metric proportional to M^2 with first derivatives of the part of the external-universe metric proportional to $\mathscr{R}^{-2}$, i.e., terms of the form $M^2/r^2\times r^2/\mathscr{R}^2$. Terms of the form $M/r\times r^2/\mathscr{R}^2$, which have the correct powers of M, $\mathscr{R}$, and r will not contribute to $\dot{P}^i$ at order M^2 because the general form (2.4b) of the final answer involves the spin direction s^i which is absent from the metric at order M. Terms of the form $M/r\times r^2/\mathscr{R}^2$, which might be thought to dominate the M^2 result as $r\propto M^\epsilon$ goes to zero, vanish because the final answer involves the spin direction s^i which is absent from the metric at order M. Put differently, the M/r part of the Kerr geometry is spherically symmetric so the $M/r\times r^2/\mathscr{R}^2$ terms give no preferred direction for $\dot{P}^i$ to point. Put yet differently, the external-universe geometry at order $1/\mathscr{R}^2$ transforms as a quadrupole under rotations; a black-hole vector or higher multipole is needed to couple to this external quadrupole to give a vectorial $\dot{P}^i$, but vectors (s^i) and higher multipoles (e.g., s^is^j) enter the Kerr geometry only at order M^2 and higher, not at order M. For any of these same three reasons, terms of the form $M/r\times M/r\times r^2/\mathscr{R}^2$ also give a vanishing contribution to $\dot{P}_i$.

From a computational point of view the important consequence of the above argument is that one can compute $\dot{P}^i$ without first solving Einstein's equation for the external universe's $Mr/\mathscr{R}^2$ deformations of the Kerr metric. The unperturbed Kerr M^2/r^2 terms and the unperturbed external-universe $r^2/\mathscr{R}^2$ terms alone, inserted in Eq. (2.3), suffice to give the leading-order correction to geodesic motion. To carry out this calculation efficiently we shall exploit the freedom in the choice of gauge remaining after the modest constraints imposed by the form of the expansion (2.6). For the external universe we use the deDonder gauge discussed in the Appendix, in which the metric has the expansion

$$g_{00}^{(0)} = -1 - \mathscr{E}_{ij}x^i x^j + \cdots , \tag{2.7a}$$

$$g_{0i}^{(0)} = -\tfrac{2}{3}\epsilon_{ijk}\mathscr{B}^j{}_l x^k x^l + \cdots , \tag{2.7b}$$

$$g_{ij}^{(0)} = \delta_{ij}(1 - \mathscr{E}_{kl}x^k x^l) + \cdots . \tag{2.7c}$$

For the Kerr geometry we use "rectangular" Boyer-Lindquist coordinates (t,x^1,x^2,x^3), which are related to the usual "polar" ones (t,r,θ,ϕ) [MTW Eq. (33.2)] by the usual flat-space transformation between polar and rectangular coordinates.[30] In these coordinates the order M^2/r^2 part of $g_{00}^{[0]}$ vanishes. The order M^2/r^2 part of $g_{ij}^{[0]}$ is even in s^i and therefore cannot contribute to a final result of the form (2.4b). The only relevant part of the Kerr metric is therefore

$$g_{0i}^{[0]} = -\frac{2M^2\chi}{r^2}(\vec{s}\times\vec{n})_i + O\left(\frac{M^3}{r^3}\right), \tag{2.8}$$

where n^i is the unit radial vector x^i/r.

With these elaborate preparations a straightforward calculation of the $M^2/r^2\times r^2/\mathscr{R}^2$ contribution to (2.3b)

$$\dot{P}^i = -\mathscr{B}^i{}_j\mathscr{S}^j , \tag{2.9}$$

which is expression (2.4b) with $\alpha(\chi)=\chi$ and is also expression (1.9b).

D. Derivation of the law of precession, $\dot{\mathscr{S}}^i$

We next turn to the evaluation of the surface integral for $\dot{\mathscr{S}}^i$. A little study of the tableau (2.6) convinces one that if one is not careful one *will* have to solve Einstein's equation for the $Mr/\mathscr{R}^2$ corrections to the Kerr-plus-external-universe metric in order to evaluate the pseudotensor and thence $\dot{\mathscr{S}}^i$ to order M^3. For example, there are products of the form $M/r\times M/r\times Mr/\mathscr{R}^2$ and $M^2/r^2\times Mr/\mathscr{R}^2$ which are of the correct order in M, $\mathscr{R}$, and r to contribute in the evaluation of (2.3c). Further, there are terms like $M/r\times r^2/\mathscr{R}^2$ and $M^2/r^2\times r^2/\mathscr{R}^2$ which would dominate the order-M^3 result if r shrinks as M^ϵ. At order M the Kerr metric contains no information on the orientation of the black hole so that, similarly to the argument for linear momentum, products involving coefficients of only this order or less, such as $M/r\times M/r\times Mr/\mathscr{R}^2$, cannot contribute to the final vector $\dot{\mathscr{S}}^i$. In a general gauge the M^2/r^2 terms in the Kerr metric will contain both vector terms (in the g_{0i} components) and quadrupole terms (in the g_{00} and g_{ij} components). This is the case, for example, in the "rectangular" Boyer-Lindquist coordinates described above.[30,31] Parity considerations rule out any contribution of the vector M^2/r^2 parts to the final result (they are odd under parity, the quadrupole lowest-order external-universe metric is even, and the result is even). There remain, however, the quadrupole M^2/r^2 terms which *can* couple to the quadrupole $Mr/\mathscr{R}^2$ terms to enter into the final result for $\dot{\mathscr{S}}^i$. Thus, in a general gauge, and in rectangular Boyer-Lindquist coordinates in particular, it is necessary to solve Einstein's equation for the $Mr/\mathscr{R}^2$ corrections in (2.6) in order to derive the lowest-order-in-M law of precession of a black hole.

To leading order in M the precession of the black hole arises from a coupling between the hole's intrinsic quadrupole moment and the curvature of the external geometry, as Eq. (2.4c) shows. The physical information contained in the $Mr/\mathscr{R}^2$ term in the metric is not this, but rather the lowest-order effect of the black hole on the metric of the external universe. This has nothing physically to do with the precession. One might reasonably expect, therefore, to be able to avoid a laborious iteration of Einstein's equation to obtain the $Mr/\mathscr{R}^2$ term, when computing $\dot{\mathscr{S}}^i$ at the lowest order in M. One can. The occurrence of quadrupole terms in the M^2/r^2 part of the Kerr metric is not a gauge-invariant result. There exist gauges in which they are absent. These are the ACMC-1 ["asymptotically Cartesian and mass centered to order $(1/r)^{l+1}$"] coordinates of Ref. 14. A derivation of the equation of precession which starts with the Kerr metric expressed in an ACMC-1 coordinate system will give zero contribution from the $M^2/r^2\times Mr/\mathscr{R}^2$ terms and thus will not require an iteration of Einstein's equation. Perhaps even more importantly there will be no contributions to $\dot{\mathscr{S}}^i$ from terms like $M^2/r^2\times r^2/\mathscr{R}^2$ which would dominate if r shrinks as M^ϵ. The leading-order equation of motion is of order M^3.

In an ACMC-1 coordinate system the Kerr metric takes the form[32]

$$g_{00}^{[0]} = -1 + \frac{2M}{r} - \frac{3M^3\chi^2(\vec{n}\cdot\vec{s})^2}{r^3} + O\left(\frac{M^5}{r^5}\right), \tag{2.10a}$$

$$g_{0i}^{[0]} = -\frac{2M^2\chi}{r^2}(\vec{s}\times\vec{n})_i + O\left(\frac{M^4}{r^4}\right), \tag{2.10b}$$

$$g_{ij}^{[0]} = \delta_{ij} + \frac{2M}{r}n_i n_j + \frac{M^2}{r^2}[\chi^2\delta_{ij} + (4-2\chi^2)n_i n_j] + 8\frac{M^3}{r^3}n_i n_j + \frac{M^3\chi^2}{r^3}\{2[1-(\vec{s}\cdot\vec{n})^2]\delta_{ij} + [-6-5(\vec{s}\cdot\vec{n})^2]n_i n_j - 2s_i s_j + 8(\vec{s}\cdot\vec{n})n_{(i}s_{j)}\} + O\left(\frac{M^4}{r^4}\right), \tag{2.10c}$$

$$(-g^{[0]})^{1/2} = 1 + \frac{1}{2}\frac{M^2\chi^2}{r^2} + O\left(\frac{M^4}{r^2}\right). \tag{2.10d}$$

[The complicated M^3/r^3 term in g_{ij} does not actually enter the calculation when the deDonder gauge (2.7) is used for $g^{(0)}$.] By adding Eqs. (2.10) for the Kerr metric to Eqs. (2.7) for the external-universe metric, calculating terms in $(-g)t^{ij}_{\rm LL}$ of order M^3/r^3(quadrupole) $\times 1/\mathscr{R}^2$(quadrupole), and evaluating the surface integral in (2.3c), one arrives at the following result for the equation of black-hole precession to lowest nonvanishing order in M:

$$\dot{\mathscr{S}}^i = -M^3\chi^2\epsilon^i{}_{jk}\mathscr{E}^k{}_l S^j S^l . \tag{2.11}$$

This is expression (2.4c) with $\beta(\chi) = -\chi^2$ and is also expression (1.9c')

E. Uncertainties in the definitions of mass, momentum, and angular momentum

The order of magnitude of the uncertainties in the definitions of the mass M, momentum P^j, and angular momentum $\mathscr{S}^j$ of the black hole in an external universe may be estimated by either of two procedures. One may study the tableau (2.6) to see how these quantities, as defined by $g^{[0]}$, can be altered by moving into $g^{[0]}$ terms from $\mathscr{R}^{-2}g^{[2]}$. Alternatively, and analogously to the derivation of the laws of motion, one can compute the changes in the surface integrals (2.2) for M, P^j, and $\mathscr{S}^j$ caused by including $\mathscr{R}^{-2}g^{[2]}$ terms in the computation of their integrands.

The dependence of the uncertainties in M, P^j, and $\mathscr{S}^j$ on the curvature of the geometry of the external universe and on the spin direction of the black hole can be determined, as in Sec. II B, from transformation properties under rotations, parity, and time reversal. The dependence on M and r can be found from dimensional arguments. The results of such an analysis are

$$(\text{uncertainty in } M) \sim M^3 \mathscr{E}_{ij} S^i S^j , \tag{2.12a}$$

$$(\text{uncertainty in } P^i) \sim M^3 \epsilon^i{}_{jk} \mathscr{B}^j{}_l S^k S^l , \tag{2.12b}$$

$$(\text{uncertainty in } \mathscr{S}^i) \sim M^{3+\epsilon} \mathscr{E}^i{}_j S^j \Big|_{\epsilon=1} . \tag{2.12c}$$

The uncertainty in $\mathscr{S}^i$ depends on ϵ, but ϵ can be pushed as close to unity as one likes. The rate of change of the uncertainty (2.12a) in M, due to time changing $\mathscr{E}_{ij}$, is of the same order as the rate of change of M itself, Eq. (2.4a). However, the rates of change of the uncertainties (2.12b) and (2.12c) in P^i and $\mathscr{S}^i$ are of higher order than the curvature-induced changes (2.4b) and (2.4c) in P^i and $\mathscr{S}^i$ themselves. Because the rates of change of the uncertainties in the definitions are smaller than those deduced from the leading-order laws of motion and precession, these leading-order equations have a definite meaning.

F. Comments on the derivations

The above analysis shows that the methods of matched asymptotic expansions provide an efficient tool for deriving the leading-order corrections to geodesic motion [Eq. (2.9)] and Fermi-Walker spin transport [Eq. (2.11)] provided suitable care is exercised in the choice of gauge and in not calculating terms which cannot contribute to the final answer. The reader may find the arguments organizing the calculation in this way excessively intricate. An appreciation of their value can perhaps be acquired by contemplating the task of computing $(-g)t^{ij}_{\rm LL}$ [MTW, Eq. (20.22)] in full detail for the superposed and nonlinearly interacting Kerr and external-universe metrics in an arbitrarily rather than carefully chosen gauge.

The methods of matched asymptotic expansions raise a number of mathematical issues whose resolution would lead to increased confidence in the results presented here. The forms of the expansions (2.1), (2.5), and (2.6) have been motivated physically rather than demonstrated mathematically. Are these expansions rigorously and generally valid in some precise sense?[33] Can a more rigorous and geometrical meaning be given to the approximate extraction of the body's mass, momentum, and angular momentum from the family of spacetimes generated by varying the mass of the body while keeping parameters of the external universe fixed? Can the derivation of the equations of motion be carried out in a manifestly gauge-invariant way?[34] Are the results obtained the first orders in an always well-defined procedure of successive approximation? We believe that the answer to all these questions is "yes," and that it is a problem of considerable interest to give a rigorous justification of the approximate situations discussed here.

III. DERIVATION OF LAWS OF MOTION AND PRECESSION FOR AN ARBITRARY BODY

A. Foundations

The derivation of laws of motion and precession in Sec. II was limited for clarity and familiarity to the case of a Kerr black hole. We now turn to the general case of the motion and precession of an arbitrary, isolated body ($L \ll \mathscr{R}$, $L \ll \mathscr{L}$, $L \ll \mathscr{T}$), with slowly time-changing moments ($T \ll \mathscr{T}$), interacting with an arbitrary external universe. Our analysis will follow the pattern of Sec. II but we will augment it with the multipole-moment formalism of Thorne.[14,35]

In the general case, because of the large number of dimensionless expansion parameters in the buffer region and their specific forms (M/r, L/r, r/T, $r/\mathscr{R}$, $r/\mathscr{L}$, $r/\mathscr{T}$), there are two pairs of complicated matched asymptotic expansions occurring at once: $(M/r, L/r)\times(r/T)$ and $(M/r, L/r, r/T)\times(r/\mathscr{R}, r/\mathscr{L}, r/\mathscr{T})$. However, we shall neither spell out the details of the expansion tableaux [analogs of Eq. (2.6)] nor the details of the limits and the matchings involved in the analysis. Rather, with confidence gained from the analysis of Sec. II, we shall proceed more simply and directly to the final, more general results.

In the buffer region around our arbitrary, isolated body we introduce coordinates (t,x^j) which are nearly Lorentz and precisely deDonder (harmonic), and which at time $t=0$ are centered on the body (so there is no dipole term $\mathscr{I}_j x^j/r^3$ in g_{00}) and coincide with the body's asymptotic rest frame (so there is no spatial momentum term $-4P_j/r$ in g_{0j}). We further require, as in Sec. II, that as time

passes these coordinates remain local asymptotically inertial (so there is no acceleration term $a_j x^j$ in g_{00} and no rotation term $\epsilon_{jkl}x^k\omega^l$ in g_{0j}); the slowness of all time variations guarantees that this can be achieved by adjustments of gauge which keep the gauge deDonder. Such a coordinate system, in the body's "buffer region" and the outer parts of its "neighborhood," makes precise the concept of "the body's local asymptotic rest frame" used in Sec. I.

In these coordinates we shall characterize gravity by the contravariant metric density:

$$\mathfrak{g}^{\alpha\beta}\equiv(-g)^{1/2}g^{\alpha\beta}\ ,\quad g\equiv\det||g_{\mu\nu}||\ , \tag{3.1}$$

where $g^{\alpha\beta}$ and $g_{\mu\nu}$ are the components of the metric, and we shall denote by $-\bar{h}^{\alpha\beta}$ the small deviations of $\mathfrak{g}^{\alpha\beta}$ from the Minkowski metric $\eta^{\alpha\beta}\equiv\text{diag}(-1,1,1,1)$:

$$\mathfrak{g}^{\alpha\beta}\equiv\eta^{\alpha\beta}-\bar{h}^{\alpha\beta}\ . \tag{3.2}$$

The "isolation" of the body guarantees that $|\bar{h}^{\alpha\beta}|\ll 1$ throughout the buffer region. In the limit that only linear corrections to $\eta^{\alpha\beta}$ are included, $\bar{h}^{\alpha\beta}$ is the trace-reversed metric perturbation, but in the nonlinear regime it is not.

We shall split $\bar{h}^{\alpha\beta}$ up into three parts: one $\bar{h}_B^{\alpha\beta}$ associated with the body (analog of $g^{[0]}-\eta$ in Sec. II), a second $\bar{h}_E^{\alpha\beta}$ associated with the external universe (analog of $g^{(0)}-\eta$ in Sec. II), and a third $\bar{h}_I^{\alpha\beta}$ associated with interactions between the body and the external universe [analog of interior terms in Sec. II's tableau (2.6)]:

$$\bar{h}^{\alpha\beta}\equiv\bar{h}_B^{\alpha\beta}+\bar{h}_E^{\alpha\beta}+\bar{h}_I^{\alpha\beta}\ . \tag{3.3}$$

We shall study each of these parts in turn, and then insert them into the surface integrals (2.3) to derive the laws of motion and precession.

B. The body's gravitational field $\bar{h}_B^{\alpha\beta}$

The body's field $\bar{h}_B^{\alpha\beta}$ is defined to have the same form as it would have if the external universe were flat spacetime.[14] This form is fully determined in the buffer zone by two families of multipole moments, "mass moments" M, $\mathscr{I}_{jk}$, $\mathscr{I}_{jkl}$, $\mathscr{I}_{jklm}$, ..., and "current moments," $\mathscr{S}_j$, $\mathscr{S}_{jk}$, $\mathscr{S}_{jkl}$, $\mathscr{S}_{jklm}$, ... These moments are purely spatial tensors which are symmetric and trace-free ("STF") on all pairs of indices. They are independent of spatial position, and M and $\mathscr{S}_j$ are truly constant (except for radiation-reaction-induced changes which we shall neglect), but $\mathscr{I}_{jk}$, $\mathscr{I}_{jkl}$, ..., $\mathscr{S}_{jk}$, $\mathscr{S}_{jkl}$, ... may vary with time t. The body's mass M is the "mass monopole moment," $\mathscr{I}_{jk}$ is the "mass quadrupole moment," and $\mathscr{I}_{jkl}$ is the "mass octupole moment,"...; the body's angular moment $\mathscr{S}_j$ is the "current dipole moment"; $\mathscr{S}_{jk}$ is the "current quadrupole moment," and $\mathscr{S}_{jkl}$ is the "current octupole moment,".... The moments for a Kerr black hole have been computed by Hansen,[36] and Gürsel[37] has translated them into Thorne's conventions. The lowest few moments are given in Eq. (1.7) above; see also Sec. XI D of Ref. 14. The moments of any body have magnitudes

$$\begin{gathered}|\mathscr{I}_{a_1a_2\cdots a_l}|\lesssim ML^l,\quad |\mathscr{S}_{a_1a_2\cdots a_l}|\lesssim ML^l\ ,\\ \left|\frac{d}{dt}\mathscr{I}_{a_1\cdots a_l}\right|\lesssim\frac{ML^l}{T},\quad \left|\frac{d}{dt}\mathscr{S}_{a_1\cdots a_l}\right|\lesssim\frac{ML^l}{T}\quad\text{for } l\geq 2\ ,\end{gathered} \tag{3.4}$$

where M, L, and T are the length scales defined in Eqs. (1.1) and (1.2).

If $r=[(x^1)^2+(x^2)^2+(x^3)^2]^{1/2}$ is the distance from the body, then the region $r\ll T$ is the "near zone" of the body and $r\gg T$ is its "wave zone." We demand that T be very large ($T\gg L$) so that the buffer region can be contained entirely within the near zone, $r\ll T$. If $L\gg M$ the body has weak internal gravity; if $L\sim M$ its internal gravity is strong. In either case we shall confine our buffer region to the weak-gravity zone $r\gg M$ far outside the body's surface $r\gg L$.

Thorne (Ref. 14, Sec. IX) exhibits the general structure of the body's gravitational field $\bar{h}_B^{\alpha\beta}$ as a spherical-harmonic expansion and a simultaneous power-series expansion in the three dimensionless variables: M/r ("nonlinearity expansion"), L/r ("distance-from-body expansion"), and r/T ("time-derivative expansion"). (Thorne uses the notation R for L and $\lambda\!\!\!^{-}$ for T.) It will turn out that, to derive the body's equations of motion and precession to the accuracy of Eqs. (1.9), we must know $\bar{h}_B^{\alpha\beta}$ with accuracy

$$|\ \text{errors in } \bar{h}_B^{\alpha\beta}\ |\ll\frac{ML^2}{r^3}\ . \tag{3.5}$$

Gürsel (the Appendix of Ref. 37) gives $\bar{h}_B^{\alpha\beta}$ to the required accuracy (but note that the sign of his $\bar{h}^{0j}$ is reversed)

$$\bar{h}_B^{00}=\frac{4M}{r}+\frac{7M^2}{r^2}+\frac{8M^3}{r^3}+\frac{6\mathscr{I}_{jk}n^jn^k}{r^3}\ , \tag{3.6a}$$

$$\bar{h}_B^{0j}=+\frac{2\epsilon^j{}_{kl}\mathscr{S}^kn^l}{r^2}+\frac{2\epsilon^j{}_{kl}M\mathscr{S}^kn^l}{r^3}+\frac{4\epsilon^j{}_{kl}\mathscr{S}^k{}_mn^ln^m}{r^3}\ , \tag{3.6b}$$

$$\bar{h}_B^{ij}=\frac{M^2}{r^2}n^in^j\ . \tag{3.6c}$$

From Sec. IX of Ref. 14 (or by dimensional analysis) we infer that the largest corrections to these $\bar{h}_B^{\mu\nu}$ have the following forms and magnitudes:

$$\frac{M^4}{r^4}\sim\left[\frac{ML^2}{r^3}\right]\left[\frac{M}{L}\right]^2\left[\frac{M}{r}\right],$$
$$\left|\frac{\mathscr{S}_{jk}M}{r^4}\right|\lesssim\left|\frac{\mathscr{I}_{jk}M}{r^4}\right|\lesssim\left[\frac{ML^2}{r^3}\right]\left[\frac{M}{r}\right],$$
$$\left|\frac{\mathscr{S}_{jkl}}{r^4}\right|\lesssim\left|\frac{\mathscr{I}_{jkl}}{r^4}\right|\lesssim\left[\frac{ML^2}{r^3}\right]\left[\frac{L}{r}\right],$$
$$\left|\frac{\mathscr{S}_j M^2}{r^4}\right|\lesssim\left[\frac{ML^2}{r^3}\right]\left[\frac{M}{L}\right]\left[\frac{M}{r}\right], \tag{3.7}$$
$$\left|\frac{\mathscr{S}_j\mathscr{S}_k}{r^4}\right|\lesssim\left[\frac{ML^2}{r^3}\right]\left[\frac{M}{r}\right],$$
$$\left|\frac{d\mathscr{I}_{jk}/dt}{r^2}\right|\lesssim\left[\frac{ML^2}{r^3}\right]\left[\frac{r}{T}\right],$$
$$\left|\frac{d\mathscr{S}_{jk}/dt}{r^2}\right|\lesssim\left[\frac{ML^2}{r^3}\right]\left[\frac{r}{T}\right].$$

Since $M/L\lesssim 1$, $M\ll r$, $L\ll r$, and $r\ll T$ these corrections are all of the allowed magnitude (3.5).

C. The gravitational field of the external universe

In the Appendix to this paper we use the methods of Ref. 14 to sketch out the general structure of the gravitational field $\bar{h}_E^{\alpha\beta}$ of the external universe in the body's buffer zone. That field is determined by two families of STF moments: electric-type moments $\mathscr{E}_{a_1\cdots a_l}$ and magnetic-type moments $\mathscr{B}_{a_1\cdots a_l}$; but whereas the body's mass moments $\mathscr{I}_{a_1\cdots a_l}$ begin at monopole order (M) and its current moments $\mathscr{S}_{a_1\cdots a_l}$ begin at dipole order ($\mathscr{S}_j$), both families of external moments begin at quadrupole order, $\mathscr{E}_{jk}$ and $\mathscr{B}_{jk}$. The external moments are independent of radius r but may vary with time t. In order of magnitude

$$|\mathscr{E}_{a_1\cdots a_l}|\lesssim\frac{1}{\mathscr{R}^2\mathscr{L}^{l-2}},\quad|\mathscr{B}_{a_1\cdots a_l}|\lesssim\frac{1}{\mathscr{R}^2\mathscr{L}^{l-2}},$$
$$\left|\frac{d}{dt}\mathscr{E}_{a_1\cdots a_l}\right|\lesssim\frac{1}{\mathscr{R}^2\mathscr{L}^{l-2}\mathscr{T}}, \tag{3.8}$$
$$\left|\frac{d}{dt}\mathscr{B}_{a_1\cdots a_l}\right|\lesssim\frac{1}{\mathscr{R}^2\mathscr{L}^{l-2}\mathscr{T}},$$

where $\mathscr{R}$, $\mathscr{L}$, and $\mathscr{T}$ are the length scales of Eq. (1.3) with $\mathscr{L}\lesssim\mathscr{R}$, $\mathscr{L}\lesssim\mathscr{T}$.

It will turn out that, to derive the body's equations of motion and precession to the accuracy of Eqs. (1.9), we must know $\bar{h}_E^{\alpha\beta}$ with accuracy:

$$|\text{ errors in }\bar{h}_E^{\alpha\beta}|\ll\frac{r^2}{\mathscr{R}^2}. \tag{3.9}$$

In the Appendix we show that to the required order and with a specialization of the deDonder gauge, $\bar{h}_E^{\alpha\beta}$ is

$$\bar{h}_E^{00}=-2\mathscr{E}_{jk}x^jx^k, \tag{3.10a}$$
$$\bar{h}_E^{0j}=\tfrac{2}{3}\epsilon^j{}_{kl}\mathscr{B}^k{}_m x^l x^m, \tag{3.10b}$$
$$\bar{h}_E^{ij}=0. \tag{3.10c}$$

D. The interaction field

The full gravitational field $\bar{h}^{\mu\nu}$ must satisfy the deDonder gauge conditions

$$\bar{h}^{\mu\nu}{}_{,\nu}=0 \tag{3.11a}$$

and the vacuum Einstein field equations[38]

$$\eta^{\alpha\beta}\bar{h}^{\mu\nu}{}_{,\alpha\beta}=W^{\mu\nu}, \tag{3.11b}$$
$$W^{\mu\nu}\equiv-16\pi(-g)t_{\rm LL}^{\mu\nu}-\bar{h}^{\mu\alpha}{}_{,\beta}\bar{h}^{\nu\beta}{}_{,\alpha}+\bar{h}^{\mu\nu}{}_{,\alpha\beta}\bar{h}^{\alpha\beta}. \tag{3.11c}$$

Here $t_{\rm LL}^{\mu\nu}$ is the Landau-Lifshitz pseudotensor [Eq. (2.3d)].

The body field $\bar{h}_B^{\mu\nu}$ by itself satisfies the gauge conditions and field equations (3.11), as does the external field by itself. However, when both are present in $\bar{h}^{\mu\nu}$ the nonlinearity of the field equations (3.11b) requires that an interaction field $\bar{h}_I^{\mu\nu}$ also be present. In principle we could iterate the gauge conditions and field equations (3.11a) and (3.11b) to obtain $\bar{h}_I^{\mu\nu}$. Such an iteration, keeping the coordinates inertial (so there is no acceleration term a_jx^j in g_{00} and no rotation term $\epsilon_{jkl}x^k\omega^l$ in g_{0j}) would produce, via the gauge conditions (3.11a), secularly changing terms of the form $4\dot{P}_jt/r$ and $2\epsilon_{jkl}\dot{\mathscr{S}}^k tn^l/r^2$ in $\bar{h}_I^{0j}$, where $\dot{P}^j$ and $\dot{\mathscr{S}}^k$ would be the expressions given on the right-hand sides of Eqs. (1.9b) and (1.9c); cf. Sec. IX G of Ref. 14. Since, in deDonder gauge, the body's momentum and angular momentum always show up in precisely this form in $\bar{h}^{0j}$ (cf. Secs. IX and X of Ref. 14), from these $\bar{h}_I^{0j}$ we would infer that the interactions produce the time changes of the body's momentum and angular momentum given by Eqs. (1.9b) and (1.9c).

E. Derivation of $\dot{M}$, $\dot{P}^j$, and $\dot{\mathscr{S}}^j$ from surface integrals

We have not actually carried out this iterative solution of the field equations. Instead, we have computed $\dot{P}^j$ and $\dot{\mathscr{S}}^j$ and also $\dot{M}$ (which is zero to our accuracy) by the mathematically equivalent but computationally simpler route of evaluating the buffer-region surface integrals (2.3). The remainder of this section is a demonstration that the surface integrals (2.3) give the claimed results: Eqs. (1.9b) and (1.9c) for $\dot{P}^j$ and $\dot{\mathscr{S}}^j$, and zero (to accuracy $ML/\mathscr{R}^2$) for $\dot{M}$.

To avoid unnecessary calculations we first list all possible final answers that we might get: All final answers must be constructed as products of at least one $\mathscr{E}_{a_1\cdots a_l}$ or $\mathscr{B}_{a_1\cdots a_l}$ or time derivative thereof with at least one M, $\mathscr{S}_j$, $\mathscr{I}_{a_1\cdots a_l}$, or $\mathscr{S}_{a_1\cdots a_l}$ or time derivative thereof; and the products can involve the Levi-Cività tensor ϵ_{ijk}. If we assume that $T\ll\mathscr{R}^2/M$ or $\mathscr{T}\ll\mathscr{R}^2/M$, the largest dimensionless scalars that can be so constructed are

$$\dot{M}\sim(\dot{\mathscr{I}}^{jk}\,\&\,\dot{\mathscr{S}}^{jk})(\mathscr{E}_{jk}\,\&\,\mathscr{B}_{jk})\,\&\,(\mathscr{I}^{jk}\,\&\,\mathscr{S}^{jk})(\dot{\mathscr{E}}_{jk}\,\&\,\dot{\mathscr{B}}_{jk}), \tag{3.12a}$$

and thus the final answer for $\dot{M}$ can involve only these terms plus corrections of higher order. Here "&" means "and a term of the form," as in Eq. (2.6). Similarly the largest dimensionless vectors that can be so constructed are

$$\dot{P}^j \sim (\mathscr{E}^j{}_k \,\&\, \mathscr{B}^j{}_k)\mathscr{S}^k \,, \tag{3.12b}$$

and thus the final answer for $\dot{P}^j$ can only involve these two terms plus higher-order corrections. Finally the largest vectors with dimension length that can be so constructed are

$$\dot{\mathscr{S}}^j \sim (\mathscr{E}^j{}_k \,\&\, \mathscr{B}^j{}_k)M\mathscr{S}^k \,\&\, \epsilon^j{}_{ab}(\mathscr{E}^a{}_c \,\&\, \mathscr{B}^a{}_c)(\mathscr{I}^{cb} \,\&\, \mathscr{S}^{cb}) \,, \tag{3.12c}$$

and thus the final answer for $\dot{\mathscr{S}}^j$ can only involve these terms plus higher-order corrections.

Some of the terms in (3.12a), (3.12b), and (3.12c) can be ruled out by parity and time-reversal considerations, which we describe in this paragraph in more detail than most readers will want: Under spatial reflections (for which $\dot{P}^i \to +\dot{P}^i$, $\mathscr{S}^i \to -\mathscr{S}^i$, $\dot{\mathscr{S}}^i \to -\dot{\mathscr{S}}^i$, $\mathscr{I}_{jk} \to +\mathscr{I}_{jk}$, $\mathscr{S}_{jk} \to -\mathscr{S}_{jk}$, $\mathscr{E}_{jk} \to \mathscr{E}_{jk}$, $\mathscr{B}_{jk} \to -\mathscr{B}_{jk}$, and $\epsilon_{ijk} \to -\epsilon_{ijk}$) the following terms do not transform correctly and are thus ruled out: $\dot{\mathscr{I}}^{jk}\mathscr{B}_{jk}$, $\mathscr{I}^{jk}\dot{\mathscr{B}}_{jk}$, $\dot{\mathscr{S}}^{jk}\mathscr{E}_{jk}$, and $\mathscr{S}^{jk}\dot{\mathscr{E}}_{jk}$ in (3.12a); $\mathscr{E}^j{}_k\mathscr{S}^k$ in (3.12b); and $\mathscr{B}^j{}_kM\mathscr{S}^k$, $\epsilon^j{}_{ab}\mathscr{E}^a{}_c\mathscr{S}^{cb}$, and $\epsilon^j{}_{ab}\mathscr{B}^a{}_c\mathscr{I}^{cb}$ in (3.12c). Under time reversal (for which $\dot{M} \to -\dot{M}$, $\dot{P}^i \to +\dot{P}^i$, $\dot{\mathscr{S}}^i \to +\dot{\mathscr{S}}^i$, $\mathscr{S}^i \to -\mathscr{S}^i$, $\mathscr{I}_{jk} \to +\mathscr{I}_{jk}$, $\mathscr{S}_{jk} \to -\mathscr{S}_{jk}$, $\mathscr{E}_{jk} \to +\mathscr{E}_{jk}$, $\mathscr{B}_{jk} \to -\mathscr{B}_{jk}$, and $\epsilon_{ijk} \to +\epsilon_{ijk}$) the following terms do not transform correctly and are thus ruled out: $\dot{\mathscr{I}}^{jk}\mathscr{B}_{jk}$, $\mathscr{I}^{jk}\dot{\mathscr{B}}_{jk}$, $\dot{\mathscr{S}}^{jk}\mathscr{E}_{jk}$, and $\mathscr{S}^{jk}\dot{\mathscr{E}}_{jk}$ in (3.12a); $\mathscr{E}^j{}_k\mathscr{S}^k$ in (3.12b); and $\mathscr{E}^j{}_kM\mathscr{S}^k$, $\epsilon^j{}_{ab}\mathscr{E}^a{}_c\mathscr{S}^{cb}$, and $\epsilon^j{}_{ab}\mathscr{B}^a{}_c\mathscr{I}^{cb}$ in (3.12c). (In the actual computations the parity-incorrect terms are removed by the angular integration of the surface integrals (2.3). Consider, for example, the term $\dot{\mathscr{S}}^j \sim \mathscr{B}^j{}_kM\mathscr{S}^k$. Because of the way that the Levi-Cività tensor ϵ_{ijk} enters into $\bar{h}^{\mu\nu}_B$ and $\bar{h}^{\mu\nu}_E$ [Eqs. (3.6) and (3.10)] and because there are no ϵ_{ijk}'s in the expressions for $t^{\mu\nu}_{\rm LL}$ and $W^{\mu\nu}$ in terms of $\bar{h}^{\mu\nu}$ [MTW Eq. (20.22), and Eq. (3.11c) above], any term $\dot{\mathscr{S}}^j \sim \mathscr{B}^j{}_kM\mathscr{S}^k$ must arise in Eq. (2.3c) from a $t^{jk}_{\rm LL}$ of the form

$$(-g)t^{jk}_{\rm LL} \sim \frac{M}{r^3}\epsilon_{\cdot a\cdot}\mathscr{S}_a\epsilon_{\cdot b\cdot}\mathscr{B}_b \,. \tag{3.13}$$

Here each subscript dot is a j or a k or is contracted into an n^l or is contracted into another subscript dot. When (3.13) is inserted into Eq. (2.3c) the resulting integrand will involve an odd number of n's and will thus integrate to zero.) (Similarly, in the actual computations, the time-reversal-incorrect terms are absent because they never show up at all in $(-g)t^{jk}_{\rm LL}$; considerations of temporal-index counting rule them out. Consider, for example, the term $\dot{\mathscr{S}}^j \sim \mathscr{E}^j{}_kM\mathscr{S}^k$: $\dot{\mathscr{S}}^j$ is generated, in the surface integral (2.3c), by $t^{ij}_{\rm LL}$ which has no temporal indices. All of the processes involved in computing $t^{ij}_{\rm LL}$ [iteration of the Einstein equations and deDonder gauge conditions (3.11) to get nonlinear terms in $\bar{h}^{\mu\nu}_B$ and $\bar{h}^{\mu\nu}_E$ from linear terms (Sec. IX of Ref. 12 and the Appendix of this paper); further iteration to get $\bar{h}^{\mu\nu}_I$ from $\bar{h}^{\mu\nu}_B$ and $\bar{h}^{\mu\nu}_E$, and computation of $t^{ij}_{\rm LL}$ from $\bar{h}^{\mu\nu}=\bar{h}^{\mu\nu}_B+\bar{h}^{\mu\nu}_E+\bar{h}^{\mu\nu}_I$] involve changes in the number of temporal indices by an even integer 2, 4, . . .; for example, $\bar{h}^{ij}_{I,pq}\delta^{pq}=\bar{h}^{ij}_{I,00}+W^{ij}$, and $\bar{t}^{ij}_{\rm LL} \sim -\frac{1}{8}\mathfrak{g}^{ij}\mathfrak{g}^{00}\mathfrak{g}_{00}\mathfrak{o}_{00}\bar{h}^{00}{}_{,0}\bar{h}^{00}{}_{,0}$. Thus $t^{ij}_{\rm LL}$ must arise ultimately from a product of one or more linearized $\bar{h}^{\mu\nu}_E$ with one or more linearized $\bar{h}^{\mu\nu}_B$, and that product must have an even number of temporal indices. However, $\mathscr{E}_{jk}$ originates in the linearized part of $\bar{h}^{00}_E$, M originates in the linearized $\bar{h}^{00}_B$, $\mathscr{S}_l$ originates in the linearized $\bar{h}^{0j}_B$, and the product of these has an odd number of temporal indices, five, by contrast with $t^{ij}_{\rm LL}$ which has an even number, zero. Thus, no $\mathscr{E}_{jk}M\mathscr{S}_l$ term can appear in $\bar{t}^{ij}_{\rm LL}$, and correspondingly none can appear in $\dot{\mathscr{S}}^j$.)

After eliminating from (3.12a), (3.12b), and (3.12c) the terms ruled out by parity and time-reversal invariance, we obtain for the possible forms of $\dot{M}$, $\dot{P}^j$, and $\dot{\mathscr{S}}^j$

$$\dot{M} \sim \dot{\mathscr{I}}^{jk}\mathscr{E}_{jk} \,\&\, \dot{\mathscr{S}}^{jk}\mathscr{B}_{jk} \,\&\, \mathscr{I}^{jk}\dot{\mathscr{E}}_{jk} \,\&\, \mathscr{S}^{jk}\dot{\mathscr{B}}_{jk} \,, \tag{3.14a}$$

$$\dot{P}^j \sim \mathscr{B}^j{}_k\mathscr{S}^k \,, \tag{3.14b}$$

$$\dot{\mathscr{S}}^j \sim \epsilon^j{}_{ab}\mathscr{E}^a{}_c\mathscr{I}^{cb} \,\&\, \epsilon^j{}_{ab}\mathscr{B}^a{}_c\mathscr{S}^{cb} \,. \tag{3.14c}$$

Zhang[23] has recently performed the surface integration (2.3a) to obtain the numerical coefficients of the four terms in $\dot{M}$ [Eq. (3.14a)]. His final answer, after averaging over time, is given by Eq. (1.15).

The possible final answers (3.14b) and (3.14c) for $\dot{P}^j$ and $\dot{\mathscr{S}}^j$ are linear in each of $\mathscr{E}_{jk}$, $\mathscr{B}_{jk}$, $\mathscr{S}_j$, $\mathscr{S}_{jk}$, and $\mathscr{I}_{jk}$. Thus, we only need those parts of $t^{ij}_{\rm LL}$ which are linear in these quantities, and they can be obtained from the linearized and truncated metric [Eqs. (3.6) and (3.10)]

$$\bar{h}^{00} = -4\psi \,, \quad \bar{h}^{0j} = -A_j \,, \quad \bar{h}^{ij} = 0 \,; \tag{3.15a}$$

$$\psi = -\frac{3}{2}\frac{\mathscr{I}_{jk}n^jn^k}{r^3} + \tfrac{1}{2}\mathscr{E}_{jk}x^jx^k \,; \tag{3.15b}$$

$$A_j = -2\frac{\epsilon_{jkl}\mathscr{S}^kn^l}{r^2} - 4\frac{\epsilon_{jkl}\mathscr{S}_{km}n^ln^m}{r^3} - \tfrac{2}{3}\epsilon_{jkl}\mathscr{B}^k{}_mx^lx^m \,, \tag{3.15c}$$

in which the moments are all regarded as time independent. Inserting Eq. (3.15a) into MTW Eq. (20.22) for $(-g)t^{ij}_{\rm LL}$ in terms of $\mathfrak{g}^{\alpha\beta}=\eta^{\alpha\beta}-\bar{h}^{\alpha\beta}$ and keeping only terms of quadratic order, we obtain the standard result for a time-independent field in the linearized approximation to general relativity:

$$(-g)t^{ij}_{\rm LL} = \frac{1}{16\pi}[4g^ig^j + H^iH^j - \tfrac{1}{2}\delta^{ij}(4\vec{\mathrm{g}}^2 + \vec{\mathrm{H}}^2)] \,, \tag{3.16a}$$

where $\vec{\mathrm{g}}$ is the Newtonian acceleration and $\vec{\mathrm{H}}$ is the gravitational analog of the magnetic field ("gravitomagnetic field")

$$\vec{\mathrm{g}} = -\vec{\nabla}\psi \,, \quad \vec{\mathrm{H}} = \vec{\nabla}\times\vec{\mathrm{A}} \,. \tag{3.16b}$$

(Here we use three-dimensional flat-space notation.)

By inserting expressions (3.15b) and (3.15c) for ψ and A_j into Eqs. (3.16) for the pseudotensor and performing

the integrations (2.3b) and (2.3c) we obtain the claimed results

$$\frac{dP^j}{dt} = -\mathscr{B}^j{}_a \mathscr{S}^a , \tag{3.17a}$$

$$\frac{d\mathscr{S}^j}{dt} = -\epsilon^j{}_{ab} \mathscr{I}^a{}_c \mathscr{E}^{cb} - \tfrac{4}{3} \epsilon^j{}_{ab} \mathscr{S}^a{}_c \mathscr{B}^{cb} \tag{3.17b}$$

[cf. Eqs. (1.9)].

F. Accuracy of definitions of M, P^j, and $\mathscr{S}^j$

We conclude this section with a discussion of the accuracy to which the body's M, P^i, and $\mathscr{S}^i$ are determined in the presence of the complicated external universe. As in Sec. II, M, P^i, and $\mathscr{S}^i$ can be determined by surface integrals of the form (2.2a), (2.2b), and (2.2c); and as in Sec. II the uncertainties in M, P^i, and $\mathscr{S}^i$ are of order the nonlinear contributions to those surface integrals, which result from the coupling of the body metric to the external-universe metric. Considerations of dimensionality, parity, time reversal, and lining up of indices tell us the forms of those differences [analogs of Eqs. (2.12)]:

$$(\text{uncertainty in } M) \sim \mathscr{E}_{ij} \mathscr{I}^{ij} \,\&\, \mathscr{B}_{ij} \mathscr{S}^{ij} = O\left(\frac{ML^2}{\mathscr{R}^2}\right) , \tag{3.18a}$$

$$(\text{uncertainty in } P^i) \sim \epsilon_{ijk} \mathscr{B}^{ja} \mathscr{I}_a{}^k \,\&\, \epsilon_{ijk} \mathscr{E}^{ja} \mathscr{S}_a{}^k = O\left(\frac{ML^2}{\mathscr{R}^2}\right) , \tag{3.18b}$$

$$(\text{uncertainty in } \mathscr{S}^i) \sim M^2 \mathscr{E}^{ij} \mathscr{S}_j = O\left(\frac{M^3 L}{\mathscr{R}^2}\right) . \tag{3.18c}$$

IV. EQUATIONS OF MOTION AND PRECESSION FOR A SYSTEM OF SEVERAL BODIES

When the "external universe" through which the body of interest moves is so massive that it is affected negligibly by the body's presence ("test body"), the conversion of the body's laws of motion and precession (1.11) into explicit equations of motion and precession is completely straightforward. The body's world line is a curve in the external universe described by the fixed $g^{(0)}$, and the laws of motion become differential equations for this curve. Not so straightforward are systems where the body's back action on the external universe is important. In this section we study, as an example of such a system, several bodies of comparable mass orbiting each other. In subsection A we use our single-body laws of motion and precession (1.11) as a foundation for deriving the several-body equations of motion and precession for this example, and in subsection B we specialize our several-body equations to a black-hole binary system, and thereby recover D'Eath's[8] equations of motion and precession.

A. Derivation of several-body equations of motion and precession

Consider a system of N bodies each with mass $\lesssim M$ and size $\lesssim L$, orbiting each other with separations $\gtrsim \mathscr{L}$ and velocities $\lesssim (M/\mathscr{L})^{1/2}$, where M, L, and $\mathscr{L}$ are constants that characterize the system and $M \ll \mathscr{L}$, $L \ll \mathscr{L}$. For such a system the equations of motion and precession can be expanded in powers of $M/\mathscr{L}$ ("post-Newtonian expansion"). When one of the bodies is a black hole, the coupling of its multipole moments to the external curvature will first show up at post$^{1.5}$-Newtonian order [fractional corrections $\sim (M/\mathscr{L})^{3/2}$ to Newtonian motion; fractional errors of post-post-Newtonian order, $(M/\mathscr{L})^2$]. For this reason we shall seek post$^{1.5}$-Newtonian accuracy when deriving the equations of motion and precession.

In order to simplify our analysis, we shall require that all the bodies in the system be sufficiently compact that

$$L \lesssim (M\mathscr{L})^{1/2} . \tag{4.1}$$

Then the spin-curvature "force" $-\mathscr{S}^a \mathscr{B}_a{}^j$ of Eq. (1.9b) [with $|\mathscr{B}_a{}^j| \lesssim (M/\mathscr{L}^3)(M/\mathscr{L})^{1/2}$ and $|\mathscr{S}^j| \lesssim ML$] and the quadrupole-curvature "force" $-\frac{1}{2} \mathscr{I}_{ab} \mathscr{E}^{abj}$ of Eq. (1.12) (with $|\mathscr{I}_{ab}| \lesssim ML^2$ and $|\mathscr{E}^{abj}| \lesssim M/\mathscr{L}^4$) can both be as large as post-Newtonian magnitude $M^3/\mathscr{L}^3$ (though they are post$^{1.5}$-Newtonian for a black hole). Thus, these $-\mathscr{S}^a \mathscr{B}_a{}^j$ and $-\frac{1}{2} \mathscr{I}_{ab} \mathscr{E}^{abj}$ forces must be included in our analysis. On the other hand, all higher-order moment-curvature coupling forces are of post-post-Newtonian magnitude or smaller and thus can be ignored. Similarly, the quadrupole-curvature torque $-\epsilon^j{}_{ab} \mathscr{I}^a{}_c \mathscr{E}^{cb}$ of Eq. (1.9c) can be as large as post-Newtonian magnitude $M^3/\mathscr{L}^2$ (post$^{1.5}$-Newtonian for a black hole) and must be included, but all other moment-curvature coupling torques are of post-post-Newtonian magnitude or smaller and can be ignored. Thus, in our analysis we must keep the effects of each body's mass M, angular momentum $\mathscr{S}_j$, and mass quadrupole moment $\mathscr{I}_{jk}$, but we can ignore all other moments.

The single-body laws of motion and precession (1.9) and (1.12) are only one ingredient in the derivation of the several-body equations of motion and precession for this system. A second key ingredient is a spacetime metric for the "external region," which joins the buffer zones of the bodies to each other. In order to obtain post$^{1.5}$-Newtonian equations of motion and precession, we shall need an external metric $g^{(0)}$ that satisfies the Einstein field equations to post$^{1.5}$-Newtonian order.

As is well known,[4] an approximate several-body metric cannot satisfy the Einstein field equation to post^n-Newtonian order unless the world lines that appear in it satisfy the post$^{(n-1)}$-Newtonian equations of motion.[39] Thus, what we really need for $g^{(0)}$ is not a true post$^{1.5}$-Newtonian metric, but rather something we might call an "incipient" post$^{1.5}$-Newtonian metric: a metric in which appear the world lines of the bodies of our system, and which has the property that it will come to satisfy the full post$^{1.5}$-Newtonian Einstein equations as soon as the world lines are constrained to satisfy the equations of motion.

Einstein, Infeld, and Hoffman[20] (EIH) have given an in-

cipient, post$^{1.5}$-Newtonian external metric (errors of post^2-Newtonian order) for a system of bodies with masses but no spins or quadrupole moments [see Eq. (39.63) of MTW]. It is straightforward to graft onto that incipient metric terms due to the spins and quadrupole moments of the bodies, since assumption (4.1) makes those terms be no larger than post-Newtonian magnitude and thus be devoid of nonlinearities at the desired post$^{1.5}$-Newtonian level of accuracy. After this grafting the EIH incipient metric takes the form

$$g_{jk}=\delta_{jk}\left[1+2\sum_A \frac{M_A}{r_A}\right]+O\left[\frac{M^2}{\mathscr{L}^2}\right], \tag{4.2a}$$

$$g_{0j}=-4\sum_A \frac{M_A}{r_A}v_{Aj}-2\sum_A \frac{\epsilon_{jkl}\mathscr{S}_A{}^k n_A{}^l}{r_A{}^2}+O\left[\frac{M^{5/2}}{\mathscr{L}^{5/2}}\right], \tag{4.2b}$$

$$\begin{aligned} g_{00}=&-1+2\sum_A \frac{M_A}{r_A}-2\left[\sum_A \frac{M_A}{r_A}\right]^2+3\sum_A \frac{M_A v_A{}^2}{r_A}\\ &-2\sum_A\sum_{B\neq A}\frac{M_A M_B}{r_A r_{AB}}+\frac{\partial^2}{\partial t^2}\sum_A M_A r_A\\ &+3\sum_A \frac{\mathscr{I}_{Ajk}n_A{}^j n_A{}^k}{r_A{}^3}\\ &+4\sum_A \frac{\epsilon_{jkl}v_A{}^j\mathscr{S}_A{}^k n_A{}^l}{r_A{}^2}+O\left[\frac{M^3}{\mathscr{L}^3}\right]. \end{aligned} \tag{4.2c}$$

Here capital letters label the bodies; M_A, $\mathscr{S}_A{}^j$, and $\mathscr{I}_A{}^{jk}$ are the mass and the spatial coordinate components of the angular momentum and quadrupole moment of body A; the center of attraction of the gravitational field of body A at time t has spatial coordinates $x_A{}^j(t)$, i.e., $x_A{}^j(t)$ is the world line of body A; and we define

$$\begin{aligned} &v_A{}^j\equiv\frac{d}{dt}x_A{}^j, \quad v_{AB}{}^j\equiv v_A{}^j-v_B{}^j,\\ &n_A{}^j\equiv\frac{x^j-x_A{}^j}{r_A}, \quad r_A\equiv[\delta_{jk}(x^j-x_A{}^j)(x^k-x_A{}^k)]^{1/2}, \\ &n_{AB}{}^j\equiv\frac{x_A{}^j-x_B{}^j}{r_{AB}}, \quad r_{AB}\equiv[\delta_{jk}(x_A{}^j-x_B{}^j)(x_A{}^k-x_B{}^k)]^{1/2}. \end{aligned} \tag{4.3}$$

The equations of motion for the bodies determine the world lines $x_A{}^j(t)$ of their centers of attraction in the external metric; the equations of precession together with constitutive relations [e.g., Eqs. (1.7) for a black hole and Ref. 22 for a rigidly rotating neutron star] determine the evolution of the spins $\mathscr{S}_A{}^j(t)$ and quadrupole moments $\mathscr{I}_A{}^{jk}(t)$; and the equations of motion and precession convert the incipient metric (4.2) into a full solution of the Einstein equations, accurate to post$^{1.5}$-Newtonian order.

To derive the equations of motion and precession for body K from the laws of motion and precession (1.11a), (1.11b), and (1.12) we must first identify the "external-universe metric" $g^{(0)K}_{\mu\nu}$ seen by body K in its buffer region. The most accurate candidate for that metric is the full EIH incipient metric (4.2) with all terms removed that are divergent as $r_K\to 0$ (i.e., with all "body-K terms" removed):

$$g^{(0)K}_{jk}=\delta_{jk}\left[1+2\sum_{A\neq K}\frac{M_A}{r_A}\right]+O\left[\frac{M^2}{\mathscr{L}^2}\right], \tag{4.4a}$$

$$g^{(0)K}_{0j}=-4\sum_{A\neq K}\frac{M_A}{r_A}v_{Aj}-2\sum_{A\neq K}\frac{\epsilon_{jkl}\mathscr{S}_A{}^k n_A{}^l}{r_A{}^2}+O\left[\frac{M^{5/2}}{\mathscr{L}^{5/2}}\right], \tag{4.4b}$$

$$\begin{aligned} g^{(0)K}_{00}=&-1+2\sum_{A\neq K}\frac{M_A}{r_A}-2\left[\sum_{A\neq K}\frac{M_A}{r_A}\right]^2+3\sum_{A\neq K}\frac{M_A v_A{}^2}{r_A}-2\sum_{A\neq K}\sum_{B\neq A}\frac{M_A M_B}{r_A r_{AB}}+\frac{\partial^2}{\partial t^2}\sum_{A\neq K}M_A r_A\\ &+3\sum_{A\neq K}\frac{\mathscr{I}_{Ajk}n_A{}^j n_A{}^k}{r_A{}^3}+4\sum_{A\neq K}\frac{\epsilon_{jkl}v_A{}^j\mathscr{S}_A{}^k n_A{}^l}{r_A{}^2}+O\left[\frac{M^6}{\mathscr{L}^6}\right]. \end{aligned} \tag{4.4c}$$

A less accurate candidate is the EIH incipient metric with M_K set to zero:

$$g'^{(0)K}_{jk}=g^{(0)K}_{jk}, \quad g'^{(0)K}_{0j}=g^{(0)K}_{0j}, \quad g'^{(0)K}_{00}=g^{(0)K}_{00}+2\sum_{A\neq K}\frac{M_A M_K}{r_A r_{AK}}. \tag{4.5}$$

Note that these two candidates differ by a term $2\sum_{A\neq K}M_A M_K/r_A r_{AK}$ which, when expanded in powers of r_K, has the form

$$g'^{(0)K}_{00}-g^{(0)K}_{00}=2\sum_{A\neq K}\frac{M_A M_K}{r_{AK}{}^2}+\left[2\sum_{A\neq K}\frac{M_A}{r_{AK}{}^3}n^j_{AK}n^j_K\right]M_K r_K+\left[2\sum_{A\neq K}\frac{M_A}{r_{AK}{}^4}(3n^i_{AK}n^j_{AK}n^i_K n^j_K-1)\right]M_K r_K{}^2+\cdots. \tag{4.6}$$

If we were to transform to a reference frame in the buffer region of body K that falls freely in the external metric (the "local asymptotic rest frame" of Secs. II and III), the first two terms on the right-hand side would disappear, while the third would remain unchanged. Note that this remaining third term has the form $Mr^2/\mathcal{R}^3$, in the notation of the tableau (2.6) [where $M=M_K$, $r=r_K$, and $\mathcal{R}\sim(r_{AK}{}^3/M_A)^{1/2}$ is regarded formally as the same as $\mathcal{L}\sim r_{AK}$]. As is discussed in the next to the last paragraph of Sec. II B, this $Mr^2/\mathcal{R}^3$ term can be moved into and out of the external-universe metric $g^{(0)}$ without invalidating the laws of motion derived in Secs. II and III. However, the accuracy of the resulting equations of motion for body K will depend crucially on whether this $Mr^2/\mathcal{R}^3$ term is included in $g^{(0)}$ or not. Failing to include it, i.e., using $g'^{(0)K}_{\mu\nu}$ (Eq. 4.5) for $g^{(0)}$, would produce "test-body equations of motion" which ignore the back action of body K on the other bodies of the system. Including it, as we shall, i.e., using $g^{(0)K}_{\mu\nu}$ [Eq. (4.4)] for $g^{(0)}$ will produce the full post$^{1.5}$-Newtonian equations of motion, including the effects of back action.

Having thus identified expressions (4.4) as the $g^{(0)}$ for body K, we can now convert the laws of motion and precession for body K into equations of motion and precession. The laws are given by Eqs. (1.11) augmented by the frame-independent version of (1.12):

$$P_K{}^{\alpha}{}_{|\beta}u_K{}^{\beta}=-\mathscr{B}_K{}^{\alpha\beta}\mathscr{S}_{K\beta}-\tfrac{1}{2}\mathscr{E}_K{}^{\alpha}{}_{\beta\gamma}\mathscr{I}_K{}^{\beta\gamma}\,, \tag{4.7a}$$

$$\mathscr{S}_{K\alpha|\beta}u_K{}^{\beta}=-\epsilon_{\mu\alpha\beta\gamma}\mathscr{I}_K{}^{\beta\delta}\mathscr{E}_{K\delta}{}^{\gamma}u_K{}^{\mu}\,. \tag{4.7b}$$

Here $u_K{}^{\beta}$ is the four-velocity of the world line $x_K{}^{j}(t)$ in the external metric $g^{(0)K}_{\mu\nu}$ seen by K; $P_K{}^{\alpha}=M_K u_K{}^{\alpha}$ is the four-momentum of K in that metric; "$|$" denotes the covariant derivative in that metric; and $\mathscr{E}_{K\alpha\beta}$, $\mathscr{B}_{K\alpha\beta}$, and $\mathscr{E}_{K\alpha\beta\gamma}$ are expressed in terms of the Riemann curvature tensor $R_{K\alpha\mu\beta\nu}$ of $g^{(0)K}_{\mu\nu}$ by the frame-independent versions of Eqs. (1.4a) and (1.13a) evaluated on the world line of K:

$$\mathscr{E}_{K\alpha\beta}=R_{K\alpha\mu\beta\nu}u_K{}^{\mu}u_K{}^{\nu}\,, \tag{4.8a}$$

$$\mathscr{B}_{K\alpha\beta}=\tfrac{1}{2}\epsilon_{\mu\alpha\rho\sigma}R_K{}^{\rho\sigma}{}_{\beta\nu}u_K{}^{\mu}u_K{}^{\nu}\,, \tag{4.8b}$$

$$\mathscr{E}_{K\alpha\beta\gamma}=[R_{K\alpha\mu\beta\nu|\sigma}u_K{}^{\mu}u_K{}^{\nu}(\delta^{\sigma}{}_{\gamma}+u_K{}^{\sigma}u_{K\gamma})]^{S}\,. \tag{4.8c}$$

Here the superscript S denotes symmetrization on all free indices, $\alpha\beta\gamma$. Note that the fact that $\mathscr{S}_K{}^{\alpha}$ and $\mathscr{I}_K{}^{\alpha\beta}$ are purely spatial in the asymptotic rest frame of K, $\mathscr{S}_K{}^{\alpha}u_{K\alpha}=\mathscr{I}_K{}^{\alpha\beta}u_{K\beta}=0$, implies that to post$^{1.5}$-Newtonian accuracy

$$\mathscr{S}_K{}^{0}=\mathscr{S}_K{}^{j}v_{Kj}\,, \tag{4.9a}$$

$$\mathscr{I}_K{}^{i0}=\mathscr{I}_K{}^{ij}v_{Kj}\,,\quad \mathscr{I}_K{}^{00}=\mathscr{I}_K{}^{ij}v_{Ki}v_{Kj}\,, \tag{4.9b}$$

and similarly for $\mathscr{E}_{K\alpha\beta}$, $\mathscr{B}_{K\alpha\beta}$, and $\mathscr{E}_{K\alpha\beta\gamma}$.

It is now straightforward, though moderately tedious, to evaluate the single-body equation of motion (4.7a) for our several-body system. The result is

$$M_K\frac{d^2\vec{x}_K}{dt^2}=\vec{F}_K{}^{(\mathrm{EIH})}+\vec{F}_K{}^{(\mathrm{Q})}+\vec{F}_K{}^{(\mathrm{SO})}+\vec{F}_K{}^{(\mathrm{SS})}\,. \tag{4.10}$$

Here $\vec{F}_K{}^{(\mathrm{EIH})}$ is the "force" term of Einstein, Infeld, and Hoffman[20] [Eq. (39.64) of MTW]

$$\vec{F}_K{}^{(\mathrm{EIH})}=\sum_{A\neq K}\frac{M_AM_K}{r_{AK}{}^2}\vec{n}_{AK}\left[1-4\sum_{B\neq K}\frac{M_B}{r_{BK}}-\sum_{C\neq A}\frac{M_C}{r_{CA}}\left[1-\frac{1}{2}\frac{r_{AK}}{r_{CA}}\vec{n}_{AK}\cdot\vec{n}_{CA}\right]+\vec{v}_K{}^2+2\vec{v}_A{}^2-4\vec{v}_A\cdot\vec{v}_K-\tfrac{3}{2}(\vec{v}_A\cdot\vec{n}_{AK})^2\right]$$
$$-\sum_{A\neq K}\vec{v}_{AK}\frac{M_KM_A}{r_{AK}{}^2}\vec{n}_{AK}\cdot(3\vec{v}_A-4\vec{v}_K)+\tfrac{7}{2}\sum_{A\neq K}\sum_{C\neq A}\vec{n}_{CA}\frac{M_KM_AM_C}{r_{AK}r_{CA}{}^2}\,. \tag{4.11a}$$

$\vec{F}_K{}^{(\mathrm{Q})}$ is the quadrupolar force term

$$\vec{F}_K{}^{(\mathrm{Q})}=\sum_{A\neq K}\frac{M_K}{r_{AK}{}^4}\left[3\overleftrightarrow{\mathscr{I}}_A\cdot\vec{n}_{KA}-\tfrac{15}{2}\vec{n}_{KA}(\vec{n}_{KA}\cdot\overleftrightarrow{\mathscr{I}}_A\cdot\vec{n}_{KA})\right]+\sum_{A\neq K}\frac{M_A}{r_{AK}{}^4}\left[3\overleftrightarrow{\mathscr{I}}_K\cdot\vec{n}_{KA}-\tfrac{15}{2}\vec{n}_{KA}(\vec{n}_{KA}\cdot\overleftrightarrow{\mathscr{I}}_K\cdot\vec{n}_{KA})\right]\,, \tag{4.11b}$$

with the first summation arising from geodesic motion of body K in the quadrupolar part of the external metric (4.4c) and the second arising from the nongeodesic quadrupole-curvature coupling force (4.7a). $\vec{F}_K{}^{(\mathrm{SO})}$ is the "spin-orbit" force

$$\vec{F}_K{}^{(\mathrm{SO})}=\sum_{A\neq K}\frac{M_K}{r_{AK}{}^3}[6\vec{n}_{KA}(\vec{\mathscr{S}}_A\times\vec{n}_{KA}\cdot\vec{v}_{KA})+4\vec{\mathscr{S}}_A\times\vec{v}_{KA}-6(\vec{\mathscr{S}}_A\times\vec{n}_{KA})(\vec{v}_{KA}\cdot\vec{n}_{KA})]$$
$$+\sum_{A\neq K}\frac{M_A}{r_{AK}{}^3}[6\vec{n}_{KA}(\vec{\mathscr{S}}_K\times\vec{n}_{KA}\cdot\vec{v}_{KA})+3\vec{\mathscr{S}}_K\times\vec{v}_{KA}-3(\vec{\mathscr{S}}_K\times\vec{n}_{KA})(\vec{v}_{KA}\cdot\vec{n}_{KA})]\,, \tag{4.11c}$$

with the first summation arising from geodesic motion of body K in the spin part of the external metric (4.4b) and (4.4c) and the second summation arising from the nongeodesic spin-curvature coupling force (4.7a). Finally, $\vec{F}_K{}^{(\mathrm{SS})}$ is the "spin-spin" force

$$\vec{F}_K{}^{(\mathrm{SS})}=\sum_{A\neq K}\frac{1}{r_{AK}{}^4}[-3\vec{n}_{KA}(\vec{\mathscr{S}}_K\cdot\vec{\mathscr{S}}_A)-3\vec{\mathscr{S}}_K(\vec{n}_{KA}\cdot\vec{\mathscr{S}}_A)-3\vec{\mathscr{S}}_A(\vec{n}_{KA}\cdot\vec{\mathscr{S}}_K)+15\vec{n}_{KA}(\vec{n}_{KA}\cdot\vec{\mathscr{S}}_K)(\vec{n}_{KA}\cdot\vec{\mathscr{S}}_A)]\,, \tag{4.11d}$$

which arises entirely from the nongeodesic spin-curvature coupling force (4.7a). Throughout Eqs. (4.10) and (4.11) the notation is that of flat-space, three-dimensional vector analysis with each vector (e.g., $d^2\vec{x}_K/dt^2$, $\vec{n}_{KA}$, $\vec{\mathscr{S}}_K$) representing contravariant components in the metric (4.4) or (4.2) (e.g., $d^2x_K{}^j/dt^2$, $n_{KA}{}^j$, $\mathscr{S}_K{}^j$).

All of the "forces" (4.11) have been derived previously in one context or another, e.g., $\vec{F}_K{}^{(\mathrm{EIH})}$ by Einstein, Infeld, and Hoffman[20] for binary systems with weakly gravitating bodies [their Eq. (17.2); MTW Eq. (39.64)]; $\vec{F}_K{}^{(\mathrm{Q})}$ by authors lost in antiquity for Newtonian systems and by Barker and O'Connell[21] for post-Newtonian binary systems with weakly gravitating bodies [their Eqs. (55) and (56)]; $\vec{F}_K{}^{(\mathrm{SO})}$ by Damour[9] for binary systems with strongly gravitating bodies [his Eq. (3)] and by Barker and O'Connell[21] for binary systems with weakly gravitating bodies [their Eqs. (52) and (53), which differ from our $\vec{F}_K{}^{(\mathrm{SO})}$ because of a different "spin supplementary condition"]; and $\vec{F}_K{}^{(\mathrm{SS})}$ by Barker and O'Connell[21] for weakly gravitating binaries [their Eq. (54)].

Turn next to the precession law (4.7b). In order to bring it into a simple form we shall study *not* the evolution of the coordinate components $\mathscr{S}_K{}^j$ of $\vec{\mathscr{S}}_K$ in the coordinates of (4.4), but rather the evolution of the components $\mathscr{S}_K{}^{\hat{j}}$ on the orthonormal spatial basis vectors $\vec{e}_{\hat{j}}$ of the local asymptotic rest frame of body K:

$$\vec{e}_{\hat{j}}=v_K{}^j\frac{\partial}{\partial t}+\left[1-\sum_{A\neq K}\frac{M_A}{r_{AK}}\right]\frac{\partial}{\partial x^j}+\tfrac{1}{2}v_K{}^jv_K{}^l\frac{\partial}{\partial x^l}\,. \tag{4.12}$$

This is the procedure followed in Sec. 40.7 of MTW for a spinning test particle moving in a post-Newtonian gravitational field. Note that $\mathscr{S}_K{}^{\hat{j}}$ and $\mathscr{S}_K{}^j$ differ only by fractional corrections of order $M/\mathscr{L}$:

$$\mathscr{S}_K{}^{\hat{j}}=\left[1+\sum_{A\neq K}\frac{M_A}{r_{AK}}\right]\mathscr{S}_K{}^j-\tfrac{1}{2}v_K{}^jv_K{}^l\mathscr{S}_{Kl}+O\left(\frac{M^2}{\mathscr{L}^2}\mathscr{S}_K{}^l\right)\,; \tag{4.13}$$

and since such fractional corrections are negligible in the equations of motion (4.10) and (4.11) and in the metrics (4.2) and (4.4), we are free to regard the spins $\mathscr{S}_K{}^j$ and $\vec{\mathscr{S}}_K$ which appear there as actually equal to $\mathscr{S}_K{}^{\hat{j}}$.

By following the computational procedure of Sec. 40.7 of MTW one can fairly easily bring the law of precession (4.7b), for $\mathscr{S}_K{}^{\hat{j}}$ in the metric (4.4) of the "external universe as seen by body K," into the following form:

$$\frac{d\vec{\mathscr{S}}_K}{dt}=(-\tfrac{1}{2}\vec{H}_K+\tfrac{3}{2}\vec{v}_K\times\vec{g}_K)\times\vec{\mathscr{S}}_K-(\epsilon_{iab}\mathscr{S}_K{}^a{}_c\mathscr{E}_K{}^{cb})\vec{e}_{\hat{i}}\,. \tag{4.14}$$

Here, as in the equations of motion (4.10) and (4.11), the notation is that of flat-space vector analysis but with $\vec{\mathscr{S}}_K$ having components $\mathscr{S}_K{}^{\hat{j}}$ rather than $\mathscr{S}_K{}^j$. The quantities $\vec{H}_K$ and $\vec{g}_K$ are the "gravitomagnetic field" and Newtonian acceleration at K's location in its "external metric"

$$H_K{}^i=\epsilon_{iab}g_{0b,a}^{(0)K}\,,\quad \vec{g}_K=\vec{\nabla}(\tfrac{1}{2}g_{00}^{(0)K})\,. \tag{4.15}$$

A straightforward evaluation of Eq. (4.14) using (4.15) and (4.4) gives

$$\frac{d\vec{\mathscr{S}}_K}{dt}=\left[\vec{\Omega}_K^{(\mathrm{GM})}+\vec{\Omega}_K^{(\mathrm{geod})}\right]\times\vec{\mathscr{S}}_K+\vec{N}^{(\mathrm{Q})}\,. \tag{4.16}$$

Here $\vec{\Omega}_K^{(\mathrm{GM})}$ is the angular velocity of "gravitomagnetic" (or "Lens-Thirring") precession

$$\vec{\Omega}_K^{(\mathrm{GM})}=\sum_{A\neq K}\frac{1}{r_{AK}{}^3}[-\vec{\mathscr{S}}_A+3\vec{n}_{KA}(\vec{n}_{KA}\cdot\vec{\mathscr{S}}_A)]\,, \tag{4.17a}$$

which arises entirely from the gravitomagnetic term $\vec{H}_K$ of Eq. (4.14). $\vec{\Omega}_K^{(\mathrm{geod})}$ is the angular velocity of "geodetic precession":

$$\vec{\Omega}_K^{(\mathrm{geod})}=\sum_{A\neq K}\frac{M_A}{r_{AK}{}^2}(2\vec{v}_A-\tfrac{3}{2}\vec{v}_K)\times\vec{n}_{KA}\,, \tag{4.17b}$$

the first term of which arises from $\vec{H}_K$ and the second from $\frac{3}{2}\vec{v}_K\times\vec{g}_K$ in Eq. (4.14). Finally, $\vec{N}^{(\mathrm{Q})}$ is the quadrupole-curvature torque

$$\vec{N}^{(\mathrm{Q})}=-\sum_{A\neq K}3\frac{M_A}{r_{AK}{}^3}\vec{n}_{KA}\times\overleftrightarrow{\mathscr{I}}_K\cdot\vec{n}_{KA}\,, \tag{4.17c}$$

which arises from the last term of Eq. (4.14).

The quadrupole-curvature torque (4.17c) is well known from Newtonian theory, where it produces the general precession of the Earth's spin axis (MTW, Exercise 16.4). The gravitomagnetic and geodetic precessions are well known for a test body orbiting a massive companion (e.g., Papapetrou[19]) and for a binary system made of weakly gravitating bodies [e.g., Barker and O'Connell,[21] Eq. (41)].

B. Specialization to a black-hole binary system

When specialized to a binary system made of two black holes (labeled "K" and "A"), the equations of motion and precession (4.10), (4.11), (4.16), and (4.17) reduce to a form which agrees with that of D'Eath.[8] The EIH "force" (4.10a) takes the form

$$\vec{F}_K{}^{(\mathrm{EIH})}=\frac{M_AM_K}{r_{AK}{}^2}\vec{n}_{AK}\left[1-\frac{4M_A+5M_K}{r_{AK}}+\vec{v}_K{}^2+2\vec{v}_A{}^2-4\vec{v}_K\cdot\vec{v}_A-\tfrac{3}{2}(\vec{v}_A\cdot\vec{n}_{AK})^2\right]+\vec{v}_{KA}\frac{M_AM_K}{r_{AK}{}^2}\vec{n}_{AK}\cdot(3\vec{v}_A-4\vec{v}_K) \tag{4.18a}$$

[D'Eath,[8] Eq. (7.1) with the factor $(3\mu_1)^2$ corrected to read $3\mu_1{}^2$; EIH,[20] Eq. (17.2)]. Because of the very small quadrupole moment of a black hole (Eq. 1.7), the quadrupole force (4.11b) is of post-post-Newtonian magnitude and thus is below the accuracy of our analysis:

$$\vec{F}_K^{(Q)} = O\left[\frac{M^4}{\mathscr{L}^4}\right] . \tag{4.18b}$$

The spin-orbit force (4.11c), by virtue of Eq. (1.7), is of post$^{1.5}$-Newtonian magnitude and takes the form

$$\vec{F}_K^{(SO)} = \frac{M_A{}^2 M_K \chi_A}{r_{AK}{}^3}[6\vec{n}_{KA}(\vec{s}_A\times\vec{n}_{KA}\cdot\vec{v}_{KA})+4\vec{s}_A\times\vec{v}_{KA}-6(\vec{s}_A\times\vec{n}_{KA})(\vec{v}_{KA}\cdot\vec{n}_{KA})]$$
$$+\frac{M_A M_K{}^2 \chi_K}{r_{AK}{}^3}[6\vec{n}_{KA}(\vec{s}_K\times\vec{n}_{KA}\cdot\vec{v}_{KA})+3\vec{s}_K\times\vec{v}_{KA}-3(\vec{s}_K\times\vec{n}_{KA})(\vec{v}_{KA}\cdot\vec{n}_{KA})] \tag{4.18c}$$

[D'Eath, Eq. (6.7)]. The spin-spin force is of post-post-Newtonian magnitude and is thus below the accuracy of our analysis

$$\vec{F}_K^{(SS)} = O\left[\frac{M^4}{\mathscr{L}^4}\right] . \tag{4.18d}$$

The gravitomagnetic precession (4.17a) is of post$^{1.5}$-Newtonian magnitude and takes the form

$$\vec{\Omega}_K^{(GM)} = \frac{M_A{}^2 \chi_A}{r_{AK}{}^3}[-\vec{s}_A+3\vec{n}_{KA}(\vec{n}_{KA}\cdot\vec{s}_A)] \tag{4.19a}$$

[D'Eath, Eq. (6.6)]. The geodetic precession (4.17b) is of post-Newtonian magnitude and takes the form

$$\vec{\Omega}_K^{(geod)} = \frac{M_A}{r_{AK}{}^2}(2\vec{v}_A-\tfrac{3}{2}\vec{v}_K)\times\vec{n}_{KA} \tag{4.19b}$$

[D'Eath, Eq. (6.6)]. Finally, the quadrupole-curvature torque (4.17c) is of post$^{1.5}$-Newtonian magnitude and takes the standard precession form (Eq. 1.9b′)

$$\vec{N}_K^{(Q)} = \vec{\Omega}_K^{(T)}\times\vec{\mathscr{S}}_K , \tag{4.19c}$$

where $\vec{\Omega}_K^{(T)}$ is the angular velocity of torqued precession

$$\vec{\Omega}_K^{(T)} = \frac{3M_A M_K \chi_K}{r_{AK}{}^3}\vec{n}_{KA}(\vec{n}_{KA}\cdot\vec{s}_K) \tag{4.19d}$$

[D'Eath, Eq. (6.6)].

Notice (as Damour has pointed out to us) that in the limit where hole K is infinitesimal compared to hole A, the deviations from geodesic motion due to spin-curvature coupling [the second piece of (Eq. 4.18c)] are of magnitude

$$-\mathscr{S}_K{}^a\mathscr{B}_{Ka}{}^j \sim \left[\frac{M_K}{\mathscr{L}}\right]\frac{M_K M_A}{\mathscr{L}^2}\left[\frac{M_A}{\mathscr{L}}\right]^{1/2} .$$

This is of order the errors that one would make if one were to compute the small hole's motion ignoring its back action on the large hole ("test-body approximation"; use of $g'^{(0)K}_{\mu\nu}$ [Eq. (4.5)] for the external metric rather than $g^{(0)K}_{\mu\nu}$ [Eq. (4.4)]). Thus, for a small black hole (by contrast with small-mass bodies with larger sizes and spins), the test-body approximation must restrict attention to geodesic motion and neglect the spin-curvature coupling force.

V. DISCUSSION: THE PRECESSION OF BLACK HOLES IN THE REAL UNIVERSE

We conclude this paper with a discussion of the laws of motion and precession (1.11a) and (1.11b) for black holes in astrophysical contexts. We do not know of any way that the gravitational force $P^\alpha{}_{|\beta}u^\beta = -\mathscr{B}^{\alpha\beta}\mathscr{S}_\beta$ on a black hole might be detected astronomically in the foreseeable future. However, the tidal torque $\mathscr{S}_{\alpha|\beta}u^\beta = -\epsilon_{\mu\alpha\beta\gamma}\mathscr{I}^{\beta\delta}\mathscr{E}_\delta{}^\gamma u^\mu$ [Eq. (1.11b)] might conceivably be detected by its influence on the shapes of jets that emerge from the nuclei of some galaxies and quasars:[40] In several fashionable models for jet production, a supermassive black hole is at the jets' origin and the directions of the two jets are tied to the hole's rotation axis, so if the hole precesses, the jets develop an S-like shape. Jets with such shapes are observed, in fact, by radio astronomers; and the length scales of their S-shapes are $\sim 10^3$–10^7 light years corresponding to time scales for precession of $\sim 10^3$–10^9 years (depending on the speeds of the jets).[40]

For a Kerr black hole, with mass M, rotation parameter χ, and spin direction $\vec{s}$, the tidal torque (1.11b) takes the form (1.9c′):

$$\frac{d\vec{\mathscr{S}}}{dt} = \vec{\Omega}_T\times\vec{S} , \quad \Omega_T{}^i = -\mathscr{E}^i{}_j M\chi s^j . \tag{5.1}$$

Thus, the hole precesses with a "torqued angular velocity" $\vec{\Omega}_T$ that is proportional to its rotation parameter χ and to the external electric-type curvature. The time scale $1/|\vec{\Omega}_T|$ for the torqued precession can be in the observed range (though not easily), as the following example shows.

Consider a supermassive hole at the center of a dense elliptical star cluster. In order of magnitude $\mathscr{E}_{jk}$ will be $\alpha G\rho$, where α is a dimensionless measure of the deviations of the cluster from sphericity ($\alpha\simeq 1$ for a pancake-shaped cluster), G is Newton's gravitation constant (set to unity elsewhere in this paper), and ρ is the mean mass density in the cluster. The resulting torqued precession rate will be

$$\Omega_T \sim \alpha G\rho M\chi = \alpha\left[\frac{1}{4\times 10^7\ \text{yr}}\right]\left[\frac{\rho}{10^9 M_\odot/\text{ly}^3}\right]\left[\frac{M}{10^9 M_\odot}\right]\chi . \tag{5.2}$$

Thus, for a very massive hole surrounded by a very dense and rather nonspherical cluster, the torqued precession could be interestingly large.[41] However, for more typical situations it will be negligible.

One should keep in mind that a black hole can precess relative to the "distant stars" (but not relative to inertial frames in its buffer region) even when its torqued angular velocity vanishes: When the hole lives in a complicated external universe, pure Fermi-Walker transport of the hole's spin (zero torque) can produce geodetic precession [second of the three terms in Eq. (4.14)] and gravitomagnetic precession [first term of Eq. (4.14)]. For example, if the hole is in a circular orbit of radius b around an external body of mass $M_E \gg M$, its orbital motion gives rise to the geodetic precession of its spin, $d\vec{\mathscr{S}}/dt = \vec{\Omega}_{\rm geod} \times \vec{\mathscr{S}}$, relative to inertial frames far from the system (at $r \gg b$). Here

$$\vec{\Omega}_{\rm geod} = \frac{3M_E}{2b}\left(\frac{M_E}{b^3}\right)^{1/2}\vec{\mathrm{n}}\ , \tag{5.3}$$

[Eq. (4.19b)], where $\vec{\mathrm{n}}$ is the normal to the orbit. Similarly, the spin angular momentum $\vec{J}_E$ of the external body gives rise to gravitomagnetic precession with

$$\vec{\Omega}_{\rm GM} = \frac{(3\vec{J}_E\cdot\hat{r})\hat{r} - \vec{J}_E}{b^3}\ , \tag{5.4}$$

[Eq. (4.19a)], where $\hat{r}$ is the unit radial vector from the external body to the hole. For comparison, the dominant tidal field $\mathscr{E}_{jk}$ of the external body is

$$\mathscr{E}_{jk} = \frac{M_E}{b^3}(-3\hat{r}_j\hat{r}_k + \delta_{jk}); \tag{5.5}$$

and this tidal field produces a torqued precession of the hole relative to its local asymptotic inertial frame ($M_{\rm hole} \ll |\vec{\mathrm{r}} - \vec{\mathrm{r}}_{\rm hole}| \ll b$) and also relative to distant inertial frames ($r \gg b$) with angular velocity [Eqs. (5.1) and (4.19d)]

$$\vec{\Omega}_T = -3\chi\frac{MM_E}{b^3}\cos\theta\hat{r}\ . \tag{5.6}$$

Here θ is the angle between the hole's spin direction $\vec{\mathrm{s}}$ and the radial direction $\hat{r}$. The relative magnitudes of these three precessions, "torqued," "gravitomagnetic," and "geodetic," are

$$|\vec{\Omega}_T| : |\vec{\Omega}_{\rm GM}| : |\vec{\Omega}_{\rm geod}| \simeq \chi\left(\frac{M}{M_E}\frac{M}{b}\right)^{1/2} : \left(\frac{R_E}{b}\right)^{1/2} : 1\ , \tag{5.7}$$

where R_E is the external body's radius, and we have assumed that the body is rotating as rapidly as possible (centrifugal force of order self-gravity force) so as to make the gravitomagnetic precession as large as possible. Note that for this orbital situation $\Omega_T \ll \Omega_{\rm GM} \lesssim \Omega_{\rm geod}$.

This conclusion, that torqued precession is negligible compared to geodetic and gravitomagnetic, may seem surprising to someone imbued with a Newtonian viewpoint. After all, torqued precession is a Newtonian effect, while the geodetic and gravitomagnetic precessions are post-Newtonian. However, for a black hole the torqued precession is strongly suppressed (to post$^{1.5}$-Newtonian magnitude) by the small size of the hole and the consequent smallness of the hole's quadrupole moment. Begelman, Blandford, and Rees,[42] being well aware of this, have identified geodetic precession as the astrophysically most important precession that a black hole can undergo, and have suggested it as the most likely cause for the S-shaped distortions of jets in galactic nuclei and quasars.

It is only when geodetic and gravitomagnetic precession are strongly suppressed below their values for binary systems that the tidal torque $d\mathscr{S}^j/dt = -\epsilon^j{}_{ab}\mathscr{I}^{ac}\mathscr{E}_c{}^b$ will be the dominant source of precession. An example is a hole at rest at the center of an elliptical star cluster which has negligible angular momentum [Eq. (5.2) above].

As small as the tidally torqued precession may be in practice, it is enormously large compared to the tidally-induced spin-down of a black hole.[43] The spin-down was of great importance historically because it gave us new insight into the behaviors of black-hole horizons. In the "membrane viewpoint" on black holes,[44] which grew out of that insight, an external gravitational field $\mathscr{E}_{jk}$ raises a tide on the hole's horizon, and as the hole rotates the resulting rate of shear of the horizon works against the horizon's viscosity to dissipate rotational energy. The rates of loss of rotational energy and angular momentum, in order of magnitude, are

$$\frac{dM}{dt} = \vec{\Omega}_H\cdot\frac{d\vec{\mathscr{S}}}{dt} \sim (M^3\Omega_H\mathscr{E}_{jk})^2\ , \tag{5.8}$$

where $\vec{\Omega}_H$ is the hole's angular velocity, which has magnitude of order χ/M. Note that the spin-down rate, $\Omega_{\rm SD} \sim |\vec{\mathscr{S}}|^{-1}|d\vec{\mathscr{S}}/dt|$, has magnitude

$$\Omega_{\rm SD} \sim M^3/\mathscr{R}^4\ . \tag{5.9}$$

Thus, it is smaller than the torqued precession rate by

$$\frac{\Omega_{\rm SD}}{\Omega_T} \sim \frac{1}{\chi}\frac{M^2}{\mathscr{R}^2}\ . \tag{5.10}$$

For the extreme star cluster of Eq. (4.2) (highly nonspherical with mean star density 10^9 solar masses per cubic light year) and a black hole of 10^9 solar masses, $\Omega_{\rm SD}/\Omega_T \sim 10^{-12}\chi^{-1}$. Thus, for any reasonable amount of rotation ($\chi \gg 10^{-12}$), the tidally torqued precession is far stronger than the spin-down.

The tidally torqued precession was overlooked in the black-hole analyses of the 1970's (Ref. 43) because those analyses focused primarily on the physics of the horizon, and the dragging of inertial frames is so strong near the horizon that changes in the direction of the angular-momentum vector relative to the inertial frames of the buffer region do not show up there.[45] On the other hand, analyses in the buffer region (the method of this paper) are capable of revealing both the torqued precession and the spin-down. The spin-down will show up in the buffer region as a flow of angular momentum due to the nonlinear interaction of the external field $\mathscr{E}_{jk}$ with that part of the hole's quadrupole moment $\mathscr{I}_{jk}$ which arises from

tidal distortion of the horizon by the external field. Although we have not carried through the details of such a calculation, a somewhat similar calculation was performed by Press,[42] for spin-down of a hole interacting with an external scalar field, shortly before horizon techniques were devised by Hawking and Hartle[42] for computing spin-down.

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APPENDIX

Thorne (Sec. IX of Ref. 14) has developed an algorithm for constructing the metric density $\mathfrak{g}^{\alpha\beta}$ (and from it the metric $g_{\alpha\beta}$) in the weak-gravity near zone ($M \ll r$, $L \ll r$, $r \ll T$) outside any body that lives alone in asymptotically flat spacetime. That algorithm constructs $\mathfrak{g}^{\alpha\beta}$ as deDonder-gauge power-series expansions in M/r, L/r, and r/T with expansion coefficients that are the body's mass moments $\mathscr{I}_{a_1\cdots a_l}$ and current moments $\mathscr{J}_{a_1\cdots a_l}$ and their time derivatives; cf. Eq. (3.6), which is not carried to high enough order to exhibit the time derivatives.

This appendix sketches an analogous algorithm for constructing the metric density $\mathfrak{g}^{\alpha\beta}$ in the vacuum neighborhood ($r \ll \mathscr{R}$, $r \ll \mathscr{L}$, $r \ll \mathscr{T}$) of a timelike geodesic in an arbitrary spacetime.

As in Ref. 14 our algorithm will take as its starting point the general solution to the stationary, linearized, vacuum Einstein equations and deDonder gauge conditions [the linearized, time-independent limit of Eqs. (3.2) and (3.11)]:

$$\mathfrak{g}^{\alpha\beta}=\eta^{\alpha\beta}-\bar{h}^{\alpha\beta}_{\rm LS}\ ,\quad \bar{h}^{\alpha\beta}_{{\rm LS},jk}\delta^{jk}=0\ ,\quad \bar{h}^{\alpha j}_{{\rm LS},j}=0\ . \tag{A1}$$

In Ref. 14 the starting solution was required to be well behaved as $r\to\infty$. Here it must be well behaved as $r\to 0$; and, in fact, because $r=0$ is a timelike geodesic, it can be chosen to have $\bar{h}^{\alpha\beta}_{\rm LS}\sim r^2$ near $r=0$ (see, e.g., Sec. 8.6 of MTW). The general solution to (A1) with $\bar{h}^{\alpha\beta}_{\rm LS}\sim r^2$ can be brought into the following form by a careful adjustment of gauge:

$$\bar{h}^{00}_{\rm LS}=-\sum_{l=2}^{\infty}\frac{4}{l(l-1)}\mathscr{E}_{A_l}X_{A_l}\ , \tag{A2a}$$

$$\bar{h}^{0j}_{\rm LS}=-\sum_{l=2}^{\infty}\frac{2}{3(l-1)}\epsilon_{jpq}\mathscr{B}_{qA_{l-1}}x_pX_{A_{l-1}}\ , \tag{A2b}$$

$$\bar{h}^{ij}_{\rm LS}=0\ . \tag{A2c}$$

Here the notation is that of Ref. 14 (Sec. I C): $\mathscr{E}_{A_l}$ and $\mathscr{B}_{A_l}$ are constant STF tensors with the subscript A_l a shorthand notation for $a_1a_2\cdots a_l$; $x_j\equiv x^j$; $X_{A_l}\equiv x_{a_1}x_{a_2}\cdots x_{a_l}$; and the indices are treated as though the coordinates were precisely Cartesian (down indices are equivalent to up indices and repeated indices are summed). [This general linearized stationary solution can be derived, e.g., (i) by writing down the general solution $\bar{h}^{\mu\nu}_{\rm LS}$ of Laplace's equation, well behaved at $r=0$, in STF notation with ten independent families of moments; (ii) by then imposing the deDonder gauge conditions to get rid of four families of moments; and (iii) by then performing gauge transformations with generators ξ_μ that are solutions of Laplace's equation to get rid of four more families of moments.] Our normalization of the moments is designed to make the linearized, stationary Riemann tensor look nice:

$$R_{j0k0}=\sum_{l=2}^{\infty}\mathscr{E}_{jkA_{l-2}}X_{A_{l-2}}\ , \tag{A3a}$$

$$R_{ijk0}=\sum_{l=2}^{\infty}[\epsilon_{ijq}\mathscr{B}_{qkA_{l-2}}X_{A_{l-2}} + \tfrac{2}{3}(l-2)\epsilon_{pq[i}\mathscr{B}_{j]qkA_{l-3}}x_pX_{A_{l-3}}]\ , \tag{A3b}$$

$$R_{ipjq}=\sum_{l=2}^{\infty}(\delta_{ij}\mathscr{E}_{pqA_{l-2}}+\delta_{pq}\mathscr{E}_{ijA_{l-2}} -\delta_{iq}\mathscr{E}_{jpA_{l-2}}-\delta_{jp}\mathscr{E}_{iqA_{l-2}})X_{A_{l-2}}\ . \tag{A3c}$$

Note that the moments $\mathscr{E}_{A_l}$ determine the "electric part" of the Riemann tensor R_{i0j0} and the moments $\mathscr{B}_{A_l}$ determine the "magnetic part" R_{ijk0}; thus $\mathscr{E}_{A_l}$ are called "electric-type moments" and $\mathscr{B}_{A_l}$ are called "magnetic-type." Alternatively, by analogy with the external field of an isolated body, $\mathscr{E}_{A_l}$ are the "mass moments" and $\mathscr{B}_{A_l}$ the "current moments" of the Riemann tensor.

The general solution of the fully nonlinear, dynamical, vacuum Einstein equations and deDonder gauge conditions [Eqs. (3.2) and (3.11)] can be constructed from the linearized, stationary starting solution (A2) by iteration. In the iteration one regards the moments $\mathscr{E}_{A_l}$ and $\mathscr{B}_{A_l}$ as having the magnitudes and time derivatives (3.8) (with "$\sim$" rather than "$\lesssim$"); and one expands $\bar{h}^{\mu\nu}$ in powers of $r/\mathscr{R}$ ("nonlinearity expansion"), $r/\mathscr{L}$ ("distance-from-origin" expansion), and $r/\mathscr{T}$ ("time-derivative expansion"), and in spherical harmonics:

$$\bar{h}^{\mu\nu}=\sum_{pnul}\gamma^{\mu\nu}_{pnul}\ . \tag{A4a}$$

Here

$$\gamma^{\mu\nu}_{pnul}\sim\left[\left[\frac{r}{\mathscr{R}}\right]^{2p}\left[\frac{r}{\mathscr{L}}\right]^{n}\left[\frac{r}{\mathscr{T}}\right]^{u}\right]_{\text{order } l}\ , \tag{A4b}$$

and l is the spherical-harmonic order. In terms of this expansion, the Einstein equations and deDonder gauge conditions read

$$\nabla^2\gamma^{\mu\nu}_{pnul}=\frac{\partial^2}{\partial t^2}\gamma^{\mu\nu}_{pn(u-2)l}+w^{\mu\nu}_{pnul}\ , \tag{A5a}$$

$$\frac{\partial}{\partial x^j}\gamma^{\mu j}_{pnul}=-\frac{\partial}{\partial t}\gamma^{\mu 0}_{pn(u-1)l}\ . \tag{A5b}$$

Here $w^{\mu\nu}_{pnul}$ is the "*pnul*" part of the nonlinear field $W^{\mu\nu}$ (3.11c); it is constructed entirely from terms of lower order than "*pnu*." Because all quantities appearing on the right-hand sides of Eqs. (A5) are of lower order in *pnu* than quantities on the left, these equations form the foundation for an iterative solution for $\bar{h}^{\mu\nu}$.

The starting point for the iteration is the linearized, stationary solution (A2):

$$\gamma^{00}_{1(l-2)0l} = -\frac{4}{l(l-1)} \mathscr{E}_{A_l} X_{A_l} \quad \text{for } l \geq 2 , \tag{A6a}$$

$$\gamma^{0j}_{1(l-2)0l} = -\frac{2}{3(l-1)} \epsilon_{jpq} \mathscr{B}_{qA_{l-1}} x_p X_{A_{l-1}} \quad \text{for } l \geq 2 , \tag{A6b}$$

$$\text{all other } \gamma^{\mu\nu}_{1n0l} = 0 , \tag{A6c}$$

with $\mathscr{E}_{A_l}$ and $\mathscr{B}_{A_l}$ now regarded as slowly varying functions of time.

The metric density $\mathfrak{g}^{\mu\nu} \equiv \eta^{\mu\nu} - \bar{h}^{\mu\nu}$ that results from this iterative algorithm will be a power series in radius r. The first few terms in this power series (too few to show the effects of nonlinearities) are

$$\mathfrak{g}^{00} = -1 + 2\mathscr{E}_{jk} x^j x^k + \tfrac{2}{3} \mathscr{E}_{jkl} x^j x^k x^l + O(r^4) , \tag{A7a}$$

$$\mathfrak{g}^{0j} = \tfrac{2}{3} \epsilon^j{}_{kl} \mathscr{B}^l{}_m x^k x^m + O(r^3) , \tag{A7b}$$

$$\mathfrak{g}^{ij} = \delta^{ij} + O(r^3) . \tag{A7c}$$

The electric and magnetic parts of the corresponding Riemann tensor are

$$R_{i0j0} = \mathscr{E}_{ij} + \mathscr{E}_{ijk} x^k + \tfrac{1}{3} (\epsilon_{jpq} \dot{\mathscr{B}}^q{}_i + \epsilon_{ipq} \dot{\mathscr{B}}^q{}_j) x^p + O(r^2) , \tag{A8a}$$

$$R_{ijk0} = \epsilon_{ijl} \mathscr{B}^l{}_k + O(r) ; \tag{A8b}$$

and thus $\mathscr{E}_{ij}$, $\mathscr{E}_{ijk}$, and $\mathscr{B}_{jk}$ can be expressed as follows in terms of the Riemann tensor and its covariant gradient, evaluated at $r = 0$:

$$\mathscr{E}_{ij} = R_{i0j0} , \tag{A9a}$$

$$\mathscr{E}_{ijk} = (R_{i0j0|k})^S \equiv \tfrac{1}{3} (R_{i0j0|k} + R_{j0k0|i} + R_{k0i0|j}) , \tag{A9b}$$

$$\mathscr{B}_{ij} = \tfrac{1}{2} \epsilon_i{}^{pq} R_{pqj0} . \tag{A9c}$$

Xiao-He Zhang at Caltech is currently studying the mathematical details, structure, and consequences of the iterative algorithm sketched in this appendix.

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[27] Recall from the discussion following Eq. (1.14) that the $\dot{M}$ of Eq. (2.4a) will not produce a physically meaningful change in M (i.e., a change larger than its uncertainty) unless the external time scale $\mathscr{T}$ is of order the time scale T for changes in the hole's moments, i.e., for changes in χ or s^i. In writing down Eq. (2.4a) we assumed that χ and s^i were constants, i.e., $T = \infty$; so in this approximation physical meaningfulness requires $\mathscr{T} = T = \infty$ which implies $\dot{\mathscr{E}}_{ij} = 0$ and $\dot{M} = 0$. In actuality s^i will change due to precession on a time scale $T \sim \mathscr{R}^2/M$ [Eq. (1.10) with $L \sim M$], so expression (2.4a) and analogous terms with $\dot{\mathscr{S}}^i$ present will produce a meaningful $\dot{M}$ of magnitude $\dot{M} \sim M^3/\mathscr{R}^2$ (T and $\mathscr{T}$) $\sim M^4/\mathscr{R}^4$.

[28] R. Penrose (private communication).

[29]Such logarithmic terms appear in the deDonder-gauge near-zone expansion of the metric of a dynamically evolving body in asymptotically flat spacetime (see Sec. IX F of Ref. 14), but they do not appear when the body is stationary [for a flawed proof see Eq. (10.4b) of Ref. 14; for a correct proof see T. Damour and L. Blanchet (unpublished)], and they do not appear in the deDonder-gauge expansion of the external-universe metric $g^{(0)}$ [see X.-H. Zhang (unpublished)].

[30]For the Kerr metric expanded in powers of $1/r$ in rectangular Boyer-Lindquist coordinates see Eq. (2.5) of D'Eath, Ref. 5.

[31]For a discussion of the quadrupole terms see Sec. XI D of Ref. 14.

[32]Reference 14, Eq. (11.26), with some change of notation.

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[34]For example, by replacing pseudotensor arguments by ones using Penrose's quasilocal quantities? See R. Penrose, Proc. R. Soc. London **A381**, 53 (1982).

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